Dynamical Systems Theory in Machine Learning & Data Science

Lecturers: Daniel Durstewitz
Tutors: Manuel Brenner, Janik Fechtelpeter, Max Thurm, Lukas Eisenmann, Florian Hess
WS2022/23

Exercise 3

To be uploaded before the exercise group on November 16, 2022

1 Linearization Fail

Note that for any $a \in \mathbb{R}$ the following system has a fixed point at $(0, 0)$:

$$\dot{x} = -y + (ax - y)(x^2 + y^2)$$
$$\dot{y} = x + (x + ay)(x^2 + y^2)$$

1. Show that the linearization technique predicts (0, 0) to be a center.
2. To prove that it is not always a true center, transfer the system into polar coordinates. This means that you have to find equivalent differential equations for $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$
3. Find $a \in \mathbb{R}$ such that $(0, 0)$ is a stable spiral, unstable spiral, or center.
4. For each of these cases, find the stable set Ω^s and the unstable set Ω^u.

2 Cycles

A cycle or closed orbit is a trajectory $x(t)$ in a dynamical system such that there are $t_2 > t_1$ with $x(t_2) = x(t_1)$, and for all $t \in (t_1, t_2)$, $x(t) \neq x(t_1)$. A center is a fixed point such that all trajectories sufficiently close to it are cycles.
For each of the following cases, decide if the system either has no cycle or at least one. Explain.

1. Consider the system:
$$\dot{x} = y, \quad \dot{y} = -2x^3$$
Hint: show that the origin is a linear center, and that $E(x, y) = x^4 + y^2$ is constant in time (i.e. along trajectories). Use both to show that the origin is a true center.
2. The system
$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$
has a fixed point $x^\star = (0, 0)$, which the linearization technique predicts to be a center. Let $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$ for all $x, y \in \mathbb{R}$. Such a system is called reversible.
3. There exists a continuously differentiable function $V : \mathbb{R}^2 \to \mathbb{R}$ such that (a) $\frac{dV}{dt} \equiv 0$, and (b) the system can be written as $\dot{x} = -\nabla V$.
4. Assume $\dot{x} = f(x)$ in the $\mathbb{R}^2$ space has no fixed point in the set $C = \{x \in \mathbb{R}^2 | 1 \leq ||x|| \leq 2\}$, and that there exists a trajectory x such that for all $t \in \mathbb{R}$, $x(t) \in C$ (For this task, an intuitive explanation suffices. A rigorous one follows in the next lecture).

3 Example Systems

For each of the following conditions, provide an example of a dynamical system defined in $\mathbb{R}^2$ which fulfills them (with proof that it does, the proof may be graphical). The examples have to be nontrivial, i.e. no derivative can be 0 everywhere. Use either cartesian, polar, or cylinder coordinates. If you think that no such system exists, explain why.

1. A stable fixed point at $(-1, 1)$ and an unstable fixed point at $(1, 1)$. No other fixed points.
2. A linear system with a fixed point other than the origin, and no line attractor (a.k.a. non-isolated fixed point).
3. Two saddle nodes, a stable and an unstable node. The system may have more fixed points.
4. A half-stable "saddle" cycle around the origin.
5. A conservative system with two cycles (hint: think of a physical system which has a conserved quantity).
6. Four saddle nodes at $(\pm\sqrt{2}, \pm\sqrt{2})$ and a half-stable cycle with radius 2 around $(0, \sqrt{2})$. No other fixed points.
7. Infinitely many fixed points, but no line attractor.