Dynamical Systems Theory in Machine Learning & Data Science

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Exercise 3

To be uploaded before the exercise group on November 17, 2021

1 Limit Cycles

You don't always need to use polar coordinates to find limit cycles. Consider the system

$$\dot{x} = -x + ay + x^2 y, \quad \dot{y} = b - ay - x^2 y$$

where $a, b > 0$. This is a model of glycolysis, the transformation of sugar (y) to ADP (x), which fuels cells. This is why we only consider $x, y \geq 0$ too. Show that this system has a limit cycle, such that the sugar transformation "comes in waves".

1. Find equations for the nullclines and draw a phase portrait. For each of the subregions of $\mathbb{R}^2$ that are created by the nullclines, determine the signs of the flow.
2. Find conditions on a, b such that the (single) fixed point is unstable.
3. Let C be the closed set bordered by the pentagon defined by (1) the x axis, (2) the y axis, (3) $y = b/a$, (4) $y = b/a + b - x$ and (5) $x = z$ where z is defined as the intersection of $y = b/a + b - x$ and the x nullcline (no need to determine this value exactly, it's ugly). Let Γ be all trajectories going through the border of C at time t. Show that for all $\epsilon > 0, \gamma \in \Gamma : \gamma(t + \epsilon) \in int(C)$ (there are no trajectories moving out of C and none moving along its boundary).
4. Argue why, if the fixed point is unstable, there must be an attracting limit cycle.

2 FitzHugh-Nagumo Model

This model as defined in the lecture imitates the generation of neuronal action potentials:

$$\dot{v} = v - \frac{1}{3}v^3 - w + I, \quad \tau\dot{w} = v - a - bw$$

with $a \in \mathbb{R}, b \in (0, 1), I \geq 0, \tau \gg 1$. v mimics the membrane potential and w a recovery outward current.

1. Write down equations for the nullclines for $a = 0, b = 0.5$. Calculate the fixed point and classify its stability as a function of the input current I.
2. Make a plot with the nullclines for $I = 0, \tau = 10$, and a trajectory starting from any initial conditions except the origin. To plot the trajectory, use a numerical integrator (e.g. in Python `scipy.integrate.solve_ivp`). Please attach the code.

3. On your plot, you should see a limit cycle, which partly follows the v nullcline. Make an educated guess about the time the trajectory flows along the nullcline compared to the time it spends apart from it, e.g. $t_{nullcline} \approx O(xyz)$.

4. Numerically ascertain that the cycle vanishes for some $a < 0$, but returns if you increase I. Find a condition for a and I that guarantees the existence or non-existence of the cycle.