Convergence analysis for non-compatible fracture flow discretizations

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Abstract

Simulation of the underground water flow in real 3D domains requires also modeling of the flow in 2D fractures and their 1D intersections. For each dimension we consider the continuity equation and Darcy's law as a model of the stationary saturated flow. The water flux between individual dimensions is assumed to be proportional to the pressure difference. The classical discretization schemes requires the alignment of computational meshes between dimensions. We present a mixed-hybrid formulation of the problem and two approximations of the communication terms that relax the alignment condition. We perform convergence analysis of these approximations.

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1. Introduction

In this work we shall make an analysis of mixed-hybrid formulations of a linear elliptic equations on several domains of different dimension. For the sake of clarity we present our ideas on the simple 2D-1D case, but we shall prove abstract results that can be used also for general cases. Let us consider a 2D domain $\Omega_2 \subset \mathbf{R}^2$ splitted into two subdomains by a 1D fracture $\Omega_1 \subset \Omega_2$. We denote $\tilde{\Omega}_2 = \Omega_2 \setminus \Omega_1$ and $\tilde{\Omega}_1 = \Omega_1$. To avoid technical difficulties we assume that Ω_2 have polygonal boundary and Ω_1 is a straight line. The flow on the domain Ω_d ($d = 1, 2$) is described by the velocity $\boldsymbol{u}_d$ and the pressure p_d. These state variables has to satisfy Darcy's law

$$\boldsymbol{u}_d = -\mathbb{K}_d \nabla p_d \quad \text{on } \tilde{\Omega}_d \tag{1}$$

and the continuity equation

$$\mathrm{div} \boldsymbol{u}_d = F_d \quad \text{on } \tilde{\Omega}_d, \tag{2}$$

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where $\mathbb{K}_d$ is (tensor of) the hydraulic conductivity, $F_2 = f_2$ and $F_1 = f_1 + q$ are water sources, while q denotes the outflow from 2D domain. We consider a "non-separating" crack, which means that the pressure is continuous across the crack and the sum of outflow from the walls of the fracture is equal to the fracture inflow, namely

$$\begin{aligned} p_2^+ = p_2^- \quad \text{on } \Omega_1, \\ [\boldsymbol{u}_2]_{\Omega_1} := (\boldsymbol{u}_2^+ \cdot \boldsymbol{n}^+ + \boldsymbol{u}_2^- \cdot \boldsymbol{n}^-) = q. \end{aligned}$$

Since the pressure is continuous we can prescribe

$$q = \sigma(p_2|_{\Omega_1} - p_1), \tag{3}$$

where σ is the water transfer coefficient, through our analysis we $\sigma = 1$. The system is completed by the boundary conditions

$$\begin{aligned} p_d = p^D \quad \text{on } \Gamma_d^D, \\ \boldsymbol{u}_d \cdot \boldsymbol{n} = u^N \quad \text{on } \Gamma_d^N. \end{aligned}$$

where open set Γ_d^D is the Dirichlet part of the boundary and $\Gamma_d^N = \partial\Omega_d \setminus \Gamma_d^D$ is the Neumann part of the boundary.
A direct discretization of the presented problem leads to a compatibility condition for the meshing of individual domains, namely triangles of Ω_2 should be aligned with Ω_1. The main reason is presence of the trace of the pressure p_2 in (3). While this is not very restrictive in the case of few domains, it should be intractable in the case of hundreds or thousends of interacting domains. To relax the compatibility condition we propose to approximate the trace of p_2 by a suitable linear operator $T(p_2)$. In Section 2 we define abstract mixed-hybrid formulations of the original problem and of the problem with approximated trace. We shall show the existence and uniqueness of the solution and we will prove an abstract error estimate. In Section **??** we present several applications of our abstract results for particular choice of the trace approximating operator T.

2. Abstract mixed-hybrid formulations

Let $\mathcal{P} = \{\Omega_d^i\}$, $i \in I_d$ be a decomposition of domain Ω_d into disjoint subdomains that satisfy the alignment condition

$$\Omega_1 \subset \Gamma_2 \setminus \partial\Omega_2, \qquad \text{where } \Gamma_d := \bigcup_{i \in I_d} \partial\Omega_d^i \tag{4}$$

is union of subdomain boundaries. Assuming that all indeces are unique, we denote a common index set $I = I_1 \cup I_2$. On the decomposition $\mathcal{P}$, we introduce the velocity space

$$V = V_2 \times V_1 = \prod_{i \in I_2} H(\mathrm{div}, \Omega_2^i) \times \prod_{i \in I_1} H(\mathrm{div}, \Omega_1^i). \tag{5}$$

Further we introduce the space of the pressure and its trace on $\Gamma_d \setminus \Gamma_d^D$

$$Q = Q_2 \times Q_1 \times \mathring{Q}_2 \times \mathring{Q}_1, \tag{6}$$
$$Q_d = L^2(\Omega_d), \quad \mathring{Q}_d = \{\varphi \in H^{1/2}(\Gamma_d) \,|\, \varphi = 0 \text{ on } \Gamma_d^D\}.$$

For the components of $\boldsymbol{u} \in V$ and $p \in Q$, we will use notation $\boldsymbol{u} = (\boldsymbol{u}_2, \boldsymbol{u}_1)$ and $p = (p_2, p_1, \mathring{p}_2, \mathring{p}_1)$ respectively. On these spaces we shall define mixed-hybrid solution similarly as in [3] or [1], but using the language of the book [2] by Brezzi and Fortin.

Definition 2.1. *We say that the pair $(\boldsymbol{u}, p) \in V \times Q$ is a solution of problem $P(\mathcal{P})$ on the partitioning $\mathcal{P}$ if it satisfies a saddle point problem*

$$a(\boldsymbol{u}, \boldsymbol{w}) + b(\boldsymbol{w}, p) = \langle G, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in V, \tag{7}$$
$$b(\boldsymbol{u}, q) - c(p, q) = \langle F, q \rangle \qquad \forall q \in Q, \tag{8}$$

where the bilinear forms on the left-hand side are

$$a(\boldsymbol{u}, \boldsymbol{w}) = \sum_{d=1,2} \sum_{i \in I_d} \int_{\Omega_d^i} \boldsymbol{u}_d \mathbb{K}_d^{-1} \boldsymbol{w}_d,$$
$$b(\boldsymbol{u}, q) = \sum_{d=1,2} \sum_{i \in I_d} \left(\int_{\Omega_d^i} -\mathrm{div} \boldsymbol{u}_d q_d + \int_{\partial \Omega_d^i} (\boldsymbol{u}_d \cdot \boldsymbol{n}) \mathring{q}_d \right),$$
$$c(p, q) = \int_{\Omega_1} (p_1 - \mathring{p}_2)(q_1 - \mathring{q}_2),$$

and the linear functionals on the right-hand side have the form

$$\langle G, \boldsymbol{w} \rangle = -\sum_{d=1,2} \sum_{i \in I_d} \int_{\partial \Omega_d^i} (\boldsymbol{w} \cdot \boldsymbol{n}) \tilde{P}_d,$$
$$\langle F, q \rangle = -\sum_{d=1,2} \int_{\Omega_d} f_d q_d + \sum_{d=1,2} \int_{\Gamma_d^N} u_d^N \mathring{q}_d,$$

where $\tilde{P}_d \in H^{1/2}(\Gamma_d)$ is an extension of the Dirichlet condition $P_d \in H^{1/2}(\Gamma_d^D)$. Consequently the full trace of the unknown pressure on Γ_d is given by $\mathring{p}_d + \tilde{P}_d$.

Note that in definition of bilinear form $c(p, q)$, we need trace $\mathring{p}_2$ on Ω_1, which is available because of the compatibility condition (4). In order to relax this condition, we shall define MH-solution to a problem $P_T(\mathcal{P})$ with interdomain flux:

$$q_T = T(p_2, \mathring{p}_2) - p_1,$$

where trace of the pressure is approximated by a linear operator T:

$$T : Q_2 \times \mathring{Q}_2 \to L^2(\Omega_1), \tag{9}$$

which is continuous and maps constants to constants, i.e.

$$\text{if} \quad p_2 = \mathring{p}_2 = \pi, \ \pi \in \mathbf{R} \quad \text{then} \quad T(p_2, \mathring{p}_2) = \pi. \tag{10}$$

We will also write Tp as an abbreviated form of $T(p_2, \mathring{p}_2)$.

Definition 2.2. *We say that the pair* $(\boldsymbol{u}, p) \in V \times Q$ *is a solution of problem* $P_T(\mathcal{P})$ *if it satisfies a saddle point problem*

$$a(\boldsymbol{u}, \boldsymbol{w}) + b(\boldsymbol{w}, p) = \langle G, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in V, \tag{11}$$

$$b(\boldsymbol{u}, q) - c_T(p, q) = \langle F, q \rangle \qquad \forall q \in P, \tag{12}$$

where

$$c_T(p, q) = \int_{\Omega_1} (p_1 - T(p_2, \mathring{p}_2))(q_1 - T(q_2, \mathring{q}_2))$$

and the other forms are same as in Definition 2.1.

Remark 2.3. *Even if we consider Problem* $P_T(\mathcal{P})$ *as an approximation of Problem* $P(\mathcal{P})$*. It is in fact generalization of Problem* $P(\mathcal{P})$*, since the later one is a particular case of the first one with* $T(p_2, \mathring{p}_2) = \mathring{p}_2$*.*

Remark 2.4. *Further we can observe, that Definition 2.2 does not change if we test only by* q *from* H^1*. More specificaly it is enough to test by* $q_d = \varphi$*,* $\mathring{q}_d = Tr(\varphi)$*, where*

$$\varphi \in H_d := \{ f \in H^1(\Omega_d) \,|\, f = 0 \text{ on } \Gamma_d^D \}. \tag{13}$$

Indeed for any $q_d \in L^2(\Omega_d)$ one can use standard molifiers and cut-off functions to construct a sequence

$$f_n \in \prod_{i \in I_d} H^1_0(\Omega^i_d), \quad f_n \longrightarrow q_d \text{ in } L^2(\Omega_d).$$

Similarly for any $\mathring{q}_d \in \mathring{P}_d$, there exists $g \in H_d$ with trace on Γ equal to $\mathring{q}_d$. Multiplying it by cut-off functions concentrating on Γ, we can find a sequence $g_n \in H_d$ with common trace $\mathrm{Tr}_\Gamma(g_n) = \mathring{q}_d$ and converging to zero in $L^2(\Omega_d)$. For the sum of these sequences we have

$$\big(f_n + g_n, \mathrm{Tr}_\Gamma(f_n + g_n)\big) \longrightarrow (q_d, \mathring{q}_d) \quad \text{ in } Q_d \times \mathring{Q}_d.$$

Our next goal is to show existence and uniqueness of the solution of Problem P_T. To this end we shall use following abstract existence result:

Theorem 2.5. *[2, Theorem 1.2] Assume that* $a(\,\cdot\,,\,\cdot\,)$*,* $b(\,\cdot\,,\,\cdot\,)$*,* $c(\,\cdot\,,\,\cdot\,)$ *are continuous bilinear forms on* $V \times V$*, on* $V \times Q$*, and on* $Q \times Q$ *respectively. Assume further that* $a(\,\cdot\,,\,\cdot\,)$ *is coercive on* V*, assume that operator*

$$B : V \to Q', \quad \langle B(\boldsymbol{u}), q \rangle = b(\boldsymbol{u}, q)$$

is closed in Q'*, and assume that* $c(\,\cdot\,,\,\cdot\,)$ *is possitive definite, symmetric, and coercive on* $Ker B^t$*. Then for every* $g \in V'$ *and every* $f \in Q'$ *the system*

$$a(\boldsymbol{u}, \boldsymbol{w}) + b(\boldsymbol{w}, p) = \langle g, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in V, \tag{14}$$

$$b(\boldsymbol{u}, q) - c(p, q) = \langle f, q \rangle \qquad \forall q \in Q, \tag{15}$$

has a solution $(\boldsymbol{u}, p)$ *which is unique in* $V \times Q/M$*, where*

$$M = \operatorname{Ker} B^t \cap \operatorname{Ker} C; \quad \langle C(p), \cdot \rangle := c(p, \cdot).$$

Moreover we have the bound

$$\|\boldsymbol{u}\|_V + \|p\|_{Q/\operatorname{Ker} B^t} \le K(\|f\|_{V'} + \|g\|_{Q'}), \tag{16}$$

where K *is a nonlinear function of* $\|a\|$*, of* $\|c\|$*, and of the coercivity constants for* a*,* c*. The function* K *is bounded on bounded subsets.*

To apply this theorem, we have to check the necessary properties for the particular forms a, b, c_T, F, and G. The form $a(\cdot, \cdot)$ is continuous and coercive if and only if $\mathbb{K}_d(\cdot)$ is bounded and uniformly positive definite a.e. The closeness of B is more delicate. Let us assume that there is at least one non-empty Dirichlet boundary Γ_d^D. Without loss of generality let $\Gamma_2^D \neq \emptyset$ and $\Gamma_1^D = \emptyset$. We shall prove the reprezentation

$$\operatorname{Im} B = M := \{\varphi \in Q' \,|\, \langle \varphi_1, 1\rangle + \langle \mathring{\varphi}_1, 1\rangle = 0\}. \tag{17}$$

Taking zero q_2, $\mathring{q}_2$ and q_1, $\mathring{q}_1$ equal to one in the definition of $b(\cdot, \cdot)$, one can directly verify inclusion $\operatorname{Im} B \subset M$. On the other hand, for every $\varphi \in M$, there exist U_2 in H_2, see its definition (13), and $U_1 \in H_1 = H^1(\Omega_1)$ such that

$$\int_{\Omega_d} \nabla U_d \cdot \nabla \psi_d = \int_{\Omega_d} \varphi_d \psi_d + \int_{\Gamma_d} \mathring{\varphi}_d \psi_d \qquad \forall \psi_d \in H_d. \tag{18}$$

It is easy to check that $\boldsymbol{u} = (\nabla U_2, \nabla U_1)$ satisfies $b(\boldsymbol{u}, q) =< \varphi, q >$ for all $q \in Q$, which yields $\operatorname{Im} B \supset M$.
Because of the reprezentation (17), the codimension of $\operatorname{Im} B$ is 1 and therefore B is closed in Q'. At the presence of both Dirichlet boundaries, the operator B is even surjective since (18) has solution for any $\varphi \in Q'$. For closed operator B one has $(\operatorname{Ker} B^t) \perp \operatorname{Im} B$ which together with (17) implies the reprezentation

$$\operatorname{Ker} B^t = \{q \in Q \,|\, q_d = \mathring{q}_d = const.\} \tag{19}$$

whenever Γ_d^D is empty or even

$$\operatorname{Ker} B^t = \{0\}$$

when both Dirichlet bounderies are present.
Knowladge of $\operatorname{Ker} B^t$ allows us to check properties of the form c. For $p \in \operatorname{Ker} B^t$ we have $p_1 = 0$, $p_2 = \mathring{p}_2 = C$ or $p_1 = C$, $p_2 = \mathring{p}_2 = 0$, possibly with $C = 0$. Since T maps constants to constants, cf. (10), we conclude

$$c(p,p) = |\Omega_1| C^2 \ge \tilde{C} \|p\|_Q^2,$$

i.e. coercivity of c on $\operatorname{Ker} B^t$ with a constant $\tilde{C}$ dependent only on the problem geometry. By the same token

$$\operatorname{Ker} B^t \cap \operatorname{Ker} C = \{0\}.$$

Finally the functionals G and F are bounded in V' and Q' provided

$$\tilde{P}_d \in H^{1/2}(\Gamma_d),\ f_d \in L^2(\Omega_d),\ \text{and } \boldsymbol{u}_d^N \in L^2(\Gamma_d^N), \quad \text{for } d = 1, 2. \tag{20}$$

Having verified all assumptions of Theorem 2.5 we can claim its applied version.

Theorem 2.6. *Let $\tilde{P}_d$, f_d, and $\boldsymbol{u}_d^N$ be given functions satifying* (20)*. Assume that $\mathbb{K}_d$, $d = 1, 2$ are given tensor functions $\Omega_d \to \mathbf{R}^{d\times d}$, which are bounded and uniformly possitive definite, i.e.*

$$|\mathbb{K}_d(\boldsymbol{x})| \le \alpha,\ \text{and } |\mathbb{K}_d(\boldsymbol{x})^{-1}| \le \beta, \quad \text{for any } \boldsymbol{x} \in \Omega_d.$$

Then there exists a solution $(\boldsymbol{u}, p) \in V \times Q$ of the problem $P_T(\mathcal{P})$, which is unique in the same space. Moreover we have the bound

$$\|\boldsymbol{u}\|_V + \|p\|_Q \le K \sum_{d=1,2} \Big(\|\tilde{P}_d\|_{H^{1/2}(\Gamma_d)} + \|f_d\|_{L^2(\Omega_d)} + \|u_d^N\|_{L^2(\Gamma_d^N)} \Big) \tag{21}$$

where K is a nonlinear function of α and β, which is bounded on bounded subsets.

Compared to (16), we have the pressure bounded even in Q since a possible constant shift of the pressure of the non-Dirichlet domain is controled through the form c. In fact we can prove higher regularity of the pressure.

Proposition 2.7. *Let $(\boldsymbol{u}, p)$ be a solution to the problem $P_T(\mathcal{P})$ then p_d belongs to H_d and*

$$\|p_d\|_{H_d} \le C\big(\|\boldsymbol{u}\|_V + \|p\|_Q\big), \quad d = 1, 2. \tag{22}$$

Moreover

$$\mathring{p}_d + \tilde{P}_d = \mathrm{Tr}_{\Gamma_d}(p_d) \quad \text{and} \quad \mathbb{K}^{-1}\boldsymbol{u}_d = -\nabla p_d \quad \text{for } d = 1, 2. \tag{23}$$

Proof. Taking $\boldsymbol{w} \in H_0^1(\Omega_d)$ as a test function in (11) the integrals over the interior faces cancels out and we end up with

$$\int_{\Omega_d} \boldsymbol{u}_d \mathbb{K}_d^{-1} \boldsymbol{w} - \int_{\Omega_d} p_d \mathrm{div} \boldsymbol{w} = 0.$$

Thus the weak gradient of p_d exists, is equal to $\mathbb{K}^{-1}\boldsymbol{u}_d$ and consequently bounded in $L^2(\Omega_d)$. This confirms (22). Further for any $\varphi \in L^2(\Gamma_d)$, we can find a valid test function $w \in V_d$ with trace φ on Γ_d. Multiplying $\boldsymbol{w}$ by a sequence of cutoff functions concentrating on Γ_d, we obtain a sequence $\boldsymbol{w}_n$ converging to zero in $L^2(\Omega_d)$. Using this as a test function in (11), integrating by parts and passing to the limit we get

$$\int_{\Gamma_d} \mathring{q}_d \varphi - \int_{\Gamma_d} p_d \varphi = - \int_{\Gamma_d} \tilde{P}_d \varphi,$$

which gives $p_d = \mathring{p}_d + \tilde{P}_d$ on Γ_d.

Next, we prove that Definiton 2.2 is in some sense independent of the partitioning. Let $\mathcal{P} = \{\Omega_d^i\}$ be a partitioning of domains Ω_d, $d = 1, 2$. Chose two neigbouring subdomains $K_1 = \Omega_d^j$ and $K_2 = \Omega_d^k$ sharing the boundary $S = \partial K_1 \cap \partial K_2$. The joined subdomain K is the smallest open superset of K_1 and K_2. We denote $\tilde{\mathcal{P}}$ the partitioning that contains K instead of K_1 and K_2. Now we state following eqivalence of solutions on decompositions $\mathcal{P}$ and $\tilde{\mathcal{P}}$.

Proposition 2.8. *Assume that operator T does not depend on $\mathring{q}_2$ with support on S, i.e.*

$$T(0, \mathring{q}_2) = 0, \quad \text{for } \mathring{q}_s \in H_0^{1/2}(S). \tag{24}$$

Then for any choice of K_1, K_2 the pair $(\boldsymbol{u}, p) \in V \times Q$ is solution of $P_T(\mathcal{P})$ if and only if it is solution of $P_T(\tilde{\mathcal{P}})$.

Proof. Let us denote $\tilde{V}$, $\tilde{Q}$, $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ the spaces and the forms of the problem $P_T(\tilde{\mathcal{P}})$ corresonding to the objects in Definition 2.2 of problem $P_T(\mathcal{P})$. First assume that $(\boldsymbol{u}, p)$ is solution of problem $P_T(\mathcal{P})$. Testing (12) by $\mathring{q} \in H_0^{1/2}(S)$ we get

$$\int_{\partial K_1} (\boldsymbol{u} \cdot \boldsymbol{n})\mathring{q} + \int_{\partial K_2} (\boldsymbol{u} \cdot \boldsymbol{n})\mathring{q} = 0, \tag{25}$$

because of the condition (24). Consequently $\boldsymbol{u}$ have continuous normal trace on S and it belongs in $\tilde{V}$. In the view of Proposition 2.7 we can consider p as member of $\tilde{Q}$. From its definition a is equal to $\tilde{a}$ on $V \times V$. Further $b = \tilde{b}$ on $\tilde{V} \times Q$ since (25) holds for any $\boldsymbol{u} \in \tilde{V}$. Finally, from condition (24) we have $c = \tilde{c}$ on $Q \times Q$. The right-hand sides are same. Because Accordingly $(\boldsymbol{u}, p)$ as a member of $\tilde{V} \times \tilde{P}$ is also the solution of $P_T(\tilde{\mathcal{P}})$.
Conversly for any solution $(\tilde{\boldsymbol{u}}, \tilde{p})$ of $P_T(\tilde{\mathcal{P}})$, there exists a unique solution $(\boldsymbol{u}, p)$ of the problem $P_T(\mathcal{P})$, which we have just proved to be also solution of $P_T(\tilde{\mathcal{P}})$ which is also unique, thus $(\tilde{\boldsymbol{u}}, \tilde{p}) = (\boldsymbol{u}, p)$.

The main result of this section is comparison of the solution to the P and P_T problems on different partitioning.

Theorem 2.9. *Let $(\boldsymbol{u}, p) \in V \times Q$ be the solution of the problem $P(\mathcal{P})$ and $(\tilde{\boldsymbol{u}}, \tilde{p}) \in \tilde{V} \times \tilde{Q}$ be the solution of the problem $P_T(\tilde{\mathcal{P}})$, where T is a continuous linear operator satisfying* (10) *and*

$$\|\operatorname{Tr} q - Tq\|_{L^2(\Omega_1)} \le \|\operatorname{Tr} - T\| \|q\|_{H_2}. \tag{26}$$

Assume further that at least one of Dirichlet boundaries Γ_d^D is non-empty. Then

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^2(\Omega_2) \times L^2(\Omega_1)} + \|p - \tilde{p}\|_{H_2 \times H_1} \le K \|\tilde{p}\|_{H_2 \times H_1}, \tag{27}$$

where

$$K = \frac{C}{\|\mathbb{K}^{-1}\|_{L^\infty}}$$

*Furthermore, one can use Raviart-Thomas elements to construct approximation spaces and use again theory from [**?**] to get an $O(h)$ estimate:*

$$\left\|\boldsymbol{u}-\boldsymbol{u}_h\right\|_V + \left\|p-p_h\right\|_P \le Ch\left(\left\|\boldsymbol{u}\right\|_{H^1} + \left\|p\right\|_{H^1}\right). \tag{28}$$

Note that according to the regularity theory for the linear elliptic equations one can expect $p \in W^{2,p}$ and $\boldsymbol{u} \in W^{1,p}$ for all $1 < p < \infty$, provided data from L^∞ and certain beter regularity of the boundary conditions.

3. Mixed-hybrid formulation on non-compatible meshes

In practical applications, we frequently use statistically generated fractures. In this situation, it could be nearly impossible to produce a regular mesh satisfying the alignment condition (4). In order to relax this condition, we have to construct an approximation of the trace of the pressure on the fracture Ω_1. Let $\{\Omega_2^i\}$, $i \in I_2$ be a regular triangular decomposition of Ω_2 with diameter of elements bounded by h and let

$$\Omega_1^i = \Omega_2^i \cap \Omega_1,$$

be elements of the induced decomposition of Ω_1. On these decompositions we consider spaces V^h and P^h similar to (5) and (6). Splitting further the triangles intersecting Ω_1, we obtain an aligned decomposition of Ω_2 on which we can build spaces V and P. Finally, we denote Ω_{12} the union of Ω_2-subdomains intersecting Ω_1

$$\Omega_{12} = \bigcup_{i \in I_1} \Omega_2^i, \quad \textit{where } I_1 = \{i \in I_2 | \Omega_1^i \neq \emptyset\}.$$

Let Π be a continuous linear operator from P_h to the functions piecewise continuous on subdomains Ω_2^i, $i \in I_1$. Then we can define an approximation T of the trace operator for the functions $p \in P_h$ by the formula $T(p) = Tr_{\Omega_1}(\Pi(p))$, locally on each triangle Ω_2^i, $i \in I_1$. In particular, we can use an average approximation

$$\Pi_0|_{\Omega_2^i}(p) = \frac{1}{|\Omega_2^i|}\int_{\Omega_2^i} p_2 \,\mathrm{d}\boldsymbol{x}, \quad \forall i \in I_1$$

or a piecewise linear approximation Π_1 such that

$$\int_S \Pi_1|_{\Omega_2^i}(p) = \int_S \mathring{p}_2$$

for every side S of the triangle Ω_2^i, $i \in I_1$.
The operator T can be used to extend the trace component $\mathring{p}_2$ of the space P_h on the fracture Ω_1. Consequently, we can treat the space P_h as a subspace of P with a norm

$$\left\|f\right\|_P = \left\|f\right\|_{P_h} + \left\|Tf\right\|_{H^{1/2}(\Omega_1)}. \tag{29}$$

*In view of this convention, one can use Definition 1 with spaces $V_h \subset V$ and $P_h \subset P$ to introduce a semi-discrete solution $(\boldsymbol{u}_h, p_h)$. Then again, Theorem 1.2 in [**?**] implies the existence and uniqueness of the solution.*

*Now we want to compare the MH-solution $(u, p) \in V \times P$ to the semi-discrete solution $(\boldsymbol{u}_h, p_h) \in V_h \times P_h$. Following the proof of Proposition 2.11 in [**?**], we can show*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - p_h\|_P \leq C\big(\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \inf_{q_h \in P_h} \|p - q_h\|_P\big). \tag{30}$$

Taking $\boldsymbol{v}_h = \boldsymbol{u} - \boldsymbol{u}^*$, where $\boldsymbol{u}^*$ is a divergence free extension of $[\boldsymbol{u}]_{\Omega_1}$ to Ω_{21}, we can bound the first term by $O(h)$ times L^2-norm of $[\boldsymbol{u}]_{\Omega_1}$. Further, we take q_h equal to the projection of p on the space P_h, then according to (33) *we get*

$$\|p - q_h\|_P = \|p - Tp\|_{H^{1/2}(\Omega_1)} \leq C\Big(\sum_{i \in I_1} \|p - \Pi p\|^2_{H^1(\Omega_2^i)}\Big)^{\frac{1}{2}}.$$

*If Π preserves polynomials up to the order k, the standard approximation estimates (see [**?**] Theorem 16.2) leads to*

$$\|p - \Pi p\|_{1,\Omega_2^i} \leq C|\Omega_2^i|^{\frac{1}{2}-\frac{1}{q}} h^k |p|_{k,q,\Omega_2^i}.$$

Since we assume regular triangulation, we have $|\Omega_2^i| \leq Ch^N$ and $|I_1| \leq Ch^{1-N}$, where $N = 2$ is dimension of Ω_2. Then we conclude

$$\|p - Tp\|_{H^{1/2}(\Omega_1)} \leq Ch^{N(\frac{1}{2}-\frac{1}{q})+k}\Big(\sum_{i \in I_1} |p|^2_{k+1,q,\Omega_2^i}\Big)^{\frac{1}{2}} \leq Ch^\alpha |p|_{k+1,q,\Omega_2}$$

where

$$\alpha = N\Big(\frac{1}{2} - \frac{1}{q}\Big) + k + (1 - N)\frac{q-2}{2q} = k + \frac{1}{2} - \frac{1}{q}. \tag{31}$$

4. ***Conclusion***

We have proposed two approximations of the original MH-problem on the non-aligned meshes. The first is based on the operator Π_0, which preserves only polynomials of zero order. Hence, $\alpha < 1/2$ for all q and we obtain a suboptimal convergence compared to (32). *In the later case, the approximation is based on the operator Π_1, which preserves polynomials of the first order. We get $\alpha = 1$ for $q = 2$ and an optimal convergence rate $O(h)$. In both cases, we have to assume certain regularity of the exact solution. Although we did our analysis only in 2D case with simple geometry, the abstract formulation and the results hold also for more complicated geometries and 3D-2D communication.*

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