Convergence analysis for non-compatible fracture flow discretizations

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1 Introduction

We consider a porous media flow in a 3D domain which contains a network of 2D polygonal fractures and their 1D intersections – chanels. The 3D volume $\tilde{\Omega}_3$ is splitted by the fracture planes into subdomains Ω_3^i, $i \in I_3$; the set of open 2D polygons $\tilde{\Omega}_2$ is broken into open convex subpolygons Ω_2^i, $i \in I_2$ by the lines of their intersections; the set $\tilde{\Omega}_1$ of these 1D intersections is docomposed into line segments Ω_1^i, $i \in I_1$ by their cross points Ω_0^i, $i \in I_0$.

In particular, we have

$$\Omega_{d-1} \subset \partial\Omega_d \setminus \partial\tilde{\Omega}_d \quad \mathrm{for} d = 1, 2, 3; \tag{1}$$

where Ω_d is union of Ω_d^i, $i \in I_d$.

On every subdomain Ω_d^i, we consider linear flow equations

$$\mathrm{div}\boldsymbol{u} = f; \quad \boldsymbol{u} = -\mathbb{K}\nabla p, \tag{2}$$

where the velocity $\boldsymbol{u}$ and the pressure p are unknowns, while given data are the volume density f of the water sources, and the hydraulic permeability tensor $\mathbb{K}$. The velocity $\boldsymbol{u}$ and the permeability tensor $\mathbb{K}$ live on the tangent space of the particular subdomain.

Water exchange between subdomains is given by pressure gradient. Whenever Ω_d^i is intersection of manifolds Ω_{d+1}^j, $j \in I_i$, we prescribe Newton-like condition

$$\boldsymbol{u}_j(\boldsymbol{x}) \cdot \boldsymbol{n}_j(\boldsymbol{x}) = \sigma_j(\boldsymbol{x})(p_j(\boldsymbol{x}) - p_i(\boldsymbol{x}), \quad \forall j \in I_i,\ \boldsymbol{x} \in \Omega_d^i$$

where $\boldsymbol{u}_j(\boldsymbol{x})$ and $p_j(\boldsymbol{x})$ are traces of the velocity and the pressure on the boundary of manifold Ω_{d+1}^i, $p_i(\boldsymbol{x})$ is pressure on Ω_d^i, and $\sigma_j(\boldsymbol{x})$ is the water transfer coefficient of the manifold Ω_{d+1}^j. The additional source $\tilde{f}_d(\boldsymbol{x})$ on the manifold Ω_d^i, $d = 1, 2$ has a form

$$\tilde{f}_d(x) = \sum_{j \in I_i} \boldsymbol{u}_j(x) \cdot \boldsymbol{n}_j(x).$$

On the exterior boundary $\Gamma_d = \partial\Omega \cap \partial\Omega_d$ of the domain Ω_d, we prescribe general Newton boundary condition

$$\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) = \alpha_d(\boldsymbol{x})(p(\boldsymbol{x}) - P_d(\boldsymbol{x})),$$

where α_d and P_d are given data.

In order to obtain good approximation of the velocity field for the transport model, we use discretization based on mixed-hybrid formulation. We denote $\tilde{\Omega}_d^i$, $i \in \tilde{I}_d$ a further decomposition of Ω_d^i, $i \in I_d$ and we denote

$$\Gamma_d^i = \overline{\Omega_d^i} \setminus \bigcup_{i \in \tilde{I}_d} \tilde{\Omega}_d^i.$$

the interior and boundary edges of the decomposed set Ω_d^i. Then we introduce spaces

$$V = V_3 \times V_2 \times V_1 = \prod_{d \in 3,2,1} \prod_{j \in \tilde{I}_d} H(\mathrm{div}, \tilde{\Omega}_d^j) \tag{3}$$

$$P = P_3 \times P_2 \times P_1 \times \mathring{P}_3 \times \mathring{P}_2 \times \mathring{P}_1, \tag{4}$$

$$P_d = L^2(\Omega_d), \quad \mathring{P}_d = \prod_{i \in I_d} H^{1/2}(\Gamma_d^i).$$

In following, we use notation $\boldsymbol{u}_d^j$ for components of $\boldsymbol{u} \in V$ and p_d, $\mathring{p}_d^i$ for components of $p \in P$. Noting, that on Ω_d^i, $d \in 1,2$, we have one pressure p_d on the fracture p_d and possibly several traces $\mathring{p}_d^i$ of the pressure on neighbouring subdomains.

For every $j \in \tilde{I}_d$ we denote

$$i(j) \in I_d \;:\; \Omega_d^j \subset \Omega_d^{i(j)}$$

the index of "mother" domain of Ω_d^j. Now we define mixed-hybrid solution as follows

Definition 1 *We say that pair $(\boldsymbol{u}, p) \in V \times P$ is MH-solution of the problem if it satisfy abstract saddle point problem*

$$a(\boldsymbol{u}, \boldsymbol{w}) + b(\boldsymbol{w}, p) = 0 \qquad \forall \boldsymbol{w} \in V, \tag{5}$$
$$b(\boldsymbol{u}, q) - c(p, q) = \langle F, q \rangle \qquad \forall q \in P, \tag{6}$$

where

$$\begin{aligned}
a(\boldsymbol{v}, \boldsymbol{w}) &= \sum_{d=1,2,3} \sum_{j \in \tilde{I}_d} \int_{\tilde{\Omega}_d^j} \boldsymbol{v}_d^j \mathbb{K}_d^{-1} \boldsymbol{w}_d^j, \\
b(\boldsymbol{v}, q) &= \sum_{d=1,2,3} \sum_{j \in \tilde{I}_d} \left(\int_{\tilde{\Omega}_d^j} -\mathrm{div} \boldsymbol{v}_d^j q_d + \int_{\partial \tilde{\Omega}_d^j} (\boldsymbol{v}_d^j \cdot \boldsymbol{n}) \mathring{q}_d^{i(j)} \right), \\
c(p, q) &= \sum_{d=1,2,3} \sum_{j \in \tilde{I}_{d-1}} \left(\int_{\tilde{\Omega}_{d-1}^j} (p_{d-1} - \mathring{p}_d^j)(q_{d-1} - \mathring{q}_d^j) + \int_{\Gamma_d} \alpha_d \mathring{p}_d \mathring{q}_d \right), \\
\langle F, q \rangle &= - \sum_{d=1,2,3} \left(\int_{\Omega_d} f_d p_d + \int_{\Gamma_d} \alpha_d P_d \mathring{q}_d \right).
\end{aligned}$$

To simplify formula for $c(p, q)$, we have formally set $\tilde{I}_0 = \emptyset$ since the first integral is taken only over 1D and 2D communication interfaces, while the second term corresponds to the Newton boundary condition for every dimension. In the term $c(p, q)$, the symbols $\mathring{p}$ and $\mathring{q}$ should be understood as elemnets of $L^2 \supset H^{1/2}$.

In the following we consider lowest order approximation of the MH-formulation. Assume that Ω_d are polyhedrons triangulated by simplexes $\tilde{\Omega}_d^j$, $j \in \tilde{I}_d$. Then, we approximate the space $H(\mathrm{div}, \tilde{\Omega}_d^j)$

by the Raviart-Thomas space $RT_0(\tilde{\Omega}_d^j)$ (see [?]) and the spaces $L^2(\Omega_d)$ and $H^{1/2}(\Gamma_d^i)$ by piecewise constant functions on elements and their edges respectively. (for details see [?]).

We consider linear model for a flow in saturated porous media

$$\boldsymbol{u}_d = -\mathbb{K}_d \nabla p_d \quad \text{on } \tilde{\Omega}_d \tag{7}$$

and the continuity equation

$$\mathrm{div}\boldsymbol{u}_d = F_d \quad \text{on } \tilde{\Omega}_d, \tag{8}$$

For the sake of clarity we present ideas on the simple 2D-1D case. Let us consider a 2D domain $\Omega_2 \subset \mathbf{R}^2$ splitted into two subdomains by a 1D fracture $\Omega_1 \subset \Omega_2$. We denote $\tilde{\Omega}_2 = \Omega_2 \setminus \Omega_1$ and $\tilde{\Omega}_1 = \Omega_1$. To avoid technical difficulties we assume that Ω_2 have polygonal boundary and Ω_1 is a straight line. The flow on the domain Ω_d ($d = 1, 2$) is described by the velocity $\boldsymbol{u}_d$ and the pressure p_d. These state variables has to satisfy Darcy's law where $\mathbb{K}_d$ is (tensor of) the hydraulic conductivity, $F_2 = f_2$ and $F_1 = f_1 + q$ are water sources, while q denotes the outflow from 2D domain. We consider a non-separating crack, which means that the pressure is continuous across the crack and the sum of outflow from the walls of the fracture is equal to the fracture inflow, namely

$$\begin{aligned} p_2^+ &= p_2^- \quad \text{on } \Omega_1, \\ [\boldsymbol{u}_2]_{\Omega_1} &:= (\boldsymbol{u}_2^+ \cdot \boldsymbol{n}^+ + \boldsymbol{u}_2^- \cdot \boldsymbol{n}^-) = q. \end{aligned}$$

Since the pressure is continuous we can prescribe

$$q = \sigma(p_2|_{\Omega_1} - p_1),$$

where σ is an interchange coefficient, we take $\sigma = 1$. The system is completed by the boundary conditions

$$\begin{aligned} p_d &= p^D \quad \text{on } \Gamma_d^D, \\ \boldsymbol{u}_d \cdot \boldsymbol{n} &= u^N \quad \text{on } \Gamma_d^N. \end{aligned}$$

where Γ_d^D is Dirichlet and Γ_d^N Neumann part of the boundary $\partial\Omega_d$.

2 Mixed-hybrid formulation on aligned meshes

Let $\{\Omega_d^i\}$, $i \in I_d$ be a decomposition of domain Ω_d into disjoint subdomains that satisfy the alignment condition

$$\Omega_1 \subset \Gamma_2, \quad \Gamma_d := \bigcup_{i\in I_d} \partial\Omega_d^i \setminus \partial\Omega_d, \tag{9}$$

where Γ_d is union of interior faces. We multiply the equations (7), (8) by suitable test functions and integrate over the individual subdomains. We integrate by parts in (7) and we treat traces of the pressure as an independent variable (for details see [3]). Finally we obtain mixed-hybrid formulation of the problem specified in the previous section.

Theorem 2 *Definition 1. We say that $(\boldsymbol{u}, p)$ is MH-solution of the problem if the composed velocity $\boldsymbol{u}$ and the composed pressure p belong to the spaces*

$$\boldsymbol{u} = (\boldsymbol{u}_2, \boldsymbol{u}_1) \in V = V_2 \times V_1 = \prod_{i\in I_2} H(\mathrm{div}, \Omega_2^i) \times \prod_{i\in I_1} H(\mathrm{div}, \Omega_1^i) \tag{10}$$

$$p = (p_2, p_1, \overline{p}_2, \overline{p}_1) \in P = P_2 \times P_1 \times \overline{P}_2 \times \overline{P}_1, \tag{11}$$

$$P_d = L^2(\Omega_d), \quad \overline{P}_d = H^{1/2}(\Gamma_d \cup \Gamma_d^N).$$

and they satisfy abstract saddle point problem

$$a(\boldsymbol{u}, \boldsymbol{w}) + b(\boldsymbol{w}, p) = \langle G(p^D), \boldsymbol{w}\rangle \qquad \forall \boldsymbol{w} = (\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_1) \in V, \tag{12}$$

$$b(\boldsymbol{u}, q) - c(p, q) = \langle F(f, u^N), q\rangle \quad \forall q = (q_2, q_1, \overline{q}_2, \overline{q}_1) \in P, \tag{13}$$

where

$$a(\boldsymbol{v},\boldsymbol{w}) = \sum_{d=1,2}\sum_{i\in I_d}\int_{\Omega_d^i} \boldsymbol{v}_d \mathbb{K}_d^{-1}\boldsymbol{w}_d$$

$$b(\boldsymbol{v},q) = \sum_{d=1,2}\sum_{i\in I_d}\int_{\Omega_d^i} -\mathrm{div}\boldsymbol{v}_d q_d + \int_{\Gamma_d}[\boldsymbol{v}_d]\overline{q}_d + \int_{\Gamma_d^N}(\boldsymbol{v}_d\cdot\boldsymbol{n})\overline{q}_d$$

$$c(p,q) = \int_{\Omega_1}(p_1-\overline{p}_2)(q_1-\overline{q}_2).$$

Note that in definition of bilinear form $c(p,q)$, we need trace $\overline{p}_2$ on Ω_1, which is available because of the condition (9).

The existence and uniqueness of the MH-solution is a direct corollary of Theorem 1.2 in [1]. Furthermore, one can use Raviart-Thomas elements to construct approximation spaces and use again theory from [1] to get an $O(h)$ estimate:

$$\left\|\boldsymbol{u}-\boldsymbol{u}_h\right\|_V + \left\|p-p_h\right\|_P \le Ch\left(\left\|\boldsymbol{u}\right\|_{H^1} + \left\|p\right\|_{H^1}\right). \qquad (14)$$

Note that according to the regularity theory for the linear elliptic equations one can expect $p\in W^{2,p}$ and $\boldsymbol{u}\in W^{1,p}$ for all $1<p<\infty$, provided data from L^∞ and certain beter regularity of the boundary conditions.

3 Mixed-hybrid formulation on non-compatible meshes

In practical applications, we frequently use statistically generated fractures. In this situation, it could be nearly impossible to produce a regular mesh satisfying the alignment condition (9). In order to relax this condition, we have to construct an approximation of the trace of the pressure on the fracture Ω_1. Let $\{\Omega_2^i\},\ i\in I_2$ be a regular triangular decomposition of Ω_2 with diameter of elements bounded by h and let

$$\Omega_1^i = \Omega_2^i \cap \Omega_1,$$

be elements of the induced decomposition of Ω_1. On these decompositions we consider spaces V^h and P^h similar to (10) and

(11). Splitting further the triangles intersecting Ω_1, we obtain an aligned decomposition of Ω_2 on which we can build spaces V and P. Finally, we denote Ω_{12} the union of Ω_2-subdomains intersecting Ω_1

$$\Omega_{12} = \bigcup_{i \in I_1} \Omega_2^i, \quad \text{where } I_1 = \{i \in I_2 | \Omega_1^i \neq \emptyset\}.$$

Let Π be a continuous linear operator from P_h to the functions piecewise continuous on subdomains Ω_2^i, $i \in I_1$. Then we can define an approximation T of the trace operator for the functions $p \in P_h$ by the formula $T(p) = Tr_{\Omega_1}(\Pi(p))$, locally on each triangle Ω_2^i, $i \in I_1$. In particular, we can use an average approximation

$$\Pi_0|_{\Omega_2^i}(p) = \frac{1}{|\Omega_2^i|} \int_{\Omega_2^i} p_2 \, \mathrm{d}\boldsymbol{x}, \quad \forall i \in I_1$$

or a piecewise linear approximation Π_1 such that

$$\int_S \Pi_1|_{\Omega_2^i}(p) = \int_S \overline{p}_2$$

for every side S of the triangle Ω_2^i, $i \in I_1$.

The operator T can be used to extend the trace component $\overline{p}_2$ of the space P_h on the fracture Ω_1. Consequently, we can treat the space P_h as a subspace of P with a norm

$$\|f\|_P = \|f\|_{P_h} + \|Tf\|_{H^{1/2}(\Omega_1)}. \tag{15}$$

In view of this convention, one can use Definition 1 with spaces $V_h \subset V$ and $P_h \subset P$ to introduce a semi-discrete solution $(\boldsymbol{u}_h, p_h)$. Then again, Theorem 1.2 in [1] implies the existence and uniqueness of the solution.

Now we want to compare the MH-solution $(u, p) \in V \times P$ to the semi-discrete solution $(\boldsymbol{u}_h, p_h) \in V_h \times P_h$. Following the proof of Proposition 2.11 in [1], we can show

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - p_h\|_P \leq C\big(\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_V + \inf_{q_h \in P_h} \|p - q_h\|_P \big). \tag{16}$$

Taking $\boldsymbol{v}_h = \boldsymbol{u} - \boldsymbol{u}^*$, where $\boldsymbol{u}^*$ is a divergence free extension of $[\boldsymbol{u}]_{\Omega_1}$ to Ω_{21}, we can bound the first term by $O(h)$ times L^2-norm of $[\boldsymbol{u}]_{\Omega_1}$. Further, we take q_h equal to the projection of p on the space P_h, then according to (15) we get

$$\left\|p - q_h\right\|_P = \left\|p - Tp\right\|_{H^{1/2}(\Omega_1)} \le C\Big(\sum_{i\in I_1} \left\|p - \Pi p\right\|^2_{H^1(\Omega_2^i)}\Big)^{\frac{1}{2}}.$$

If Π preserves polynomials up to the order k, the standard approximation estimates (see [2] Theorem 16.2) leads to

$$\left\|p - \Pi p\right\|_{1,\Omega_2^i} \le C|\Omega_2^i|^{\frac{1}{2}-\frac{1}{q}} h^k |p|_{k,q,\Omega_2^i}.$$

Since we assume regular triangulation, we have $|\Omega_2^i| \le Ch^N$ and $|I_1| \le Ch^{1-N}$, where $N = 2$ is dimension of Ω_2. Then we conclude

$$\left\|p - Tp\right\|_{H^{1/2}(\Omega_1)} \le Ch^{N(\frac{1}{2}-\frac{1}{q})+k}\Big(\sum_{i\in I_1} |p|^2_{k+1,q,\Omega_2^i}\Big)^{\frac{1}{2}} \le Ch^{\alpha}|p|_{k+1,q,\Omega_2}$$

where

$$\alpha = N\Big(\frac{1}{2} - \frac{1}{q}\Big) + k + (1-N)\frac{q-2}{2q} = k + \frac{1}{2} - \frac{1}{q}. \qquad (17)$$

4 Conclusion

We have proposed two approximations of the original MH-problem on the non-aligned meshes. The first is based on the operator Π_0, which preserves only polynomials of zero order. Hence, $\alpha < 1/2$ for all q and we obtain a suboptimal convergence compared to (14). In the later case, the approximation is based on the operator Π_1, which preserves polynomials of the first order. We get $\alpha = 1$ for $q = 2$ and an optimal convergence rate $O(h)$. In both cases, we have to assume certain regularity of the exact solution. Although we did our analysis only in 2D case with simple geometry, the abstract formulation and the results hold also for more complicated geometries and 3D-2D communication.

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References

[1] F. Brezzi, M. Fortin, *Mixed and Hybrid Finite Element Methods.* Springer Verlag, Berlin/Heidelberg, 1991

[2] P.G. Ciarlet, J.L. Lions, *Handbook of Numerical Analysis : Finite element method (Part 1).* North-Holland, Amsterdam, 1991

[3] J. Maryška, M. Rozložník, M. Tůma. *Mixed-hybrid finite element approximation of the potential fluid flow problem.* J. Comput. Appl. Math., Vol. 63, (1995), pp. 383–392.