TECHNICAL UNIVERSITY OF LIBEREC
Faculty of Mechatronics, Informatics
and Interdisciplinary Studies

# Partition of unity methods for approximation of point water sources in porous media

*Pavel Exner, Jan Březina* | *5th February 2014*

⋆ **doctoral seminar**

# Outline

# Motivation

problems in FE (finite element) models:

- cracks in material
- quantity interfaces, membrane model
- **point sources – fluid flow**, heat flow, electrostatics

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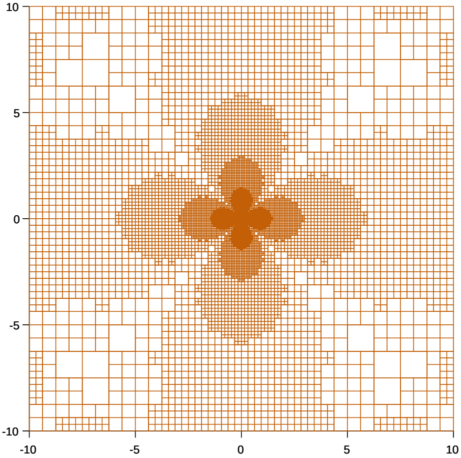
# Motivation – solution

- local refinement of a mesh can be difficult and leads to large computational costs

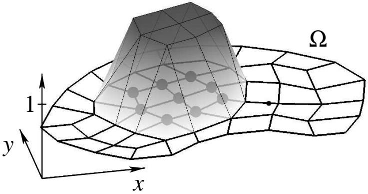


# Motivation – solution

- local refinement of a mesh can be difficult and leads to large computational costs

- local character of the solution can be known
  $\rightarrow$ using FE shape functions as PU (Partition of Unity) method we can incorporate the known information into FE solution

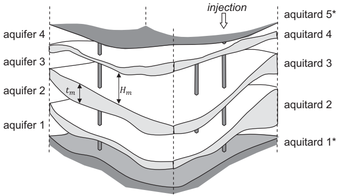
$$PU: \quad \sum_{i \in \mathcal{N}} \varphi_i(\mathbf{x}) = 1$$




- model based on

  R. Gracie, J.R. Craig, *Modelling well leakage in multilayer aquifer systems using the extended finite element method*. 2010

# Model of pressure in a multi-aquifer system




Darcy's law and continuity eq.:

$$\mathbf{u} = -\mathbf{T}\nabla h$$

$$\nabla \cdot \mathbf{u} = f$$

$\Omega^m \subset \mathbb{R}^2$ – $m$-th aquifer
$B_w^m$ – cross-section of the well $w$ and aquifer $m$

$$\Theta^m = \Omega^m - \bigcup_{w=1}^{W} B_w^m$$

$\rightarrow$ Poisson equation:

$$-T^m \Delta h^m = f^m \qquad \text{in } \Theta^m \qquad (1)$$

boundary conditions:

$$\begin{aligned} \mathbf{u}^m \cdot \mathbf{n}^m &= 0 && \text{on } \Gamma_N^m \\ h^m &= h_D^m && \text{on } \Gamma_D^m \\ \mathbf{u}^m \cdot \mathbf{n}^m &= \sigma_w^m (h^m - H_w^m) && \text{on } \Gamma_T^m \end{aligned}$$

well edges boundary:

$$\Gamma_T^m = \bigcup_{w=1}^{W} \partial B_w^m$$

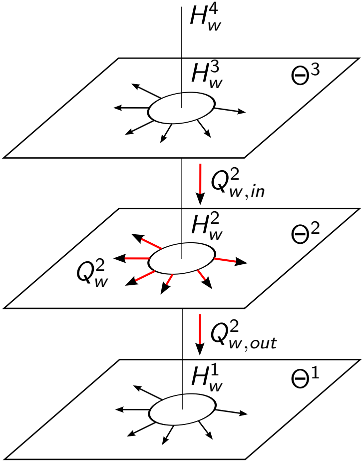
# Communication among wells and aquifers

$$Q_w^m = Q_{w,in}^m - Q_{w,out}^m \qquad (2)$$

$$\begin{aligned} Q_w^m &= \text{flow into aquifer} \\ Q_{w,in}^m &= \text{flow from upper aquifer} \\ Q_{w,out}^m &= \text{flow into lower aquifer} \end{aligned}$$

$$\begin{aligned} Q_w^m &= -\int_{\partial B_w^m} \sigma_w^m \left(h^m(\mathbf{x}) - H_w^m\right) \mathrm{d}\mathbf{x} \\ Q_{w,in}^m &= -c_w^{m+1}\left(H_w^m - H_w^{m+1}\right) \\ Q_{w,out}^m &= -c_w^m\left(H_w^{m-1} - H_w^m\right) \end{aligned}$$

$\boxed{\text{we are looking for } (h^m, H_w^m)}$



# Compared methods

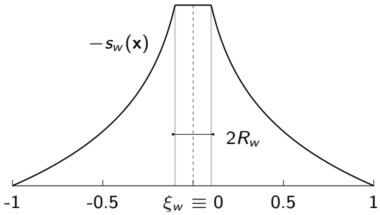
1. h-adaptive linear finite element method (FEM)
2. corrected XFEM – eXtended FEM (implementing ramp function)
3. corrected XFEM (implementing both ramp function and shift)
4. SGFEM (Stable Generalized FEM)

- implementation done with **deal.II**
  (C++ library, no enrichment technique support sofar, quadrilateral cells)

# Enrichment function

$$s_w(\mathbf{x}) = \begin{cases} \log(r_w(\mathbf{x})), & r_w > R_w \\ \log(R_w), & r_w \leq R_w \end{cases}$$

$$r_w(\mathbf{x}) = \|\mathbf{x} - \xi_w\| = \sqrt{(x - x_w)^2 + (y - y_w)^2}$$




- index set for nodes inside an enriched area:

$$\mathcal{N}_w = \{k \in \mathbb{N} : \|\mathbf{x}_k - \xi_w\| \leq r_{enr}\}, \qquad \mathbf{x}_k \text{ is a node of the triangulation}$$

- solution of pressure with a single local enrichment (standard XFEM):

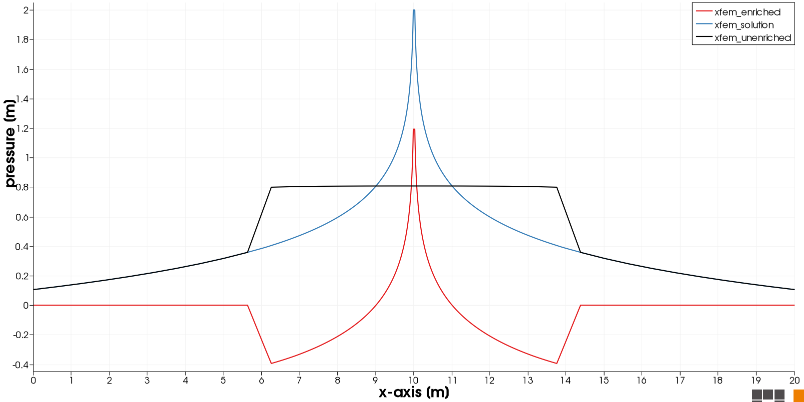
$$h(\mathbf{x}) = \underbrace{\sum_{j \in \mathcal{N}} \alpha_j \, \varphi_j(\mathbf{x})}_{\text{FEM}} + \underbrace{\sum_{k \in \mathcal{N}_w} \beta_k \, \varphi_k(\mathbf{x}) \, s(\mathbf{x})}_{\text{enrichment}}$$

# corrected XFEM – ramp function

ramp function $g(\mathbf{x}) = \sum_{u \in \mathcal{N}} q_u \varphi_u(\mathbf{x}) \quad q_u \in \{0, 1\}$

pressure in the aquifer:

$$h(\mathbf{x}) = \sum_{j \in \mathcal{N}} \alpha_j \, \varphi_j(\mathbf{x}) + \sum_{k \in \mathcal{N}_w} \beta_k \, \varphi_k(\mathbf{x}) \, \underline{g(\mathbf{x})} \, s(\mathbf{x})$$

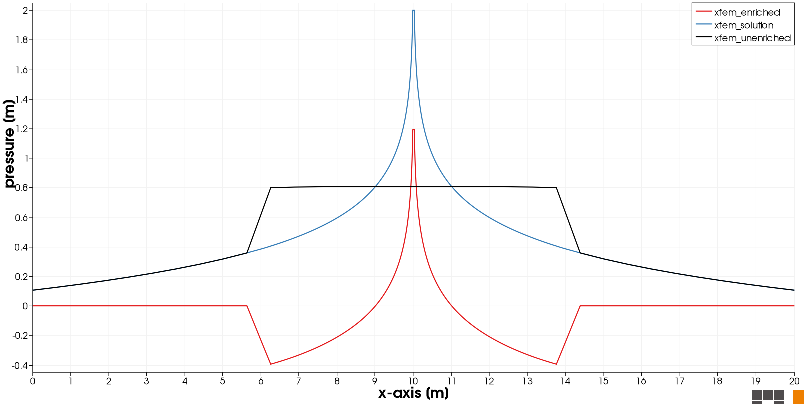


# corrected XFEM – ramp function

ramp function $g(\mathbf{x}) = \sum_{u \in \mathcal{N}} q_u \varphi_u(\mathbf{x}) \quad q_u \in \{0, 1\}$

pressure in $m$-th aquifer with multiple wells:

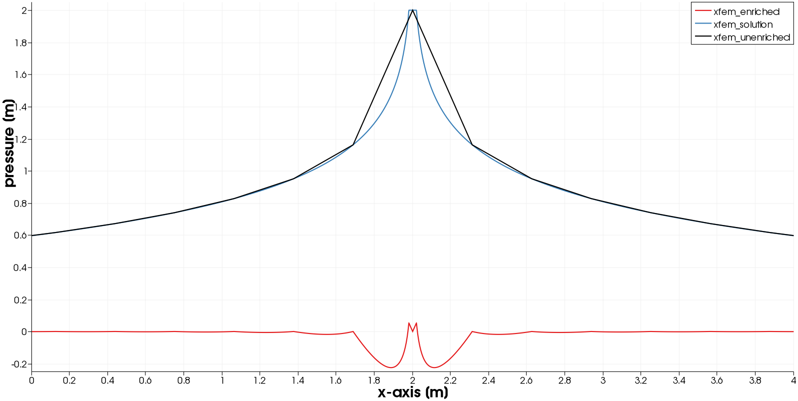
$$h^m(\mathbf{x}) = \sum_{j \in \mathcal{N}} \alpha_j^m \, \varphi_j^m(\mathbf{x}) + \sum_{w \in \mathcal{W}} \sum_{k \in \mathcal{N}_w} \beta_{w,k}^m \, \varphi_k^m(\mathbf{x}) \, \underline{g_w^m(\mathbf{x})} \, s_w^m(\mathbf{x})$$



# corrected XFEM – ramp function, shift

both ramp function and shift
pressure in the aquifer:

$$h(\mathbf{x}) = \sum_{j \in \mathcal{N}} \alpha_j \, \varphi_j(\mathbf{x}) + \sum_{k \in \mathcal{N}_w} \beta_k \, \varphi_k(\mathbf{x}) \, \underline{g(\mathbf{x})} \, \underline{\left[ s(\mathbf{x}) - s(\mathbf{x_k}) \right]}$$

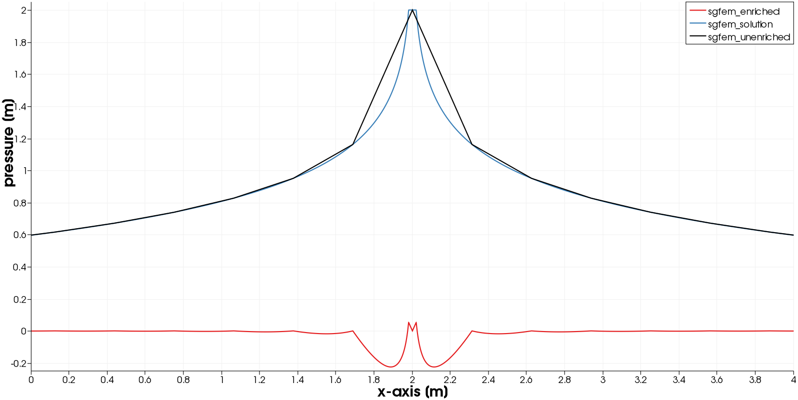


# SGFEM

piecewise linear finite element interpolant $\mathcal{I}(s)(\mathbf{x}) = \sum_{u \in \mathcal{N}} \varphi_u(\mathbf{x}) s(\mathbf{x}_u)$

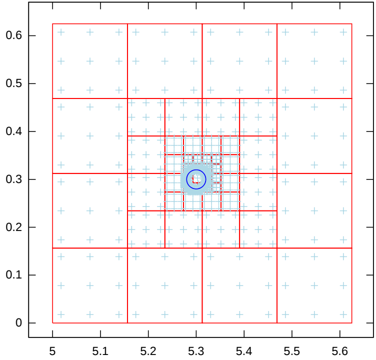
pressure in the aquifer: pressure in $m$-th aquifer:

$$h(\mathbf{x}) = \sum_{j \in \mathcal{N}} \alpha_j \varphi_j(\mathbf{x}) + \sum_{k \in \mathcal{N}_w} \beta_k \varphi_k(\mathbf{x}) \underline{\left[ s(\mathbf{x}) - \mathcal{I}(s)(\mathbf{x}) \right]}$$

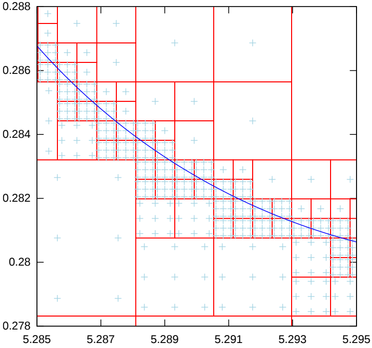


# Adaptive integration on enriched cells

- higher order quadrature on cells without a well
- adaptive element refinement on cells with a well
- fixed number of refinements (12 chosen)

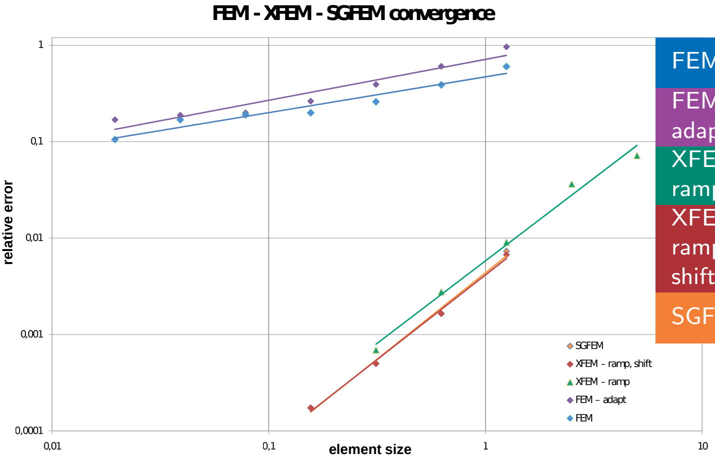



(a) refined element with a well inside



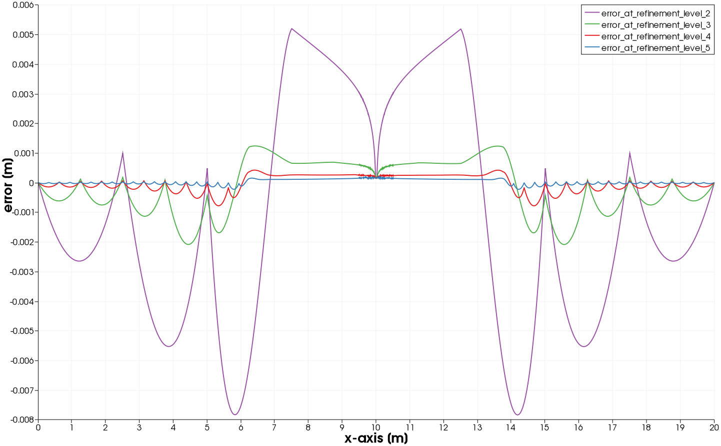

(b) detail of a well edge

# Convergence of methods




| | |
|---|---|
| FEM | $O(h^{0.37})$ |
| FEM adapt. | $O(h^{0.42})$ |
| XFEM ramp | $O(h^{1.71})$ |
| XFEM ramp, shift | $O(h^{1.76})$ |
| SGFEM | $O(h^{1.79})$ |

# Error distribution in SGFEM



# Solving linear system with CG

| number of CG iterations | | | | |
|---|---|---|---|---|
| DoFs (enriched) | XFEM ramp | XFEM ramp, shift | SGFEM | DoFs (enriched) |
| 18 (9) | 16 | 13 | 6 | 10 (1) |
| 34 (9) | 26 | 19 | 12 | 26 (1) |
| 106 (25) | 74 | 42 | 29 | 90 (9) |
| 358 (69) | 246 | 82 | 58 | 326 (37) |
| 1274 (185) | 959 | 155 | 118 | 1218 (129) |
| 4838 (613) | * | 286 | 239 | 4734 (509) |
| 18910 (2269) | * | 567 | 488 | 18702 (2061) |

– CG tolerance set $10^{-12}$, Jacobi preconditioner applied

* did not converged after 4000 iterations.

# Conlusion

- logarithmic function can approximate point sources well
- corrected XFEM
  - ramp function only – convergence rate is lower, system matrix is worse conditioned
  - ramp function and shift – convergence is almost optimal (in agreement with Gracie, Craig)
- SGFEM
  - linear system solved with less iterations than in XFEM case
  - it is proved (by Babuška and Banerjee) that the system matrix is not worse conditioned than the corresponding FEM matrix
  - no more degrees of freedom on blending elements

# Future research

- choice of the enrichment radius
- improve the adaptive integration on enriched elements
- use more complicated model for convergence measurements
- deal with hanging nodes in the enriched area

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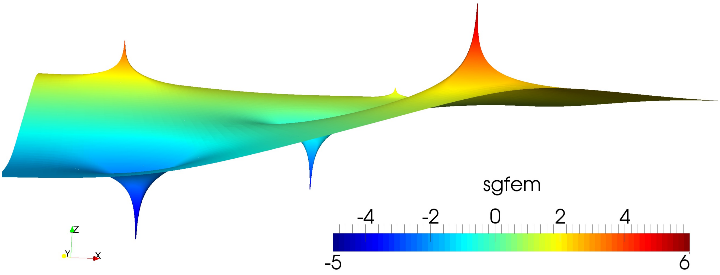
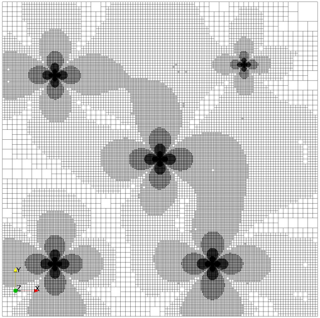
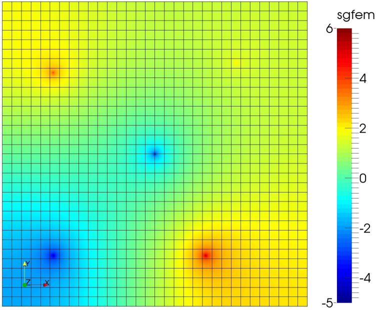
- enrichment in mixed hybrid method
  - enrichment only in pressure
  - enrichment both in pressure and velocity
- implementation in Flow123d
- use enrichment to better approximate pressure and velocity near fractures
- non-compatible meshes, 3D-1D and 2D-1D connections

# Multiple wells problem



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Thank You for Your attention.