Sparse Approximation of FEM Matrix for Sheet Current Integro-Differential Equation

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In the report we consider the problem of effective numerical solution of 2D sheet current integro-differential equations. Typically the equations for sheet current are derived from Maxwell equations for magnetic potential. The equation contain weak singularity in the kernel and for normal conductors has the form

$$R_s \vec{J}(r) + \frac{j\omega}{4\pi} \iint\limits_S \frac{\vec{J}(r')}{|r - r'|} ds = -\nabla \Phi(r), \quad \Delta \Phi(r) = 0, \quad r \in S. \tag{1}$$

Here S is 2D domain, $r = (x, y)$. $\vec{J}(r)$, $\Phi(r)$ are phasors for sheet current and voltage. ω is frequency, $\mathrm{Re}(R_s) \geq 0$, $\mathrm{Im}(R_s) \geq 0$- thin plate resistance [1]. For (1), well developed numerical techniques exist [1]. In many cases, the numerical solution of (1) is very time consuming even if GMRES and Fast Multipoles Method for kernel approximation is used [1].

We are interested in the solution of very similar to (1) task for superconductors [3]. In our case equations are real and does not depend on frequency. The equations we solve are singular and follows from Biot-Savart formula and has the form

$$\lambda_\perp \left(\nabla \times \vec{J}(r) \right)_z + \frac{1}{4\pi} \iint_S \left(\vec{J}(r) \times (\nabla_r \cdot \frac{1}{|r - r_0|}) \right)_z dxdy = 0, \tag{2}$$

where $r = (x, y)$, $r_0 = (x_0, y_0)$, $r, r_0 \in S$, $\lambda_\perp$ is real coefficient. Equation (2) must be fulfilled by the boundary conditions for $\vec{j}_s(r)$. On the part of the boundary of S they reduce to $J_{s,n} = 0$, on the rest of the boundary the non-zero value of $J_{s,n}$ is specified. Equation (1) can be reduced to form similar to (2) using 2D rotor operation $(\nabla\times)_{xy}$.

To solve (2) we first reduce it using stream function $\psi(r)$. Then we obtain the problem very similar to first boundary problem for Laplace equation:

$$-\lambda_\perp \Delta \psi(r_0) + \frac{1}{4\pi} \iint_S \left(\nabla \psi(r), \nabla_r \frac{1}{|r - r_0|} \right) dxdy = 0, \quad \psi(r) = F(r), \quad r \in \partial S. \tag{3}$$

The function $F(r)$ is completely defined by boundary condition for current. Equation (3) for$\psi(r)$ is hypersingular with kernel $1/|r - r_0|^3$. It is known that operator in (3) is positive and self-adjoint [2] (if $\psi(r) = 0$ on the boundary of S).

To solve (3) numerically we use finite element method based on triangular grids and linear elements [3]. The matrix in finite element method is dense die to integral operator. Matrix elements are

$$a(u_i^h, u_j^h) = \lambda_\perp \iint_{S_i \cap S_j} (\nabla u_i^h, \nabla u_j^h) ds + + \frac{1}{4\pi} \iint_{S_i} ds \iint_{S_j} \frac{(\nabla u_j^h(r'), \nabla u_i^h(r))}{|r - r'|} ds', \tag{4}$$

u_i^h, u_j^h FEM basis functions. The method of calculation of quadruple integrals is given in [3].

Die to integral term in (3) FEM reduces to fully populated symmetric matrix. The dimension of this matrix for practical problems can be large ($10^3 - 10^4$ finite elements). This fact is the key limitation of the method. The problem is to evaluate a method for large tasks. We suggest a simple method for sparsifying the FEM matrix. We simply drop small elements. Symmetry of matrix is preserved. Positive definetness also can be preserved.

Consider matrix element for integral operator (4). Diagonal matrrix elements are large and positive but matrix is not diagonally dominant. Near-diagonal elements can be of any sign. If $S_i \bigcap S_j = \emptyset$ then elements are negative and

$$b_{ij} = \iint_{S_i} ds \iint_{S_j} \frac{(\nabla u_j^h(r'), \nabla u_i^h(r))}{|r - r'|} ds' = - \iint_{S_i} ds \iint_{S_j} \frac{u_j^h(r') u_i^h(r)}{|r - r'|^3} ds' \tag{5}$$

Let h be the estimation of diameters of triangles in the mesh. Then $m_{ij} h = |r_i - r_j|$ and from (5) follows

$$|b_{ij}| \leq M \frac{h}{m_{ij}^3}. \tag{6}$$

Thus matrix elements quickly decay if distance between finite elements grows. For equation with weak singularity this decaying is of first order only.

In the report, we present set of results of numerical experiments with sparse FEM matrix approximation and direct linear system solver. The conclusion is that the simple method can solve large problems and overcome memory and CPU time limitations of dense matrix technique.

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