Finding all Convex Cuts of a Plane Graph in Polynomial Time

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Finding all convex cuts of a plane graph in polynomial time*

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Abstract

Convexity is a notion that has been defined for subsets of $\mathbb{R}^n$ and for subsets of general graphs. A convex cut of a graph $G = (V, E)$ is a 2-partition $V_1 \dot{\cup} V_2 = V$ such that both V_1 and V_2 are convex, i. e., shortest paths between vertices in V_i never leave V_i, $i \in \{1, 2\}$. Finding convex cuts is $\mathcal{NP}$-hard for general graphs. To characterize convex cuts, we employ the Djoković relation, a reflexive and symmetric relation on the edges of a graph that is based on shortest paths between the edges' end vertices.

It is known for a long time that, if G is bipartite and the Djoković relation is transitive on G, i. e., G is a partial cube, then the cut-sets of G's convex cuts are precisely the equivalence classes of the Djoković relation. In particular, any edge of G is contained in the cut-set of exactly one convex cut. We first characterize a class of plane graphs that we call *well-arranged*. These graphs are not necessarily partial cubes, but any edge of a well-arranged graph is contained in the cut-set(s) of at least one convex cut. Moreover, the cuts can be embedded into the plane such that they form an arrangement of pseudolines, or a slight generalization thereof. Although a well-arranged graph G is not necessarily a partial cube, there always exists a partial cube that contains a subdivision of G.

We also present an algorithm that uses the Djoković relation for computing all convex cuts of a (not necessarily plane) bipartite graph in $\mathcal{O}(|E|^3)$ time. Specifically, a cut-set is the cut-set of a convex cut if and only if the Djoković relation holds for any pair of edges in the cut-set.

We then characterize the cut-sets of the convex cuts of a general graph H using two binary relations on edges: (i) the Djoković relation on the edges of a subdivision of H, where any edge of H is subdivided into exactly two edges and (ii) a relation on the edges of H itself that is not the Djoković relation. Finally, we use this characterization to present the first algorithm for finding all convex cuts of a plane graph in polynomial time.

Keywords: Convex cuts, Djoković relation, partial cubes, plane graphs, bipartite graphs

1 Introduction

A *convex k-partition* of an undirected graph $G = (V, E)$ is a partition $(V_1, \ldots, V_k)$ of V such that the subgraphs of G induced by $V_1, \ldots V_k$ are all convex. A convex subgraph of G, in turn, is a subgraph S of G such that for any pair of vertices v, w in S all shortest paths from v to w in G are fully contained in S. The vertex set of a convex subgraph is called *convex set*.

A *convex cut* of G is a convex 2-partition of G. If G has a convex k-partition, then G is said to be *k-convex*. Artigas et al. [1] showed that, for a given $k \geq 2$, it is $\mathcal{NP}$-complete to decide whether a (general) graph is k-convex. Moreover, given a bipartite graph $G = (V, E)$ and an integer $l < |V|$, it is $\mathcal{NP}$-complete to decide whether there exists a convex set with at least l vertices [10].

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There exists a different notion of convexity for plane graphs. A plane graph is called convex if all of its faces are convex polygons. This second notion is different and *not* object of our investigation. The notion of convexity in acyclic directed graphs, motivated by embedded processor technology, is also different [2]. There, a subgraph S is called convex if there is no *directed* path between any pair v, w in S that leaves S. In addition to being directed, these paths do not have to be *shortest* paths as in our case.

Applications that potentially benefit from convex cuts include data-parallel numerical simulations on graphs. Here the graph is partitioned into parts that have nearly the same number of vertices [3, 5]. For some linear solvers used in these simulations, the *shape* of the parts, in particular short boundaries, small aspect ratios, but also connectedness and smooth boundaries, plays a significant role [13]. Convex subgraphs of meshes typically admit these properties. Another example is the preprocessing of road networks for shortest path queries by partitioning according to natural cuts [7]. The definition of a natural cut is not as strict as that of a convex cut, but they have a related motivation.

Due to the importance of graph partitions in theory and practice [3], it is natural to ask whether the time complexity of finding convex cuts is polynomial for certain types of input graphs. In this paper, we will see that polynomial-time algorithms exist for a sub-class of plane graphs and for bipartite graphs. Specifically, a cut-set is the cut-set of a convex cut of a bipartite graph if and only if the Djoković relation holds for any pair of edges in the cut-set.

We also characterize the cut-sets of the convex cuts of a general graph H in terms of two binary relations, each on a different kind of edges: the edges of a subdivision of H, where any edge of H is subdivided into two edges and (ii) the edges of H itself. The relation on the first kind of edges is the Djoković relation (see Section 2), and the relation on the second kind of edges, denoted by τ, is such that $e \; \tau \; f$ iff the distance between any end vertex of e to any end vertex of f is the same.

1.1 Related work

Artigas et al. [1] show that every connected chordal graph $G = (V, E)$ is k-convex, for $1 \leq k \leq |V|$. They also establish conditions on $|V|$ and p to decide whether the pth power of a cycle is k-convex. Moreover, they present a linear-time algorithm that decides whether a cograph is k-convex.

Our methods for characterizing and finding convex cuts of a plane graph G are motivated by the work in Chepoi et al. [6] who defined alternating cuts and specified conditions under which alternating cuts are convex cuts and vice versa. Our approach is more myopic, though. We call a face F of G even [odd] if $|E(F)|$ is even, and an alternating path is one that cuts an even face F such that the number of vertices in the two parts of $E(F)$ is equal. In an odd face the alternating path makes a slight left or right turn so that the number of vertices in the two parts of $E(F)$ differ by one. As in [6], when following an alternating path through the faces of G, a left [right] turn must be compensated by a right [left] turn as soon as this is possible.

Plane graphs usually have alternating cuts that are not convex and convex cuts that are not alternating. Proposition 2 in [6] characterizes the set of plane graphs for which the alternating cuts coincide with the convex cuts in terms of a condition on the boundary of *any* alternating cut. In this paper we represent the alternating cuts as plane curves that we call embedded alternating paths (EAPs)—an EAP partitions G exactly like the alternating cut it represents. In contrast to [6], however, we focus on the *intersections* of the EAPs (i. e., alternating cuts).

If any pair of EAPs intersects at most once, we have a slight generalization of what is known as an *arrangement of pseudolines*. The latter arise in discrete geometry, computational geometry, and in the theory of matroids [4]. Duals of arrangements of pseudolines are known to be partial cubes (see Section 2), a fact that has been applied to graphs before by Eppstein [11], for example. For basics on partial cubes we rely on Ovchinnikov's survey [14]. The following basic fact about partial cubes is crucial for our method to find convex cuts: partial cubes are precisely the graphs that are bipartite and on which the Djoković relation [9] (defined in Section 2) is transitive. For a characterization of planar partial cubes see Peterin [15].

1.2 Paper outline and contribution

In Section 3 we first represent (myopic versions of) the alternating cuts of a plane graph $G = (V, E)$, as defined in [6], by EAPs. The main work here is on the proper embedding. We then study the case of G

being *well-arranged*, as we call it, i. e., the case in which the EAPs form an arrangement of pseudolines, or a slight generalization thereof. We show that the dual $G_{\mathcal{E}}$ of such an arrangement is a partial cube and reveal a one-to-one correspondence between the EAPs of G and the convex cuts of $G_{\mathcal{E}}$. Specifically, the edges of $G_{\mathcal{E}}$ intersected by an EAP form the cut-set of a convex cut of $G_{\mathcal{E}}$. Conversely, the cut-set of any convex cut of $G_{\mathcal{E}}$ is the set of edges intersected by a unique EAP of G. From (i) the one-to-one correspondence between the EAPs of G and the convex cuts of $G_{\mathcal{E}}$ and (ii) the construction of $G_{\mathcal{E}}$ we derive that the EAPs also define convex cuts of G.

In Section 4 we specify an $\mathcal{O}(|E|^3)$-time algorithm to find all convex cuts of a not necessarily plane bipartite graph. The fact that we can compute all convex cuts in bipartite graphs in polynomial time is no contradiction to the $\mathcal{NP}$-completeness of the decision problem whether the largest convex set in a bipartite graph has a certain size [10]. Indeed, for a cut to be convex, *both* subgraphs have to be convex, whereas the complement of a convex set is not required to be a convex set itself.

In Section 5 we characterize the cut-sets of the convex cuts of a general graph H in terms of the Djoković relation and τ. The results of Section 5 are then used in Section 6 to derive new necessary conditions for convexity of cuts of a *plane* graph G. As in the case of well-arranged graphs, we iteratively proceed from an edge on the boundary of a face F of G to another edge on the boundary "opposite" of F. This time, however, "opposite" is with respect to the Djoković relation on a subdivision of G. Thus we arrive at a polynomial-time algorithm that finds all convex cuts of G. We correct an error in a preliminary version [12] of this paper. The running time is now $\mathcal{O}(|V|^7)$ instead of $\mathcal{O}(|V|^3)$.

2 Preliminaries

Unless stated otherwise, $G = (V, E)$ is a finite, undirected, unweighted, and two-connected graph that is free of self-loops. Two-connectedness is not a limitation for the problem of finding convex cuts because a convex cut cannot cut through more than one block of G, and self-loops have no impact on the convex cuts. For $e \in E$ with end points u, v ($u \neq v$) we sometimes write $e = \{u, v\}$ even when e is not necessarily determined by u and v due to parallel edges. We use the term *path* in the general sense: a path does not have to simple, and it can be a cycle.

If G is plane, V is a set of points in $\mathbb{R}^2$, and E is a set of plane curves that intersect only at their end points which, in turn, make up V. The unbounded face of G is denoted by F_∞. For a face F of G, we write $E(F)$ for the set of edges that bound F. Our definitions and results on plane graphs are invariant to topological isomorphism [8] which, in conjunction with two-connectedness, is equivalent to combinatorial isomorphism [8]. Since any plane graph is combinatorially isomorphic to a plane graph whose edges are line segments [8], we can always resort to the case of straight edges without loss of generality. We do so especially in our illustrations.

We denote the standard metric on G by $d_G(\cdot, \cdot)$. In this metric the distance between $u, v \in V$ amounts to the number of edges on a shortest path from u to v. A subgraph $S = (V_S, E_S)$ of a (not necessarily plane) graph H is an *isometric* subgraph of H if $d_S(u, v) = d_H(u, v)$ for all $u, v \in V_S$.

Following Djoković [9] and using Ovchinnikov's notation [14], we set

$$W_{xy} = \{w \in V : d_G(w, x) < d_G(w, y)\} \quad \forall \{x, y\} \in E. \tag{1}$$

Let $e = \{x, y\}$ and $f = \{u, v\}$ be two edges of G. The Djoković relation θ on G's edges is defined as follows:

$$e \ \theta \ f \Leftrightarrow f \text{ has one end vertex in } W_{xy} \text{ and one in } W_{yx}. \tag{2}$$

The Djoković relation is reflexive, symmetric [17], but not necessarily transitive. The vertex set V of G is partitioned by W_{ab} and W_{ba} if and only if G is bipartite.

A *partial cube* $G_q = (V_q, E_q)$ is an isometric subgraph of a hypercube. Interested readers find more details on partial cubes in Ovchinnikov's survey [14]; we state a few important results for the sake of self-containment. Partial cubes and θ are related in that a graph is a partial cube if and only if it is bipartite and θ is transitive. Thus, θ is an equivalence hypercube. For a survey on partial cubes see Ovchinnikov [14]. Partial cubes and θ are related in that a graph is a partial cube if and only if it is bipartite and θ is transitive. Thus, θ is an equivalence relation on E_q, and the equivalence classes are cut-sets of G_q. Moreover, the cuts defined by these cut-sets are precisely the convex cuts of G_q. If (V_q^1, V_q^2) is a convex cut, the (convex) subgraphs induced by V_q^1 and V_q^2 are called *semicubes*. If θ gives rise to k equivalence classes $E_q^1, \ldots E_q^k$, and thus k pairs (S_a^i, S_b^i) of semicubes, where the ordering of the semicubes in the pair is arbitrary, one can derive a *Hamming labeling* $b : V_q \mapsto \{0,1\}^k$ by setting

$$b(v)_i = \begin{cases} 0 & \text{if } v \in S_a^i \\ 1 & \text{if } v \in S_b^i \end{cases} \tag{3}$$

In particular, $d_{G_q}(u,v)$ amounts to the Hamming distance between $b(u)$ and $b(v)$ for all $u, v \in V_q$. This is a consequence of the fact that the corners of a hypercube have such a labeling and that G_q is an isometric subgraph of a hypercube.

3 Partial cubes from embedding alternating paths

In Section 3.1 we define a multiset of (not yet embedded) alternating paths of a graph G. Section 3.2 is devoted to embedding the alternating paths into $\mathbb{R}^2$ and to the definition of well-arranged graphs. In Section 3.3 we study the dual of an embedding of alternating paths and show that it is a partial cube whenever G is well-arranged.

3.1 Alternating paths

Intuitively, an embedded alternating path P runs through a face F of G such that the edges through which P enters and leaves F are opposite—or nearly opposite because, if $|E(F)|$ is odd, there is no opposite edge, and P has to make a slight turn to the left or to the right. The exact definitions leading up to (not yet embedded) alternating paths are as follows.

Definition 3.1 (Opposite edges, left, right, unique opposite edge)**.** *Let $F \neq F_\infty$ be a face of G, and let $e, f \in E(F)$. Then e and f are called* opposite edges *of F if the lengths of the two paths induced by $E(F) \setminus \{e, f\}$ differ by at most one. If the two paths have different lengths, f is called the* left [right] opposite edge *of e if starting on e and running clockwise around F, the shorter [longer] path comes first. Otherwise, e and f are called* unique opposite edges*.*

Definition 3.2 (Alternating path graph $A(G) = (V_A, E_A)$)**.** *The* alternating path graph $A(G) = (V_A, E_A)$ *of $G = (V, E)$ is the (non-plane) graph with $V_A = E$ and E_A consisting of all two-element subsets $\{e, f\}$ of E such that e and f are opposite edges of some face $F \neq F_\infty$.*

The alternating path graph defined above will provide the edges for the multiset of alternating paths defined next. We resort to a *multiset* for the sake of uniformity, i.e., to ensure that any edge of G is contained in exactly two alternating paths (see Figure 1a).

Definition 3.3 ((Multiset $\mathcal{P}(G)$ of) alternating paths in $A(G)$)**.** *A* maximal *path $P = (v_A^1, e_A^1, v_A^2, \ldots e_A^{n-1}, v_A^n)$ in $A(G) = (V_A, E_A)$ is called* alternating *if*

(i) v_A^i and v_A^{i+1} are opposite for all $1 \leq i \leq n-1$ and

(ii) if v_A^{i+1} is the left [right] opposite of v_A^i, and if j is the minimal index greater than i such that v_A^j and v_A^{j+1} are not unique opposites (and j exists at all), then v_A^{j+1} is the right [left] opposite of v_A^j.

We (arbitrarily) select one path from each pair formed by an alternating path P and the reverse of P. The multiset $\mathcal{P}(G)$ contains all selected paths: the multiplicity of P in $\mathcal{P}(G)$ is two if v_A^{i+1} is a unique opposite of v_A^i for all $1 \leq i \leq n-1$, and one otherwise.

3.2 Embedding of alternating paths

In this section we embed the alternating paths of a plane graph G into $\mathbb{R}^2$. We may assume that the edges of G are straight line segments (see Section 2). An edge $\{e, f\}$ of an alternating path turns into a non-self-intersecting plane curve with one end point on e and the other end point on f. An alternating path with multiplicity $m \in \{1, 2\}$ gives rise to m embedded paths. Visually, we go from Figure 1a to Figure 1b.

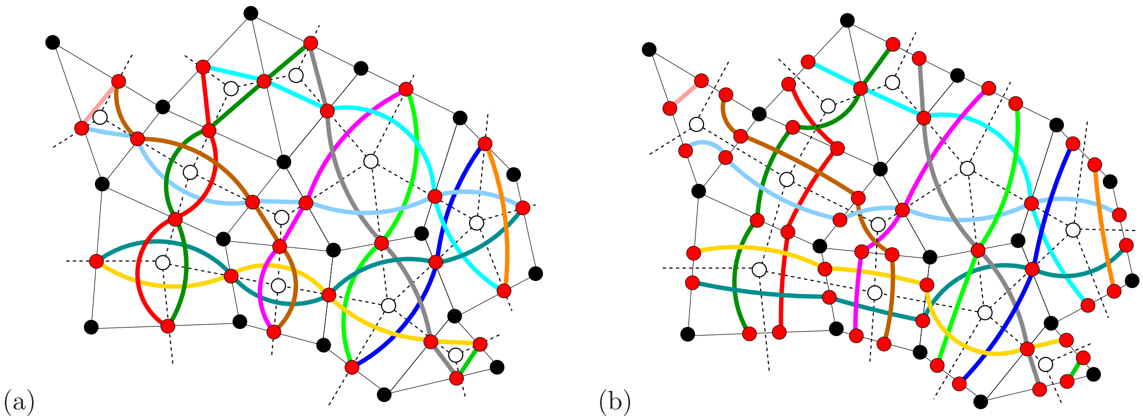


Figure 1: Primal graph: black vertices, thin solid edges. Dual graph: white vertices, dashed edges. (a) Multiset $\mathcal{P}(G)$ of alternating paths: Red vertices, thick solid lines. The paths in $\mathcal{P}(G)$ are colored. In this ad-hoc drawing the two alternating paths that share a vertex, i. e., an edge of G, go through the same (red) point on the edge. (b) Collection $\mathcal{E}(G)$ of EAPs: Red vertices, thick solid colored lines.

Note that any edge e of G is contained in exactly two alternating paths. For any e that separates two bounded odd faces we predetermine a point s on e's interior and require that both alternating paths containing e must run through s. If e does not separate two odd faces, we predetermine two points, s_1 and s_2, on e's interior and require that one alternating path runs through s_1 and the other one runs through s_2. We refer to s, s_1 and s_2 as *slots* of e. If $P = (v_A^1, e_A^1, v_A^2, \ldots e_A^{n-1}, v_A^n)$ is an alternating path, let $F_i = F_i(P)$ be the ith (bounded) face of G that will be traversed by embedded P, i. e., the (unique) face with $v_A^i, v_A^{i+1} \in E(F_i)$. Since we required that G is two-connected, we have $v_A^i \neq v_A^{i+1}$. Thus, if v_A^i has two slots, there exists a well-defined left and right slot from the perspective of standing on v_A^i and looking into F_i, $1 \leq i < n$. Finally, left and right on v_A^n is from the perspective of looking into F_∞.

The overall scenario is that we embed the alternating paths one after the other, where the order of the paths is arbitrary. The following rules for an individual path P then determine which slots are occupied by which alternating paths. For an example of slot choice see Figure 2a,b. The variable $a(P)$ is zero if and only if P makes no left and no right turn. Otherwise, $a(P)$ indicates the preference for the next slot at any time.

1. If P has no left turn and no right turn, set $a(P)$ to 0. Otherwise, if the first turn of P is a left [right] turn, set $a(P)$ to l [r].

2. Let s_l [s_r] be the left [right] slot on v_A^1. If $a(P) = 0$, choose a vacant slot (arbitrarily if both slots are vacant). If $a(P) = l$ [$a(P) = r$], occupy the left [right] slot if that slot is still vacant. Otherwise, occupy the alternative slot and set $a(P) = r$ [$a(P) = l$].

3. Assume that we have found slots for $v_A^1, \ldots v_A^i$.

   - If $a(P) = 0$ and the slot occupied on v_A^i was the left [right] slot, then occupy the left [right] slot on v_A^{i+1}.
   - If v_A^{i+1} has only one slot, then occupy it (single slots can be occupied by two paths). If $(a(P) = l)$ [$(a(P) = r)$], then set $(a(P) = r)$, [$(a(P) = l)$].
   - If v_A^{i+1} has two slots and $a(P) = l$ [$a(P) = l$], then occupy the left [right] slot, if vacant. Otherwise, occupy the alternative slot and set $a(P) = r$ [$a(P) = l$].

The embedding of the alternating paths will be such that two paths that share a point p will always cross at p, and not just touch (see Proposition 3.4 and Figure 2c). We are not interested in the exact course of the embedded alternating paths (EAPs), but only in their *intersection pattern*, i. e., whether certain pairs of EAPs cross in a certain face or on a certain edge. The intersection pattern is not going to be unique,

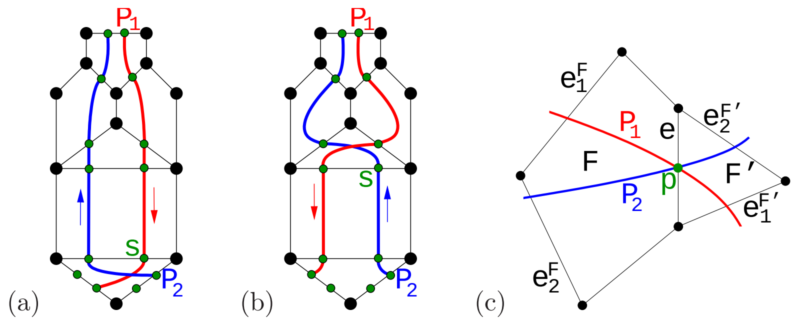


Figure 2: (a,b) Slots are colored in green, and slot conflicts occur at s.(a) P_1 picked the slots before P_2. (b) P_2 picked the slots before P_1. (c) Illustration to proof of Proposition 3.4.

but our central definition, i. e., that of a well-arranged graph, will be invariant to ambiguities of intersection patterns (see Proposition 3.6).

Next we formulate rules for embedding a single edge $\{e, f\}$ of an alternating path into $\mathbb{R}^2$. If F is the unique face of G with $\{e, f\} \subset E(F)$, we embed $\{e, f\}$ into $\overline{F} = F \cup E(F)$. To this end, we first represent $\overline{F}$ by a *regular* filled polygon $\overline{F_r}$ with the same number of sides. We then embed $\{e, f\}$ into $\overline{F_r}$ as a line segment L between two points on the sides of $\overline{F_r}$. Due to the Jordan-Schönflies theorem [16], there exists a homeomorphism $\hbar : \overline{F_r} \mapsto \overline{F}$. The embedding of $\{e, f\}$ into $\overline{F}$ is then given by $\hbar(L)$. Since $\hbar$ is a homeomorphism, the intersection pattern of the line segments in $\overline{F}$ coincides with that in $\overline{F_r}$ (see Figures 3a and 4a).

Local embedding rules. The local rules for embedding an alternating path into $\overline{F_r}$ are as follows.

1. The part of an EAP that runs through F_r is a line segment, and EAPs cannot coincide in F_r.

2. An EAP can intersect $e \in E(F_r)$ only in e's relative interior, i. e., not at e's end vertices.

3. Let $F_r \neq F_\infty$ be an even face of G, let e, f be unique opposite edges in $E(F_r)$, and let P_1, P_2 be the two alternating paths that contain the edge $\{e, f\}$ ($P_1 = P_2$ if and only if the multiplicity of P_1 is two). Then the parts of embedded P_1 and P_2 that run through $\overline{F_r}$ must form a pair of distinct parallel line segments (see Figure 3a).

4. Let $F \neq F_\infty$ be an odd face of G, let $e \in E(F_r)$, and let P_1, P_2 be the two alternating paths that contain the vertex e. If e also bounds an even bounded face or F_∞, embedded P_1, P_2 must intersect e at two distinct points (see the upper left edge of the hexagon in Figures 3b,c). If the other face is a bounded odd face, embedded P_1, P_2 must cross at a point on e (see the upper right edge of the hexagon in Figures 3b,c).

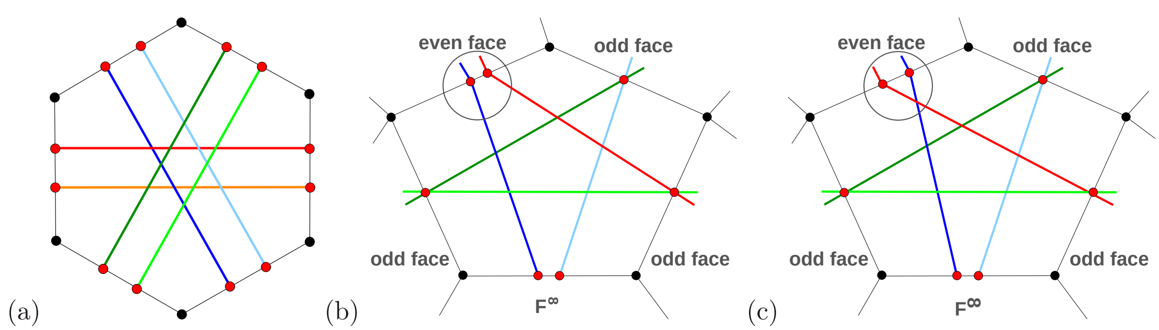


Figure 3: (a) Intersection pattern in a regular hexagon (b,c) Two intersection patterns in a regular pentagon (see gray circle for difference). In (b) we have no slot conflict on the upper left edge of the hexagon, and in (c) we have a slot conflict on the upper left edge.

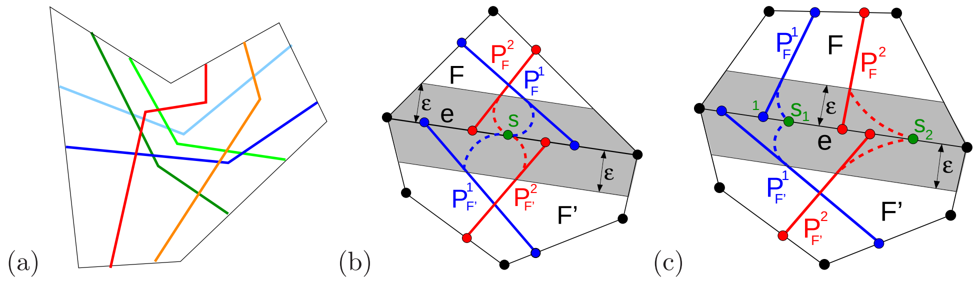


Figure 4: (a) Hexagonal face with the same intersection pattern as in Figure 3a. (b,c) Bending of alternating paths is indicated by the dashed colored lines. For the notation see the text. (b) Bending toward a single slot s on e. (c) Bending toward two slots s_1, s_2 on e.

5. A slot conflict on an edge e of G can occur only if e separates a bounded odd face F from a face that is not both bounded and odd. Let $\{e, f\}$, $\{e, f'\}$ be the two edges (of alternating paths) occupying the two slots of e, and let F_r be the regular polygon that represents F. Then $\{e, f\}$ and $\{e, f'\}$ cross inside F_r if and only if there is a slot conflict on e (see Figures 3b,c).

We map the embedded edges (of alternating paths) from any $\overline{F_r}$ into $\overline{F}$ using a homeomorphism from $\overline{F_r}$ onto $\overline{F}$. The following is about tying the loose ends of the locally embedded paths, which all sit on edges of G, so as to arrive at a *global* embedding of the alternating paths (see Figures 4b,c). Let $e \in E(G)$, and let F, F' be the faces of G that are bounded by e. We have two locally embedded paths P_F^1, P_F^2 in $\overline{F}$ and two locally embedded paths $P_{F'}^1, P_{F'}^2$ in $\overline{F'}$ that all hit e. We bend the four paths toward their predetermined slots. The bending operations can be done such that the intersection patterns do not change in the interiors of F and F'. Indeed, recall that the paths are homeomorphic to straight line segments. Thus, there exists $\epsilon > 0$ such that all locally embedded paths in F and F' other than P_F^1, P_F^2, $P_{F'}^1$, and $P_{F'}^2$ keep a distance greater than ϵ to e. The bending, in turn, can be done such that it affects P_F^1, P_F^2, $P_{F'}^1$, and $P_{F'}^2$ only in an ϵ-neighborhood of e.

Proposition 3.4. *If two EAPs share a point p, they cross at p and not just touch.*

Proof. Consider two EAPs P_1 and P_2 that share a point p. If p sits in a face F of G, P_1 and P_2 cross at p because (i) $P_1 = \hbar(L_1)$ and $P_2 = \hbar(L_2)$ for a homeomorphism $\hbar : F_r \mapsto F$, (ii) $L_1 \neq L_2$ cannot touch without crossing, and (iii) homeomorphisms preserve crossings and non-crossings of curves.

If p sits on (the interior of an edge) $e \in E$, the two faces separated by e, and denoted by F and F', must be finite and odd. As illustrated in Figure 2c, P_1 [P_2] enters F through an edge e_F^1 [e_F^2] in $E(F) \setminus E(F')$, runs from F into F' via e, and then leaves F' via an edge $e_{F'}^1$ [$e_{F'}^2$] in $E(F') \setminus E(F)$. Since F and F' are finite and odd, we have $e_F^1 \neq e_F^2$ and $e_{F'}^1 \neq e_{F'}^2$. Without loss of generality, e is the left [right] opposite of e_F^1 [e_F^2]. Then, due to item (ii) in Definition 3.3, $e_{F'}^1$ [$e_{F'}^2$] is the right [left] opposite of e. Thus, before reaching point p on e, P_1 is left of P_2, and after leaving p, P_1 is right of P_2. In other words, P_1 and P_2 cross at p. □

EAPs like the ones in Figure 1b are special in that they form an arrangement in the following sense.

Definition 3.5 (Arrangement of alternating paths)**.** *A collection of all EAPs in G is called an arrangement of embedded alternating paths if (i) none of the EAPs crosses itself and (ii) no pair of EAPs crosses twice.*

We will now see that Definition 3.5 does not depend on the particular collection of EAPs, i. e., that it is actually a definition for G.

Proposition 3.6. *If one collection of EAPs in G is an arrangement of alternating paths, then any collection of EAPs in G is an arrangement of alternating paths.*

Proof. Definition 3.5 depends only on the intersection pattern of the EAPs. Different intersection patterns, in turn, can only arise from different solutions of slot conflicts.

No EAP P can have a slot conflict with itself, because this would mean that P would traverse a face twice, a contradiction because then all the other EAPs that traverse the face would cross P twice.

Thus, it suffices to consider slot conflicts between different EAPs. Due to (i) P_1 and P_2 being alternating paths and (ii) the way we assigned the slots, the intersection pattern of two EAPs P_1 and P_2 that cross edges of G from the same side is unique. If, however, an edge e of G is crossed by P_1 in one direction, and by P_2 in the opposite direction, and if $e \in E(F)$ for a bounded odd face F, the intersection pattern of P_1 and P_2 depends on whether P_1 or P_2 was embedded first. This case is illustrated in Figures 2a,b.

It remains to show that the above ambiguity in intersection patterns does not turn an arrangement into a non-arrangement or vice versa. Indeed, if F is the only bounded odd face traversed by P_1 and P_2, then P_1 and P_2 do not cross in F, anyway.

Now assume that there exists another bounded odd face $\hat{F}$ traversed by P_1 and P_2. Examples for F, $\hat{F}$ are the lower and central triangular faces in Figures 2a,b. We denote by P_1^* the reverse of P_1. Without loss of generality we assume that P_1^* [P_2] turns left [right] on F. Due to the slot conflict on $E(F)$ (resulting in the crossing of P_1^* and P_2 in F), P_1^* then runs on the right side of P_2^*. Since P_1^*, P_2 are alternating paths, P_1^* [P_2] has to take a right [left] turn in $\hat{F}$. Thus, P_1^* and P_2 diverge into different faces behind $\hat{F}$ without crossing in $\hat{F} \cup E(\hat{F})$.

If the slot conflict on $E(F)$ had been avoided, there would be no crossing in F, and P_1^* would run on the left side of P_2^* (see Figure 2b). Then P_1^* and P_2 would cross in $\hat{F}$. Since the intersection pattern before F and behind $\hat{F}$ is not affected by the ambiguity in F, the total number of crossings between P_1 and P_2 is not affected, either. □

Proposition 3.6 justifies the following definition.

Definition 3.7 (Well-arranged graph)**.** *We call a plane graph G* well-arranged *if its collections of EAPs are arrangements of alternating paths.*

3.3 Partial cubes from well-arranged graphs

Arrangements of alternating paths are generalizations of arrangements of pseudolines [4]. The latter are known to have duals that are partial cubes [11]. In this section we will see that the dual of an arrangement of alternating paths is a partial cube, too.

Notation 3.8. *From now on $\mathcal{E}(G)$ denotes a collection of EAPs.*

The purpose of the following is to prepare the definition of $\mathcal{E}(G)$'s dual (see Definition 3.10).

Definition 3.9 (Domain $D(G)$ of G, facet of $\mathcal{E}(G)$, adjacent facets)**.** *The domain $D(G)$ of G is the set of points covered by the vertices, edges and bounded faces of G. A facet of $\mathcal{E}(G)$ is a (bounded) connected component (in $\mathbb{R}^2$) of $D(G) \setminus (\bigcup_{e \in E(G)} e \cup \bigcup_{v \in V(G)} v)$. Two facets of $\mathcal{E}(G)$ are adjacent if their boundaries share more than one point.*

In the following, DEAP stands for Dual of Embedded Alternating Paths.

Definition 3.10 (DEAP graph $G_{\mathcal{E}}$ of G)**.** *A DEAP graph $G_{\mathcal{E}}$ of G is a plane graph that we obtain from G by placing one vertex into each facet of $\mathcal{E}(G)$ and drawing edges between a pair (u, v) of these vertices if the facets containing u and v are adjacent in the sense of Definition 3.9. A vertex of $G_{\mathcal{E}}$ can also sit on the boundary of a face as long as it does not sit on an EAP from $\mathcal{E}(G)$ (for an example see the black vertex on the upper left in Figure 5a).*

Due to the intersection pattern of the EAPs in G's bounded faces, as specified in Section 3.2 and illustrated in Figure 3, there are the following three kinds of vertices in $V(G_{\mathcal{E}})$.

Definition 3.11 (Primal, intermediate and star vertex of $G_{\mathcal{E}}$)**.**

- ***Primal vertices:*** *Vertices which represent a facet that contains a (unique) vertex v of G in its interior or on its boundary. As we do not care about the exact location of $G_{\mathcal{E}}$'s vertices, we may assume that the primal vertices of $G_{\mathcal{E}}$ are precisely the vertices of G.*
- ***Intermediate vertices****: The neighbors of the primal vertices in $G_{\mathcal{E}}$.*
- ***Star vertices*** *The remaining vertices in $G_{\mathcal{E}}$.*

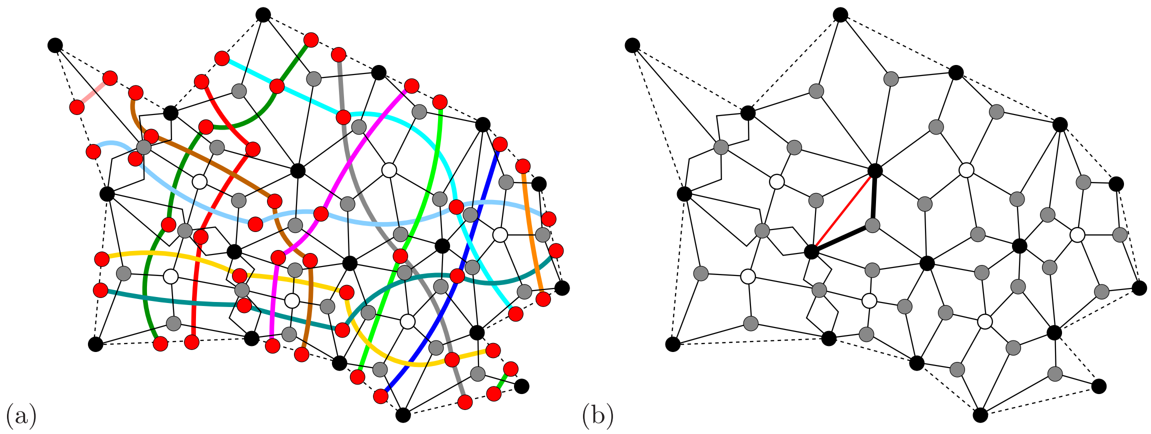


Figure 5: DEAP graph $G_{\mathcal{E}}$ of the primal graph G shown in Figure 1a. (a) Collection $\mathcal{E}(G)$ of EAPs: Red vertices, thick solid colored lines. DEAP graph $G_{\mathcal{E}}$: Black, gray and white vertices, thin black solid lines. The black, gray and white vertices are the primal, intermediate and star vertices, respectively. The dashed polygonal line delimits $D(G)$. (b) $G_{\mathcal{E}}$ only. The red edge, however, is an edge of G. The path formed by the two bold black edges is an example of a path in $G_{\mathcal{E}}$ of length two that connects two primal vertices that are adjacent in G via an intermediate vertex in $G_{\mathcal{E}}$.

For an example of a DEAP graph see Figure 5, where the black, gray and white vertices correspond to the primal, intermediate, and star vertices, respectively.

Definition 3.12. *Let P be an EAP of G, and let D_1, D_2 denote the two connected components $D(G) \setminus P$. The cut of $G = (V, E)$ given by P is (V_1, V_2) with $V_i = \{v \in V \mid v \in D_i\}$, and the cut of $G_{\mathcal{E}} = (V_{\mathcal{E}}, E_{\mathcal{E}})$ given by P is (V_1, V_2) with $V_i = \{v \in V_{\mathcal{E}} \mid v \in D_i\}$.*

Theorem 3.13. *The DEAP graph $G_{\mathcal{E}}$ of a well-arranged plane graph G is a partial cube, the convex cuts of which are precisely the cuts given by the EAPs of G.*

Proof. We denote the Hamming distance by $h(\cdot, \cdot)$. To show that $G_{\mathcal{E}} = (V_{\mathcal{E}}, E_{\mathcal{E}})$ is a partial cube, it suffices to specify a labeling $l : V_{\mathcal{E}} \mapsto \{0, 1\}^n$ for some $n \in \mathbb{N}$ such that $d_{G_{\mathcal{E}}}(u, v) = h(l(u), l(v))$ for all $u, v \in V_{\mathcal{E}}$ (see Section 2).

Let $\mathcal{E}(G) = \{P_1, \ldots, P_n\}$. The length of the binary vectors will be n. The entry of $l(v)$ indicates v's position with respect to the paths in $\mathcal{E}(G)$. Specifically, we arbitrarily select one component of $D(G) \setminus P_i$ and set the ith entry of $l(v)$ to one if the face represented by v is part of the selected component (zero otherwise).

It remains to show that $d_{G_{\mathcal{E}}}(u, v) = h(l(u), l(v))$ for any pair $u \neq v \in V$. Since on any path of length k from u to v in $G_{\mathcal{E}}$ it holds that $h(l(u), l(v)) \leq k$, we have $d_{G_{\mathcal{E}}}(u, v) \geq h(l(u), l(v))$.

We assume $u \neq v$ (the case $u = v$ is trivial). To see that $d_{G_{\mathcal{E}}}(u, v) = h(l(u), l(v))$, it suffices to show that u has a neighbor u' such that $h(l(u'), l(v)) < h(l(u), l(v))$ (because then there also exists u'' such that $h(l(u''), l(v)) < h(l(u'), l(v))$ and so on until v is reached in exactly $h(l(u), l(v))$ steps).

Let F_u denote the facet of $\mathcal{E}(G)$ that is represented by u, and $I(u)$ denote the set of indices of EAPs in $\mathcal{E}(G)$ that bound F_u.

1. If u has only one neighbor u', then $I(u) = \{k\}$ for some k, and the only vertex in one of the components of $D(G) \setminus P_k$ is u. For an example of u' see the black vertex in the upper left corner of Figure 5b. Since $l(u)$ and $l(u')$ differ only at position k, it must hold that $h(l(u'), l(v)) < h(l(u), l(v))$.

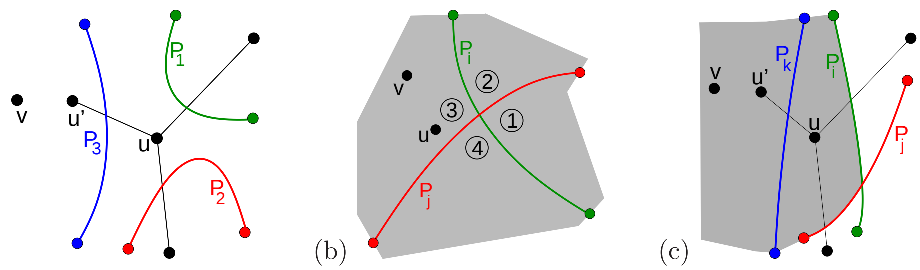
2. If u has at least two neighbors, we first assume that none of the P_k with $k \in I(u)$ cross each other (see Figure 6a). Then u is uniquely determined by the entries of $l(u)$ at the positions given by $I(u)$. Indeed, F_u is then bounded by non-intersecting paths EAPs that run from a point on the border of $D(G)$ to another point on the border of $D(G)$, and only a vertex inside F_u can have the same entries in $l(\cdot)$ as $l(u)$ at the positions given by $I(u)$. Thus, since u is the only vertex in F_u and since $u \neq v$,

$l(u)$ and $l(v)$ must differ at a position k^* in $I(u)$ and, from the perspective of u, we find u' in the face on the other side of P_{k^*}.

3. The remaining case is that u has at least two neighbors and there exists at least one pair $(i, j) \in I(u) \times I(u)$, $i \neq j$, such that P_i crosses P_j. Let C denote the set of all such pairs. For any pair $(i, j) \in C$ the path P_i crosses the path P_j exactly once, because $\mathcal{E}(G)$ is an arrangement of alternating paths. Thus P_i and P_j subdivide $D(G)$ into four regions (see Figure 6b), each of which is characterized by one of the four $0/1$ combinations of vertex label entries at i and at j. We may assume that v is contained in the same region as u for each pair $(i, j) \in C$ (otherwise we choose u' on the other side of P_i or P_j and are done). Let R be the intersection of all these regions, one region per pair in C.

   If all $i \in I(u)$ are contained in at least one pair of C, we are done. Indeed this means that $R = F_u$ and thus that u is uniquely determined by the entries of $l(u)$ at the positions given by $I(u)$. We can then proceed as above. The remaining case is that there exist $k \in I(u)$ such that P_k does not intersect any P_j with $j \in I(u), j \neq k$ (see Figure 6c). Recall that we assumed $u \neq v$, $u, v \in R$, i.e., u and v are separated by P_k. Hence, the entries of $l(u)$ and $l(v)$ differ at a position $k \in I(u)$, and u' with $h(l(u'), l(v)) < h(l(u), l(v))$ can be reached from u by crossing P_k.

So far we have shown that $G_{\mathcal{E}}$ is a partial cube and that the cut of $G_{\mathcal{E}}$ given by any P_i is precisely the cut that determines the i-th entry of $l(u)$ for all $u \in V_{\mathcal{E}}$. Thus, the cuts of $G_{\mathcal{E}}$ given by the P_i, $1 \leq i \leq n$, are precisely the n convex cuts of $G_{\mathcal{E}}$. □



(a) (b) (c)

Figure 6: Subgraph of $G_{\mathcal{E}}$: black vertices and black edges. Subset of EAPs: colored vertices and edges. (a) Illustration to item 2 in proof of Theorem 3.13. (b,c) Illustrations to item 3 in proof of Theorem 3.13. The shaded regions in (b) and (c) indicate $D(G)$ and R, respectively.

Recall that the cut-sets of the convex cuts of a partial cube are precisely the equivalence classes of the Djoković relation (see Section 2). Thus, Theorem 3.13 yields the following.

Corollary 3.14. *If G is well-arranged, there exists a one-to-one correspondence between the EAPs of G and the equivalence classes of the Djoković relation on $G_{\mathcal{E}}$. Specifically, an equivalence class of the Djoković relation on $G_{\mathcal{E}}$ is given by the set of edges intersected by an EAP of G, and vice versa.*

3.4 Convex Cuts from Alternating Paths

In case of G being well-arranged, the subgraph defined below will serve as a stepping stone for relating the convex cuts of $G_{\mathcal{E}}$ to those of G (see Lemma 3.16).

Definition 3.15 (Subgraph $\widetilde{G_{\mathcal{E}}}$ of $G_{\mathcal{E}}$)**.** *$\widetilde{G_{\mathcal{E}}}$ is the graph obtained from $G_{\mathcal{E}}$ by deleting all star vertices and replacing parallel edges by single edges.*

In the following recall that we identified the primary vertices in $G_{\mathcal{E}}$ with the vertices of G.

Lemma 3.16. *If G is well-arranged, then $d_G(u, v) = \frac{1}{2} d_{\widetilde{G_{\mathcal{E}}}}(u, v) = \frac{1}{2} d_{G_{\mathcal{E}}}(u, v)$.*

Proof.

- For any face F of G, let G^F $[G^F_\mathcal{E}]$ denote the subgraph of G $[G_\mathcal{E}]$ that is contained in $\overline{F}$. From G being well-arranged follows that G^F is well arranged, and Theorem 3.13 yields that $G^F_\mathcal{E}$ is a partial cube.
- Let u, v be vertices of G^F. Then, due to the EAPs being embedded alternating paths, there exists a path P from u to v in G that (i) contains only edges from $E(F)$ and (ii) crosses any of the EAPs through F at most once. Hence, there exists a path $\widetilde{P_\mathcal{E}}$ from u to v in $\widetilde{G_\mathcal{E}}$ (see Definition 3.15) that (i) is a subdivision of P with every other vertex being an intermediate vertex and (ii) also crosses any of the EAPs through F at most once. Since $G^F_\mathcal{E}$ is a partial cube, $\widetilde{P_\mathcal{E}}$ is a shortest path in $G^F_\mathcal{E}$. Its length is $2d_G(u, v)$.
- Let u, v be vertices of G. Using the previous item repeatedly, we get that there exists a shortest path $\widetilde{P_\mathcal{E}}^*$ from u to v in $\widetilde{G_\mathcal{E}}$ whose length is $d_{G_\mathcal{E}}(u, v) = d_{\widetilde{G_\mathcal{E}}}(u, v)$. On $\widetilde{P_\mathcal{E}}^*$ the vertices in G alternate with intermediate vertices, and for any edge $\{x, y\}$ in G there exists a path of length two between x and y with the vertex in the middle being an intermediate vertex. Thus, $d_G(u, v) = \frac{1}{2}d_{\widetilde{G_\mathcal{E}}}(u, v) = \frac{1}{2}d_{G_\mathcal{E}}(u, v)$.

□

Proposition 3.17. *If G is well-arranged, any cut given by an EAP of G is convex.*

Proof. Let (V_1, V_2) be the cut of G given by an EAP P of $G = \{V, E\}$. Without loss of generality let $u_1, v_1 \in V_1$. We have to show that any shortest path from u_1 to v_1 contains only vertices from V_1.

We assume the opposite, i. e., that there exists a shortest path S from u_1 to v_1 in G that contains a vertex from V_2, i. e., a vertex on the other side of the EAP P. This shortest path can be turned into a path $S_\mathcal{E}$ in $G_\mathcal{E}$ with twice the length by inserting a vertex from $v' \setminus V$ between any pair of consecutive vertices on S. Lemma 3.16 then yields that $S_\mathcal{E}$ is a shortest path in $G_\mathcal{E}$ that crosses the EAP P twice. This is a contradiction to Theorem 3.13. □

4 Convex cuts of bipartite graphs

Let $H' = (V, E)$ be a bipartite but not necessarily plane graph. As mentioned in Section 2, any edge $e = \{a, b\}$ of H' gives rise to a cut of H' into W_{ab} and W_{ba}. The cut-set of this cut is $C_e = \{f \in E \mid e\ \theta'\ f\}$, where θ' is the Djoković relation on H'. Note that $C_e = \{f = \{u, v\} \in E \mid d_{H'}(a, u) = d_{H'}(b, v)\}$. In the following we characterize the cut-sets of the *convex* cuts of H'. This characterization is key to finding all convex cuts of a bipartite graph in $\mathcal{O}(|E|^3)$ time.

Lemma 4.1. *Let $H' = (V, E)$ be a bipartite graph, and let $e \in E$. Then C_e is the cut-set of a convex cut of H' if and only if $f\ \theta'\ \hat{f}$ for all $f, \hat{f} \in C_e$.*

Proof. "⇐" Let $e = \{a, b\}$, and assume that the cut with cut-set C_e is not convex. Then there exists a shortest path $P = \{v_1, \dots v_n\}$ with both end vertices in, say, W_{ab} such that P has a vertex in W_{ba}. Let i be the smallest index such that $v_i \in W_{ba}$, and let j be the smallest index greater than i such that $v_j \in W_{ab}$. Then $f = \{v_{i-1}, v_i\}$ and $\hat{f} = \{v_{j-1}, v_j\}$ are contained in C_e. We use now a result by Ovchinnikov [14, Lemma 3.5], which states that no pair of edges on a shortest path are related by θ', i. e., $f\ \theta'\ \hat{f}$ does *not* hold.

"⇒" Let $f = \{u, v\}$ and $\hat{f} = \{\hat{u}, \hat{v}\}$ be edges in C_e such that $f\ \theta'\ \hat{f}$ does not hold. Without loss of generality assume $u, \hat{u} \in W_{ab}$, $v, \hat{v} \in W_{ba}$ and $d_{H'}(u, \hat{u}) < d_{H'}(v, \hat{v})$. Due to H' being bipartite, both distances are even or both are odd. Hence, $d_{H'}(v, \hat{v}) - d_{H'}(u, \hat{u}) \geq 2$. Consider the path $\hat{P}$ from v via f to u, then along a shortest path from u to $\hat{u}$ and finally from $\hat{u}$ to $\hat{v}$ via $\hat{f}$. This path has length $d_{H'}(u, \hat{u}) + 2 \leq d_{H'}(v, \hat{v})$. Thus, $\hat{P}$ is a shortest path from v to $\hat{v}$ (and $d_{H'}(v, \hat{v}) - d_{H'}(u, \hat{u}) = 2$). The path $\hat{P}$ is a shortest path from $v \in W_{ba}$ via $u, \hat{u} \in W_{ab}$ to $\hat{v} \in W_{ba}$, so that C_e is not the cut-set of a convex cut. □

Lemma 4.1 suggests to determine the convex cuts of H' as sketched in Algorithm 1 by checking for each cut-set C_{e^i} if the cut-sets of the contained edges f^j all coincide.

Theorem 4.2. *All convex cuts of a bipartite graph can be found using $\mathcal{O}(|E|^3)$ time and $\mathcal{O}(|E|)$ space.*

Algorithm 1 Find all cut-sets of convex cuts of a bipartite graph H'

1: **procedure** EVALUATECUTSETS(bipartite graph H')
 $\triangleright$ Computes C_{e^i} for each edge e^i and stores in $isConvex[i]$ if C_{e^i} is the cut-set of a convex cut
2: Let $e^1, \ldots e^m$ denote the edges of H'; initialize all m entries of the array $isConvex$ as $true$
3: **for** $i = 1, \ldots, m$ **do**
4: Determine $C_{e^i} = \{f^j \mid e^i \ \theta \ f^j\}$
5: **for all** $f^j \neq e^i$ **do**
6: Determine $C_{f^j} = \{g^k \mid f^j \ \theta \ g^k\}$
7: **if** $C_{f^j} \neq C_{e^i}$ **then**
8: $isConvex[i] := false$
9: **break**
10: **end if**
11: **end for**
12: **end for**
13: **end procedure**

Proof. We use Algorithm 1. The correctness follows from Lemma 4.1 and the symmetry of the Djoković relation. Regarding the running time, observe that for any edge e^i, the (not necessarily convex) cut-set C_{e^i} can be determined using breadth-first search to compute the distances of any vertex to the end vertices of e^i. Thus, any C_{e^i} can be determined in time $\mathcal{O}(|E|)$. We mark [unmark] the edges in C_{e^i} when entering [exiting] the inner for-loop, which has $\mathcal{O}(|E|))$ iterations.

The time complexity $\mathcal{O}(|E|))$ of an iteration of the inner loop is due to the calculation of C_{f^j}. The test whether $C_{f^j} = C_{e^i}$ is done on the fly: any time a new edge of C_{f^j} is found, we only check if the edge has a mark. We have $C_{f^j} = C_{e^i}$ if and only if all new edges are marked. This follows from the fact that no proper subset of C_{e^i} can be a cut-set (the subgraphs induced by W_{ab} and W_{ba} are connected). Hence, Algorithm 1 runs in $\mathcal{O}(|E|^3)$ time. Since no more than two cut-sets (with $\mathcal{O}(|E|)$ edges each) have to be stored at the same time, the space complexity of Algorithm 1 is $\mathcal{O}(|E|)$. $\square$

A simple loop-parallelization over the edges in line 3 leads to a parallel running time of $\mathcal{O}(|E|^2)$ with $\mathcal{O}(|E|)$ processors. If one is willing to spend more processors and a quadratic amount of memory, then even faster parallelizations are possible. Since they use standard PRAM results, we forgo their description.

5 Convex cuts of general graphs

In Theorem 5.3 of this section we characterize the cut-sets of the convex cuts of a general graph H in terms of two binary relations on edges: the Djoković relation and the relation τ (for τ see Definition 5.1). We will use Theorem 5.3 in Section 6 to find all convex cuts of a *plane* graph.

While the relation τ is applied to the edges of H, the Djoković relation is applied to the edges of a bipartite subdivision H' of H. Specifically, H' is the graph that one obtains from H by subdividing each edge of H into two edges. An edge e in H that is subdivided into edges e_1, e_2 of H' is called *parent* of e_1, e_2, and e_1, e_2 are called *children* of e.

Definition 5.1 (Relation τ). *Let $e = \{u_e, v_e\}$ and $f = \{u_f, v_f\}$ be edges of H. Then, $e \ \tau \ f$ iff $d_H(u_e, u_f) = d_H(v_e, v_f) = d_H(u_e, v_f) = d_H(v_e, u_f)$.*

The next lemma follows directly from the definition of θ' and τ.

Lemma 5.2. *If $e \ \tau \ f$, then none of the children of e is θ'-related to a child of f.*

Theorem 5.3. *A cut of H with cut-set C is convex if and only if for all $e, f \in C$ it holds either $e \ \tau \ f$ or that there exists a child e' of e and a child f' of f such that $e' \ \theta' \ f'$.*

To simplify the proof of Theorem 5.3, we first establish the following result.

Lemma 5.4. *Let $e = \{u_e, v_e\}$ and $f = \{u_f, v_f\}$ be edges of H. Then the following is equivalent.*

(i) There exists a child $\{a', b'\}$ of e with a' closer to u_e than b' and a child $\{c', d'\}$ of $f = \{u_f, v_f\}$ with c' closer to u_f than d' such that $d_{H'}(a', c') = d_{H'}(b', d')$.

(ii) $e \; \tau \; f$ or there exists a child e' of e and a child f' of f with $e' \; \theta' \; f'$.

Proof. We denote by w'_e $[w'_f]$ the vertex of H' that subdivides e $[f]$. Without loss of generality we assume that $d_H(u_e, u_f) \geq d_H(v_e, v_f)$ (see Figures 7a,b).

To prove "$(i) \Rightarrow (ii)$", let $d_{H'}(a', c') = d_{H'}(b', d')$.

- We first assume $d_H(u_e, v_f) \neq d_H(u_e, u_f)$ and $d_H(v_e, u_f) \neq d_H(u_e, u_f)$ (see Figure 7a). Then our assumption $d_H(u_e, v_f) \geq d_H(u_e, u_f)$, in conjunction with the fact that w'_e and w'_f both have degree two, imply that there exists a shortest path from w'_e via v_e and v_f to w'_f. The equality $d_{H'}(a', c') = d_{H'}(b', d')$ yields $d_H(u_e, u_f) = d_H(v_e, v_f)$. Indeed, $d_H(u_e, u_f) > d_H(v_e, v_f)$ would mean that there exists a shortest path from u_e via v_e to u_f, a contradiction to $d_{H'}(a', c') = d_{H'}(b', d')$ (recall that a' is closer to u_e than b' and that c' is closer to u_f than d').

  From $d_H(u_e, v_f) \neq d_H(u_e, u_f)$, $d_H(v_e, u_f) \neq d_H(u_e, u_f)$, and $d_{H'}(a', c') = d_{H'}(b', d')$ we conclude $a' = u_e$, $b' = w'_e$, $c' = w'_f$ and $d' = v_f$. In particular, $\{a', b'\} \; \theta \; \{c', d'\}$.

- If $d_H(u_e, v_f) \neq d_H(u_e, u_f)$ and $d_H(v_e, u_f) = d_H(u_e, u_f)$ holds (see Figure 7b), there must exist a shortest path from v_e to u_f via v_f and w'_f because otherwise the prerequisite $d_{H'}(a', c') = d_{H'}(b', d')$ would not hold. Then $d_H(v_e, v_f) = d_H(v_e, u_f) - 1 = d_H(u_e, u_f) - 1$. Thus $d_{H'}(a', c') = d_{H'}(b', d')$ implies $a' = u_e$, $b' = w'_e$, $c' = v_e$ and $d' = w'_f$. In particular, $\{a', b'\} \; \theta \; \{c', d'\}$.

- If $d_H(u_e, v_f) \neq d_H(u_e, u_f)$ and $d_H(v_e, u_f) = d_H(u_e, u_f)$, then we can proceed as in the previous item and conclude that we cannot have $d_{H'}(a', c') = d_{H'}(b', d')$ or that $d_H(u_e, u_f) = d_H(v_e, v_f) - 1$, a contradiction to our assumption $d_H(u_e, u_f) \geq d_H(v_e, v_f)$.

- If $d_H(u_e, v_f) = d_H(u_e, u_f)$ and $d_H(v_e, u_f) = d_H(u_e, u_f)$, then $e \; \tau \; f$. Indeed, due to the definition of τ, it suffices to show that $d_H(u_e, u_f) = d_H(v_e, v_f)$. We assume the opposite, i.e., that $d_H(u_e, u_f) > d_H(v_e, v_f)$ (recall that we assume $d_H(u_e, u_f) \geq d_H(v_e, v_f)$). Thus, $d_H(u_e, u_f) > d_H(v_e, v_f) \geq d_H(v_e, u_f) - 1$, i.e., $d_H(u_e, u_f) \geq d_H(v_e, u_f)$. If $d_H(u_e, u_f) = d_H(v_e, u_f)$, and thus $d_H(v_e, v_f) < d_H(u_e, u_f) = d_H(v_e, u_f) = d_H(u_e, v_f)$, there exists a shortest path from v_e via v_f to u_f, a contradiction to $d_{H'}(a', c') = d_{H'}(b', d')$. Otherwise, $d_H(u_e, u_f) > d_H(v_e, u_f) = d_H(u_e, u_f)$. If $d_H(v_e, v_f) < d_H(v_e, u_f)$, there exists a shortest path from v_e via v_f to u_f (recall that $d_H(u_e, u_f) > d_H(v_e, v_f)$), a contradiction to $d_{H'}(a', c') = d_{H'}(b', d')$. The case $d_H(v_e, v_f) \geq d_H(v_e, u_f)$ does not occur since, due to $d_H(v_e, u_f) = d_H(u_e, v_f)$, we would get $d_H(v_e, v_f) \geq d_H(u_e, u_f)$.

To prove "$(ii) \Rightarrow (i)$", assume that $e' \; \theta \; f'$ for a child e' of e and a child f' of f. Then the end vertices of a', b' of e' and c', d' of f' fulfill $d_{H'}(a', c') = d_{H'}(b', d')$ by definition of θ'. If $e \; \tau \; f$, then we set $a' = u_e$, $b' = w'_e$, $c' = w'_f$, and $d' = v_f$ as in Figure 7a.

The "either" in the claim follows from Lemma 5.2. □

Proof of Theorem 5.3.
To prove necessity, let C be the cut-set of a *convex* cut that partitions V into V_1 and V_2, and let $e, f \in C$ (see Figure 7). Thanks to Lemma 5.4 it suffices to find a child $\{a', b'\}$ of e and a child $\{c', d'\}$ of f such that $d_{H'}(a', c') = d_{H'}(b', d')$. Let w'_e $[w'_f]$ denote the vertex of H' that subdivides e $[f]$. Without loss of generality we assume $u_e, u_f \in V_1$ and $v_e, v_f \in V_2$. Since C is the cut-set of a convex cut, we know that $d_H(u_e, u_f)$ and $d_H(v_e, v_f)$ differ by at most one.

1. If $d_H(v_e, v_f) = d_H(u_e, u_f)$, let $\{a', b'\} = \{u_e, w'_e\}$ and $\{c', d'\} = \{w'_f, v_f\}$ (this is the case illustrated in Figure 7a). Then, due to the degrees of w'_e and w'_f being two, $d_{H'}(a', c') = d_{H'}(u_e, w'_f) = d_{H'}(w'_e, v_f) = d_{H'}(b', d')$.

2. If $d_H(u_e, u_f)$ and $d_H(v_e, v_f)$ differ by exactly one, we may assume without loss of generality that $d_H(v_e, v_f) = d_H(u_e, u_f) + 1$. Set $\{a', b'\} = \{w'_e, v_e\}$ and $\{c', d'\} = \{w'_f, v_f\}$. Then, due to the degrees of w'_e and w'_f being two, $d_{H'}(a', c') = d_{H'}(w'_e, w'_f) = d_{H'}(v_e, v_f) = d_{H'}(b', d')$.

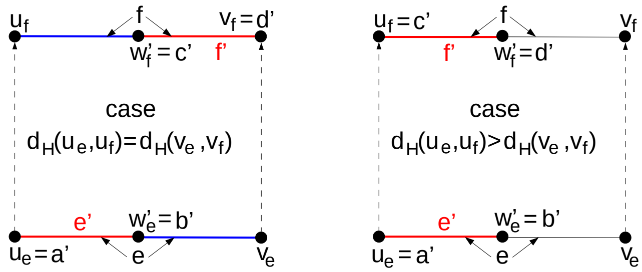


Figure 7: Illustrations to proof of Theorem 5.3. (a) If $e\ \tau\ f$ does not hold, at least one child of e is θ'-related to a child of f, e. g., $e'\ \theta'\ f'$. (b) $e\ \tau\ f$ does not hold, and exactly one child of e is related to a child of f (here $e'\ \theta'\ f'$).

Conversely, to prove sufficiency, let C be the cut-set of a cut that partitions V into V_1 and V_2. We distinguish the two cases of the prerequisite.

- Case 1 ($e\ \tau\ f$): Then we have $d_H(u_e, u_f) = d_H(v_e, v_f)$ by definition of τ.
- Case 2 (there exists a child e' of e and a child f' of f such that $e'\ \theta'\ f'$): As above we assume without loss of generality that $u_e, u_f \in V_1$ and $v_e, v_f \in V_2$. There are four possibilities for the positions of e' and f' within e and f, only two of which need to be considered due to symmetry.

  1. $e' = \{u_e, w'_e\}$ and $f' = \{w'_f, v_f\}$. Since the degrees of w'_e and w'_f are two, and since $e'\ \theta'\ f'$, any shortest path from u_e to w'_f runs via u_f, and any shortest path from w'_e to v_f runs via v_e. Hence, $d_{H'}(u_e, u_f) = d_{H'}(u_e, w'_f) - 1 = d_{H'}(w'_e, v_f) - 1 = d_{H'}(v_e, v_f)$.
  2. $e' = \{u_e, w'_e\}$ and $f' = \{u_f, w'_f\}$. In this case $d_{H'}(u_e, u_f) = d_{H'}(w'_e, w'_f) = d_{H'}(v_e, v_f) \pm 2$.

To summarize Case 1 and Case 2, we always have $d_{H'}(u_e, u_f) = d_{H'}(v_e, v_f) \pm 2$.

Due to $d_{H'}(u, v) = 2d_H(u, v)$ for all vertices u, v of H, we have that $d_H(u_e, u_f) = d_H(v_e, v_f) \pm 1$ for all $e = \{u_e, v_e\}, f = \{u_f, v_f\}$ in the cut-set C. Hence, any shortest path with end vertices in V_1 [V_2] stays within V_1 [V_2], i. e., C is the cut-set of a convex cut.

6 Convex cuts of plane graphs

In this section $G = (V, E)$ is a plane graph with the restrictions formulated in Section 2. Recall that the restrictions are not essential for finding convex cuts.

We search for cut-sets of convex cuts of G by brachiating from an edge e_0 of G via a bounded face F_0 of G, i. e., $e_0 \in E(F_0)$, to an edge e_1 on $E(F_0) \cap E(F_1)$ for some bounded face F_1 of G, and so on. Theorem 5.3 in this paper and Lemma 2 in [6] restrict and thus guide the brachiating. The latter lemma says that for the cut-set C of any convex cut and any bounded face F we have that $|C \cap E(F)|$ equals zero or two. Our approach to finding (cut-sets of) convex cuts through brachiating suggests the following notation.

Notation 6.1. *C denotes a cut-set of a cut of G and is written as a non-cyclic or cyclic (simple cycle) sequence $(e_0, \ldots, e_{|C|-1})$. If C is non-cyclic, there exist bounded faces $F_0, \ldots, F_{|C|-2}$ of G such that $e_{i-1} \in E(F_{i-1}) \cap E(F_i)$. If C is cyclic there exist bounded faces $F_0, \ldots, F_{|C|-1}$ such that $e_{i-1} \in E(F_{i-1}) \cap E(F_i)$, and indices are modulo $|C|$.*

In particular, $C \cap E(F_\infty) = \{e_0, e_{|C|-1}\}$ for non-cyclic C and $C \cap E(F_\infty) = \emptyset$ for cyclic C.

Analogous to Section 5, $G' = (V', E')$ denotes the (plane bipartite) graph that one obtains from G by placing a new vertex into the interior of each edge of G.

Definition 6.2 (e_0^l, e_0^r, C_l', C_r', C_l, C_r, C_τ)**.** *The left [right] child of e_0 when standing on e_0 and looking into F_0 is denoted by e_0^l [e_0^r]. Furthermore, C_l' [C_r'] denotes the set of edges in E' that are θ'-related to e_0^l [e_0^r]. Recall that C_l' and C_r' are cut-sets of cuts of G'. Thus, they induce cut-sets of G denoted by C_l and C_r. Generally "left" and "right" w. r. t. an edge e_i is from the perspective of standing on e_i and looking into F_i. Finally, $C_\tau = \{e \in E \mid e_0 \ \tau \ e\}$.*

6.1 Embedding of cuts

In this section we first represent a cut of G through e_0 with cut-set C by a simple path or simple cycle $\gamma(C)$ in the *line graph* (sometimes referred to as *edge graph*) L_G of G. We then embed the edges of L_G that we need for representing cuts. In particular, all $\gamma(C)$ turn into simple non-closed or closed curves.

Definition 6.3 ($L_G(V^L, E^L)$, cut $\gamma(C)$)**.** *$L_G = (V^L, E^L)$ denotes the line graph of G, i. e., $V^L = E$. Using Notation 6.1, we define $\gamma(C)$ to be the path in L_G whose edge set is*

$$E^L(C) = \{\{e_{i-1}, e_i\}\} \tag{4}$$

If C is non-cyclic [cyclic], $\gamma(C)$ is a maximal simple path [simple cycle] in L_G.

An edge $\{e, \hat{e}\}$ of L_G can be part of $\gamma(C)$ for some C only if there exists a face F of G such that $e, \hat{e} \in E(F)$. To embed such an edge we proceed basically as in Section 3.2. Let p and $\hat{p}$ denote the midpoints of e and $\hat{e}$, respectively. Furthermore, let F^r denote a regular polygon with the same number of sides as F, and let $\hbar : \overline{F^r} \mapsto \overline{F}$ be a homeomorphism (recall that $\overline{F} = F \cup E(F)$). We embed $\{e, \hat{e}\}$ as $\hbar(L)$, where L is the line segment between $\hbar^{-1}(p)$ and $\hbar^{-1}(\hat{p})$. Thus, the vertices of embedded $\gamma(C)$ are all midpoints of edges of G, and embedded $\gamma(C)$ is a curve that subdivides $D(G)$ into two connected components (for $D(G)$ see Definition 3.9).

6.2 Restrictive conditions for convex cuts

Any cut-set of a convex cut through e_0 must be contained in $C_l \cup C_r \cup C_\tau$. This follows from Theorem 5.3, i. e., the fact that for any e_i in C it must hold that either $e_0 \ \tau \ e_i$ or that there exists a child of e_0 and a child of e_i that are θ'-related.

The next lemma tells us that, on a local level, we have to deal only with θ' and not with τ.

Lemma 6.4. *If $e_{i-1} \ \tau \ e_i$, then C cannot be the cut-set of a convex cut.*

Proof. Let $e_{i-1} = \{u_{i-1}, v_{i-1}\}$, $e_i = \{u_i, v_i\}$. Without loss of generality we assume that u_{i-1} is on the same side of the convex cut as u_i and that v_{i-1} is on the same side of the convex cut as v_i (see Figure 8a). Then $e_{i-1} \ \tau \ e_i$ and $e_{i-1}, e_i \in E(F_{i-1})$ imply that any shortest path from u_{i-1} to v_i intersects any shortest path from v_{i-1} to u_i at a vertex that we denote by w. Without loss of generality we may assume that w is on the same side of the convex cut as u_{i-1}. Due to $e_{i-1} \ \tau \ e_i$ we have $d_G(w, u_i) = d_G(w, v_i)$. Thus, there exists a shortest path from v_i via w to v_{i-1} that starts and ends on the side of v_{i-1}, but contains the vertex w, which is on the side of u_{i-1}. Hence the cut cannot be convex. $\square$

The observation below will lead to more restrictive conditions for convex cuts.

Observation 6.5. *The case distinction in the proof of Theorem 5.3 yields the following for plane graphs. If a child e_i' of e_i is θ'-related to a child e_j' of e_j with $j \neq i$, then exactly one of the next two cases holds.*

a) θ' induces a one-to-one correspondence between the two children of e_i and the two children of e_j (see Figure 7a). If e_i' is the left [right] child of e_i, then e_j' is the right [left] child of e_j.

b) The pair e_i', e_j' is the only pair of θ'-related children (see Figure 7b). In particular, e_i' and e_j' must be on the same side of the cut, and there exists a shortest path from the end vertex of e_i that is also the end vertex of e_i' via e_i to the end vertex of e_j that is not an end vertex of e_j'. If e_i' is the left [right] child of e_i, then e_j' is the left [right] child of e_j.

All we know about the cut (V_1, V_2) in the next lemma is that a pair of edges has certain children that are θ' related. Still, (V_1, V_2) tells us that certain convex cuts cannot exist.

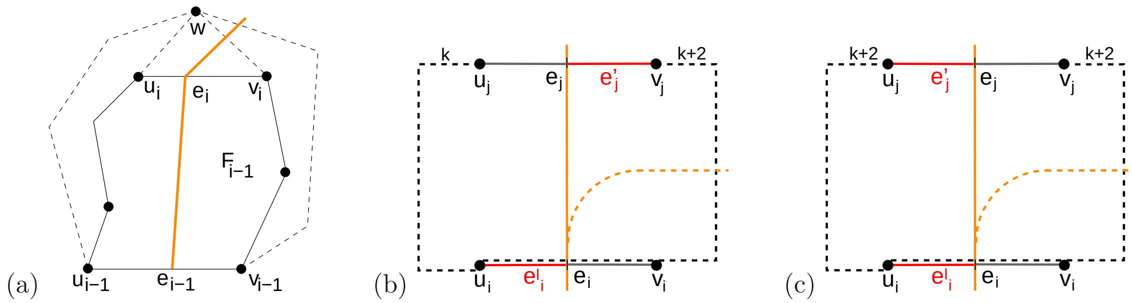


Figure 8: (a) Illustration to proof of Lemma 6.4. The dotted black edges indicate shortest paths in G, and the orange zigzag line indicates a non-convex cut. (b, c) Illustrations to proof of Lemma 6.6. Curved gray lines indicate paths on $E(F_{i-1})$, and gray line segments indicate children of edges on $E(F_{i-1})$. The red children are θ'-related. The cut with cut-set C is indicated by the solid orange curve, and the cut indicated by the dashed orange curve cannot exist.

Lemma 6.6. *Let C be the cut-set of an embedded cut (V_1, V_2) such that the left [right] child of e_i is θ'-related to a child of e_j for some i, j with $j > i$. Then there exists no (embedded) convex cut with e_i in its cut-set that runs right [left] of (V_1, V_2).*

Proof. Without loss of generality we assume that the left child of e_i, denoted by e_i^l, is θ'-related to a child of e_j, denoted by e'_j. We denote the left and right end vertex of e_i [e_j] by u_i and v_i [u_j and v_j], respectively. Due to Observation 6.5, one of the two following cases must hold.

1. $d_{G'}(u_i, u_j) = d_{G'}(v_i, v_j) =: k$, and e'_j is the right child of e_j. This is the case illustrated in Figure 8b. The shortest path from u_i to v_j in G' cannot be shorter than $k+2$, because this would entail $d_{G'}(u_i, v_j) = k$ and thus $v_j \in W_{u_i, v_i}$, a contradiction to $e_i^l \; \theta' e'_j$. Hence, there exists a shortest path P' in G' from u_i via v_i to v_j (with length $k+2$). A cut with e_i in its cut-set that runs right of (V_1, V_2) is crossed twice by P'. Hence the cut is not convex.

2. $d_{G'}(u_i, u_j) = d_{G'}(v_i, v_j) + 2$, and e'_j is the left child of e_j. This is the case illustrated in Figure 8c. From $e_i^l \; \theta' \; e'_j$ follows again that there exists a shortest path P' in G' from u_i via v_i to v_j (with length $k+2$), and the claim follows as in the item above.

□

We will now see that embedded C_l and C_r border all embedded convex cuts through e_0.

Proposition 6.7.

1. *The embedded cut with cut-set C_l runs on the right side of the embedded cut with cut-set C_r (except on $C_l \cap C_r$, where the embedded cuts touch).*

2. *Any embedded convex cut runs between the embedded cut with cut-set C_l and the embedded cut with cut-set C_r, i. e., no part of the convex cut runs right of C_l or left of C_r.*

Proof.

1. We use Observation 6.5b: for any $e^l \in C_l \setminus C_r$ there exists a shortest path P in G' from the left end vertex of e_0 via e_0 and the right end vertex of e_0 towards the right end vertex of e^l. The assumption that parts of C_l run on the left side of C_r lead to a contradiction. Indeed, this would entail that there exists a shortest path P as above which also crosses C_r via an edge $e^r \in C_r$, i. e., P contains a shortest path from the left end vertex of e_0^r via e_0^r and the right end vertex of e_0^r (which equals the right end vertex of e_0) and further on via the right end vertex of e^r to the left end vertex of e^r — a contradiction to $e^r \in C_r$.

2. A consequence of a special case of Lemma 6.6, i. e., the case $i = 0$.

□

The following proposition reveals that the edges of G that are not in $C_l \cup C_r$, i. e., the edges in C_τ, may serve as unique sequences of stepping stones for convex cuts that move from C_l to C_r or vice versa.

Proposition 6.8. *Let $C = (e_0, \dots, e_{|C|-1})$ be the cut-set of a convex cut through e_0. Then the following holds. If $e_{i-1} \in C_l$ [$e_{i-1} \in C_r$] and $e_i \in C_\tau$, then there exists $j > i$ such that $e_i, \dots, e_{j-1} \in C_\tau$ and $e_j \in C_r$ [$e_j \in C_l$]. Moreover, any cut-set of a convex cut through e_0 that coincides with C on $e_0, e_1, \dots, e_i$ must coincide with C on $e_0, e_1, \dots, e_j$.*

Proof. Without loss of generality we assume $e_{i-1} \in C_l$.

1. Let $e_0 = \{u_0, v_0\}$ and $e_i = \{u_i, v_i\}$, let P_u [P_v] be a shortest path from u_i [v_i] to u_0 in G, and let e^u [e^v] be the first edge on P_u [P_v] (see Figure 9a). Then $e^u, e^v \in C_r$.

   Without loss of generality we show that $e^u \in C_r$. Let w_0 [w_i] denote the vertex of G' that subdivides e_0 [e^u]. To prove $e^u \in C_r$, it suffices to show $\{w_i, u_i\}\theta'\{u_0, w_0\}$. Indeed, by definition of τ, we have that the length of P_u equals $d_G(u_i, v_0)$. Since the degrees of w_i and w_0 are two, a shortest path from w_i to w_0 runs via u_0 or via v_0. In both cases the distance is $2d_{G'}(u_i, v_0)$. Thus, $\{w_i, u_i\}$ is θ'-related to e_0^r, i. e., $e^u \in C_r$.

2. The following case distinction yields $e^v = e_j$ for some $j > i$ (see Figure 9b).

   (a) The embedded convex cut with cut-set C crosses embedded C_r. This case cannot occur due to Lemma 6.6.

   (b) C contains e^u. Then $\{w_i, u_i\}$ is a left child of e^u w. r. t. C. Since e_0^r is a right child of e_0 w. r. t. C, and e_0^r is θ'-related to $\{w_i, u_i\}$ (see item 2a), Observation 6.5a yields that e_0^l and the right child of e^u are θ'-related, too. This is a contradiction to $e^u \in C_r$ (see item 1).

   (c) The remaining case is that C contains $e^v \in C_r$, i. e., $e^v = e_j$ for some $j > i$.

3. It remains to show that the extension from e_i to e_j is unique (see Figure 9c). Indeed, let y_i be the end vertex of e_j that is not v_i. The path from u_i to y_i via v_i has length two and contains two edges of C. Due to the cut being convex, there exists an edge $\hat{e}$ from u_i to y_i, i. e., the edges e_i, e_j and $\hat{e}$ form a triangle.

   If $d_G(x, v_i) < d_G(x, y_i)$, then x must be on the same side of the cut as v_0. Indeed, assume that x is on the other side, i. e., the side of u_0. Then, due to $e_0\ \tau\ e_i$, i. e., $d_G(u_0, u_i) = d_G(u_0, v_i) = d_G(v_0, u_i) = d_G(v_0, v_i)$, there exists a shortest path from x via v_i to v_0 that crosses the convex cut twice, a contradiction.

   If $d_G(x, v_i) \geq d_G(x, y_i)$, then x must be on the same side of the convex cut as u_0. Indeed, assume that x is on the other side, i. e., the side of v_0. Then, due to $e_0\ \tau\ e_i$, there exists a path from x via y_i and u_0 to v_0 that is not longer than alternative paths of x via u_i or v_i to v_0. The path via y_i thus is a shortest path from x to v_0 that crosses the convex cut twice, a contradiction.

   To summarize, the side of any vertex x in the triangle is unique i. e., the extension from e_i to e_j is unique.

□

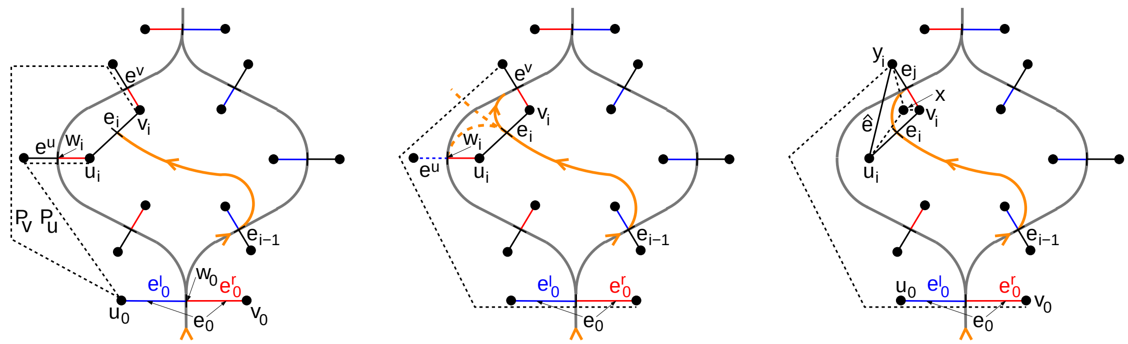


Figure 9: Illustrations to the proof of Lemma 6.8. The cuts defined by C_l and C_r are indicated by the two gray curves that bifurcate at e_0 and merge at the top. Vertices of G $[G']$ are marked as filled circles [bars], and edges of G are shown as solid line segments between filled circles, possibly consisting of two colors indicating the children. Colors of children indicate membership to C'_l and C'_r. The orange arrowheads and the orange line indicate the cut defined by C. The black edge e_i is τ-related to e_0. (a) The dashed black zigzag lines indicate shortest paths P_u and P_v from the left end vertex u_0 of e_0 to the left end vertex u_i of e_i, and from u_0 to the right end vertex v_i of e_i, respectively. (b) Dashed orange lines indicate potentially convex cuts that turn out to be non-convex because of the shortest paths indicated by the dashed black lines (see the proof of Lemma 6.8). (c) There exists an edge $\hat{e} = \{u_i, y_i\}$. Furthermore, if $d_G(x, v_i) \geq d_G(x, y_i)$, then there exists a shortest path from x via y_i and u_0 to v_0.

6.3 Intersection pattern of embedded convex cuts

In Section 6.1 we represented a cut of G through e_0 by an embedded simple path or simple cycle $\gamma(C)$ in the line graph L_G of G. In this section we study the intersection pattern of a pair of embedded convex cuts of G through e_0. More formally, if C and $\hat{C}$ are cut-sets of convex cuts of G through e_0, we study the patterns in $\mathbb{R}^2$ that are formed by the curves $\gamma(C)$ and $\gamma(\hat{C})$.

The boundary of any face F_L of $\gamma(C) \cup \gamma(\hat{C})$ constitutes a cyclical cut of G (see Figure 10). Let $v^L \in V^L$ be a vertex on the boundary of F_L. The vertex v^L is the midpoint of an edge $e \in C$. In particular, one end vertex of e must be contained in the interior of F_L. Thus we have proven the following.

Lemma 6.9. *Any face of $\gamma(C) \cup \gamma(\hat{C})$ contains at least one vertex of G.*

Lemma 6.4 and the case of Lemma 6.6 in which e_i and e_j sit on the boundary of the same face of G yield Proposition 6.10 (see also Figures 8b,c). It states that all embedded convex cuts through e_0 which have reached a face F on the midpoint of an edge on $E(F)$ can cut through F in at most two ways.

Proposition 6.10. *Let $C = (e_0, \dots e_{|C|-1})$ be the cut-set of a convex cut of G through e_0. Then the following holds. For all e_i in C there exists $f_i \in E(F_{i-1})$ (possibly $e_i = f_i$) such that $\hat{e}_j \in \{e_i\} \cup \{f_i\}$ for the cut-set $\hat{C} = (e_0, \hat{e}_1, \dots \hat{e}_{|\hat{C}|-1})$ of any convex cut with $\hat{e}_{j-1} = e_{i-1}$ and $|\{e_0, \hat{e}_1, \dots \hat{e}_{j-1}\} \cap F_{i-1}| = 1$.*

Definition 6.11 ($\gamma(C)$ touches $\gamma(\hat{C})$, $\gamma(C)$ crosses $\gamma(\hat{C})$, overlap, crossing $M_{C,\hat{C}}$)**.** *Let C and $\hat{C}$ be cut-sets of cuts of G through e_0. We say that $\gamma(C)$ touches [crosses] $\gamma(\hat{C})$ on the maximal common curve $M_{C,\hat{C}}$ of $\gamma(C)$ and $\gamma(\hat{C})$ if the part of $\gamma(C)$ directly before $M_{C,\hat{C}}$ is on the same side [on the other side] of $\gamma(\hat{C})$ as the part of $\gamma(C)$ directly after $M_{C,\hat{C}}$. The curve $M_{C,\hat{C}}$ is called an overlap of $\gamma(C)$ and $\gamma(\hat{C})$. If $\gamma(C)$ crosses $\gamma(\hat{C})$ on $M_{C,\hat{C}}$, we refer to $M_{C,\hat{C}}$ as the crossing of $\gamma(C)$ and $\gamma(\hat{C})$.*

For examples of touching and crossing cuts see Figure 10. The following proposition describes the intersection pattern of a pair of embedded convex cuts of G through e_0.

Proposition 6.12. *Let $C, \hat{C}$ be cut-sets of convex cuts of G through e_0. Then $\gamma(C)$ cannot touch $\gamma(\hat{C})$, and $\gamma(C)$ and $\gamma(\hat{C})$ can have at most one crossing.*

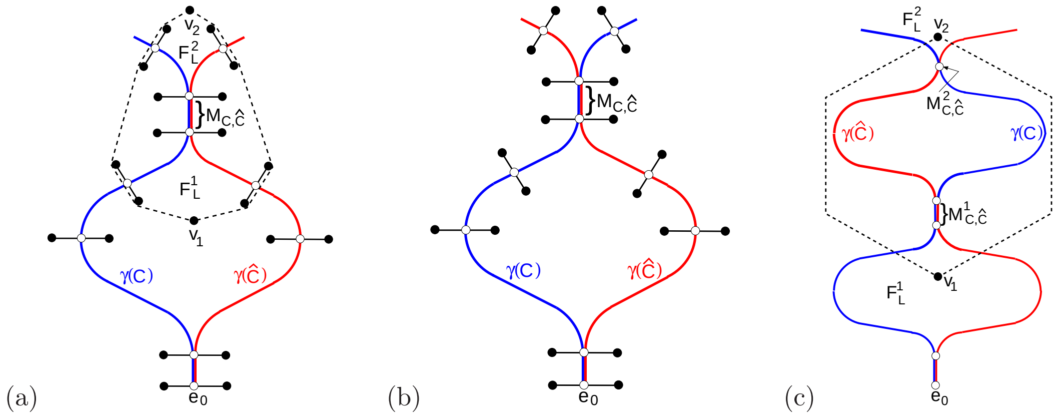


Figure 10: (a) Illustration to proof of Proposition 6.12. The embedded cuts $\gamma(C)$ and $\gamma(\hat{C})$ are indicated by the blue and red curves, respectively. Vertices and edges of G are marked as filled circles and solid black lines, respectively. Shortest paths in G are shown as dotted black lines. Note that shortest paths in G can use only vertices marked as filled circles. Vertices of S are shown as open circles. (a, b, c) Illustration to the proof of Proposition 6.12. (a) $\gamma(C)$ touches $\gamma(\hat{C})$ on the maximal common curve $M_{C,\hat{C}}$ of $\gamma(C)$ and $\gamma(\hat{C})$. (b) $\gamma(C)$ crosses $\gamma(\hat{C})$ on $M_{C,\hat{C}}$. (c) $\gamma(C)$ and $\gamma(\hat{C})$ have crossings $M^1_{C,\hat{C}}$ and $M^2_{C,\hat{C}}$.

Proof.

- Assume that $\gamma(C)$ touches $\gamma(\hat{C})$. Let F^1_L and F^2_L be the faces formed by the parts of $\gamma(C)$ and $\gamma(\hat{C})$ before and after $M_{C,\hat{C}}$, respectively (see Figure 10a). Lemma 6.9 yields that there exist vertices $v_1 \in V \cap F^1_L$ and $v_2 \in V \cap F^2_L$. Any shortest path from v_1 to v_2 either crosses $\gamma(C)$ or $\gamma(\hat{C})$ twice, a contradiction to C and $\hat{C}$ being cut-sets of convex cuts.
- Assume that $\gamma(C)$ and $\gamma(\hat{C})$ have crossings $M^1_{C,\hat{C}} \neq M^2_{C,\hat{C}}$, and that there is no crossing between $M^1_{C,\hat{C}}$ and $M^2_{C,\hat{C}}$. Let F^1_L and F^2_L be the faces formed by the parts of $\gamma(C)$ and $\gamma(\hat{C})$ before $M^1_{C,\hat{C}}$ and after $M^2_{C,\hat{C}}$, respectively (see Figure 10c). Lemma 6.9 yields that there exist vertices $v_1 \in V \cap F^1_L$ and $v_2 \in V \cap F^2_L$. As in the previous item, any shortest path from v_1 to v_2 either crosses $\gamma(C)$ or $\gamma(\hat{C})$ twice, a contradiction to C and $\hat{C}$ being cut-sets of convex cuts.

□

6.4 Upper bound on number of convex cuts through e_0

To find an upper bound on the number of convex cuts of G through e_0 we start by assuming that there exists at least one such cut with cut-set $\hat{C}$. The necessary conditions for convex cuts in Section 6.2 and Proposition 6.12 impose constraints on the other candidates for convex cuts through e_0. In particular, Proposition 6.12 implies the following. The first overlap of $\gamma(C)$ and $\gamma(\hat{C})$ is always the one that contains the midpoint of e_0. It always exists. If there is a second overlap, and this second overlap is not a crossing, it must be the last overlap, and it must contain the midpoint of $e_{|C|-1}$. This follows from the fact that $\gamma(C)$ and $\gamma(\hat{C})$ cannot touch. If the second overlap exists and constitutes a crossing, there may or may not be another overlap. If there exists such a third overlap, it must be the last one, and it must contain $e_{|C|-1}$ (since $\gamma(C)$ and $\gamma(\hat{C})$ cannot touch, and they cannot cross twice). Thus, we have proven the following.

Proposition 6.13 (At most three overlaps). *Let C and $\hat{C}$ be cut-sets of convex cuts of G through e_0. Then $\gamma(C)$ and $\gamma(\hat{C})$ can have at most three overlaps.*

Definition 6.14 (Fork and corresponding join of $\gamma(C)$ and $\gamma(\hat{C})$, detour on $\gamma(C)$). *Let C and $\hat{C}$ be cut-sets of cuts of G through e_0. Furthermore, let M_1 and M_2 be two consecutive overlaps of $\gamma(C)$ and $\gamma(\hat{C})$, i. e.,* *there is no overlap of $\gamma(C)$ and $\gamma(\hat{C})$ of M_1 and M_2. Then the last point of M_1 and the first point of M_2 are called fork and corresponding join of $\gamma(C)$ and $\gamma(\hat{C})$. The sub-path of $\gamma(C)$ between a fork and a corresponding join is called detour on $\gamma(C)$ around $\gamma(\hat{C})$.*

Proposition 6.15 (Unique detours)**.** *Let C, C^* and $\hat{C}$ be cut-sets of convex cuts of G through e_0, let p^f and p^j be a fork and a corresponding join of $\gamma(C)$ and $\gamma(\hat{C})$, as well as of $\gamma(C^*)$ and $\gamma(\hat{C})$. If $\{p^f\}$ is not a crossing, then the corresponding detours on $\gamma(C)$ and on $\gamma(C^*)$ coincide.*

Proof.

1. p^f is the midpoint of an edge of G. Indeed, an edge of $\gamma(C)$, $\gamma(C^*)$ or $\gamma(\hat{C})$ takes the form $\{f_{i-1}, f_i\}$, where f_{i-1}, f_i are edges of G. Here the indices reflect the order of the corresponding cut-sets (see Notation 6.1). Moreover, there exists a face F_{i-1} of G with $f_{i-1}, f_i \in F_{i-1}$. The *embedded* edge $\{f_{i-1}, f_i\}$ is a curve from the midpoint of f_{i-1} through the interior of F_{i-1} to the midpoint of f_i. The embedding is such that two embedded edges that cross in a face of G must cross at a single point (see Section 6.1). Thus, the condition that the join $\{p^f\}$ is not a crossing implies that p^f is the midpoint of an edge of G.

2. The first edge of the detour on $\gamma(C)$ around $\gamma(\hat{C})$ from p^f to p^j must coincide with the first edge of the detour on $\gamma(C^*)$ around $\gamma(\hat{C})$. Indeed, let e^f be the edge with midpoint p^f. Let the successors of e^f in the cut-sets C, C^* and $\hat{C}$ be denoted by s, s^* and $\hat{s}$, respectively. Since p^f and p^j are a fork and a corresponding join of $\gamma(C)$ and $\gamma(\hat{C})$, as well as of $\gamma(C^*)$ and $\gamma(\hat{C})$, the edges $\{e^f, s\}$, $\{e^f, s^*\}$ and $\{e^f, \hat{s}\}$ all go through the same face of G. Proposition 6.10 yields that at least two of the three embedded edges must coincide. Thus, the first edge of the detour on $\gamma(C)$ around $\gamma(\hat{C})$ from p^f and p^j must coincide with the first edge of the detour on $\gamma(C^*)$ around $\gamma(\hat{C})$.

3. Analogous to Definition 6.2, let C_l^f $[C_r^f]$ be the set of edges of G that have a child which is θ'-related to the left [right] child of e^f. Without loss of generality we assume that $\gamma(C)$ runs right of $\gamma(\hat{C})$. Lemmas 6.4 and Proposition 6.10 then yield that (i) the edge in C directly after e^f, denoted by e is contained in C_l^f and (ii) the edge in $\hat{C}$ directly after e^f, denoted by $\hat{e}$, is contained in C_r^f.

   We assume the opposite of the claim, i. e., that $\gamma(C)$ and $\gamma(C^*)$ fork at or behind e and join at or before p^j. Then Proposition 6.7 and Proposition 6.8 imply that one of the embedded cuts continues on C_l^f, while the other one switches from C_l^f to C_r^f. If the switching embedded cut hits p^j as soon as it reaches C_r^f, the other embedded cut has missed p^j, a contradiction to $p^j \in \gamma(C)$ and $p^j \in \gamma(C^*)$. If the switching embedded cut does not hit p^j as soon as it reaches C_r^f, the embedded cut $\gamma(\hat{C})$ has already switched from C_r^f to C_l^f, and must thus have hit $\gamma(C)$ or $\gamma(C^*)$ before p^j, a contradiction.

□

Proposition 6.16. *An upper bound for the number of convex cuts of G through e_0 is $|E|^4$.*

Proof. We may assume that there exists a convex cut through e_0 with cut-set $\hat{C}$.

1. *Number of the convex cuts $\gamma(C)$ of G through e_0 that do not cross $\hat{C}$.* Proposition 6.12 yields that there can be at most one fork and corresponding join of $\gamma(\hat{C})$ and $\gamma(C)$. The number of $\gamma(C)$ is thus bounded by the number of detours around $\gamma(\hat{C})$. Proposition 6.15 yields that the number of detours cannot surmount the number of forks of $\gamma(\hat{C})$ and $\gamma(C)$ times the number of corresponding joins of $\gamma(\hat{C})$ and $\gamma(C)$, i. e., at most $|E|(|E|-1)/2$.

2. *Number of the convex cuts $\gamma(C)$ of G through e_0 that cross $\hat{C}$.* We first select a sub-path $M_{\hat{C}}$ of $\gamma(\hat{C})$ and determine the number of $\gamma(C)$ that cross $\gamma(\hat{C})$ on $M_{\hat{C}}$. Let $\gamma(C)$ be such a path. $\gamma(C)$ joins $\gamma(\hat{C})$ at the first point of $M_{\hat{C}}$, denoted by p^j. Using Proposition 6.12 we get that $\gamma(C)$ coincides with $\gamma(\hat{C})$ between e_0 and the last point of $M_{\hat{C}}$, with the exception of at most one detour before $M_{\hat{C}}$. We already know that p^j is the join of a detour. Using Proposition 6.15 we get that the number of detours before $M_{\hat{C}}$ is less than $|E|$. The same holds for the number of detours behind $M_{\hat{C}}$. Thus, the number of $\gamma(C)$ that cross $\gamma(\hat{C})$ on $M_{\hat{C}}$ is less than $|E|^2$. The number of non-empty sub-paths $M_{\hat{C}}$ of $\gamma(\hat{C})$, in turn, amounts to $|E|(|E|-1)/2$. Hence, the number of $\gamma(C)$ that cross $\gamma(\hat{C})$ is less than $(|E|^4 - |E|^3)/2$.

The total number of convex cuts of G through e_0 thus cannot surmount $|E|^4$. □

6.5 Algorithm for finding all convex cuts

We search for convex cuts of G using a subgraph S of $L_G = (V^L, E^L)$ (for L_G see Definition 6.3).

Definition 6.17 (Search graph $S = (V_S, E_S)$)**.** *We set* $E_S = \{\{e, f\} \in E^L \mid \{e, f\} \in E(F)$ *for some face* F *of* G *and* $e'\ \theta'\ f'$ *for children* e' *of* e *and* f' *of* $f\}$*. The search graph* S *is the subgraph of* L_G *that is induced by* E_S*.*

Definition 6.18. *We say that* $v_S, w_S \in V_S$ *are compatible, if (i) a child of* v_S *is* θ' *related to a child of* w_S *or (ii)* $v_S\ \tau\ w_S$*.*

Theorem 5.3 and Proposition 6.4 yield the following characterization of convex cuts in terms of the search graph S.

Lemma 6.19. *Let* C *be a non-cyclic [cyclic] cut-set of a cut of* G *through* e_0*. Then the cut is convex if and only if* $\gamma(C)$ *is a maximal path [cycle] in* S *such that any pair of vertices* $v_S \neq w_S$ *on the path [cycle] is compatible.*

If the cut-set C of a cut of G is non-cyclic, we have $C \cap E(F_\infty) = \{e_0, e_{|C|-1}\}$, and if C is cyclic, we have $C \cap E(F_\infty) = \emptyset$.

The two matrices defined next will allow us to check in constant time whether two vertices of S are compatible. We build a $|E| \times |E|$ matrix A_τ with boolean entries such that $A_\tau(i, j)$ is true if and only if edge i is τ-related to edge j. Likewise, we build a $(2|E|) \times 2(|E|)$ matrix $A_{\theta'}$ with boolean entries such that $A_{\theta'}(i, j)$ is true if and only if edge i in G' is θ'-related to edge j in G'.

Our algorithm for finding (the cut-sets of) all convex cuts of G consists of two steps: find the non-cyclic cut-sets starting at each $e_0 \in E(F_\infty)$ and then find the cyclic ones starting at each $e_0 \notin E(F_\infty)$. In both steps we carry along and extend paths $(e_0, \ldots, e_k)$ of S as long as all its vertices are pairwise compatible. If $e_0 \in E(F_\infty)$, there exists only one bounded face F_0 whose boundary contains e_0 and the candidates for e_1. If $e_0 \notin E(F_\infty)$, there exist two such faces, and we arbitrarily declare one of them to be F_0.

If, after starting at $e_0 \in E(F_\infty)$, a path we carry along has reached $E(F_\infty)$ again, we have found a non-cyclic oriented cut-set of a convex cut and store it (recall that the vertices of the path are pairwise compatible). When we are done with e_0, i. e., when none of the paths that we carry along can be extended, we insert e_0 into a tabu list for further searches (we have already found all convex cuts through e_0). The tabu list ensures that we do not end up with two copies of a convex cut, i. e., one for each orientation. Any non-cyclic cut-set of a convex cut is found by our algorithm since it carries along one orientation of any pairwise compatible non-cyclic cut-set starting at e_0.

If, after starting at $e_0 \notin E(F_\infty)$, a path we carry along has reached e_0 again, we have found a cyclic cut-set of a convex cut and store it. Conversely, any cyclic cut-set of a convex cut is found by our algorithm since it carries along one orientation of any pairwise compatible cyclic cut-set starting at e_0. Here, the orientation is given by the choice of F_0 (see above). Again, we insert e_0 into a tabu list.

Algorithm 2 Finding the cut-sets C of all convex cuts of a plane graph G

1: Build the search graph and the matrices A_τ and $A_{\theta'}$.

2: For any start vertex e_0 of S with $e_0 \in E(F_\infty)$ perform a breadth-first-traversal (BFT) starting at e_0. The first path carried along is (e_0). For any new vertex v_S of S that is visited by the BFT and that is not in the tabu list, and for any path and cycle carried along, use the matrices A_τ and $A_{\theta'}$ to check whether v_s is compatible with the path or the cycle. Whenever $E(F_\infty)$ is reached, store C and put e_0 into the tabu list.

3: For any $e_0 \notin E(F_\infty)$ declare one of the two bounded faces with e_0 on their boundaries to be F_0. Proceed as in the case $e_0 \in E(F_\infty)$, except that (i) when at e_0, the BFT is restricted such that only edges in $E(F_0)$ are found and (ii) C is stored only if e_0 is reached. Finally, put e_0 into the tabu list.

Theorem 6.20. *Algorithm 2 finds all convex cuts of* $G = (V, E)$ *using* $\mathcal{O}(|V|^7)$ *time and* $\mathcal{O}(|V|^5)$ *space.*

Proof. Algorithm 2 is correct due to Lemma 6.19 and the fact that we find any maximal path and cycle in S with pairwise compatible vertices exactly once.

To build the matrix A_τ, we iterate over all vertices v_S of S and identify all vertices of S that are τ-related to v_S. For a given vertex v_S this can be done in $\mathcal{O}(|V|)$ time. Indeed, if $e = \{u, v\}$ is the edge in G that equals v_S, we can compute the distances of any $w \in V$ to u and v in $\mathcal{O}(|V|)$, e. g., by using BFT. For any $f \in E$ we can then determine in constant time whether $e \; \tau \; f$.

To build the matrix $A_{\theta'}$, we proceed as above, except that we use G' instead of G. The running time for computing A_τ and $A_{\theta'}$ is $\mathcal{O}(|E|^2)$, and A_τ and $A_{\theta'}$ take $\mathcal{O}(|E|^2)$ space.

Any time the BFT reaches a new vertex v_S of S, the paths and cycles carried along need to be checked for compatibility with v_s. Using the matrices A_τ and $A_{\theta'}$, this takes $\mathcal{O}(|E|)$ time per path. According to Proposition 6.16 the number of convex cuts of G through a starting edge e_0 is bounded by $|E|^4$. This is also the maximal number of paths that we carry along and that need to be checked. Hence, processing v_S takes time $\mathcal{O}(|E|)^5$. Finding all convex cuts through a starting edge can then be done in $\mathcal{O}(|E|^6)$ time and, since there are $|E|$ starting edges, total running time is $\mathcal{O}(|E|^7)$. Storing the $|E|^4$ paths and cycles takes $\mathcal{O}(|E|^5)$ space.

The time and space requirements of (constructing) the search graphs and the tabu lists are below the time and space requirements specified so far. The claim now follows from $\mathcal{O}(|E|) = \mathcal{O}(|V|)$ which, in turn, is a consequence of G being plane. $\square$

7 Conclusions

We have presented an algorithm for finding all convex cuts of a plane graph in polynomial time. To the best of our knowledge, it is the first polynomial-time algorithm for this task. We have also presented an algorithm that computes all convex cuts of a not necessarily plane but bipartite graph in cubic time.

Both algorithms are based on binary, symmetric, but generally not transitive relations on edges. In the case of a plane graph G we employed two relations: (i) the Djoković relation on the edges of a subdivision of G and (ii) another relation on the edges of G. In case of a bipartite graph it was sufficient to employ the Djoković relation on the graph's edges.

To prove that the number of convex cuts of a plane graph is not exponential, we employed results on the intersection pattern of convex cuts that are based on a specific embedding of the cuts. Thus, a connection to the first part of the paper arises, where we defined a sub-class of plane graphs via the intersection patterns of certain embedded cuts (which all turned out to be convex). In particular, the transition from the sub-class to general plane graphs is reflected by a generalization of the intersection patterns of convex cuts from arrangements of pseudolines to patterns where forks and joins of convex cuts are possible.

The characterization of convex cuts of general graphs, as given by Theorem 5.3, was instrumental in finding all convex cuts of a bipartite or a plane graph in polynomial time. We reckon that this new characterization of convex cuts of general graphs also helps when devising polynomial-time algorithms for finding convex cuts in graphs from other classes.

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