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Computer simulation of the inverse problem of electrocardiography: use of properties of harmonic functions

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Abstract

The problem of reconstructing the pattern of heart excitation from body surface potentials is simulated. The problem is well known as the inverse problem of electrocardiography and in a general case this problem has a non-unique solution. The relationship of the problem with the inverse problem of potential theory is shown. From this relationship a new excitation propagation model for the heart ventricles is developed. The model is based on a classical multidipole cardiogenerator, but is stable, flexible, and provides a unique solution for the inverse problem of electrocardiography. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Here, we consider the problem of reconstructing the pattern of the heart excitation from the body surface potentials. By observing the dynamics of the changes in heart excitation, a cardiologist can diagnose various cardiac irregularities and, most importantly, can determine their anatomical location. The fundamental significance of the problem has been emphasized in [1].

We represent the pattern as a time sequence of domains of the heart excitation. From the mathematical point of view, the current domain of the heart excitation is the support of the volume function of electrostatics sources inside the heart, specifically, the domain of non-zero values of this function. It is known [2] that in the general case there is an endless set of functions generating equal external potentials. Thus, the problem has a non-unique solution and belongs to the wide class of ill-posed problems [3].

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Recent studies deal with a relaxed problem: to reconstruct activation times at the ventricular surface (i.e. both the epi- and endocardial surface) from the body surface potentials [4,5]. This problem has a unique solution [4], but obviously it is not sufficient to determine precise anatomical locations of diseased tissue.

Let us consider the simplest biophysical model of the process:

BP1. The body and environment form an infinitely wide homogeneous medium.
BP2. The heart excitation domain is a set of dipoles.

Then mathematical model of such a process is described with the following equation:

$$\frac{1}{4\pi\sigma}\int_{P(t)} \frac{p(\vec{r},t)}{\|\vec{r}-\vec{s}\|_{R_3}}\,\mathrm{d}r = u(\vec{s},t), \tag{1}$$

where $t \in [0,T]$ is time, $p(\vec{r},t)$ is unknown volume function of electrostatics sources with domain of definition $P(t)$, $u(\vec{s},t)$ is potentials given on surface $\partial U(t)$ of the body $U(t)$ with error $\delta > 0$, σ is a specific conductivity of the medium (we should reconstruct support of $p(\vec{r},t)$ so we can make $\sigma = 1/4\pi$). Also we assume $p(\vec{r},t)$ and $u(\vec{s},t)$ belong to the Hilbert spaces. Support $P'(t) = \operatorname{supp} p(\vec{r},t)$ is unknown, but for any time $P'(t) \subset P(t)$, and because of BP2 we have

$$\int_{P'(t)} p(\vec{r},t)\,\mathrm{d}r = \int_{P(t)} p(\vec{r},t)\,\mathrm{d}r = 0. \tag{2}$$

Thus, the problem is to reconstruct $P'(t)$ (domain of the heart excitation). In [6,7] Eq. (1) was numerically solved considering the functional properties of $p(\vec{r},t)$ (for example (2)), $u(\vec{s},t)$ and geometrical properties of $P(t)$, $\partial U(t)$. Unfortunately, these methods only reduce the set of solutions and cannot resolve non-uniqueness.

2. Inverse problem of potential theory

The inverse problem of potential theory is described with following equation:

$$\int_{P} \frac{p(\vec{r})}{\|\vec{r}-\vec{s}\|_{R_3}}\,\mathrm{d}r = u(\vec{s}), \quad \vec{s} \in U = R_3/P. \tag{3}$$

If function $p(\vec{r})$ is arbitrary, then Eq. (3) has a non-unique solution.

There are many papers regarding the conditions of uniqueness for the solution of (3). Among them we can point to two main approaches. The first [8–10] uses various restrictions for P and $p(\vec{r})$, thus providing uniqueness for the solution of (3). Let P be a simply connected domain, then in [10] the following theorem has been proved:

Theorem 1. *If $p(\vec{r})$ is harmonic function on P, then Eq. (3) has a unique solution.*

The second approach [11] applies a unique approximation of $p(\vec{r})$. Let P be a restricted and simply connected domain, then in [11] the following theorem has been proved:

Theorem 2. *If $p(\vec{r}) \in H^4(P)$, $\partial P \in C^2$ then for $p(\vec{r})$ there is a unique representation $p(\vec{r}) = p_1(\vec{r}) + p_2(\vec{r}) + h(\vec{r})$, where $p_1(\vec{r})$ is a harmonic function on P, $p_2(\vec{r})$ is a biharmonic function on P, $h(\vec{r})$ is an orthogonal to any harmonic function on P, and ∂P is boundary of P.*

Theorem 2 allows us to derive an approximate function of $p(\vec{r})$ to within the norm of $h(\vec{r})$.

3. Main result

Let us consider Eq. (3) under the conditions of Theorem 1 where $\vec{s} \in U = O/I$, $P \subset I \subset O$, surfaces ∂O, $\partial I \in C^2$ and the maximum distance between them equals to $\varepsilon > 0$, and the minimum distance between them is greater than 0.

Is it possible to use Theorem 1 in this case? The following theorem gives us a positive answer to this question:

Theorem 3. *If $p_1(\vec{r}) \neq p_2(\vec{r})$ then for arbitrary $U = O/I$ there is $\vec{s} \in U$ such that $u_1(\vec{s}) \neq u_2(\vec{s})$, where $u_1(\vec{s})$ and $u_2(\vec{s})$ are potentials generated with $p_1(\vec{r})$ and $p_2(\vec{r})$, respectively.*

Proof. By contradiction, let us assume $p_1(\vec{r}) \neq p_2(\vec{r})$, but assume there is a $U' = O'/I'$ such that $\forall \vec{s} \in U'$ $u_1(\vec{s}) = u_2(\vec{s})$.

Let us consider an arbitrary $U'' \subset O'$ such that $\partial U'' \subset U'$, $\partial U'' \in C^2$ and the minimum distance between $\partial U''$ and $\partial O'$, $\partial I'$ is greater than ε/M, $M > 0$. Then using Green's formula [12], we get

$$\begin{aligned}
&\int_{U''} \frac{p_1(\vec{r}) - p_2(\vec{r})}{\|\vec{r} - \vec{s}\|_{R_3}} \, \mathrm{d}r \\
&\quad = \int_{\partial U''} \left[(u_1(\vec{q}) - u_2(\vec{q})) \frac{\partial 1/\|\vec{q} - \vec{s}\|_{R_3}}{\partial \vec{n}} - \frac{1}{\|\vec{q} - \vec{s}\|_{R_3}} \frac{\partial}{\partial \vec{n}} (u_1(\vec{q}) - u_2(\vec{q})) \right] \mathrm{d}q \\
&\quad = - \int_{\partial U''} \left[\frac{1}{\|\vec{q} - \vec{s}\|_{R_3}} \frac{\partial}{\partial \vec{n}} (u_1(\vec{q}) - u_2(\vec{q})) \right] \mathrm{d}q.
\end{aligned}$$

From the determination of the normal derivative and our assumptions we get

$$- \int_{\partial U''} \left[\frac{1}{\|\vec{q} - \vec{s}\|_{R_3}} \frac{\partial}{\partial \vec{n}} (u_1(\vec{q}) - u_2(\vec{q})) \right] \mathrm{d}q = 0,$$

where $\forall \vec{s} \in R_3/\bar{U}''$, $\vec{n}$ is an internal normal to $\partial U''$.

Thus, we have

$$\int_{U''} \frac{p_1(\vec{r}) - p_2(\vec{r})}{\|\vec{r}-\vec{s}\|_{R_3}}\,\mathrm{d}r = 0, \quad \forall \vec{s} \in R_3/\bar{U}''.$$

It is known [8] that in this case function $p_1(\vec{r}) - p_2(\vec{r})$ is orthogonal to any harmonic function on U''. Since $p_1(\vec{r})$, $p_2(\vec{r})$ satisfy the conditions of Theorem 1 then they are harmonic functions on $P_1 = \operatorname{supp} p_1(\vec{r})$, $P_2 = \operatorname{supp} p_2(\vec{r})$ and zero functions on R_3/P_1, R_3/P_2, respectively. So we have:

$$\begin{cases} \displaystyle\int_{U''} (p_1(\vec{r}) - p_2(\vec{r}))p_1(\vec{r})\,\mathrm{d}r = 0 \\ \displaystyle\int_{U''} (p_1(\vec{r}) - p_2(\vec{r}))p_2(\vec{r})\,\mathrm{d}r = 0 \end{cases}$$

$$\to \int_{U''} (p_1(\vec{r}) - p_2(\vec{r}))p_1(\vec{r})\,\mathrm{d}r - \int_{U''} (p_1(\vec{r}) - p_2(\vec{r}))p_2(\vec{r})\,\mathrm{d}r = 0$$

$$\to \int_{U''} (p_1(\vec{r}) - p_2(\vec{r}))^2\,\mathrm{d}r = 0.$$

Finally, $p_1(\vec{r}) - p_2(\vec{r}) = 0$, $\forall \vec{r} \in P$ and we get a contradiction with the initial conditions of the theorem. $\square$

Using Theorem 3 and the continuity of harmonic functions, we can approximate $u(\vec{s})$ on $U = O/I$ over the maximum distance $\varepsilon \to 0$. Another method is to use Green's formula or multipole expansion for the calculation $u(\vec{s})$ on $U = O/I$ for arbitrary $\varepsilon > 0$.

Thus, if $p(\vec{r}, t)$ is under the conditions of Theorem 1 and $\partial U(t) \in C^2$ then Eq. (1) has a unique solution.

Let us consider Eq. (1) under the following restrictions:

$$\begin{cases} p^+(\vec{r},t) \text{ is harmonic and positive, } \forall \vec{r} \in \operatorname{supp} p^+(\vec{r},t) = P^+(t) \subset P(t), \\ p^-(\vec{r},t) \text{ is harmonic and negative, } \forall \vec{r} \in \operatorname{supp} p^-(\vec{r},t) = P^-(t) \subset P(t), \\ P^+(t) \cap P^-(t) = \emptyset, \\ P^+(t) \cup P^-(t) = P'(t) = \operatorname{supp} p(\vec{r},t) \subset P(t), \\ p(\vec{r},t) = p^+(\vec{r},t),\ \forall \vec{r} \in P^+(t), \\ p(\vec{r},t) = p^-(\vec{r},t),\ \forall \vec{r} \in P^-(t), \\ \displaystyle\int_{P(t)} p(\vec{r},t)\,\mathrm{d}r = 0, \\ \displaystyle\int_{P(t)} \frac{p^+(\vec{r},t)}{\|\vec{r}-\vec{s}\|_{R_3}}\,\mathrm{d}r = u^+(\vec{s},t), \\ \displaystyle\int_{P(t)} \frac{p^-(\vec{r},t)}{\|\vec{r}-\vec{s}\|_{R_3}}\,\mathrm{d}r = u^-(\vec{s},t), \\ u(\vec{s},t) = u^+(\vec{s},t) + u^-(\vec{s},t) = u^+(\vec{s},t) + \begin{cases} u^-(\vec{s},t), & t = 0, \\ -u^+(\vec{s},t-1), & t \in [1,T], \end{cases} \end{cases} \tag{4}$$

where $t \in [0, T]$, $u^+(\vec{s}, 0)$, $u(\vec{s}, t)$, $U(t)$, $P(t)$ are given, $P(t)$ is simply connected domain, $P(t) \subset U(t)$, $U(t)$ is simply connected domain and $\vec{s} \in \partial U(t) \in C^2$. Then Eq. (1) can be reduced to the following system of equations:

$$\begin{cases} u^+(\vec{s}, t) = \begin{cases} u^+(\vec{s}, t), & t = 0, \\ u(\vec{s}, t) - u^+(\vec{s}, t-1), & t \in [1, T], \end{cases} \\ \int_{P(t)} \dfrac{p^+(\vec{r}, t)}{\|\vec{r} - \vec{s}\|_{R_3}} \, \mathrm{d}r = u^+(\vec{s}, t), \\ \int_{P(t)} \dfrac{p^-(\vec{r}, t)}{\|\vec{r} - \vec{s}\|_{R_3}} \, \mathrm{d}r = u(\vec{s}, t) - u^+(\vec{s}, t). \end{cases} \tag{5}$$

From Theorems 1 and 3, it follows that system (5) has a unique solution and excitation domain $P'(t)$ is approximated with $\operatorname{supp} p^+(\vec{r}, t) \cup \operatorname{supp} p^-(\vec{r}, t)$ where $p^+(\vec{r}, t)$ and $p^-(\vec{r}, t)$ are solutions of (5). Thus the inverse problem of the electrocardiography under conditions (4) has a unique solution.

From the biophysical point of view, condition (4) describes the multidipole model [1], but with the following substantial differences:

1. Solving (5) we use Tikhonov's regularization method, therefore our model is stable.
2. We do not initially set positions, directions and activation times of the dipoles, therefore our model is flexible.
3. Based on direction of the excitation propagation for the ventricles [1] we introduce the restriction that negative charges of the dipoles from $P'(t)$ are close with respect to position and absolute value to the positive charges of dipoles from $P'(t-1)$; hence, $u^-(\vec{s}, t) = -u^+(\vec{s}, t-1)$ (see last row of (4)).
4. To solve (5) we must know the initial $u^+(\vec{s}, 0)$. From our knowledge of ventricle excitation, we establish that it begins at the endocardial surface. If the endocardial surface $P(0)$ is approximated with Liapunov's surface [12], then Eq. (1) has a unique solution [10]. Using this solution, it is easy to calculate $u^+(\vec{s}, 0)$.

4. Computer simulations

In computer experiments, we identify an extending 2D excitation domain (Fig. 2). In Fig. 1, the dots represent two grids approximating the surface $\partial U(t)$(64×64 grid) and domain $P(t)$(16×16 grid), respectively. In Fig. 2, the sequence of propagation of the excitation domain ($t \in [0, 2]$) is shown.

Solving Eq. (1) directly (without considering restriction (4)) we get results shown in Fig. 3, where one can see an incorrect reconstruction of the excitation domain.

Now let us assume that $u^+(\vec{s}, 0)$ is known and (4) is true, then we can solve system (5) and we get the results shown in Fig. 4, where one can see a correct reconstruction of the excitation domain.

Comparing Figs. 4 and 3, one can see that our approach (Fig. 4) allows an accurate reconstruction of the 2D excitation domain in contrast with the classical approach (Fig. 3).

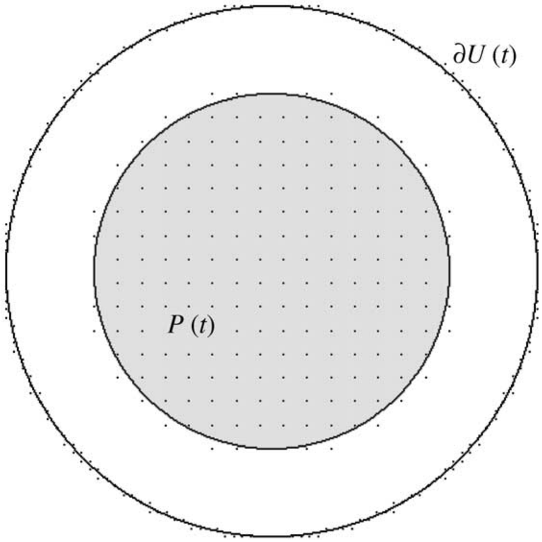

Fig. 1. Dots represent two grids approximating the surface $\partial U(t)$ (64×64 grid) and domain $P(t)$ (16×16 grid), respectively.

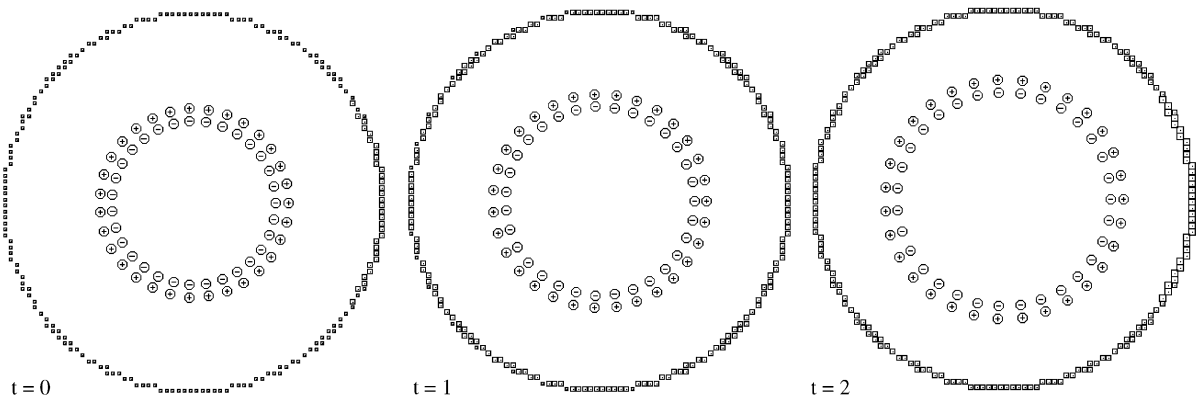

Fig. 2. Sequence of propagation of the excitation domain ($t \in [0, 2]$).

5. Future goals

The results of the paper can be extended in the following directions:

1. Computer simulations for 3D space.
2. Implementation of a realistic model of excitation propagation taking into account the inhomogeneous, dynamic and restricted medium.

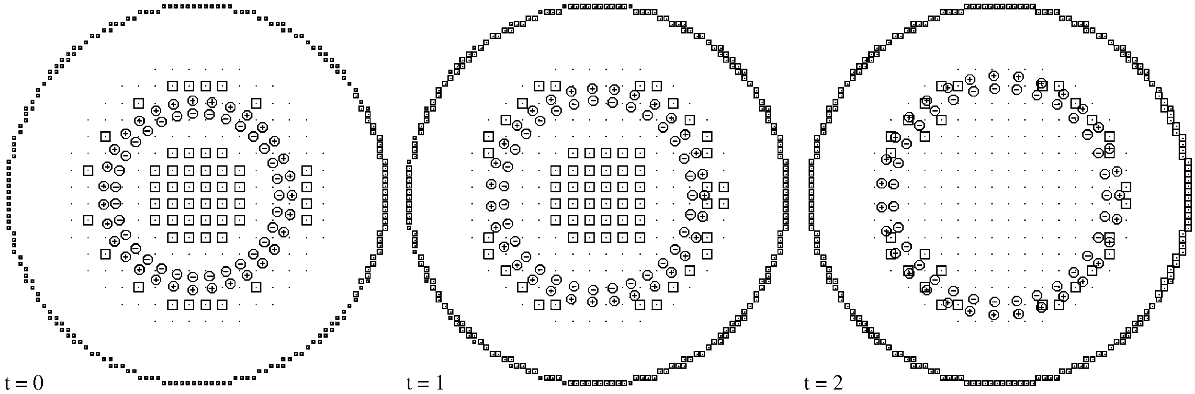

Fig. 3. Eq. (1) is solved. Grid knots belonging to the solution $P'(t)$ (50% threshold separation) are surrounded with squares.

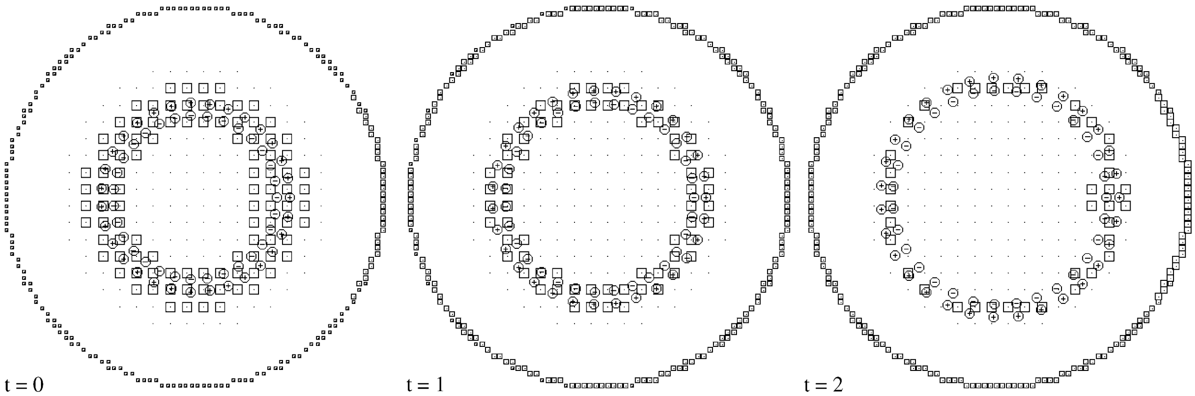

Fig. 4. System (5) is solved. Grid knots belonging to the solution $P'(t)$ (50% threshold separation) are surrounded with squares.

6. Conclusions

The relationship between the inverse problem of electrocardiography and the inverse problem of potential theory is shown. On the basis of this relationship, a new model for heart excitation propagation is suggested. This model is based on a classical multidipole cardiogenerator, but is stable, flexible, and provides a unique solution of the inverse problem of electrocardiography. Computer simulations have demonstrated the advantages of the presented approach.

7. Summary

The reconstruction of the pattern of heart excitation from potentials given on the body surface is simulated. The problem is well known as the inverse problem of electrocardiography and in a

general case this problem has a non-unique solution. The relationship of the problem with the inverse problem of potential theory is shown, and a new excitation propagation model for the heart ventricles is suggested. The model is based on the classical multidipole cardiogenerator, but is stable, flexible, and provides a unique solution of the inverse problem of electrocardiography. Computer simulations have demonstrated the advantages of this new approach.

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