ДЗ 7. Тензорный анализ.

№1. $S_{ij} = S_{ji}$.

$$S'_{ij} = \alpha_{ik}\alpha_{jl}S_{kl} = \alpha_{ik}\alpha_{jl}S_{lk} = S'_{ji}$$

№2. т.д. $\Pi'_{ij} = \alpha_{ik}\alpha_{jl}\Pi_{kl} \equiv \alpha\Pi\alpha^T = \Pi'$

$$\Pi'_{ij} = \alpha_{ik}\alpha_{jl}\Pi_{kl} = \alpha_{ik}\Pi_{kl}\alpha^T_{lj} = (\alpha\Pi\alpha^T)_{ij}$$

№3. $\alpha = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$; $\varepsilon_{ij} = -\varepsilon_{ji}$; $\varepsilon_{12} = 1$

$\varepsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\varepsilon'_{ij} = \alpha\varepsilon_{ij}\alpha^T$; $\alpha^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\varepsilon'_{ij} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \\ -\cos^2\theta - \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

№4. Тензоры II ранга размерности n A_{ij} и B_{ij}

1) $C_{ijkl} = A_{ij}B_{kl} = \sum\limits_{a,m}^{n} \alpha_{ia}\alpha_{jm}A_{am} \cdot \sum\limits_{p,q} \alpha_{kp}\alpha_{lq}B_{pq} =$

$= \sum\limits_{a,m,p,q} \alpha_{ia}\alpha_{jm}\alpha_{kp}\alpha_{lq}A_{am}B_{pq}$ – тензор IV ранга

2) $D_{il} = \sum\limits_{j} C_{ijjl} \equiv \sum\limits_{j} A_{ij}B_{jl} = \sum\limits_{j}\sum\limits_{k,m,p,q} \alpha_{ik}\overbrace{\alpha_{jm}\alpha_{jp}}^{\delta_{mp}}\alpha_{lq}A_{km}B_{pq}$

$$= \sum_{\substack{k,m\\p,q}} \alpha_{ik} \alpha_{\ell q} \delta_{mp} A_{km} B_{pq} = \alpha_{ik} \alpha_{\ell q} A_k B_q = \alpha_{ik} \alpha_{\ell q} D_{kq} = D'_{i\ell}$$

3) $D = \sum_i D_{ii} = \sum \alpha_{ik} \alpha_{i\ell} D_{k\ell} = \delta_{k\ell} D_{k\ell} = D$

№5. $\varphi(r) = \varphi(x_1, x_2, x_3)$, $D_{ij} = \dfrac{\partial \varphi}{\partial x_i \partial x_j}$,

$x'_i = x'_i(x_1, x_2, x_3)$ — новая сис-ма координат

$\dfrac{\partial \varphi}{\partial x'_i} = \sum_{k=1}^{3} \dfrac{\partial \varphi}{\partial x_k} \dfrac{\partial x_k}{\partial x'_i}$ x_k — старые координаты, $\dfrac{\partial x_k}{\partial x'_i}$ — якобиан преобр-я коорд-т

$$\frac{\partial^2 \varphi}{\partial x'_i \partial x'_j} = \frac{\partial}{\partial x'_j}\left(\frac{\partial \varphi}{\partial x'_i}\right) = \frac{\partial}{\partial x'_j}\left(\sum_{k=1}^{3} \frac{\partial \varphi}{\partial x_k} \frac{\partial x_k}{\partial x'_i}\right) = \frac{\partial x_k}{\partial x'_i} \frac{\partial}{\partial x'_j} \frac{\partial \varphi}{\partial x_k} +$$

$$+ \frac{\partial \varphi}{\partial x_k} \frac{\partial}{\partial x'_j} \frac{\partial x_k}{\partial x'_i} = \frac{\partial x_k}{\partial x'_i} \frac{\partial x_\ell}{\partial x'_j} \frac{\partial^2 \varphi}{\partial x_k \partial x_\ell} + \dots$$

$$D'_{ij} = \frac{\partial x_i}{\partial x'_i} \frac{\partial x_j}{\partial x'_j} D_{ij}$$

№6.

$$A_{ij} = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & w_2 \end{pmatrix}; \quad A_{ij} = -A_{ji}, \text{ т.д. } \vec{w} = (w_1, w_2, w_3) \text{ — вектор}$$

$$P'_{ij} = \alpha_{ik} \alpha_{j\ell} P_{k\ell} = -\alpha_{ik} \alpha_{j\ell} P_{\ell k} = -\alpha_{j\ell} \alpha_{ik} P_{\ell k} = -P'_{ji}$$

$$w'_3 = P'_{12} = \alpha_{1k} \alpha_{2\ell} P_{k\ell} = w_1(\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22}) +$$
$$+ w_2(\alpha_{13}\alpha_{21} - \alpha_{11}\alpha_{23}) + w_3(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) =$$
$$= w_1 \alpha_{31} + w_2 \alpha_{32} + w_3 \alpha_{33}$$

$$w'_i = \alpha_{ik} w_k$$

№ 7.

1) $\sum_j A_{ij}\delta_{jk} = A_{i1}\delta_{1k} + A_{i2}\delta_{2k} + \ldots + A_{in}\delta_{nk} = A_{ik}$

$\sum_i A_{ij}\delta_{ik} = \sum_i A_{ij}\delta_{ki} = = A_{jk}$

$\sum_{ij} A_{ij}\delta_{ij} = A$

2) $\sum_k \delta_{ik}\delta_{kj} = \delta_{ij}$ $\qquad \sum_{ik}\delta_{ik}\delta_{ki} = \sum_{ik}\delta_{ik}\delta_{ik} = \delta = 1$

$\sum_{ik}\delta_{ik}\delta_{ik} = \delta = 1$

№ 8. Т.д. $\varepsilon_{ij}\varepsilon_{lm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Т.к. $\varepsilon_{ijk}\varepsilon_{lmn} = \det\begin{pmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{pmatrix} \Rightarrow$

$\varepsilon_{ij}\varepsilon_{lm} = \det\begin{pmatrix}\delta_{il} & \delta_{im}\\ \delta_{jl} & \delta_{jm}\end{pmatrix} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

антисимметричность в правой части сохр-ся т.к. при перестановке строк или столбцов местами, знак определителя меняется

проверим для $i = l = 1$, $j = m = 2$:

$\varepsilon_{12}\varepsilon_{12} = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} = 1$

№9. 1) $\det\begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) +$

$+ \delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$

2) $\sum_k \varepsilon_{ijk}\varepsilon_{lmk} = \sum_k \varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{ij}\varepsilon_{lm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

3) $\sum_{jk} \varepsilon_{ijk}\varepsilon_{ljk} = \sum_{jk} \varepsilon_{ijk}\varepsilon_{klj} = \sum_j \varepsilon_{ij}\varepsilon_{lj} = \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} =$

$= 3\delta_{il} - \delta_{il} = 2\delta_{il}$

№10. $A_{ij} = -A_{ji}$, $S_{ij} = S_{ji}$, т.д. $\sum_{ij} A_{ij} S_{ij} \equiv 0$

$\left.\begin{array}{l} A_{ij}S_{ij} = -A_{ji}S_{ij} = -C \\ A_{ij}S_{ij} = A_{ij}S_{ji} = C \end{array}\right\} C = -C \Rightarrow C = 0$

№11. 1) $b_i = \sum_j \Pi_{ij} a_j$

$b_i' = \sum_j \Pi_{ij}' a_j' = \sum_j \sum_{kl} \alpha_{ik} \alpha_{jl} \Pi_{kl} \alpha_{jl} a_l =$

$= \alpha_{ik} \sum_l \alpha_{jl} \Pi_{kl} a_l = \alpha_{ik} b_k$

2) $c_j = \sum_i \Pi_{ij} a_i$

$c_j' = \sum_i \Pi_{ij}' a_i' = \sum_i \alpha_{ik}\alpha_{jl}\Pi_{kl}\alpha_{im} a_m =$

$= \alpha_{jl} \sum_k \alpha_{ik} \Pi_{kl} a_k = \alpha_{jl} c_l$