Problem 4.1.

$\varphi(z) = \sqrt[3]{z}$

$\varphi_0(\underset{z_0}{-1}) = e^{i\frac{\pi}{3}}$

$\varphi(1), \varphi(i+0), \varphi(i-0) = ?$

$$\varphi(z) = \sqrt[3]{\frac{z}{z_0} z_0} = \sqrt[3]{\frac{z}{z_0}} \cdot \varphi_0 = $$
$$= \varphi_0 \cdot \left(\frac{|z|}{|z_0|} \frac{e^{i \arg z}}{e^{i \arg z_0}} \right)^{1/3} = \varphi_0 \left| \frac{z}{z_0} \right|^{1/3} e^{\frac{i}{3} \Delta \arg z}$$

$$\Delta \arg (z-1) = \pi$$

$$\varphi(1) = e^{i\frac{\pi}{3}} \left| \frac{1}{-1} \right|^{1/3} \cdot e^{\frac{i\pi}{3}} = e^{\frac{i2\pi}{3}}$$

$$\Delta \arg (i+0) = \frac{3\pi}{2}$$

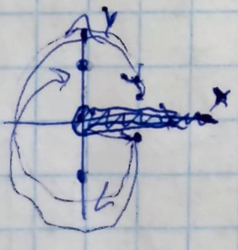
$$\varphi(i+0) = e^{i\frac{\pi}{3}} \left| \frac{i}{-1} \right|^{1/3} \cdot e^{\frac{i}{3}\frac{3\pi}{2}} = e^{i\left(\frac{\pi}{3} + \frac{3\pi}{6}\right)} = e^{i\frac{5\pi}{6}}$$

$$\Delta \arg (i+0) = -\frac{\pi}{2}$$

$$\varphi(i-0) = e^{i\frac{\pi}{3}} \left| \frac{i}{1} \right|^{1/3} \cdot e^{-\frac{i}{3}\frac{\pi}{2}} = e^{+i\frac{\pi}{6}}$$

$\varphi(z) = \ln z$

$\varphi_0(\underset{z_0}{1-i0}) = 0$



$\varphi(1+i0), \varphi(i), \varphi(-i) = ?$

$$e^{\varphi(z)} = z$$

$$e^{\varphi(z)} = \left| \frac{z}{z_0} \right| e^{i \Delta \arg z_0} \cdot e^{\varphi_0}$$

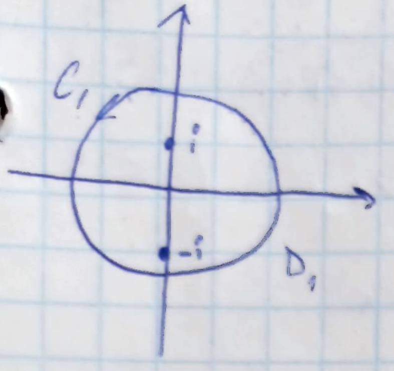
$$\varphi(z) = \ln \left| \frac{z}{z_0} \right| + i \Delta \arg_{z z_0} + \varphi_0$$

$$\varphi(1+i0) = \ln \left| \frac{1}{1} \right| + i(-2\pi) + 0 = -2\pi i$$

$$\varphi(i) = \ln \left| \frac{i}{1} \right| + i\left(-\frac{3\pi}{2}\right) + 0 = -\frac{3\pi}{2} i$$

$$\varphi(-i) = i\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} i$$

Problem 4.3.

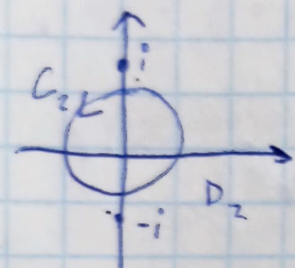


$f(z) = \sqrt{1+z^2}$

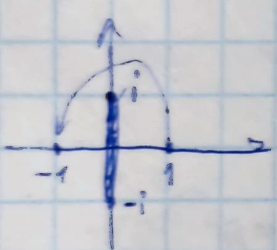
$I_1 = \oint_{C_1} \sqrt{1+z^2}\, dz = 2\pi i \operatorname{res}_{\infty} f(z) =$

$= + 2\pi i \cdot \frac{1}{2} = + \pi i$

$\sqrt{1+z^2} = z\sqrt{1+\frac{1}{z^2}} = z\left(1 + \frac{1}{2z^2} + \dots\right)$



$I_2 = \oint_{C_2} \sqrt{1+z^2}\, dz = 2\pi i \operatorname{res} f(z) = 0$

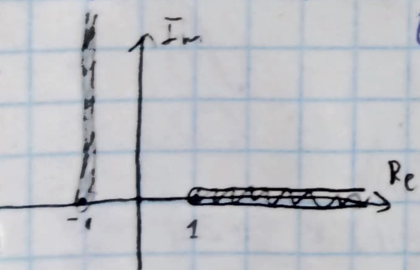


$f(z) = \sqrt{1+z^2}$, $f(1) = \sqrt{2}$

$f(-1) = \sqrt{2}\sqrt{\frac{2}{2}}\, e^{\frac{i}{2}\left(\frac{3\pi}{2} + \frac{\pi}{2}\right)} = -\sqrt{2}$

Problem 4.6.

$\varphi(z) = \sqrt[3]{1+z^2}$, $\varphi(0) = 1$; $\varphi(3i) = ?$



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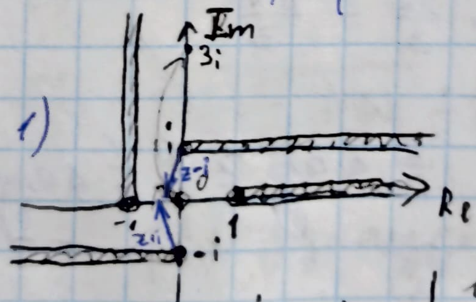
1)



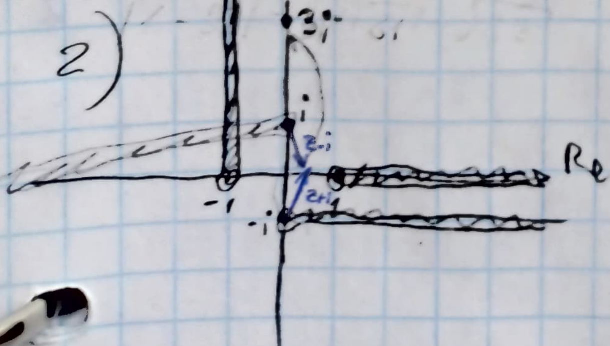
$\Delta \arg(z-i) = -\pi$

$\Delta \arg(z+i) = 0$

$\Delta \arg g = -\pi$

$\varphi(3i) = \left|\frac{1-9}{1}\right|^{1/3} e^{-\frac{i\pi}{3}} \cdot 1 = 2e^{-i\frac{\pi}{3}}$

2)



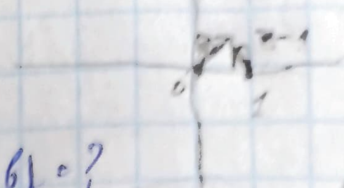
$\Delta \arg(z-i) = \pi$

$\Delta \arg(z+i) = 0$

$\Delta \arg g = \pi$

$\varphi(3i) = 8^{1/3} e^{\frac{i\pi}{3}} \cdot 1 = 2e^{\frac{i\pi}{3}}$

Problem 4.9



$f(z) = z^a (z-1)^b$. How many branch points $N(a,b) = ?$

$N(1,1) = 0$; $\quad 0: \Delta \arg z + \Delta \arg (z-1) = 2\pi + 0 = 2\pi$

$1: \Delta \arg z + \Delta \arg (z-1) = 0 + 2\pi = 2\pi$

$\infty: \Delta \arg z + \Delta \arg (z-1) = 2\pi + 2\pi = 4\pi$

$N(1; \frac{1}{2})$

$$\left.\begin{aligned} 0&: \Delta \arg z + \tfrac{1}{2} \Delta \arg (z-1) = 2\pi + 0 = 2\pi \\ 1&: \Delta \arg z + \tfrac{1}{2} \Delta \arg (z-1) = 0 + \pi = \pi \\ \infty&: 2\pi + \pi = 3\pi \end{aligned}\right\} \Rightarrow N = 2$$

$N(\frac{1}{2}; \frac{1}{3})$

$$\left.\begin{aligned} 0&: \tfrac{1}{2} \Delta \arg z + \tfrac{1}{3} \Delta \arg (z-1) = \pi + 0 = \pi \\ 1&: \tfrac{1}{2} \Delta \arg z + \tfrac{1}{3} \Delta \arg (z-1) = \frac{2\pi}{3} \\ \infty&: \pi + \frac{2\pi}{3} = \frac{5\pi}{3} \end{aligned}\right\} N = 3$$

$N(\frac{2}{3}; \frac{1}{3})$

$$\left.\begin{aligned} 0&: \tfrac{2}{3} \Delta \arg z + \tfrac{1}{3} \Delta \arg (z-1) = \tfrac{2}{3} \cdot 2\pi = \frac{4\pi}{3} \\ 1&: \frac{2\pi}{3} \\ \infty&: \tfrac{2}{3} \cdot 2\pi + \tfrac{1}{3} \cdot 2\pi = 2\pi \end{aligned}\right\} N = 2$$

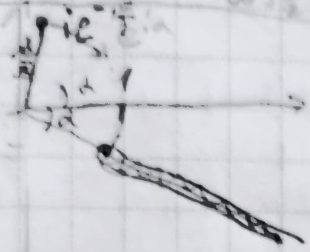
$\varphi_1(z)=\sqrt{z-e^{-i\alpha}}$; $\varphi_2(z)=\ln(z-e^{-i\alpha})$; $\alpha\in(0,\frac{\pi}{2})$

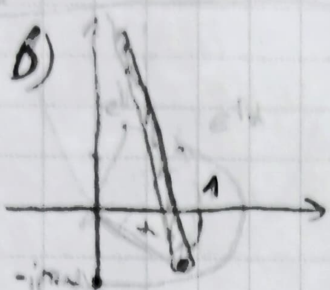
$\varphi_1(0)=ie^{-\frac{i\alpha}{2}}$ $\varphi_2(0)=-i\pi-i\alpha$

$= i\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}$ $=-i(\pi+\alpha)$

$\varphi_{1,2}(e^{i\alpha})=?$

$\varphi_{1,2}(i)=?$

a) 

б) 

$g_1=z-e^{-i\alpha}$

$g_2=z-e^{-i\alpha}$

a) 1. $z=e^{i\alpha}$ $\Delta\arg g=-(\frac{\pi}{2}-\frac{\alpha}{2}-\alpha)==-\frac{\pi-3\alpha}{2}$

$=\cos\alpha-i\sin\alpha$

$\varphi_1(e^{i\alpha})=ie^{-i\frac{\alpha}{2}}\cdot\left|\frac{e^{i\alpha}-e^{-i\alpha}}{-e^{-i\alpha}}\right|^{\frac{1}{2}}\cdot e^{\frac{i}{2}\frac{3\alpha-\pi}{2}}=\left|\frac{e^{-i\alpha}(e^{2}-1)}{-e^{i\alpha}}\right|^{\frac{1}{2}}ie^{\frac{i}{2}(\frac{3\alpha-\pi}{2}-\alpha)}=$

$=\sqrt{e^2-1}\cdot i\cdot e^{\frac{i}{2}(\frac{\alpha-\pi}{2})}$

$z=i$: $\Delta\arg g=\frac{\alpha}{2}$

$\varphi_1(i)=ie^{-i\frac{\alpha}{2}}\left|\frac{i-e^{-i\alpha}}{-e^{-i\alpha}}\right|^{\frac{1}{2}}e^{\frac{i}{2}\frac{\alpha}{2}}=2^{\frac{1}{4}}ie^{i\frac{\alpha}{2}(\frac{1}{2}-1)}=2^{\frac{1}{4}}ie^{-\frac{i\alpha}{4}}$

2. $z=e^{i\alpha}$ $\Delta\arg g=\frac{3\alpha-\pi}{2}$

$\varphi_2(e^{i\alpha})=\ln g(e^{i\alpha})=\ln\left|\frac{e^{i\alpha}-e^{-i\alpha}}{-e^{-i\alpha}}\right|+ie^{-\frac{i\alpha}{2}}+i\frac{3\alpha-\pi}{2}=$

$=\ln(e^2-1)+i(\frac{3\alpha-\pi}{2}+e^{-\frac{i\alpha}{2}})$

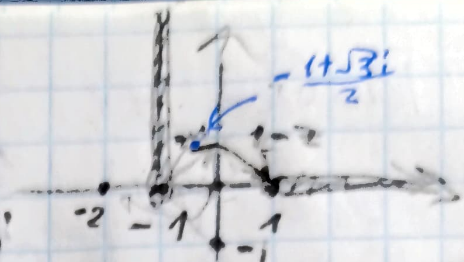
$\varphi_2(i)=\ln\left|\frac{i-e^{-i\alpha}}{-e^{-i\alpha}}\right|+ie^{-i\frac{\alpha}{2}}+i\frac{\alpha}{2}$

b) $z=i$ $\Delta\arg g=0$

$\varphi_2(i)=\ln\left|\frac{i-e^{-i\alpha}}{-e^{-i\alpha}}\right|-i\pi-i\alpha$

$\varphi_1(i)=ie^{-i\frac{\alpha}{2}}\left|\frac{i-e^{-i\alpha}}{-e^{-i\alpha}}\right|$

Problem 4.5.



$\varphi(z) = \ln(1-z^2), \varphi(0) = -2\pi i$

$g(z) = (1-z)(1+z)$ | $\varphi(-2)$; $\varphi(-i)$; $\varphi\left(\frac{-1+\sqrt{3}i}{2}\right)$

$\varphi(-2)$: $\Delta \arg(1-z) = 0$

$\Delta \arg(1+z) = -\pi$

$\Delta \arg g = -\pi$

$\varphi(-2) = \ln\left|\frac{1-4}{1}\right| - 2\pi i - \pi i = \ln 3 - 3\pi i$

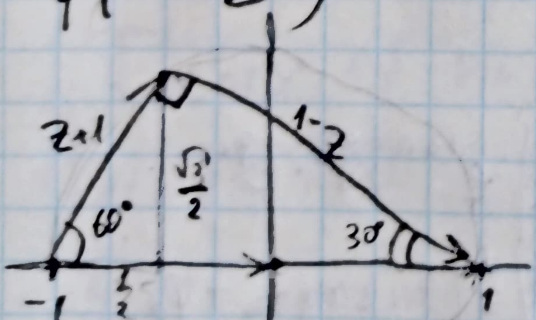
$\varphi(-i)$: $\Delta \arg(1-z) = \frac{\pi}{4}$, $\Delta \arg(1-z) = -\frac{\pi}{4}$; $\Delta \arg g = 0$

$\varphi(-i) = \ln\left|\frac{1+1}{1}\right) - 2\pi i + 0 = \ln 2 - 2\pi i$

$\varphi\left(\frac{-1+\sqrt{3}i}{2}\right)$: $\Delta \arg(z+1) = \frac{\pi}{3}$; $\Delta \arg(1-z) = -\frac{\pi}{6}$

$\Delta \arg g = \frac{\pi}{6}$

$\varphi\left(\frac{-1+\sqrt{3}i}{2}\right) = \ln\left|\frac{1+1+\sqrt{3}i}{1}\right| + i\frac{-\pi}{6} - 2\pi i =$



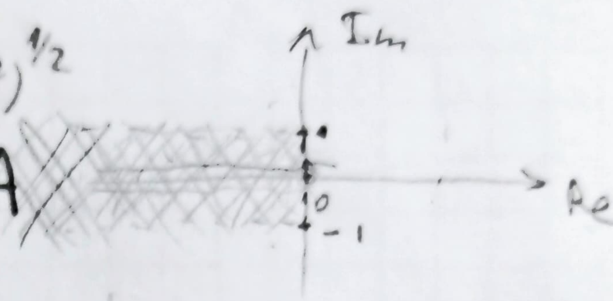
$\left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \left(\frac{1-2\sqrt{3}i-3}{4}\right) = -1-\sqrt{3}i$

$\frac{\pi}{6} - 2\pi = \frac{\pi - 12\pi}{6}$

$= \ln 7 - \frac{11\pi}{6} i$

$f(x) = \ln[(1+x^2)^{1/2}]$; $g(x) = (1+x^2)^{1/2}$

$h(x) = 1+x^2$



$(1+x^2)^{1/2} > 0$ para $x>0$

$A: \{x: 0>x>-\infty\} \cup \{ix: 1>x>-1\}$

$F(z)$: $f(x) = F(x)$ para real $x>0 \vee A$

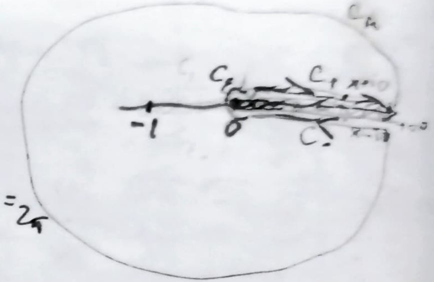
$$F(x) = \frac{1}{2}\ln(1+x^2)$$

$$\lim_{\varepsilon\to 0} F(\varepsilon) = \lim_{\varepsilon\to 0} \frac{1}{2}\left(\varepsilon^2 - \frac{\varepsilon^4}{2} + \dots\right) = 0$$

$$\lim_{\varepsilon\to 0} F(\varepsilon e^{\frac{3\pi i}{4}}) = \lim_{\varepsilon\to 0} \frac{1}{2}\ln(1+\varepsilon^2 e^{\frac{3\pi i}{2}}) = \lim_{\varepsilon\to 0} \frac{1}{2}\left(\varepsilon^2 e^{\frac{3\pi i}{2}} - \frac{\varepsilon^4 e^{3\pi i}}{2}\right)$$

$F(\varepsilon e^{\frac{3\pi i}{4}}) =$

$I=\int_0^\infty \frac{dx}{(x+1)\sqrt{x}}$ $\quad x=-1,\ x=0\ ;\quad f(z)=\frac{1}{(z+1)\sqrt{z}}\ ,\quad f(z)_{z\to\infty}=\frac{1}{z^{3/2}}$



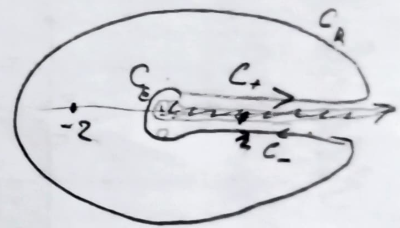
$$\oint=\overset{I}{\int_{C_+}}+\int_{C_-}+\overset{0}{\int_{C_R}}+\overset{0}{\int_{C_\varepsilon}}=2\pi i\operatorname*{res}_{-1} f(z)=2\pi i\,\frac{1}{\sqrt{z}}\Big|_{-1}=\frac{2\pi i}{e^{\frac{i\pi}{2}}}=\frac{2\pi i}{i}=2\pi$$

$$g(z)=z^{-\frac{1}{2}}\ ,\quad g(x-i0)=g(x+i0)\cdot\left|\frac{x-i0}{x+i0}\right|^{-\frac{1}{2}}\cdot e^{-i\frac{1}{2}2\pi}=g(x+i0)\cdot e^{-\pi i}=-g(x+i0)$$

$$\int_{C_-}=\int_{+\infty}^{0}\frac{dx}{(x+1)\,g(x-i0)}=\int_0^{+\infty}\frac{dx}{(x+1)\,g(x+i0)}=I$$

$$I+I=2\pi\ \Rightarrow\ I=\underline{\pi}$$

$\int_0^\infty\frac{dx}{(x^2+4)\sqrt[3]{x}}$ $\quad x=\pm 2\quad f(z)=\frac{1}{(z^2+4)\sqrt[3]{z}}\ ;\ g(z)=z^{-\frac{1}{3}}$



$$\oint=\overset{I}{\int_{C_+}}+\int_{C_-}+\overset{0}{\int_{C_R}}+\overset{0}{\int_{C_\varepsilon}}=2\pi i\operatorname*{res}_{-2} f(z)=2\pi i\,\frac{1}{(z-2)\sqrt[3]{z}}\Big|_{2}=\frac{8\pi i}{2\cdot 4\sqrt[3]{2}}=\frac{\pi i}{2\sqrt[3]{2}}$$

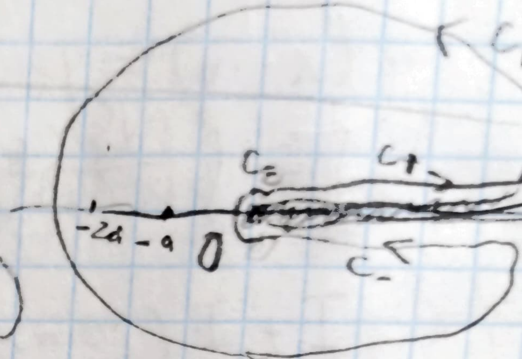
$$g(x-i0)=g(x+i0)\left|\frac{x-i0}{x+i0}\right|^{-\frac{1}{3}}e^{-i\frac{2\pi}{3}}=g(x+i0)\cdot e^{-i\frac{2\pi}{3}}=g(z)\Big(\overset{-\frac{1}{2}}{\cos\frac{2\pi}{3}}-i\overset{\frac{\sqrt{3}}{2}}{\sin\frac{2\pi}{3}}\Big)=$$

$$\int_{C_-}=\int_{+\infty}^{0}\frac{dx}{(x^2+4)\sqrt[3]{x}}\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=\int_0^{+\infty}\frac{dx}{(x^2+4)\sqrt[3]{x}}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\int_0^{+\infty}\frac{(1+i\sqrt{3})\,dx}{2(x^2+4)\sqrt[3]{x}}=\frac{1+i\sqrt{3}}{2}I$$

$$I\left(1+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)=\frac{\pi i}{2\sqrt[3]{2}}$$

$$I=\frac{\pi i}{\sqrt[3]{2}\,(3+i\sqrt{3})}$$

8) $I = \int_0^\infty \frac{x^\alpha \, dx}{(x+a)(x+2a)}$ $\quad -1 < \alpha < 1, \; a > 0$



$$f(z) = \frac{z^\alpha}{(z+a)(z+2a)}$$

$$\oint = \int_{C_+} - \int_{C_-} + \int_{C_R}^{\to 0} + \int_{C_\varepsilon}^{\to 0} = 2\pi i \left(\operatorname{res}_{a} f(z) + \operatorname{res}_{2a} f(z) \right)$$

$$g(x-i0) = g(x+i0) \left| \frac{x-i0}{x+i0} \right|^\alpha e^{i\alpha 2\pi} = g(x+i0) \cdot e^{i\alpha 2\pi}$$

$$\operatorname{res}_{a} f(z) = \frac{z^\alpha}{z+2a} \Big|_{a} = \frac{a^\alpha}{3a}$$

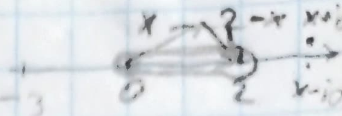
$$\operatorname{res}_{2a} f(z) = \frac{z^\alpha}{z+a} \Big|_{2a} = \frac{(2a)^\alpha}{3a}$$

$$I + I e^{i\alpha 2\pi} = 2\pi i \left(\frac{a^\alpha + 2^\alpha a^\alpha}{3a} \right)$$

$$I = \frac{2\pi i \, a^{\alpha-1} (1+2^\alpha)}{3(1+e^{i\alpha 2\pi})}$$

№ 28.25.

$g(z)=\sqrt{z(2-z)}$



1) $I=\int_0^2 \frac{\sqrt{x(2-x)}}{x+3}\,dx$; $f(z)=\frac{\sqrt{z(2-z)}}{z+3}$

$$f(z)=f(z_0)\left|\frac{g(z)}{g(z_0)}\right|^{1/2} e^{\frac{i}{2}\Delta\arg g}$$

$$f(x+i0)=f(z_0)\cdot e^{-\frac{i}{2}\pi}=-i\,\frac{\sqrt{x(2-x)}}{x+3}$$

$$f(x-i0)=f(z_0)\,e^{-i\frac{3\pi}{2}}=i\,\frac{\sqrt{x(2-x)}}{x+3}$$

$$\operatorname*{res}_{z=-3} f_0(z)=\sqrt{z(2-z)}\,\exp\left(\frac{i}{2}\pi\right)\Big|_{-3}=\sqrt{-15}\,i=-\sqrt{15}$$

При $z\to\infty$: $f(z)=\frac{\sqrt{z(2-z)}}{z+3}\exp\left(\frac{i}{2}(-\pi)\right)=i(-i)=1$

$$f(z)=\frac{\sqrt{z(2-z)}}{z+3}=i\,\frac{\sqrt{1-\frac{2}{z}}}{1+\frac{3}{z}}=i\left(1-\frac{2}{2z}+\dots\right)\left(1-\frac{3}{z}+\dots\right)=$$

$$=i\left(1-\frac{4}{z}+\dots\right)\Rightarrow C_{-1}=-4i\ ;\ \operatorname*{res}_{\infty} f(z)=4i$$

$$\oint f(z)\,dz=2\pi i\left(\operatorname*{res}_{-3} f(z)+\operatorname*{res}_{\infty} f(z)\right)=2\pi i\left(-\sqrt{15}+4i\right)$$

$$\oint f(z)\,dz=\int_0^2 f(x+i0)\,dx+\int_2^0 f(x-i0)\,dx=$$

$$=-i\,2I=2\pi i\left(4i-\sqrt{15}\right)$$

$$I=\pi\left(\sqrt{15}-4i\right)$$