Problem 5.1

$$I_0(z) = \int_0^\infty t^z e^{-t} dt$$

$$I_1(z) = \frac{1}{z+1} \int_0^\infty t^{z+1} e^{-t} dt$$

$$I_2(z) = \frac{1}{(z+1)(z+2)} \int_0^\infty t^{z+2} e^{-t} dt$$

$$I_3(z) = \frac{1}{(z+3)(z+2)(z+1)} \int_0^\infty t^{z+3} e^{-t} dt$$

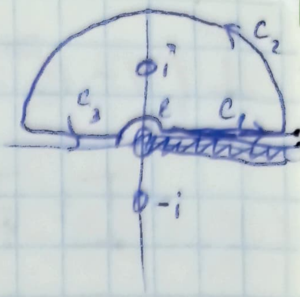
$$\operatorname*{res}_{z=-3} I_3(z) = \frac{1}{(z+2)(z+1)} \int_0^\infty t^{z+3} e^{-t} dt \Big|_{z=-3} =$$

$$= \frac{1}{2} \int_0^\infty e^{-t} dt = \frac{1}{2}(-e^{-\infty}+1) = \frac{1}{2}$$

Problem 6.1.

$$I = \int_0^\infty \frac{\ln x \, dx}{x^2+1} \qquad f(z) = \frac{\ln z}{z^2+1} = \frac{\ln z}{(z-i)(z+i)}$$

$$g(z) = \ln z$$



$$\oint = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_\varepsilon} = 2\pi i \operatorname{res}_i f(z) = 2\pi i \left.\frac{\ln z}{z+i}\right|_i = 2\pi i \frac{\ln i}{2i} = i\frac{\pi^2}{2}$$

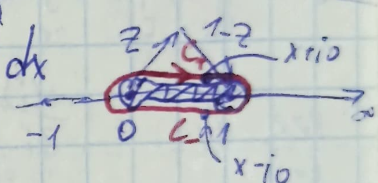
$$g(-x_0+i0) = \ln\left|\frac{-x_0+i0}{x_0+i0}\right| + \ln(x_0+i0) + i\pi = \ln x_0 + i\pi$$

$$I_1 = \int_0^\infty \frac{\ln(x_0+i0)}{x^2+1}dx, \quad I_3 = \int_\infty^0 \frac{\ln(-x_0+i0)}{x^2+1}dx = \int_0^\infty \frac{\ln x\,dx}{x^2+1} + i\pi\int_0^\infty \frac{dx}{1+x^2}$$

$$\oint = 2I_1 + i\pi \int_0^\infty \frac{dx}{x^2+1} = i\frac{\pi^2}{2} \Rightarrow I_1 = 0$$

Problem 6.2.

$$I(\alpha) = \int_0^1 \frac{x^\alpha(1-x)^{2-\alpha}}{x+1}dx$$



$$f(z) = z^\alpha(1-z)^{2-\alpha} \qquad f_1(z) = \frac{f(z)}{z+1}$$

$$f(x+i0) > 0, \quad 0 < x < 1$$

$$\oint = \int_{C_+} + \int_{C_-} = \int_0^1 \frac{f(x+i0)}{x+1}dx + \int_1^0 \frac{f(x-i0)}{x+1}dx = I(1-e^{2\pi\alpha i}) = 2\pi i \sum \operatorname{res} f_1$$

$(x-i0)$:
$\Delta\arg z = 0$; $\Delta\arg(1-z) = -2\pi$; $\Delta\arg f = 2\pi\alpha = \varphi_n$; $f(x-i0) = f(x+i0)e^{2\pi\alpha i}$

$$\operatorname{res}_{-1} f_1(z) = f(z)|_{-1} = 2^{2-\alpha}e^{i\pi\alpha}$$

(-1): $\Delta\arg z = \pi$; $\Delta\arg(1-z) = 0$; $\Delta\arg f = \pi\alpha$; $f(-1) = 1^\alpha 2^{2-\alpha}e^{i\pi\alpha}$

$$\operatorname{res}_\infty f(z) = -e^{i\pi\alpha}\frac{8-5\alpha+\alpha^2}{2}$$

(∞): $\Delta\arg z = 0$; $\Delta\arg(1-z) = -\pi$; $\Delta\arg f = \pi\alpha - 2\pi$; $\underset{x\to\infty}{f(x)} = |x|^\alpha|1-x|^{2-\alpha}e^{i\pi\alpha}$

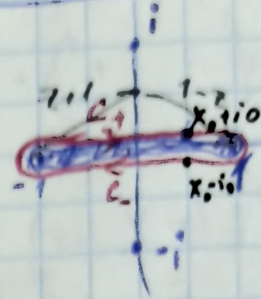
$$f = \frac{x^\alpha(x-1)^{2-\alpha}}{x+1} = \frac{x^\alpha x^{2-\alpha}(1-\frac{1}{x})^{2-\alpha}}{x(1+\frac{1}{x})} \simeq x\left(1 - \frac{2-\alpha}{x} + \frac{(2-\alpha)(1-\alpha)}{2x}\right)\left(1-\frac{1}{x}+\frac{1}{x^2}\right) =$$

$$= x\left(1 + \dots + \frac{2-\alpha}{x^2} + \frac{2-3\alpha+\alpha^2}{2x^2} + \frac{1}{x^2} + \dots\right) = x\left(\dots + \frac{8-5\alpha+\alpha^2}{2x^2} + \dots\right)$$

$$I\left(1-e^{2\pi i\alpha}\right)=e^{i\pi\alpha}(-2i\sin\pi\alpha)I=2\pi i\,e^{i\pi\alpha}\left(\frac{4}{2^\alpha}-\frac{8-5\alpha+\alpha^2}{2}\right),$$

$$I=\frac{\pi}{\sin\pi\alpha}\left(\frac{8-5\alpha+\alpha^2}{2}-\frac{4}{2^\alpha}\right)=\frac{\pi}{\sin\pi\alpha}\left(\frac{\alpha^2}{2}-\frac{5}{2}\alpha+4-2^{2-\alpha}\right)$$

Problem 6.3. $\displaystyle I(\alpha)=\int_{-1}^{1}\frac{(1-x)^\alpha(1+x)^{1-\alpha}}{x^2+1}\,dx$



$$f(z)=(1-z)^\alpha(1+z)^{1-\alpha};\qquad f_1(z)=\frac{f(z)}{(z-i)(z+i)}$$

(x_0-i0): $\Delta\arg(1-z)=-2\pi$
$\Delta\arg(1+z)=0$
$\Delta\arg f=-2\pi\alpha$

$$f(x_0-i0)=f(x+i0)\,e^{-2\pi i\alpha}$$

$$\oint=\int_{C_+}+\int_{C_-}=\int_{-1}^{1}\frac{f(x_0+i0)}{x^2+1}dx+\int_{1}^{-1}\frac{f(x_0-i0)}{x^2+1}dx=I\left(1-e^{-2\pi i\alpha}\right)=2\pi i\sum\operatorname{res}f_1$$

$$\operatorname*{res}_{z=i}f_1(z)=\frac{f(i)}{2i}=\frac{\sqrt2\,e^{i\left(\frac{\pi}{4}-\frac{\pi}{2}\alpha\right)}}{2i}$$

i: $\Delta\arg(1-z)=-\frac{\pi}{4}$
$\Delta\arg(1+z)=\frac{\pi}{4}$
$\Delta\arg f=\frac{\pi}{4}(1-\alpha)-\frac{\pi}{4}\alpha=\frac{\pi}{4}-\frac{\pi}{2}\alpha$

$$f(i)=|1-i|^\alpha|1+i|^{1-\alpha}e^{i\left(\frac{\pi}{4}-\frac{\pi}{2}\alpha\right)}=\sqrt2^{\,\alpha}\sqrt2^{\,1-\alpha}e^{i\left(\frac{\pi}{4}-\frac{\pi}{2}\alpha\right)}=\sqrt2\,e^{i\left(\frac{\pi}{4}-\frac{\pi}{2}\alpha\right)}$$

$$\operatorname*{res}_{z=-i}f_1(z)=\frac{f(-i)}{-2i}=\frac{\sqrt2\,e^{-i\left(\frac{\pi}{4}-\frac{\pi}{2}\alpha\right)}}{-2i}$$

$-i$: $\Delta\arg(1-z)=\frac{\pi}{4}$
$\Delta\arg(1+z)=-\frac{\pi}{4}$
$\Delta\arg f=\frac{\pi}{4}\alpha-\frac{\pi}{4}(1-\alpha)=\frac{\pi}{2}\alpha-\frac{\pi}{4}$

$$f(-i)=\sqrt2\,e^{-i\left(\frac{\pi}{4}-\frac{\pi}{2}\alpha\right)}$$

$$\operatorname*{res}_{\infty}f_1(z)=-e^{-i\pi\alpha}$$

$$\underset{x\to\infty}{f(x)}=\left|\cancel{\frac{f(x)}{f(x_0+i0)}}\right|f(x_0+i0)\,e^{-\pi i\alpha}=|1-x|^\alpha|1+x|^{1-\alpha}e^{-i\pi\alpha}=$$

∞: $\Delta\arg(1-z)=-\pi$
$\Delta\arg(1+z)=0$
$\Delta\arg f=-\pi\alpha$

$$=(x-1)^\alpha(1+x)^{1-\alpha}e^{-i\pi\alpha}$$

$$f_1=\frac{(x-1)^\alpha(x+1)^{1-\alpha}}{x^2+1}=\frac{x^\alpha x^{1-\alpha}\left(1-\frac{1}{x}\right)^\alpha\left(1+\frac{1}{x}\right)^{1-\alpha}}{x^2\left(1-\frac{1}{x}\right)}=\frac{1}{x}\left(1+\ldots\right)$$

$$\operatorname*{res}_{z=i} f_1(z) + \operatorname*{res}_{z=-i} f_1(z) = \sqrt{2}\left(\frac{e^{i(\frac{\pi}{4}-\frac{\pi}{2}\alpha)} - e^{-i(\frac{\pi}{4}-\frac{\pi}{2}\alpha)}}{2i}\right) =$$

$$= \sqrt{2}\,\sin\left(\frac{\pi}{4}-\frac{\pi}{2}\alpha\right) = \sqrt{2}\left(\sin\frac{\pi}{4}\cos\frac{\pi}{2}\alpha - \cos\frac{\pi}{4}\sin\frac{\pi}{2}\alpha\right) =$$

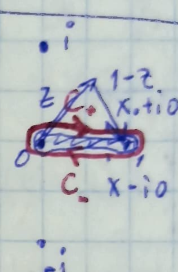
$$= \cos\frac{\pi}{2}\alpha - \sin\frac{\pi}{2}\alpha$$

$$I(1-e^{-2\pi i\alpha}) = I e^{-\pi i\alpha}(2i\sin\pi\alpha) = 2i\pi\cdot(\cos\tfrac{\pi}{2}\alpha - \sin\tfrac{\pi}{2}\alpha$$

$$-e^{i\ldots}$$

$$I = \frac{\pi}{\sin\pi\alpha}\left(\cos\frac{\pi\alpha}{2} + \sin\frac{\pi\alpha}{2} - 1\right)$$

Problem 6.5. $I = \int_0^1 \ln\frac{1-x}{x}\frac{dx}{x^2+1}$



$$g(z) = \frac{1-z}{z} \qquad f(z) = \ln g(z)\cdot\frac{1}{(z-i)(z+i)}$$

(x_0-i0): $\Delta\arg z = 0$

$\Delta\arg(1-z) = -2\pi$ $\qquad f(x_0-i0) = \ln|1| + \ln\left(\frac{1-x_0}{x_0}\right) - 2\pi i$

$\Delta\arg g = -2\pi$

$$\oint = \int_{C_+} + \int_{C_-} = I + I + 2\pi i\int_0^1\frac{dx}{x^2+1} = 2\pi i\sum\operatorname{res} f(z)$$

$$\operatorname*{res}_{i} f(z) = \frac{1}{2i}\ln g(i) = \frac{1}{2i}\left(\ln\sqrt{2} - i\frac{3\pi}{4}\right)$$

i: $\Delta\arg z = \frac{\pi}{2}$ $\qquad \ln g(i) = \ln\left|\frac{\frac{1-i}{i}}{\frac{1-x_0}{x_0}}\right| + \cancel{\ln\left(\frac{1-x_0}{x_0}\right)} - i\frac{3\pi}{4} =$

$\Delta\arg(1-z) = -\frac{\pi}{4}$

$\Delta\arg g = -\frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}$ $\qquad \ln|-i-1| - i\frac{3\pi}{4} = \ln\sqrt{2} - i\frac{3\pi}{4}$

$$\operatorname*{res}_{-i} f(z) = -\frac{1}{2i}\ln g(-i) = -\frac{1}{2i}\left(\ln\sqrt{2} + i\frac{3\pi}{4}\right)$$

$-i$: $\Delta\arg z = -\frac{\pi}{2}$

$\Delta\arg(1-z) = \frac{\pi}{4}$ $\qquad \ln g(-i) = \ln\left|\frac{1+i}{-i}\right| + i\frac{3\pi}{4} = \ln\sqrt{2} + i\frac{3\pi}{4}$

$\Delta\arg g = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$

$$\operatorname*{res}_{\infty} f(z) = +i\pi$$

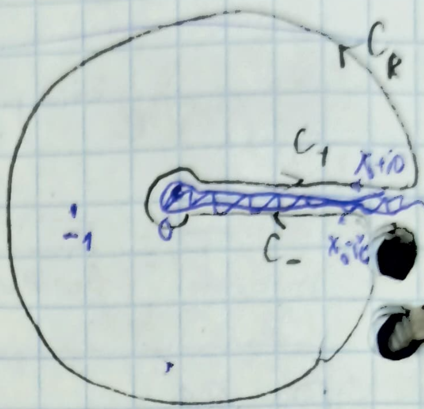
∞: $\Delta\arg z = 0$; $\Delta\arg(1-z) = -\pi$; $\Delta\arg g = -\pi$ $\qquad f(x)_{x\to\infty} = \ln\left|\frac{1-x}{x}\right|^{0} - i\pi = -i\pi$

$$\ln\frac{1-x}{x}\cdot\frac{1}{1^2+1} = \ln\left(-\left(1-\frac{1}{x}\right)\right)\cdot\frac{1}{1+x^2} = \left(\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{3x^3}+\dots\right)\left(1-x^2+x^4-\dots\right)$$

$$2I + 2\pi i\int_0^1 \frac{dx}{x^2+1} = 2\pi i\left(\frac{1}{2i}\left(\ln\sqrt{2} - i\frac{3\pi}{4} + \ln\sqrt{2} + i\frac{3\pi}{4}\right) + i\pi\right)$$

$$I = \frac{\pi}{8}\ln 2$$

Problem $\overset{6.8}{5.7.}$ $\quad I = PV\int_0^\infty \frac{\sqrt{x}\,dx}{x^2-1}$



$f(z) = \frac{\sqrt{z}}{(z-1)(z+1)} \qquad g(z) = \sqrt{z}$

$$\oint = \int_{C_+} + \int_{C_-} + \int_{C_R}^{\to 0} = 2\pi i \operatorname*{res}_{-1} f(z)$$

$$f(x_0 - i0) = f(x_0+i0)\,e^{i\pi} = -f(x_0+i0)$$

$$\int_0^\infty f(x_0+i0)\,dx + \int_\infty^0 f(x_0-i0) = 2I$$

$$\operatorname*{res}_{-1} f(z) = \frac{g(z)}{2}\Big|_{-1} = \frac{i}{2}$$

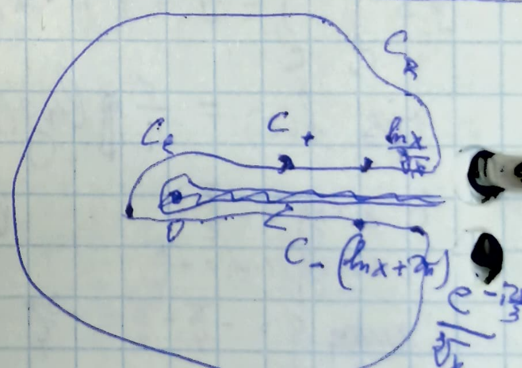
$$g(-1) = g(1)\cdot e^{i\frac{\pi}{2}} = i$$

$$2I = 2\pi i\cdot\frac{i}{2}$$

$$I = \frac{\pi}{2}$$

Problem 6.4.

$$I = \int_0^\infty \frac{\ln x}{\sqrt[3]{x}\,(x+1)^2} \qquad f(z) = \frac{\ln z}{\sqrt[3]{z}\,(z+1)^2}$$



$$\int x^{-\frac13}\ln x\,dx = \left|\begin{matrix} x^{-\frac13}dx = dv & u = \ln x \\ v = \frac32 x^{\frac23} & du = \frac{dx}{x}\end{matrix}\right| =$$

$$= \frac32 x^{\frac23}\ln x - \int x^{-\frac43}dx = \frac32 x^{\frac23}\ln x + 3x^{-\frac13} = x^{-\frac13}\left(\frac32 x\ln x + 3\right)$$

$$\oint = \int_{C_+} + \int_{C_-}$$

$$\int_{C_-} = \int_{+\infty}^0 \frac{(\ln x + 2\pi i)\,e^{-i\frac{2\pi}{3}}}{\sqrt[3]{x}\,(x+1)^2}\,dx = -e^{-i\frac{2\pi}{3}}I - 2\pi i\int_0^{+\infty}\overbrace{\frac{e^{-i\frac{2\pi}{3}}\,dx}{\sqrt[3]{x}\,(x+1)^2}}^{J}$$

$$\oint = I\left(1-e^{-i\frac{2\pi}{3}}\right) - 2\pi i\, e^{-i\frac{\pi}{3}} J = 2\pi i \operatorname*{res}_{z=-1} f(z) = 2\pi i \frac{d}{dz}\left(z^{-\frac{1}{3}}\ln z\right)$$

$$= 2\pi i\left(-\frac{1}{3} z^{-\frac{4}{3}}\ln z + z^{-\frac{4}{3}}\right)\Big|_{-1} = 2\pi i\left(\frac{1}{3}\pi i\, e^{-\frac{i\pi}{3}} - e^{-\frac{i\pi}{3}}\right)$$

$\ln(-1) = \pi i$; $g_1(-1) = e^{-\frac{i\pi}{3}}$

$g_1 = z^{-\frac{1}{3}}$

$$\oint = e^{-i\frac{\pi}{3}} I \sin\frac{\pi}{3}\cdot 2i - 2\pi i\, e^{-i\frac{2\pi}{3}} J = 2\pi i\, e^{-\frac{i\pi}{3}}\left(\frac{\pi i}{3}-1\right) \Big| \cdot \frac{e^{\frac{\pi i}{3}}}{2i}$$

$$I \sin\frac{\pi}{3} - \pi e^{-\frac{i\pi}{3}} J = \pi\left(\frac{\pi i}{3}-1\right)$$

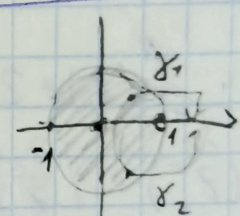
Im: $\pi \sin\frac{\pi}{3} J = \frac{\pi^2}{3} \Rightarrow J = \frac{\pi}{3}\frac{2}{\sqrt{3}}$

Re: $I\cdot\frac{\sqrt{3}}{2} - \pi \overset{\frac{1}{2}}{\cos\frac{\pi}{3}}\cdot\frac{\pi}{3}\frac{2}{\sqrt{3}} = -\pi$

$$I = \left(-\pi + \frac{\pi^2}{3\sqrt{3}}\right)\cdot\frac{2}{\sqrt{3}} = \frac{2}{9}\pi^2 - \frac{2}{\sqrt{3}}\pi$$

$f_\gamma(2)$

Problem 5.2.



$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)(n+2)} = \sum_{n=0}^{\infty}\frac{z^n}{n+1} - \sum_{n=0}^{\infty}\frac{z^n}{n+2} =$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1} - \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^{n+2}}{n+2} = \frac{1}{z}\sum_{n=1}^{\infty}\frac{z^n}{n} - \frac{1}{z^2}\left(\sum_{n=1}^{\infty}\frac{z^n}{n} - z\right) =$$

$$= -\frac{\ln(1-z)}{z} + \frac{1}{z^2}\left(\ln(1-z)+z\right) = \frac{\ln(1-z)+z-z\ln(1-z)}{z^2}$$

γ_1: $\ln(2) = \ln\left|\frac{1-2}{x_0}\right| + \ln x_0 - i\pi = -i\pi$

$$f_{\gamma_1}(2) = \frac{-i\pi+2+2i\pi}{4} = \frac{1}{2} + \frac{\pi i}{4}$$

γ_2: $\ln(2) = \ln\left|\frac{1-2}{x_0}\right| + \ln x_0 + i\pi = \pi i$

$$f_{\gamma_2}(2) = \frac{\pi i + 2 - 2\pi i}{4} = \frac{1}{2} - \frac{\pi i}{4}$$

Problem 5.3.

$F(z,t) = z^3 - 3z^2 + t = 0$

$z' = -\frac{F'_t}{F'_z} = -\frac{1}{3z^2-6z}$

$F'_z = 3z^2 - 6z$

$F'_t = 1$

$3z^2 - 6z = 0$

$z(z-2) = 0$

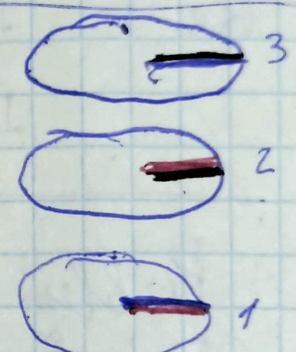
$z = 0 \quad z = 2$

$t = 3z^2 - z^3$

$t_0(0) = 0$

$t_1(2) = 3\cdot\overset{12}{4} - 8 = 4$

Problem 5.4.



$f(z) = z^{1/3}$; $\Pi_{z_1} = \Pi_{z_2} = \Pi_{z_3} = -2$

$f(z_1) = 2^{1/3} e^{\frac{i\pi}{3}}$

$f(z_2) = \left|\frac{z_2}{z_1}\right|^{1/3} e^{i\frac{2\pi}{3}} \cdot 2^{1/3} \cdot e^{\frac{i\pi}{3}} = 2^{1/3} e^{i\pi} = -2^{1/3}$

$f(z_3) = \left|\frac{z_3}{z_2}\right|^{1/3} e^{i\frac{2\pi}{3}} \cdot 2^{1/3} e^{i\pi} = e^{\frac{i5\pi}{3}} 2^{1/3} = 2^{1/3} e^{-\frac{i\pi}{3}}$

$g(z) = \sqrt{1-z^2}$. $\Pi_{z_1} = \Pi_{z_2} = 2i$; $\Pi_{z_3} = -2i$

$g(z_1) = 5^{1/2}$

$g(z_2) = \left|\frac{1-z_2}{1-z_1}\right|^{1/2} \cdot e^{-\frac{2\pi i}{2}} \cdot 5^{1/2} = e^{-\pi i} 5^{1/2} = -5^{1/2}$

$g(z_3) = \left|\frac{1-z_3}{1-z_2}\right|^{1/2} \cdot e^{0} \cdot e^{-\pi i} 5^{1/2} = -5^{1/2}$