№ 2202.

1) $\int\limits_{\partial D} \frac{dz}{z^3(z^{10}-2)}$ $(D: |z|<2)$

$\int\limits_{\partial D} f(z)\,dz = -2\pi i \operatorname{res}_{\infty} f(z) = c_{-1} = 0$

$\cancel{\frac{1}{z^3(z^{10}-2)} = \frac{1}{z^3(z^5-\sqrt{2})(z^5+\sqrt{2})}}$

4) $\int\limits_{\partial D} \frac{z^3}{z+1} e^{1/z}\,dz$ $(D: |z|<2)$

$\int\limits_{\partial D} = 2\pi i \operatorname{res}_{-1} f(z) = 2\pi i \cdot z^3 e^{\frac{1}{z}}\Big|_{z=-1} = -\frac{2\pi i}{e}$

2) $\int\limits_{\partial D} \frac{z^2 \sin^2 \frac{1}{z}}{(z-1)(z-2)}\,dz$ $(D: |z|<3)$

$\int\limits_{\partial D} f(z)\,dz = 2\pi i\left(\operatorname{res}_1 f(z) + \operatorname{res}_2 f(z)\right)$

$\operatorname{res}_{z=1} f(z) = \frac{z^2 \sin^2 \frac{1}{z}}{z-2}\Big|_{z=1} = -\sin^2 1$

$\operatorname{res}_{z=2} f(z) = \frac{z^2 \sin^2 \frac{1}{z}}{z-1}\Big|_{z=2} = 4\sin^2 \frac{1}{2}$

$\frac{z^2}{(z-1)(z-2)} = \frac{z^2}{z^2(1-\frac{1}{z})(1-\frac{2}{z})} = (1+\frac{1}{z})(1+\frac{2}{z})$

$1+\frac{3}{z}$

$\sin^2 \frac{1}{z} = \left(\frac{1}{z} - \frac{1}{z^3 \cdot 6} + \ldots\right)^2$

$\int\limits_{\partial D} f(z)\,dz = c_{-1} = 0$

5) $\int\limits_{\partial D} \sin \frac{z}{z+1}\,dz$ $(D: |z|>3)$

$z=-1$ - сущ. особая т. - вне обл.

$f(z) = \sin\left(\frac{z}{z+1}\right) = \sin\left(1-\frac{1}{z+1}\right) = \sin 1 \cos\left(\frac{1}{z+1}\right) - \cos 1 \sin\left(\frac{1}{z+1}\right) =$

$= \sin 1\left(1 - \frac{1}{2(z+1)^2} + \ldots\right) - \cos 1\left(\frac{1}{z+1} - \ldots\right)$

$\frac{1}{2\pi i}\oint = \sum_{in} \operatorname{res} = -\sum_{out} \operatorname{res} = -c_{-1} \Rightarrow \oint = 2\pi i(-c_{-1}) = 2\pi i \cos 1$

6) $\int\limits_{\partial D} z \sin \frac{z+1}{z-1}\,dz$ $(D: |z|<2)$

$\sin\left(\frac{z+1}{z-1}\right) = \sin\left(1+\frac{2}{z-1}\right) = \sin 1 \cos\left(\frac{2}{z-1}\right) + \cos 1 \sin\left(\frac{2}{z-1}\right) =$

$= \sin 1\left(1 - \frac{2}{(z-1)^2} + \ldots\right) + \cos 1\left(\frac{2}{z-1} - \ldots\right)$

$f(z) = (1+z-1)\left(\sin 1\left(1 - \frac{1}{(z-1)^2} + \ldots\right) + \cos 1\left(\frac{2}{z-1} - \ldots\right)\right) = \sin 1 +$

$+ \cos 1 \cdot 2 \cdot 1 - \sin 1 \cdot \frac{2}{z-1} + \ldots \Rightarrow c_{-1} = 2\cos 1 - 2\sin 1$

$\oint = 2\pi i\, c_{-1} = 4\pi i(\cos 1 - \sin 1)$

7) $\int\limits_{\partial D} \sin \frac{1}{z-1}\,dz$ $(D: |z-1|>1)$

$\sin \frac{1}{z-1} = \left(\frac{1}{z-1} - \ldots\right) \Rightarrow c_{-1} = 1$

$\oint = -2\pi i\, c_{-1} = -2\pi i$

8) $\int\limits_{\partial D} \exp \frac{1}{1-z} \frac{dz}{z}$ $(D: |z-2|+|z+2|<6)$

$z=1$: $\exp \frac{1}{1-z} = 1 + \frac{1}{1-z} + \ldots \Rightarrow c_{-1} = 1$

$\oint = 2\pi i\, c_{-1} = 2\pi i$

13) $\int\limits_{\partial D} \frac{z\,dz}{e^{z^2}-1}$ $(D: |z|>4)$ $e^{z^2} = 1 \Rightarrow z=0$

$\operatorname{res}_0 f(z) = \frac{z^2}{e^{z^2}-1}\Big|_{z=0} = \frac{z^2}{1+z^2-1} = 1$ $\oint = -2\pi i \operatorname{res} = -2\pi i$ $(-10\pi i)$?

№ 20.21.

1) $\frac{z}{(z+2)^2} = \frac{1}{z+2} - \frac{2}{(z+2)^2}$

$\frac{A}{z+2} + \frac{B}{(z+2)^2} \Rightarrow Az + 2A + B = z \quad \begin{cases} A = 1 \\ 2A+B=0 \end{cases} \quad \begin{matrix} A = 1 \\ B = -2 \end{matrix}$

2) $\frac{e^z+1}{e^z-1} = \frac{e^{z-2k\pi i}+1}{e^{z-2k\pi i}-1} = \frac{1+1}{1+(z-2k\pi i)-1} = \frac{2}{z-2k\pi i}$

3) $\frac{z-1}{\sin^2 z} = \frac{z-1}{z^2} = \frac{1}{z} - \frac{1}{z^2}$

№ 2102

1) $\frac{1}{z+z^3} = \frac{1}{z(1+z^2)} = \frac{1}{z(z+i)(z-i)}$

$\operatorname{res}_0 f(z) = \frac{1}{1+z^2} = 1$; $\operatorname{res}_i f = \frac{1}{2i \cdot i} = -0{,}5$ $\quad \operatorname{res}_{-i} = \frac{1}{-2i(-i)} = -0{,}5$

$f(z) = \frac{h(z)}{g(z)} \quad h(z_0) \neq 0$

$\operatorname{res}_{z_0} f(z) = \frac{h(z_0)}{g'(z_0)}$

2) $\frac{z^2}{1+z^4} = \frac{z^2}{(z^2+i)(z^2-i)}$; $z^4 = -1$

$z^4 = e^{i(\pi + 2\pi k)}$

$z = e^{\frac{i(\pi+2\pi k)}{4}}$

$\operatorname{res}_{e^{i\frac{\pi}{4}}} f(z) = \frac{e^{i\frac{\pi}{2}}}{4e^{i\frac{3\pi}{4}}} = \frac{1}{4} e^{-\frac{i\pi}{4}} = \frac{1-i}{4\sqrt{2}}$

$\operatorname{res}_{e^{-i\frac{\pi}{4}}} f(z) = \frac{e^{-i\frac{\pi}{2}}}{4e^{-\frac{3\pi}{4}}} = \frac{1}{4} e^{\frac{i\pi}{4}} = \frac{1+i}{4\sqrt{2}}$

$\operatorname{res}_{e^{i\frac{3\pi}{4}}} f(z) = \frac{e^{i\frac{3\pi}{2}}}{4e^{i\frac{9\pi}{4}}} = \frac{1}{4} e^{-i\frac{3\pi}{4}} = -\frac{1+i}{4\sqrt{2}}$

$\operatorname{res}_{e^{-i\frac{3\pi}{4}}} f(z) = \frac{e^{-i\frac{3\pi}{2}}}{4e^{-i\frac{9\pi}{4}}} = \frac{1}{4} e^{i\frac{3\pi}{4}} = \frac{i-1}{4\sqrt{2}}$

3) $\frac{z^2}{(1+z)^3} \quad z = -1$ - п 3п

$\operatorname{res}_{-1} f(z) = \frac{1}{2} \frac{d^2}{dz^2} z^2 \Big|_{-1} = \frac{d}{dz} z = 1$

4) $\frac{1}{(z^2+1)^3} \quad z = i \quad z = -i$ - п 3 п

$\operatorname{res}_i f(z) = \frac{1}{2} \frac{d^2}{dz^2} (z+i)^{-3} \Big|_i = -\frac{3}{2} \frac{d}{dz} (z+i)^{-4} \Big|_i = 6 (z+i)^{-5} \Big|_{z=i} = \frac{6}{32i} = \frac{3}{16i}$

$\operatorname{res}_{-i} f(z) = -\operatorname{res}_i f(z) = -\frac{3}{16i}$

5) $\frac{1}{(z^2+1)(z-1)^2} \quad z = 1$ - п 2п ; $z = i$, $z = -i$ - п 1п

$\operatorname{res}_i f(z) = \frac{1}{(z+i)(z-1)^2} \Big|_i = \frac{1}{2i(-1-2i+1)} = \frac{1}{4}$

$\operatorname{res}_{-i} f(z) = \frac{1}{(z-i)(-i-1)^2} \Big|_{-i} = \frac{1}{-2i(-1+2i+1)} = \frac{1}{4}$

$\operatorname{res}_1 f(z) = \frac{d}{dz} \frac{1}{z^2+1} \Big|_1 = -\frac{1}{2}$

№ 21.02.

7) $\frac{1}{\sin \pi z}$, $\operatorname*{res}_{k} f(z) = \frac{1}{\pi \cos \pi k} = \frac{(-1)^k}{\pi}$, $k \in \mathbb{Z}$

8) $\operatorname{ctg} \pi z = \frac{\cos \pi z}{\sin \pi z}$, $\operatorname*{res}_{k} f(z) = \frac{\cos \pi z}{\pi \cos \pi z} = \frac{1}{\pi}$

9) $\operatorname{th} z = \frac{\operatorname{sh} z}{\operatorname{ch} z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1}$ $\quad e^{2z} = -1$

$2z = i(\pi + 2\pi n) \Rightarrow z = \frac{i(\pi + 2\pi n)}{2}$, $n \in \mathbb{Z}$.

$$\operatorname*{res}_{\frac{i(\pi+2\pi n)}{2}} f(z) = \frac{e^{2z} - 1}{2e^{2z}}\Bigg|_{z_n} = \frac{1}{2} - \frac{1}{2e^{2\frac{i(\pi+2\pi n)}{2}}} = \frac{1}{2} + \frac{1}{2} = 1$$

10) $\operatorname{cth}^2 \pi z = \frac{\operatorname{ch}^2 \pi z}{\operatorname{sh}^2 \pi z} = \left(\frac{e^{\pi z} + e^{-\pi z}}{e^{\pi z} - e^{-\pi z}}\right)^2 = \left(\frac{e^{2\pi z} + 1}{e^{2\pi z} - 1}\right)^2$ $\quad e^{2\pi z} = 1$

$2\pi z = i2\pi k$

$z = ik$, $k \in \mathbb{Z}$

п2п

$$\operatorname*{res}_{z=ik} f(z) = \frac{d}{dz}\left(\frac{e^{2\pi z}+1}{e^{2\pi z}-1}\right)^2 (z-ik)^2\Bigg|_{ik} \cancel{2\frac{(e^{2\pi z}+1)}{(e^{2\pi z}-1)^3}(-4\pi e^{2\pi z})}\Bigg|_{ik}$$

$$\cancel{\frac{d}{dz}\left((e^{2\pi z}+1)(e^{2\pi z}-1)^{-1}\right) = 2\pi e^{2\pi z}(e^{2\pi z}-1)^{-1} - 2\pi e^{2\pi z}(e^{2\pi z}-1)^{-2}(e^{2\pi z}+1)} =$$

$$= \cancel{\frac{2\pi e^{2\pi z}(e^{2\pi z}-1) - 2\pi e^{2\pi z}(e^{2\pi z}+1)}{(e^{2\pi z}-1)^2} = \frac{-4\pi e^{2\pi z}}{(e^{2\pi z}-1)^2}}$$

11) $\frac{\cos z}{(z-1)^2}$, $z = 1$ - п2п

$$\operatorname*{res}_{z=1} f(z) = \frac{d}{dz} \cos z\Big|_1 = -\sin 1$$

12) $e^{\frac{1}{e^z+1}}$ $\quad e^z + 1 = 0 \Rightarrow e^z = -1 \Rightarrow z = i(\pi + 2\pi n)$, $n \in \mathbb{Z}$

$$\operatorname*{res}_{i(\pi+2\pi n)} f(z) = e^{\frac{1}{e^z}}\Big|_{i(\pi+2\pi n)} = -1$$

13) $\frac{\sin \pi z}{(z-1)^3}$ $\quad z = 1$ - п2п

$$\operatorname*{res}_{1} f(z) = \frac{d^2}{dz^2} \frac{\sin \pi z}{z-1}\Bigg|_1 = 0$$

$$\frac{d^2}{dz^2}(z-1)^{-1}\sin \pi z = \frac{d}{dz}\left(-(z-1)^{-2}\sin \pi z + \pi \cos \pi z (z-1)^{-1}\right) = 2(z-1)^{-3}\sin \pi z -$$

$$- (z-1)^{-2} \pi \cos \pi z - \pi^2 \sin \pi z (z-1)^{-1} - \pi \cos \pi z (z-1)^{-2}$$

14) $\frac{1}{\sin z^2}$ $\quad z^2 = \pi n$

$z = \pm\sqrt{\pi n}$

$$\operatorname*{res}_{\sqrt{\pi n}} f(z) = \frac{1}{\cos z^2 \cdot 2z}\Bigg|_{\sqrt{\pi n}} = \frac{1}{\cos \pi n \cdot 2\sqrt{\pi n}} = \frac{(-1)^n}{2\sqrt{\pi n}}$$

$$\operatorname*{res}_{0} f(z) = \frac{1}{\cos z^2 \cdot 2z}\Big|_0 = 0 ?$$

11.28.07

3) $\displaystyle\int_{-\infty}^{\infty} \frac{(x^3+5x)\sin x}{x^4+10x^2+9}\,dx$

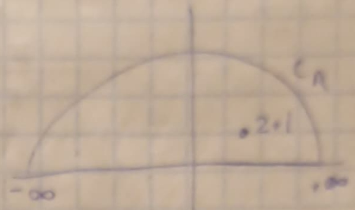
$$f(z) = \operatorname{Im} \frac{z^3+5z}{(z^2+9)(z^2+1)} e^{iz}$$

$$\oint = \int_{-\infty}^{+\infty} + \int_{C_R}^{\to 0} = \operatorname{Im} 2\pi i \left(\operatorname*{res}_{i} f(z) + \operatorname*{res}_{3i} f(z)\right) = 2\pi\left(\frac{1}{4e} + \frac{1}{8e^3}\right) = \pi\left(\frac{1}{2e} + \frac{1}{4e^3}\right) = \frac{2e^2+1}{4e^3}\pi$$

$$\operatorname*{res}_{i} f(z) = \frac{(i^3+5i)e^{i \cdot i}}{8 \cdot 2i} = \frac{1}{4e}$$

$$\operatorname*{res}_{3i} f(z) = \frac{(3i)^3+5 \cdot 3i}{6i\,(-9+1)} e^{-3} = \frac{-9+15}{-6 \cdot 8e^3} = \frac{1}{8e^3}$$

5) $\displaystyle\int_{-\infty}^{\infty} \frac{(x-1)\cos 2x}{x^2-4x+5}\,dx$



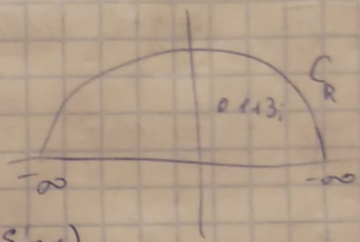
$$f(z) = \frac{(z-1)e^{i2z}}{z^2-4z+5}$$

$$\oint = \int_{-\infty}^{+\infty} + \int_{C_R}^{\to 0} = \operatorname{Re}\left(2\pi i \operatorname*{res}_{2+i} f(z)\right) = \frac{\pi}{e^2}(-\sin 4 + \cos 4)$$

$$\operatorname*{res}_{2+i} f(z) = \left.\frac{(z-1)e^{i2z}}{z-2+i}\right|_{2+i} = \frac{1+i}{2i} e^{4i-2} = \left(\frac{1}{2i} + \frac{1}{2}\right)(\cos 4 + i\sin 4)\,e^{-2} =$$

$$= \frac{1}{2e^2}\left(\frac{\cos 4}{i} + \sin 4 + \cos 4 + \frac{i\sin 4}{i}\right)$$

7) $\displaystyle\int_{-\infty}^{+\infty} \frac{x\cos x}{x^2-2x+10}\,dx$ $\qquad f(z) = \dfrac{z e^{iz}}{z^2-2z+10}$



$$\oint = \int_{-\infty}^{+\infty} + \int_{C_R}^{\to 0} = \operatorname{Re}\left(2\pi i \operatorname*{res}_{1+3i} f(z)\right) = \frac{\pi}{e^3}\left(\frac{\cos 1}{3} - \sin 1\right)$$

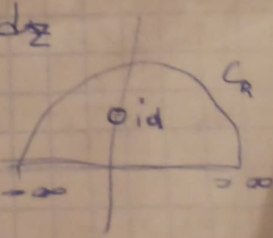
$$\operatorname*{res}_{1+3i} f(z) = \left.\frac{z e^{iz}}{z-1+3i}\right|_{1+3i} = \frac{(1+3i)e^{i-3}}{6i} = \left(\frac{1}{6i} + \frac{1}{2}\right)(\cos 1 + i\sin 1)e^{-3} =$$

$$= \frac{1}{2e^3}\left(\frac{\cos 1}{3i} + \cos 1 + \frac{\sin 1}{3} + i\sin 1\right)$$

13) $\displaystyle\int_0^{\infty} \frac{\cos x}{(x^2+a^2)^3}\,dx$, $\operatorname{Re} a > 0$

$$\int_0^{\infty} \frac{\cos x}{(x^2+a^2)^3}\,dx = \frac{1}{2}\int_0^{+\infty} \frac{e^{ix}}{(x^2+a^2)^3}\,dx + \frac{1}{2}\int_0^{\infty} \frac{e^{-ix}}{(x^2+a^2)^3}\,dx = \left|\begin{matrix} -x = z \\ dz = -dx \end{matrix}\right| = \frac{1}{2}\int_0^{-\infty} \frac{e^{iz}}{(z^2+a^2)^3}(-dz)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iz}}{(z^2+a^2)^3}\,dz$$



$$\oint = \int_{-\infty}^{\infty} + \int_{C_R}^{\to 0} = 2\pi i \operatorname*{res}_{ia} f(z) = \frac{\pi}{e^a} \cdot \frac{(a^2+3a+3)}{16a^5}$$

$$\operatorname*{res}_{ia} f(z) = \left.\frac{d^2}{dz^2}\frac{e^{iz}}{(z+ia)^3}\right|_{ia} = e^{iz}\left.\left((z+ia)^{-3} - 6(z+ia)^{-4} + 12(z+ia)^{-5}\right)\right|_{ia} =$$

$$= e^{-a}\left((2ia)^{-3} - 6 \cdot (2ia)^{-4} + 12(2ia)^{-5}\right) =$$

N20.24

5) $\frac{(z^2+1)^2}{z^3+6^2} = \frac{(z^2+1)^2}{z^3\left(1+\left(\frac{6}{z}\right)^2\right)} = \frac{z^3}{1+\left(\frac{6}{z}\right)^2} + \frac{2z}{1+\left(\frac{6}{z}\right)^2} + \frac{1}{z^3\left(1+\left(\frac{6}{z}\right)^2\right)} = z^3\left(1-\left(\frac{6}{z}\right)^2+\frac{1}{2}\left(\frac{6}{z}\right)^4 - \dots\right) +$

$+ 2z\left(1-\left(\frac{6}{z}\right)^2+\frac{1}{2}\left(\frac{6}{z}\right)^4 - \dots\right) + \frac{1}{z^3}\left(1-\left(\frac{6}{z}\right)^2+\frac{1}{2}\left(\frac{6}{z}\right)^4\right) \Rightarrow z^3 \to \infty$ при $z \to z_0$

N21.03

1) $\frac{z^4+1}{z^6-1} \approx \frac{1}{z^2} \Rightarrow C_{-1} = 0$ $\operatorname{res} f(z) = ?\ 0$

$z^6-1 = (z-1)(z^2+z+1)(z+1)(z^2-z+1)$

$\frac{A}{z-1} + \frac{Bz+C}{z^2+z+1} + \frac{D}{z+1} + \frac{Ez+F}{z^2-z+1} \Rightarrow A(z^5+z^4+z^3+z^2+z+1) + (Bz+C)(z^4-z^3+z-1) +$
$+ D(z^5-z^4+z^3-z^2+z-1) + (Ez+F)(z^4+z^3-z-1)$

$z^5: A+D+B+E=0$
$z^4: A+C-B-D+F+E=1$
$z^3: A-C+D+F=0$
$z^2: A+B-D+E=0$
$z: A-B+C+D-E-F=0$
$z^0: A-C-D-F=1$

$$\begin{array}{cccccc} A & B & C & D & E & F \end{array}$$
$$\left(\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & -1 & 1 \end{array}\right)$$

2. $\cos\pi\frac{z+2}{2z} = \approx 1 - \left(\frac{\pi}{2}\right)^2\left(\frac{z^2+4z+4}{z^2}\right) \Rightarrow C_{-1} = -\pi^2 \Rightarrow \operatorname{res}_\infty f(z) = \pi^2$

3. $\frac{\sin\frac{1}{z}}{z-1} = \frac{1}{z-1}\left(\frac{1}{z} - \frac{1}{z^3 3!}\right) \Rightarrow C_{-1} = 0 = \operatorname{res}_\infty f(z)$

4. $\frac{\cos^2\frac{\pi}{z}}{z+1} = \frac{1}{z+1}\left(1-\left(\frac{\pi}{z}\right)^2 + \dots\right)^2 \Rightarrow C_{-1} = 1 \Rightarrow \operatorname{res}_\infty f(z) = -1$

5. $\frac{(z^{10}+1)\cos\frac{1}{z}}{(z^5+2)(z^6-1)} \approx \frac{z^{10}}{z^{11}} \Rightarrow C_{-1} = 1 \Rightarrow \operatorname{res}_\infty f(z) = -1$

6. $z\cos^2\frac{\pi}{z} \approx z\left(1-\left(\frac{\pi}{z}\right)^2\right)^2 = z\left(1-2\frac{\pi^2}{z^2}+\left(\frac{\pi}{z}\right)^4\right) \Rightarrow C_{-1} = -2\pi^2 \Rightarrow \operatorname{res}_\infty f(z) = 2\pi$