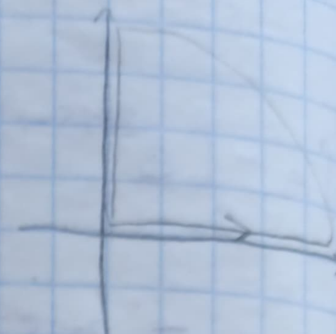
hw3

1.1. $C=\int_0^\infty \frac{\ln t}{t}\sin t\,dt$; $f(a)=\int_0^\infty t^{a-1}\sin t\,dt$

$f'(a)=\int \frac{t^a \ln t}{t}\sin t\,dt$; $C=f'(0)$



$f(a)=\mathrm{Im}\,F(a)$; $F(a)=\int_0^\infty t^{a-1}e^{it}\,dt$

$0=\oint_C z^{a-1}e^{iz}\,dz = F(a) - e^{\frac{i\pi a}{2}}\Gamma(a)$

$\Gamma(a)=\int_0^\infty e^{-t}t^{a-1}\,dt$

$F(a)=e^{\frac{i\pi a}{2}}\Gamma(a) = \left(\cos\frac{\pi a}{2}+i\sin\frac{\pi a}{2}\right)\int_0^\infty t^{a-1}e^{-t}\,dt$

$f(a)=\mathrm{Im}\,F(a)=\sin\frac{\pi a}{2}\cdot\int_0^\infty e^{-t}t^{a-1}\,dt$

$f'(a)=\frac{\pi}{2}\cos\frac{\pi a}{2}\int_0^\infty e^{-t}t^{a-1}\,dt+\sin\frac{\pi a}{2}\cdot\int_0^\infty e^{-t}t^{a-1}\,dt$

$C=f'(0)=\frac{\pi}{2}\int_0^\infty e^{-t}t^{-1}\,dt = -\frac{\pi}{2}\gamma$

1.2. $\underset{\mathrm{Re}\,\nu>-1}{I(\nu)}=\int_0^{\frac{\pi}{2}}\sin^\nu x\,dx = \left|\begin{matrix} t=\sin x\\ dt=\cos x\,dx\end{matrix}\right| = \int_0^1 t^\nu(1-t^2)^{-\frac12}\,dt =$

$=\int_0^1 (t^2)^{\frac{\nu}{2}}(1-t^2)^{-\frac12}\cdot\frac{dt^2}{2t} = \frac12\int_0^1 (t^2)^{\frac{\nu}{2}-\frac12}(1-t^2)^{-\frac12}\,dt =$

$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\,dt$ $\quad a-1=\frac{\nu}{2}-\frac12 \quad b-1=-\frac12$

$a=\frac{\nu}{2}+\frac12 \quad b=\frac12$

$= \frac12 B\left(\frac{\nu}{2}+\frac12;\frac12\right) = \frac12\frac{\Gamma(\frac\nu2+\frac12)\Gamma(\frac12)}{\Gamma(\frac\nu2+\frac12+\frac12)} = \boxed{\frac12\frac{\Gamma(\frac\nu2+\frac12)\cdot\sqrt\pi}{\Gamma(\frac\nu2+1)}}$

$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

№1.3. $h(a)=\int_0^1\frac{t^{a-1}}{1+t}\,dt = \left|\frac{1}{1+t}=\sum_{n=0}^\infty(-1)^n t^n\right| = \int_0^1 t^{a-1}\sum_{n=0}^\infty(-1)^n t^n\,dt=$

$=\sum_{n=0}^\infty(-1)^n\int_0^1 t^{a+n-1}\,dt = \sum_{n=0}^\infty (-1)^n \frac{t^{a+n}}{a+n}\Big|_0^1 = \sum_{n=0}^\infty(-1)^n\frac{1}{a+n}=$

$=\frac12\left[\psi\left(\frac{1+a}{2}\right)-\psi\left(\frac a2\right)\right]$

№1.4. $a>0;\ b>0$ $H(a,b)=\int_0^1 \frac{t^{a-1}-t^{b-1}}{1+t}\cdot\frac{dt}{\ln t}$ — сходится ; $\int_0^1 \frac{t^{a-1}}{1+t}\frac{dt}{\ln t}$ — расходится

$$\frac{d}{da}H(a,b)=\int_0^1 \frac{t^{a-1}\ln t}{(1+t)\cdot\ln t}dt=\int_0^1\frac{t^{a-1}}{1+t}dt=h(a)=\frac{1}{2}\psi\left(\frac{1}{2}+\frac{a}{2}\right)-\frac{1}{2}\psi\left(\frac{a}{2}\right)$$

$$\frac{d}{db}H(a,b)=\int_0^1 \frac{-t^{b-1}\ln t}{(1+t)\ln t}dt=-\int_0^1\frac{t^{b-1}}{1+t}dt=-h(b)=\frac{1}{2}\psi\left(\frac{b}{2}\right)-\frac{1}{2}\psi\left(\frac{1}{2}+\frac{b}{2}\right)$$

$H(1,1)=0$ $\qquad \int\psi(x)dx=\int\frac{\Gamma'(x)}{\Gamma(x)}dx=\ln(\Gamma(x))$

$$H(a,b)=\int h(a)\,da=\frac{1}{2}\int\left[\psi\left(\frac{a}{2}+\frac{1}{2}\right)-\psi\left(\frac{a}{2}\right)\right]da=\frac{1}{2}\ln\frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}+H(b)$$

$$H(a,b)=\int -h(b)\,db=\frac{1}{2}\int\left[\psi\left(\frac{b}{2}\right)-\psi\left(\frac{b}{2}+\frac{1}{2}\right)\right]db=\frac{1}{2}\ln\frac{\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}+H(a)$$

$$\boxed{H(a,b)=\frac{1}{2}\ln\frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\cdot\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\cdot\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}}$$

№2. $I(\alpha)=\frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)}\int_0^\infty x^\alpha(1+x)^{2\alpha}(2+x)^{3\alpha}e^{-x}dx=$

$f(x)=x^\alpha(1+x)^{2\alpha}(2+x)^{3\alpha}e^{-x}$; $x\to 0$, $f(x)\to x^\alpha\cdot 2^{3\alpha}e^{-x}$

$$=\frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)}\int_0^\infty\left(x^\alpha(1+x)^{2\alpha}(2+x)^{3\alpha}e^{-x}-x^\alpha 2^{3\alpha}e^{-x}\right)dx+\underbrace{\frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)}\int_0^\infty x^\alpha 2^{3\alpha}e^{-x}dx}_{2^{3\alpha}\cdot\Gamma(\alpha+1)}$$

$$I(-1)=\frac{1}{\cancel{\Gamma(0)}}\int_0^{\infty}x^{-1}e^{-x}\left(\frac{1}{(1+x)^2}\cdot\frac{1}{(2+x)^3}-\frac{1}{8}\right)dx+\frac{1}{8}\frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha+1}{2}\right)}=\frac{1}{16}$$

$$\Gamma(2z)\sqrt{\pi}=2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

$$\frac{\Gamma(2z)}{\Gamma(z)}=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma\left(z+\frac{1}{2}\right)=\frac{2^{-1}}{\sqrt{\pi}}\cdot\sqrt{\pi}=\frac{1}{2}$$

$z=\frac{\alpha+1}{2}$ $\qquad z+\frac{1}{2}=\frac{\alpha}{2}+1$

$\alpha=-1$ $\qquad \Gamma\left(z+\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

hw3 №3. $G(in) = \sum_{k=1}^{\infty}\left(\frac{1}{-a+ik+in} - \frac{1}{-a-ik+in} + \frac{2i}{k}\right) = \Big| \psi(z+1)+\gamma = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \dots\right.$

$= i\sum_{k=1}^{\infty}\left(\frac{2}{k} - \frac{1}{ia+k+n} - \frac{1}{-ia+k-n}\right) = i\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+ia+n} + \frac{1}{k} - \frac{1}{k-ia-n}\right) =$

$= i\left[\gamma + \psi(1+ia+n) + \gamma + \psi(1-ia-n)\right]$

№4. $L(z) = \frac{1}{\Gamma(z)}\int_0^{\infty}\frac{x^{z-1}e^{-x}}{1+e^{-2x}}dx$

$L(1) = \int_0^{\infty}\frac{e^{-x}}{1+e^{-2x}}dx = \left|\begin{matrix} t = e^x \\ dt = e^x dx\end{matrix}\right| = \int_1^{\infty}\frac{t^{-1}}{1+t^{-2}}\frac{dt}{t} = \int_1^{\infty}\frac{dt}{1+t^2} = \operatorname{arctg} t\Big|_1^{\infty} = \frac{\pi}{4}$

$L(z) = \frac{1}{\Gamma(z)}\int_0^{\infty}\frac{x^{z+1}}{e^x+e^{-x}}dx = \left|\begin{matrix} u = \frac{1}{e^x+e^{-x}} & dv = x^{z+1}dx \\ du = \frac{e^x-e^{-x}}{(e^x+e^{-x})^2}dx & v = \frac{x^z}{z}\end{matrix}\right| =$

$= \frac{1}{\Gamma(z)}\left(\frac{1}{z}\frac{x^z}{e^x+e^{-x}}\Big|_0^{\infty} - \frac{1}{z}\int_0^{\infty}x^z u'dx\right) = -\frac{1}{z\cdot\Gamma(z)}\int_0^{\infty}x^z u'dx = -\frac{1}{\Gamma(z+1)}\int_0^{\infty}x^z u' dx$

$L(0) = -\int_0^{\infty}u'dx = -\left(\frac{1}{e^x+e^{-x}}\right)\Big|_0^{\infty} = \frac{1}{2}$

$L(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^z}$, $S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$