Holonomy of branched projective structures

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Projective structures

Let Σ be a connected oriented compact surface of genus $g \geqslant 0$ without boundary.
Branched projective structure: atlas of charts $(\varphi_\alpha, U_\alpha)_\alpha$ into $\mathbb{CP}^1$, with transition maps that are Möbius transformations: $\mathrm{PSL}_2(\mathbb{C}) = \{\frac{az+b}{cz+d}\}$.
Branched charts:

$$\begin{tikzcd} U_\alpha \subset \Sigma \arrow[r, "\varphi_\alpha"] \arrow[d] & V_\alpha \subset \mathbb{CP}^1 \arrow[d] \\ D(0,1) \arrow[r, "z \mapsto z^n"] & D(0,1). \end{tikzcd}$$

Transition maps:

$$\forall \alpha, \beta \ \exists h \in \mathrm{PSL}_2(\mathbb{C}) \ \varphi_\alpha = h \circ \varphi_\beta.$$

The space of projective structures $\mathcal{BP}(\Sigma)$ is naturally stratified: we may fix the number of branch points and their orders.

$$\mathcal{BP}(\Sigma) = \bigcup_{k \geqslant 0} \bigcup_{n_i \geqslant 1} \mathcal{P}(n_1, \ldots, n_k).$$

An example

A surface obtained by gluing the sides of a polygon with translations is called a *translation surface*.

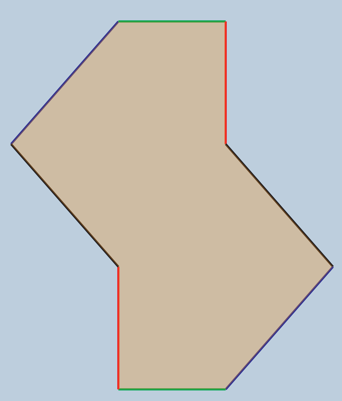


Figure 1: Genus 2 translation surface.

Let $\mathcal{H}(n_1, \ldots, n_k)$ be the subset of $\mathcal{P}(n_1, \ldots, n_k)$ consisting of translation surfaces.

Developing map and holonomy

The analytic continuation of a chart leads to the *developing map* $\mathrm{dev} : \widetilde{\Sigma} \to \mathbb{CP}^1$ that satisfies:

$$\forall z \in \widetilde{\Sigma} \ \forall \gamma \in \pi_1(\Sigma) \ \exists! \rho(\gamma) \in \mathrm{PSL}_2(\mathbb{C}),$$

$$\mathrm{dev}(\gamma \cdot z) = \rho(\gamma) \cdot \mathrm{dev}(z).$$

We thus define the *holonomy* $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$. Another choice of chart conjugates the holonomy, thus there is a well-defined map

$$\mathrm{Hol} : \mathcal{BP}(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C}).$$

Statement of the problem

A celebrated theorem of Gallo, Kapovich and Marden in [GKM00] characterizes the holonomies of unbranched projective structures.

Gallo Kapovich Marden. A homomorphism $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ is the holonomy of a unbranched projective structure if and only if it is non-elementary and lifts to $\mathrm{SL}_2(\mathbb{C})$.

Problem. *Characterize the holonomies of projective structures with fixed branch data. In other words describe* $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

Statement of the theorem

We give a list of conditions that a representation must satisfy to be the holonomy of a projective structure with fixed branch points, [LF21] for more information on these conditions and notations.

Theorem. *A representation* $\rho : \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ *is the holonomy of a projective structure in* $\mathcal{P}(n_1, \ldots, n_k)$ *if and only if it satisfies the 6 conditions below.*

Condition 1. $\sum_i n_i$ is even if and only if ρ lifts to $\mathrm{SL}_2(\mathbb{C})$.

Condition 2. If ρ is elementary then $\sum_i n_i \geqslant 2g-2$ with strict inequality if ρ is conjugated into $\mathrm{PSU}(2)$.

Condition 3. If ρ has finite image of order n, then

$$n(2 - 2g + \sum_i n_i) \geqslant 2 \max_i (n_i + 1).$$

Condition 4. If $\sum_i n_i = 2g-2$ and ρ is Euclidean, then $\mathrm{Vol}(\rho) > 0$.

Condition 5. If $\sum_i n_i = 2g-2$ and $\rho : \pi_1(\Sigma) \to \mathrm{Aff}(\mathbb{C})$ is such that $|\mathrm{Li} \circ \rho(\pi_1(\Sigma))| = n < \infty$ and that $\Lambda = \{z_0 \in \mathbb{C} \mid z + z_0 \in \rho(\pi_1(\Sigma))\}$ is a lattice in $\mathbb{C}$, then

$$n\mathrm{Vol}(\rho) \geqslant \max_i (n_i + 1) \mathrm{Vol}(\mathbb{C}/\Lambda).$$

Condition 6. If $g = 2$ and ρ is dihedral and $\rho \in \mathrm{Hol}(\mathcal{P}(2))$, then ρ is affine.

Ideas of the proof

Our goal is to *geometrize* ρ: find a projective structure with fixed branch data and holonomy ρ.

Step 1: Mapping class group action

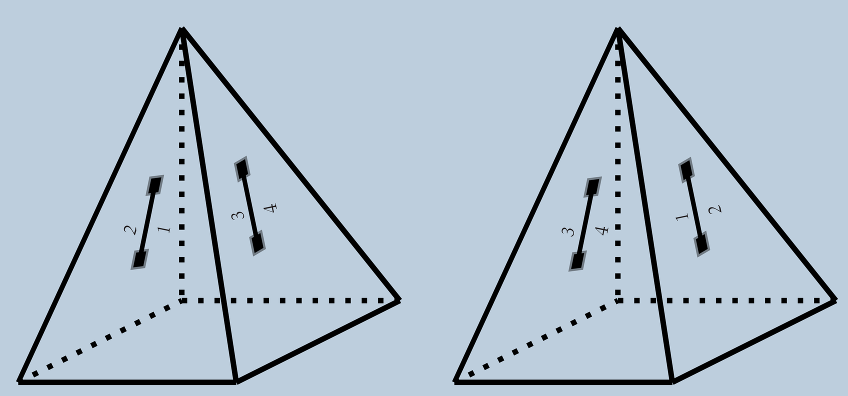
The group $\mathrm{Homeo}^+(\Sigma)$ acts on $\mathcal{P}(n_1, \ldots, n_k)$ and modifies the holonomy by precomposing it with the element it induces in $\mathrm{Out}^+(\pi_1(\Sigma))$. Hence ρ is geometric if and only if every representation in its orbit under $\mathrm{Mod}(\Sigma) \simeq \mathrm{Out}^+(\pi_1\Sigma)$ is. Therefore we want to describe the orbits $\mathrm{Mod}(\Sigma) \cdot \rho$ and find special representives of those that are easier to geometrize. For example in [GKM00] the authors show that a non-elementary ρ admits a pants decomposition such that the restriction of ρ to each pair pants in an isomorphism onto a Schottky group. However there is a gap in this proof and the representations that do not admit such a decomposition are classified in [LF19]. In particular in [LF21] we describe the orbits of representations with finite image.

Proposition. *Suppose* $g \geqslant 2$. *Let G be a finite subgroup of* $\mathrm{PSL}_2(\mathbb{C})$.

$$|\mathrm{Mod}(\Sigma) \backslash \{\pi_1(\Sigma) \twoheadrightarrow G\}/G| = \begin{cases} 1 & \text{if } G = \mathbb{Z}/n\mathbb{Z}, \mathrm{D}_{2n+1} \\ 2 & \text{if } G = \mathrm{D}_{2n}, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5 \end{cases}.$$

Step 2: Constructions

The study of the mapping class group action reduces the problem to geometrizing a small number of representations. *Example:* The datum of a projective structure with cyclic holonomy of order n amounts to the datum of a structure modeled on the pyramid with n sides. We cut and glue pyramids along segments to produce a genus g surface with desired branch data and cyclic holonomy.



A particular case

As part of the proof of our main result, we characterize the holonomies, also called period maps, of translation surfaces with fixed branched points in [LF20].

Theorem. *A representation* $\chi : \pi_1(\Sigma) \to \mathbb{C}$ *is the period map of a translation surface in* $\mathcal{H}(n_1, \ldots, n_k)$ *if and only if:*

- $\mathrm{Vol}(\chi) > 0$
- $\mathrm{Vol}(\chi) \geqslant (\max_i n_i + 1)\mathrm{Vol}(\mathbb{C}/\Lambda)$ *if* $\Lambda = \chi(\pi_1(\Sigma))$ *is a lattice in* $\mathbb{C}$.

Reference

[GKM00] Daniel Gallo, Michael Kapovich, and Albert Marden. The monodromy of groups of Schwarzian equations on closed Riemann surfaces. *Ann. of Math. (2)*, 151(2):625-704, 2000

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[LF20] Thomas Le Fils. Periods of abelian differentials with prescribed singularities, to appear in *Int. Math. Res. Not. IMRN.*

[LF19] Thomas Le Fils. Pentagon representations and complex projective structures on closed surfaces, to appear in *Ann. Inst. Fourier.*