Deterministic Optimal Control in Discrete Time: Literature Review

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PRELIMINARY AND INCOMPLETE

The problem at hand is of the form

$$\sup_{(x_t)_t,(u_t)_t} \sum_{t=0}^{\infty} f_t(x_t, u_t) \text{ s.t } x_{t+1} = g_t(x_t, u_t),\ x_0 \text{ given} \tag{OC}$$

where $(x_t, u_t) \in W_t \subset X \times U$ for some metric spaces X and U. This is a deterministic problem in infinite dimension. It often appears in economics in optimal growth problems.

This literature review aims at covering the following:

1. Under what conditions does problem (OC) have a solution?
2. Can we reformulate this problem with dynamic programming? What does it buy us?
3. Does duality theory apply to (OC)?

1 Existence of solutions

From Keerthi and Gilbert (1985), if (OC) satisfies the following $\forall t$:

(a) W_t is a closed subset of $X \times U$.

(b) $g_t : W_t \to \mathbb{R}$ is continuous.

(c) $f_t : W_t \to \mathbb{R}$ is lowersemicontinuous and there exists a sequence of real numbers $(\alpha_t)_t$ such that $|\sum_t \alpha_t| < +\infty$ and $f_t(x, u) \geq \alpha_t$ whenever $(x, u) \in W_t$.

(d) Given any compact set $P \subset X$, the set $S_t(P) = \{(x,u) \in W_t | x \in P\}$ is compact in $X \times U$.

(e) There exists an adminissible sequence pair with finite objective function.

Then problem (OC) has a solution. Condition (c) is satisfied in particular if the objective function has the following shape:

$$f_t(x_t, u_t) = \beta^t \tilde{f}_t(x_t, u_t) \tag{DO}$$

where $0 \leq \beta < 1$ is a discount factor and $\tilde{f}_t$ is uniformely bounded on W_t. Note that (b) and (c) are general conditions: in particular there is no requirement that the objective or the constraints are convex.

One can also turn to general convex optimization theory to lay out conditions for existence of a solution, such as Combettes and Bauschke (2017). We will come back to it in section 3.

Note that in macroeconomics, optimal growth model are often formulated in a slightly different way: the sequence pair $(x_t, u_t)_t$ should satisfy an inequality constraint, instead of an equality. Inada conditions are then required to ensure the set of feasible solutions is bounded.

2 Dynamic Programming

See Bertsekas (1995)

3 Duality Theory

Duality theory is widely used in convex optimization in finite dimensions. Can we extend it to infinite dimension? Dechert (1982) shows that the answer is yes, when the problem is of the form:

$$\sup_{(x_t)_t} \sum_{t=0}^{\infty} f_t(x_t) \text{ s.t } h_t(x_t) \leq 0 \tag{1}$$

where f_t and h_t are convex, and a Slater condition is satisfied, i.e. $\exists x$ s.t $\sup |x| < \infty$ and

$$\sup_t g_t(x) < 0 \tag{2}$$

Note that Slater condition can never be satisfied in problem (OC) if the $(g_t)_t$ are non-linear: by definition the equality constraints make it fail.

Combettes and Bauschke (2017) also offer a exhaustive analysis of duality in Hilbert spaces. In particular in their section 19.3, they show how to derive the dual of a convex optimization problem, where the constraints take the shape of a bounded linear operator.

References

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