EBA 3650 : Quantitative Economics

Impact of a negative externality on the society welfare and the optimal taxation combination (Ramsey)

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Introduction

The purpose of this paper is to explore the impact of a negative externality of production in the general equilibrium model, and to use Ramsey's taxation to find the new market efficiency and optimize welfare for society.

Economists define a negative externality when a third party is burdened with costs resulting from an activity they have no influence over, such as consumption, production, and investment decisions of individuals, households, and firms. This indirect effect can be tiny, but when they are large it can be problematic because the impact on the welfare of society is deeper. This is then one of the main reasons why governments need to intervene in the economic sphere (Helbling, T., 2020). Moreover, in the event of a negative externality, the market efficiency is lost because these third parties don't take into account the external costs, culminating in differences between private returns or costs and costs or returns to society as a whole.

When producers are causing this effect on society, it means the Marginal Social Cost (MSC) is greater than the Marginal Private Cost (MPC). A solution to balance this equation out is for the government to influence the taxation of the goods producing such externality. That's when Ramsey's rule steps in: to minimize the total excess burden (deadweight loss), taxation rates should be adjusted so the percentage reduction in the quantity demanded of each good demanded is the same (*Efficient and Equitable Taxation*, 2017). The government should then allocate the taxes such that it maximizes the utility of consumers while generating the required level of tax revenue, and this optimization determines the most efficient set of commodity taxes - thus causing every good to have the same proportional reduction in compensated demand (Oxford Reference, 2009). This approach is strongly connected to Pigou's theory: a per unit tax imposed on a good to correct the inefficiency of the market outcome caused by the negative externality.

To develop this analysis, the paper will first go through a base case scenario: an efficient market with two distinct goods that clear at given prices and quantities, considering consumers, firms and government as actors in this model. The government sets a specific target revenue, which is to be assumed given in this paper, and an optimal taxation will be found through Ramsey's method in order to maximize the utility of the society.

At a second moment, a negative externality will be introduced on only one of the goods. The burden will be carried out by the society, thus facing a decrease in the utility (the incremental satisfaction received when consuming a good or service). The next step is to understand how the government can use Ramsey's theory once again to optimize the allocation of taxes between the goods while keeping the same target taxation revenue, for a maximum welfare (the new utility deducted of the impact of the negative externality).

Finally, a comparison between these two scenarios will be drawn when calculating the demand elasticity of each good. The reaction of a singular customer's demand shouldn't change from each scenario because the externality is not considered on a daily basis by the end-user. However, when the aggregate overview faces a change in an economic factor - taxes in this case, which directly impacts prices, there will be a more visible variation in the change of consumption of a good in relation to a change in its price.

Part 1: General Equilibrium Model of Production in an Efficient Market

In this part, the analysis will focus on a general equilibrium model comprised by the demand of consumers, defined by the elasticity of goods and the work (or labor, used as source of income); and the production of two distinct goods, c_1 and c_2, produced by Firm 1 and Firm 2, respectively. Consumers and Firms operate at an equilibrium price and quantity, at a given taxation established by the government with t_1 on good c_1, and t_2 on good c_2.

To combine the two inputs of consumption into an aggregate quantity for society, the model assumes the Constant Elasticity of Substitution (CES) utility function. It is an aggregator function with constant elasticity of substitution (the percentage change in the two inputs used in response to a percentage change in their prices, or the substitution of one good for another).

Assuming the labor m is the numeraire, then the wage is equal to 1 and e is the demand elasticity of each good, with $e_1 \neq e_2 \neq 0$:

$$u(c_1, c_2, m) = \frac{1}{1+\frac{1}{e_1}} c_1^{1+\frac{1}{e_1}} \cdot \frac{1}{1+\frac{1}{e_2}} c_2^{1+\frac{1}{e_2}} - m \tag{1}$$

Consumers work to get m, in addition to a profit π given back by the firms to society in exchange of the tax revenue charged by the government, and they distribute this total endowment between the goods according to the prices p_1 and p_2, as follows:

$$(p_1 + t_1)c_1 + (p_2 + t_2)c_2 = m + \pi \tag{2}$$

The expression above represents the budget constraint. It is a linear regression with a downward slope that shows all possible combinations of two different products that a consumer can afford to buy, exhausting all their income. On the other hand, an indifference curve illustrates all different combinations of the two goods that yield the same level of satisfaction (i.e., utility).

An ordinary demand will be obtained when the budget constraint is tangent to the indifference curve, meaning an optimal solution. In other words, when the marginal rate of substitution (MRS, characterized by the budget constraint) is equal to the indifference curve - the utility function, it is possible to find the associated optimal demand of each good.

Therefore, by rearranging the constraint in the r.h.s. of equation 2 and replacing m in equation 1, it is possible to derive the demand function of c_1 and c_2 from the utility model, such that:

$$Demand\, c_i = c_i^{\frac{1}{e_i}} - (p_i + t_i) - \frac{1}{n} \cdot c_1 \tag{3}$$

This represents the Marshallian demand function: given a budget and a particular set of prices, consumers allocate their spending between good 1 and good 2 such that the total utility is maximized.

In contrast to the Marshallian function, the Hicksian demand theory relates the price and quantity of a particular good while keeping constant the prices and quantities of the other goods, at a same utility level (Gallego, 2017).

As the target is to keep the budget fixed (using labor as numeraire) and maximize utility, the Marshallian function is preferred in lieu of the Hicksian curve (which aims at minimizing the cost for a given level of utility) (Gallego, 2017).

A Cobb-Douglas function will be used in regard to the production. It establishes the relationship between the production output and input factors of a firm (McKenzie, T., 2020). The inputs are defined by labor, m, and capital, k. Each of these factors of production is taken to an exponent level that determines the returns to scale on the factor inputs (the

scale of production). The sum of these exponents must be equal to 1 in this case, to assume a linear and homogeneous production function, meaning a relative change in production is equal to the proportional change in the factors (Toppr, 2022).
Assuming each firm has a capital stock, k_1 and k_2, and buys the labor m from the consumers, the aggregate production function is given by:

$$c_i = k_i^{\alpha} \cdot m_i^{1-\alpha} \tag{4}$$

The profit of the firm is then determined by how many units c it produces at a given price, subtracting the labor m used as input:

$$\pi_i = p_i \times k_i^{\alpha} \cdot m_i^{1-\alpha} - m \tag{5}$$

Because the labor m is assumed to be the constraint to society, the marginal productivity of this input determines the rate at which production changes with respect to the amount of work. Therefore, the supply function is obtained by taking a partial derivative of this last equation with regards to m. The first order condition (F.O.C.) for the firm is then:

$$Supply\, c_i = (1-\alpha)p_i k_i^{\alpha} m_i^{1-\alpha} - 1 \tag{6}$$

The product market will clear and find its equilibrium when the quantity demanded is equal to the quantity supplied at a given price for each good. Using the F.O.C.s found above for the firms and for the consumers, it can be inferred a general equilibrium model for each good is as follows:

$$Market\, Clearance\, for\, good\, c_i = c_i - (1-\alpha)k_i^{\alpha} m_1^{1-\alpha} \tag{7}$$

This study will assume a demand sensibility of e_1 and e_2, capital availability of k_1 and k_2 at a given output elasticity of α. Using these parameters, it is possible to calculate c_1, c_2, m_1, m_2, p_1 and p_2 for the demand, supply, and market clearance equations.
t_1 and t_2 will be optimized through the bisection method such that it maximizes the aggregate utility. Ramsey's taxation will be used to achieve this objective. This theory initially started with finding a solution for the problem of how to set the commodity taxes in order to minimize the inefficiency of the tax burden in a society while keeping the taxpayers / consumers as well off as possible.

The government has to generate revenue to improve infrastructures, as for example roads, hospitals, and schools. Hence, the State needs to impose taxes to raise revenue. Such a policy can lead to a decrease in overall welfare because it will increase the buyers' price and reduce the price that the sellers will be receiving. Consequently, it will decrease the stable equilibrium, which will have a ripple effect on the production, demand for labor, the income going into every household, and their expenditures. Taxation will create an additional, inefficient cost that will diminish the utility of taxpayers and will escalate the deadweight loss. Thus, Ramsey taxation will help allocate the resources by finding the optimal tax distribution for every commodity or good.

Ramsey taxation denotes the following assumptions:

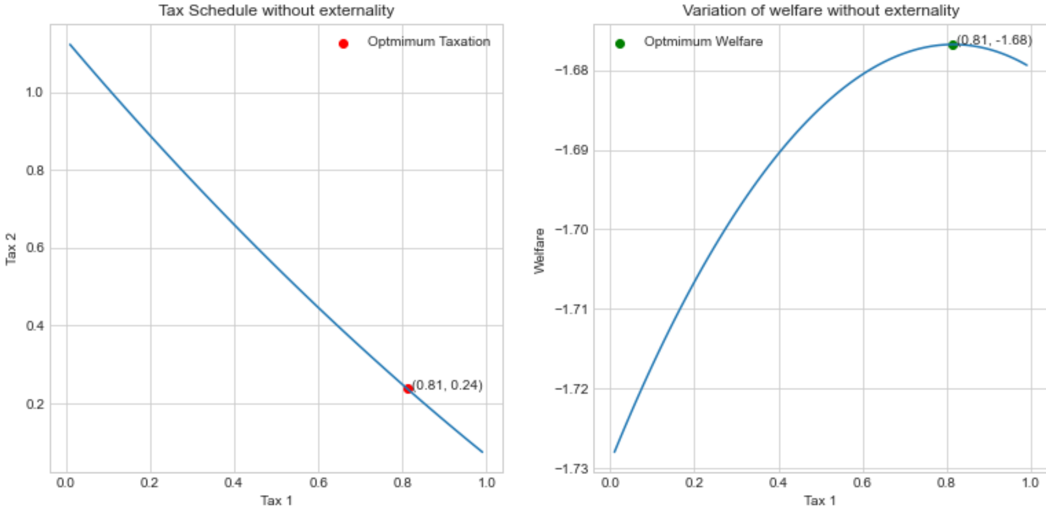
1. The utility function u(c_1, c_2, m) is the net utility of producing and consuming the goods.
2. With tax levied on goods, the stable equilibrium equals $(p + t)$ because taxation will shift the supply curve inwards and thus will change the equilibrium quantity and price.
3. The equilibrium is the area where the utility is maximized.
4. The tax revenue is equal to $(t_1 \cdot c_1 + t_2 \cdot c_2)$.
5. Economy is perfectly competitive.

For illustration purposes, the following values are assumed for the parameters above, with $\alpha = 0.3$:

$$e_1 = -0.1;\, e_2 = -0.2;\, k_1 = 2;\, k_2 = 4$$

The objective now is to find the optimal tax allocation in this general equilibrium model without any externality, for a given tax revenue of what is assumed to be 1.0. With the bisection method, a parameter for the tax allocation on good 1 will be created, using an array of evenly spaced 100 samples within the range of 0.01 and 0.99. Then, t_1 will be iterated 100 times through the bisection method in combination with the chosen 2 initial points for the tax revenue. Assuming these 2 points are between -1 and 1.2, it will be possible to find the optimal tax allocation for good 2. An illustration of how this works in Python, the chosen programming language for this paper, can be further seen in the Appendix.
In sum, the bisection rule is a root-finding method that takes two known values, preferably a negative and a positive one. Intuitively, the process will iterate several times within the given interval until the function is equal to zero. Once the model reaches such combination of t_2 for given t_1, that will represent the most favorable tax distribution. This allocation aims at maximizing the utility of consumers by minimizing the total loss of social welfare while generating a tax revenue equivalent to the government's taxation target.

After running the solver, the code provides all plausible scenarios of tax schedule between t_1 and t_2 such that it returns the same target taxation revenue, and the maximum utility obtained for consumers at the calculated t_1. The results are illustrated below:



The conclusion here is that the optimal tax combination is 0.81 for good 1 and 0.24 for good 2. Taking a look at the graph on the right, it is seen that for such tax levels, the solvers found an optimal utility of -1.68.

With the given combination of taxes and utility, it is now feasible to get the optimized value of the consumption for c_1 and c_2, labor and prices. These are considered as the stable equilibrium for quantities, prices and labor. Therefore, with the tax distribution of 0.81 for good 1, the equilibrium quantity is of 0.94 at the price of 1.03. Conversely, at the tax distribution of 0.24 for good 2, the equilibrium quantity is of 1 at the price of 0.79.

Part 2: General Equilibrium with the Presence of a Negative Externality on Production

After defining the efficient market in under ordinary conditions, the analysis will now focus on the assumption that good c_1 is causing a negative externality on good c_2 and that the burden is faced by society.

Under this occurence, the utility - now welfare, will be penalized by the aggregate consumption of c_1 because it harms consumers. The equation then becomes as follows:

$$w(c_1, c_2, m) = \frac{1}{1+\frac{1}{e_1}} c_1^{1+\frac{1}{e_1}} \cdot \frac{1}{1+\frac{1}{e_2}} c_2^{1+\frac{1}{e_2}} - m - C_1 \tag{8}$$

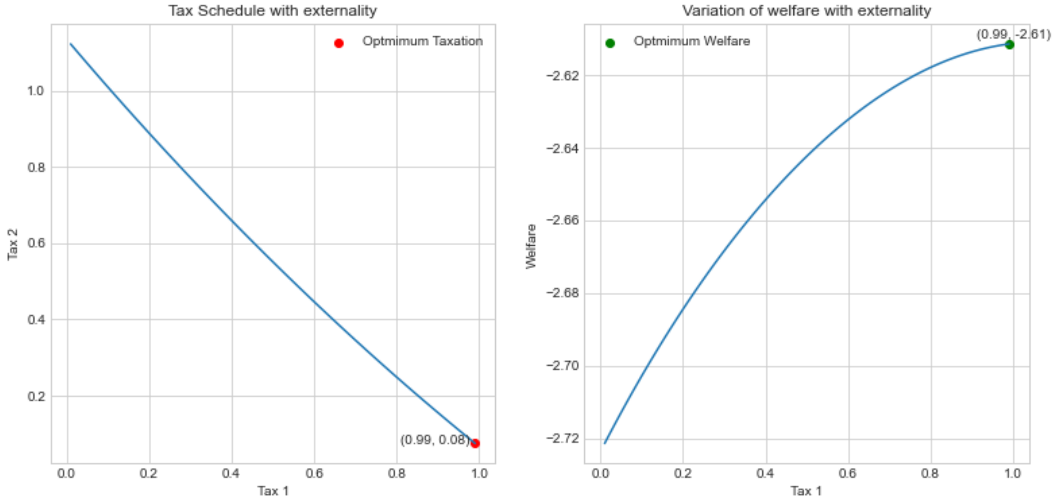
The function above represents the welfare for the whole population. If combined with the endowment constraint as illustrated in Part 1, this would be the demand function in the presence of a negative externality caused by good 1:

$$demand\, c_i = c_i^{\frac{1}{e_i}} - (p_i + t_i) - \frac{1}{n} \cdot c_1 \, with\, i = 0, 1, 2, ..., n \tag{9}$$

By analyzing this new expression, it is plausible to say that the larger the size of the population n is, the lower the impact of the externality will be at the unitary level - essentially close to zero, because consumers alone don't think about the negativity when purchasing each good. Hence, this additional externality effect of c_1 seen in the formula above is essentially close to zero at a significant volume of samples and the demand function is similar to the one presented in the general equilibrium model in the efficient market.

To put into another perspective, the consumer doesn't take into account the negative externality effect of how much less to consume of good 1. They will keep consuming the same amounts of c_1 and c_2 found in Part 1. However, there will be a negative effect on welfare if the whole society as a unit is taken into consideration and therefore a reduced optimal utility from the one evaluated in the previous chapter.
Assuming the supply function is to be held identical as the F.O.C. found in the general equilibrium model of the previous stage, meaning factors of production are kept unchanged and assigned according to the same return to scale determined earlier, the solution is to introduce a government intervention through taxation to correct the negative externality impact whilst maximizing the welfare of society. The output of supply will be given according to the new price, which now has to be pondered based on the updated tax schedule allocation (total price of good i = $p_i + t_i$).
Combining the supply and the demand functions as described above, which are fundamentally identical to what was found in Part 1, the new tax allocation can be calculated by applying Ramsey's theory one more time, but now under the current circumstance: for the same given tax revenue, it is the welfare function w(c1,c2,m) that must be optimized.
Keeping the same assumptions as in the general equilibrium model for e_1, e_2, k_1, k_2, the solver will run the same method as applied in the first scenario. The difference now is that the aggregate consumption of good 1 will be deducted from the utility function, which is now worse off due to the negative externality.

The bisection method and root finding solver will find a new combination of taxes such that every household will be as better off as possible, and the deadweight loss will be minimized. As per the result, the welfare is maximized when consuming good 1 if its tax allocation t_1 equals 0.99, and if the tax distribution t_2 for good 2 is 0.08. In relative terms, good 1 has a tax rate of 0.96, while good 2 has 0.09. The results were computed by dividing the tax over the price.



The graphs above represent the tax allocation and welfare optimum for the calculated t_1 under the general equilibrium model with the presence of a negative externality.
The figure on the left side remains the same as to Part 1 because the linear regression depicted represents all the available combinations of t_1 and t_2 that together, yields a total tax revenue of 1 set as target revenue for the government. With an externality event, the solver optimized that almost the full totality of the tax schedule should be allocated to good 1, which is causing such impact on the welfare.

By looking at the graph on the right, it can be observed that the society is indeed less well-off compared to the first scenario as the maximum value for the welfare function of -2.61 units is much lower than that of the utility function (-1.68 units). This is because the effect of the negative externality is very small at the individual level, but it has a significant impact when taken to the integral or aggregate level. The welfare then is deducted from the total consumption of the good causing the negative externality, c_1.
In regard to the taxation theory used to conduct this study, it is valid to point that Ramsey and Pigou taxation are closely related. While the first searches for the optimal taxation to maximize the welfare of society under an externality (by minimizing the deadweight loss), the Pigouvian tax is defined as a per-unit tax, imposed on a good, to correct the inefficiency of the market outcome caused by a negative externality. Therefore, once the Ramsey taxation is calculated, the Pigouvian taxation is inferred by simply taking the ratio of the total revenue outcome from t_1 by c_1, and similarly for t_2 and c_2. Under this situation, the Pigou taxation for good 1 is \$ 1.06 per unit, and \$ 0.07 per unit for good 2.

Part 3: Elasticity

The results of the optimal taxation structure will be analyzed using the Inverse Elasticity Rule (IER). This is to validate if it is feasible to minimize the deadweight loss while generating the government's tax revenue target in both scenarios (Economics in Many Lessons, 2021).

The IER under Ramsey taxation aims at checking if the taxes were allocated on equal percentage change and moreover, if the tax structure is efficient. The inverse elasticity assumes that the demand for each good depends only on its own price and has no cross-price effect (Oxford Reference, 2009). It also indicates that the optimal tax rates to raise a specific and required tax revenue are inversely proportional to the goods' elasticity of demand.

First, it is necessary to check the inverse elasticity by taking the tax rates for each good in relative terms (Sandmo, 1975). To calculate this, the following formula will be used :

$$Relative\,tax\,rate\,for\,good\,i\; = \frac{t_i}{p_i} \tag{10}$$

Where t_i is the optimal tax allocation for each good and p_i is its price. The output is a markup percentage: the tax rate added to the buyer's price for such items.

The next step is to confirm whether the tax allocation is efficient as per Ramsey Rule (Economics in Many Lessons, 2021). The following form must be satisfied :

$$\frac{tax\,rate\,of\,good\,1}{tax\,rate\,of\,good\,2} = \frac{elasticity\,of\,demand\,of\,good\,2}{elasticity\,of\,demand\,of\,good\,1} \tag{11}$$

The table below shows a comparison of the results between the two firms, assuming the inputs given throughout this paper and using the formulas (1) and (2) above to calculate the elasticity:

Table 1: Scenario 1- without externality

	Firm 1	**Firm 2**
Consumption	0.94	1
Price	1.03	0.79
Tax Rate	0.81	0.24
Relative Tax	0.79	0.3
Elasticity	-0.1	-0.2

In Scenario 1, where externality doesn't exist, the relative tax rate for Firm 1 is 0.49 higher when compared to Firm 2. Goods with relatively inelastic demand should be levied with higher tax rates because the quantity demanded will not drastically change when the price increases. This is proven as Firm 1 has lower elasticity of demand than Firm 2.

When goods have elastic demand, it attracts lower taxes because the extent of price change will be greater if compared to goods that have low elasticity. Elastic goods have a higher substitute effect that, if not minimized, will result in increased distortions and, consequently, in increased deadweight loss. Therefore, imposing lower taxes can diminish this effect.

Looking at the efficiency of the optimal tax structure using formula (10), the ratio of relative tax rates is of 2.6 (or 0.79 / 0.30). This is not far from 2 (a 30% variation), which is the ratio of elasticity of demand. Thus, the rate of optimal taxation is close to the range of -0.2 to -0.1, which are the demand elasticities of the firms. However, it is relatable that the optimal taxation rate is also dependent on the supply elasticities due to the difference in the ratios.
As per the results, it can be concluded that a t_1 of 0.81 a t_2 of 0.24 are the optimal tax distribution to generate the target tax revenue set by the government when externality doesn't exist.

Table 2: Scenario 2 - with externality

	Firm 1	**Firm 2**
Consumption	0.93	1.03
Price	1.03	0.8
Tax Rate	0.99	0.08
Relative Tax	0.96	0.09
Elasticity	-0.1	-0.2

In scenario 2, where the externality exists, Firm 1 is levied with an even higher relative tax rate compared to Firm 2: 0.96 and 0.09 respectively, which corresponds to a difference of 0.87. In this case, since Firm 1 is polluting and has the lowest elasticity, it attracted a much larger tax. By symmetry, Firm 2 is compensated with lower tax because it is indirectly affected by the pollution created by Firm 1, and it has high elasticity. The changes in the individual demand of the two goods is not considered significant because comparing Table 1 (without externality) and Table 2 (with externality), prices remain purely the same and changes in quantities consumed are immaterial. However, the adjustment on the tax allocation is highly significant as it reflects the impact of externality.

When considering the optimality of the tax allocation, the ratio of the relative tax rates is 10.21, which is much larger than the ratio of the elasticities of demand of 2. Thus, the optimal tax distribution is also affected by the marginal social damage (externality) and supply elasticity. However, formula (2) didn't hold in this case due to the constraint of externality. Under this circumstance, it is a combination of Ramsey and Pigou's theory. Nevertheless, the optimal tax allocation when there is externality is of 0.99 and 0.08 for good 1 and good 2, respectively.

Based on the calculations, there is a significant relationship between the elasticity of the goods and their tax rates. These taxes are inversely proportional to their elasticity of demand. In both scenarios, a firm with inelastic demand attracts high taxes. But a polluting firm with low elasticity will receive even larger taxes.

Conclusion

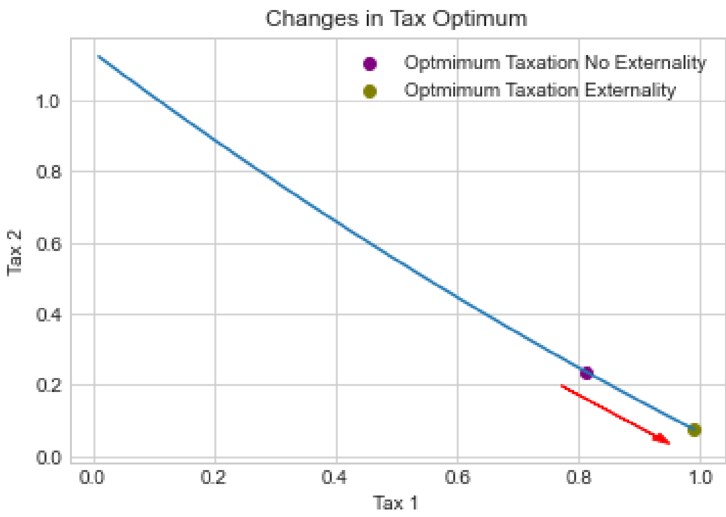
Drawing a comparison of supply and demand as described in Part 1 and Part 2, it is reasonable to say that there are no significant changes at the individual level. This is because the externality doesn't influence the daily decision of consumption of goods 1 and 2 of a household alone. However, when taking into consideration the aggregate scenario, there will be a negative and significant impact on the welfare of society.

In the presence of an externality caused by the production side, the marginal social cost (MSC) deviates from the equilibrium at the marginal private cost (MPC) due to the existence of the externality cost. Graphically, this will result in the shifting of the supply curve MSC to the left. This means that the costs charged for the goods of the polluting firm will increase based on the additional tax allocation necessary to compensate for the harm it is causing to society.

The optimal tax schedule is obtained from Ramsey's concept that aims at maximizing the utility and the welfare of society, or in other words, minimizing the deadweight loss caused by the externality while keeping the tax revenue set by the government stable. Assuming inputs for the elasticity and capital are kept constant in both scenarios, there will be changes in the outputs of aggregate quantities and total prices charged (i.e., including taxation) due to the updated t_1 and t_2.

In the general equilibrium model illustrated in Part 1, the optimal tax allocation was distributed as t_1 0.81 for c_1 and as t_2 0.24 for c_2. In the second part, under the presence of negative externality, almost 100% of the taxation was assessed to good 1 solely.

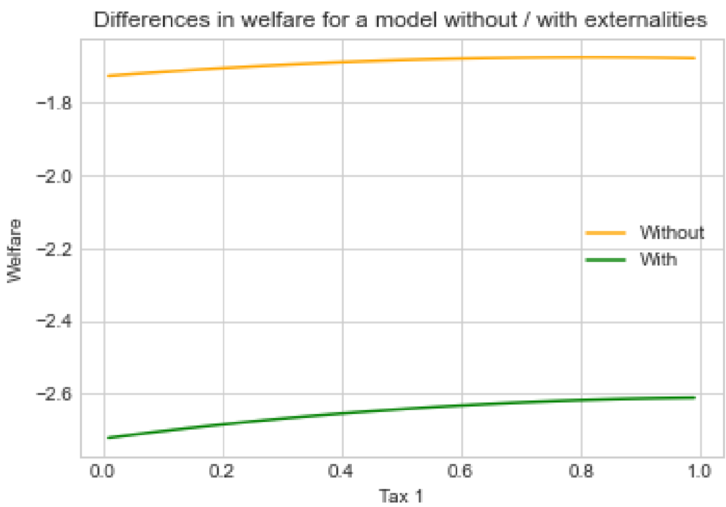
This change in the optimal tax schedule is illustrated in the figure below:



With or without the presence of externality, the firm with inelastic demand will get a higher tax rate. In the case of c_1 with externality, its tax rate is directly related to the average weight of inverse elasticity and marginal social damage. Since good 1 is an externality-creating commodity and is more inelastic, the solver concludes that welfare is maximized at such a distribution of taxes and the target taxation revenue of the government is achieved.

If it was the case of good 1 being taxed more in than 100%, or to more than 1 (which was set as the target taxation revenue in this exercise), it means that good 2 would be receiving a subsidy from the government on behalf of good 1.

The sole difference between the utility and welfare is caused by the impact of the negative externality incurred by the consumption of good 1 and therefore characterized by its aggregate quantity. This is demonstrated in the graph below, where the variation between the two functions is kept constant among all iterations:



In summary, the demand elasticity of goods regulates the optimal taxes calculated via Ramsey's solver for a fixed taxation revenue. If a good is inelastic, then the tax charged should be higher. And similarly, if a good is very elastic, then the taxes charged should be lower.

For the assumed values in this paper, the optimal utility was of -1.68 and the optimal welfare was of -2.61. This difference of 0.93 is the same as the value of c_1 because of the negative impact caused by good 1.

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