

EBA3650: Quantitative economics

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Root Finding

Root finding refers to the general problem of searching for a solution of an equation $F(x) = 0$ for some function F . If we want to optimise a function $f(x)$ then we need to find critical points and therefore solve the equation $f'(x) = 0$.

Example quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Source

Bisection Method

The algorithm applies to any continuous function $f(x)$ on an interval a, b where the value of the function $f(x)$ changes sign from a to b . The idea is simple: divide the interval in two, a solution must exist within one subinterval, select the subinterval where the sign of $f(x)$ changes and repeat.

Algorithm

The bisection method procedure is:

1. Choose a starting interval $[a_0, b_0]$ such that $f(a_0)f(b_0) < 0$
2. Compute $f(m_0)$ where $m_0 = (a_0 + b_0)/2$ is the midpoint.
3. Determine the next subinterval $[a_1, b_1]$:
 - (a) If $f(a_0)f(m_0) < 0$, then let $[a_1, b_1]$ be the next interval with $a_1 = a_0$ and $b_1 = m_0$.
 - (b) If $f(b_0)f(m_0) < 0$, then let $[a_1, b_1]$ be the next interval with $a_1 = m_0$ and $b_1 = b_0$.
4. Repeat (2) and (3) until the interval $[a_N, b_N]$ reaches some predetermined length.
5. Return the midpoint value $m_N = (a_N + b_N)/2$

A solution of the equation $f(x)$ on an interval a,b is guaranteed by the Intermediate Value Theorem provided $f(x)$ is continuous on $[a,b]$ and $f(a)f(b) < 0$. In other words, the function changes sign over the interval and therefore must equal 0 at some point in the interval $[a,b]$.

Python Implementation

Write a function called `bisection` which takes 4 input parameters `f`, `a`, `b` and `N` and returns the approximation of a solution of $f(x) = 0$ given by N iterations of the bisection method. If $f(a_N)f(b_N) > 0$ at any point in the iteration (caused either by a bad initial interval or rounding error in computations), then print "Bisection method fails." and return `None`.

```

1 def bisection(f,a,b,N):
2     '''Approximate solution of f(x)=0 on interval [a,b] by
3     bisection method.
4
5     Parameters
6     -----
7     f : function
8         The function for which we are trying to approximate a
9         solution f(x)=0.
10    a,b : numbers
11        The interval in which to search for a solution. The
12        function returns
13        None if f(a)*f(b) >= 0 since a solution is not
14        guaranteed.
15    N : (positive) integer
16        The number of iterations to implement.
17
18    Returns
19    -----
20    x_N : number
21        The midpoint of the Nth interval computed by the
22        bisection method. The
23        initial interval [a_0,b_0] is given by [a,b]. If f(m_n)
24        == 0 for some
25        midpoint m_n = (a_n + b_n)/2, then the function returns
26        this solution.
27        If all signs of values f(a_n), f(b_n) and f(m_n) are the
28        same at any
29        iteration, the bisection method fails and return None.

```

```

23     Examples
24     -----
25     >>> f = lambda x: x**2 - x - 1
26     >>> bisection(f,1,2,25)
27     1.618033990263939
28     >>> f = lambda x: (2*x - 1)*(x - 3)
29     >>> bisection(f,0,1,10)
30     0.5
31     '''
32     if f(a)*f(b) >= 0:
33         print("Bisection method fails.")
34         return None
35     a_n = a
36     b_n = b
37     for n in range(1,N+1):
38         m_n = (a_n + b_n)/2
39         f_m_n = f(m_n)
40         if f(a_n)*f_m_n < 0:
41             a_n = a_n
42             b_n = m_n
43         elif f(b_n)*f_m_n < 0:
44             a_n = m_n
45             b_n = b_n
46         elif f_m_n == 0:
47             print("Found exact solution.")
48             return m_n
49         else:
50             print("Bisection method fails.")
51             return None
52     return (a_n + b_n)/2

```

Source

Secant method

The secant method is very similar to the bisection method except instead of dividing each interval by choosing the midpoint the secant method divides each interval by the secant line connecting the endpoints. The secant method always converges to a root of $f(x)$ is continuous on $[a, b]$ and $f(a)f(b) < 0$.

Secant line formula

Let $f(x)$ be a continuous function on $[a, b]$ and $f(a)f(b) < 0$. A solution of the equation $f(x) = 0$ for $x \in [a, b]$ is guaranteed by the Intermediate Value Theorem. Consider the line connecting the endpoint values $(a, f(a))$ and $(b, f(b))$. The line connecting these two points is called the secant line and is given by the formula

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

The point where the secant line crosses the x -axis is

$$0 = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

which we solve for x

$$x = a - f(a) \frac{b - a}{f(b) - f(a)}$$

Algorithm

The secant method procedure is:

1. Choose a starting interval $[a_0, b_0]$ such that $f(a_0)f(b_0) < 0$
2. Compute $f(x_0)$ where x_0 is given by the secant line :

$$x_0 = a_0 - f(a_0) \frac{b_0 - a_0}{f(b_0) - f(a_0)}$$

3. Determine the next subinterval $[a_1, b_1]$:
 - (a) If $f(a_0)f(x_0) < 0$, then let $[a_1, b_1]$ be the next interval with $a_1 = a_0$ and $b_1 = x_0$.
 - (b) If $f(b_0)f(x_0) < 0$, then let $[a_1, b_1]$ be the next interval with $a_1 = x_0$ and $b_1 = b_0$.
4. Repeat (2) and (3) until the interval $[a_N, b_N]$ reaches some predetermined length.
5. Return the value x_N , the x -intercept of the N th subinterval.

A solution of the equation $f(x) = 0$ on an interval a, b is guaranteed by the Intermediate Value Theorem provided $f(x)$ is continuous on $[a, b]$ and $f(a)f(b) < 0$. In other words, the function changes sign over the interval and therefore must equal 0 at some point in the interval $[a, b]$.

Python Implementation

Write a function called `secant` which takes 4 input parameters `f`, `a`, `b` and `N` and returns the approximation of a solution of $f(x) = 0$ given by N iterations of the secant method. If $f(a_N)f(b_N) > 0$ at any point in the iteration (caused either by a bad initial interval or rounding error in computations), then print "Secant method fails." and return `None`.

```
55 def secant(f,a,b,N):
56     '''Approximate solution of f(x)=0 on interval [a,b] by the
57     secant method.
58
59     Parameters
60     -----
61     f : function
62         The function for which we are trying to approximate a
63         solution f(x)=0.
64     a,b : numbers
65         The interval in which to search for a solution. The
66         function returns
67         None if f(a)*f(b) >= 0 since a solution is not
68         guaranteed.
69     N : (positive) integer
70         The number of iterations to implement.
71
72     Returns
73     -----
74     m_N : number
75         The x intercept of the secant line on the the Nth
76         interval
77         m_n = a_n - f(a_n)*(b_n - a_n)/(f(b_n) - f(a_n))
78         The initial interval [a_0,b_0] is given by [a,b]. If f(
79         m_n) == 0
80         for some intercept m_n then the function returns this
81         solution.
82         If all signs of values f(a_n), f(b_n) and f(m_n) are the
83         same at any
84         iterations, the secant method fails and return None.
85
86     Examples
87     -----
88     >>> f = lambda x: x**2 - x - 1
89     >>> secant(f,1,2,5)
90     1.6180257510729614
```

```

83     '''
84     if f(a)*f(b) >= 0:
85         print("Secant method fails.")
86         return None
87     a_n = a
88     b_n = b
89     for n in range(1,N+1):
90         m_n = a_n - f(a_n)*(b_n - a_n)/(f(b_n) - f(a_n))
91         f_m_n = f(m_n)
92         if f(a_n)*f_m_n < 0:
93             a_n = a_n
94             b_n = m_n
95         elif f(b_n)*f_m_n < 0:
96             a_n = m_n
97             b_n = b_n
98         elif f_m_n == 0:
99             print("Found exact solution.")
100             return m_n
101         else:
102             print("Secant method fails.")
103             return None
104     return a_n - f(a_n)*(b_n - a_n)/(f(b_n) - f(a_n))

```

Source

Newton Method

Newton's method is a root finding method that uses linear approximation. In particular, we guess a solution x_0 of the equation $f(x) = 0$, compute the linear approximation of $f(x)$ at x_0 and then find the x -intercept of the linear approximation.

Newton's formula

Let $f(x)$ be a differentiable function. If x_0 is near a solution of $f(x) = 0$ then we can approximate $f(x)$ by the tangent line at x_0 and compute the x -intercept of the tangent line. The equation of the tangent line at x_0 is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

The x -intercept is the solution x_1 of the equation

$$0 = f'(x_0)(x_1 - x_0) + f(x_0)$$

and we solve for x_1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If we implement this procedure repeatedly, then we obtain a sequence given by the recursive formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which (potentially) converges to a solution of the equation $f(x) = 0$.

Python Implementation

Write a function called `newton` which takes 5 input parameters `f`, `Df`, `x0`, `epsilon` and `maxiter` and returns the approximation of a solution of $f(x) = 0$ given by Newton's method.

The function may terminate in 3 ways:

1. If `abs(f(xn)) < epsilon`, the algorithm has found an approximate solution and returns `xn`.
2. If `f'(xn) == 0`, the algorithm stops and returns `None`.
3. If the number of iterations exceeds `maxiter`, the algorithm stops and returns `None`.

```
107 def newton(f,Df,x0,epsilon,max_iter):
108     '''Approximate solution of f(x)=0 by Newton's method.
109
110     Parameters
111     -----
112     f : function
113         Function for which we are searching for a solution f(x)
114         =0.
115     Df : function
116         Derivative of f(x).
117     x0 : number
118         Initial guess for a solution f(x)=0.
119     epsilon : number
120         Stopping criteria is abs(f(x)) < epsilon.
121     max_iter : integer
122         Maximum number of iterations of Newton's method.
```

```

122
123     Returns
124     -----
125     xn : number
126         Implement Newton's method: compute the linear
127         approximation
128         of f(x) at xn and find x intercept by the formula
129             x = xn - f(xn)/Df(xn)
130         Continue until abs(f(xn)) < epsilon and return xn.
131         If Df(xn) == 0, return None. If the number of iterations
132         exceeds max_iter, then return None.
133
134     Examples
135     -----
136     >>> f = lambda x: x**2 - x - 1
137     >>> Df = lambda x: 2*x - 1
138     >>> newton(f,Df,1,1e-8,10)
139     Found solution after 5 iterations.
140     1.618033988749989
141     '''
142     xn = x0
143     for n in range(0,max_iter):
144         fxn = f(xn)
145         if abs(fxn) < epsilon:
146             print('Found solution after',n,'iterations.')
147             return xn
148         Dfxn = Df(xn)
149         if Dfxn == 0:
150             print('Zero derivative. No solution found.')
151             return None
152         xn = xn - fxn/Dfxn
153     print('Exceeded maximum iterations. No solution found.')
154     return None
155

```

Source

Lecture Code: Newton Solver, we try to find x such that $f(x) = 0$.

```

1 # Newton Solver
2 def our_newton_solver(funcname,startvalue,arglist):
3     ''' Parameters:
4         funcname = Function to optimize
5         startvalue = Value to start the resesrch of optimal
        value

```

```

6         arglist = optimal values for the function
7     Returns:
8         Optimal value for which the function is solved'''
9     current=startvalue
10    fval = funcname(current, arglist)
11    grad = (funcname(current+0.5*1e-5, arglist)-funcname(current
12    -0.5*1e-5, arglist))*1e+5
13    while (abs(fval)>1e-8):
14        current = current - fval/grad
15        fval = funcname(current, arglist)
16        grad = (funcname(current+0.5*1e-5, arglist)-funcname(
17        current-0.5*1e-5, arglist))*1e+5
18    return current

```

Lecture Code: Newton Maximizer, we try to find x such that $f'(x) = 0$.

```

1    # Newton Maximizer
2    def our_newton_maximizer(funcname, startvalue, arglist):
3        ''' Parameters:
4            funcname = Function to optimize
5            startvalue = Value to start the resesrch of optimal
6            value
7            arglist = optimal values for the function
8        Returns:
9            Optimal value for which the function is maximized'''
10    current=startvalue
11    fval = funcname(current, arglist)
12    grad = (funcname(current+0.5*1e-5, arglist)-funcname(current
13    -0.5*1e-5, arglist))*1e+5
14    secgrad1 = (funcname(current+0.5*1e-5+0.5*1e-5, arglist)-
15    funcname(current-0.5*1e-5+0.5*1e-5, arglist))*1e+5
16    secgrad2 = (funcname(current+0.5*1e-5-0.5*1e-5, arglist)-
17    funcname(current-0.5*1e-5-0.5*1e-5, arglist))*1e+5
18    secderiv = (secgrad1-secgrad2)*1e+5
19    while (abs(grad)>1e-8):
20        current = current - grad/secderiv
21        fval = funcname(current, arglist)
22        grad = (funcname(current+0.5*1e-5, arglist)-funcname(
23        current-0.5*1e-5, arglist))*1e+5
24        secgrad1 = (funcname(current+0.5*1e-5+0.5*1e-5, arglist)-
25        funcname(current-0.5*1e-5+0.5*1e-5, arglist))*1e+5
26        secgrad2 = (funcname(current+0.5*1e-5-0.5*1e-5, arglist)-
27        funcname(current-0.5*1e-5-0.5*1e-5, arglist))*1e+5
28        secderiv = (secgrad1-secgrad2)*1e+5

```

```

23     return current
24

```

Utility Functions

There are several classes of utility functions that are frequently used to generate demand functions.

- One of the most common is the Cobb-Douglas utility function, which has the form

$$u(x, y) = x^a y^{1-a} \text{ with } a \in [0, 1]$$

- Another common form for utility is the Constant Elasticity of Substitution (CES) utility function. This function has the form

$$u(x, y) = (ax^r + by^r)^{1/r}$$

- A third common utility function is quadratic, which has the form

$$u(x, y) = 2ax - (b - y)^2$$

Cobb-Douglas Utility Function

Constant Elasticity of Substitution (CES)

The constant elasticity of substitution applied to utility can use the formula

$$u(x, y) = (ax^r + by^r)^{1/r} \text{ where } -\infty < r < 1 \text{ and } r \neq 0$$

where r is the substitution parameter.

Marginal rate of substitution (MRS) is computed by

$$MRS = -\frac{a}{b} \left(\frac{x}{y} \right)^{r-1}$$

The demand functions are computed

$$x(p_x, p_y, I) = \frac{p_x^{1/(r-1)}}{p_x^{r/(r-1)} + p_y^{r/(r-1)}} \cdot I$$

$$y(p_x, p_y, I) = \frac{p_y^{1/(r-1)}}{p_x^{r/(r-1)} + p_y^{r/(r-1)}} \cdot I$$

where (p_x, p_y, I) are price of good x, price of good y and income.

Conquences of variations of r :

- If $r \rightarrow 0$ then $u(x, y) \rightarrow$ Cobb Douglas Utility Function

$$u(x, y) = x^a y^{1-a}$$

- If $r \rightarrow -\infty$ then $u(x, y) \rightarrow$ Leontief utility Function (inputs are perfect complements)

$$u(x, y) = \text{Min}(ay, bx)$$

- If $r \rightarrow 1$ then $u(x, y) \rightarrow$ Linear Production (inputs are perfect substitutes)

$$u(x, y) = ay + bx$$

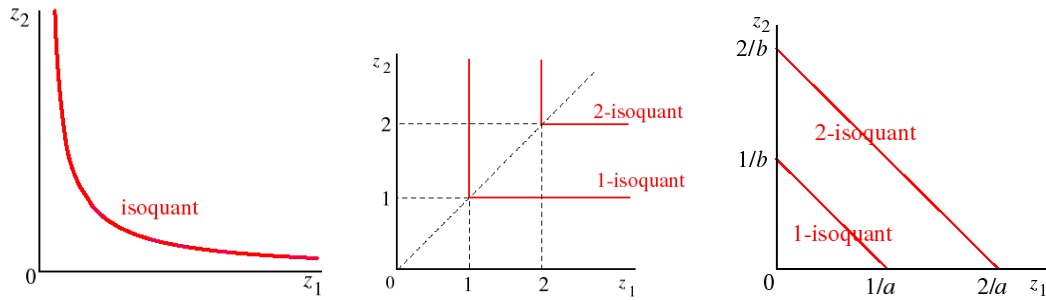


Figure 1: Utility Curves

Quasilinear Utility Functions

Utility function that is independent of the income effect.

$$u(x, y) = v(x) + y$$

where v is an arbitrary function that is strictly increasing if good x is desired.

Indifference curve for α utility level:

$$v(x) + y = \alpha$$

$$y = \alpha - v(x)$$

Marginal rate of substitution is computed by

$$MRS = \frac{\partial u}{\partial x} / \frac{\partial u}{\partial y}$$

where $\frac{\partial u}{\partial x} = v'(x)$ and $\frac{\partial u}{\partial y} = 1$

Therefore $MRS = v'(x)$

Slutsky decomposition: Income and substitution effects

Slutsky decomposition is the total effect of substitution and income.

Normal Goods

Normal goods are goods for which demand increases when income increases. In the Slutsky decomposition, the income and substitution effect reinforce each other when the good's price changes.

Income and substitution effects cause an increase in demand when prices decrease.

Income Inferior Goods

Demand reduced with higher income.

Substitution and Income effects oppose each other. When income increases, demand decreases, the substitution effect is the same as normal goods.

Giffon Goods

Extreme income-inferiority, the income effect may be larger in size than the substitution effect, causing quantity demanded to fall as own-prices rise.

Slutsky's decomposition of the effect of a price change into a pure substitution effect and an income effect thus explains why the law of downwards-sloping demand is violated for extremely income-inferior goods.

Microeconomy

The intertemporal utility function

A plan for consumption in the periods $0, 1, \dots, T-1$ is denoted $c_{t=0}^{T-1}$, where c_t is the consumption in period t . We say the plan has time horizon T. Period 0 ('the initial period') need not refer to the 'birth' of the household but is just an arbitrary period within the lifetime of the household .

We assume that preferences of a household/consumer can be represented by a time-separable intertemporal utility function with a constant utility discount rate and no utility from leisure (the latter assumption implies that the labour supply curve of the household in each period is inelastic, ie whatever the changes of prices of labour and leisure the supply of labour stays the same). The time-separability itself just means that the intertemporal utility function is additive.

In addition, we assume geometric utility discounting, meaning that utility obtained t periods ahead is converted into a present equivalent by multiplying by the discount factor $(1+q)^{-t}$, where q is a constant utility discount rate. Hence, $u_t(c_t) = u(c_t)(1+q)^{-t}$, where $u(c)$ is a time-independent period utility function. Together, these two assumptions amount to

$$U(c_0, c_1, \dots, c_{T-1}) = u(c_0) + \frac{u(c_1)}{(1+q)} + \dots + \frac{u(c_{T-1})}{(1+q)^{T-1}} = \sum_{t=0}^{T-1} \frac{u(c_t)}{(1+q)^t}$$

The period utility function is assumed to satisfy $u'(c) > 0$ and $u''(c) < 0$. The number $1+q$ tells how many units of utility in the next period the household insists on "in return" for a decrease of one unit of utility in the current period. So, a $q > 0$ will reflect that if the chosen level of consumption is the same in two periods, then the individual always appreciates a marginal unit of consumption higher the earlier it arrives. This explains why q is named the rate of time preference or rate of impatience.

The utility discount factor, $\frac{1}{1+q}$, indicates how many units of utility the household is at most willing to give up in period 0 to get one additional unit of utility in period t .

Now we assume that the consumer has a utility function over consumption today (time period 0, $t = 0$) and the future (time period 1, $t = 1$). Say,

$$u(c_{t=0}, c_{t=1}) = c_{t=0}^\alpha + \frac{1}{1+q} * c_{t=1}^\alpha$$

where α is a parameter between 0 and 1, say 0.5, and q is a given number for instance 0.04. The consumer has some income/endowment (e or I) in both periods but can also save or borrow money. Savings s is given by

$$s = e_0 - c_{t=0}$$

and consumption in period 2 is given by

$$c_{t=1} = e_1 + s \times (1 + r)$$

where r is the interest rate (s will be negative if the consumer borrows money.) Since e_1 is in tomorrow's money, we have to multiply the savings we bring forward by $(1 + r)$ to We can put this into our numerical framework by substituting for s , giving a budget constraint

$$c_{t=1} = e_1 + (e_0 - c_{t=0}) \times (1 + r)$$

(Source)

Code: Visualise the changes in consumption combination (indifference curve) when the interest rate varies.

```

1 def f(rs, arglist):
2     e = arglist[0] ; q = arglist[1]
3     for element in range(len(rs)):
4         r = rs[element]
5
6     def utility(c1, c2):
7         #e = 0.5 ; q = 0.04
8         return np.power(c1, e) + (1/(1+q)) * np.power(c2, e)
9
10    def utility_budget(c1, ip_list):
11        I1 = ip_list[0] ; I2 = ip_list[1] ; r = ip_list[2]
12        c2 = I2 + (I1 - c1) * (1+r)
13        return utility(c1, c2)
14
15    def demand(I1, I2, r):
16        c1 = newton_max(utility_budget, 0.1, [I1, I2, r])
17        c2 = I2 + (I1 - c1) * (1+r)
18        return c1, c2
19
20    def indiff_dist(c2, mylist):
21        c1 = mylist[0] ; utility_level = mylist[1]

```



```

22     utitility_achieved = utility(c1,c2)
23     return utitility_achieved - utility_level
24
25 def indifference(c1, util):
26     return newton_add(indiff_dist, 1, [c1,util])
27
28 def indirect_utility(I1, I2, r):
29     c11, c22 = demand(I1, I2, r)
30     return utility(c11, c22)
31
32 # initialise quantiites of good 1 (x), income (I), and
33 # prices of the two goods
34 c1 = np.linspace(1,30,100) ; I1 = 10 ; I2 = 12
35
36 # get indirect utility curve here
37 utility_level = indirect_utility(I1, I2, r)
38 c2 = np.zeros(len(c1))
39
40 for idx in range(len(c1)):
41     c2[idx] = indifference(c1[idx], utility_level)
42
43 #Compute the optimal demand for each r
44 dem = []
45 dem= demand(I1,I2,r)
46
47 # plot and make nice
48 plt.figure(figsize = (6,4)) ; plt.plot(c1, c2, label = f'
49 Indifference curve with r = {np.round(r,2)}')
50 plt.plot(c1, I2 + (I1 - c1)*(1+r), color = 'red', label = '
51 budget line')
52 plt.scatter(dem[0],dem[1], label = "Optimum demand", color =
53 'black')
54 plt.text(dem[0], dem[1], '({}, {})'.format(np.round(dem
55 [0],2), np.round(dem[1],2)))
56 plt.xlabel('quantity of goods consumed today') ; plt.ylabel(
57 'quantity of good s consumed tomorow')
58 plt.legend(loc = 'upper right')
59 plt.title('Evolution of indifference curves when interest
60 rates varie')
61 plt.xlim(0,30) ; plt.ylim(0,30)
62 plt.show()

```

Code: Visualise Slutsky decomposition to determine consumer's behaviour.

```

1     def utility(c1, c2):
2         e = 0.5 ; q = 0.04
3         return np.power(c1,e)+(1/(1+q))*np.power(c2,e)
4
5     def utility_budget(c1, ip_list):
6         I1 = ip_list[0] ; I2 = ip_list[1] ; r = ip_list[2]
7         c2 = I2 + (I1 - c1)*(1+r)
8         return utility(c1, c2)
9
10    def demand(I1, I2, r):
11        c1 = newton_max(utility_budget, 0.1, [I1, I2, r])
12        c2 = I2 + (I1 - c1)*(1+r)
13        return c1, c2
14
15    def indiff_dist(c2, mylist):
16        c1 = mylist[0] ; utility_level = mylist[1]
17        utility_achieved = utility(c1,c2)
18        return utility_achieved - utility_level
19
20    def indifference(c1, util):
21        return newton_add(indiff_dist, 1, [c1,util])
22
23    def indirect_utility(I1, I2, r):
24        c11, c22 = demand(I1, I2, r)
25        return utility(c11, c22)
26
27    def diff_indirect_utilities(income_to_be_found,longlist):
28        I1 = longlist[0] ; I2 = longlist[1] ; r1 = longlist[2] ; r2
29        = longlist[3]
30        return indirect_utility(I1, I2, r1) - indirect_utility(I1,
31        income_to_be_found,r2)
32
33    # define the parameters
34    I1 = 12 ; I2 = 8 ; r1 = 0.1 ; r2 = 0.85
35
36    # Give optimal income level for period 2 when prices change
37    a = newton_add(diff_indirect_utilities, 1, [I1, I2, r1, r2])
38    # print(a)
39
40    c1 = np.linspace(0.1, 10, 100) # quantity of good today
41
42    # plot with the new budget line as well
43    plt.figure(figsize = (12, 8))

```

```

43
44 # budget curve 1 : Before changes
45 plt.plot(c1, I2 + (I1 - c1)*(1+r1), color = 'pink', label = '
    budget curve 1')
46
47 # budget curve2 : After changes
48 plt.plot(c1, a + (I1 - c1)*(1+r2), color = 'red', label = '
    budget curve with a')
49
50 # get the utility levels
51 u_bar = indirect_utility(I1, I2, r1)
52 u_bar2 = indirect_utility(I1, I2, r2)
53
54 #print(u_bar, u_bar2)
55
56 # express utility max as a function of quantity of good tomorrow
57 c21 = np.zeros((100,1)) ; c22 = np.zeros((100,1))
58
59 for idx in range(len(c1)):
60     c21[idx] = indifference(c1[idx], u_bar)
61     c22[idx] = indifference(c1[idx], u_bar2)
62
63 plt.plot(c1, c21, label = 'indifference curve 1', color = 'green
    ')
64 plt.plot(c1, c22, label = 'indifference curve 2', color = 'blue'
    )
65
66 # find utility-maximising consumption bundle
67 dems1 = demand(I1, I2, r1) ; plt.scatter(dems1[0], dems1[1],s
    =50,alpha=0.5,color='black')
68 dems2 = demand(I1, I2, r2) ; plt.scatter(dems2[0], dems2[1],s
    =50,alpha=0.5,color='black')
69
70 # make nice
71 # plt.vlines(dems1[0], 8, 10.5, linestyle = 'dotted', color = '
    k')
72 # plt.vlines(dems2[0], 8, 10.5, linestyle = 'dotted', color = '
    k')
73 plt.annotate('A', xy = (dems1[0], dems1[1]), xycoords='data',
    rotation = 30, size=15)
74 plt.annotate('B', xy = (dems2[0], dems2[1]), xycoords='data',
    rotation = 30, size=15)
75
76 plt.vlines(dems1[0],0,dems1[1],linestyle='dotted')

```

```

77 plt.vlines(dems2[0],0,dems2[1],linestyles='dotted')
78 plt.hlines(dems1[1],0,dems1[0], linestyles='dotted')
79 plt.hlines(dems2[1],0,dems2[0], linestyles='dotted')
80 plt.ylim(0,40) ; plt.xlim(0,10)
81 plt.title('Income and Substitution effect: Consequences on
consumption in time')
82 plt.xlabel('consumption today, c1') ; plt.ylabel('consumption
tomorrow, c2')
83 plt.legend(loc = 'upper right')

```

Then to determine the consumer's behaviour you compute the savings for the first period (today).

```

1 s = I1 - dems1[0]
2 # where dems1[0] is the consumer of today's goods

```

If the savings are negative then the consumer is a borrower, if savings are positive then the consumer is a saver. To visualise both cases you can play with the values of Incomes (I1 and I2) and the rates (r1 and r2).

Monopoly

A monopoly consists of one firm that produces a unique product or service with no close substitutes. Entry into the market is blocked, which gives the firm market power (i.e., the power to raise price above marginal cost).

Some definitions,

- Average revenue = market demand curve.
- Marginal revenue: change in revenue resulting from a one-unit increase in output.
 - $MR > 0 \rightarrow$ Revenue increases
 - $MR < 0 \rightarrow$ Revenue decreases
 - When the demand curve is downward sloping, the price ($= AR$) is above MR for all units sold at the same price.
 - No supply curve in monopolistic markets. That is, there is a one-to-one relationship between price and the quantity produced

Profit Maximization for Monopolists

All profit maximizing firms, regardless of the structure of the markets in

which they sell, maximize profits by setting output such that marginal revenue equals marginal cost.

Monopolists are ‘price makers’ in the sense that they can set their prices by picking a point on the demand curve. Note that they are not free of constraint; the demand curve dictates the maximum price they can charge for every quantity level. The task for the profit-maximizing monopolist is to determine which point on the demand curve maximizes their profits. A downward sloping demand curve will mean that the firm faces a trade-off: they can sell more but must lower their price to do so. This means that the marginal revenue of a monopolist will depend on their output decision.

The total revenue for a firm is the amount of goods they sell, Q , multiplied by the price at which they sell the goods, p . Note that Q is both the firm’s output and the market output as there is only the one firm supplying the market.

$$TR = p \times Q$$

The firm’s marginal revenue is the change in total revenue from the sale of one more unit of their output. Thus a firm that earns ΔTR extra revenue from the sale of ΔQ more units has a marginal revenue of:

$$MR = \frac{\partial TR}{\partial Q}$$

Dividing both sides by Q gives us an expression for marginal revenue:

$$MR = p + Q \frac{\Delta p}{\Delta Q}$$

The first part, p , is the extra revenue the firm gets from selling an extra unit of the good. The second part, $Q \frac{\Delta p}{\Delta Q}$, which is negative, is the decrease in revenue from the fact that increasing output marginally, lowers the price the firm receives for all of its output. Since the second part is negative we know that the MR curve is below the demand curve for every Q since demand curve relates price and quantity.

A profit maximizing firm chooses output such that the marginal cost of producing that output exactly matches the marginal revenue it gets from selling that output. The intersection of the marginal revenue and marginal cost curves occurs at point Q^M and the highest price at which the monopolist

can sell exactly Q^M is p^M . At Q^M the total revenue is simply $(p^M) \times (Q^M)$, and total cost is $(AC) \times (Q^M)$ since AC is just $\frac{TC}{Q}$.

Suppose the demand for a good produced by a monopolist is $p = A - BQ$. Note that this is an inverse demand curve a demand curve written with price as a function of quantity. We can find the marginal revenue curve by first noting that $TR = p \times Q$. Thus $TR = (A - BQ) \times Q$, or $TR = AQ - BQ^2$. In other words, for an inverse demand curve, $p = A - BQ$, the marginal revenue curve is $MR = A - 2BQ$.

(Source)

Expenditure functions

The expenditure function gives the minimum amount of money an individual needs to spend to achieve some level of utility, given a utility function and the prices of the available goods.

Say the utility we want to achieve is α for n number of commodities at a price vector p . The expenditure function is then:

$$e(p, \alpha) = \min_{x \in \geq(\alpha)} p \cdot x$$

where $x \in \geq \alpha$ means that we have consumption bundles that give at least the same utility level as α .

Properties of the expenditure function :

- The function is continuous in p and u
- The function is nondecreasing in p and strictly increasing in u provided $p \gg 0$
- The function is concave in p
- If the utility function is quasi-concave, the Shephard's Lemma can be applied to the expenditure function.

Shephard's Lemma

The lemma states that if the indifference curves of the expenditure or cost function are convex, then the cost minimizing point of a given good i with price p_i is unique.

The idea is that a consumer will buy a unique ideal amount of each item to minimize the price for obtaining a certain level of utility given the price of goods in the market. (Source)

(Source)

Hicksian demand function

A consumer's Hicksian demand function or compensated demand function for a good is his quantity demanded as part of the solution to minimizing his expenditure on all goods while delivering a fixed level of utility.

Essentially, a Hicksian demand function shows how an economic agent would react to the change in the price of a good, if the agent's income was compensated to guarantee the agent the same utility previous to the change in the price of the good—the agent will remain on the same indifference curve before and after the change in the price of the good.

The demand curve we usually refer to is the Marshallian demand function. The Marshallian demand function and the Hicksian demand function are related :

$$h(p, u) = x(p, e(p, u)),$$

where $e(p, u)$ is the expenditure function ; and by:

$$h(p, v(p, w)) = x(p, w),$$

where $v(p, w)$ is the indirect utility function (which gives the utility level of having a given wealth under a fixed price regime).

The two demand functions are still difference in the sense that Marshallian demand comes from the Utility Maximization Problem, while the Hicksian Demand comes from the Expenditure Minimization Problem.

Compensation of prices changes

The Hicksian demand function isolates the substitution effect by supposing the consumer is compensated with exactly enough extra income after the price rise to purchase some bundle on the same indifference curve. If the

Hicksian demand function is steeper than the Marshallian demand, the good is a normal good; otherwise, the good is inferior. Hicksian demand always slopes down.

Markian demand function

Marshallian demand curves show the effect of price changes on quantity demanded. As the price of a good rises, ordinarily, the quantity of that good demanded will fall, but not in every case. The price rise has both a substitution effect and an income effect. The substitution effect is the change in quantity demanded due to a price change that alters the slope of the budget constraint but leaves the consumer on the same indifference curve (i.e., at the same level of utility). The substitution effect always is to buy less of that good. The income effect is the change in quantity demanded due to the effect of the price change on the consumer's total buying power.

(Source)