

# EBA3650: Quantitative economics

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This file was created to contain explanations and code examples for the course Quantitative Economics.

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## Root Finding

Root finding refers to the general problem of searching for a solution of an equation  $F(x) = 0$  for some function  $F$ . If we want to optimise a function  $f(x)$  then we need to find critical points and therefore solve the equation  $f'(x) = 0$ .

Example quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Source

## Bisection Method

The algorithm applies to any continuous function  $f(x)$  on an interval  $a, b$  where the value of the function  $f(x)$  changes sign from  $a$  to  $b$ . The idea is simple: divide the interval in two, a solution must exist within one subinterval, select the subinterval where the sign of  $f(x)$  changes and repeat.

## Algorithm

The bisection method procedure is:

1. Choose a starting interval  $[a_0, b_0]$  such that  $f(a_0)f(b_0) < 0$
2. Compute  $f(m_0)$  where  $m_0 = (a_0 + b_0)/2$  is the midpoint.
3. Determine the next subinterval  $[a_1, b_1]$  :
  - (a) If  $f(a_0)f(m_0) < 0$ , then let  $[a_1, b_1]$  be the next interval with  $a_1 = a_0$  and  $b_1 = m_0$ .
  - (b) If  $f(b_0)f(m_0) < 0$ , then let  $[a_1, b_1]$  be the next interval with  $a_1 = m_0$  and  $b_1 = b_0$ .
4. Repeat (2) and (3) until the interval  $[a_N, b_N]$  reaches some predetermined length.
5. Return the midpoint value  $m_N = (a_N + b_N)/2$

A solution of the equation  $f(x)$  on an interval  $a,b$  is guaranteed by the Intermediate Value Theorem provided  $f(x)$  is continuous on  $[a,b]$  and  $f(a)f(b) < 0$ . In other words, the function changes sign over the interval and therefore must equal 0 at some point in the interval  $[a,b]$ .

## Python Implementation

Write a function called `bisection` which takes 4 input parameters `f`, `a`, `b` and `N` and returns the approximation of a solution of  $f(x) = 0$  given by  $N$  iterations of the bisection method. If  $f(a_N)f(b_N) > 0$  at any point in the iteration (caused either by a bad initial interval or rounding error in computations), then print "Bisection method fails." and return `None`.

```

1 def bisection(f,a,b,N):
2     '''Approximate solution of f(x)=0 on interval [a,b] by
3     bisection method.
4
5     Parameters
6     -----
7     f : function
8         The function for which we are trying to approximate a
9         solution f(x)=0.
10    a,b : numbers
11        The interval in which to search for a solution. The
12        function returns
13        None if f(a)*f(b) >= 0 since a solution is not
14        guaranteed.
15    N : (positive) integer
16        The number of iterations to implement.
17
18    Returns
19    -----
20    x_N : number
21        The midpoint of the Nth interval computed by the
22        bisection method. The
23        initial interval [a_0,b_0] is given by [a,b]. If f(m_n)
24        == 0 for some
25        midpoint m_n = (a_n + b_n)/2, then the function returns
26        this solution.
27        If all signs of values f(a_n), f(b_n) and f(m_n) are the
28        same at any
29        iteration, the bisection method fails and return None.

```

```

23     Examples
24     -----
25     >>> f = lambda x: x**2 - x - 1
26     >>> bisection(f,1,2,25)
27     1.618033990263939
28     >>> f = lambda x: (2*x - 1)*(x - 3)
29     >>> bisection(f,0,1,10)
30     0.5
31     '''
32     if f(a)*f(b) >= 0:
33         print("Bisection method fails.")
34         return None
35     a_n = a
36     b_n = b
37     for n in range(1,N+1):
38         m_n = (a_n + b_n)/2
39         f_m_n = f(m_n)
40         if f(a_n)*f_m_n < 0:
41             a_n = a_n
42             b_n = m_n
43         elif f(b_n)*f_m_n < 0:
44             a_n = m_n
45             b_n = b_n
46         elif f_m_n == 0:
47             print("Found exact solution.")
48             return m_n
49         else:
50             print("Bisection method fails.")
51             return None
52     return (a_n + b_n)/2

```

Source

## Secant method

The secant method is very similar to the bisection method except instead of dividing each interval by choosing the midpoint the secant method divides each interval by the secant line connecting the endpoints. The secant method always converges to a root of  $f(x)$  is continuous on  $[a, b]$  and  $f(a)f(b) < 0$ .

## Secant line formula

Let  $f(x)$  be a continuous function on  $[a, b]$  and  $f(a)f(b) < 0$ . A solution of the equation  $f(x) = 0$  for  $x \in [a, b]$  is guaranteed by the Intermediate Value Theorem. Consider the line connecting the endpoint values  $(a, f(a))$  and  $(b, f(b))$ . The line connecting these two points is called the secant line and is given by the formula

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

The point where the secant line crosses the  $x$ -axis is

$$0 = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

which we solve for  $x$

$$x = a - f(a) \frac{b - a}{f(b) - f(a)}$$

## Algorithm

The secant method procedure is:

1. Choose a starting interval  $[a_0, b_0]$  such that  $f(a_0)f(b_0) < 0$
2. Compute  $f(x_0)$  where  $x_0$  is given by the secant line :

$$x_0 = a_0 - f(a_0) \frac{b_0 - a_0}{f(b_0) - f(a_0)}$$

3. Determine the next subinterval  $[a_1, b_1]$  :
  - (a) If  $f(a_0)f(x_0) < 0$ , then let  $[a_1, b_1]$  be the next interval with  $a_1 = a_0$  and  $b_1 = x_0$ .
  - (b) If  $f(b_0)f(x_0) < 0$ , then let  $[a_1, b_1]$  be the next interval with  $a_1 = x_0$  and  $b_1 = b_0$ .
4. Repeat (2) and (3) until the interval  $[a_N, b_N]$  reaches some predetermined length.
5. Return the value  $x_N$ , the  $x$ -intercept of the  $N$ th subinterval.

A solution of the equation  $f(x) = 0$  on an interval  $a, b$  is guaranteed by the Intermediate Value Theorem provided  $f(x)$  is continuous on  $[a, b]$  and  $f(a)f(b) < 0$ . In other words, the function changes sign over the interval and therefore must equal 0 at some point in the interval  $[a, b]$ .

## Python Implementation

Write a function called `secant` which takes 4 input parameters `f`, `a`, `b` and `N` and returns the approximation of a solution of  $f(x) = 0$  given by  $N$  iterations of the secant method. If  $f(a_N)f(b_N) > 0$  at any point in the iteration (caused either by a bad initial interval or rounding error in computations), then print "Secant method fails." and return `None`.

```
55 def secant(f,a,b,N):
56     '''Approximate solution of f(x)=0 on interval [a,b] by the
57     secant method.
58
59     Parameters
60     -----
61     f : function
62         The function for which we are trying to approximate a
63         solution f(x)=0.
64     a,b : numbers
65         The interval in which to search for a solution. The
66         function returns
67         None if f(a)*f(b) >= 0 since a solution is not
68         guaranteed.
69     N : (positive) integer
70         The number of iterations to implement.
71
72     Returns
73     -----
74     m_N : number
75         The x intercept of the secant line on the the Nth
76         interval
77         m_n = a_n - f(a_n)*(b_n - a_n)/(f(b_n) - f(a_n))
78         The initial interval [a_0,b_0] is given by [a,b]. If f(
79         m_n) == 0
80         for some intercept m_n then the function returns this
81         solution.
82         If all signs of values f(a_n), f(b_n) and f(m_n) are the
83         same at any
84         iterations, the secant method fails and return None.
85
86     Examples
87     -----
88     >>> f = lambda x: x**2 - x - 1
89     >>> secant(f,1,2,5)
90     1.6180257510729614
```

```

83     '''
84     if f(a)*f(b) >= 0:
85         print("Secant method fails.")
86         return None
87     a_n = a
88     b_n = b
89     for n in range(1,N+1):
90         m_n = a_n - f(a_n)*(b_n - a_n)/(f(b_n) - f(a_n))
91         f_m_n = f(m_n)
92         if f(a_n)*f_m_n < 0:
93             a_n = a_n
94             b_n = m_n
95         elif f(b_n)*f_m_n < 0:
96             a_n = m_n
97             b_n = b_n
98         elif f_m_n == 0:
99             print("Found exact solution.")
100             return m_n
101         else:
102             print("Secant method fails.")
103             return None
104     return a_n - f(a_n)*(b_n - a_n)/(f(b_n) - f(a_n))

```

Source

## Newton Method

Newton's method is a root finding method that uses linear approximation. In particular, we guess a solution  $x_0$  of the equation  $f(x) = 0$ , compute the linear approximation of  $f(x)$  at  $x_0$  and then find the  $x$ -intercept of the linear approximation.

### Newton's formula

Let  $f(x)$  be a differentiable function. If  $x_0$  is near a solution of  $f(x) = 0$  then we can approximate  $f(x)$  by the tangent line at  $x_0$  and compute the  $x$ -intercept of the tangent line. The equation of the tangent line at  $x_0$  is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

The  $x$ -intercept is the solution  $x_1$  of the equation

$$0 = f'(x_0)(x_1 - x_0) + f(x_0)$$



and we solve for  $x_1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If we implement this procedure repeatedly, then we obtain a sequence given by the recursive formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which (potentially) converges to a solution of the equation  $f(x) = 0$ .

## Python Implementation

Write a function called `newton` which takes 5 input parameters `f`, `Df`, `x0`, `epsilon` and `maxiter` and returns the approximation of a solution of  $f(x) = 0$  given by Newton's method.

The function may terminate in 3 ways:

1. If `abs(f(xn)) < epsilon`, the algorithm has found an approximate solution and returns `xn`.
2. If `f'(xn) == 0`, the algorithm stops and returns `None`.
3. If the number of iterations exceeds `maxiter`, the algorithm stops and returns `None`.

```
107 def newton(f,Df,x0,epsilon,max_iter):
108     '''Approximate solution of f(x)=0 by Newton's method.
109
110     Parameters
111     -----
112     f : function
113         Function for which we are searching for a solution f(x)
114         =0.
115     Df : function
116         Derivative of f(x).
117     x0 : number
118         Initial guess for a solution f(x)=0.
119     epsilon : number
120         Stopping criteria is abs(f(x)) < epsilon.
121     max_iter : integer
122         Maximum number of iterations of Newton's method.
```

```

122
123     Returns
124     -----
125     xn : number
126         Implement Newton's method: compute the linear
127         approximation
128         of f(x) at xn and find x intercept by the formula
129             x = xn - f(xn)/Df(xn)
130         Continue until abs(f(xn)) < epsilon and return xn.
131         If Df(xn) == 0, return None. If the number of iterations
132         exceeds max_iter, then return None.
133
134     Examples
135     -----
136     >>> f = lambda x: x**2 - x - 1
137     >>> Df = lambda x: 2*x - 1
138     >>> newton(f,Df,1,1e-8,10)
139     Found solution after 5 iterations.
140     1.618033988749989
141     '''
142     xn = x0
143     for n in range(0,max_iter):
144         fxn = f(xn)
145         if abs(fxn) < epsilon:
146             print('Found solution after',n,'iterations.')
147             return xn
148         Dfxn = Df(xn)
149         if Dfxn == 0:
150             print('Zero derivative. No solution found.')
151             return None
152         xn = xn - fxn/Dfxn
153     print('Exceeded maximum iterations. No solution found.')
154     return None
155

```

Source

Lecture Code: Newton Solver, we try to find  $x$  such that  $f(x) = 0$ .

```

1 # Newton Solver
2 def our_newton_solver(funcname,startvalue,arglist):
3     ''' Parameters:
4         funcname = Function to optimize
5         startvalue = Value to start the resesrch of optimal
        value

```

```

6         arglist = optimal values for the function
7     Returns:
8         Optimal value for which the function is solved'''
9     current=startvalue
10    fval = funcname(current, arglist)
11    grad = (funcname(current+0.5*1e-5, arglist)-funcname(current
12    -0.5*1e-5, arglist))*1e+5
13    while (abs(fval)>1e-8):
14        current = current - fval/grad
15        fval = funcname(current, arglist)
16        grad = (funcname(current+0.5*1e-5, arglist)-funcname(
17        current-0.5*1e-5, arglist))*1e+5
18    return current

```

Lecture Code: Newton Maximizer, we try to find  $x$  such that  $f'(x) = 0$ .

```

1    # Newton Maximizer
2    def our_newton_maximizer(funcname, startvalue, arglist):
3        ''' Parameters:
4            funcname = Function to optimize
5            startvalue = Value to start the resesrch of optimal
6            value
7            arglist = optimal values for the function
8        Returns:
9            Optimal value for which the function is maximized'''
10    current=startvalue
11    fval = funcname(current, arglist)
12    grad = (funcname(current+0.5*1e-5, arglist)-funcname(current
13    -0.5*1e-5, arglist))*1e+5
14    secgrad1 = (funcname(current+0.5*1e-5+0.5*1e-5, arglist)-
15    funcname(current-0.5*1e-5+0.5*1e-5, arglist))*1e+5
16    secgrad2 = (funcname(current+0.5*1e-5-0.5*1e-5, arglist)-
17    funcname(current-0.5*1e-5-0.5*1e-5, arglist))*1e+5
18    secderiv = (secgrad1-secgrad2)*1e+5
19    while (abs(grad)>1e-8):
20        current = current - grad/secderiv
21        fval = funcname(current, arglist)
22        grad = (funcname(current+0.5*1e-5, arglist)-funcname(
23        current-0.5*1e-5, arglist))*1e+5
24        secgrad1 = (funcname(current+0.5*1e-5+0.5*1e-5, arglist)-
25        funcname(current-0.5*1e-5+0.5*1e-5, arglist))*1e+5
26        secgrad2 = (funcname(current+0.5*1e-5-0.5*1e-5, arglist)-
27        funcname(current-0.5*1e-5-0.5*1e-5, arglist))*1e+5
28        secderiv = (secgrad1-secgrad2)*1e+5

```

```

23     return current
24

```

## Utility Functions

There are several classes of utility functions that are frequently used to generate demand functions.

- One of the most common is the Cobb-Douglas utility function, which has the form

$$u(x, y) = x^a y^{1-a} \text{ with } a \in [0, 1]$$

- Another common form for utility is the Constant Elasticity of Substitution (CES) utility function. This function has the form

$$u(x, y) = (ax^r + by^r)^{1/r}$$

- A third common utility function is quadratic, which has the form

$$u(x, y) = 2ax - (b - y)^2$$

### Cobb-Douglas Utility Function

### Constant Elasticity of Substitution (CES)

The constant elasticity of substitution applied to utility can use the formula

$$u(x, y) = (ax^r + by^r)^{1/r} \text{ where } -\infty < r < 1 \text{ and } r \neq 0$$

Marginal rate of substitution (MRS) is computed by

$$MRS = -\frac{a}{b} \left( \frac{x}{y} \right)^{r-1}$$

The demand functions are computed

$$x(p_x, p_y, I) = \frac{p_x^{1/(r-1)}}{p_x^{r/(r-1)} + p_y^{r/(r-1)}} \cdot I$$

$$y(p_x, p_y, I) = \frac{p_y^{1/(r-1)}}{p_x^{r/(r-1)} + p_y^{r/(r-1)}} \cdot I$$

where  $(p_x, p_y, I)$  are price of good x, price of good y and income.  
Conquences of variations of  $r$ :

- If  $r \rightarrow 0$  then  $u(x, y) \rightarrow$  Cobb Douglas Utility Function

$$u(x, y) = x^a y^{1-a}$$

- If  $r \rightarrow -\infty$  then  $u(x, y) \rightarrow$  Leontief utility Function (inputs are perfect complements)

$$u(x, y) = \text{Min}(ay, bx)$$

- If  $r \rightarrow 1$  then  $u(x, y) \rightarrow$  Linear Production (inputs are perfect substitutes)

$$u(x, y) = ay + bx$$

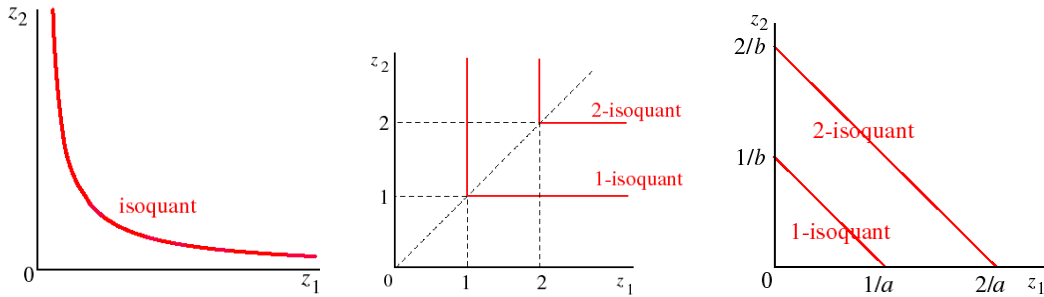


Figure 1: Utility Curves

## Quasilinear Utility Functions

Utility function that is independent of the income effect.

$$u(x, y) = v(x) + y$$

where  $v$  is an arbitrary function that is strictly increasing if good  $x$  is desired.

Indifference curve for  $\alpha$  utility level:

$$v(x) + y = \alpha$$

$$y = \alpha - v(x)$$

Marginal rate of substitution is computed by

$$MRS = \frac{\partial u}{\partial x} / \frac{\partial u}{\partial y}$$

$$\text{where } \frac{\partial u}{\partial x} = v'(x) \text{ and } \frac{\partial u}{\partial y} = 1$$

$$\text{Therefore } MRS = v'(x)$$

## **Slutsky decomposition: Income and substitution effects**

Slutsky decomposition is the total effect of substitution and income.

### **Normal Goods**

Normal goods are goods for which demand increases when income increases. In the Slutsky decomposition, the income and substitution effect reinforce each other when the good's price changes.

Income and substitution effects cause an increase in demand when prices decrease.

### **Income Inferior Goods**

Demand reduced with higher income.

Substitution and Income effects oppose each other. When income increases, demand decreases, the substitution effect is the same as normal goods.

### **Giffon Goods**

Extreme income-inferiority, the income effect may be larger in size than the substitution effect, causing quantity demanded to fall as own-prices rise.

Slutsky's decomposition of the effect of a price change into a pure substitution effect and an income effect thus explains why the law of downwards-sloping demand is violated for extremely income-inferior goods.