Real and Functional Analysis

Master Degree Program in Mathematical Engineering
a.y. 2021-2022, group M-Z

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Topics (tentative detailed description)

(*) theorem with proof; (**) theorem with a partial proof

1. *Set Theory.* Power set, collection of sets, sequence of sets; lim inf, lim sup, lim of sequence of sets, cover and subcover of sets, characteristic function of a set. Relations, equivalence and order relations. Equipotent sets, cardinality, Schröder-Bernstein theorem, Cantor theorem. Infinite and finite sets, countable and uncountable sets, continuum hypothesis. Axiom of choice. Partially ordered sets, chains, maximal element, upper bounds, Zorn's lemma.

2. *Metric and Topological Spaces.* Distance function, metric spaces. Examples: $\mathbb{R}^n$, ℓ^p, $C^k([a, b])$. Balls, interior, boundary and exterior points, open and closed sets, accumulation points (or cluster points), closure of a set. Sequences, Cauchy sequences, convergent sequences. Completeness, completion of a metric space. Theorem of nested balls. Functions (or mapping) between two metric spaces, continuous functions, Lipschitz functions. Dense set, separable spaces. Nowhere dense set. Sets of first and second category. Baire's category theorem (*), corollary (*). Compact metric spaces, sequentially compact metric spaces, totally bounded metric spaces. Characterization of compact metric spaces. Ascoli-Arzelà theorem (*), corollary (*). Introduction to topological spaces.

3. *Measure.* Algebra, $\sigma-$algebra, measurable space, measurable sets, $\sigma-$algebra generated by a set, Borel $\sigma-$algebra, Borel sets, generation of $\mathcal{B}(\mathbb{R})$ and of $\mathcal{B}(\overline{\mathbb{R}})$. Measure, finite measure, $\sigma-$finite measure, measure space. Properties of a measure (additivity, monotonicity, $\sigma-$additivity, continuity) (*), Borel-Cantelli Lemma (*). Sets of zero measure, negligible sets, properties true almost everywhere. Complete measure space, completion of a measure space (**). Outer measure μ^*. Generation of an outer measure (*). Carathéodory's condition and μ^*-measurable sets, its equivalent form (*). If a set has zero outer measure, then it is μ^*-measurable (*). Measure induced by an outer measure. Lebesgue measure in $\mathbb{R}$. Every countable set is Lebesgue-measurable (*), every Borel set is Lebesgue measurable (*), the translate of a measurable set is measurable (*). Excision property (*). Regularity of the Lebesgue measure (**). The ternary Cantor set and its properties. Every measurable set of positive measure contains a non-measurable subset (Vitali set) (*). Lebesgue measure in $\mathbb{R}^N$.

4. *Measurable functions.* Measurable functions between two measurable spaces. Measurability of the composite function (*), equivalent notion of measurable functions (*), Lebesgue measurable functions, Borel measurable functions. Every continuous function is both Borel and Lebesgue measurable (*). Measurability for real valued functions (*), measurability of $\sup_n f_n$, $\inf_n f_n$, $\max\{f, g\}$, $\min\{f, g\}$, $\limsup_n f_n$, $\liminf_n f_n$ (*). Measurability of $f + g, fg$ (*). Measurability of $f^{\pm}$ and of $|f|$ (*). Simple functions, step functions. The simple approximation theorem (**); $\operatorname{esssup} f, \operatorname{essinf} f$, their properties (*), essentially bounded functions, $\mathcal{L}^\infty$. The Vitali-Lebesgue function. Existence of a Lebesgue measurable set which is not a Borel set (*), $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is not complete (*).

5. *The Lebesgue Integral.* Integral of nonnegative simple functions, properties. Measure defined by the integral of a nonnegative simple function. Integral of nonnegative measurable functions, properties. Chebychev's inequality (*). Finite integral and a.e. finite functions (*). Vanishing lemma for nonnegative measurable functions (*). Monotone convergence theorem (*). Fatou's lemma (*). Integration of series of nonnegative functions (*). Measure defined by the integral of a nonnegative measurable function (*). Integrals and sets of zero measure (*). Integrable functions, the Lebesgue integral. The set $\mathcal{L}^1$. Membership to $\mathcal{L}^1$ of $f, f^{\pm}, |f|$ (*). $\mathcal{L}^1$ is a vector space (*). Vanishing lemma for $\mathcal{L}^1$ functions (*). Functions equal a.e. have the same integral (*). Lebesgue's dominated convergence theorem (*), its alternative form (*). Integration of series (*). Comparison between Riemann and Lebesgue integrals. Radon-Nikodym's theorem. Uniqueness of the derivative of a measure (*), properties of the derivative. The spaces L^1 and L^∞. L^1 and L^∞ are metric spaces (*).

6., 7. *Types of Convergence and Product Measure.* Various types of convergence for sequence of functions: point-wise convergence, uniform convergence, a.e. convergence, convergence in L^1 and in L^∞, convergence in measure. Two equivalent formulations of convergence a.e. (*). Convergence in measure implies convergence a.e. up to subsequences. On space of finite measure, convergence a.e. implies convergence in measure (*), the converse implication is not true. Convergence in L^1 implies convergence in measure (*), the converse implication is not true. Almost uniform convergence, the Severini-Egorov theorem. Product measure space. $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$ is the completion of $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$. Tonelli theorem, Fubini theorem.

8. *Functions of Bounded Variations and Absolutely Continuous Functions.* Lebesgue points. First fundamental theorem of calculus (*), total variation of a function, functions of bounded variation, the space $BV([a,b])$. Properties of the total variation, positive and negative variations. Characterization of BV functions (Jordan decomposition) (*). Monotone functions are differentiable a.e. , properties of the derivative of monotone functions (*). Absolutely continuous functions, the space AC. Absolute continuity of the integral (*), the integral function is absolutely continuous (*). Characterization of AC functions (**). Second fundamental theorem of calculus (*). Integration by parts, change of variables.

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1. *Banach Spaces.* Normed spaces. Sequences, bounded sequences, Cauchy sequences, convergent sequences. Series. Schauder basis. Dense set, separable space. Stone-Weierstrass theorem. $C^0([a,b])$ is separable (*). Completeness, Banach spaces. Examples: $\mathbb{R}^N, C^k([a,b]), L^1$, L^∞, ℓ^p, $BV([a,b])$, $AC([a,b])$. Completeness and convergence of series (*). Compactness. Riesz lemma (*), Riesz theorem (*). Equivalent norms, all norms are equivalent when the dimension is finite (*).

2. *Lebesgue Spaces.* The sets $\mathcal{L}^p$ and L^p, L^p is a vector space (*), Young's inequality (*), Hölder's inequality (*), Minkowski inequality (*). Inclusion of L^p spaces (*), L^p is a Banach space (Riesz-Fisher theorem) (*). Convergence in L^p implies convergence in measure (*). Lusin theorem. Simple functions with support of finite measure are dense in $L^p(\mathbb{R})$ for $p \in [1, +\infty)$ (*), $C^0_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $p \in [1, +\infty)$ (*), L^p is separable for $p \in [1, +\infty)$ (*), $L^\infty(\mathbb{R})$ is not separable (*). ℓ^p spaces and their main properties.

3. *Linear Operators.* Linear operators, bounded operators, continuous operators. Characterization of linear bounded operators (*). $\mathcal{L}(X,Y)$, operator norm, equivalent definitions (*), $\mathcal{L}(X,Y)$ is a Banach space (*). Isometries, injections. Banach-Steinhaus theorem (or uniform boundedness principle) (*), corollary (*). Open mapping theorem ~~(*)~~, bounded inverse mapping theorem (*), corollary (*). Closed operators, closed operators and closed graph (*), closed graph theorem (*).

4. *Duality and Reflexivity.* Dual space of a normed space. Isomophism between two Banach spaces and their dual spaces. Characterization of linear functionals (*). Duality of L^p spaces for $p \in [1, +\infty)$. Sublinear functionals, Hahn-Banach theorem (dominated extension form) (*), Hahn-Banach theorem (continuous extension) (*), corollaries (*). Dual space and separability (*). The dual of $L^\infty(\mathbb{R}^N)$ is not $L^1(\mathbb{R}^N)$ (*). Bidual space, canonical embedding. The canonical embedding is linear and preserves the norm (*). Reflexive spaces. James theorem (*). Properties of reflexive spaces (*). Uniformly convex Banach spaces, Milman-Pettis theorem.

5, 6 *Weak Convergence and Compact Operators.* Weak convergence in Banach spaces. Strong convergence implies weak convergence (*), the weak limit is unique (*), weak convergent sequences are bounded (*). Lower semicontinuity of the norm w.r.t. weak convergence (*), convergence of the composition of strongly convergent operators with weakly convergent arguments (*), linear bounded operators are weak-weak continuous (*). A sufficient condition for weak convergence in reflexive spaces (*). Weak* convergence and its properties. Banach-Alaoglu theorem (*), corollary (*). Eberlein-Smulyan theorem. Compact operators. Compact operators are bounded (*). Finite-rank operators. Compact operators and bijectivity (*). Characterization of compact operators (*). $K(X, Y)$ is a Banach space (*).

Hilbert spaces. Spaces with inner product, Cauchy-Schwarz inequality, Hilbert space. Examples: L^2, ℓ^2. Pythagoras theorem, parallelogram identity (*). Convex subsets. Minimal distance theorem from convex closed subsets (*). Orthogonal spaces and their properties (*). Projection theorem (*). Projections are linear and bounded operators (*). The dual of a Hilbert space. Riesz representation theorem (*). Orthonormal bases. Two orthonormal bases are equipotent, orthogonal dimension. Bessel inequality (*), abstract Fourier series and Parseval identity (*). In a separable Hilbert space; an orthonormal basis converges weakly, but not strongly (*). Riesz-Fisher theorem (*).

Linear Operators on Hilbert spaces. Norm of a bounded linear operator on a Hilbert space (*), symmetric operators. Norm of a symmetric operator (*). Eigenvalues, eigenvectors, eigenspaces. Eigenvectors associated to different eigenvalues of a symmetric operator. Eigenvalues of compact symmetric linear operators on Hilbert spaces (*), Eigenspaces of compact symmetric linear operators have finite dimension (*). Spectrum and resolvent. Spectral theorem (*). Fredholm's alternative (*).

Lecture Notes

- M. Grasselli, *Course Lecture Notes, a,y, 2020/21* (slides available on WeBeep)
- V. Pata, *Appunti del Corso di Analisi Reale e Funzionale* (available on WeBeep)
- F. Punzo, *Course Lecture Notes* (handwritten with GoodNotes, available on WeBeep lesson by lesson)

Biblography

Exercises

- M. Muratori, F. Punzo, N. Soave, *Esercizi Svolti di Analisi Reale e Funzionale*, Esculapio (2021)

Theory: some significant references for learning more

- S. C. Bose, *Introduction to Functional Analysis*, Macmillan (1992)
- H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer (2011)
- S. Kesavan, *Functional Analysis*, Hindustan Book Agency (2009)
- A. N. Kolmogorov, S. Fomin, *Elementi di Teoria delle Funzioni e di Analisi Funzionale*, Editori Riuniti (2012)
- H.L. Royden, P. M. Fitzpatrick, *Real Analysis*, Pearson Education (2010)
- A. Tesei, *Istituzioni di Analisi Superiore*, Bollati Boringhieri (1997)