f **Lebesgue** measurable $\Leftarrow$ $f=g$ a.e. and g is continuous

f **Borel** measurable $\Leftarrow$ $f(x) = \begin{cases} f_1(x) & x \in A \\ f_2(x) & x \in B \end{cases}$ measurable if:

$$\left[\underbrace{\{x \in A : f_1(x) > \alpha\}}_{\substack{A \text{ measurable} \\ f_1 \text{ measurable}}} \cup \underbrace{\{x \in B : f_2(x) > \alpha\}}_{\substack{B \text{ measurable} \\ f_2 \text{ measurable}}} \text{ measurable} \right]$$

$$\Longleftrightarrow f(x) = \underbrace{f_1(x)\, \mathbb{1}_A(x)}_{\substack{\text{measurable iff} \\ f_1 \text{ meas.} \\ A \text{ meas.}}} + \underbrace{f_2(x)\, \mathbb{1}_B(x)}_{\substack{\text{measurable iff} \\ f_2 \text{ meas.} \\ B \text{ meas.}}} \text{ measurable}$$

E.g. $f(x) = \begin{cases} 2^x & x \in [0,1] \\ e^x - x^2 & x \notin [0,1] \end{cases} = \begin{cases} f_1(x) & x \in A \\ f_2(x) & x \in B \end{cases} = f_1(x)\, \mathbb{1}_{[0,1]}(x) + f_2(x)\, \mathbb{1}_{\mathbb{R} \setminus [0,1]}(x)$

f_1 is continuous $\Rightarrow$ Borel/Lebesgue measurable
$[0,1] \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$ $\Big\}$ $f_1 \mathbb{1}_A$ meas.

f_2 is continuous $\Rightarrow$ Borel/Lebesgue measurable
$\mathbb{R} \setminus [0,1] \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$ $\Big\}$ $f_2 \mathbb{1}_B$ meas.

$\Big\}$ f meas.

$\mu : \mathcal{A} \to [0, +\infty]$: (1) $\mu(\emptyset) = 0$

(2) finitely additive: $\{E_j\}_{j=1}^n$ disjoint $\Rightarrow \mu(\cup_{j=1}^n E_j) = \sum_{j=1}^n \mu(E_j)$

(3) continuous among increasing sequences:

$\{E_n\}_n \nearrow \;\Rightarrow\; \mu(\cup_{n=1}^\infty E_n) = \lim_{n\to\infty} \mu(E_n)$

$\Longrightarrow$ μ is a measure.

In fact, let $\{A_n\}_n$ be disjoint and let $E_n = \cup_{j=1}^n A_j$. Then: $\{E_n\}_n \nearrow$ and $\cup_n E_n = \cup_n A_n$ $\Longrightarrow$ $\mu(\cup_n A_n) = \mu(\cup_n E_n) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(\cup_{j=1}^n A_j)$

$= \lim_{n\to\infty} \sum_{j=1}^n \mu(A_j) = \sum_{j=1}^\infty \mu(A_j)$

DCT : check that $g \in L^1([a,b])$? $\quad g \in C([a,b]) \Longrightarrow g \in L^1([a,b])$

DCT : $\lim_{n\to\infty} \int_0^1 nx\, e^{-nx}\, dx$?

$h(t) = t e^{-t}$ is continuous and $\lim_{t\to\infty} t e^{-t} = 0$

$\Longrightarrow$ $0 \le h(t) \le M \quad \forall t \ge 0$

$\Longrightarrow$ $0 \le h(nx) = nx\, e^{-nx} \le M \quad \forall x \in [0,1],\ \forall n \in \mathbb{N}$

$\Longrightarrow$ **DCT** : $\lim_{n\to\infty} \int_0^1 nx\, e^{-nx}\, dx = \int_0^1 \lim_{n\to\infty} nx\, e^{-nx}\, dx = 0$

$(g(x) = M \mathbb{1}_{[0,1]} \in L^1([0,1]))$

$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx \Longrightarrow \lim_{n\to\infty} \int_0^1 f_n(x)\, dx + \lim_{n\to\infty} \int_1^\infty f_n(x)\, dx$

$1 + x \le e^x \quad \forall x \ge 0$

$1 - x \le e^{-x} \quad \forall x \ge 0$

$\Longrightarrow$ $(1 + \frac{x}{n})^n \le (e^{\frac{x}{n}})^n = e^x$

$(1 - \frac{x}{n})^n \le e^{-x}$

$\{(1 + \frac{x}{n})^n\}_n \nearrow \quad (\{(1 - \frac{x}{n})^n\}_n$ is **not** $)$

$f_n \to f$ in L^1? $\int_X |f_n - f| \, d\mu$, **not** $\int f_n - f \, d\mu$! check for $f_n - f \geq 0$ (more frequently: $f_n \geq 0$)

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pointwise $\neq$ pointwise **a.e.** WARNING $f_n(x) = \cos(x^n) \to \begin{cases} 1 & x \in [0,1) \\ \cos(1) & x = 1 \end{cases}$ pointwise

$\to 1$ pointwise a.e. in $[0,1]$

$f_n \to f$ in L^1? Try to $\int_X |f_n - f| \, d\mu$ + **DCT**

E.g. $f_n(x) = \cos(x^n)$ $x \in [0,1]$, $f(x) = 1 \Rightarrow \int_0^1 |\cos(x^n) - 1| \, dx$:

$|\cos(x^n) - 1| \leq 2 \quad \forall x \in [0,1], \forall n \in \mathbb{N}$

and $2 \in L^1([0,1]) \Rightarrow$ DCT

$\lim_{n\to\infty} \int_0^1 |f_n - f| dx = \int_0^1 \lim_{n\to\infty} |f_n - f| dx = 0$

$f_n(x)$ $x \in \mathbb{R} \Rightarrow x < 0, x = 0, x > 0$ (**WARNING**: maybe $x < 1, -1 \leq x \leq 1, x > 1$)

E.g. $f_n(x) = n \mathbb{1}_{[0,\frac{1}{n}]}(x)$ $x \in \mathbb{R} \Rightarrow$ $x < 0 \Rightarrow f_n \to 0$ $(f_n(x) = 0 \ \forall x < 0)$

$x = 0 \Rightarrow f_n(x) = n \to \infty$

$x > 0 \Rightarrow \forall x > 0 \ \exists \bar{n} \in \mathbb{N}: \frac{1}{n} < x \quad \forall n \geq \bar{n}$

and so: $f_n(x) = 0 \quad \forall n \geq \bar{n}$

and so: $f_n \to 0 \quad \forall x > 0$

E.g. convergence in **measure** (if $f_n \not\to f$ in L^1)

$f_n(x) = n \mathbb{1}_{[0,\frac{1}{n}]}(x)$ $x \in \mathbb{R} \Rightarrow$ $E_n := \{|f_n(x) - f(x)| \geq \varepsilon\}$ ($\varepsilon > 0$ fixed)

$= \{|f_n(x)| \geq \varepsilon\}$

$\subseteq [0, \frac{1}{n}]$

$\Rightarrow \lambda(E_n) \leq \lambda([0, \frac{1}{n}]) = \frac{1}{n} \to 0$

$\frac{1}{n^2} \mathbb{1}_{[1,n]}(x) \leq \frac{1}{x^2} \mathbb{1}_{[1,n]}(x) \leq \frac{1}{x^2} \in L^1((1,\infty))$

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f monotone? $\Rightarrow$ **E.g.** $f(x) = \mathbb{1}_{[0,\frac{1}{2}]}(x)$ $x \in [0,1]$:

forall $0 \leq x \leq y \leq 1$:

- $0 \leq x \leq y \leq \frac{1}{2}$: $f(x) = f(y) = 1 \Rightarrow f(x) \geq f(y)$
- $0 \leq x \leq \frac{1}{2} < y \leq 1$: $f(x) = 1, f(y) = 0 \Rightarrow f(x) \geq f(y)$
- $\frac{1}{2} < x \leq y \leq 1$: $f(x) = f(y) = 0 \Rightarrow f(x) \geq f(y)$

$f \in AC([a,b])$: $f' \in L^1([a,b])$?

If it is not easy to do "$\leq$" with an L^1 function, remember that:

$f \in C([a,b]) \Rightarrow \exists M > 0 : 0 \leq f(x) \leq M$

$\Rightarrow$ If $f'(x) = f(x) \cdot g(x)$ for some $g(\cdot)$:

$|f(x) g(x)| = |f'(x)| \leq M \cdot |g(x)|$ $(\Rightarrow g \in L^1([a,b])$?$)$

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f integrable in $[a,b]$? $f \in C([a,b]) \Rightarrow f \in L^1([a,b])$

$f_n \to f$ in measure? $[f_n \to f$ a.e. , $\mu(X) < \infty] \Rightarrow f_n \to f$ in measure

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(**E.g.**) $F_\alpha(x) = \sum_{k=1}^\infty \frac{1}{x^\alpha + k^\alpha}$ $x \geq$: $\alpha > 1$ for which $F_\alpha \in L^1([0,\infty))$?

Since $\frac{1}{x^\alpha + k^\alpha} \in \mathcal{M}_+ \Rightarrow \int_0^\infty F_\alpha \, d\mu = \sum_{k=1}^\infty \int_0^\infty \frac{1}{x^\alpha + k^\alpha} dx = \sum_{k=1}^\infty \frac{k}{k^\alpha} \underbrace{\int_0^\infty \frac{1}{1+y^\alpha} dy}_{:= C_\alpha < \infty}$

$\Rightarrow \int_0^\infty F_\alpha \, d\mu = \sum_{k=1}^\infty \frac{C_\alpha}{k^{\alpha-1}} < \infty \iff \alpha - 1 > 1$

$\alpha > 2$

$f_n \in L^1$? First f_n must be measurable, then $|f_n| \le g \in L^1$

$f_n \to f$ in L^1? It's enough that $|f_n| \le g \in L^1 \xRightarrow{DCT} \int_X |f_n - f| \, d\mu \to 0$

$x_n \to 0$ in ℓ^2 ? **E.g.** $x_n : x_n^{(k)} = \frac{1}{n+k} \Rightarrow \|x_n\|_2^2 = \sum_{k=1}^{\infty} \left|\frac{1}{n+k}\right|^2 = \sum_{k=n+1}^{\infty} \left|\frac{1}{k}\right|^2$

$$\Rightarrow \lim_{n\to\infty} \|x_n\|_2^2 = \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \left|\frac{1}{k}\right|^2 = \lim_{n\to\infty} \sum_{k=1}^{\infty} \left|\frac{1}{k}\right|^2 - \sum_{k=1}^{n} \left|\frac{1}{k}\right|^2 = 0$$

$f_n \to f$ in L^p ? f_n must be L^p $\forall n$, f must be L^p **!!** $f \in C([a,b]) \Rightarrow f \in L^p$

$f_n \to 0$ in $L^\infty([a,b])$? If $f_n \in C([a,b]) \Rightarrow \|f_n\|_\infty = \operatorname{ess\,sup}_{x\in[a,b]} |f_n| = \sup_{x\in[a,b]} |f_n| \geq \dots \to 0$

only because $f \in C([a,b])$, $[a,b]$ compact

$T : X \to Y$ **injective** $\Leftrightarrow [\forall x,y \in X : x \neq y \Rightarrow T(x) \neq T(y)]$

$\Leftrightarrow [\forall x,y \in X : T(x) = T(y) \Rightarrow x = y]$

E.g. $T : L^\infty([0,1]) \to L^\infty([0,1])$: $T(f) = e^{-x} \int_0^x e^y f(y)\, dy$ injective?

Let $f, g \in L^\infty([0,1])$ be such that $T(f) = T(g)$

$\Rightarrow e^{-x} \int_0^x e^y f(y)\, dy = e^{-x} \int_0^x e^y g(y)\, dy$

$\Rightarrow \int_0^x e^y f(y)\, dy = \int_0^x e^y g(y)\, dy$ $\quad e^{-x} \geq 0$

since $\lambda([0,1]) = 1 < \infty \Rightarrow L^\infty([0,1]) \subseteq L^1([0,1])$

$\Rightarrow f, g \in L^1([0,1])$

since $e^y \in C([0,1]) \Rightarrow e^y$ bounded in $[0,1]$

$\Rightarrow f(y)e^y,\ g(y)e^y \in L^1$

$\Rightarrow$ **1FTC** :

$f \in L^1$: $F(x) = \int_0^x f(t)\,dt$ is differentiable a.e. and $F' = f$ a.e.

$\int_0^x e^y f(y)\, dy$, $\int_0^x e^y g(y)\, dy$ diff. a.e. and :

$e^y f(y) = \left(\int_0^x e^y f(y)\,dy\right)' = \left(\int_0^x e^y g(y)\,dy\right)' = e^y g(y)$

$\Rightarrow f(y) = g(y)$ a.e.

$f_n \rightharpoonup 0$ in $L^p((0,\infty))$? $(p > 1,\ p \neq \infty)$

E.g. $f_n(x) = \mathbb{1}_{[n,n+1]}(x)$

$f_n \rightharpoonup 0$ in $L^p \Leftrightarrow \int_0^\infty f_n g \, d\lambda \to 0 \quad \forall g \in L^q$

$$\left|\int_0^\infty f_n g\, dx\right| \leq \int_0^\infty |f_n g|\, dx = \int_n^{n+1} |g|\, dx \leq \left(\int_n^{n+1} 1^p dx\right)^{1/p} \left(\int_n^{n+1} |g|^q dx\right)^{1/q} = 1 \cdot \left(\int_0^\infty |g|^q \mathbb{1}_{[n,n+1]}(x)\, dx\right)^{1/q}$$

Hölder

- $|g|^q \mathbb{1}_{[n,n+1]}(x) \longrightarrow 0$
- $\left||g|^q \mathbb{1}_{[n,n+1]}(x)\right| \leq |g|^q \in L^1$

$\Rightarrow$ **DCT** :

$$\lim_{n\to\infty} \int_0^\infty |g|^q \mathbb{1}_{[n,n+1]}\, dx = \int_0^\infty \lim_{n\to\infty} |g|^q \mathbb{1}_{[n,n+1]}\, dx = 0$$

$$\Rightarrow \lim_{n\to\infty} \int_0^\infty f_n g\, dx = 0$$

$f_n \not\rightharpoonup 0$ in $L^1((0,\infty))$? $\quad g \equiv 1 \Rightarrow \int_0^{+\infty} \mathbb{1}_{[n,n+1]} \cdot 1\, dx = 1 \not\to 0$

$\|T_N - T\|_2$: $\quad (T_N(x))^{(n)} = \begin{cases} \frac{n}{1+n^2} x^{(n)} & n \leq N \\ 0 & n > N \end{cases}$

$$\|T_N(x) - T(x)\|_2^2 = \sum_{n=N+1}^{\infty} \left|\frac{n}{n^2+1} x^{(n)}\right|^2 \leq \left(\sup_{n \geq N+1}\left(\frac{n}{n^2+1}\right)\right) \|x\|_2^2$$

$$\lim_{N\to\infty} \|T_N - T\|_2 = \lim_{N\to\infty}\left(\sup_{n\geq N+1}\left(\frac{n}{n^2+1}\right)\right) = \lim_{N\to\infty} \frac{N}{N^2+1} = 0$$
