Exam questions I – Discrete Optimization

Problem 1

A company must optimize its telecommunication network. Let V be the set of nodes and A the set of links (arcs). Let (s_k, t_k), with $k \in K$, denote the $|K|$ origin-destination pairs that requested the service and $d_k \geq 0$ the amount of data (in GB) to be sent from node s_k to node t_k. The network capacity, which is initially zero, can be increased by installing (activating) on each arc appropriate communication devices. Each such device has a capacity of 1 GB and a cost of c_{ij}, for each arc $(i, j) \in A$. Let u_{ij} be the maximum amount of capacity (in GB) that can be installed (activated) on each arc $(i, j) \in A$. Finally let s_{ij} be the cost for routing on arc $(i, j) \in A$ one unit amount of data.

a) Assume that, for each origin-destination pair (s_k, t_k), the data must be routed along a single path (to avoid delay issues at destination). Give a mixed integer linear programming formulation for the problem of determining how to install the capacity on the arcs of the network and how to route the demands so as to minimize the total routing and installation costs, while satisfying the demands of all origin-destination pairs.

b) To protect the network from failures, we require that there exist at least two link-disjoint paths, using only arcs with non zero capacity, from each origin s_k to each destination t_k, namely, the graph composed of arcs with non zero capacity must be biconnected. How can we modify the formulation to account for this robustness requirement? What is the size of the resulting formulation?

c) How can we extend the formulation of point b) to account for at least two node-disjoint paths?

Problem 2

Suppose that three products with given demands d_i, with $i = 1, \ldots, 3$, have to be loaded onto a tanker. The tanker has five cargo holes and products cannot be blended into a cargo hole. The following table indicates the capacities k_j, with $j = 1, \ldots, 5$, of the five cargo holes.

Cargo hole	1	2	3	4	5
Capacity k_j (ton)	5400	5600	2200	1800	3400

All demands need not be delivered. But in case a demand is not delivered a penalty cost p_i, with $i = 1, \ldots, 3$, per ton must be paid. The following table indicates for each product the demands d_i, the maximum amount not delivered m_i and the penalty cost p_i per ton.

Product	Demand d_i (ton)	Max non-deliveries m_i (ton)	Penalty cost p_i (per ton)
91-octane	4500	800	0.5
95-octane	6000	1800	0.4
96-octane	6800	1000	0.3

Give an mixed integer linear programming formulation for the problem of deciding how to allocate the products to the cargo holes so as to minimize the penalty costs while respecting the loading constraints.

Problem 3

Consider the minimum cost flow problem: given a directed graph $G = (V, A)$ with a capacity k_{ij} and a unit cost c_{ij} associated to each arc $(i, j) \in A$ and a demand/availability b_i for each node $i \in V$ (b_i positive for sources, negative for destinations, zero for intermediate nodes), determine a feasible flow of minimum total cost which satisfies all the demands. To guarantee feasibility, we assume that $\sum_{i \in V} b_i = 0$.

- Give an integer linear programming formulation for the problem.
- Show that it is an ideal formulation, after clearly explaining the meaning of such a statement.
- Explain why the shortest path problem (Given a directed graph $G = (V, A)$ with a cost c_{ij} for each arc $(i, j) \in A$, and two prescribed nodes s and t, determine a path of minimum toal cost from s to t) is a special case of the minimum cost flow problem.
- Explain why the the maximum flow problem (Given a directed graph $G = (V, A)$ with a capacity u_{ij} for each arc $(i, j) \in A$, a source s and a sink t, determine a feasible flow of maximum value from s to t) is also a special case of the minimum cost flow problem.

Problem 4

Consider the Symmetric Traveling Salesman Problem: given an undirected graph $G = (V, E)$ with a cost c_e associated to each edge $e \in E$, determine an Hamiltonian cycle, i.e., a cycle which visits each node exactly once, of minimum total cost.

- Give two integer linear programming formulations for the problem and indicate the size of the formulation in terms of the number of nodes in the graph.
- Which relation exists between the linear relaxations of these two formulations? Explain and motivate your answer.

Problem 5

Consider the feasible region of a generic binary knapsack problem, namely $X = \{\mathbf{x} \in \{0, 1\}^n : \sum_{j=1}^{n} a_j x_j \leq b\}$ where all coefficients a_j e b are positive.

- State the definition of a cover inequality and explain why it is a valid inequality for X.
- Describe the separation problem for the cover inequalities and explain how it can be solved.

- Consider the specific feasible region $X = \{\mathbf{x} \in \{0,1\}^6 \; : \; 12x_1 + 9x_2 + 7x_3 + 5x_4 + 5x_5 + 3x_6 \leq 14\}$, list all the minimal cover inequalities that are valid for X, and apply the lifting procedure to the cover inequality contained the variables x_3, x_5 and x_6.

Problem 6

Consider a set of n candidate sites where a depot can be open and a set of m clients distributed across a given area. Suppose there is a fixed opening cost f_j for each candidate site j, with $1 \leq j \leq n$, and a profit c_{ij} when the whole demand of client i is satisfied by the depot located in candidate site j, with $1 \leq i \leq m$ and $1 \leq j \leq n$. In the Uncapacitated Facility Location problem (UFL) variant considered here, we have to decide in which candidate sites to open depots and how to satisfy the demand of each client so as to maximize the total profit minus the total fixed costs.

- Give two integer linear programming formulations for the problem.
- Indicate which formulation is stronger (provides a tighter bound) and motivate your answer.
- Describe in detail the Lagrangian relaxation for this problem where the demand constraints are relaxed. Clearly explain how the Lagrangian subproblem and Lagrangian dual can be solved.
- Apply the Lagrangian relaxation method to the instance with $m = 6$ clients, $n = 5$ locations, the fixed location costs $f = (2, 4, 5, 3, 3)$, and the client-location profit matrix

$$(c_{5ij}) = \begin{pmatrix} 6 & 2 & 1 & 3 & 5 \\ 4 & 10 & 2 & 6 & 1 \\ 3 & 2 & 4 & 1 & 3 \\ 2 & 0 & 4 & 1 & 4 \\ 1 & 8 & 6 & 2 & 5 \\ 3 & 2 & 4 & 8 & 1 \end{pmatrix}.$$

Start with the multiplier vector $\mathbf{u}_0^t = (5, 6, 3, 2, 6, 4)$.

Problem 7

Describe the general idea of the cutting plane methods for Integer Linear Programming problems. Illustrate with an example: describe a specific problem and at least one class of valid inequalities for it. Indicate how it is possible to combine the generation of cutting planes and the Branch-and-Bound method. Explain which are the main advantages of such a Branch-and-Cut approach.

Problem 8

Describe the Lagrangian relaxation method for Integer Linear Programming problems. Illustrate with an example. State the central result about the strength of the bound obtained by solving the Lagrangian dual problem, and describe how the Lagrangian dual problem can be solved.

Problem 9

Similar open questions concerning the other methods covered in the course...

Exam questions – Nonlinear Optimization

Problem 1

The LIGO company produces 1000 types of different pieces ($1 \leq j \leq 1000$) and with these pieces creates 100 types of different boxes ($1 \leq i \leq 100$), which are sold in toy stores. The following information is available: the number a_{ij} of pieces of type j in each box of type i, the unit cost c_j for producing the pieces of type j, the maximum production capacity m_j for pieces of type j, and the unit selling price p_i of each box of type i. There is a discount on the selling price proportional to the amount of boxes sold (with an appropriate proportionality constant α) for orders up to 2000 boxes, and a fixed cost (of 30%) for the order exceeding 2000 boxes. Suppose that LIGO can sell all produced boxes.
Give a nonlinear optimization model to plan the production so as to maximize the total profit (selling price minus costs).

Problem 2

Consider the nonlinear optimization problem

$$\begin{array}{rl} \min & x_1 + x_2 \\ \text{s.t.} & x_1^2 + x_2^2 \leq 4 \\ & x_1^2 + x_2^2 \geq 1 \\ & x_1, x_2 \in \mathbb{R} \end{array}$$

- establish whether it is a convex optimization problem,
- draw the feasible region,
- establish whether the constraint qualification assumption is satisfied at all points of the feasible region,
- solve the problem by determining all the candidate points satisfying the first order optimality conditions,
- give the Lagrangian function associated to the problem.

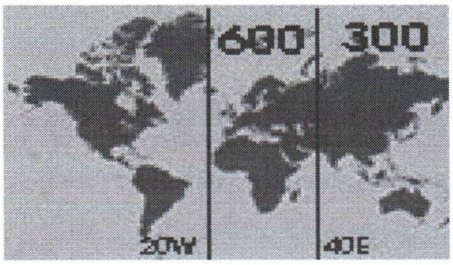
All answers must be well motivated.

Problem 3

Describe the gradient method for unconstrained optimization problems, and provide an illustarive example. Discuss the convergence properties of the method, its advantages and disadvantages. Briefly describe one type of approach that has been proposed to circumvent the above-mentioned disadvantages.

Problem 4

The AirB company must build 5 airline maintenance centers that serve the Euro-Asian area. The cost to build each center is equal to 300 millions euro in Europe (between 20°W and 40°E) and to 150 millions euro in Asia (between 40°E and 160°E).



Each center can serve up to 60 airplanes per year. The centers must serve the airports where most of the clients of AirB are concentrated, as detailed in the following table

Aeroporto	Coordinates		N. airplanes
London Heathrow	51°N	0°W	30
Frankfurt	51°N	8°E	35
Lisboa	38°N	9°W	12
Zürich	47°N	8°E	18
Roma Fiumicino	41°N	12°E	13
Abu Dhabi	24°N	54°E	8
Moskva Sheremetyevo	55°N	37°E	15
Vladivostok	43°N	132°E	7
Sydney	33°S	151°E	32
Tokyo	35°N	139°E	40
Johannesburg	26°S	28°E	11
New Dehli	28°N	77°E	20

which reports, for each airport, the geographic coordinates (latitude δ_i and longitude φ_i) and the expected number n_i of airplanes per year needing maintenance.

The total cost of a maintenance center is given by the construction cost plus the expected service cost. The service cost of each airplane depends linearly on the distance that it has to travel to reach the maintenance center, with a proportionality constant c euro/Km. Assume that Earth is a perfect sphere and that the shortest path between two points specified by the geographic coordinates (δ_i, φ_i) and (δ_j, φ_j) is given by:

$$d(\delta_i, \varphi_i, \delta_j, \varphi_j) = 2r \text{ asin} \sqrt{\sin^2\left(\frac{\delta_i - \delta_j}{2}\right) + \cos\delta_1 \cos\delta_2 \sin^2\left(\frac{\varphi_i - \varphi_j}{2}\right)},$$

where r is Earth radius, namely 6371Km.

Give a nonlinear optimization model to decide where to locate the 5 maintenance centers and how to distribute the airplanes among these centers so as to minimize the overall costs.

Problem 5

Consider the nonlinear optimization problem

$$\begin{array}{ll} \min & 2x_1^2 + 6x_1x_2 + x_2^2 \\ \text{s.t.} & 3x_1 + x_2 \geq 4 \\ & x_1 + x_2 \geq 12 \\ & x_1, x_2 \in \mathbb{R} \end{array}$$

- draw the feasible region,
- establish whether the constraint qualification assumption is satisfied at all points of the feasible region,
- determine all the points satisfying the Karush-Kuhn-Tucker conditions,
- indicate and explain why these conditions are necessary and/or sufficient,
- write the Lagrangian dual problem.

Problem 6

An industry must build a cylindrical silos (bin) to be positioned on one of its bases in a storage building. The storage bulding has a rectangular floor plan of size 20×10 meters and has a roof leaning along the side of 10 meters, which has a maximum height of 5 meters and a minimum height of 3 meters. The silos must be built with a thin and flexible plastic material that can be cut, modelled and firmly stuck. 200 m^2 of such material are available.

Give a nonlinear optimization formulation for the problem of determining the dimensions of the silos so as to maximize the amount of liquid that it can contain.

Problem 7

Consider the optimization problem

$$\begin{array}{ll} \min & (x_1 - \frac{9}{4})^2 + (x_2 - 2)^2 \\ \text{s.t.} & -x_1^2 + x_2 \geq 0 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{array}$$

- Draw the feasible region.
- Determine graphically the global minimum (optimal solution) and the global maximum.
- At which points of the feasible region is the constraint qualification assumption satisfied? Motivate the answer.
- Determine all the points satisfying the Karush-Kuhn-Tucker conditions.
- Indicate whether the above optimality conditions are necessary and/or sufficient. Explain why.

- Write the Lagrangian function of the problem.

Problem 8

An international company selling a single type of product must decide how to serve n markets, specified by the coordinates (a_j, b_j) and the demand r_j with $1 \le j \le n$, from m depots with known capacity c_i, $1 \le i \le m$. For logistic reasons each depot can serve at most k markets, with $k < n$. The service cost of a market j from a depot i is proportional not only to the distance between i and j but also to the amount of product transported.

Give a mathematical programming model for the problem of determining where to locate the depots and how to serve the markets so as to minimize the total costs while satisfying all demands.

Problem 9

Consider the following optimization problem

$$\begin{array}{rl} \min & x_1^3 + x_2^2 \\ \text{s.t.} & x_1^2 + x_2^2 = 9 \\ & x_1, x_2 \ge 0, \end{array}$$

- indicate at which points of the feasible region the constraint qualification assumption is satisfied,
- determine the points satisfying the Karush-Kuhn-Tucker optimality conditions,
- indicate whether these conditions are necessary and/or sufficient,
- write the dual problem,
- state the main properties of the dual and describe how it can be solved.

The answers must be well motivated.

Problem 10

Describe the quadratic penalty method for constrained nonlinear optimization problems, and provide an illustrative example. State and discuss the main properties of this method and the limitations when the penalty terms are simply added to the objectve function. Briefly describe the approach that has been proposed to circumvent such limitations.

Problem 11

Similar (open) questions concerning the other methods for unconstrained or constrained nonlinear optimization problems covered in the course...