Stochastic dynamical models

July 8^{th}, 2020

- Pocket calculators without wifi connection function are allowed. Pocket calculators with wifi connection function are NOT allowed, even if set in airplane mode.
- The teacher's or your own lecture notes are allowed, other notes are not.
- You can not ask questions. If you think that there is an error or something ambiguous, explain why and write your solution accordingly.
- Solutions can be written either in English or Italian (also French and Spanish).

Exercises

Exercise 1. A post office has two counters. The number of customers in the office is modeled by a discrete time Markov chain $(X_n)_{n\geq 0}$ with set of states $\mathbb{N}$. More precisely, states are:

0 No customer in the office,
1 Only a customer being served at a counter,
2 Two customers both being served at a counter,
m ($m \geq 3$) Two customers both being served at a counter and $m-2$ in a queue.

Suppose that:

- If a counter is busy at time k, with probability $1/2$ it will finish the job and call another customer from the queue (if any),
- The number of new customers entering in the office at any time is a random variable with Bernoulli distribution $B(1,p)$ ($0 < p < 1$),
- The number of customers served at each of the two counters and the number of new customers arriving at any time are independent random variables.

(1) Write the transition matrix of the Markov chain $(X_n)_{n\geq 0}$,
(2) Classify states, find classes and establish if they are recurrent or transient.
(3) Suppose $X_0 = 0$, how much time is needed, on average, before there are at least 3 customers in the office?
mhm (4) Compute the probability that one finds both counters are busy when arriving in the system, namely $\lim_{n\to\infty} \mathbb{P}\{X_n \geq 2\}$.
(5) Suppose that I enter in the office and find $N \geq 1$ people waiting in the queue (namely, counters are busy and I am the $N+1$ customer waiting for a free counter). Let $(Y_n)_{n\geq 0}$ be the Markov chain defined by Y_n = my position in the queue at time n (0 = I have been called to a counter; 1 = I am the next one who will be called to a counter; 2 = there is only one person that will be called to a counter before ...). Write the transition matrix of the Markov chain $(Y_n)_{n\geq 0}$. How much time do I spend in the queue before being called at a counter?

Exercise 2. Let $(X_t)_{t\geq 0}$ be the continuous time Markov chain with state space $\{0,1,2,3\}$ and transition matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

(1) Classify states.
(2) Write the transition matrix P_t at all times $t > 0$
(3) Let $f : \{0,1,2,3\} \to \mathbb{R}$, $f(0) = 1, f(1) = 0,\ f(2) = f(3) = -1$. Show that the process $(M_t)_{t\geq 0}$ defined by

$$M_t := f(X_t) - \int_0^t \mathbb{1}_{\{X_s > 0\}} ds$$

is a martingale.
(4) Compute the mean arrival time in 0 starting from 3 applying the stopping theorem.
(5) Is the martingale $(M_t)_{t\geq 0}$ convergent as $t \to \infty$? In which sense?

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- 2 counters
- X_n = number of customers at time n
- $\mathbb{P}(\text{finish}) = \frac{1}{2}$
- new costumer $\sim Be(p)$

1. For $m \geq 2$:

$$\mathbb{P}(X_{n+1} = m \mid X_n = m) = \mathbb{P}(\text{no arrivals, no finish}) + \mathbb{P}(\text{1 arrival, 1 finish})$$
$$\overset{\perp}{=} (1-p)\left(\tfrac{1}{2}\right)^2 + p\left(\tfrac{1}{2}\right)^2 \cdot 2$$
$$\overset{\perp}{=} \tfrac{1}{4}(1-p) + \tfrac{1}{2}p = \frac{1+p}{4}$$

$$\mathbb{P}(X_{n+1} = m+1 \mid X_n = m) = \mathbb{P}(\text{1 arrival, no finish})$$
$$\overset{\perp}{=} p\left(\tfrac{1}{2}\right)^2 = \frac{p}{4}$$

$$\mathbb{P}(X_{n+1} = m-1 \mid X_n = m) = \mathbb{P}(\text{one arrival, two finish}) + \mathbb{P}(\text{one finish, no arrival})$$
$$\overset{\perp}{=} p\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 \cdot 2 \cdot (1-p)$$
$$\overset{\perp}{=} \tfrac{1}{4}p + \tfrac{1}{2}(1-p) = \frac{2-p}{4}$$

$$\mathbb{P}(X_{n+1} = m-2 \mid X_n = m) = \mathbb{P}(\text{two finish, no arrivals})$$
$$\overset{\perp}{=} \left(\tfrac{1}{2}\right)^2(1-p) = \frac{1-p}{4}$$

For $m = 1$:

$$\mathbb{P}(X_{n+1} = 1 \mid X_n = 1) = \mathbb{P}(\text{no arrivals, no finish}) + \mathbb{P}(\text{one finish, one arrival})$$
$$\overset{\perp}{=} (1-p)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right) \cdot p$$
$$\overset{\perp}{=} \tfrac{1}{2} - \tfrac{1}{2}p + \tfrac{1}{2}p = \frac{1}{2}$$

$$\mathbb{P}(X_{n+1} = 2 \mid X_n = 1) = \mathbb{P}(\text{no finish, one arrival})$$
$$\overset{\perp}{=} \left(\tfrac{1}{2}\right)p = \tfrac{1}{2}p$$

$$\mathbb{P}(X_{n+1} = 0 \mid X_n = 1) = \mathbb{P}(\text{finish, no arrival})$$
$$\overset{\perp}{=} \tfrac{1}{2}(1-p)$$

For $m = 0$:

$$\mathbb{P}(X_{n+1} = 0 \mid X_n = 0) = \mathbb{P}(\text{no arrival}) = (1-p)$$
$$\mathbb{P}(X_{n+1} = 1 \mid X_n = 0) = \mathbb{P}(\text{arrival}) = p$$

$$P = \frac{1}{4}\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \\ \end{array}\begin{bmatrix} 4(1-p) & 4p & 0 & 0 & 0 & \cdots & \cdots & & \cdots \\ 2(1-p) & 2 & 2p & 0 & 0 & \cdots & \cdots & & \cdots \\ 1-p & 2-p & 1+p & p & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1-p & 2-p & 1+p & p & & & & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & \end{bmatrix}$$

2.

The MC is irreducible since all of the states communicate.

The average variation is:

$$\mathbb{E}[X_{n+1} - X_n] = -2\,\mathbb{P}(m \to m-2) - 1\,\mathbb{P}(m \to m-1) + 0\,\mathbb{P}(m \to m) + 1\,\mathbb{P}(m \to m+1)$$

$$\stackrel{!}{=} -2\left(\frac{1-p}{4}\right) - \left(\frac{2-p}{4}\right) + \frac{p}{4}$$

$$\stackrel{!}{=} \frac{-2+2p-2+p+p}{4}$$

$$\stackrel{!}{=} \frac{4p-4}{4} = p-1 \implies$$ we expect the MC to be recurrent

We check it through:

$(X_n)_{n\geq 0}$ irreducible, if $\exists (y_j)_j$ s.t.: $\sum_{k\in E} P_{jk} y_k \leq y_j$, $\lim_{n\to\infty} y_n = +\infty$ then $(X_n)_{n\geq 0}$ is recurrent.

We try with $y_j = j$: $\quad (f(j) = j)$

$$(Pf)(j) = \begin{cases} p & j=0 \\ \frac{1}{2} + p & j=1 \\ j + p - 1 & j \geq 2 \end{cases}$$

and so $\sum_k P_{jk} y_k \leq y_j$ only for $j \geq 2$ (and we can neglect at most one term, not two).

However if we consider $f(j) = j^2$:

$$(Pf)(j) = \begin{cases} p & j=0 \\ \frac{1}{2} + p & j=1 \\ j^2 - 2j(1-p) - p + 3 & j \geq 2 \end{cases}$$

$\implies (Pf)(j) \leq f(j) - g(j)$ with $g(j) = 2j(1-p) + p - 3$ for $j \geq 2$

and $g(j) \to \infty$ as $j \to \infty$

$\implies$ so the MC is recurrent and we proved that is positive recurrent

3. T_3 starting from $X_0 = 0$.

Mean absorption in 3 starting from $i=0$?

$$\begin{cases} w_0 = 1 + (1-p)\,w_0 + p\,w_1 \\ w_1 = 1 + \frac{1}{2}(1-p)\,w_0 + \frac{1}{2}w_1 + \frac{1}{2}p\,w_2 \\ w_2 = 1 + \frac{1}{4}(1-p)\,w_0 + \frac{1}{4}(2-p)\,w_1 + \frac{1}{4}(1+p)\,w_2 + \frac{1}{4}p\,w_3 \\ w_3 = 0 \end{cases}$$

and we stop here since we go to 3 the first time (starting from 0) only from 0, 1, 2, so we don't need w_j $j > 3$

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3. $\Longrightarrow w_0 = \frac{3(1+p+p^2)}{p^3}$, $w_1 = \frac{3+3p+2p^2}{p^3}$, $w_2 = \frac{3+2p+p^2}{p^3}$

starting from 0, w_0 is the mean absorption

4. $\lim_{n\to\infty} \mathbb{P}(X_n \geq 2)$

We look for the invariant distr.

$\pi P = \pi$:

$$\begin{cases} \pi_0 = (1-p)\pi_0 + \frac{1}{2}(1-p)\pi_1 + \frac{1-p}{4}\pi_2 \\ \pi_1 = p\pi_0 + \frac{1}{2}\pi_1 + \frac{2-p}{4}\pi_2 + \frac{1-p}{4}\pi_3 \\ \pi_2 = \frac{1}{2}p\pi_1 + \frac{1+p}{4}\pi_2 + \frac{2-p}{4}\pi_3 + \frac{1-p}{4}\pi_4 \\ \pi_j = \frac{p}{4}\pi_{j-1} + \frac{1+p}{4}\pi_j + \frac{2-p}{4}\pi_{j+1} + \frac{1-p}{4}\pi_{j+2} \qquad j \geq 3 \end{cases}$$

General j:

$$p + (p-3)x + (2-p)x^2 + (1-p)x^3 = 0$$

$$x_{1/2/3} = 1, \ \frac{2p-3 \pm \sqrt{9-8p}}{2(1-p)}$$

$$\pi_j = A + \left(\frac{2p-3+\sqrt{9-8p}}{2(1-p)}\right)^j B + \left(\frac{2p-3-\sqrt{9-8p}}{2(1-p)}\right)^j C$$

$A = 0$

$\left|\frac{2p-3+\sqrt{9-8p}}{2(1-p)}\right| \overset{?}{<} 1 \Longrightarrow 2p-3+\sqrt{9-8p} \overset{?}{<} 2-2p$
$\Longrightarrow [\dots] \quad 0 < 16(p-1)^2$
$\Longrightarrow$ ok

$\left|\frac{2p-3-\sqrt{9-8p}}{2(1-p)}\right| \overset{?}{<} 1 \Longrightarrow 3-\cancel{2p}+\sqrt{9-8p} \overset{?}{<} 2-\cancel{2p}$
$\Longrightarrow \sqrt{9-8p} \overset{?}{<} -1$ no

$$\Longrightarrow \pi_j = B\left(\frac{2p-3+\sqrt{9-8p}}{2(1-p)}\right)^j$$

From here:

$$\pi_3 = \frac{p}{4}\pi_2 + \frac{1-p}{4}\pi_3 + \frac{2-p}{4}\pi_4 + \frac{1-p}{4}\pi_5$$

$$\Longrightarrow \pi_2 = \frac{3+p}{p}\pi_3 + \frac{p-2}{p}\pi_4 + \frac{p-1}{p}\pi_5$$

$\Longrightarrow$ we use $\pi_2 \uparrow$ to get: π_1:

$\pi_2 = \frac{1}{2}p\pi_1 + \dots$

Then we use π_2, π_1 to get π_0 and we've done (after we finish we normalize to find B)

5. At each time $j \geq 2$:

$\mathbb{P}(j \to j-2) = \mathbb{P}(\text{both get free}) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\mathbb{P}(j \to j-1) = \mathbb{P}(\text{one get free}) = \left(\frac{1}{2}\right)^2 \cdot 2 = \frac{1}{2}$

$\mathbb{P}(j \to j) = \mathbb{P}(\text{none get free}) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\mathbb{P}(0 \to 0) = 1$

$\mathbb{P}(1 \to 0) = \mathbb{P}(\text{one or two get free}) = \mathbb{P}(\text{one}) + \mathbb{P}(\text{two}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$\mathbb{P}(1 \to 1) = \mathbb{P}(\text{none get free}) = \frac{1}{4}$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots \\ 3/4 & 1/4 & 0 & 0 & \cdots & \cdots \\ 1/4 & 1/2 & 1/4 & 0 & \cdots & \cdots \\ 0 & 1/4 & 1/2 & 1/4 & \cdots & \cdots \\ \vdots & & & & & \end{bmatrix} = P$$

Mean abs. time in 0:

$$\begin{cases} w_0 = 0 \\ w_1 = 1 + \frac{3}{4} w_0 + \frac{1}{4} w_1 \\ w_j = 1 + \frac{1}{4} w_{j-2} + \frac{1}{2} w_{j-1} + \frac{1}{4} w_j & j \geq 2 \end{cases}$$

$\Longrightarrow w_1 = 1 + \frac{1}{4} w_1 \quad \Longrightarrow \quad w_1 = \frac{4}{3}$

w_j: $\quad 3w_j - 2w_{j-1} - w_{j-2} = 4$

Homogeneous: $\quad w_j \Rightarrow \quad 3x^2 - 2x - 1 = 0 \quad \Rightarrow \quad w_j = A + \left(-\frac{1}{3}\right)^j B$

Particular: $\quad w_j = D \quad \to \quad 3D - 2D - D = 4 \qquad$ <u>no</u>

$w_j = j \quad \Longrightarrow \quad 3j - 2(j-1) - (j-2) = 4$

$\cancel{3j} - \cancel{2j} + 2 - \cancel{j} + 2 = 4 \qquad$ <u>ok</u>

Complete: $\quad w_j = A + \left(-\frac{1}{3}\right)^j B + j$

Considering $\quad w_0 = 0, \quad w_1 = \frac{3}{4} \quad \Longrightarrow \quad w_j = \frac{1}{4}\left(1 - \left(\frac{1}{3}\right)^j\right) + j$

$\Longrightarrow \quad w_N = \frac{1}{4}\left(1 - \left(-\frac{1}{3}\right)^N\right) + N$

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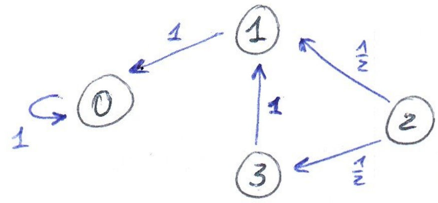
#2

1. Discrete skeleton from $Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



{0} recurrent
1,2,3 transient
If we leave 3,2 we never go back to them

2. BKE : $p_{ij}'(t) = \sum_k q_{ik}\, p_{kj}(t)$

$p_{00}'(t) = 0 \implies p_{00}(t) = c$ and since $p_{00}(0) = 1 \Rightarrow c = 1$

$p_{00}(t) = 1$, $p_{01}(t) = p_{02}(t) = p_{03}(t) = 0$

$p_{10}'(t) = 1\, p_{00}(t) + (-1)\, p_{10}(t) = -p_{10}(t) + 1$

quelle in blue sono standard e quindi non riportate

Homogeneous : $p_{10}(t) = ke^{-t}$
Complete : $p_{10}(t) = ke^{-t} + c$

$p_{10}'(t) = -ke^{-t} = -ke^{-t} - c + 1 \implies c = 1$

$p_{10}(0) = 0 \implies k + 1 = 0 \implies k = -1$

$p_{10}(t) = 1 - e^{-t}$

$p_{11}'(t) = 1\, p_{01}(t) + (-1)\, p_{11}(t) = -p_{11}(t)$

$p_{11}(t) = ke^{-t} \implies p_{11}(0) = 1 \Rightarrow k = 1 \implies p_{11}(t) = e^{-t}$

$p_{12}(t) = p_{13}(t) = 0$

$$\begin{cases} p_{20}'(t) = 1\, p_{10}(t) - 2\, p_{20}(t) + 1\, p_{30}(t) \implies [..]\ \ p_{20}(t) = 1 - te^{-t} - e^{-t} \\ p_{21}'(t) = 1\, p_{11}(t) - 2\, p_{21}(t) + 1\, p_{31}(t) \implies [..]\ \ p_{21}(t) = te^{-t} \\ p_{22}'(t) = \cancel{1\, p_{12}(t)} - 2\, p_{22}(t) + \cancel{1\, p_{32}(t)} \implies p_{22}(t) = e^{-2t} \\ p_{23}'(t) = \cancel{1\, p_{13}(t)} - 2\, p_{23}(t) + 1\, p_{33}(t) \implies [..] : p_{23}(t) = e^{-t} - e^{-2t} \end{cases}$$

$$\begin{cases} p_{30}'(t) = 1\, p_{10}(t) - 1\, p_{30}(t) \implies [..] : p_{30}(t) = 1 - te^{-t} - e^{-t} \\ p_{31}'(t) = 1\, p_{11}(t) - 1\, p_{31}(t) \implies [..] : p_{31}(t) = te^{-t} \\ p_{32}'(t) = \cancel{1\, p_{12}(t)} - p_{32}(t) \implies p_{32}(t) = 0 \\ p_{33}'(t) = \cancel{1\, p_{13}(t)} - p_{33}(t) \implies p_{33}(t) = e^{-t} \end{cases}$$

3. $f\left(\begin{bmatrix}0\\1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}1\\0\\-1\\-1\end{bmatrix}$: $Qf\left(\begin{bmatrix}0\\1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}0 & 0 & 0 & 0\\1 & -1 & 0 & 0\\0 & 1 & -2 & 1\\0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} = \mathbb{1}_{\{X_s > 0\}}$

since $(Qf)(X_s) = \mathbb{1}_{\{X_s>0\}} \implies M_t = f(X_t) - \int_0^t (Qf)(X_s)\,ds$

$\quad\quad\quad\quad \downarrow f(X_t) - \int_0^t \mathbb{1}_{\{X_s>0\}}\,ds$

is a martingale

4. stopping theorem: $\mathbb{E}_3[M_0] = \mathbb{E}_3[M_t] = \mathbb{E}_3[M_{T\wedge t}]$

starting from 3, mean time to go to 0?

$\mathbb{E}_3[T_0]$ = mean time $3 \to 0$

$\mathbb{E}_3[M_0] = \mathbb{E}_3[f(X_0)] = 1 \quad (= f(X_0))$

$\mathbb{E}_3[M_T] = \mathbb{E}_3[f(X_T) - \int_0^T ds] = \mathbb{E}_3[-1 - T] = 1 - \mathbb{E}_3[T]$

$\Rightarrow \mathbb{E}_3[T_0] = 2$

5. $M_t = f(X_t) - \int_0^t \mathbb{1}_{\{X_s>0\}}\,ds$

$t \to \infty \quad M_t \to f(X_0) - \underbrace{\int_0^\infty \mathbb{1}_{\{X_s>0\}}\,ds}$

total time spent in the transient states

(since $\int_0^\infty \mathbb{1}_{\{X_s>0\}}\,ds < 0$ then all the M_t converges in L^1)