Homework:

$$X_i | \lambda_i \overset{iid}{\sim} \text{Poiss}(\lambda_i t_i) \quad i = 1, .., 10 \qquad \lambda_1, .., \lambda_{10} | \beta \overset{iid}{\sim} \text{gamma}(\alpha, \beta)$$

$$\left( f(x_i | \lambda_i) = \frac{(t_i \lambda_i)^{x_i}}{x_i!} e^{-\lambda_i t_i} \right. \qquad \beta \sim \text{gamma}(\gamma, \delta) \qquad (t_i \text{ fixed})$$

---

$$\pi(\lambda_1, .., \lambda_{10}, \beta) = \pi(\lambda_1, .., \lambda_{10} | \beta)\, \pi(\beta)$$

$$\pi(\lambda_1, .., \lambda_{10}, \beta | \underline{x}) \propto \underbrace{\left[ \prod_{i=1}^{10} f(x_i | \lambda_i) \right]}_{\text{likelihood}} \cdot \underbrace{\left[ \prod_{i=1}^{10} \pi(\lambda_i | \beta) \right]}_{\text{prior } \lambda_i | \beta} \underbrace{\left[ \pi(\beta) \right]}_{\text{prior } \beta}$$

$$\propto \left[ \prod_{i=1}^{10} \frac{(t_i \lambda_i)^{x_i}}{x_i!} e^{-\lambda_i t_i} \right] \left[ \prod_{i=1}^{10} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda_i^{\alpha-1} e^{-\beta \lambda_i} \right] \left[ \frac{\delta^\gamma}{\Gamma(\gamma)} \beta^{\gamma-1} e^{-\delta\beta} \right] \mathbb{1}_{[0,\infty)}(\beta)$$

$$\propto \left( \prod_{i=1}^{10} \frac{(t_i \lambda_i)^{x_i}}{x_i!} \lambda_i^{\alpha-1} \right) e^{-\sum_{i=1}^{10} \lambda_i t_i - \beta \sum_{i=1}^{10} \lambda_i} \left( \frac{\beta^\alpha}{\Gamma(\alpha)} \right)^{10} \frac{\delta^\gamma}{\Gamma(\gamma)} \beta^{\gamma-1} e^{-\delta\beta} \mathbb{1}_{[0,\infty)}(\beta)$$

without multiplicative constants

$$\propto \left( \prod_{i=1}^{10} \lambda_i^{x_i + \alpha - 1} \right) e^{-\sum_{i=1}^{10} \lambda_i (t_i + \beta)} \beta^{10\alpha + \gamma - 1} e^{-\delta\beta} \mathbb{1}_{[0,\infty)}(\beta)$$

$$\mathbb{1}_{[0,\infty)}(\lambda_i)$$

- $[\lambda_i | \beta, \underline{x}] \propto \lambda_i^{x_i + \alpha - 1} e^{-\lambda_i (t_i + \beta)} \quad \forall \implies \text{gamma}(x_i + \alpha,\ t_i + \beta)$
- $[\beta | \underline{\lambda}, \underline{x}] \propto e^{-\sum_{i=1}^{10} \lambda_i \cdot \beta} e^{-\delta\beta} \cdot \beta^{10\alpha + \gamma - 1} \mathbb{1}_{[0,\infty)}(\beta)$

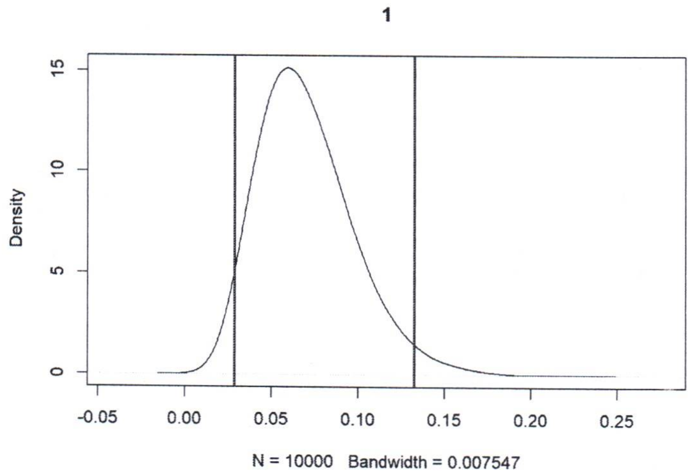
  $\propto e^{-\beta(\delta + \sum_{i=1}^{10} \lambda_i)} \beta^{10\alpha + \gamma - 1} \times \mathbb{1}_{[0,\infty)}(\beta) \implies \text{gamma}(10\alpha + \gamma,\ \delta + \sum_{i=1}^{10} \lambda_i)$

# [Bayesian statistic] Homework

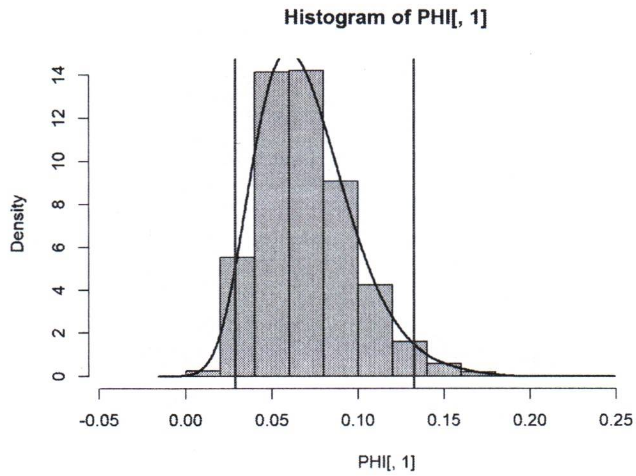
```
library(coda)
```

```
# ---------------------------------------------------
# Model
# ---------------------------------------------------
#     X_i|lambda_i ~ Poiss(lambda_i*t_i)
#    lambda_i|beta ~ gamma(alpha, beta)
#             beta ~ gamma(gamma, delta)

# ---------------------------------------------------
# Data
# ---------------------------------------------------
x = c(5,1,5,14,3,19,1,1,4,22)
t = c(94.32, 15.72, 62.88, 125.76, 5.24, 31.44, 1.05, 1.05, 2.10, 10.48)

alpha = 1.8
gamma = 0.01
delta = 1
n     = 10

# ---------------------------------------------------
# Gibbs sampler MCMC of S draws
# ---------------------------------------------------
S   = 10000
PHI = matrix(nrow=S, ncol=11)      # cols: [lambda_1, .., lambda_10, beta]
phi = c(x/t, 1)
PHI[1,] = phi                      # Initial point

for(s in 2:S){
  # generate new values for lambda_i
  for(i in 1:10){
    phi[i] = rgamma(1, x[i]+alpha, t[i]+phi[11])
  }

  # generate new value for beta
  phi[11]  = rgamma(1, 10*alpha + gamma, delta + sum(phi[1:10]))
  PHI[s,] = phi
}
```

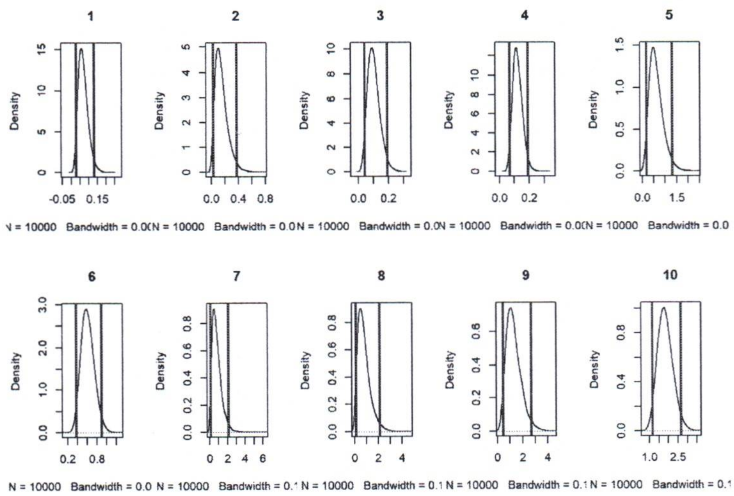
```
#####################################################
# LAMBDA's
#####################################################
# Lambda_1 posterior distribution + CI_95%
# --------------------------------------------------
par(mfrow=c(1,1))
plot(density(PHI[,1], adj=2), xlim=c(min(PHI[,1])-0.05,max(PHI[,1])+0.05), main=1)
abline(v=quantile(PHI[,1], prob=c(0.025, 0.975)), lwd=2, col='red')
```



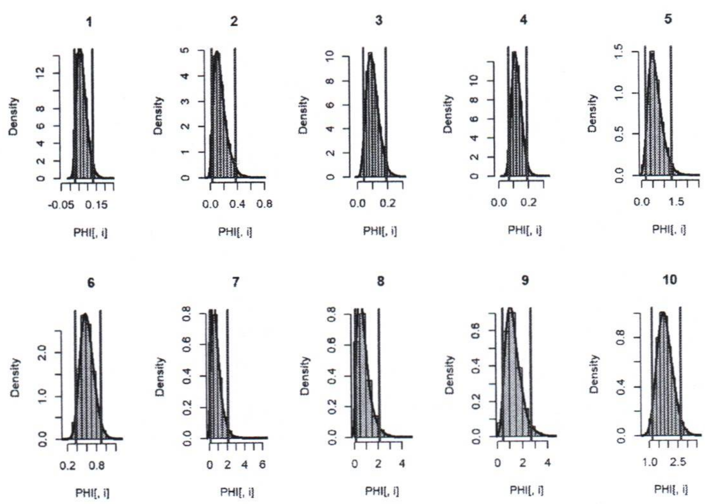

```
hist(PHI[,1], prob=T,xlim=c(min(PHI[,1])-0.05,max(PHI[,1])+0.05) )
lines(density(PHI[,1], adj=2), xlim=c(min(PHI[,1])-0.05,max(PHI[,1])+0.05), main=1, lwd=2)
abline(v=quantile(PHI[,1], prob=c(0.025, 0.975)), lwd=2, col='red')
```




```
# All lambdas posterior distributions + CI_95%
# ---------------------------------------------------
par(mfrow=c(2,5))
for(i in 1:10){
  plot(density(PHI[,i], adj=2), xlim=c(min(PHI[,i])-0.05,max(PHI[,i])+0.05), main=i)
  abline(v=quantile(PHI[,i], prob=c(0.025, 0.975)), lwd=2, col='red')
}
```




```
for(i in 1:10){
  hist(PHI[,i], prob=T,xlim=c(min(PHI[,i])-0.05,max(PHI[,i])+0.05), main=i)
  lines(density(PHI[,i], adj=2), xlim=c(min(PHI[,i])-0.05,max(PHI[,i])+0.05), main=1, lwd=2)
  abline(v=quantile(PHI[,i], prob=c(0.025, 0.975)), lwd=2, col='red')
}
```



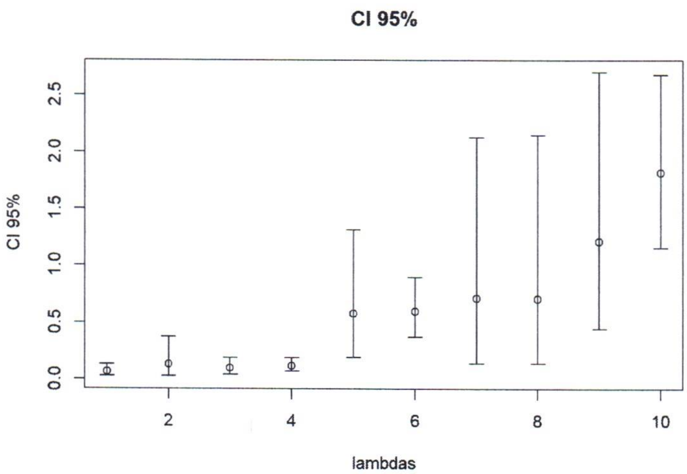

```
# CI for all lambdas
# ---------------------------------------------------
CI = matrix(nrow=10, ncol=3)
for(i in 1:10){
  CI[i,] = as.numeric(quantile(PHI[,i], c(0.025, 0.5, 0.975)))
}
CI
```

```
##              [,1]       [,2]      [,3]
##  [1,] 0.02878828 0.06646910 0.1324100
##  [2,] 0.02912226 0.13491459 0.3754291
##  [3,] 0.04166341 0.09978045 0.1930109
##  [4,] 0.07047315 0.12044136 0.1906064
##  [5,] 0.19510065 0.57963271 1.3161677
##  [6,] 0.37875440 0.60298801 0.8968136
##  [7,] 0.14569989 0.71364306 2.1313073
##  [8,] 0.14398170 0.71077697 2.1514248
##  [9,] 0.44803443 1.21487029 2.7027229
## [10,] 1.15762287 1.81820377 2.6790596
```

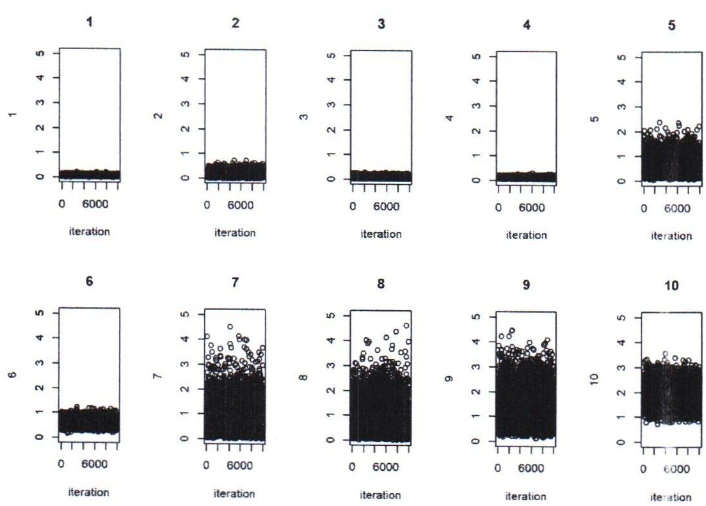
```
par(mfrow=c(1,1))
require(plotrix)
```

```
## Loading required package: plotrix
```

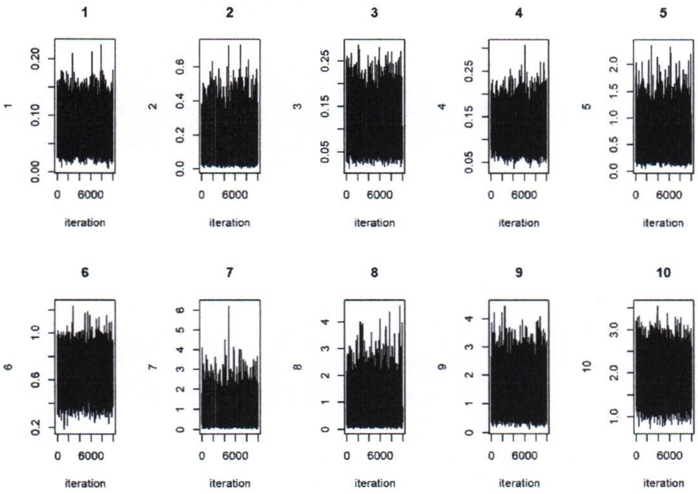
```
plotCI(1:10, CI[,2], ui=CI[,3], li=CI[,1], xlab='lambdas', ylab='CI 95%', main="CI 95%")
```



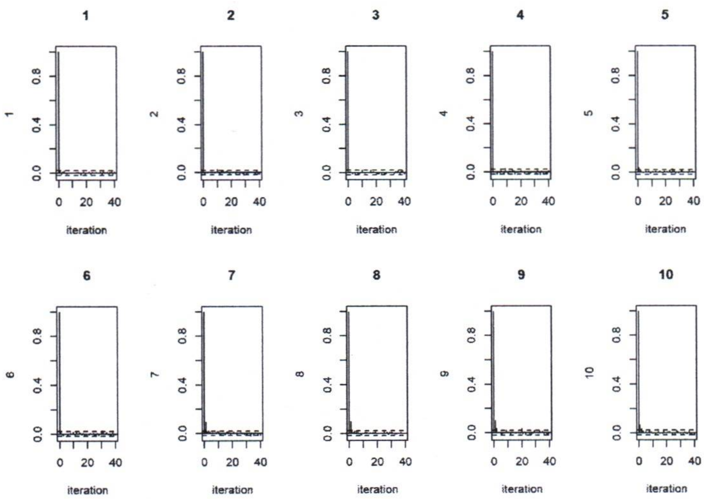

```
# Convergence diagnostic
# ---------------------------------------------------
par(mfrow=c(2,5))
for(i in 1:10){
  plot(PHI[,i], xlab='iteration', ylab=i, ylim=c(0,5), main=i)
}
```




```
par(mfrow=c(2,5))
for(i in 1:10){
  plot(ts(PHI[,i]), xlab='iteration', ylab=i, main=i)
}
```




```
par(mfrow=c(2,5))
for(i in 1:10){
  acf(PHI[,i], xlab='iteration', ylab=i, main=i)
}
```



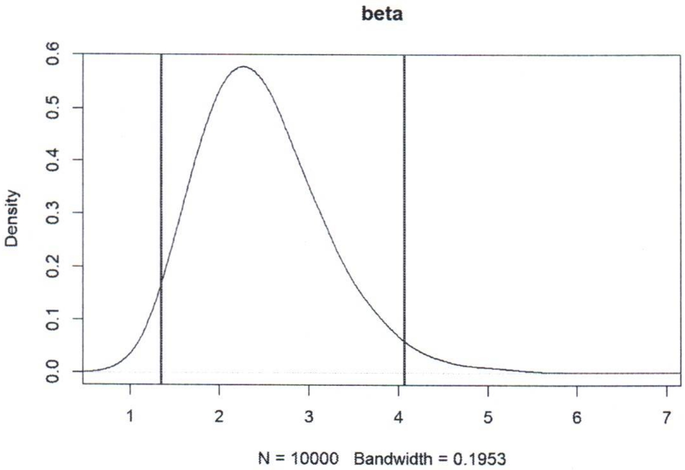

```
effectiveSize(PHI[,1:10])
```

```
##       var1      var2      var3      var4      var5      var6      var7      var8
##  9490.711 10000.000 10000.000 10043.975  9005.596  9513.875  7819.586  7914.399
##       var9     var10
##  7707.240  8279.221
```

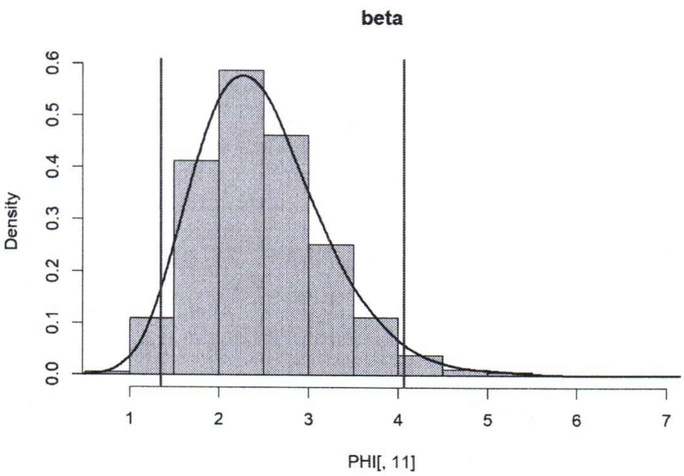
```
effectiveSize(PHI[,1:10])/S    # in percentage
```

```
##      var1      var2      var3      var4      var5      var6      var7      var8
## 0.9490711 1.0000000 1.0000000 1.0043975 0.9005596 0.9513875 0.7819586 0.7914399
##      var9     var10
## 0.7707240 0.8279221
```

```
#####################################################
# BETA
#####################################################
# Beta posterior distribution + CI_95%
# ---------------------------------------------------
par(mfrow=c(1,1))
plot(density(PHI[,11], adj=2), xlim=c(min(PHI[,11])-0.05,max(PHI[,11])+0.05), main='beta')
abline(v=quantile(PHI[,11], prob=c(0.025, 0.975)), lwd=2, col='red')
```



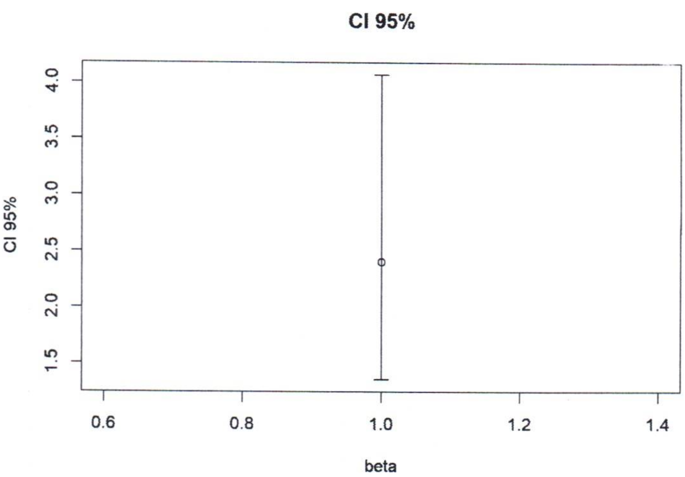

```
hist(PHI[,11], prob=T,xlim=c(min(PHI[,11])-0.05,max(PHI[,11])+0.05), main='beta' )
lines(density(PHI[,11], adj=2), xlim=c(min(PHI[,11])-0.05,max(PHI[,11])+0.05), main=1, lwd=2)
abline(v=quantile(PHI[,11], prob=c(0.025, 0.975)), lwd=2, col='red')
```



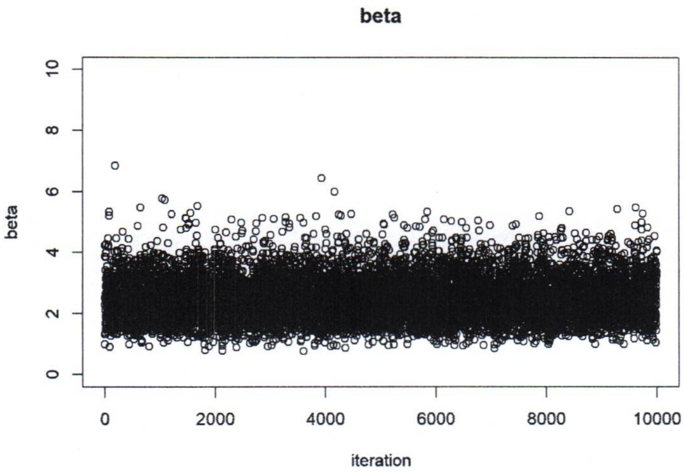

```
# CI for beta
# --------------------------------------------------
CI_beta = as.numeric(quantile(PHI[,11], c(0.025, 0.5, 0.975)))
CI_beta
```

```
## [1] 1.354245 2.400607 4.065896
```

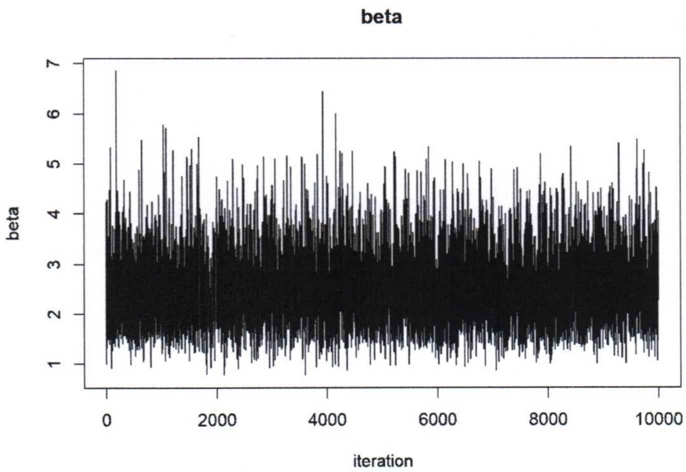
```
par(mfrow=c(1,1))
require(plotrix)
plotCI(1, CI_beta[2], ui=CI_beta[3], li=CI_beta[1],
       xlab='beta', ylab='CI 95%', main="CI 95%")
```



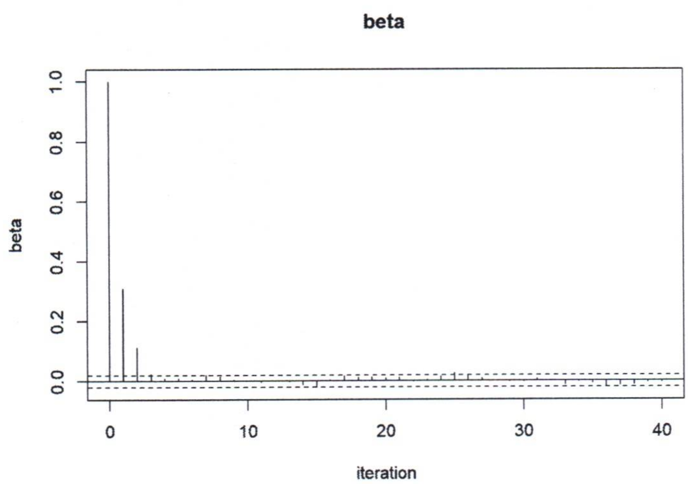
```
# Convergence diagnostic
# ----------------------------------------------------
par(mfrow=c(1,1))
plot(PHI[,11], xlab='iteration', ylab='beta', ylim=c(0,10), main='beta')
```




```
plot(ts(PHI[,11]), xlab='iteration', ylab='beta', main='beta')
```




```
acf(PHI[,11], xlab='iteration', ylab='beta', main='beta')
```




```
effectiveSize(PHI[,11])
```

```
##     var1
## 5265.657
```

```
effectiveSize(PHI[,11])/S
```

```
##      var1
## 0.5265657
```