Stochastic dynamical models

January 18$^{\text{th}}$, 2020

- Pocket calculators without wifi connection function are allowed. Pocket calculators with wifi connection function are NOT allowed, even if set in airplane mode.
- The teacher's or your own lecture notes are allowed, other notes are not.
- You can not ask questions. If you think that there is an error or something ambiguous, explain why and write your solution accordingly.
- Solutions can be written either in English or Italian (also French and Spanish).

Exercises

Exercise 1. Consider the Markov chain $(X_n)_{n\geq 0}$ with state space $E = \{1,2,3,4,5,6\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 & 0 & 0 \\ 0 & 0 & 0 & 1/6 & 1/3 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/4 & 0 & 3/4 \end{pmatrix} \quad \begin{bmatrix} 0 \\ 17 \\ 14 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(1) Classify the states of the Markov chain.
(2) Find the invariants distributions.
(3) Find the absorption probability associated to each recurrence class.
(4) Compute $\lim_{n\to\infty} p_{24}^{(n)}$.
(5) Let $f : \{1,2,3,4,5,6\} \to \mathbb{R}$ be the function

$$f(1) = 0, \ f(2) = 17, \ f(3) = 14, \ f(4) = f(5) = f(6) = 0.$$

Show that the process $(M_n)_{n\geq 0}$ defined by $M_0 = f(X_0)$ and

$$M_n = f(X_n) + 5\sum_{k=0}^{n-1} \mathbb{1}_{\{X_k \in \{2,3\}\}}$$

for $n \geq 1$ is a martingale.
(6) Applying the stopping theorem compute the mean time spent by the systems in the set of states $\{2,3\}$ starting from 3.

Exercise 2. Two identical machines operate continuously unless they are broken. A repairer is available if necessary to fix the broken machines. The repair time has an exponential distribution with mean 1/2 day. Once repaired, the life time of the machine before its next break has an exponential distribution with mean 1 day. Assume that repair times and break times of the machines are independent and the repairer repairs only one machine at a time. Let X_t the number of machine out of order at time t.

(1) Find the transition rate matrix of the Markov chain $(X_t)_{t\geq 0}$.
(2) Find the invariant distribution.
(3) Find the matrices P_t of the transition semigroup of $(X_t)_{t\geq 0}$.

Calcoli & metodo

The following exercise is for students of the 2019-2020 academic year only!

Exercise 3. Let $(Z_n)_{n\geq 1}$ be a sequence of independent random variables with Gaussian standard distribution $N(0,1)$, let us consider the partial sums $S_n = \sum_{k=1}^{n} Z_k$ and define

$$X_0 = 1, \qquad X_n = e^{\theta S_n - \theta^2 n/2} \quad n \geq 1$$

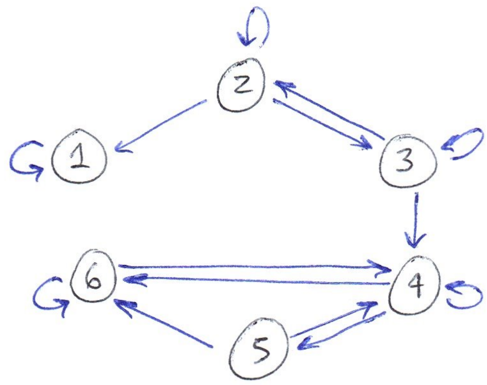
where θ is a real parameter.

(1) Show that $(X_n)_{n\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ with $\mathcal{F}_n := \sigma\{Z_1, \dots Z_n\}$.
(2) Show that $(X_n)_{n\geq 0}$ is a discrete time homogeneous Markov process and determine the transition kernel.

18/01/2020

#1

1.



{1} recurrent class
{2,3} transient class
{4,5,6} recurrent class

2. All the invariant distributions are of the form:

$$\lambda(1,0,0,0,0,0) + (1-\lambda)(0,0,0,\pi_4,\pi_5,\pi_6)$$

$\pi_4; \pi_5, \pi_6$:

$$\begin{cases} \frac{1}{3}\pi_4 = \pi_5 & \longrightarrow \quad \pi_5 = \frac{1}{3}\pi_4 \\ \frac{1}{2}\pi_4 + \frac{1}{2}\pi_5 + \frac{3}{4}\pi_6 = \pi_6 & \\ \frac{1}{6}\pi_4 + \frac{1}{2}\pi_5 + \frac{1}{4}\pi_6 = \pi_4 & \longrightarrow \quad \pi_6 = \frac{8}{3}\pi_4 \\ \pi_4 + \pi_5 + \pi_6 = 1 & \end{cases}$$

$$\pi_4 = 1 - \pi_5 - \pi_6 = 1 - \left(\frac{1}{3} + \frac{8}{3}\right)\pi_4 \implies \pi_4 = \frac{1}{4}, \quad \pi_5 = \frac{1}{12}, \quad \pi_6 = \frac{2}{3}$$

3. Absorption in 1:

$V_1 = 1$

$V_4 = V_5 = V_6 = 0$

$V_2 = p_{21} + p_{22}V_2 + p_{23}V_3 = \frac{1}{4} + \frac{1}{2}V_2 + \frac{1}{4}V_3$

$V_3 = \cancel{p_{31}} + p_{32}V_2 + p_{33}V_3 = \frac{1}{5}V_2 + \frac{2}{5}V_3$

$$\Rightarrow \begin{cases} \frac{1}{2}V_2 = \frac{1}{4} + \frac{1}{4}V_3 & \longrightarrow \quad 2V_2 = 1 + V_3 \longrightarrow \quad 6V_3 = 1 + V_3 \\ \frac{3}{5}V_3 = \frac{1}{5}V_2 & \longrightarrow \quad V_2 = 3V_3 \end{cases}$$

$$\Rightarrow [V_1, V_2, V_3, V_4, V_5, V_6] = \left[1, \frac{3}{5}, \frac{1}{5}, 0, 0, 0\right]$$

Abs. in {4,5,6}:

$V_1 = 0, \quad V_4 = V_5 = V_6 = 1$

$$\begin{cases} V_2 = p_{23}V_3 + p_{22}V_2 = \frac{1}{4}V_3 + \frac{1}{2}V_2 \\ V_3 = p_{34} + p_{32}V_2 + p_{33}V_3 = \frac{2}{5} + \frac{1}{5}V_2 + \frac{2}{5}V_3 \end{cases}$$

$$\Rightarrow \begin{cases} 4V_2 = V_3 + 2V_2 \\ 5V_3 = 2 + V_2 + 2V_3 \end{cases} \Rightarrow \begin{cases} 2V_2 = V_3 \\ 3V_3 = 2 + V_2 \end{cases} \rightarrow \quad 6V_2 = 2 + V_2$$

$$\Rightarrow [V_1, V_2, V_3, V_4, V_5, V_6] = \left[0, \frac{2}{5}, \frac{4}{5}, 1, 1, 1\right]$$

4. $\boxed{\lim_{n\to\infty} p_{ij}^{(n)} = v_i \pi_j \qquad i \text{ transi } t,\ j \text{ recurrent}}$!

$$p_{24}^{(n)} \longrightarrow v_2 \pi_4 = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}$$

5. $f(X_n) = f\left(\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}\right) = \begin{bmatrix}0\\17\\14\\0\\0\\0\end{bmatrix}$

$$Pf(X_n) = \begin{bmatrix}0\\ 1/2\cdot 17 + 1/4 \cdot 14\\ 1/5 \cdot 17 + 2/5 \cdot 14\\ 0\\0\\0\end{bmatrix} \qquad Pf(X_n) - f(X_n) = \begin{bmatrix}0\\(*)\\((*))\\0\\0\\0\end{bmatrix}$$

$$(*):\ \frac{1}{2}\,17 + \frac{1}{4}\,14 - 17 = \frac{34+14-68}{4} = \frac{-20}{4} = -5$$

$$((*)):\ \frac{1}{5}\cdot 17 + \frac{2}{5}\cdot 14 - 14 = \frac{17+28-14\cdot 5}{5} = \frac{17+28-70}{5} = -\frac{25}{5} = -5$$

$$\Longrightarrow\ Pf(X_n) - f(X_n) = -\mathbb{1}_{\{2,3\}}(X_n)\cdot 5 \ \Longrightarrow\ \text{Martingale}$$

6. $\mathbb{E}_3[M_0] = \mathbb{E}_3[f(X_0)] = \mathbb{E}_3[f(3)] = 14$ mean time spent in $\{2,3\}$ starting from 3

Entering in $T^1 = \{1\}$:

$$\mathbb{E}_3[M_{T^1}] \overset{!}{=} \mathbb{E}_3\left[f(X_{T^1}) + 5\sum_{k=0}^{T^1-1} \mathbb{1}_{\{X_k \in \{2,3\}\}}\right]$$
$$\overset{!}{=} \mathbb{E}_3[f(1) + 5T^1]$$
$$\overset{!}{=} \mathbb{E}_3[5T^1]$$

$$\Longrightarrow \mathbb{E}_3[M_0] = \mathbb{E}_3[M_{T^1}] \Rightarrow 5\,\mathbb{E}_3[T^1] = 14 \Rightarrow \mathbb{E}_3[T^1] = \frac{14}{5}$$

Entering in $T^2 = \{4,5,6\}$:

$$\mathbb{E}_3[M_{T^2}] = \mathbb{E}_3[f(4) + 5T^2] = \mathbb{E}_3[5T^2] \Rightarrow \mathbb{E}_3[T^2] = \frac{14}{5}$$

2

- repair time (service time) $\sim \mathcal{E}(2)$
- duration without breaking $\sim \mathcal{E}(1)$
- repairing 1 at the time

0 1 2

$(X_t)_{t\geq 0}$: X_t = # machines being broken

1. $0 \to 0,1$: $q_{01} = 2$, $q_{00} = -2$, $q_{02} = 0$

$1 \to 0,2,1$: $q_{10} = 2$, $q_{12} = 1$, $q_{11} = -3$

$2 \to 1,2$: $q_{20} = 0$, $q_{21} = 1\cdot 2$, $q_{22} = -1\cdot 2$

it breaks the first or the second

$$Q = \begin{bmatrix}-2 & 2 & 0\\ 2 & -3 & 1\\ 0 & 2 & -2\end{bmatrix}$$

18/01/2020 (2)

#2 (2)

2. $$\left.\begin{array}{l} -2\pi_0 + 2\pi_1 = 0 \\ 2\pi_0 - 3\pi_1 + 2\pi_2 = 0 \\ \pi_1 - 2\pi_2 = 0 \\ \pi_1 + \pi_2 + \pi_0 = 1 \end{array}\right\} \Longrightarrow \quad \pi_0 = \frac{2}{5}, \quad \pi_1 = \frac{2}{5}, \quad \pi_2 = \frac{1}{5}$$

3. $$Q = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 2 & -2 \end{bmatrix} \qquad \text{FKE:} \quad p_{ij}'(t) = \sum_k p_{ik}(t)\, q_{kj}$$

$$p_{00}'(t) = q_{00}\, p_{00}(t) + q_{10}\, p_{01}(t) + q_{20}\, p_{02}(t)$$
$$= -2 p_{00}(t) + 2 p_{01}(t)$$

$$p_{00}(t) = e^{-2t} + e^{-2t} \int_0^t e^{2s} p_{01}(s)\, ds$$

$$p_{01}'(t) = p_{00}(t)\, q_{01} + p_{01}(t)\, q_{11} + p_{02}(t)\, q_{21}$$
$$= 2 p_{00}(t) - 3 p_{01}(t) + 2 p_{02}(t)$$
$$= 2\left(1 - p_{01}(t)\right) - 3 p_{01}(t)$$
$$= 2 - 5 p_{01}(t)$$

$$p_{01}(t) = k e^{-5t} + A$$
$$p_{01}'(t) = \cancel{-5ke^{-5t}} = 2 - \cancel{5ke^{-5t}} - 5A$$
$$A = \frac{2}{5}$$
$$p_{01}(0) = 0 = k + \frac{2}{5} \Rightarrow \quad k = -\frac{2}{5}$$
$$p_{01}(t) = -\frac{2}{5} e^{-5t} + \frac{2}{5} = \frac{2}{5}\left(1 - e^{-5t}\right)$$
$$p_{00}(t) = e^{-2t} + e^{-2t} \int_0^t e^{2s}\, \frac{2}{5}\left(1 - e^{-5s}\right) ds$$
$$= e^{-2t} + e^{-2t}\, \frac{2}{5} \int_0^t e^{2s} - e^{-3s}\, ds$$
$$= e^{-2t} + e^{-2t} \cdot \frac{2}{5} \left[\frac{1}{2} e^{2s} + \frac{1}{3} e^{-3s}\right]_0^t$$
$$= e^{-2t} + \frac{2}{5} e^{-2t} \left[\frac{1}{2} e^{2t} + \frac{1}{3} e^{-3t} - \frac{1}{2} - \frac{1}{3}\right]$$
$$= e^{-2t} + \frac{1}{5} + \frac{2}{15} e^{-5t} + \frac{2}{5}\left(-\frac{1}{2} - \frac{1}{3}\right) e^{-2t}$$
$$= \frac{1}{5} + \frac{2}{15} e^{-5t} + \frac{2}{3} e^{-2t}$$

[.. calcoli ?? ..]

3

$S_n = \sum_{k=1}^{n} Z_k$, $\quad Z_k \sim N(0,1)$

$S_{n+1} = Z_{n+1} + S_n$

$$X_n = \begin{cases} 1 & n=0 \\ e^{\theta S_n - \theta^2 \frac{n}{2}} & n \geq 1 \end{cases}$$

1. $$\mathbb{E}[X_{n+1} \mid \sigma(Z_1, \dots, Z_n)] = \mathbb{E}\left[e^{\theta S_{n+1} - \theta^2 \frac{n+1}{2}} \mid \sigma(\dots)\right]$$
$$= \mathbb{E}\left[e^{\theta(S_n + Z_{n+1})} e^{-\theta^2 \frac{n+1}{2}} \mid \sigma(\dots)\right]$$
$$= e^{\theta S_n - \theta^2 \frac{n}{2}} e^{-\frac{\theta^2}{2}} \mathbb{E}\left[e^{\theta Z_{n+1}}\right]$$
$$= e^{\theta S_n - \theta^2 \frac{n}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}} dx$$
$$= e^{\theta S_n - \theta^2 \frac{n}{2}} = X_n$$

by the projective property...

2. $$\mathbb{P}_n(X_n, A) = \mathbb{E}[\mathbb{1}_A(X_{n+1}) \mid \sigma(\dots)]$$
$$= \mathbb{E}\left[\mathbb{1}_A\left(e^{\theta Z_{n+1} - \frac{\theta^2}{2}} X_n\right) \mid \sigma(\dots)\right]$$
$$= \int_{\mathbb{R}} \mathbb{1}_A\left(e^{\theta x - \frac{\theta^2}{2}} X_n\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$e^{\theta x - \frac{\theta^2}{2}} X_n = y$$

$$\theta x - \frac{\theta^2}{2} = \ln\left(\frac{y}{X_n}\right) \implies x = \frac{\theta}{2} + \frac{\ln(y/X_n)}{\theta}$$

$$dx = \frac{1}{\theta y} dy$$

$$P_n(x, A) = \int_{\mathbb{R}} \mathbb{1}_A(y) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y/x)}{\theta} + \frac{\theta}{2}\right)^2} \frac{1}{\theta y} dy$$