Stochastic dynamical models, Mathematical Engineering 2019/2020 - SDM **6**

theorical

Exercise 1.

1) Let $\mathcal{D} = \{\Omega, \emptyset\}$. Show that
$$E[X|\mathcal{D}] = E[X]$$

2) Let A be an event such that $P(A) > 0$ and let $\mathcal{D}$ be the $\sigma-$algebra $\{A, A^c, \Omega, \emptyset\}$. Find $E[X|\mathcal{D}]$, where X is a random variable.

Exercise 2. Let B be a continuous brownian motion and let η be a random variable $N(\mu, \rho^2)$ independent of B. For every $t \geq 0$ we set
$$Y_t = \eta t + \sigma B_t$$

1) Find $E[Y_t|\eta]$.

2) Find $E[Y_t|B_t]$.

3) Find $E[Y_t|B_s]$ for $0 \leq s \leq t$.

Exercise 3. Let B be a brownian motion and let
$$X_t = B_t - tB_1, \quad 0 \leq t \leq 1$$

For $0 \leq s \leq t \leq 1$:

a) Show that the random variable X_s and $X_t - \frac{1-t}{1-s} X_s$ are independent.

b) Find $E[X_t|X_s]$.

Exercise 4. Let B be a continuous brownian motion. Let

1) Let $X_t = B_t^3, \quad t \geq 0$. Find $E[X_t|X_s]$ for $0 \leq s \leq t$.

2) Let $Y_t = x(1-t) + yt + B_t - tB_1$, for $0 \leq t \leq 1$. Find $E[Y_t|B_s]$ for $0 \leq s \leq t \leq 1$.

#1 (#6)

1. $D = \{\Omega, \emptyset\}$, $X : (\Omega, D, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

$Z := \mathbb{E}[X|D] : \Omega \to \mathbb{R}$ $\quad$ Z is D-measurable $\Rightarrow$ $Z^{-1}(A) \in D \quad \forall A \in \mathcal{B}_{\mathbb{R}}$

$A = \{x\}$, $Z^{-1}(A) = Z^{-1}(\{x\}) = \{w \in \Omega : Z(w) = x\} = \begin{cases} \emptyset \\ \Omega \end{cases}$

$\Longrightarrow$ Z is a constant

$\Longrightarrow$ $\mathbb{E}[X|D] =$ constant $\Longrightarrow$ $\mathbb{E}\big[\mathbb{E}[X|D]\big] = \mathbb{E}[X|D]$ (constant)

but $\mathbb{E}\big[\mathbb{E}[X|D]\big] = \mathbb{E}[X]$

2. $A : \mathbb{P}(A) > 0$, $D = \{A, A^c, \emptyset, \Omega\}$, $\mathbb{E}[X|D] = ?$

- on A: $\mathbb{E}[X \mathbb{1}_A] = \mathbb{E}\big[\mathbb{E}[X|D]\mathbb{1}_A\big] = \mathbb{E}[X|D]\,\mathbb{E}[\mathbb{1}_A]$

$$\mathbb{E}[X|D] = \frac{\mathbb{E}[X\mathbb{1}_A]}{\mathbb{P}(A)}$$

- on A^c: $\mathbb{E}[X\mathbb{1}_{A^c}] = \mathbb{E}[X|D]\,\mathbb{E}[\mathbb{1}_{A^c}] \to \mathbb{E}[X|D] = \dfrac{\mathbb{E}[X\mathbb{1}_{A^c}]}{\mathbb{P}(A^c)}$

#2

$\left.\begin{array}{l} B \text{ continuous brownian motion} \\ \eta \sim N(\mu, \rho^2) \end{array}\right\} \perp\!\!\!\perp$

$Y_t = \eta t + \sigma B_t$

1. $\mathbb{E}[Y_t|\eta] = \mathbb{E}[Y_t|\sigma(\eta)] = t\,\mathbb{E}[\eta|\sigma(\eta)] + \sigma\,\mathbb{E}[B_t|\sigma(\eta)]$

$= t\eta + \sigma \cdot 0 = t\eta$

2. $\mathbb{E}[Y_t|B_t] = \mathbb{E}[\eta t|B_t] + \mathbb{E}[\sigma B_t|B_t] = \mathbb{E}[\eta t] + \sigma B_t$

$= \mu t + \sigma B_t$

3. $\mathbb{E}[Y_t|B_s] = \mathbb{E}[\eta t] + \mathbb{E}[\sigma B_t|B_s] = \mu t + \sigma\,\mathbb{E}[B_t - B_s + B_s|B_s]$

$(0 \le s \le t)$

$= \mu t + \sigma\,\mathbb{E}[B_t - B_s|B_s] + \sigma\,\mathbb{E}[B_s|B_s]$

$= \mu t + \sigma\,\mathbb{E}[B_t - B_s] + \sigma B_s$

$= \mu t + \sigma B_s$

#3

B brownian, $\quad X_t := B_t - tB_1 \quad 0 \le t \le 1$ $\quad \mathbb{E}[X_t] = \mathbb{E}[B_t] - t\,\mathbb{E}[B_1] \underset{\downarrow}{=} 0$

a. $\begin{bmatrix} X_s \\ X_t - \frac{1-t}{1-s} X_s \end{bmatrix} \sim N\left(\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}, \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}\right)$

$$\operatorname{Cov}\left(X_s, X_t - \tfrac{1-t}{1-s}X_s\right) = \mathbb{E}\left[X_sX_t - \tfrac{1-t}{1-s}X_s\cdot X_s\right] - \mathbb{E}[X_s]\,\mathbb{E}\left[X_t - \tfrac{1-t}{1-s}X_s\right]$$

$$= \mathbb{E}[X_sX_t] - \mathbb{E}\left[\tfrac{1-t}{1-s}X_s^2\right] - \cancel{\mathbb{E}[X_s]\,\mathbb{E}\left[X_t - \tfrac{1-t}{1-s}X_s\right]}$$

$$= \mathbb{E}[(B_s - sB_1)(B_t - tB_1)] - \left(\tfrac{1-t}{1-s}\right)\mathbb{E}[X_s^2]$$

$$= \mathbb{E}[B_sB_t] - s\,\mathbb{E}[B_tB_1] - t\,\mathbb{E}[B_sB_1] + st\,\mathbb{E}[B_1^2] - \left(\tfrac{1-t}{1-s}\right)\operatorname{Var}(X_s)$$

$$= s - st - \cancel{ts + st} - \left(\tfrac{1-t}{1-s}\right)\left[\operatorname{Var}(B_s) + s^2\operatorname{Var}(B_1)\right]$$

$$= s - st - \left(\tfrac{1-t}{1-s}\right)(s + s^2)$$

$$= s(1-t) - \frac{(1-t)}{(1-s)}(1+s)\cdot s$$

$$= 0 \implies \perp\!\!\!\perp$$

b. $$\mathbb{E}[X_t \mid X_s] = \mathbb{E}\left[X_t - \left(\tfrac{1-t}{1-s}\right)X_s \,\middle|\, X_s\right] + \mathbb{E}\left[\left(\tfrac{1-t}{1-s}\right)X_s \,\middle|\, X_s\right]$$

$$= \mathbb{E}\left[X_t - \left(\tfrac{1-t}{1-s}\right)X_s\right] + \left(\tfrac{1-t}{1-s}\right)X_s = \left(\tfrac{1-t}{1-s}\right)X_s$$

#4

1. $X_t = B_t^3$: $\quad \mathbb{E}[X_t \mid X_s] = \mathbb{E}[B_t^3 \mid \sigma(B_s)] = \mathbb{E}[(B_t - B_s + B_s)^3 \mid \sigma(B_s)]$

$$= \mathbb{E}[(B_t - B_s)^3] + B_s^3 + 3\,\mathbb{E}[(B_t - B_s)^2 B_s \mid B_s] + 3\,\mathbb{E}[(B_t - B_s)B_s^2 \mid B_s]$$

$$= \mathbb{E}[(B_t - B_s)^3] + B_s^3 + 3B_s\,\mathbb{E}[(B_t - B_s)^2] + 3B_s^2\,\mathbb{E}[B_t - B_s]$$

$$= 0 + B_s^3 + 3B_s(t-s) + 3B_s^2\cdot 0$$

$$= B_s^3 + 3(t-s)B_s$$

$B_t - B_s \sim N(0, t-s)$
$\mathbb{E}[B_t - B_s] = \mathbb{E}[(B_t - B_s)^3] = 0$
$\mathbb{E}[(B_t - B_s)^2] = t-s$

2. $Y_t = x(1-t) + yt + B_t - tB_1 \quad , \quad \mathbb{E}[Y_t \mid B_s]$?

$$\mathbb{E}[Y_t \mid B_s] = x(1-t) + yt + \mathbb{E}[B_t - tB_1 \mid B_s]$$

$$= x(1-t) + yt + \mathbb{E}[B_t - B_s] + B_s - t\,\mathbb{E}[B_1 - B_s] - tB_s$$

$$= x(1-t) + yt + B_s(1-t) + (t-s) - t(1-s)$$

$$= (x + B_s)(1-t) + yt + ts - s$$