Stochastic dynamical models

August 29th, 2019

EXERCISES

Exercise 1. A bug moves randomly in a square divided into 9 boxes

1	2	3
6	5	4
7	8	9

At each unit time it randomly moves to one of the nearest boxes with the same probabilities (for example, starting from box 1 at any time n it can jump to box 2, 5 or 6 with probability $1/3$ at time $n+1$).

Let $(X_n)_{n\geq 0}$ be the discrete time Markov chain in which X_n is the position (box) of the bug at time n.

(1) Write the transition matrix of the Markov chain. Is it irreducible?

(2) Find all the invariant densities.

$\big[$ Hint: assume $\pi = (3/40, *, 3/40, *, 8/40, *, 3/40, *, 3/40)$ $\big]$

(3) Compute $\lim_{n\to\infty} \mathbb{E}\left[X_n \mid X_0 = i \right]$

(4) Write (do not solve!) the system of equations for computing the expectation of the first passage time in the state 9 starting from the state i.

(5) Let $f : \{1,2,3,4,5,6,7,8,9\} \to \mathbb{R}$ be the function

$$f(i) = \begin{cases} i-1 & \text{for } i = 1,2,3 \\ 1 & \text{for } i = 4,5,6 \\ 9-i & \text{for } i = 7,8,9 \end{cases}$$

Show that the process $(Y_n)_{n\geq 0}$ defined by

$$Y_n = f(X_n) - n + \sum_{k=1}^{n-1} f(X_k)$$

is a martingale.

Exercise 2. Let $(X_t)_{t\geq 0}$ be a homogeneous time continuous Markov chain with state space $\mathbb{N} = \{0, 1, \dots\}$, rate matrix

$$Q = \begin{bmatrix} -3 & 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 5 & -8 & 1 & 2 & 0 & 0 & 0 & \dots \\ 6 & 5 & -14 & 1 & 2 & 0 & 0 & \dots \\ 0 & 6 & 5 & -14 & 1 & 2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

i.e., for all $m \geq 2$,

$$q_{mn} = \begin{cases} 6 & \text{if } n = m-2 \\ 5 & \text{if } n = m-1 \\ -14 & \text{if } n = m \end{cases} \qquad q_{mn} = \begin{cases} 1 \text{ if } n = m+1 \\ 2 \text{ if } n = m+2 \\ 0 \text{ otherwise.} \end{cases}$$

(1) Classify states. How about irreducibility?
(2) Is it recurrent or transient?
(3) Compute the invariant densities.
[Hint. $6x^4 + 5x^3 - 14x^2 + x + 2 = (x-1)(2x-1)(3x+1)(x+2)$]
(4) Let T be the stopping time $T = \inf\{\, t \geq 0 \mid X_t \leq 1 \,\}$ (*first arrival time* in the set of states $\{0, 1\}$). Compute $\mathbb{P}_n\{\, X_T = 1 \,\} = \mathbb{P}\{\, X_T = 1 \mid X_0 = n\}$ for all n, up to three constants *which do not need to be computed explicitly to avoid a long computation.* probability of absorption in 1
(5) For all $m > 4$, let f_m be the function $f(0) = 11/3$, $f(1) = 1$, $f(2) = 5$ and $f_m(x) = x^2 \wedge m^2 = \min\{\, x^2, m^2 \,\}$ for all $x \geq 3$. Show that the process $(M_t)_{t\geq 0}$ with M_t defined by

$$\begin{aligned} M_t &:= f_m(X_t) - \int_0^t \frac{100}{3} \mathbb{1}_{\{X_s=1\}} ds - \int_0^t Y_s(m) ds \\ Y_s(m) &= (38 - 24X_s)\mathbb{1}_{\{\, 2\leq X_s \leq m-2 \,\}} + (60 - 28m)\mathbb{1}_{\{\, X_s=m-1 \,\}} \\ &\quad + (29 - 34m)\mathbb{1}_{\{\, X_s=m \,\}} + (6 - 12m)\mathbb{1}_{\{\, X_s=m+1 \,\}} \end{aligned}$$

is a martingale and deduce the inequality

$$\mathbb{E}_n \left[X_t^2 \wedge m^2 - \int_0^t \frac{100}{3} \mathbb{1}_{\{X_s=1\}} ds \right] \leq n^2.$$

(where $\mathbb{E}_n[\,\cdot\,] := \mathbb{E}[\,\cdot \mid X_0 = n])$ for all $n \geq 2$.
(6) Let T be the above defined first arrival time in $\{0, 1\}$. Show that

$$\sup_{t\geq 0} \mathbb{E}_n \left[X_{T\wedge t}^2 \right] \leq n^2$$

(7) Let $f : \mathbb{N} \to \mathbb{N}$ be the function $f(n) = n$. Show that the process $(M_t)_{t\geq 0}$ with M_t defined by

$$M_t = f(X_t) - \int_0^t \left(5\, \mathbb{1}_{\{X_s=0\}} - 12\, \mathbb{1}_{\{X_s\geq 2\}}\right) ds$$

is a martingale.
(8) Applying the martingale stopping theorem compute $\mathbb{E}_n[T]$ for all $n \geq 2$ up to the three constants in (4) (all steps must be rigorously justified).

29/08/2019

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1.

	1	2	3	4	5	6	7	8	9
1		1/3			1/3	1/3			
2	1/5		1/5	1/5	1/5	1/5			
3		1/3		1/3	1/3				
4		1/5	1/5		1/5			1/5	1/5
5	1/8	1/8	1/8	1/8		1/8	1/8	1/8	1/8
6	1/5	1/5			1/5		1/5	1/5	
7					1/3	1/3		1/3	
8				1/5	1/5	1/5	1/5		1/5
9				1/3	1/3			1/3	

Irreducible: starting from any state we can reach any state in a finite time with strictly positive prob.

2. $\pi_1 = 3/40$ $\qquad \pi_2$?

$\pi_3 = 3/40$ $\qquad \pi_4$?

$\pi_5 = 8/40$ $\qquad \pi_6$?

$\pi_7 = 3/40$ $\qquad \pi_8$?

$\pi_9 = 3/40$

$$\frac{1}{5}\pi_2 + \frac{1}{8}\pi_5 + \frac{1}{5}\pi_6 = \pi_1 \qquad \frac{1}{5}\pi_2 + \frac{1}{40} + \frac{1}{5}\pi_6 = \frac{3}{40}$$

$$\frac{1}{5}\pi_2 + \frac{1}{5}\pi_4 + \frac{1}{8}\pi_5 = \pi_3 \qquad \frac{1}{5}\pi_2 + \frac{1}{5}\pi_4 + \frac{1}{40} = \frac{3}{40}$$

$$\frac{1}{8}\pi_5 + \frac{1}{5}\pi_6 + \frac{1}{5}\pi_8 = \pi_7 \qquad \frac{1}{40} + \frac{1}{5}\pi_6 + \frac{1}{5}\pi_8 = \frac{3}{40}$$

$$\frac{1}{5}\pi_4 + \frac{1}{8}\pi_5 + \frac{1}{5}\pi_8 = \pi_9 \qquad \frac{1}{5}\pi_4 + \frac{1}{40} + \frac{1}{5}\pi_8 = \frac{3}{40}$$

$$\left.\begin{aligned} 8\pi_2 + 8\pi_6 &= 2 \\ 8\pi_2 + 8\pi_4 &= 2 \\ 8\pi_6 + 8\pi_8 &= 2 \\ 8\pi_4 + 8\pi_8 &= 2 \end{aligned}\right] \quad \pi_4 = \pi_6, \quad \pi_2 = \frac{1}{4} - \pi_6, \quad \pi_8 = \frac{1}{4} - \pi_6$$

$$\frac{1}{3}\pi_1 + \frac{1}{5}\pi_2 + \frac{1}{8}\pi_5 + \frac{1}{3}\pi_7 + \frac{1}{5}\pi_8 = \pi_6 \implies [\,..\,] \Rightarrow \pi_6 = \frac{5}{40}$$

$$\pi_4 = \frac{5}{40}, \quad \pi_2 = \pi_8 = \frac{10-5}{40} = \frac{5}{40}$$

3. $\lim_{n\to\infty} E[X_n \mid X_0 = i]$?

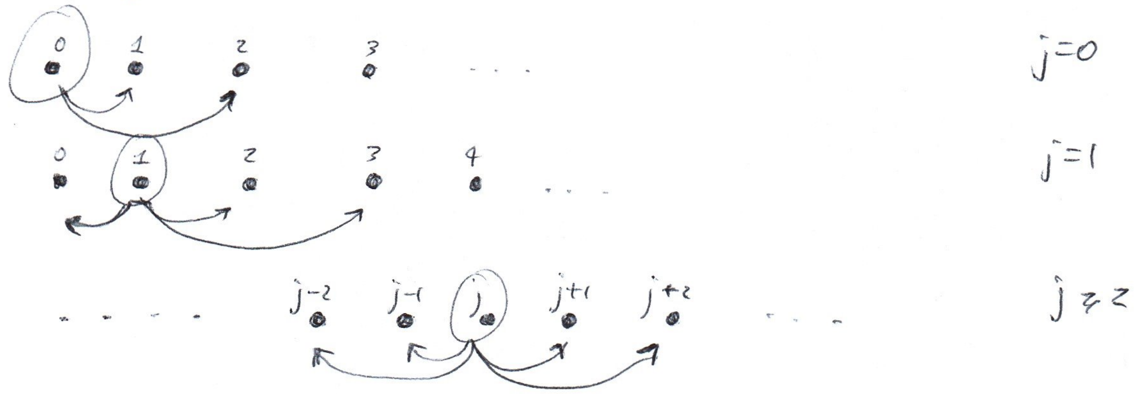
$$E[X_n \mid X_0 = i] = \sum_{k=1,..,9} k \cdot P(X_n = k \mid X_0 = i)$$
$$\xrightarrow{} \sum_{k=1,..,9} k \cdot \pi_k = .. = \mathbf{5}$$

4. $W_i = 1 + \sum_{j \in T} p_{ij} W_j$ ...

5. Use the theorem

#2

1. $$P = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 5/8 & 0 & 1/8 & 2/8 & 0 & 0 & 0 & 0 & \cdots \\ 3/7 & 5/14 & 0 & 1/14 & 1/7 & 0 & 0 & 0 & \cdots \\ 0 & 3/7 & 5/14 & 0 & 1/14 & 1/7 & 0 & 0 & \cdots \\ 0 & 0 & 3/7 & 5/14 & 0 & 1/14 & 1/7 & 0 & \cdots \\ \vdots \end{bmatrix}$$



Irreducible MC

2. $$\frac{2}{8} y_{j-2} + \frac{1}{4} y_{j-1} + \frac{5}{14} y_{j+1} + \frac{3}{7} y_{j+2} = y_j - x_j$$

$$= j^2\left(\frac{2}{8} + \frac{1}{4} + \frac{5}{14} + \frac{3}{7}\right) - j\left(\frac{2}{7}\right) + c = y_j - x_j \quad \checkmark$$

$\Longrightarrow$ recurrent

moreover $\exists! \, (\pi_j)_j$

3. $$\begin{cases} -3\pi_0 + 5\pi_1 + 6\pi_2 = 0 \\ \pi_0 - 8\pi_1 + 5\pi_2 + 6\pi_3 = 0 \\ 2\pi_{j-2} + \pi_{j-1} - 14\pi_j + 5\pi_{j+1} + 6\pi_{j+2} = 0 \qquad j \geq 2 \end{cases}$$

$$2 + x - 14x^2 + 5x^3 + 6x^4 = (x-1)(2x-1)(3x+1)(x+2)$$

$$\pi_j = A + B\left(\frac{1}{2}\right)^j + C\left(-\frac{1}{3}\right)^j + D(-2)^j$$

$$A, D = 0$$

$$\pi_j = B\left(\frac{1}{2}\right)^j + C\left(-\frac{1}{3}\right)^j$$

$$\Longrightarrow -3\pi_0 + 5\pi_1 + 6\pi_2 = -3B - 3C + \frac{5}{2}B - \frac{5}{3}C + \frac{3}{2}B + \frac{2}{3}C = 0$$

$$\Rightarrow B\left(-3 + \frac{5}{2} + \frac{3}{2}\right) + C\left(-3 - \frac{5}{3} + \frac{2}{3}\right) = 0$$

$$B = 4C \longrightarrow \pi_j = 4C\left(\frac{1}{2}\right)^j + C\left(-\frac{1}{3}\right)^j = C\left(4\left(\frac{1}{2}\right)^j + \left(\frac{1}{3}\right)^j\right)$$

$$\sum_{j=0}^{+\infty} \pi_j = 1 \Longrightarrow \sum_{j=0}^{+\infty}\left(4C\left(\frac{1}{2}\right)^j + C\left(-\frac{1}{3}\right)^j\right) = 1$$

$$= 4C\sum_{j\geq 0}\left(\frac{1}{2}\right)^j + C\sum_{j\geq 0}\left(-\frac{1}{3}\right)^j = 4C \cdot 2 + \frac{3}{4}C = \frac{35}{4}C = 1 \Rightarrow C = \frac{4}{35}, \quad B = \frac{16}{35}$$

29/08/2020 (2)

#2 (2)

4. probability of abs_in 1
(hitting prob. in 1)

$$\begin{cases} V_0 = 0, \quad V_1 = 1 \\ V_2 = \frac{5}{14} + \frac{1}{14} V_3 + \frac{1}{7} V_4 \\ V_j = \frac{3}{7} V_{j-2} + \frac{5}{14} V_{j-1} + \frac{1}{14} V_{j+1} + \frac{1}{7} V_{j+2} \qquad j \geq 3 \end{cases}$$

$$\frac{3}{7} + \frac{5}{14} x - x^2 + \frac{1}{14} x^3 + \frac{1}{7} x^4 = 0$$

$$6 + 5x - 14x^2 + x^3 + 2x^4 = 0$$

$$x = \frac{1}{z} \Rightarrow \quad 6z^4 + 5z^3 - 14z^2 + z + 2 = 0$$

$$x_{1/2/3/4} = 1, 2, -3, -\frac{1}{2}$$

$$\Rightarrow \quad V_j = A + B(2)^j + C(-3)^j + D\left(-\frac{1}{2}\right)^j$$

$B = C = 0$ since $V_j < \infty$ (it's a probability)

$$V_j = A + D\left(-\frac{1}{2}\right)^j$$

5.