Stochastic dynamical models, Mathematical Engineering 2019/2020 - SDM 3

Exercise 1. A friend of mine convinced me to play head and tail with a non-fair coin: At every toss, the probability of obtaining head is 1/3. The rules are the following:
- My intial capital is N euros;
- At each toss, my bet is one euro;
- If the result is head, I recover the one euro plus two more;
- If the result is tail, I lose the euro used for my bet;
- I stop playing when I have no more money, or I have earned at least the double of my capital (i.e. $2N$ or more).

Let $(X_n)_{n\geq 0}$ the Markov chain on $I = \{0, 1, \ldots, 2N, 2N+1\}$ such that $X_0 = N$, and
$X_n =$ My capital after the n-th toss.

1) Write the transition matrix of the Markov chain.

! 2) What is the probability to lose all my money? Suggestion: It may be useful to know that the solutions of the linear system

$$\begin{cases} x + z = 1 \\ x + ay + bz = 0 \\ x + (a+1)y - 2bz = 0 \end{cases}$$

are

$$x = -\frac{(3a+1)b}{1-(3a+1)b}, \quad y = \frac{3b}{1-(3a+1)b}, \quad z = \frac{1}{1-(3a+1)b}$$

Exercise 2. A circus acrobat is doing his balancing act on the rope. The length of the rope is exactly (2N + 1), indicated with integers $\{0, 1, 2, \ldots, 2N-1, 2N\}$. The acrobat successfully completes his performance when he manages to reach any of the two ends 0 or $2N$ of the rope. Today, however, our acrobat has lifted his elbow a little too much before the show, and so at this moment he is dangerously lurching (i.e. oscillating) on the rope. More precisely, he is so drunk that:
- with each step he takes there is a probability equal to $1/5$ he falls below and therefore fails his number;
- if instead he manages to stay in balance, it is equally probable that he takes a step forward or backward.

Suppose that at the initial time the acrobat is exactly in the middle of the string (i.e. in position N).

1) Model the state of the acrobat with a suitable discrete Markov chain (suggestion: Indicate with -1 the state "the acrobat fell down").

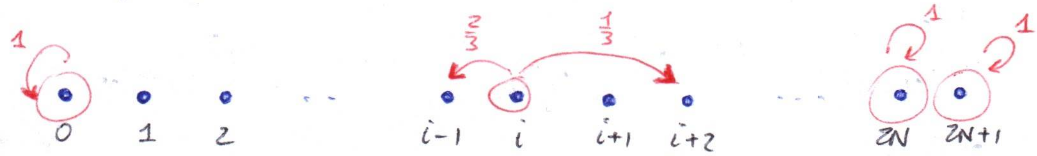
2) What is the probability that the acrobat can successfully finish his number?

3) How long does the number of acrobatics lasts on average (independently of its success or not)?

#1 (#3)

- $\mathbb{P}(H) = \frac{1}{3}$
- $X_0 = \text{initial capital} = N$, $X_{n+1} = \begin{cases} X_n + 2 & H \\ X_n - 1 & T \end{cases}$
- end : $\begin{cases} X_\cdot \geq 2N \\ X_\cdot = 0 \end{cases}$

1.



$$p_{ij} = \begin{cases} 1 & i=j = 0, 2N, 2N+1 \\ \frac{2}{3} & i \notin R \quad j = i-1 \\ \frac{1}{3} & i \notin R \quad j = i+2 \\ 0 & \text{otherwise} \end{cases} \qquad R := \{0, 2N, 2N+1\}$$

$$\begin{array}{c|ccccccccccccccc}
 & 0 & 1 & 2 & 3 & & i-1 & i & i+1 & i+2 & & 2N-1 & 2N & 2N+1 \\
\hline
0 & 1 & 0 & 0 & 0 & \cdots & & & & & & & & \\
1 & 2/3 & 0 & 0 & 1/3 & \cdots & & & & & & & & \\
2 & 0 & 2/3 & 0 & 0 & \cdots & & & & & & & & \\
3 & 0 & 0 & 2/3 & 0 & \cdots & & & & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & & & \\
i-1 & & & & & \cdots & 0 & 0 & \frac{1}{3} & 0 & \cdots & & & \\
i & & & & & \cdots & \frac{2}{3} & 0 & 0 & \frac{1}{3} & \cdots & & & \\
i+1 & & & & & \cdots & 0 & \frac{2}{3} & 0 & 0 & \cdots & & & \\
i+2 & & & & & \cdots & 0 & 0 & \frac{2}{3} & 0 & \cdots & & & \\
 & & & & & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
2N-1 & & & & & & & & & & \cdots & 0 & 0 & \frac{1}{3} \\
2N & & & & & & & & & & \cdots & 0 & 1 & 0 \\
2N+1 & & & & & & & & & & \cdots & 0 & 0 & 1
\end{array} = P$$

2. Probability of losing all the money = absorption in 0

$$V_i = p_{i0} + \sum_{j \in T} p_{ij} V_j \Longrightarrow \begin{cases} V_0 = 1 \\ V_{2N} = V_{2N+1} = 0 \\ V_1 = \frac{2}{3} + \frac{1}{3} V_3 \\ V_i = \frac{2}{3} V_{i-1} + \frac{1}{3} V_{i+2} \quad i \geq 2 \end{cases}$$

$$\frac{1}{3} V_{i+2} - V_i + \frac{2}{3} V_{i-1} = 0 \Longrightarrow \frac{1}{3} x^3 - x^2 + \frac{2}{3} = 0 \Longrightarrow x_{1/2} = 1, \ x_3 = -2$$

$$\Longrightarrow V_i = A + Bi + C(-2)^i \qquad i \geq 2 :$$

Can we write V_1 as it?

$$\begin{cases} V_2 = A + 2B + 4C \\ V_1 = A + B - 2C \\ V_4 = A + 4B + 16C \\ V_2 = \frac{2}{3} V_1 + \frac{1}{3} V_4 \end{cases} \longrightarrow A + 2B + 4C = \frac{2}{3}(A + B - 2C) + \frac{1}{3}(A + 4B + 16C)$$

$$3A + 6B + 12C = 3A + 6B + 12C \quad V$$

Can we express V_0 as it?

$$V_1 = A + B - 2C \overset{?}{=} \frac{2}{3}V_0 + \frac{1}{3}V_3 = \frac{2}{3}(A+C) + \frac{1}{3}(A+3B-8C) \quad \checkmark$$

$$\Longrightarrow \quad \begin{aligned} V_0 &= A + C = 1 \\ V_{2N} &= A + (2N)B + C(-2)^{2N} = 0 \\ V_{2N+1} &= A + (2N+1)B + C(-2)^{2N+1} = 0 \end{aligned} \qquad \begin{cases} x + z = 1 \\ x + ay + bz = 0 \\ x + (a+1)y - 2bz = 0 \end{cases}$$

$$\Longrightarrow \quad A = \frac{(-2)^{2N}(6N+1)}{1-(-2)^{2N}(6N+1)}, \quad B = \frac{3(-2)^{2N}}{1-(-2)^{2N}(6N+1)}, \quad C = \frac{1}{1-(-2)^{2N}(6N+1)}$$

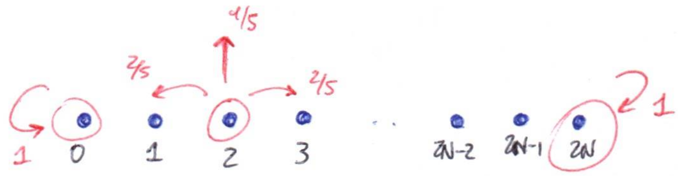
$$V_N = A + BN + C(-2)^N$$

$x = A$, $y = B$, $z = C$; $a = 2N$, $b = (-2)^{2N}$

#2

- $E = \{0, 1, 2, \ldots, 2N-1, 2N\}$: $0/2N$ end points $\quad E = E \cup \{-1\}$
- each step he can fail with probability $= \frac{1}{5}$
- if he doesn't fall he goes $\leftarrow / \rightarrow$ with the same probability
- $X_0 = N$

1. $$p_{ij} = \begin{cases} 1 & i = j \in R = \{0, 2N, -1\} \\ 1/5 & i \notin R,\ j = -1 \\ 2/5 & i \notin R,\ j = i \pm 1 \end{cases}$$



2. Probability of finishing in $0 / 2N$?

$=$ prob. of absorption in $\{0, 2N\}$

$$V_i = p_{i0} + p_{i(2N)} + \sum_{j \in T} p_{ij} V_j \quad : \quad \begin{cases} V_{-1} = 0 \\ V_0 = V_{2N} = 1 \\ V_1 = \frac{2}{5} + \frac{2}{5}V_2 \\ V_{2N-1} = \frac{2}{5} + \frac{2}{5}V_{2N-2} \\ V_i = \frac{2}{5}V_{i-1} + \frac{2}{5}V_{i+1} \quad i \in \{2, \ldots, 2N-3\} \end{cases}$$

$$5V_i = 2V_{i-1} + 2V_{i+1} \quad \leftarrow$$

$$2x^2 - 5x + 2 = 0 \qquad x_{1/2} < \begin{matrix} 2 \\ \frac{1}{2} \end{matrix}$$

$$V_i = B(2)^i + A\left(\tfrac{1}{2}\right)^i \qquad i \in \{2, \ldots, 2N-3\}$$

Can we write V_0 and V_1 like this?

$$2V_3 - 5V_2 + 2\underline{V_1} \overset{?}{=} 0 \quad : \quad \frac{1}{4}A + 16B - \frac{5}{4}A - 20B + A + 4B = 0 \quad \checkmark$$

$$2V_2 - 5V_1 + 2\underline{V_0} \overset{?}{=} 0 \quad : \quad \frac{A}{2} + 8B - \frac{5}{2}A - 10B + 2A + 2B = 0 \quad \checkmark$$

$$\Longrightarrow \text{ we impose: } \quad \begin{aligned} V_0 &= 1 \qquad & A + B &= 1 \\ V_{2N} &= 1 \qquad & \left(\tfrac{1}{2}\right)^{2N}A + (2)^{2N}B &= 1 \end{aligned}$$

$$\Longrightarrow (A, B) = \left(2^{2N}\frac{2^{2N}-1}{2^{4N}-1}, \frac{2^{2N}-1}{2^{4N}-1}\right)$$

$$\Longrightarrow V_N = A\left(\tfrac{1}{2}\right)^N + B(2)^N$$

#2 (#3)

3. How long does he last on average?
= mean of the hitting time of $C = \{-1, 0, 2N\}$:

$$\begin{cases} w_i = 1 + \sum_{j \in T} p_{ij} w_j = 1 + p_{i(i-1)} w_{i-1} + p_{i(i+1)} w_{i+1} & i \in \{2, .., 2N-2\} \\ w_1 = 1 + p_{12} w_2 \\ w_{2N-1} = 1 + p_{(2N-1)(2N-2)} w_{2N-2} \end{cases}$$

Homogeneous :
$$w_i = \frac{2}{5} w_{i-1} + \frac{2}{5} w_{i+1}$$

$$5x = 2 + 2x^2 \implies 2x^2 - 5x + 2 = 0 < \begin{matrix} 2 \\ \frac{1}{2} \end{matrix}$$

$$w_i = A\left(\frac{1}{2}\right)^i + B(2)^i$$

Complete :
$$w_i = A\left(\frac{1}{2}\right)^i + B(2)^i + 5$$

particular solution

$w_i = 5 \Rightarrow 2 \cdot 5 - 5 \cdot 5 + 2 \cdot 5 = -5$ ✓

[..]