Question 1: GAIN of a dynamical system

$$u(t) \longrightarrow \boxed{W(z)} \xrightarrow{y(t)} \qquad W(z) = \frac{C(z)}{A(z)}$$

For example: $W(z) = \dfrac{1+cz^{-1}}{1-az^{-1}}$

We want to find the mean value of the output given the mean value of the input.

- Suppose that $u(t)$ is a deterministic constant signal:

$u(t) = \bar{u} \quad \forall t$

$y(t)$? $\quad y(t) \longrightarrow \bar{y}$ since our assumption is that the system is stable

$$y(t) = W(z)\,u(t) = \frac{1+cz^{-1}}{1-az^{-1}}\,u(t)$$

$$(1-az^{-1})\,y(t) = (1+cz^{-1})\,u(t)$$

$$y(t) = a\,y(t-1) + u(t) + c\,u(t-1)$$

$$\Longrightarrow \bar{y} = a\bar{y} + \bar{u} + c\bar{u}$$

$$\Longrightarrow \bar{y} = \frac{1+c}{1-a}\,\bar{u}$$

relation between the output and the input when the input is a deterministic constant signal

$\dfrac{1+c}{1-a} \longrightarrow W(z)\big|_{z=1} := \text{GAIN } (:= \mu)$

$$\Longrightarrow \bar{y} = \mu\,\bar{u}$$

- Suppose that $u(t)$ = stationary process with mean $\bar{u}$

$$\underbrace{\mathbb{E}[y(t)]}_{\bar{y}} = \mathbb{E}[a\,y(t-1) + u(t) + c\,u(t-1)]$$

$$\bar{y} = a\bar{y} + \bar{u} + c\bar{u}$$

$$\Longrightarrow \frac{\bar{y}}{\bar{u}} = \text{gain} = W(z)\big|_{z=1}$$

Question 2: PERIODOGRAM

How can we estimate the spectrum from data?

$$\Gamma(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma(\tau)\,e^{-j\omega\tau} \qquad (*)$$

$\Longrightarrow$ We have to estimate the covariance function $\gamma(\tau)$ from data. How?

$$\gamma(0) = \frac{1}{N}\sum_{t=1}^{N} y(t)^2$$

$$\gamma(1) = \frac{1}{N}\sum_{t=1}^{N-1} y(t)\,y(t+1)$$

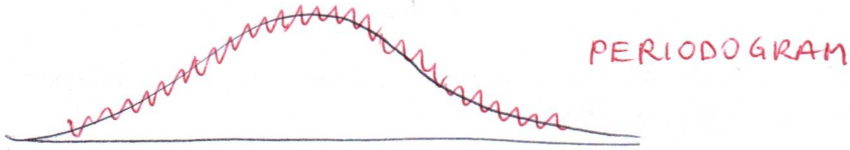
$$\gamma(2) = \frac{1}{N}\sum_{t=1}^{N-2} y(t)\,y(t+2)$$

...

$\Longrightarrow$ 1. Computation of $\gamma(\tau)$ from data
2. Troncation of formula $(*)$:

$$\Longrightarrow \sum_{\tau=-(N-1)}^{N-1} \gamma(\tau)\, e^{-j\omega\tau}$$

and we obtain:



Another way:

Data estimate of an AR or an ARMA model, then we use the formula of fundamental theorem of spectral analysis.
(:= minimum entropy estimation of $\Gamma(\omega)$)