POLITECNICO DI MILANO

Exercises on Estimation of reliability parameters from experimental data

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T=TTT (Total Time on Test)
r=number of failures

	I, fixed t_0	II, fixed r
one-sided (lower) $P(MTTF > \vartheta_1) = \alpha$	$\vartheta_1 = \frac{2T}{\chi^2_\alpha(2r+2)}$ (# of degrees of freedom; percentile)	$\vartheta_1 = \frac{2T}{\chi^2_\alpha(2r)}$
two-sided (lower and upper) $P(\vartheta_1 < MTTF < \vartheta_2) = \alpha$	$(\vartheta_1, \vartheta_2) = \frac{2T}{\chi^2_{\frac{1+\alpha}{2}}(2r+2)}, \frac{2T}{\chi^2_{\frac{1-\alpha}{2}}2r}$	$(\vartheta_1, \vartheta_2) = \frac{2T}{\chi^2_{\frac{1+\alpha}{2}}(2r)}, \frac{2T}{\chi^2_{\frac{1-\alpha}{2}}(2r)}$

α Percentile values of the $\chi^2(f)$ distribution

f \ α	0.005	0.025	0.050	0.900	0.950	0.975	0.990	0.995	0.999
1	0.0439	0.03982	0.02393	2.71	3.84	5.02	6.63	7.88	10.8
2	0.0100	0.0506	0.103	4.61	5.99	7.38	9.21	10.6	13.8
3	0.0717	0.216	0.352	6.25	7.81	9.35	11.3	12.5	16.3
4	0.207	0.484	0.711	7.78	9.49	11.1	13.3	14.9	18.5
5	0.412	0.831	1.15	9.24	11.1	12.8	15.1	16.7	20.5
6	0.676	1.24	1.64	10.6	12.6	14.4	16.8	18.5	22.5
7	0.989	1.69	2.17	12.0	14.1	16.0	18.5	20.3	24.3
8	1.34	2.18	2.73	13.4	15.5	17.5	20.1	22.0	26.1
9	1.73	2.70	3.33	14.7	16.9	19.0	21.7	23.6	27.9
10	2.16	3.25	3.94	16.0	18.3	20.5	23.2	25.2	29.6
11	2.60	3.82	4.57	17.3	19.7	21.9	24.7	26.8	31.3
12	3.07	4.40	5.23	18.5	21.0	23.3	26.2	28.3	32.9
13	3.57	5.01	5.89	19.8	22.4	24.7	27.7	29.8	34.5
14	4.07	5.63	6.57	21.1	23.7	26.1	29.1	31.3	36.1
15	4.60	6.26	7.26	22.3	25.0	27.5	30.6	32.8	37.7
16	5.14	6.91	7.96	23.5	26.3	28.8	32.0	34.3	39.3
17	5.70	7.56	8.67	24.8	27.6	30.2	33.4	35.7	40.8
18	6.26	8.23	9.39	26.0	28.9	31.5	34.8	37.2	42.3
19	6.84	8.91	10.1	27.2	30.1	32.9	36.2	38.6	43.8
20	7.43	9.59	10.9	28.4	31.4	34.2	37.6	40.0	45.3
21	8.03	10.3	11.6	29.6	32.7	35.5	38.9	41.4	46.8
22	8.64	11.0	12.3	30.8	33.9	36.8	40.3	42.8	48.3
23	9.26	11.7	13.1	32.0	35.2	38.1	41.6	44.2	49.7
24	9.89	12.4	13.8	33.2	36.4	39.4	43.0	45.6	51.2
25	10.5	13.1	14.6	34.4	37.7	40.6	44.3	46.9	52.6
26	11.2	13.8	15.4	35.6	38.9	41.9	45.6	48.3	54.1
27	11.8	14.6	16.2	36.7	40.1	43.2	47.0	49.6	55.5
28	12.5	15.3	16.9	37.9	41.3	44.5	48.3	51.0	56.9
29	13.1	16.0	17.7	39.1	42.6	45.7	49.6	52.3	58.3
30	13.8	16.8	18.5	40.3	43.8	47.0	50.9	53.7	59.7
35	17.2	20.6	22.5	46.1	49.8	53.2	57.3	60.3	66.6
40	20.7	24.4	26.5	51.8	55.8	59.3	63.7	66.8	73.4
45	24.3	28.4	30.6	57.5	61.7	65.4	70.0	73.2	80.1
50	28.0	32.4	34.8	63.2	67.5	71.4	76.2	79.5	86.7

Basic random variable	Parameter	Prior and posterior distributions of parameter	Mean and Variance of Parameter	Posterior Statistics
Binomial $p_X(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$	θ	Beta $f_\Theta(\theta) = \frac{\Gamma(q+r)}{\Gamma(q)\Gamma(r)}\theta^{q-1}(1-\theta)^{r-1}$	$E(\Theta) = \frac{q}{q+r}$ $Var(\Theta) = \frac{qr}{(q+r)^2(q+r+1)}$	$q'' = q' + x$ $r'' = r' + n - x$
Exponential $f_X(x) = \lambda e^{-\lambda x}$	λ	Gamma $f_\Lambda(\lambda) = \frac{\nu(\nu\lambda)^{k-1}e^{-\nu\lambda}}{\Gamma(k)}$	$E(\Lambda) = \frac{k}{\nu}$ $Var(\Lambda) = \frac{k}{\nu^2}$	$\nu'' = \nu' + \sum_i x_i$ $k'' = k' + n$
Normal $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ (with known σ)	μ	Normal $f_M(\mu) = \frac{1}{\sqrt{2\pi}\sigma_\mu}\exp\left[-\frac{1}{2}\left(\frac{\mu-\mu_\mu}{\sigma_\mu}\right)^2\right]$	$E(\mu) = \mu_\mu$ $Var(\mu) = \sigma_\mu^2$	$\mu_\mu'' = \frac{\bar{x}(\sigma_\mu')^2 + \mu_\mu'(\sigma^2/n)}{\sigma^2/n + (\sigma_\mu')^2}$ $\sigma_\mu'' = \sqrt{\frac{(\sigma_\mu')^2(\sigma^2/n)}{(\sigma_\mu')^2 + \sigma^2/n}}$
Normal $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$	μ, σ	Gamma-Normal $f(\mu, \sigma) = \left\{\frac{1}{\sqrt{2\pi}\sigma/\sqrt{n}}\exp\left[-\frac{1}{2}\left(\frac{\mu-\bar{x}}{\sigma/\sqrt{n}}\right)^2\right]\right\} \cdot \left\{\frac{[(n-1)/2]^{(n-1)/2}}{\Gamma[(n+1)/2]}\left(\frac{s^2}{\sigma^2}\right)^{(n+1)/2} \cdot \exp\left(-\frac{n-1}{2}\frac{s^2}{\sigma^2}\right)\right\}$	$E(\mu) = \bar{x}$ $Var(\mu) = s^2\left[\frac{n-1}{n(n-3)}\right]$ $E(\sigma) = s\sqrt{\frac{n-1}{2}}\frac{\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}$ $Var(\sigma) = s^2\left(\frac{n-1}{n-3}\right) - E^2(\sigma)$	$n'' = n' + n$ $n''\bar{x}'' = n'\bar{x}' + n\bar{x}$ $(n''-1)s''^2 + n''\bar{x}''^2$ $= [(n'-1)s'^2 + n'\bar{x}'^2]$ $+ [(n-1)s^2 + n\bar{x}^2]$
Poisson $p_X(x) = \frac{(\nu t)^x}{x!}e^{-\nu t}$	ν	Gamma $f_N(\nu) = \frac{t(\nu t)^{k-1}e^{-\nu t}}{\Gamma(k)}$	$E(\nu) = \frac{k}{t}$ $Var(\nu) = \frac{k}{t^2}$	$t'' = t' + t$ $k'' = k' + x$
Lognormal $f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} \cdot \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right]$ (with known ζ)	λ	Normal $f_\Lambda(\lambda) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{\lambda-\mu}{\sigma}\right)^2\right]$	$E(\Lambda) = \mu$ $Var(\Lambda) = \sigma^2$	$\mu'' = \frac{\mu'(\zeta^2/n) + \sigma'^2\,\overline{\ln x}}{\zeta^2/n + \sigma'^2}$ $\sigma'' = \sqrt{\frac{\sigma'^2(\zeta^2/n)}{\sigma'^2 + \zeta^2/n}}$

Exercise 1 X

A feedwater pump of an energy production plant is characterized by a constant failure rate. In order to assess the performance of the pump, a right censored test of the first type is carried out on 10 identical pumps. The duration of the test is $t_0 = 500$ hours for each pump. Table 1 reports the observed failure times.

Table 1. Results of the reliability tests on the 10 pumps.

Pump 1	Pump 2	Pump 3	Pump 4	Pump 5	Pump 6	Pump 7	Pump 8	Pump 9	Pump 10
205	99	No failure during the test time	91	458	No failure during the test time	No failure during the test time	No failure during the test time	43	35

You are required to:
1) estimate the pump failure rate using the method of maximum likelihood;
2) what is the 95% two-sided confidence interval of the pump mean time to failure.

Exercise 2 X

The failure time of a new type of industrial filter is an exponential random variable having an unknown value λ. A group of twenty filters are being monitored and, at present, their failure times are (in weeks):

1.2, 1.8*, 2.2, 4.1, 5.6, 8.4, 11.8*, 13.4*, 16.2, 21.7, 29*, 41, 42*, 42.4*, 49.3, 60.5, 61*, 94, 98, 99.2*,

where an * next to the data means that the filter is still working, whereas an unstarred data point means that the filter failed at that time.

1. What is the maximum likelihood estimate $\hat{\lambda}_{MLE}$ of λ.
2. What is the 90% two-sided confidence interval of λ.

Exercise 3 X

The number of defective rivets, D, on an airplane wing can be assumed to have a Poisson distribution with parameter λ, i.e.,

$$P(D = d) = \frac{\lambda^d e^{-\lambda}}{d!}, \quad d = 0, 1, 2, \ldots$$

A random sample of n wings is observed and $(d_1, d_2, \ldots, d_n)$ defective rivets are found.

1. What is $\hat{\lambda}_{MLE}$, the maximum likelihood estimator of λ?
2. Is this estimator unbiased?
3. Find the method-of-moments estimator of λ.

Exercise 4 X

Let p be the probability of failure on demand of a new type of relief valve used in energy production plants. Considering past experience on similar relief valves, an expert suggests that p can have only three values: $p_1 = 5 \cdot 10^{-4}$, $p_2 = 1 \cdot 10^{-3}$, $p_3 = 5 \cdot 10^{-3}$. Furthermore, the expert has observed the operation of similar valves in energy production plants for a long period of time and he proposes to use the following prior distribution for p:

$$P(p = p_1) = 0.2$$
$$P(p = p_2) = 0.6$$
$$P(p = p_3) = 0.2$$

The new type of relief valve is then used for 1 year and 2 failures to start over 500 demands are observed. You are required to:

a) update the probability distribution of p;
b) Compute the probability that the new type of valve will have 0 failures out of 3 demands

Exercise 5 ✗

The time to failure T (years) of a certain item is an exponential random variable with probability density function:

$$p(t) = \lambda e^{-\lambda t}, t > 0$$

From prior experience we are led to believe that λ is a value of an exponential random variable Λ with probability density function:

$$\pi'(\lambda) = 2e^{-2\lambda}, \lambda > 0$$

1. Estimate the item reliability at a time $t = 1$ year.
2. if we have a sample of 3 item failure times: $(t_1, t_2, t_3) = (1; 2.8; 2.2)$, find the posterior distribution of Λ, the new estimation of the item reliability at time $t = 1$ year and the 95% upper confidence limit of λ (numerical solution of the integral is not required).
3. What is the maximum likelihood estimation of λ and its 95% upper confidence limit using Frequentist statistics?

Exercise 6 ✗

Two independent and identical cables feed an important node in a power distribution network. Assume that the failure time of each cable is an exponential random variable with the (same) failure rate λ. Assuming that the following failure times have been data collected during a right censored test of the first type performed on ten cables with test duration equal to 15years:

9.1 7.7 5.5 11.7 10.6 7.4 9.1 10.7 8.9 12.5 [in years]

You are required to:

Q1a) estimate the parameter λ using the method of maximum likelihood;

Q1b) estimate the 95% two-sided confidence interval of the cable failure rate;

Q1c) repeat Q1a) following a Bayesian approach. Assume that according to an expert, the prior distribution of λ is:

$$P'(\lambda) = 0.1e^{-0.1\lambda}$$

Q1d) Consider the following maintenance strategy: each time a cable fails, a maintenance intervention (repair) is immediately performed. During maintenance, the power distribution is interrupted. The cost of the repair is 10000 €, whereas the cost of the power distribution interruption is 30000 €. You are required to estimate the average overall cost of the cables maintenance strategy in one year for the grid owner.

EXPONENTIAL
– GAMMA

$f(\cdot)$, $E[\cdot]$, $Var(\cdot)$

#1 (#9)

$T \sim \mathcal{E}(\lambda)$: $f_T(t) = \lambda e^{-\lambda}$, $F_T(t) = 1 - e^{-\lambda}$, $R_T(t) = e^{-\lambda}$

1. $L(\lambda) = \prod_{i=1}^{6} f(t_i) \prod_{j=1}^{4} R(t_j) = \left(\lambda^6 e^{-(\sum_{i=1}^{6} t_i)\lambda}\right)\left(e^{-(\sum_{j=1}^{4} t_0)\lambda}\right) = \lambda^6 e^{-\lambda\left(\sum_{i=1}^{6} t_i + 4t_0\right)}$ TTT

$\ell(\lambda) = \log(L(\lambda)) = 6\log(\lambda) - \lambda TTT$

$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{6}{\lambda} - TTT = 0 \Rightarrow \lambda = \frac{6}{TTT} = \frac{6}{2931} = 2.047 \cdot 10^{-3}$

2. $(\lambda_1^{-1}, \lambda_2^{-1}) = \left[\frac{2T}{\chi^2_{\frac{1+\alpha}{2}}(2r+2)}, \frac{2T}{\chi^2_{\frac{1-\alpha}{2}}(2r)}\right]$

$T = TTT = 2931$
$\alpha = 0.95$
$r = \#\text{failed} = 6$

$\Rightarrow$ $\chi^2_{0.975}(14) = 26.1$
$\chi^2_{0.025}(12) = 4.40$
$\Rightarrow [224.598 ; 1332.273]$
$\leq MTTF \leq$

$\Longrightarrow (\,)^{-1}$: $7.506 \cdot 10^{-4} \leq \lambda \leq 4.4524 \cdot 10^{-3}$

#2

$n = 20$, $r = n - 8 = 12$

$\lambda = \frac{r}{TTT} = \frac{12}{702.8} = 0.01707$

$\alpha = 0.90$
$\chi^2_{0.95}(26) = 38.9$
$\chi^2_{0.05}(24) = 13.8$

$\Longrightarrow \left[\frac{13.8}{2 \cdot 702{,}8}, \frac{38.9}{2 \cdot 702{,}8}\right] = [0{,}00981787, 0.027675]$

#3

$L(\lambda) = \prod_{i=1}^{n} \left(\frac{\lambda^{d_i} e^{-\lambda}}{d_i!}\right) = \frac{\lambda^{\sum d_i} e^{-n\lambda}}{\prod_{i=1}^{n}(d_i!)}$

1. $\log L(\lambda) \propto (\sum d_i)\log(\lambda) - n\lambda \longrightarrow \frac{\partial}{\partial\lambda}$: $\frac{\sum d_i}{\lambda} - n = 0 \Rightarrow \hat{\lambda}_{MLE} = \frac{\sum d_i}{n}$
2. $\mathbb{E}[\hat{\lambda}_{MLE}] = \frac{1}{n}\sum \mathbb{E}[d_i] = \frac{1}{n} n\lambda = \lambda$ ✓
3. $\mathbb{E}[D] = \frac{\sum d_i}{n}$ = first moment = $\hat{\lambda}_{MOM}$

#4

$\mathbb{P}(p_i|\text{data}) = \frac{\mathbb{P}(\text{data}|p_i)\,\mathbb{P}(p_i)}{\mathbb{P}(\text{data})}$

$\mathbb{P}(X \sim Bin(n=500, p=p_i) = 2)$

a)

	p	$\mathbb{P}(p)$	$\mathbb{P}(\text{data}\mid p)$	$\mathbb{P}(\text{data}\mid p)\,\mathbb{P}(p)$	$\mathbb{P}(p')$
p_1	$5 \cdot 10^{-4}$	0.2	0.0243	0.0048	0.0471
p_2	10^{-3}	0.6	0.0758	0.0454	0.4468
p_3	$5 \cdot 10^{-3}$	0.2	0.2570	0.0514	0.5060

b) $\mathbb{P}$(0 failures out of 3 demands) =

$= 0.0471 \underbrace{\mathbb{P}(X \sim Bi(3, p_1) = 0)}_{(1-p_1)^3} + 0.4468 \cdot \underbrace{\mathbb{P}(X \sim Bi(3, p_2) = 0)}_{(1-p_2)^3} + 0.5060 \underbrace{\mathbb{P}(X \sim Bi(3, p_3) = 0)}_{(1-p_3)^3}$

$= 0.99089$

#5

1. $R(1) = \mathbb{P}(T>1) = \int_0^\infty \mathbb{P}(T>1|\lambda)\,\pi(\lambda)\,d\lambda = \int_0^\infty e^{-\lambda}\,2e^{-2\lambda}\,d\lambda = \frac{2}{3}$

2. posterior $\propto$ likelihood $\cdot$ prior

$$\propto (\lambda^3 e^{-\lambda \Sigma t_i})(2e^{-2\lambda}) = \lambda^3 e^{-8\lambda}$$

posterior = Gamma $(\lambda=8, \alpha=4)$: $\pi(\lambda|data) = \frac{8^4 \lambda^{4-1} e^{-8\lambda}}{\Gamma(4)} = \frac{8(8\lambda)^3 e^{-8\lambda}}{\Gamma(4)}$

$$R(1) = \mathbb{P}(T>1) = \int_0^\infty \mathbb{P}(T>1|\lambda')\,\pi(\lambda')\,d\lambda' = \int_0^\infty (e^{-\lambda})\left(\frac{8^4}{6}\lambda^3 e^{-8\lambda}\right)d\lambda = 0.6243$$

Bayesian statistic upper bound:

$$\boxed{\int_0^{\lambda_{0.95}} \pi(\lambda')\,d\lambda' = 0.95 \implies \lambda_{0.95} = 0.9676}$$

!

3. $\lambda_{MLE} = \frac{n}{TTT} = \frac{3}{6} = \frac{1}{2}$

95% for MTTF : $\frac{2T}{\chi^2_{0.95}(6)} = \frac{12}{12.6} \implies \lambda_{0.95} = \frac{12.6}{12} = 1.05$

#6 (TDE)

a) exponential $\rightarrow$ $\hat{\lambda}_{MLE} = \frac{r}{TTT} = \frac{10}{93.2} = 0.107296137$

b) $\xrightarrow{\alpha=0.95}$ $\lambda \in \left[\frac{\chi^2_{\frac{1-\alpha}{2}}(2r)}{2\cdot TTT}, \frac{\chi^2_{\frac{1+\alpha}{2}}(2r+2)}{2\cdot TTT}\right] = [0.058012876, 0.183476395]$

numbers are possibly wrong

c) $\pi(\lambda) = 0.1\,e^{-0.1\lambda}$

posterior $\propto$ likelihood $\cdot$ prior

$$\propto (\lambda^{10} e^{-\lambda\cdot T})\,0.1\,e^{-0.1\lambda}$$

$$\propto 0.1\,\lambda^{10} e^{-\lambda\,93.3} \rightarrow \text{Gamma}(\lambda = 93.3, \alpha = 11)$$

$$\pi(\lambda|data) = \frac{(93.3)^{11}\lambda^{10} e^{-93.3\lambda}}{\Gamma(11)}$$

d) For a cable $\hat{\lambda}_{MLE} = 0.107296137$ $(\sim P(\lambda))$

since the system has two cables:

$$\lambda_{system} = 2\hat{\lambda}_{MLE} = 0.214592275 \quad (\sim P(\lambda))$$

If we have k failures the cost will be: $k(10'000 + 30'000)$, since k obeys a Poisson the expectation is

$\lambda = \lambda_{system}$

so the overall cost in one year is:

$$\lambda_{system}(10'000 + 30'000)\cdot 1 = 8583.690987$$