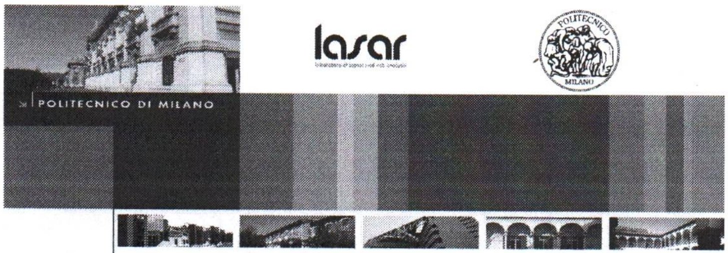

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Exercises - Probabilistic models of failure processes, Failure time distributions

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Exercise 1

Spare unit allocation

The allocation of the proper number of spare units for a single component system is of concern.

- Assume that if the operating component fails, it is instantaneously replaced by a spare unit, if available.
- Once the component has failed, it cannot be repaired.
- The rate of occurrence of components' failures is $1.67\ y^{-1}$, if the component is operating.
- Assume that the component cannot fail while in the spares depot.

The system designer's goal is to achieve a probability of system operation (reliability) of at least **0.95** at the mission time T_M of one year.

How many spare units should be allocated to achieve this goal?

Exercise 2

Peak stresses on a component

Suppose that peak stresses (i.e. stresses which exceed a certain value) occur randomly at an average rate λ. The probability that a component will survive the application of a peak stress is (1 - p) (constant). Show that the reliability of the component is $R(t) = e^{-p\lambda t}$.

Exercise 3

Traffic control 1

Suppose that, from a previous traffic count, an average of **60 cars per hour** was observed to make left turns at an intersection.

What is the probability that exactly 10 cars will be making left turns in a 10 minute interval?

Discretize the time interval of interest to approach the problem with the binomial distribution. Show that the solution of the problem tends to the exact solution obtained with the Poisson distribution as the time discretization gets finer.

Exercise 4

Traffic control 2

Suppose that it is observed that, on average, **100 cars per hour** reach an intersection. Also, it has been estimated that the probability for a car to make a left tum is **0.6**.

What is the probability that exactly 10 cars will be making left turns in a 10 minute interval?

Exercise 5

Simple system

Consider a system of two independent components with exponentially distributed failure times. The failure rates are λ_1 and λ_2, respectively.

Determine the probability that component 1 fails before component 2.

$\mathbb{P}(X_2 > X_1) \neq \mathbb{P}(X_1 < t)\,\mathbb{P}(X_2 > t)$

Exercise 6

Television picture tubes

The television picture tubes of manufacturer A have a mean lifetime of 6.5 years and a standard deviation of 0.9 years, while those of manufacturer B have a mean lifetime of 6.0 years and a standard deviation of 0.8 years. What is the probability that a random sample of 36 tubes from manufacturer A will have a mean lifetime that is at least 1 year more than the mean lifetime of a sample of 49 tubes from manufacturer B?

(Note: The statistical mean of a random sample, typically called "sample mean", is a function of the random sample and it is, therefore, a random variable itself. In fact, if I take another sample, it will in general be different from the previous one and the sample mean will take a different value. It can be shown that the expected value, or theoretical mean, of the sample mean coincides with the expected value of the underlying population distribution from which the sample was drawn whereas the variance of the sample mean turns out to be equal to the variance of the underlying population distribution, divided by the number n of values constituting the sample.)

Exercise 7

careful at modelling with poisson always t)

Capacitor

A capacitor is placed across a power source. Assume that surge voltages occur on the line at a rate of one per month and they are normally distributed with a mean value of 100 volts and a standard deviation of 15 volts. The breakdown voltage of the capacitor is 135 volts.

1. Find the mean time to failure (MTTF) for this capacitor
2. Find its reliability for a time period of one month

#1 (#3)

$N = \#$ spare units
$\lambda =$ component failure rate $= 1.67$
$T_M =$ mission time $= 1$
We have $N+1$ components (+1 because it's the one working, & spare).
The system fails if $N+1$ components fails, more precisely: if we have $N+1$ failures (in $[0, T_M]$)

$\longrightarrow$ $K =$ number of failures $\sim P(k; (0, T_M), \lambda)$

$$\mathbb{P}(K=k) = \frac{(\lambda T_M)^k e^{-(\lambda T_M)}}{k!}$$

$$R(T_M) = \mathbb{P}(K < N+1) = \sum_{i=0}^{N} \mathbb{P}(K=i) \longrightarrow$$

Trial and error:
$N=3 \longrightarrow R(T_M) = 0.910$
$N=4 \longrightarrow R(T_M) = 0.968$

$\Longrightarrow$ To have $R(T_M) \geq 0.95$ we use $N=4$

#2

$N = \#$ peak stresses in $(0,t)$ $\sim P(\lambda t)$
$\mathbb{P}(\text{surviving a peak stress}) = (1-p)$

$$R(t) = \mathbb{P}(\text{surviving until time } t \text{ at least}) = \sum_{k=0}^{+\infty} \mathbb{P}(\text{surviving } k \text{ stresses})\, \mathbb{P}(\text{having } k \text{ stresses})$$
$$= \sum_{k=0}^{+\infty} (1-p)^k \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$
$$= e^{-\lambda t} \sum_{k=0}^{+\infty} \frac{((1-p)\lambda t)^k}{k!}$$
$$= e^{-\lambda t} e^{(1-p)\lambda t} = e^{-\lambda p t}$$

#3

$\rightarrow$ Discretization 60 minutes $\Longrightarrow$ 120 intervals (30 seconds)
(we assume only 1 turn per interval) — unrealistic if the interval is of 30 seconds

$$\mathbb{P}(\text{turn in an interval}) = \frac{60}{120} = \frac{1}{2} = p$$

10 minutes = 20 intervals $\longrightarrow$ binomial $(n=20, p=\frac{1}{2})$
10 cars:

$$\mathbb{P}(X=10) = \binom{20}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{20-10} = 0.1762$$

$\rightarrow$ Discretization 60 minutes $\Longrightarrow$ 360 intervals (10 seconds)

$$\mathbb{P}(\text{turn in an interval}) = \frac{60}{360} = 0.16667 = p$$

binomial$(n=60, p=0.16667)$ (60 intervals of 10 seconds = 10 minutes)

$$\mathbb{P}(X=10) = \binom{60}{10}(0.16667)^{10}(1-0.16667)^{60-10} = 0.1370$$

$\rightarrow$ Poisson: $X \sim P(x; \lambda, (0,60)) \Longrightarrow E[X] = \lambda \cdot 60 = 60 \Longrightarrow \lambda = 1$
$Y \sim P(y; \lambda, (0,10))$ $\lambda = 1$

$$\mathbb{P}(Y=10) = \frac{e^{-10}(10)^{10}}{10!} = 0.12511$$

#4

$\sim P(k; \lambda, (0,60)) \Longrightarrow E[\,] = 60 \cdot \lambda = 100 \Longrightarrow \lambda = 5/3$
$\mathbb{P}(\text{turning left}) = 0.60 = p$
$X = \#$ arrivals in 10 minutes $\sim P(x; \lambda = 5/3, t = 10)$

$$\mathbb{P}(10 \text{ go left}) = \sum_{k=10}^{+\infty} \mathbb{P}(\text{arrives } k)\, \mathbb{P}(10 \text{ out of } k \text{ go left})$$
$$= \sum_{k=10}^{+\infty} \left(\frac{(\lambda t)^k}{k!} e^{-\lambda t}\right)\left(\binom{k}{10} p^{10} (1-p)^{k-10}\right)$$
$$= \sum_{j=0}^{+\infty} \left(\frac{e^{-\lambda t}(\lambda t)^{10} p^{10}}{10!}\right) \frac{(\lambda t(1-p))^j}{j!} = (\cdot\cdot)\, e^{(1-p)\lambda t} = \frac{1}{10!}(p\lambda t)^{10} e^{-p\lambda t}$$
$$= 0.12511$$

#5

$X_1 \sim \mathcal{E}(\lambda_1)$, $X_2 \sim \mathcal{E}(\lambda_2)$

$$\mathbb{P}(X_2 > X_1) = \int_0^{+\infty} \mathbb{P}(X_2 > t \mid X_1 = t)\, f_{X_1}(t)\, dt$$
$$= \int_0^{+\infty} e^{-\lambda_2 t}\, \lambda_1 e^{-\lambda_1 t}\, dt$$
$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Generalization:
System of n independent components with failure rates $\lambda_1, \ldots, \lambda_n$. The probability that the component j is the first to fail is:

$$\mathbb{P}(j \text{ fails first}) = \frac{\lambda_j}{\sum_{i=1}^n \lambda_i}$$

#6

$X_A \sim N(6.5, 0.9)$ $\qquad N(\mu, \sigma)$

$X_B \sim N(6.0, 0.8)$

$\mathbb{P}(\bar{X}_A^{36} \geq \bar{X}_B^{49} + 1)$:

$$\bar{X}_A^{36} \sim N\left(\mu_A, \frac{\sigma_A}{\sqrt{n_A}}\right) = N(6.5, 0.15)$$
$$\bar{X}_B^{49} \sim N\left(\mu_B, \frac{\sigma_B}{\sqrt{n_B}}\right) = N(6.0, 4/35)$$

$$T := \bar{X}_A^{36} - \bar{X}_B^{49} - 1 \sim N\left(\mu_A - \mu_B - 1, \sqrt{(\sigma_{\bar{A}^{36}})^2 + (\sigma_{\bar{B}^{49}})^2}\right) = N(-1/2, 0.1885768)$$

$$\mathbb{P}(T \geq 0) = 1 - \mathbb{P}(T < 0) = 1 - \Phi\left(\frac{1/2}{0.1885768}\right) = 1 - \Phi(2.65) = 0.0043$$

(approx. with $0.188 = 0.19$ and so $\Phi(2.63)$)

#7

$K = \#$ surge occurrence $\sim p(k; \lambda = 1, (0,t))$: $\mathbb{P}(K = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$

$M =$ magnitude of the surge $\sim N(100, 15) = N(\mu, \sigma)$

$$p := \mathbb{P}(\text{lethal surge}) = \mathbb{P}(M > 135) = 1 - \mathbb{P}(M \leq 135) = 1 - \Phi\left(\frac{135-100}{15}\right) = 1 - \overset{0.9901}{\Phi(2.33)} = 0.0099$$

2. $R(t) = \mathbb{P}(T = \text{time to failure} > t) = \mathbb{P}(\text{no lethal surge till } t)$

$$= \sum_{j=0}^{+\infty} \mathbb{P}(j \text{ surges})\, \mathbb{P}(\text{no lethal surges out of } j)$$
$$= \sum_{j=0}^{+\infty} \left(e^{-\lambda t} \frac{(\lambda t)^j}{j!}\right) (1-p)^j$$
$$= e^{-\lambda t} e^{\lambda t (1-p)} = e^{-\lambda t p}$$

$$\mathbb{R}(1 \text{ month}) = e^{-\lambda p \cdot 1} = 0.99$$

. $\mathbb{P}(T \leq t) = 1 - \mathbb{P}(T > t) = 1 - R(t) = 1 - e^{-\lambda t p} \quad \sim \mathcal{E}(\lambda p)$

$$\mathbb{E}[T] = MTTF = \frac{1}{\lambda p} = 101.01 \text{ months}$$