Stochastic dynamical models

February 5th, 2020

- Pocket calculators without wifi connection function are allowed. Pocket calculators with wifi connection function are NOT allowed, even if set in airplane mode.
- The teacher's or your own lecture notes are allowed, other notes are not.
- You can not ask questions. If you think that there is an error or something ambiguous, explain why and write your solution accordingly.
- Solutions can be written either in English or Italian (also French and Spanish).

Exercises

Exercise 1. Imagine a discrete time queue with two counters. Each counter does a certain routine job in a unit of time, i.e. each customer is served in exactly one unit of time. The number of customers arriving in each unit of time is a random variable A with distribution

$$\mathbb{P}\{A=0\}=\frac{3}{8}, \qquad \mathbb{P}\{A=2\}=\frac{1}{4}, \qquad \mathbb{P}\{A=k\}=\frac{1}{8},$$

for $k=1,3,4$. Suppose that a customer arriving at time n will *not* be served in the time interval $[n,n+1]$ even if there are free counters, he might be served in the time intervals $[n+1,n+2]$, $[n+2,n+3]$ ...

(1) Construct a discrete time Markov chain model and write the transition matrix.
(2) Is the Markov chain irreducible?
(3) Does it have an invariant distribution? (Explicit formulae are *not* required! If you think that it exists, just *prove* existence.) Is it unique?
(4) Will the queue become longer and longer or, for any initial length n, it will become empty after an a.s. finite random time with probability 1?
(5) With reference to question 4, what happens if a counter is closed? Write the Markov chain model corresponding to the single server queue.
(6) Suppose now that a counter is closed forever there is only one working counter. Let X_n be the number of customers in the system at time n and let $(X_n)_{n\geq 0}$ be the discrete time Markov chain model. Consider the sequence $(M_n)_{n\geq 0}$ of random variables defined by $M_0=m=X_0$ ($m>0$ constant) and

$$M_n = X_n - \frac{1}{2}\sum_{k=0}^{n-1}\left(3\cdot \mathbb{1}_{\{X_k=0\}} + \mathbb{1}_{\{X_k>0\}}\right).$$

Assuming that $\mathbb{E}[\,|X_n|\,]<\infty$ for all $n\geq 0$ show that $(M_n)_{n\geq 0}$ is a martingale with respect to the natural filtration of $(X_n)_{n\geq 0}$.
(7) Suppose $X_0=10$. Applying the stopping theorem compute the mean time for queue to reach state 100 assuming that it never reaches 0 before.

Exercise 2. Customers arrive randomly at an ATM cash machine. The time between two consecutive arrivals is an exponential random variable with parameter λ (customers per hour). The ATM is in a very small space that can hold only two people (one using the ATM and the other waiting and the other for her/his turn). If another customer comes and finds two people in the small space he leaves and looks for another ATM.
When a customer accesses the machine, he does some financial operation that takes a random time exponentially distributed with parameter μ (operations per hour). A customer waiting in the very small space while another one is doing his operation may get a phone call and, in this case, he leaves the queue loosing his priority. The call to a waiting customer will arrive after an exponentially distributed time with parameter α. (If he gets a call while doing his operation he ignores it). All exponential random variables are independent.

(1) Construct a Markov chain model for the random number X_t of customers in the small space at time t and write the transition rate matrix of the Markov chain $(X_t)_{t\geq 0}$.
(2) Find the invariant distribution.

Thinking now of the long time behaviour of the system answer the following.

(3) What is, on average, the fraction of customers that go to the small space, enter to the small space and eventually succeed using the machine?
(4) Suppose that the bank that owes the ATM charges b € for each operation and operation costs of the ATM machine amount to c € per hour. What relationship must fulfill the parameters of problem for the machine to be self-financing?

Suppose now $\lambda = \mu = 2$ and $\alpha = 1$

(5) Find unkown parameters θ, φ and fill the transition matrix P_t at time t

$$P_t = \left[\begin{array}{c|c|c} \frac{5}{12} + \frac{8e^{-\theta t}}{15} + \frac{e^{-\varphi t}}{20} & \frac{5}{12} - \frac{4e^{-\theta t}}{15} - \frac{3e^{-\varphi t}}{20} & \frac{1}{6} - \frac{4e^{-\theta t}}{15} + \frac{e^{-\varphi t}}{10} \\ \frac{5}{12} - \frac{4e^{-\theta t}}{15} - \frac{3e^{-\varphi t}}{20} & \frac{5}{12} + \frac{2e^{-\theta t}}{15} + \frac{9e^{-\varphi t}}{20} & \frac{1}{6} + \frac{2e^{-\theta t}}{15} - \frac{3e^{-\varphi t}}{10} \\ \frac{5}{12} - \frac{2e^{-\theta t}}{3} + \frac{e^{-\varphi t}}{4} & \cdots & \cdots \end{array}\right]$$

The following exercise is for students of the 2019-2020 academic year only!

Exercise 3. Let $(Z_n)_{n\geq 1}$ be a sequence of independent random variables with Gaussian standard distribution $N(0,1)$, and define

$$S_0 = 0, \qquad S_n = \sum_{k=1}^{n} k Z_k^2 \quad n \geq 1.$$

(1) Show that $(S_n)_{n\geq 0}$ is a discrete time Markov process. Is it time homogeneous?
(2) Determine transition kernels $P_n(\cdot,\cdot)$.

5/02/2020

#1

- two counters : each c stumer is served in a unit of time
- arriving clients : A :

$$\mathbb{P}\left(A = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3/8 \\ 1/8 \\ 1/4 \\ 1/8 \\ 1/8 \end{bmatrix}$$

1. Starting from 0 :

$p_{00} = 3/8, \quad p_{01} = 1/8, \quad p_{02} = 1/4, \quad p_{03} = 1/8, \quad p_{04} = 1/8$

starting from 1 :

$p_{10} = \mathbb{P}(\text{no arrival}) = 3/8, \quad p_{11} = \mathbb{P}(1 \text{ arrival}) = \frac{1}{8} \; \ldots$

starting from $j \geq 2$:

$\mathbb{P}_{j\,j-2} = 3/8, \quad \mathbb{P}_{j\,j-1} = \frac{1}{8}, \quad p_{jj} = 1/4, \quad p_{j\,j+1} = 1/8, \quad p_{j\,j+2} = 1/8$

$$P = \begin{array}{c} \begin{bmatrix} 3/8 & 1/8 & 1/4 & 1/8 & 1/8 & 0 & \\ 3/8 & 1/8 & 1/4 & 1/8 & 1/8 & 0 & \\ 3/8 & 1/8 & 1/4 & 1/8 & 1/8 & 0 & \\ 0 & 3/8 & 1/8 & 1/4 & 1/8 & 1/8 & \\ 0 & 0 & 3/8 & 1/8 & 1/4 & 1/8 & \\ & \ldots & \ldots & & & & \\ & & \ldots & & & & \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \end{array}$$

2. It is irreducible, all states communicate : from any position we can reach 0 and from 0 we can reach any position

3. $y_j = j^2 \Rightarrow \frac{1}{8}\left(3(j-2)^2 + (j-1)^2 + 2j^2 + (j+1)^2 + (j+2)^2\right) =$

$$= j^2 - j + \frac{9}{4} = j^2 - \left(j - \frac{9}{4}\right)$$

$\Longrightarrow \exists\, (y_j)_j, (x_j)_j$ unbounded s.t. $\sum p_{jk} y_k = y_j - x_j$ and $\lim_j x_j = \infty$, $\lim_j y_j = +\infty \Longrightarrow \exists!\, (\pi_j)_j$

with $x_0 = \ldots$, $x_1 = \ldots$

$\Longrightarrow \exists!\, (\pi_j)_j$

4. The MC is recurrent $\longrightarrow$ for any initial length n it will reach the state 0 after a random time which is not only alm. surely finite but has also finite expectation.

5. $p_{00} = \frac{3}{8}, \quad p_{01} = \frac{1}{8}, \quad p_{02} = \frac{1}{4}, \quad p_{03} = \frac{1}{8} = p_{04}$

$p_{10} = \frac{3}{8}, \quad p_{11} = \frac{1}{8} \quad p_{12} = \frac{1}{4}, \quad p_{13} = \frac{1}{8} = p_{14}$

$p_{20} = 0, \quad p_{21} = \frac{3}{8}, \quad p_{22} = \frac{1}{8}, \quad p_{23} = \frac{1}{4}, \quad p_{24} = p_{25} = \frac{1}{8}$

$\vdots$

Is it recurrent? Transient?

$$y_j = \frac{3}{8} y_{j-1} + \frac{1}{8} y_j + \frac{1}{4} y_{j+1} + \frac{1}{8} y_{j+2} + \frac{1}{8} y_{j+3}$$

$$3y_{j-1} - 7y_j + 2y_{j+1} + y_{j+2} + y_{j+3} = 0$$

$$3 - 7x + 2x^2 + x^3 + x^4 = 0$$

$$(x-1)(x^3 + 2x^2 + 4x - 3) = 0$$

$$E[X_{n+1} - X_n] = (\ )\frac{3}{8} + (\ )\frac{1}{8} + (\ )\frac{1}{4} + (\ )\frac{1}{8} + (\ )\frac{1}{8}$$

$$= -\frac{3}{8} + \frac{1}{4} + \frac{1}{4} + \frac{3}{8} = \frac{1}{2}$$

Mean number of arrivals per unit of time:

$$\frac{3}{8} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 4 = \frac{3}{2} > 1$$

$\Longrightarrow$ The MC is transient and the queue will become longer and longer

6. $M_0 = X_0$

By the known result:

$$P(X_n) = \begin{bmatrix} 3/2 \\ 3/2 \\ \vdots \\ j+\frac{1}{2} \\ \vdots \end{bmatrix}, \qquad P(X_n) - X_n = \begin{bmatrix} 3/2 \\ 1/2 \\ \vdots \\ 1/2 \\ \vdots \end{bmatrix} \begin{array}{c} (0) \\ (1) \\ \vdots \\ (j) \\ \end{array}$$

$$\Longrightarrow \quad M_n = X_n - \frac{1}{2} \sum_{k=0}^{n-1} \left(3 \mathbb{1}_{\{X_k = 0\}} + \mathbb{1}_{\{X_k > 0\}}\right)$$

$$\overset{!}{=} X_n - \sum_{k=0}^{n-1} (P(X_n) - X_n)$$

and therefore is a Martingale

7. $E[M_0] = E[f(X_0)] = E[f(10)] = 10$

$E[M_T] = E\left[100 - \sum_{k=0}^{T-1} \frac{1}{2} \mathbb{1}_{\{X_k > 0\}}\right] = 100 - \frac{1}{2} E[T]$

$E[M_0] = E[M_T] \longrightarrow E_{10}[T] = 45$

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#2

- arrivals $\sim \mathcal{E}(\lambda)$
- only two people (one working, one waiting)
- service time $\sim \mathcal{E}(\mu)$
- call for the waiter $\sim \mathcal{E}(\alpha)$

$\perp\!\!\!\perp$

1. $0 \to 1, 0$: $q_{01} = \lambda$, $q_{00} = -\lambda$

   $1 \to 2, 0, 1$: $q_{10} = \mu$, $q_{12} = \lambda$, $q_{11} = -(\lambda+\mu)$

   $2 \to 1, 2$: $q_{20} = 0$, $q_{21} = \alpha+\mu$, $q_{22} = -(\alpha+\mu)$

$$Q = \begin{array}{|c|c|c|} \hline -\lambda & \lambda & 0 \\ \hline \mu & -(\lambda+\mu) & \lambda \\ \hline 0 & \alpha+\mu & -(\alpha+\mu) \\ \hline \end{array}$$

2. $$\begin{cases} -\lambda\pi_0 + \mu\pi_1 = 0 \\ \lambda\pi_0 - (\lambda+\mu)\pi_1 + (\alpha+\mu)\pi_2 = 0 \\ \lambda\pi_1 - (\alpha+\mu)\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \qquad \begin{cases} \pi_0 = \frac{\mu}{\lambda}\pi_1 \\ \pi_2 = \frac{\lambda}{\alpha+\mu}\pi_1 \\ \left(\frac{\mu}{\lambda} + 1 + \frac{\lambda}{\alpha+\mu}\right)\pi_1 = 1 \end{cases}$$

$$\frac{\mu(\alpha+\mu) + \lambda\alpha + \cancel{\lambda\mu} + \lambda^2}{\lambda(\alpha+\mu)}\pi_1 = 1$$

$$\pi_1 = \frac{\lambda/\mu}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu(\mu+\alpha)}}, \quad \pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu(\mu+\alpha)}}, \quad \pi_2 = \frac{\frac{\lambda^2}{\mu(\mu+\alpha)}}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu(\mu+\alpha)}}$$

3. Frac people using the ATM =

   $\Rightarrow$ Frac of people entering $= \pi_0 + \pi_1$

   but for the frac π_1 we use the ATM only if we don't get a call until we can use the ATM:

$$\mathbb{P}(\text{finsh} < \text{call}) = \frac{\mu}{\alpha+\mu}$$

   $\Longrightarrow$ frac people finishing their operation at the ATM:

$$\frac{\pi_0}{\pi_0+\pi_1} + \frac{\pi_1}{\pi_0+\pi_1} \cdot \frac{\mu}{\alpha+\mu} = \frac{\mu}{\lambda+\mu}\left(1 + \frac{\lambda}{\mu+\alpha}\right)$$

4. The bank charges on average €b whenever the machine is busy $\Longrightarrow$ the average profit per hour is:

$$b(\pi_1+\pi_2) = b(1-\pi_0) = \frac{b\left(1+\frac{\lambda}{\mu}\right)}{1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu(\mu+\alpha)}}$$

   the ATM is self financing if: $b(\pi_1+\pi_2) > c$

5. $\left.\frac{dP_t}{dt}\right|_{t=0} = Q \longrightarrow [..] \quad \theta = 3, \ \varphi = 8$ $\qquad Q = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 3 & -3 \end{bmatrix}$

BKE for $p_{21}(t)$, $p_{22}(t)$

$p_{21}'(t) = 3\,p_{11}(t) + (-3)\,p_{21}(t) = -3p_{21}(t) + \frac{5}{4} + \frac{2}{5}e^{-3t} + \frac{27}{20}e^{-8t}$

Homog: $\quad p_{21}(t) = ke^{-3t}$

Full: $\quad p_{21}(t) = ke^{-3t} + A + Bte^{-3t} + Ce^{-8t}$

$p_{21}'(t) = \cancel{-3ke^{-3t}} + Be^{-3t} - 3Bte^{-3t} - 8Ce^{-8t}$

$= \cancel{-3ke^{-3t}} - \underline{3A} - 3Bte^{-3t} - 3Ce^{-8t} + \underline{\frac{5}{4}} + \frac{2}{5}e^{-3t} + \frac{27}{20}e^{-8t}$

$3A = \frac{5}{4} \Rightarrow A = \frac{5}{12}$

$B = \frac{2}{5}$

$8C = 3C - \frac{27}{20} \Rightarrow 5C = -\frac{27}{20} \Rightarrow C = -\frac{27}{100}$

$p_{21}(0) = 0 \Rightarrow k + A + C = k + \frac{5}{12} - \frac{27}{100} = 0 \Rightarrow k = \frac{11}{75}$

$\cancel{p_{21}(t) = -\frac{103}{150}e^{-3t} + \frac{2}{5} + \frac{2}{5}te^{-3t} - \frac{27}{100}e^{-8t}}$

$p_{21}(t) = \frac{11}{75}e^{-3t} + \frac{5}{12} + \frac{2}{5}e^{-3t}\,t - \frac{27}{100}e^{-8t}$

[..]

#3

$$S_n = \begin{cases} 0 & n=0 \\ \sum_{k=1}^{n} k Z_k^2 & n \geq 1 \end{cases}$$

$S_{n+1} = S_n + (n+1)Z_{n+1}^2$

$\mathbb{P}(S_{n+1} \in E_{n+1} \mid S_n \in E_n, \dots, S_0 \in E_0) = \mathbb{P}(S_n + (n+1)Z_{n+1}^2 \in E_{n+1} \mid \dots)$

$= \mathbb{P}((n+1)Z_{n+1}^2 \in E_{n+1} \setminus S_n \mid S_n \in E_n, \dots, S_0 \in E_0)$

$= \mathbb{P}((n+1)Z_{n+1}^2 \in E_{n+1} \setminus S_n \mid S_n \in E_n)$

$= \mathbb{P}(S_{n+1} \in E_{n+1} \mid S_n \in E_n)$

$P_n(S_n, A) = \mathbb{E}[\mathbb{1}_A(S_{n+1}) \mid \sigma(S_n)]$

$= \mathbb{E}[\mathbb{1}_A((n+1)Z_{n+1}^2 + S_n) \mid \sigma(S_n)]$

$= \int_{-\infty}^{+\infty} \mathbb{1}_A((n+1)x^2 + S_n) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx$

Variable change: $\quad y = (n+1)x^2 + S_n \Rightarrow x = \sqrt{\frac{y - S_n}{n+1}}$

$y > S_n \qquad dx = \frac{1}{2\sqrt{(n+1)(y-S_n)}}\,dy$

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3(2)

$$\Longrightarrow P_n(S_n, A) = \int_{S_n}^{+\infty} \mathbb{1}_A(y)\, e^{-\frac{y-S_n}{2(n+1)}} \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{(n+1)(y-S_n)}}\, dy$$

$$\Longrightarrow P_n(x, A) = \int_{x}^{+\infty} \mathbb{1}_A(y)\, e^{-\frac{y-x}{2(n+1)}} \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{(n+1)(y-x)}}\, dy$$