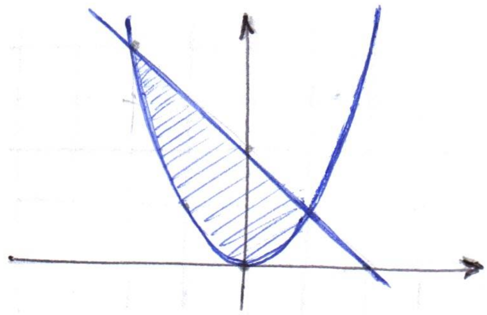
19/06/2020
problem 2

$$\min \ (x_1-2)^2 + x_2$$
$$\text{s.t.} \quad x_1^2 \le x_2 \qquad\qquad g_1(\underline{x}) = x_1^2 - x_2$$
$$x_1 + x_2 \le 2 \qquad\qquad g_2(\underline{x}) = x_1 + x_2 - 2$$

i) Draw the feasible region.



ii) At which point of the feasible region CQ assumptions are satisfied?
Both $g_1(\underline{x})$ and $g_2(\underline{x})$ are convex. Moreover $\exists \underline{x}^*$: $g_1(\underline{x}^*) < 0$ and $g_2(\underline{x}^*) < 0$
for instance $\underline{x}^* = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \longrightarrow g_1(\underline{x}^*) = \frac{1}{4} - 1 = -3/4 < 0$
$g_2(\underline{x}^*) = -2 < 0$

$\Longrightarrow$ Thanks to Slater $\forall \underline{x} \in$ feasible region is s.t. **CQ** holds

iii) Give a statement for the first order opt. conditions.
[..]

iv) Explain why the above conditions are necessary/sufficient
[..]

v) Determine all candidates points for the above conditions and identify the global optimal solution.

$$\begin{cases} \nabla f(\bar{x}) + \sum_{i \in I} u_i \nabla g_i(\bar{x}) = \underline{0} \\ u_i g_i(\bar{x}) = 0 & \forall i \in I \\ g_i(\bar{x}) \le 0 & \forall i \in I \\ u_i \ge 0 & \forall i \in I \end{cases}$$

$$\nabla f(\underline{x}) = \begin{bmatrix} 2(x_1-2) \\ 1 \end{bmatrix}, \quad \nabla g_1(\underline{x}) = \begin{bmatrix} 2x_1 \\ -1 \end{bmatrix}, \quad \nabla g_2(\underline{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 2(x_1-2) \\ 1 \end{bmatrix} + u_1 \begin{bmatrix} 2x_1 \\ -1 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\Longrightarrow \begin{cases} 2(x_1-2) + 2u_1x_1 + u_2 = 0 \\ 1 - u_1 + u_2 = 0 \end{cases} \wedge \begin{cases} u_1(x_1^2 - x_2) = 0 \\ u_2(x_1 + x_2 - 2) = 0 \end{cases}$$

We can have 4 cases:

1. $u_1 = u_2 = 0$ $\longrightarrow$ impossible because of $1 - u_1 + u_2 = 0$
2. $u_1 > 0, \ u_2 = 0$
3. $u_1 = 0, \ u_2 > 0$ $\longrightarrow$ impossible: $1 - u_1 + u_2 = 0 \Rightarrow u_2 = -1 \not> 0$
4. $u_1 > 0, \ u_2 > 0$

2. $u_1 > 0,\ u_2 = 0$:

$1 - u_1 = 0 \Rightarrow u_1 = 1$

$2(x_1 - 2) + 2u_1 = 0 \Rightarrow 2(x_1 - 2) + 2 = 0 \Rightarrow 2x_1 = 2 \Rightarrow x_1 = 1$

$x_1 + x_2 - 2 = 0 \Rightarrow x_2 = 1$

$[x_1, x_2] = [1, 1]$ is acceptable with $[u_1, u_2] = [1, 0]$

4. $u_1 > 0,\ \underline{u_2 > 0}$:

$$\begin{cases} x_1^2 - x_2 = 0 \\ x_1 + x_2 - 2 = 0 \end{cases} \Rightarrow x_1^2 + x_1 - 2 = 0 \Rightarrow x_1 = \begin{cases} -2 \\ 1 \end{cases} \Rightarrow x_2 = \begin{cases} 4 \\ 1 \end{cases}$$

If $[x_1, x_2] = [-2, 4]$:

$2(x_1 - 2) + 2u_1 x_1 + u_2 = 0 \Rightarrow -8 - 4u_1 + u_2 = 0$

$u_2 = u_1 - 1 \Rightarrow -8 - 4u_1 + u_1 - 1 = 0$

$\Rightarrow -3u_1 - 9 = 0 \Rightarrow u_1 = -3$ not acc.

If $[x_1, x_2] = [1, 1]$:

$2(x_1 - 2) + 2u_1 x_1 + u_2 = 0 \Rightarrow -2 + 2u_1 + u_2 = 0$

$\Rightarrow -2 + 2u_1 + u_1 - 1 = 0 \Rightarrow 3u_1 = 3 \Rightarrow u_1 = 1,\ \underline{u_2 = 0}$ not acc.

vi) Write the Lagrangian dual and indicate the connection between the primal problem and the dual problem.

$$L(\underline{x}, \underline{u}) = (x_1 - 2)^2 + x_2 + u_1(x_1^2 - x_2) + u_2(x_1 + x_2 - 2)$$

$$w(\underline{u}) = \min_{\underline{x} \in \mathbb{R}^2} L(\underline{x}, \underline{u})$$

Dual problem: $\max_{\underline{u} \geq 0} w(\underline{u})$

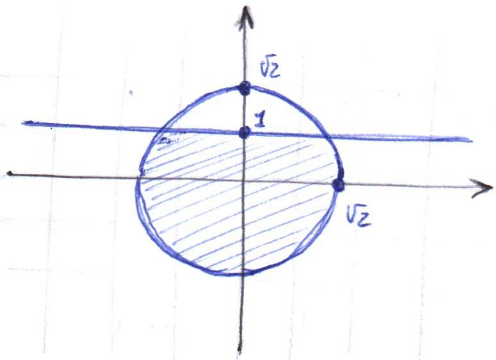
[.. notice that this is a convex problem ..]

16/07/2020
problem 2

$$\min \quad x^2 + 4x + y^2 + 4y + 8$$
$$\text{s.t.} \quad x^2 + y^2 - 2 \le 0$$
$$y - 1 \le 0$$

$g_1(\underline{x}) = x^2 + y^2 - 2$
$g_2(\underline{x}) = y - 1$

i) Draw the feasible region



ii) Points for CQ?

Both $g_1(\underline{x})$ and $g_2(\underline{x})$ are convex and $\exists \underline{x}^*$: $g_1(\underline{x}^*) < 0$ and $g_2(\underline{x}^*) < 0$

for instance $\underline{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{array}{l} g_1(\underline{x}^*) = -2 < 0 \\ g_2(\underline{x}^*) = -1 < 0 \end{array}$

$\Longrightarrow$ Thanks to Slater $\forall \underline{x} \in$ feasible region is s.t. CQ holds

iii), iv) theorical like 19/06/2020

v) Candidate points?

$$\nabla f(\underline{x}) = \begin{bmatrix} 2x + 4 \\ 2y + 4 \end{bmatrix} \qquad \nabla g_1(\underline{x}) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \qquad \nabla g_2(\underline{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} \nabla f(\bar{\underline{x}}) + \sum_{i \in I} u_i \nabla g_i(\bar{\underline{x}}) = \underline{0} \\ u_i g_i(\bar{\underline{x}}) = 0 & \forall i \\ g_i(\bar{\underline{x}}) \le 0 & \forall i \\ u_i \ge 0 & \forall i \end{cases}$$

$$\Longrightarrow \begin{cases} 2x + 4 + 2u_1 x = 0 \\ 2y + 4 + 2u_1 y + u_2 = 0 \end{cases} \wedge \begin{cases} u_1 (x^2 + y^2 - 2) = 0 \\ u_2 (y - 1) = 0 \end{cases}$$

Cases :

1. $u_1 = u_2 = 0$
2. $u_1 > 0, \ u_2 = 0$
3. $u_1 = 0, \ u_2 > 0$
4. $u_1 > 0, \ u_2 > 0$

1. $u_1 = u_2 = 0$

$$\left.\begin{array}{l} 2x + 4 = 0 \\ 2y + 4 = 0 \end{array}\right\} \Longrightarrow \begin{cases} x = -2 \\ y = -2 \end{cases} \quad \text{not acceptable}$$

2. $u_1 > 0, \; u_2 = 0$

$$\begin{cases} 2x + 4 + 2u_1 x = 0 \\ 2y + 4 + 2u_1 y = 0 \end{cases} \Longrightarrow \quad x = -\frac{2}{1+u_1}, \quad y = -\frac{2}{1+u_1}$$

$$x^2 + y^2 - 2 = 0 \Rightarrow \frac{4}{(1+u_1)^2} + \frac{4}{(1+u_1)^2} - 2 = 0$$

$$\Rightarrow 4 + 4 - 2(u_1+1)^2 = 0$$

$$\Rightarrow 8 - 2u_1^2 - 2 - 4u_1 = 0$$

$$\Rightarrow u_1^2 + 2u_1 - 3 = 0 \Longrightarrow u_{1/2} = \frac{-2 \pm \sqrt{4+4\cdot 3}}{2} = \frac{-2\pm 4}{2} \begin{cases} 1 & \text{acc.} \\ -3 & \text{not acc.} \end{cases}$$

$$u_1 = 1 \Longrightarrow [x, y] = [-1, -1]$$

3. $u_1 = 0, \; u_2 > 0$

$$2x + 4 = 0 \Rightarrow x = -2$$

$$y - 1 = 0 \Rightarrow y = 1$$

$$2y + 4 + u_2 = 0 \Longrightarrow 2 + 4 + u_2 = 0 \Rightarrow u_2 = -6 \quad \text{not acc.}$$

4. $u_1 > 0, \; u_2 > 0$

$$y = 1$$

$$x^2 + 1 - 2 = 0 \Rightarrow x^2 = 1 \Rightarrow x_{1/2} = \begin{cases} -1 \\ 1 \end{cases}$$

$[x, y] = [1, 1]$: $\quad 2(1) + 4 + 2u_1(1) = 0 \Rightarrow u_1 = -3$ not acc.

$[x, y] = [-1, 1]$: $\quad -2 + 4 - 2u_1 = 0 \Rightarrow u_1 = 1$

$\quad 2 + 4 + 2u_1 + u_2 = 0 \Rightarrow u_2 = -8$ not acc.

$\Rightarrow [-1, -1]$ global minimum

(vi) Lagrangian dual & [..]?

$$L(\underline{x}, \underline{u}) = x^2 + 4x + y^2 + 4y + 8 + u_1(x^2 + y^2 - 2) + u_2(y - 1)$$

$$W(\underline{u}) = \min_{\underline{x} \in \mathbb{R}^2} L(\underline{x}, \underline{u})$$

Lagrangian dual problem: $\max\limits_{\underline{u} \geq \underline{0}} W(\underline{u})$

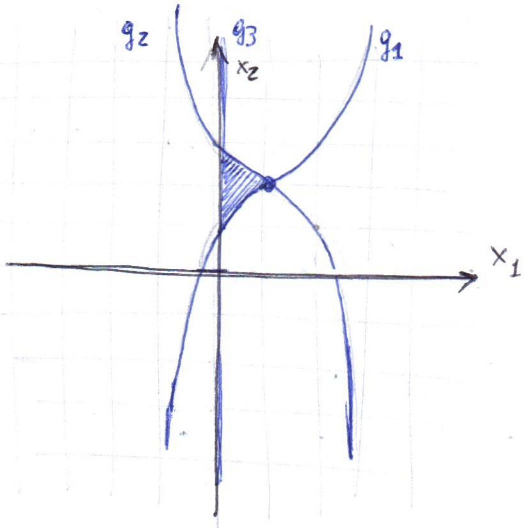
28/09/2016

problem 2

$$\min \; x_2$$
$$\text{s.t.} \quad (x_1-1)^3 + (x_2-2) \le 0$$
$$(x_1-1)^3 - (x_2-2) \le 0$$
$$x_1 \ge 0$$

$$g_1(\underline{x}) = (x_1-1)^3 + (x_2-2)$$
$$g_2(\underline{x}) = (x_1-1)^3 - (x_2-2)$$
$$g_3(\underline{x}) = -x_1$$

i) Draw the feasible region and check the points for which CQ holds.



$$\nabla g_1(\underline{x}) = \begin{bmatrix} 3(x_1-1)^2 \\ 1 \end{bmatrix}$$

$$\nabla g_2(\underline{x}) = \begin{bmatrix} 3(x_1-1)^2 \\ -1 \end{bmatrix}$$

$$\nabla g_3(\underline{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\nabla f(\underline{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\nabla g_1(\underline{x}) \overset{?}{\text{li}} \nabla g_2(\underline{x})$: $\begin{cases} a\,3(x_1-1)^2 + b\,3(x_1-1)^2 = 0 \\ a = b \end{cases} \Longrightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$ (the only one)

$[x_1, x_2] = [1,2]$ does not satisfy the sufficient condition

$\nabla g_1(x) \overset{?}{\text{li}} \nabla g_3(x)$: yes

$\nabla g_2(x) \overset{?}{\text{li}} \nabla g_3(x)$: yes

! What with $[x_1, x_2] = [1,2]$? We check with the definition:

$D(\bar{x}) = \{ [\alpha, 0] : \alpha \in \mathbb{R}^- \}$, $\quad D(\bar{x}) = \{ [\alpha, 0] : \alpha \in \mathbb{R} \}$ $\quad (\neq)$

$\Longrightarrow$ the CQ holds everywhere but in $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

ii) State the opt. cond.. Are they necessary/sufficient?

KKT are necessary and not sufficient since the problem is <u>not</u> convex.

[..]

iii) $$\begin{cases} \nabla f(\bar{x}) + \sum_{i \in I} u_i \nabla g_i(\bar{x}) = \underline{0} \\ u_i g_i(\bar{x}) = 0 \quad \forall i \in I \\ g_i(\bar{x}) \le 0 \quad \forall i \in I \\ (u_i \ge 0) \end{cases}$$

$$\begin{cases} u_1\, 3(x_1-1)^2 + u_2\, 3(x_1-1)^2 - u_3 = 0 \\ 1 + u_1 - u_2 = 0 \;\; (*) \end{cases} \wedge \begin{cases} u_1\left((x_1-1)^3 + (x_2-2)\right) = 0 \\ u_2\left((x_1-1)^3 - (x_2-2)\right) = 0 \\ u_3(-x_1) = 0 \end{cases}$$

Cases:

1. $u_1 = u_2 = u_3 = 0$ ✗ (*)
2. $u_1 > 0,\ u_2 = u_3 = 0$ ✗ (*)
3. $u_1 > 0,\ u_2 > 0,\ u_3 = 0$
4. $u_1 > 0,\ u_2 = 0,\ u_3 > 0$ ✗ (*)
5. $u_1 = 0,\ u_2 > 0,\ u_3 > 0$
6. $u_1 = 0,\ u_2 > 0,\ u_3 = 0$
7. $u_1 = 0,\ u_2 = 0,\ u_3 > 0$ ✗ (*)
8. $u_1 > 0,\ u_2 > 0,\ u_3 > 0$

3. $u_1 > 0,\ u_2 > 0,\ u_3 = 0$

$u_2 = 1 + u_1$

$(u_1+u_2)(3(x_1-1)^2) = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ not acceptable since KKT not valid

5. $u_1 = 0,\ u_2 > 0,\ u_3 > 0$

$u_2 = 1,\ x_1 = 0$

$(-1)^3 - (x_2-2) = 0 \implies x_2 = 1$

$3 - u_3 = 0 \implies u_3 = 3$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ acceptable

6. $u_1 = 0,\ u_2 > 0,\ u_3 = 0$

$u_2 = 1$

$\left.\begin{array}{l} x_1 = 1 \\ x_2 = 2 \end{array}\right\}$ not acceptable KKT not valid

8. $u_1 > 0,\ u_2 > 0,\ u_3 > 0$

$x_1 = 0$

$\left.\begin{array}{l} -1 + x_2 - 2 = 0 \\ -1 - x_2 + 2 = 0 \end{array}\right\}$ not feasible

$\implies \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ optimal solution

iv) Write the lagrangian dual and [..].

$$L(\underline{x},\underline{u}) = x_2 + u_1\left((x_1-1)^3 + (x_2-2)\right) + u_2\left((x_1-1)^3 - (x_2-2)\right) + u_3(-x_1)$$

$$w(\underline{u}) = \min_{\underline{x} \in \mathbb{R}^2} L(\underline{x},\underline{u})$$

Dual problem: $\min_{\underline{u} \geq \underline{0}} w(\underline{u})$

[.. this time the problem is not convex ...]

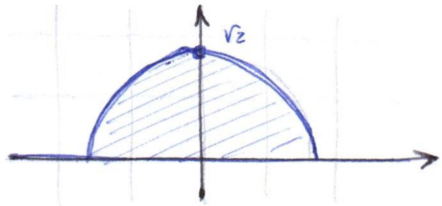
min $x_1 + x_2$

s.t. $x_1^2 + x_2^2 \le 2$ $\qquad g_1(\underline{x}) = x_1^2 + x_2^2 - 2$

$x_2 \ge 0$ $\qquad g_2(\underline{x}) = -x_2$

11/09/2013
problem 2

i) Feasible region and points for CQ?



Since both $g_1(\underline{x})$ and $g_2(\underline{x})$ are convex and $\exists\, \underline{x}^* : g_1(\underline{x}^*) < 0$ and $g_2(\underline{x}^*) < 0$

i.e. $\underline{x}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{matrix} g_1(\underline{x}^*) = -1 < 0 \\ g_2(\underline{x}^*) = -1 < 0 \end{matrix}$

$\Longrightarrow$ thanks to Slater $\forall \underline{x} \in$ feasible region is such that CQ holds.

ii) State the optimality cond [..] and find all the candidate points.

$$\begin{cases} \nabla f(\bar{\underline{x}}) + \sum_{i \in I} u_i \nabla g_i(\bar{\underline{x}}) = \underline{0} \\ g_i(\bar{\underline{x}})\, u_i = 0 \quad \forall i \\ g_i(\bar{\underline{x}}) \le 0 \quad \forall i \end{cases} \qquad (+\ u_i \ge 0\ \forall i)$$

$$\nabla f(\underline{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \nabla g_1(\underline{x}) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}, \quad \nabla g_2(\underline{x}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} 1 + 2u_1 x_1 = 0 \quad (*) \\ 1 + 2u_1 x_2 - u_2 = 0 \end{cases} \wedge \begin{cases} u_1(x_1^2 + x_2^2 - 2) = 0 \\ u_2(-x_2) = 0 \end{cases}$$

Cases:

1. $u_1 = u_2 = 0$ ✗ $(*)$
2. $u_1 > 0,\ u_2 = 0$
3. $u_1 = 0,\ u_2 > 0$ ✗ $(*)$
4. $u_1 > 0,\ u_2 > 0$

2. $u_1 > 0,\ u_2 = 0$

$x_1 = -\frac{1}{2u_1} = x_2$

$x_1^2 + x_2^2 - 2 = 0$

$\longrightarrow 1 + 1 - 2(4u_1^2) = 0$

$\Longrightarrow 2 - 8u_1^2 = 0$

$\Longrightarrow u_1 = \frac{1}{2}$

$x_2 = -1 \Longrightarrow$ not acc.

3. $u_1 = 0,\ u_2 > 0$

$u_2 = 1$

$x_2 = 0$

but $1 + \ldots = 0$ impossible

4. $u_1 > 0,\ u_2 > 0$

$x_2 = 0 \quad x_1^2 = 2 \longrightarrow x_1 = \pm\sqrt{2}$

$1 - u_2 = 0 \longrightarrow u_2 = 1$

$x_1 = \sqrt{2} \Longrightarrow 1 + 2\sqrt{2}\, u_1 = 0 \longrightarrow u_1 < 0$ not acc.

$x_1 = -\sqrt{2} \Longrightarrow 1 - 2\sqrt{2}\, u_1 = 0 \Longrightarrow u_1 = \frac{1}{2\sqrt{2}}$

$$\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/2\sqrt{2} \\ 1 \end{bmatrix}$$

iii) Dual problem and properties [..] ?

$$L(\underline{x},\underline{u}) = x_1 + x_2 + u_1(x_1^2 + x_2^2 - 2) + u_2(-x_2)$$

$$w(\underline{u}) = \min_{\underline{x} \in \mathbb{R}^2} L(\underline{x},\underline{u})$$

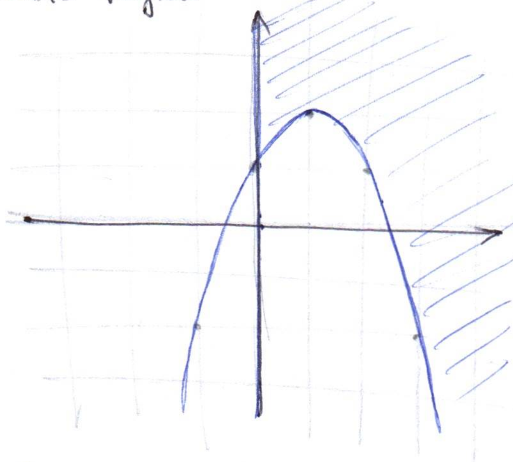
Dual problem: $\max_{\underline{u} \geq \underline{0}} w(\underline{u})$

[.. this is a convex problem ..]

11/02/2014
problem 2

$$\min \quad (x_1+1)^2 + x_2$$
$$\text{s.t.} \quad -(x_1-1)^2 - x_2 \le -2 \qquad\qquad g_1(\underline{x}) = -(x_1-1)^2 - x_2 + 2$$
$$x_1 \ge 0 \qquad\qquad g_2(\underline{x}) = -x_1$$

i) feasible region



ii) CQ?

$$\nabla g_1(\underline{x}) = \begin{bmatrix} -2(x_1-1) \\ -1 \end{bmatrix}, \quad \nabla g_2(\underline{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\nabla g_1(\underline{x}) \overset{?}{\perp\!\!\perp} \nabla g_2(\underline{x}) \quad : \begin{cases} -2a(x_1-1) - b = 0 \\ -a = 0 \end{cases} \qquad \text{they're linearly} \perp\!\!\perp \ \forall \underline{x}$$

$\Longrightarrow$ CQ holds everywhere in the feasible region

iii) theory [...]

iv) candidate points:

$$\nabla f(\underline{x}) = \begin{bmatrix} 2(x_1+1) \\ 1 \end{bmatrix} \qquad \nabla g_1(\underline{x}) = \begin{bmatrix} -2(x_1-1) \\ -1 \end{bmatrix} \qquad \nabla g_2(\underline{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} 2(x_1+1) - 2u_1(x_1-1) - u_2 = 0 \\ 1 - u_1 = 0 \quad (*) \end{cases} \wedge \begin{cases} u_1(-(x_1-1)^2 - x_2 + 2) = 0 \\ u_2(-x_1) = 0 \end{cases}$$

cases:

1. $u_1 = u_2 = 0$ × (*)
2. $u_1 > 0, u_2 = 0$
3. $u_1 = 0, u_2 > 0$ × (*)
4. $u_1 > 0, u_2 > 0$

2. $u_1 > 0, u_2 = 0$
$u_1 = 1$
$2(x_1+1) - 2(x_1-1) = 0$
$\cancel{2x_1} + 2 - \cancel{2x_1} + 2 = 0$
impossible

4. $u_1 > 0, u_2 > 0$
$u_1 = 1, \quad x_1 = 0$
$2 + 2 - u_2 = 0 \Longrightarrow u_2 = 4$
$-1 - x_2 + 2 = 0 \Longrightarrow x_2 = 1$
$\Longrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ opt.

v) Lagrangian dual.

$$L(\underline{x}, \underline{u}) = (x_1+1)^2 + x_2 + u_1(-(x_1-1)^2 - x_2 + 2) + u_2(-x_1)$$
$$w(\underline{u}) = \min_{\underline{x} \in \mathbb{R}^2} L(\underline{x}, \underline{u})$$

Dual problem: $\max_{\underline{u} \ge \underline{0}} w(\underline{u})$