MARKOV PROCESS

- $(X_n)_{n\geq 0}$ / $(X_t)_{t\geq 0}$ is a MP? $\mathbb{P}(X_{t_{m+1}} \in E_{m+1} | X_{t_m} \in E_m, \dots, X_{t_0} \in E_0) \overset{?}{=} \mathbb{P}(X_{t_{m+1}} \in E_{m+1} | X_{t_m} \in E_m)$
  - $(X_n)_{n\geq 0}$: $X_{n+1} = F(X_n, V_{n+1})$ V_{n+1} det. $\Longrightarrow$ $\mathbb{P}(X_{n+1} \in E_{n+1} | X_n = x, X_{n-1} \in E_{n-1}, \dots)$
    if $= \mathbb{P}(X_{n+1} \in E_{n+1} | X_n = x)$ ✓
  - $(X_n)_{n\geq 0}$ / $(X_t)_{t\geq 0}$: $X_n = g(B_n)$ / $X_t = g(B_t)$ $\Longrightarrow$ use def (*) and the fact that the Brownian motion is a MP

    (*) $\mathbb{P}(B_{n+1} \in E_{n+1} | B_n = x, B_{n-1} \in E_{n-1}, \dots) = \mathbb{P}(B_{n+1} - B_n + x \in E_{n+1} | B_n - B_{n-1} \in \dots)$
    $= \mathbb{P}(B_{n+1} - B_n + x \in E_{n+1})$ ⫫ increments
    $= \mathbb{P}(B_{n+1} \in E_{n+1} | B_n = x)$

- Transition Kernel? $P_t(x, A) = \mathbb{P}(X_t \in A | X_0 = x)$
  - $X_{n+1} = F(X_n, V_{n+1})$ $\Longrightarrow$ $\mathbb{P}(X_{n+1} \in A | X_n = x) = \mathbb{P}(F(x, V_{n+1}) \in A | X_n = x)$
    $= \mathbb{P}(F(x, V_{n+1}) \in A)$

    $Y := F(x, V_{n+1})$; $\mathbb{P}(Y \leq y) = \mathbb{P}(F(x, V_{n+1}) \leq y) = \mathbb{P}(V_{n+1} \leq \dots)$

    and we get $F_Y(y) = F_{..}(\dots)$

    $\longrightarrow$ $\frac{\partial F_Y}{\partial y} = f_Y(y) = \frac{\partial F_{..}(\dots)}{\partial y}$ $\Longrightarrow$ we obtain $f_Y(y)$ using the law of V_{n+1}

    $\mathbb{P}(X_{n+1} \in A | X_n = x) = \mathbb{P}(Y \in A) = \int_A f_Y(y) dy$
  - $X_t = g(B_t)$ $\Longrightarrow$ if g is a linear comb. $\Longrightarrow$ $B_t \sim N(\cdot,\cdot)$ $\Longrightarrow$ $X_t \sim N(\cdot,\cdot)$

    $$\begin{bmatrix} X_{t+s} \\ X_s \end{bmatrix} = \begin{bmatrix} g(B_{t+s}) \\ g(B_s) \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_Y \\ \mu_X \end{bmatrix}, \begin{bmatrix} \sigma_Y^2 & cov \\ cov & \sigma_X^2 \end{bmatrix}\right)$$

    $$\boxed{Y | X = x \sim N\left(\mu_Y + Cov(X,Y)\frac{(x-\mu_X)}{\sigma_X^2}, \ \sigma_Y^2 - \frac{(Cov(X,Y))}{\sigma_X^2}\right)}$$

    $\Longrightarrow$ $X_{t+s} | X_s = x \sim \dots$

    $\Longrightarrow$ $\mathbb{P}(X_{t+s} \in A | X_s = x) = P_t(x, A) = \int_A f_{X_{t+s}|X_s}(y|x)\, dy$

    $$\boxed{Y = AX, \ X \sim N(\mu_X, C) \Longrightarrow Y \sim (A\mu_X, ACA^T)}$$

- Irreducible?

  Irreducible $\Longleftrightarrow$ $[\forall x \in E, \forall A \in \mathcal{E}, \varphi(A) > 0 \Longrightarrow \exists t > 0 : P_t(x, A) > 0]$

  In our cases (most generally): $E = \mathbb{R}$, φ leb-meas. $\Longrightarrow$ $[P_t(x, A) > 0 \Longleftrightarrow \varphi(A) > 0]$

  (once we calculated $P_t(x,A)$)

- Homogeneous?

  Homogeneous $\Longleftrightarrow$ $P_t(x, A) = \mathbb{P}(X_{t+s} \in A | X_s = x)$ ⫫ s

martingale X_n

$$P_n(X_n, A) = \mathbb{E}[\mathbb{1}_A(X_{n+1}) | \sigma(X_1, \dots, X_n)]$$
$$= \mathbb{E}[\mathbb{1}_A(h(Z_{n+1}) X_n) | \sigma(\dots)]$$
$$= \int_{\mathbb{R}} \mathbb{1}_A(h(x) X_n) \cdot f_{Z_{n+1}}(x)\, dx$$
$$= \boxed{\int_{\mathbb{R}} \mathbb{1}_A(y) f_{Z_{n+1}}(\dots y)\, dy \dots}$$

$y = h(x) X_n$, $dx = \dots dy$

here X_n becomes x and we get $P_n(x, A)$

MARTINGALE

- Prove it's a martingale.
  1. $\mathbb{E}[|M_t|] < \infty \quad \forall t$
  2. M_t is $\sigma(N_t)$-meas. $\quad (N_t)_{t\geq 0}$ natural filtration
  3. $\forall s \leq t$: $\mathbb{E}[M_t \mid \sigma(N_s)] = M_s$ (**)

3. (desperate trick): $M_n = f(X_n)$, $\mathcal{D} = \sigma(X_j : j \leq n)$

if $\mathbb{E}[ZW] = \mathbb{E}(XW) \quad \forall W \ \mathcal{D}$-measurable $\Longrightarrow Z = \mathbb{E}[X \mid \mathcal{D}]$

Here $X = M_{n+1}$, $Z = M_n$, $W = g(X_0, \dots, X_n)$ generic g :

$$\mathbb{E}[M_{n+1}\, g(X_0,\dots,X_n)] = \sum_{\delta_0,\dots,\delta_{n+1}\in I} f(\delta_{n+1})\, g(\delta_0,\dots,\delta_n)\, \mathbb{P}(X_0=\delta_0, \dots, X_{n+1}=\delta_{n+1})$$

$$= \sum_{\delta_0,\dots,\delta_n \in I} \left(f(\delta_{n+1})\, p_{\delta_n \delta_{n+1}} \right) g(\delta_0,\dots,\delta_n)\, \mathbb{P}(X_0=\delta_0, \dots, X_n=\delta_n)$$

$$= [\dots] = \mathbb{E}\left[M_n\, g(X_0,\dots,X_n)\right]$$

- (M/M/K) Starting from Q, given $f(j) = k \quad j \in I$:

  is $M_t = f(X_t) - \int_0^t \dots \mathbb{1}_{\dots} \dots ds$ a martingale?

* $Qf\left(\begin{bmatrix} \vdots \\ j \\ \vdots \end{bmatrix}\right) = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \overset{?}{=} \dots \mathbb{1}_{\dots} \dots$ if so $\Longrightarrow M_t = f(X_t) - \int_0^t (Qf)(X_s)\, ds$ is a martingale

$Q \cdot f[\vdots] = [\dots][\vdots]$

In discrete time: $M_n = f(X_n) - \sum_{k=0}^{n-1} \left[Pf(X_k) - f(X_k) \right]$ (**)

- (M/M/K) <u>Average service time</u> ? Mean time for going back to $\{0\}$?

  (w/: martingale stopping theorem)

$\mathbb{E}_1[T_0] = \mathbb{E}_1[\inf\{t > 0 : X_t = 0\}]$ = mean time for going $\{1\} \to \{0\}$

Stopping theorem: $\mathbb{E}_1[M_0] = \mathbb{E}_1[M_{T\wedge t}] = \mathbb{E}_1[M_t] \quad t \geq 0$

Since $T \wedge t \xrightarrow{t \to \infty} T \Longrightarrow$ Monotone conv. theorem : $\mathbb{E}_1[T \wedge t] = \mathbb{E}_1[T]$

$\Longrightarrow$ Lebesgue's theorem : $\mathbb{E}_1[f(X_{T\wedge t})] \to \mathbb{E}_1[f(X_T)]$

$$\mathbb{E}_1[M_0] = \mathbb{E}_1\left[f(X_0) - \int_0^0 \dots\right] = \mathbb{E}_1[f(X_0)] = f(1)$$

$$\mathbb{E}_1[M_T] = \mathbb{E}_1\left[f(X_T) - \int_0^T \dots\right] = \mathbb{E}_1[f(X_T)] - \mathbb{E}\left[\int_0^T \dots\right] = f(0) - \mathbb{E}[\dots T \dots]$$

$$\Longrightarrow \mathbb{E}_1[M_0] = \mathbb{E}_1[M_T] \Longrightarrow \mathbb{E}[T] = \dots$$

(**) : $\mathbb{E}[|f(X_n)|] < \infty$ and $\mathbb{E}[|Pf(X_n)|] < \infty$

* before : " M_t is adapted to the natural filtration of the MC "

(**) If it's discrete :

" $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \dots = M_n$, and so by the projective property of the conditional expectation $\mathbb{E}[M_m \mid \mathcal{F}_s] = M_s \quad \forall m \geq s$ "

$M_t = f(X_t) - \int_0^t \mathbb{1}_{\{X_s > 0\}}\, ds \xrightarrow[\text{if } 0 \text{ only recurrent}]{t \to \infty} f(X_0) - \boxed{\int_0^\infty \mathbb{1}_{\{X_s > 0\}}\, ds}$ total time spent in the transient states (since $\int_0^\infty \mathbb{1}_{\{X_s > 0\}}\, ds < \infty$ then all the M_t converges in L^1)

CONTINUOUS MC (Discrete states space)

- starting from Q : P_t ? *

FKE : $p_{ij}'(t) = \sum_{k \in I} p_{ik}(t) q_{kj}$ / **BKE** : $p_{ij}'(t) = \sum_{k \in I} q_{ik} p_{kj}(t)$

and : $p_{ij}(0) = \delta_{ij} = \mathbb{1}_{\{i=j\}}$ (since $P_0 = \mathbf{I}$ (condition to use to determine α in P_t))

- Starting from P_t : Q ? $\quad Q = \left.\frac{dP_t}{dt}\right|_{t=0}$

- Invariant density π? $\quad \pi$ invariant $\iff 0 = \pi Q$

(**Remark**: I finite $\Rightarrow \exists (\pi_j)_j$)

- Recurrent / transient state?

i is recurrent/transient for the continuous MC $(X_t)_{t \geq 0}$ $\iff$ i is recurrent/transient for the **DISCRETE SKELETON** : $(Y_n)_{n \geq 0}$:

$$\hat{p}_{ij} = \begin{cases} \frac{q_{ij}}{-q_{ii}} & q_{ii} \neq 0, \quad i \neq j \\ 1 & q_{ii} = 0, \quad i = j \\ 0 & (q_{ii} = 0, i \neq j) \vee (q_{ii} \neq 0, i = j) \end{cases}$$

$q_{ii} = 0 \Rightarrow i$ absorbing (recurrent)

- Probability of going to j after leaving i?

If $q_{ii} \in (-\infty, 0) \Rightarrow \forall j \neq i \quad \mathbb{P}_i(X_{T_i} = j) = \frac{q_{ij}}{-q_{ii}}$

- Irreducible MC?

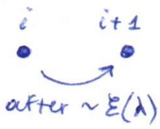
$(X_t)_{t \geq 0}$ irreducible $\iff \forall i,j \in I : \exists n \geq 1, \exists k_1, .., k_n$ s.t. $i \neq k_1, .., k_n \neq j$ and $q_{ik_1} q_{k_1 k_2} \cdots q_{k_n j} > 0$

- Average time spent in i starting from i?

$-\infty < q_{ii} < 0 \Rightarrow T_i \sim \mathcal{E}(-q_{ii})$ = exit time from i

$\mathbb{E}_i[T_i]$ = average time spent in i starting from i $= \frac{1}{-q_{ii}}$

- **POISSON PROCESS**



$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & 0 & \cdots \\ & \cdots & & & & \end{bmatrix}, \quad \hat{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ & & \cdots & & & \end{bmatrix}$$

$(N_t)_{t \geq 0}$: N_t = how many people till time t :

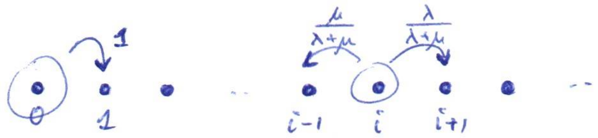
- $N_t \sim \mathcal{P}(\lambda t)$: $\mathbb{P}(N_t = k) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}$
- $N_{t+s} - N_s \sim \mathcal{P}(\lambda t)$: $\quad \mathbb{P}(N(t+s) - N(s) = k) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}$
  $N_t - N_s \sim \mathcal{P}(\lambda(t-s))$

- **M/M/K**

<u>Hp</u>:
- arriving time $\sim \mathcal{E}(\lambda)$
- permanence time $\sim \mathcal{E}(\mu)$
- k counters (servers)
- all of them are $\perp\!\!\!\perp$ r.v.
- $(X_t)_{t \geq 0}$: X_t = # customers at the time t

$$\underset{\text{general model}}{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & \cdots \\ \cdots & \cdots & \cdots & \cdots & & \end{bmatrix}$$

$$\hat{p} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \cdots \\ 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{bmatrix}$$



Starting from i we always consider the possible paths: $i \to j$
T_i = exit time from $i \sim \mathcal{E}(\cdot) \implies q_{ik} = \cdots (\cdot)$

- Recurrent/transient $\implies$ discrete skeleton + discrete MC properties
- If k is specified $\wedge$ max num. of customers:
  $\implies X_t \in \{\ldots\} \leftarrow \dim(\{\ldots\}) < \infty$
  $\implies Q$ generally different
- How many customers arrived till $t \implies N_t \neq X_t$ and N_t follows the Poisson process

Remember: when there are 2 (or +) occupied operators, the probability that one get free must consider both (or +) getting free
(same with contagion: 2 infected can both heal or infect)

- k counters, <u>mean time spent in the system?</u> (Stationarity conditions)
  W_t = time spent for a client arrived at t
  $= S_t + Y_t \mathbb{1}_{\{X_t \geq k\}}$
  $=$ (servise time) + (waiting time if there are already k customers)
  $X_t \sim \pi$ (stat. cond.) $\implies \mathbb{E}_\pi[W_t] = \mathbb{E}_\pi[S_t] + \mathbb{E}_\pi[Y_t \mathbb{1}_{\{X_t \geq k\}}]$
  $$= \frac{1}{\mu} + \sum_{\substack{j \geq k \\ j \in I}} \mathbb{E}_\pi[Y_t \mid X_t = j]\, \mathbb{P}(X_t = j)$$
  $$\frac{1}{\mu} + \sum_{\substack{j \geq k \\ j \in I}} \boxed{\mathbb{E}_\pi[Y_t \mid X_t = j]}\, \pi_j$$

if $X_t = j$, and we have k counte[rs]
we have to wait for $(j-k)+1$ counters to get free:
- one counter get free $\sim \mathcal{E}(\underbrace{\tfrac{1}{\mu} + \ldots + \tfrac{1}{\mu}}_{k \text{ times}})$
- n counters get free $\sim \Gamma(n, \underbrace{\tfrac{1}{\mu} + \ldots + \tfrac{1}{\mu}}_{k \text{ times}})$

$$\left(\mathbb{E}_\pi[W_t] = \frac{1}{\mu} + \frac{\pi_k}{k \cdot \mu} \cdot \frac{1}{\left(1 - \frac{\lambda}{\mu k}\right)^2} \right)$$

- <u>Average length of the queue?</u>
  Stationary conditions $\implies X_t \sim \pi$
  $\mathbb{E}_\pi[X_t] = \sum_i i \cdot \mathbb{P}(X_t = i) = \sum_i i \cdot \pi_i = [\ldots]$
  Little law: $\mathbb{E}_\pi[X_t] = \lambda\, \mathbb{E}_\pi[W_t]$

- $\mathbb{P}(\text{Birth} < \text{Death}) = \mathbb{P}(+1 \text{ customers faster than } -1 \text{ customer})$?
  arriving = birth $\sim \mathcal{E}(\lambda) \implies \mathbb{P}(B(i) < D(i)) = \dfrac{\lambda}{\mu + \lambda} = \underbrace{\mathbb{P}_i(N_{T_i} = j)}_{\text{probability of going to } j \text{ after leaving } i} = \dfrac{q_{ij}}{-q_{ii}}$
  (for instance $j = i+1$)

- **BROWNIAN MOTION**
  $B_t \sim N(0, \sigma^2 t)$
  $B_{t+s} - B_t \sim N(0, s\sigma^2)$
  $\mathbb{E}[(B_{t+s})(B_t)] = \sigma^2 t$
  $\mathbb{E}[B_t \mid \sigma(B_s)] = \mathbb{E}[B_t - B_s] + B_s = B_s$
  $X_t(B) \perp\!\!\!\perp X_s(B)$? $\implies \begin{bmatrix} X_t(B) \\ X_s(B) \end{bmatrix} \sim N(\ldots)$ and if $\mathrm{Cov}(X_t, X_s) = 0$
  $\implies X_t \perp\!\!\!\perp X_s$

- due moli, 4 navi
- scarico molo 1 $\sim \mathcal{E}(a)$
- scarico molo 2 $\sim \mathcal{E}(b)$
- tempo tra cui una nave lascia e torna $\sim \mathcal{E}(\lambda)$

Q: X_t = # navi al molo

	0	1	2	3	4
0	-4λ	4λ			
1	a	$-(a+3\lambda)$	3λ		
2		$a+b$	$-(a+b+2\lambda)$	2λ	
3			$a+b$	$-(a+b+\lambda)$	λ
4				$a+b$	$-(a+b)$

- 2 armchairs: every one must do both of them in order 1+2
- time at 1 $\sim \mathcal{E}(1)$
- time at 2 $\sim \mathcal{E}(1)$
- people go if 1 is empty
- if we're in 1 and done but 2 is occupied then we wait

states = situations:

0 = no people
1 = one person in the first chair
2 = one person in the second
3 = both persons (2) served
4 = person in 1 waiting, 2 be doing

	0	1	2	3	4
0	$-\frac{1}{2}$	$\frac{1}{2}$			
1		-1	1		
2	1		$-\frac{3}{2}$	$\frac{1}{2}$	
3		1		-2	1
4			1		-1

- one ATM, one place for waiting
- the waiter might get a call $(\sim \mathcal{E}(\alpha))$
- average time service $\sim \mathcal{E}(\mu)$
- arrivals $\sim \mathcal{E}(\lambda)$

0 — no people
1 — one customer using ATM
2 — one customer using ATM the other waiting

Fraction of people finishing?

Frac. people entering $= \pi_0 + \pi_1$
Who enters $\pi_0 \Rightarrow$ finish
Who enters $\pi_1 \Rightarrow$ finish if the other conclude before one call:
$P(\text{finish} < \text{call}) = \frac{\mu}{\alpha+\mu}$

frac. people finishing:

$$\frac{\pi_0}{\pi_1+\pi_0} + \frac{\pi_1}{\pi_1+\pi_0}\left(\frac{\mu}{\alpha+\mu}\right)$$

€ c /hour costs of machines
€ b for each operation
$\rightarrow$ the bank charges b€/hour whenever the machine is busy:

$$\underbrace{b(\pi_1+\pi_2)}_{\text{average profit per hour}} > c \quad \text{for self-mantainence}$$

- one ATM, max 2 customers
- arrivals $\sim \mathcal{E}(\lambda)$
- operation $\sim \mathcal{E}(\mu)$
- second operation $\sim \mathcal{E}(\alpha)$ (only p persons do it ($p = \%$))

"starting from i the jump to j occurs in an exponential time of param A with prob B" $\Rightarrow q_{ij} = AB$

States = situations:

0 = no customers
1 = one customer doing his 1st
2 = one customer doing his 2nd
3 = two customers, one is doing his 1st
4 = two customers, one is doing his 2nd

	0	1	2	3	4
0	$-\lambda$	λ			
1	$(1-p)\mu$	$-(\lambda+\mu)$	$p\mu$	λ	
2	α		$-(\alpha+\lambda)$		λ
3		$(1-p)\mu$		$-\mu$	$p\mu$
4		α			$-\alpha$

* Frac. of people using ATM? $\quad \pi_0 + \pi_1 + \pi_2$
* Prob. that a customer who succeed using ATM comes when a customer is doing his 2nd operation?

$$\frac{\pi_2}{\pi_0+\pi_1+\pi_2}$$

* How much time do we have to wait on average without knowing if the customer is doing is 1st or 2nd?

$$\frac{\pi_1}{\pi_0+\pi_1+\pi_2} \cdot T_1 + \frac{\pi_2}{\pi_0+\pi_1+\pi_2} T_2$$

probability of using ATM and coming when one is using the ATM for the 1st op.

prob. of using the ATM and coming when one is doing the 2nd

T_2 = mean time for the second op.

T_1 = mean time for the first op. and (eventualmente) for the second

$T_2 = \frac{1}{\alpha}$, $T_1 = \frac{1}{\mu} + \frac{p}{\alpha}$

DISCRETE MC

- $(X_n)_{n\geq 0}$ is a MC?
  $\mathbb{P}(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \dots, X_0=i_0)$

  - If we can, we can calculate $\frac{\mathbb{P}(X_{n+1}=j, X_n=i, \dots, X_0=i_0)}{\mathbb{P}(X_n=i, \dots, X_0=i_0)}$ and see if it depends only on i and j

  - if given i X_n is defined and X_{+1} depends only on i and j:
    $= \mathbb{P}(X_{n+1}=j \mid X_n=i) \implies$ MC
    $= p_{ij}$
  - If it's needed go with subtractions:
    $= \mathbb{P}(X_{n+1}-X_n=j-i \mid X_n=i, \dots)$
    again, if X_n depends only on $i \implies$ MC
  - if : $= [\dots] = \mathbb{P}(Z_{n+1}=(j-i)-(i-i_{n-1}) \mid S_n=(i-i_{n-1}), \dots)$
    then it depends not only on $i \longrightarrow$ (probably) <u>not</u> a MC

  Counter-example: try something like: $\mathbb{P}(X_3=3 \mid X_2=1, X_1=0)$, $\mathbb{P}(X_3=3 \mid X_2=1, X_1=1)$

I
- $\mathbb{P}(X_n=k)$? $= \pi_k^{(n)}$; $\pi^{(0)}=(\dots, \mathbb{P}(X_0=k), \dots)$, $\pi^{(n)}=P^n\pi^{(0)}$ $\quad P=QDQ^{-1}$, $P^n=QD^nQ^{-1}$

- $\mathbb{P}(X_n=k, X_m=j)$? $= \mathbb{P}(X_n=k \mid X_m=j)\,\mathbb{P}(X_m=j) \overset{H}{=} \mathbb{P}(X_{n-m}=k \mid X_0=j)\,\mathbb{P}(X_m=j)$
  $= (P^{n-m})_{jk} \cdot \mathbb{P}(X_m=j)$

- $\mathbb{P}(X_n+X_0=k)$? $= \sum_j \mathbb{P}(X_n+X_0=k \mid X_0=j)\,\mathbb{P}(X_0=j) = [\dots]$

- $\mathbb{P}(X_n=i, X_0=j \mid X_n+X_0=k)$? $= \frac{\mathbb{P}(X_n+X_0=k \mid X_0=j, X_n=i)\,\mathbb{P}(X_0=j, X_n=i)}{\mathbb{P}(X_n+X_0=k)}$

- $\mathbb{E}[X_n X_m]$? $= \sum_{i,j} ij\,\mathbb{P}(X_n=i, X_m=j)$

II
P^n not easy
- $\mathbb{P}(X_n=k)$? $= \sum_j \mathbb{P}(X_n=k \mid X_0=j)\,\mathbb{P}(X_0=j)$ then procede looking for the possible paths from $j \to \cdot \to \cdot \to \dots \to k$

- Homogeneity? For instance check: $\mathbb{P}(X_2=j \mid X_1=i) \overset{?}{=} \mathbb{P}(X_1=j \mid X_0=i)$

- closed class: C is a closed class if: $\begin{cases} i \in C \\ j \text{ accessible from } i \end{cases} \implies j \in C$

- π? $(\pi = \pi P) \wedge (\sum \pi_i = 1)$

I
- **Hitting prob.** of a recurrent state (/class) starting from transient?
  $=$ **absorption probability** $(V_i)_{i\in I}$:
  $$\boxed{V_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij}V_j}$$
  i transient
  C = set of all the recurrent states we want to consider

- Law of the **hitting time** of a recurrent state (/class) starting from T?
  $T_C = \inf\{n\geq 1 : X_n=k, k\in C\}$, $X_0=\bar{t}$
  $\mathbb{P}(T_C=n) = \mathbb{P}(X_n\in C, X_{n-1}\notin C, \dots, X_1\notin C \mid X_0=\bar{t})$
  $= \sum_{j\in C} \sum_{i_{n-1},\dots,i_1\notin C} \mathbb{P}(X_n=j \mid X_{n-1}=i_{n-1})\,\mathbb{P}(X_{n-1}=i_{n-1} \mid X_{n-2}=i_{n-2}) \dots \mathbb{P}(X_1=i_1 \mid X_0=\bar{t})$
  $= \sum_C \sum_{\notin C} p_{i_{n-1}j}\, p_{i_{n-2}i_{n-1}} \cdots p_{\bar{t}i_1}$

tempo di soggiorno (PDF)

Moreover: $\mathbb{P}(T_c = +\infty) = 1 - \mathbb{P}(T_c < +\infty) = 1 - \sum_n \mathbb{P}(T_c = n)$

stopping time

- Mean absorption time in C (recurrent) starting from $i \in T$ (trans)?

$w_i = \mathbb{E}_i[T]$ (we know that $\mathbb{E}_i[T] = \infty$ if the MC is transient or slow recurrent)

$= w_i \quad i \in T \quad : \quad \boxed{w_i = 1 + \sum_{j \in T} p_{ij} w_j} \quad i \in T$

I (Irreducible) MC recurrent? **Remark**: irreducible + finite $\Longrightarrow$ recurrent

$\mathbb{P}_0(T_0 < +\infty) = \sum_{n \geq 1} \mathbb{P}_0(T_0 = n) = \sum_{n \geq 1} \mathbb{P}(X_n = 0, X_{n-1} \neq 0, \dots, X_1 \neq 0 \mid X_0 = 0)$

$= \sum_{n \geq 1} \mathbb{P}(X_n = 0 \mid X_{n-1} \neq 0)\, \mathbb{P}(X_{n-1} \neq 0 \mid X_{n-2} \neq 0) \dots \mathbb{P}(X_1 \neq 0 \mid X_0 = 0)$

If $\mathbb{P}_0(T_0 < +\infty) = 1 \Longrightarrow$ MC recurrent

I$^{(1)}$ Existence of invariant distr. π? (Irreducible MC)

Remark: I finite $\Longrightarrow \exists (\pi_j)_j$

Irreducible MC $\Longrightarrow$ if 0 is recurrent $(\uparrow)$: $\mathbb{E}_0[T_0] = \sum_{n \geq 1} n \cdot \mathbb{P}(T_0 = n)$

If $\mathbb{E}_0[T_0] < \infty \Longrightarrow$ 0 is positive recurrent

$\Longrightarrow \exists!\ \pi$ invariant and $\pi_0 = \dfrac{1}{\mathbb{E}_0[T_0]}$

I$^{(2)}$ <u>Non</u>-existence of inv. distr. π? (Irreducible MC)

Irreducible MC <u>transient</u> : $(\pi_j)_j$ invariant $\Longleftrightarrow \pi P = \pi \Longleftrightarrow \pi P^n = \pi$

$\Longleftrightarrow \begin{cases} \pi = \pi P^n & ; \\ \pi_j = \sum_{i \geq 0} p_{ij}^{(n)} \pi_i \end{cases}$

but:

j transient $\Longrightarrow p_{ij}^{(n)} \xrightarrow{n \to \infty} 0 \quad \forall i \in I$

$\Longrightarrow \pi_j = \sum_{i \geq 0} p_{ij}^{(n)} \pi_i \xrightarrow{n \to \infty} 0 \quad \forall j$

but $\sum_j \pi_j = 1$, contraddiction $\Longrightarrow \nexists (\pi_j)_j$

II (Irreducible) MC transient? Recurrent? (not transient $\Longrightarrow$ recurrent)

if we don't know the law of T_0: $\mathbb{P}_0(T_0 = n)$

I*: (longest)

Consider the state 0:

$T_0 = \inf\{n \geq 1 : X_n = 0\}$ = hitting time of $\{0\}$

$V_i = \mathbb{P}_i(T_0 < +\infty)$ $\to$ $= 1$ i recurrent $\Longrightarrow$ MC recurrent; < 1 i transient $\Longrightarrow$ MC transient

$\boxed{V_j = p_{ji} + \sum_{k \in I \setminus \{i\}} p_{jk} V_k}$ constraints:

constraints:
- $0 \leq V_j \leq 1$
- V_j is minimal
- boundary cond $(V_1 = \dots, \dots)$

<u>Steps</u>:
- $i = 0, 1$: $V_0 = \dots, V_1 = \dots$
- $i > 1$: $V_i = \dots$
- *
- $V_i = A + B(\#)^i + C(\#)^i$
- constraints to determine $A, B, C, \dots$
- $V_0 = ?$

II*: (coolest)

$(X_n)_{n \geq 0}$ transient $\Longleftrightarrow \exists$ bounded, non-const. sol.: $\sum_{k \in I} p_{jk} y_k = y_j \quad j \in I$

($\forall j$, tranne che al più una)

<u>Steps</u>:
- $j \geq 1$: $y_j = \dots$
- *
- $y_j = A + B(\#)^j + C(\#)^j$
- $\boxed{\text{general cond. for } A, B, C, \dots \text{ to be bounded and non-const}}$ $\Longrightarrow$ we take a random value for $A, B, C \neq 0$ (usually 1) (A can be 0)

If, for example, B and C must be $= 0$ for boundness $\Longrightarrow \nexists$ sol $\Longrightarrow$ recurrent

DISCRETE MC (II)

if asked, check if the matrix is a famous one (example: birth-death)

II

Existence of invariant distr. of π? (Irreducible MC)

Irreducible $(X_n)_{n\geq 0}$ MC
$\exists (y_j)_j, (x_j)_j$ unbounded :
$\sum_{k\geq 0} p_{jk} y_k = y_j - x_j \quad \forall j$
$\longrightarrow \exists! (\pi_j)_j$ invariant

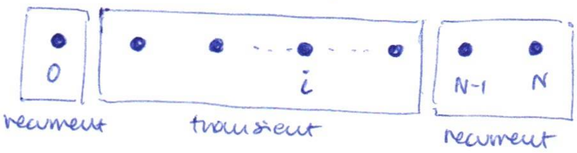
Steps:

- suppose $y_j = j \Rightarrow \sum_{k\geq 0} p_{jk} y_k = \sum_{k\geq 0} p_{jk} \cdot k = [..] = j - (..)$
- if $(..) \to \infty$ we're done
- otherwise suppose $y_j = j^2 \Rightarrow \dots$

Absorption probability in a recurrent state (general)? *

When the equation brings generic V_j (we can't manually enumerate them)

$$V_i = \sum_{j\in C} p_{ij} + \sum_{j\in T} p_{ij} V_j \qquad i \in T$$



if all the MC is transient but one class and the MC is finite $\Rightarrow$ the prob. of absorption is **1**

Steps:
Suppose we want to be abs. in $\{0\}$:

- $\begin{cases} V_0 = 1, \quad V_N = V_{N-1} = 0 \\ V_1 = .. \\ V_i = .. \qquad i \geq 2 \end{cases}$ $\longleftarrow$ we need to specify conditions near the abs. class !
- $V_i \to$ *
- $V_i = A + B(\#)^i + C(\#)^i \dots$
- can we write the specials (V_0, V_1, V_N, V_{N-1}) like V_i? if so, we impose the conditions to find $A, B, C, ..$

* DIFFERENCE EQUATION

$$x_t = a_1 x_{t-1} + .. + a_n x_{t-n} + b$$

- **Homogeneous** : $x_t = a_1 x_{t-1} + \dots + a_n x_{t-n}$
  Char. equation: $\lambda^n = a_1 \lambda^{n-1} + .. + a_n \Longrightarrow \{\lambda_1, .., \lambda_n\}$ (*)
  - (all) distinct roots :
    $x_t = c_1 \lambda_1^t + .. + c_n \lambda_n^t$
  - two equivalent roots : $\lambda_1 = \lambda_2$:
    $x_t = c_1 \lambda_1^t + t c_2 \lambda_2^t + \dots$
- **Non-homogeneous** : $x_t = a_1 x_{t-1} + .. + a_n x_{t-n} + b$
  Complete solution : $x_t =$ (homogeneous solution) + (particular sol.)
  Particular : $x_t = D \Longrightarrow D = a_1 D + .. + a_n D + b \Longrightarrow D$

$\Longrightarrow x_t =$ homogeneous $+ D$

(*) **Ex.**

$V_i = \frac{2}{5} V_{i-1} + \frac{2}{5} V_{i+1} \Longrightarrow 2V_{i+1} - 5V_i + 2V_{i-1} = 0$

$2\lambda^2 - 5\lambda + 2 = 0$

$\lambda_{1/2} < {2 \atop \frac{1}{2}} \Longrightarrow V_i = A\left(\frac{1}{2}\right)^i + B(2)^i$

$W_i = A + B(\#)^i + C(\#)^i ..$ if $\exists \lambda : \lambda = 1$

* DIFFERENTIAL EQUATION

$$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ ? & ? & ? \end{bmatrix} \Downarrow \text{BKE}$$

Ex. $p_{20}'(t) = -\frac{1}{2} p_{20}(t) + \frac{1}{2} e^{-\frac{3}{8}t} - \frac{2}{5} e^{-\frac{5}{12}t} + \frac{2}{5}$ $\quad (*)$

Homogeneous : $p_{20}'(t) = -\frac{1}{2} p_{20}(t) \implies p_{20}(t) = K e^{-\frac{1}{2}t}$

Complete : $p_{20}(t) = K e^{-\frac{1}{2}t} + A e^{-\frac{3}{8}t} + B e^{-\frac{5}{12}t} + C$

Conditions : $p_{20}'(t) = \frac{d}{dt}(\uparrow) = (*)$

$p_{20}(0) = \delta_{20} = 0 \quad (1 \text{ if } \delta_{22} / \delta_{00} / \ldots)$

Ex. $p_{11}'(t) = -2p_{11}(t) + p_{12}(t)$ $\quad (*)$

Homogeneous : $p_{11}'(t) = -2p_{11}(t) \longrightarrow p_{11}(t) = K e^{-2t}$

Complete : $p_{11}(t) = K e^{-2t} + c(t)\, e^{-2t}$

Conditions : $p_{11}'(t) = \frac{d}{dt}(\uparrow) = (*) \implies p_{12}(t) = c'(t)\, e^{-2t}$

$\implies c(t) = \int_0^t p_{12}(s)\, e^{2s}\, ds$

$p_{11}(0) = \delta_{11} = 1$

$\implies \underline{p_{11}(t)} = K e^{-2t} + e^{-2t} \int_0^t p_{12}(s)\, e^{2s}\, ds$

$p_{12}'(t) = 1 - 3p_{12}(t)$

$\uparrow$ using $p_{11}(t) + p_{12}(t) + p_{13}(t) = 1$

$\implies [\ldots \text{ Ex.} \uparrow] \implies \underline{p_{12}(t)} = \frac{1}{3}\left(1 - e^{-3t}\right)$

$\implies \underline{p_{11}(t)} = K e^{-2t} + e^{-2t} \int_0^t p_{12}(s)\, e^{2s}\, ds = [\ldots]$

$\implies p_{13}(t) = 1 - p_{11}(t) - p_{12}(t)$

$p_{00}'(t) = -2p_{00}(t) + 2p_{01}(t)$

$p_{00}(t) = e^{-2t} + e^{-2t} \int_0^t e^{2s}\, p_{01}(s)\, ds$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \to \text{FKE}$$

$p_{00}(t) = \ldots + \ldots \int p_{01}(s) \ldots ds$

$p_{01}'(t) = p_{00}(t)\, q_{01} + p_{01}(t)\, q_{11} + p_{02}(t)\, q_{21}$

$\oplus \; p_{00} + p_{01} + p_{02} = 1$

$\Downarrow$

$p_{01}'(t) = \ldots + p_{01}(t) \ldots$

$\Downarrow$

$p_{01}(t)$

$\Downarrow$

$p_{00}(t)$

$\Downarrow$

$p_{02}(t) = 1 - p_{00}(t) - p_{01}(t)$

If i, j are recurrent in the same class C:

$\lim_{n \to \infty} p_{ij}^{(n)} = \underbrace{\pi_j(C)}_{\text{invariant density restricted on } C}$

moreover for all i transient and j recurrent:

$\lim_{n \to \infty} p_{ij}^{(n)} = \underbrace{u_i(C_j)}_{\text{absorption probability in } C_j \text{ starting from } i} \pi_j(C_j)$

i transient
j recurrent

$\lim_{n \to \infty} p_{ij}^{(n)} = V_i \pi_j$

prob. of absorption in the class of j starting from i