Stochastic dynamical models, Mathematical Engineering 2019/2020 - SDM 7

Exercise 1. Let $(V_n)_{n\geq 1}$ be i.i.d real random variables with probability density function $f(x) = \frac{1}{2}e^{-|x|}$. Let

$$X_{n+1} = X_n + V_{n+1}, \quad X_0 = 0.$$

1) Is $(X_n)_{n\geq 0}$ a Markov process?
2) Write its transition kernel.
3) Is it irreducible?

Exercise 2. Let $B = (B_t)_{t\geq 0}$ be a real Brownian motion and let

$$X_n = B_n, \quad n \in \mathbb{N}.$$

1) Is $(X_n)_{n\geq 0}$ a Markov process?
2) Write its transition kernel.
3) Is it irreducible?

Exercise 3. (Time reversal of a Brownian motion) Let $B = (B_t)_{t\geq 0}$ be a Brownian motion and for $0 \leq t \leq 1$ let

$$X_t = B_{1-t}$$

1) Prove that X is a Markov process.
2) Find the transition kernel of X.
3) X is it a homogeneous Markov process?

Exercise 4. (Brownian bridge) Let B be a brownian motion and for $0 \leq t \leq 1$ let

$$X_t = B_t - tB_1$$

Prove that X is a non-homogeneous Markov process and find its transition kernel.

#1 (#7)

$(V_n)_{n\geq 1}$ iid r.v. : $f(x) = \frac{1}{2} e^{-|x|}$, $X_{n+1} = X_n + V_{n+1}$, $X_0 = 0$

1. Is $(X_n)_{n\geq 0}$ a MP?

$$\mathbb{P}(X_3 \in E_3 \mid X_2 \in E_2, X_1 \in E_1) = \mathbb{P}(X_2 + V_3 \in E_3 \mid X_2 \in E_2, X_1 \in E_1)$$

$$\Longrightarrow \mathbb{P}(X_3 \in E_3 \mid X_2 = x, X_1 \in E_1) = \mathbb{P}(x + V_3 \in E_3 \mid X_2 = x, X_1 \in E_1)$$

$$= \mathbb{P}(x + V_3 \in E_3 \mid X_2 = x)$$

$\Rightarrow$ $(X_n)_{n\geq 0}$ is a Markov Process

2. Transition kernel:

$$\mathbb{P}(X_{n+1} \in A \mid X_n = x) = \mathbb{P}(x + V_{n+1} \in A \;\; X_n = x) = \mathbb{P}(V_{n+1} + x \in A)$$

$Y := V_{n+1} + x$: $\underbrace{\mathbb{P}(Y \leq y)}_{F_Y(y)} = \mathbb{P}(V_{n+1} + x \leq y) = \underbrace{\mathbb{P}(V_{n+1} \leq y - x)}_{F_{V_{n+1}}(y-x)}$

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \frac{\partial F_{V_{n+1}}(y-x)}{\partial y} = f_{V_{n+1}}(y-x) = \frac{1}{2} e^{-|y-x|}$$

$$\mathbb{P}(X_{n+1} \in A \mid X_n = x) = \underbrace{\int_A f_Y(y)\, dy}_{\mathbb{P}(Y \in A) = \mathbb{P}(V_{n+1} + x \in A)} = \int_A \frac{1}{2} e^{-|y-x|}\, dy = P_t(x, A)$$

3. $\boxed{\text{Irreducible} \iff [\forall x \in E, \forall A \in \mathcal{E}, \varphi(A) > 0 \Rightarrow \exists t > 0 : P_t(x,A) > 0]}$

$P_t(x,A) = \frac{1}{2} \int_A e^{-|y-x|} dy$ $E = \mathbb{R}$, φ leb.-measure

$P_t(x,A) > 0 \iff \varphi(A) = 0 \Longrightarrow (X_n)_{n\geq 0}$ irreducible

#2

$B = (B_t)_{t\geq 0}$ brownian motion , $X_n := B_n$ $n \in \mathbb{N}$

1. $\mathbb{P}(X_{n+1} \in E_{n+1} \mid X_n = x, .., X_0 \in E_0) = \mathbb{P}(B_{n+1} \in E_{n+1} \mid B_n = x, B_{n-1} \in .,..., B_0 \in ..)$

$= \mathbb{P}(B_{n+1} - B_n \in E_{n+1} - x \mid B_n - B_{n-1} \in .., B_{n-1} - B_{n-2} \in ..,.)$ $\perp\!\!\!\perp$ increments

$= \mathbb{P}(B_{n+1} - B_n \in E_{n+1} - x)$ $\Rightarrow \mathbb{P}(X_{n+1} \in E_{n+1} \mid X_n = x)$

$= \mathbb{P}(B_{n+1} \in E_{n+1} \mid B_n = x)$ ✓ $\Longrightarrow$ $(X_n)_{n\geq 0}$ is a MP

2. $\mathbb{P}(X_{t+s} \in A \mid X_t = x) = \mathbb{P}(B_{t+s} \in A \mid B_t = x)$

I. $\mathbb{P}(B_{t+s} \leq y \mid B_t = x) = \mathbb{P}(B_{t+s} - B_t \leq y - x)$, $B_{t+s} - B_t \sim N(0, \sigma^2 s)$

$$\Rightarrow \mathbb{P}(B_{t+s} \in A \mid B_t = x) = \int_A \frac{1}{\sqrt{2\pi\sigma^2 s}} e^{-\frac{(y-x)}{2\sigma^2 s}}\, dy$$

$\mathbb{B}(B_{t+s} - B_t \leq y - x) = F_{B_{t+s} - B_t}(y-x)$

$\frac{\partial F_{B_{t+s} - B_t}(y-x)}{\partial y} = f_{B_{t+s} - B_t}(y-x)$

II. $\begin{bmatrix} B_t \\ B_{t+s} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 t & \sigma^2 t \\ \sigma^2 t & \sigma^2(t+s) \end{bmatrix}\right)$

$$\begin{aligned} Cov(B_t, B_{t+s}) &= \mathbb{E}[B_t B_{t+s}] - \cancel{\mathbb{E}[B_t]}\,\cancel{\mathbb{E}[B_{t+s}]} \\ &= \mathbb{E}[B_t(B_{t+s} \pm B_t)] \\ &= \mathbb{E}[B_t(B_{t+s} - B_t)] + \mathbb{E}[B_t^2] \\ &= 0 + Var(B_t) = \sigma^2 t \end{aligned}$$

$$\boxed{Y|X=s \sim N\left(\mu_Y + Cov(X,Y)\frac{1}{\sigma_X^2}(s-\mu_X),\ \sigma_Y^2 - (Cov(X,Y))^2 \frac{1}{\sigma_X^2}\right)} \ !$$

$$B_{t+s} | B_t = x \sim N\left(0 + \sigma^2 t \cdot \frac{1}{\sigma^2 t}(x-0),\ \sigma^2(t+s) - (\sigma^2 t)^2 \frac{1}{\sigma^2 t}\right)$$

$$\sim N(x, \sigma^2 s)$$

$$\mathbb{P}(B_{t+s} \in A | B_t = x) = \int_A f_{B_{t+s}|B_t}(y|x)\, dy$$

$$\stackrel{\downarrow}{=} \int_A \frac{1}{\sqrt{2\pi\sigma^2 s}} e^{-\frac{(y-x)^2}{2\sigma^2 s}}\, dy = P_s(A, x)$$

3. Irreducible? $P_t(x,A) = \int_A \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(y-x)}{2\sigma^2 t}}\, dy$, φ = Lebesgue, $E = \mathbb{R}$

$P_t(x,A) > 0 \iff \varphi(A) > 0 \Rightarrow (X_n)_{n\geq 0}$ irreducible
$(\forall x)$

#3

B Brownian, $X_t = B_{1-t}$ $\quad 0 \leq t \leq 1$

1. $0 \leq t_1 \leq t_2 \leq \dots \leq t_m \leq 1 \implies 1 \geq 1-t_1 \geq 1-t_2 \geq \dots \geq 1-t_m \geq 0$

$(:=) \quad 1 \geq s_1 \geq s_2 \geq \dots \geq s_m \geq 0$

$$\mathbb{P}(X_{t_{m+1}} \in E_{m+1} | X_{t_m} \in E_m, \dots, X_{t_1} \in E_1) = \frac{\mathbb{P}(X_{t_{m+1}} \in E_{m+1}, X_{t_m} \in E_m, \dots, X_{t_1} \in E_1)}{\mathbb{P}(X_{t_m} \in E_m, \dots, X_{t_1} \in E_1)}$$

$$= \frac{\mathbb{P}(B_{s_{m+1}} \in E_{m+1}, \dots, B_{s_1} \in E_1)}{\mathbb{P}(B_{s_m} \in E_m, \dots, B_{s_1} \in E_1)}$$

$$= \frac{\mathbb{P}(B_{s_1} \in E_{s_1} | B_{s_2} \in E_{s_2}, \dots, B_{s_{m+1}} \in E_{m+1})\, \mathbb{P}(B_{s_2} \in E_2 | B_{s_3} \in E_3, \dots, B_{s_{m+1}} \in E_{m+1}) \cdots \mathbb{P}(B_{s_{m+1}} \in E_{m+1})}{\mathbb{P}(B_{s_1} \in E_{s_1} | B_{s_2} \in E_{s_2}, \dots) \cdots \mathbb{P}(B_{s_m} \in E_m)}$$

B is a MP

$$= \frac{\cancel{\mathbb{P}(B_{s_1} \in E_1 | B_{s_2} \in E_2)}\, \cancel{\mathbb{P}(B_{s_2} \in E_2 | B_{s_3} \in E_3)} \cdots \mathbb{P}(B_{s_m} \in E_m | B_{s_{m+1}} \in E_{m+1})\, \mathbb{P}(B_{s_{m+1}} \in E_{m+1})}{\cancel{\mathbb{P}(B_{s_1} \in E_1 | B_{s_2} \in E_2)}\, \cancel{\mathbb{P}(B_{s_2} \in E_2 | B_{s_3} \in E_3)} \cdots \mathbb{P}(B_{s_m} \in E_m)}$$

$$= \frac{\mathbb{P}(B_{s_m} \in E_m | B_{s_{m+1}} \in E_{m+1})\, \mathbb{P}(B_{s_{m+1}} \in E_{m+1})}{\mathbb{P}(B_{s_m} \in E_m)}$$

$$= \frac{\mathbb{P}(B_{s_m} \in E_m, \ B_{s_{m+1}} \in E_{m+1})}{\mathbb{P}(B_{s_m} \in E_m)} = \mathbb{P}(X_{s_{m+1}} \in E_{m+1} \mid X_{s_m} \in E_m)$$

#3(#7)

2. Kernel?

$$\begin{pmatrix} X_t \\ X_{t+s} \end{pmatrix} = \begin{pmatrix} B_{1-t} \\ B_{1-(t+s)} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \sigma^2(1-t) & \sigma^2(1-(t+s)) \\ \sigma^2(1-(t+s)) & \sigma^2(1-(t+s)) \end{bmatrix} \right)$$

$$\begin{aligned} Cov(B_{1-t}, B_{1-(t+s)}) &= \mathbb{E}\left[B_{1-t} B_{1-(t+s)}\right] \\ &= \mathbb{E}\left[(B_{1-t} \pm B_{1-(t+s)}) B_{1-(t+s)}\right] \\ &= \mathbb{E}\left[(B_{1-t} - B_{1-(t+s)}) B_{1-(t+s)}\right] + \mathbb{E}\left[B^2_{1-(t+s)}\right] \\ &= Var(B_{1-(t+s)}) = (1-(t+s))\sigma^2 \end{aligned}$$

$$X_{t+s} \mid X_t = x \sim N\left(0 + \sigma^2(1-(t+s)) \frac{1}{\sigma^2(1-t)} x, \ \sigma^2(1-(t+s)) - (\sigma^2(1-(t+s)))^2 \frac{1}{\sigma^2(1-t)}\right)$$

$$\sim N\left(\frac{1-t-s}{1-t} x, \ \sigma^2 \frac{s}{1-t}(1-t-s)\right)$$

$$\mathbb{P}(X_{t+s} \in A \mid X_t = x) = \int_A \frac{1}{\sqrt{2\pi \sigma^2 \frac{s}{1-t}(1-t-s)}} e^{-\frac{(y - \frac{1-t-s}{1-t}x)^2}{2\sigma^2 \frac{s}{1-t}(1-t-s)}} dy$$

$$= P_s(x, A)$$

3. Homogeneous?

$P_t(x,A) = \mathbb{P}(X_{s+t} \in A \mid X_s = x)$ depends on s $\Rightarrow$ not homogeneous

#4

$$X_t = B_t - tB_1 \qquad 0 \leq t \leq 1$$

$$\begin{bmatrix} X_t \\ X_{t+s} \end{bmatrix} = \begin{bmatrix} B_t - tB_1 \\ B_{t+s} - (t+s)B_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(t+s) \\ 0 & 1 & -t \end{bmatrix} \begin{bmatrix} B_{t+s} \\ B_t \\ B_1 \end{bmatrix}$$

$$\boxed{Y = AX, \quad X \sim N(\mu, \Sigma) \Rightarrow Y \sim N(A\mu, A\Sigma A^T)}$$

$$X = \begin{bmatrix} B_{t+s} \\ B_t \\ B_1 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2(t+s) & \sigma^2 t & \sigma^2(t+s) \\ \sigma^2 t & \sigma^2(t) & \sigma^2 t \\ \sigma^2(t+s) & \sigma^2 t & \sigma^2 \end{bmatrix} \right)$$

$$Y = AX = \begin{bmatrix} X_t \\ X_{t+s} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, A\Sigma A^T \right)$$

$$X_{t+s} \mid X_t = y \sim N\left(\frac{1-t-s}{1-t} y, \ \sigma^2 \frac{s(1-t-s)}{1-t} \right)$$

$$\Rightarrow P_s(y,A) = \mathbb{P}(X_{t+s} \in A \mid X_t = y) = \int_A \frac{1}{\sqrt{2\pi\sigma^2 \frac{s(1-t-s)}{1-t}}} e^{-\frac{(k - \frac{1-t-s}{1-t}y)^2}{2\sigma^2 \frac{s(1-t-s)}{1-t}}} dk$$

Since it depends on t
the Markov process is <u>not</u>
an homogeneous one