Stochastic dynamical models, Mathematical Engineering 2019/2020 - SDM 4

Exercise 1.

1) Consider the Markov chain with transition matrix $P = (p_{ij})_{i,j\geq 0}$ such that

$$p_{0,0} = p_{0,1} = p_{0,2} = \frac{1}{3}$$
$$p_{i,i-1} = \frac{1}{7}, \quad p_{i,i} = p_{i,i+1} = \frac{1}{3}, \quad p_{i,i+2} = \frac{4}{21}, \quad i \geq 1$$
$$p_{ij} = 0, \quad \text{otherwise}$$

Prove that the Markov chain is transient.

2) Prove that the following Markov chain with transition matrix

$$p_{0,0} = p_{0,1} = \frac{1}{2}$$
$$p_{i,i-1} = p_{i,i+1} = p_{i,i+2} = \frac{1}{3}, \quad i \geq 1$$
$$p_{ij} = 0, \quad \text{otherwise}$$

is transient.

Exercise 2. Consider the Markov chain with transition matrix $P = (p_{ij})_{i,j\geq 0}$ such that

$$p_{0,0} = p_{0,1} = \frac{1}{2}$$
$$p_{i,i-1} = \frac{3}{4}, \quad p_{i,i+2} = \frac{1}{4}, \quad i \geq 1$$
$$p_{ij} = 0, \quad \text{otherwise}$$

has a unique invariant distribution.

Exercise 3. Consider the transition rates matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Find its associated transition matrix P_t.

#1 (#4)

1.

(alternative to (↓) : ex. 6 eserc. 2)

$$\boxed{\text{The MC is irreducible} \Longrightarrow \left\{ \begin{array}{l} (X_n)_{n\geq 0} \text{ transient} \\ \Longleftrightarrow \exists \text{ bounded, non-const. sol. of} \\ \quad \sum_{k\in E} p_{jk}\, y_k = y_j \quad \forall j \in E \end{array}\right.}$$

$j \geq 1$: $\quad y_j = p_{j(j-1)}\, y_{j-1} + p_{j(j)}\, y_j + p_{j(j+1)}\, y_{j+1} + p_{j(j+2)}\, y_{j+2}$

$$= \frac{1}{7} y_{j-1} + \frac{1}{3} y_j + \frac{1}{3} y_{j+1} + \frac{4}{21} y_{j+2}$$

$$21 y_j = 3 y_{j-1} + 7 y_j + 7 y_{j+1} + 4 y_{j+2}$$

$$4x^3 + 7x^2 - 16x + 3 = 0$$

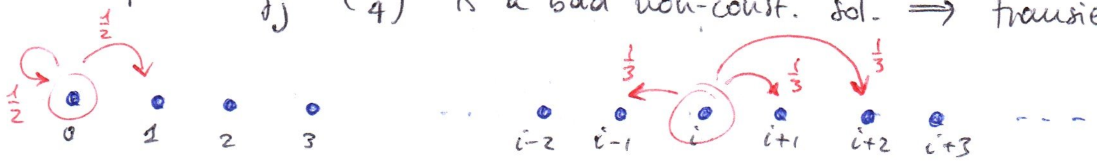
$$(x-1)(4x^2 + 11x - 3) = 0 \qquad x_{1/2/3} = -3, \tfrac{1}{4}, 1$$

$$y_j = A + B\left(\tfrac{1}{4}\right)^j + C(-3)^j$$

- bounded $\Longrightarrow$ $C = 0$
- non-const. $\Longrightarrow$ $B \neq 0$

Example: $y_j = \left(\frac{1}{4}\right)^j$ is a bdd non-const. sol. $\Longrightarrow$ transient

2.



The MC is irreducible:

$j \geq 1$: $\quad y_j = p_{j(j-1)}\, y_{j-1} + p_{j(j+1)}\, y_{j+1} + p_{j(j+2)}\, y_{j+2}$

$$= \frac{1}{3} y_{j-1} + \frac{1}{3} y_{j+1} + \frac{1}{3} y_{j+2}$$

$3 y_j = y_{j-1} + y_{j+1} + y_{j+2} \Longrightarrow$

$$3x = 1 + x^2 + x^3$$

$$x^3 + x^2 - 3x + 1 = 0$$

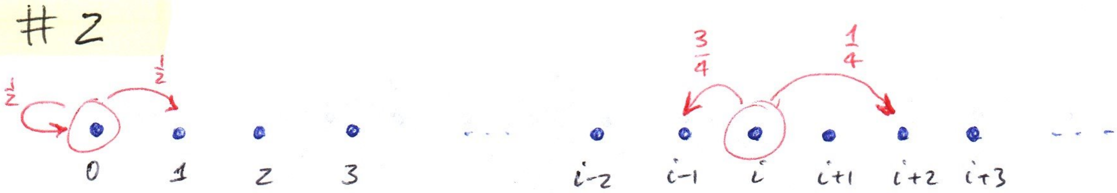
$$(x-1)(x^2 + 2x - 1) = 0 \qquad x_{1/2/3} = 1, -1 \pm \sqrt{2}$$

$$y_j = A + (-1+\sqrt{2})^j B + (-1-\sqrt{2})^j C$$

- bounded $\Longrightarrow$ $C = 0$
- non-const $\Longrightarrow$ $B \neq 0$

Ex. $y_j = (\sqrt{2}-1)^j$ is a bdd non-const. sol. $\Longrightarrow$ transient

#2



This MC is irreducible.

> **Thm.** $(X_n)_{n\geq 0}$ irreducible MC.
> $\exists (y_j)_j, (x_j)_j$ unbounded :
> $$\sum_{k\geq 0} p_{jk}\, y_k \leq y_j - x_j \quad \forall j$$
> $\Longrightarrow \exists ! (\pi_j)_j$ invariant

- suppose $y_j = j$: $\sum_{k\geq 0} p_{jk}\, y_k = \sum_{k\geq 0} p_{jk}\, k = p_{j(j-1)}(j-1) + p_{j(j+2)}(j+2) =$
  $= \frac{3}{4}(j-1) + \frac{1}{4}(j+2) = j - \frac{1}{4}$ but $x_j = \frac{1}{4}$ is not good since is bounded

- suppose $y_j = j^2$: $\sum_{k\geq 0} p_{jk}\, y_k = p_{j(j-1)}(j-1)^2 + p_{j(j+2)}(j+2)^2 =$
  $= \frac{3}{4}(j^2+1-2j) + \frac{1}{4}(j^2+4+4j)$
  $= j^2 + \frac{7}{4} - \frac{1}{2}j$
  $= j^2 - \left(\frac{1}{2}j - \frac{7}{4}\right)$ good $\left(x_j = \frac{1}{2}j - \frac{7}{4}\right)$

#3

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad P_t \ ?$$

FKE : $p_{ij}'(t) = \sum_{k\in E} p_{ik}(t)\, q_{kj}$, $p_{ij}(0) = \delta_{ij}$

- $\frac{dp_{11}(t)}{dt} = p_{11}(t) q_{11} + p_{12}(t) q_{21} + p_{13}(t) q_{31}$
  $= -2p_{11}(t) + p_{12}(t)$

$\frac{dp_{11}(t)}{dt} + 2p_{11}(t) = p_{12}(t)$; Homogeneous: $\left\{ \begin{array}{l} \frac{dp_{11}(t)}{dt} = -2p_{11}(t) \\ \frac{dp_{11}(t)}{p_{11}(t)} = -2dt \\ \tilde{p}_{11}(t) = ce^{-2t} \end{array} \right.$

#3(#4)

Complete: $p_{11}(t) = ce^{-2t} + c(t)e^{-2t}$:

$\frac{dp_{11}(t)}{dt} + 2p_{11}(t) = p_{12}(t) \Longrightarrow \cancel{-2ce^{-2t}} + c'(t)e^{-2t} \cancel{- 2c(t)e^{-2t}} + \cancel{2ce^{-2t}} + \cancel{2c(t)e^{-2t}} =$

$= p_{12}(t)$

$p_{12}(t) = c'(t)e^{-2t} \Longrightarrow c(t) = \int_0^t p_{12}(s)e^{2s}\,ds$

$\Longrightarrow p_{11}(t) = ce^{-2t} + \int_0^t p_{12}(s)e^{2s}\,ds$

but $p_{11}(0) = 1 \Longrightarrow c = 1$

$\Longrightarrow \boxed{p_{11}(t) = e^{-2t} + \int_0^t p_{12}(s)e^{2s}\,ds}$

- $\frac{dp_{12}(t)}{dt} = p_{11}(t)q_{12} + p_{12}(t)q_{22} + p_{13}(t)q_{32} = p_{11}(t) - 2p_{12}(t) + p_{13}(t)$

  $= 1 - 3p_{12}(t)$

  since $p_{13}(t) + p_{12}(t) + p_{11}(t) = 1$

Homogeneous : $\tilde{p}_{12}(t) = ce^{-3t}$

Complete : $p_{12}(t) = ce^{-3t} + k(t)e^{-3t}$

$\frac{dp_{12}(t)}{dt} + 3p_{12}(t) = 1 \Longrightarrow \cancel{-3ce^{-3t}} + k'(t)e^{-3t} \cancel{- 3k(t)e^{-3t}} + \cancel{3ce^{-3t}} + \cancel{3k(t)e^{-3t}} =$

$= 1$

$k'(t)e^{-3t} = 1 \Rightarrow k(t) = \int_0^t e^{3s}\,ds = \frac{1}{3}\left[e^{3s}\right]_0^t = \frac{1}{3}(e^{3t} - 1)$

$\Longrightarrow p_{12}(t) = ce^{-3t} + \frac{1}{3}(1 - e^{-3t})$

but $p_{12}(0) = 0 \Longrightarrow c = 0$

$\Longrightarrow \boxed{p_{12}(t) = \frac{1}{3}(1 - e^{-3t})}$

- $p_{11}(t) = e^{-2t} + \int_0^t \frac{1}{3}(1 - e^{-3s})e^{2s}\,ds = [..]$

$\Longrightarrow \boxed{p_{11}(t) = \frac{1}{6} + \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}}$

- $p_{13}(t) = 1 - p_{11}(t) - p_{12}(t) = [..]$

$\Longrightarrow \boxed{p_{13}(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}}$