Stochastic Dynamical Models

September 7, 2018

EXERCISES

Exercice 1. Let $(X_n)_{n\geq 0}$ be the discrete time Markov chain with state space $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{4} & \frac{3-2\theta}{4} & \frac{\theta}{2} & 0 & \dots \\ 0 & \frac{1}{4} & \frac{3-2\theta}{4} & \frac{\theta}{2} & \dots \\ 0 & 0 & \frac{1}{4} & \frac{3-2\theta}{4} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where $0 < \theta < 3/2$

(a) Is the Markov chain irreducible? Is the Markov chain periodic?

(b) For what values of θ is the Markov chain transient? For what values of θ is the Markov chain recurrent ?

(c) For what values of θ the Markov chain admits an invariant density? In these cases also compute explicitly all invariant densities.

we take the formula from death-birth process

(d) Let $\mathbb{P}_i$ be the conditional probability $\mathbb{P}\{\cdot \mid X_0 = i\}$ and let T the return time to 0

$$T := \inf\{n \geq 1 \mid X_n = 0\}.$$

Compute $\mathbb{E}_i[T]$ for all θ.

(e) Let $f : \mathbb{N} \to \mathbb{N}$ be the identity function $f(n) = n$. Show that the process $(M_n)_{n\geq 0}$ defined by $M_0 = 0$ and

$$M_n = X_n - \sum_{k=0}^{n-1} \left(\frac{1}{2} 1_{\{X_k=0\}} + \frac{2\theta - 1}{4} 1_{\{X_k>0\}} \right)$$

is a martingale and recover the result of (d) applying the stopping theorem.

Exercice 2. A lady's hairdresser shop has two armchairs. The armchair 1 is for shampooing and hair dyeing and armchair 2 is for haircut and styling. Service times are independent exponential random variables with parameters 1 (i.e. the mean is 1 hour). Interarrival times of potential customers are exponential random variables with parameter $1/2$ (i.e. the mean is 2 hours). A customer enters in the shop only if hairdressers at both armchairs are free (because there is only one hairdresser who can not take care of two customers at the same time).

(a) Construct a continuous time Markov chain model
(b) Which is the fraction of potential customers that actually enters in the shop in the stationary regime?

Suppose now that the hairdresser hires an assistant working at armchair 1. In this way new customers enter in the shop if the armchair 1 is free. When they finish shampooing they proceed to armchair 2 if this is free, if not they wait in armchair 1 till armchair 2 is free.

(c) Construct a continuous time Markov chain model in the new situation. Why is it possible?
(d) Which is the fraction of potential customers that actually enters in the shop?
(e) What is the average number of customers in the shop?
(f) What is the average time spent by by a customer in the shop?

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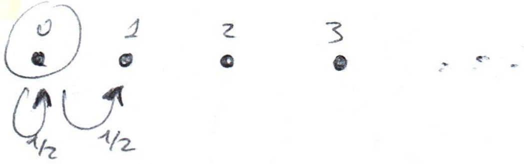
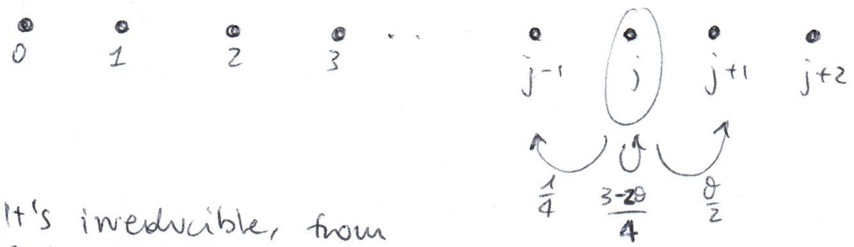
1.





It's irreducible, from every state we can reach any state (we can always get back to 0 and then reach again another state)

since $p_{ii} > 0 \ \forall i \Rightarrow$ aperiodic

2. $y_j = \frac{1}{4} y_{j-1} + \frac{3-2\theta}{4} y_j + \frac{\theta}{2} y_{j+1} \Rightarrow \quad y_{j-1} + (-2\theta - 1) y_j + 2\theta y_{j+1} = 0$

$2\theta x^2 - (2\theta + 1) x + 1 = 0 \quad \longrightarrow \quad x_{1/2} = \dots = \dfrac{2\theta + 1 \pm |2\theta - 1|}{4\theta}$

$\theta > \frac{1}{2} \Rightarrow \quad x_{1/2} \begin{cases} 1 \\ \frac{1}{2\theta} \end{cases} \Rightarrow \quad y_j = A + B\left(\frac{1}{2\theta}\right)^j$

$\frac{1}{2} < \theta < \frac{3}{2} \Rightarrow \quad 1 < 2\theta < 3 \quad \longrightarrow \quad \exists (y_j)_j \dots \Rightarrow$ transient

$\theta < \frac{1}{2} \Rightarrow \quad x_{1/2} \begin{cases} 1 \\ \frac{1}{2\theta} \end{cases} \Rightarrow \quad y_j = A + B\left(\frac{1}{2\theta}\right)^j$ but $0 < 2\theta < 1$

$\longrightarrow$ recurrent

$\theta = \frac{1}{2} \Rightarrow$ recurrent

3. $w_0 = 0$

$w_j = 1 + \frac{1}{4} w_{j-1} + \frac{3-2\theta}{4} w_j + \frac{\theta}{2} w_{j+1}$

$1 + (-2\theta - 1) x + 2\theta x^2 = 0$

$x_{1/2} = 1, \frac{1}{2\theta}$

$w_j = A + B\left(\frac{1}{2\theta}\right)^j$ (homogeneous)

$w_j = D \longrightarrow \quad D = 1 + \frac{1}{4} D + \frac{3-2\theta}{4} D + \frac{\theta}{2} D$

$4\cancel{D} = 4 + \cancel{D} + (3 - 2\theta) D + 2\theta D \quad$ X

$w_j = Dj \Rightarrow \quad 4Dj = 4 + (j-1)D + (3-2\theta) j D + 2\theta (j+1) D$

$\cancel{4Dj} = 4 + \cancel{Dj} - D + \cancel{3Dj} - \cancel{2D\theta j} + \cancel{2\theta D j} + 2D\theta$

$4 - D + 2\theta D = 0 \qquad 4 = D(2\theta - 1)(-1) \Rightarrow D = \dfrac{4}{1 - 2\theta}$

$$W_j = A + B\left(\frac{1}{2\theta}\right)^j + \frac{4}{1-2\theta}\, j$$

$$W_0 = 0 \longrightarrow A + B = 0$$

$$W_1 = 1 + \frac{1}{4}\cancel{W_0} + \frac{3-2\theta}{4} W_1 + \frac{\theta}{2} W_2$$

$$1 + \left(\frac{3-2\theta}{4} - 1\right)\left[A + \frac{1}{2\theta} B + \frac{4}{1-2\theta}\right] + \frac{\theta}{2}\left[A + \frac{1}{4\theta^2} + \frac{8}{1-2\theta}\right] = 0$$

$$\uparrow$$

$$0 = 0$$

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- 2 armchairs $\left\{ \begin{array}{ll} \text{shampooing + hair dyeing} & \sim \mathcal{E}(1) \\ \text{haircut and styling} & \sim \mathcal{E}(1) \end{array} \right.$
- arrivings $\sim \mathcal{E}(\frac{1}{2})$
- a customers enters only if both chairs are free

1. 0 = no customers : $0 \to 1, 0$: $q_{01} = \frac{1}{2}$, $q_{00} = -\frac{1}{2}$, $q_{02} = 0$

   1 = 1 customer served at the first chair : $1 \to 2, 1$: $q_{12} = 1$, $q_{11} = -1$, $q_{10} = 0$

   2 = customer at the second chair : $2 \to 0, 2$: $q_{20} = 1$, $q_{22} = -1$, $q_{21} = 0$

$$Q = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

2. $-\frac{1}{2}\pi_0 + \pi_2 = 0$

   $\frac{1}{2}\pi_0 - \pi_1 = 0$ $\qquad \pi_1 = \frac{1}{2}\pi_0 \to \pi_0 = 2\pi_1$

   $\pi_1 = \pi_2$

   $2\pi_1 + \pi_1 + \pi_1 = 1$ $\qquad \pi_1 = \frac{1}{4}$, $\pi_2 = \frac{1}{4}$, $\pi_0 = \frac{1}{2}$

3. Possible situations

   0 = both chairs free

   1 = one customer served at 1^{st}

   2 = one customer served at 2^{nd}

   3 = both customers served

   4 = one customer served at 2^{nd} while at 1^{st} waiting

$$\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & \\ \hline -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -\frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 0 & 1 & 0 & -2 & 1 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}$$

4. $\pi_2 = \frac{1}{2}\pi_0$

   $\pi_3 = \pi_1 - \pi_2$

   $\pi_3 = \pi_4$

   $\frac{1}{2}\pi_2 = 2\pi_3$

   $\pi_2 = \frac{1}{2}\pi_0$

   $5\pi_3 = \pi_1$

   $\pi_3 = \pi_4$

   $\pi_2 = 4\pi_3 \to \frac{1}{2}\pi_0 = 4\pi_3 \to \pi_3 = \frac{1}{8}\pi_0$

   $\pi_1 = \frac{1}{8}\pi_0$

   $\pi_4 = \frac{1}{8}\pi_0$

   $\pi_0 = \frac{1}{8}$