Markov Chain

Exercise lesson

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Exercise 1

A nuclear steam supply system has two turbo-generator units; unit 1 operates and unit 2 is in standby whenever both are good. The units have a constant MTTF of λ_i^{-1}, i=1 and 2, during active operation while during standby unit 2 has a MTTF of $(\lambda_2^*)^{-1}$. The repair of a unit is assumed to begin instantaneously after it fails, but its duration is random so that the instantaneous repair rates will be μ_1 and μ_2, respectively. The repairs can be done on only one unit at a time and any unit under repair will remain so until the task is completed.

1. Draw the system diagram.
2. Write the Markov equations.

Exercise 2

An alarm system is subject to both unrevealed (u) and revealed (r) faults each of which have time to occurrence which are exponentially distributed with mean values of 200 h and 100 h, respectively. If a revealed failure occurs, then the complete system is restored to the time-zero condition by a repair process which has exponentially distributed times to completion with a mean value of 10 h. If an unrevealed fault occurs, then it remains in existence until a revealed fault occurs when it is repaired along with the revealed fault.

1. What is the asymptotic unavailability of the alarm system?
2. What is the mean number of system failures in a total time of 1000 h?

Exercise 3

A new item starts operating on line. When it fails (failure rate λ_1) a partial repair is performed (repair rate μ_p) which enables the item to continue operation, but with a new failure rate $\lambda_2 > \lambda_1$. When it fails for the second time, a thorough repair (repair rate $\mu_r < \mu_p$) restores the item to the as-good-as-new state and the cycle is repeated.

1. Find the asymptotic unavailability.
2. How can you get the familiar expression for a single item under exponential failure and repair?
3. What is the asymptotic failure intensity?

Exercise 4

Two identical pumps are working in parallel logic. During normal operation both pumps are functioning. When one pump fails, the other has to do the whole job alone, with a higher load. The pumps are assumed to have exponentially distributed failure times:
$\lambda_h = 1.5 * 10^{-4}\ h^{-1}$ when the pumps are bearing half load
$\lambda_f = 3.5 * 10^{-4}\ h^{-1}$ when the pumps are bearing the full load

Both pumps may fail at the same time due to some external stresses. The failure rate with respect to this common cause failure has been estimated to be $\lambda_c = 3.0 * 10^{-5}\ h^{-1}$. This type of external stresses affects the system irrespective of how many units are working.

Repair is initiated as soon as one of the pumps fails. The mean time to repair a pump, μ^{-1}, is 15 hours. When both pumps are in the failed state, the whole system has to be shut down. In this case, the system will not be put into operation again until both pumps have been repaired. The mean downtime, ${\mu_b}^{-1}$, when both pumps are failed, has been estimated to be 25 hours.

1. Establish a state-space diagram for the system.
2. Write down the state equation in matrix format.
3. Determine the steady states probabilities.
4. Determine the percentage of time when:
   1. Both pumps are functioning
   2. Only one of the pumps is functioning
   3. Both pumps are in the failed state
5. Determine the mean number of pump repairs that are needed during a period of 5 years.
6. How many times we may expect to have a total pump failure (i.e. both pumps in a failed state at the same time) during a period of 5 years?

Exercise 5

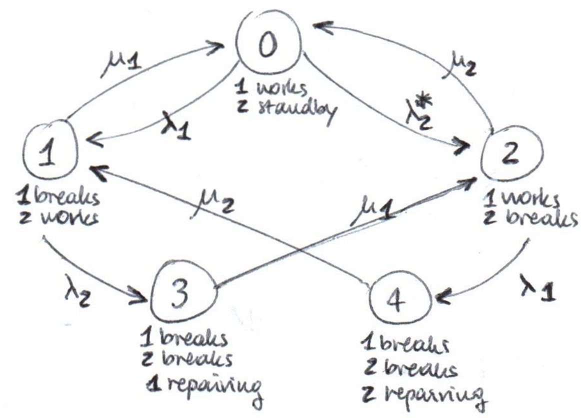
Consider a two unit standby system, with failure rate λ_a and λ_b during active operation and λ_b^* in the standby mode in which there is a switching failure probability p.

1. Draw the transition diagram.
2. Write the Markov equations.
3. Solve for the system reliability
4. reduce the reliability to the situation in which the units are identical $\lambda_a = \lambda_b = \lambda_b^* = \lambda$

#1 (#6)

$T_i \sim \mathcal{E}(\lambda_i)$ active operation
$T_2^* \sim \mathcal{E}(\lambda_2^*)$ stand-by
$T_R \sim \mathcal{E}(\mu_i)$

1.



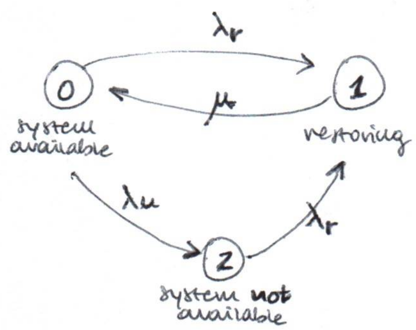
we need to consider them separately because we have 1 repair-man

2. Markov equations:

$$[P_0'(t)\ P_1'(t)\ P_2'(t)\ P_3'(t)\ P_4'(t)] = [P_0(t)\ P_1(t)\ P_2(t)\ P_3(t)\ P_4(t)] \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -(\lambda_1+\lambda_2^*) & \lambda_1 & \lambda_2^* & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 & 0 \\ \mu_2 & 0 & -(\lambda_1+\mu_2) & 0 & \lambda_1 \\ 0 & 0 & \mu_1 & -\mu_1 & 0 \\ 0 & \mu_2 & 0 & 0 & -\mu_2 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

#2

$u \sim \mathcal{E}(\lambda_u = 1/200)$
$r \sim \mathcal{E}(\lambda_r = 1/100)$
$\text{repair} \sim \mathcal{E}(\mu = 1/10)$



1. $$\underline{\underline{A}} = \begin{bmatrix} -(\lambda_r+\lambda_u) & \lambda_r & \lambda_u \\ \mu & -\mu & 0 \\ 0 & \lambda_r & -\lambda_r \end{bmatrix}$$

$$[\pi_0\ \pi_1\ \pi_2]\,\underline{\underline{A}} = \underline{0}$$

$$\begin{cases} -(\lambda_r+\lambda_u)\pi_0 + \mu\pi_1 = 0 \\ \pi_0\lambda_u - \lambda_r\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} \pi_0 = \dfrac{\mu\lambda_r}{(\lambda_u+\lambda_r)(\mu+\lambda_r)} = 0.606 \\ \pi_1 = \dfrac{\lambda_r+\lambda_u}{\mu}\pi_0 = 0.091 \\ \pi_2 = 1-\pi_0-\pi_1 = \dfrac{\lambda_u}{\lambda_r}\pi_0 = 0.303 \end{cases}$$

unavailability $= 1 -$ availability $= 1 - \pi_0 = 0.394$

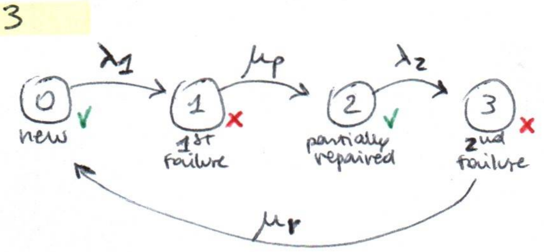
2. Frequency of system failure f = departure frequency $0\to1$ + dep. freq. $0\to2$

$$f = \nu_{01} + \nu_{02} = [A]_{01}\pi_0 + [A]_{02}\pi_0 = \pi_0(\lambda_r+\lambda_u) = 0.00909\ h^{-1}$$

Mean time in 1000 hours:

$f \cdot 1000 = 9.09$ = # system failures in 1000 hours

#3



$$\begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 \\ 0 & -\mu_p & \mu_p & 0 \\ 0 & 0 & -\lambda_2 & \lambda_2 \\ \mu_r & 0 & 0 & -\mu_r \end{bmatrix} \end{array}$$

$$\begin{cases} -\pi_0\lambda_1 + \pi_3\mu_r = 0 \\ \pi_0\lambda_1 - \pi_1\mu_p = 0 \\ \pi_1\mu_p - \pi_2\lambda_2 = 0 \\ \pi_0+\pi_1+\pi_2+\pi_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_3 = \dfrac{\lambda_1}{\mu_r}\pi_0 \\ \pi_1 = \dfrac{\lambda_1}{\mu_p}\pi_0 \\ \pi_2 = \dfrac{\lambda_1}{\lambda_2}\pi_0 \end{cases} \Rightarrow \pi_0 = \frac{\mu_p\mu_r\lambda_2}{(\lambda_1+\lambda_2)\mu_p\mu_r + (\mu_p+\mu_r)\lambda_1\lambda_2} := \frac{\mu_p\mu_r\lambda_2}{D}$$

1. $\pi_1 = \frac{\lambda_1\lambda_2\mu_r}{D}$, $\pi_3 = \frac{\lambda_1\lambda_2\mu_p}{D}$ $\Longrightarrow$ unavailability $= \pi_1 + \pi_3 = \frac{\lambda_1\lambda_2(\mu_r+\mu_p)}{D}$ (*)

2. The familiar expression for a single item under $\mathcal{E}(\cdot)$ failure and repair is:

$$\underline{\underline{A}} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \rightarrow \begin{matrix} -\lambda\pi_0 + \mu\pi_1 = 0 \\ \pi_0 + \pi_1 = 1 \end{matrix} \rightarrow \text{unavailability} = \pi_1 = \frac{\lambda}{\mu+\lambda}$$

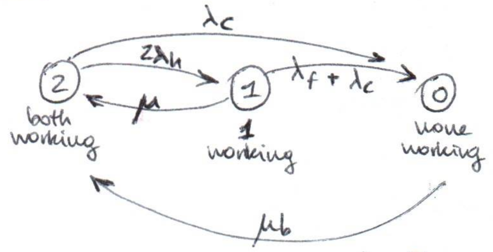
which corresponds to $\lambda_1 = \lambda_2 = \lambda$, $\mu_r = \mu_p = \mu$ in (*)

3. Asympt. failure intensity:

$$W_F = \pi_0\lambda_1 + \pi_2\lambda_2 = \frac{2\lambda_1\lambda_2\mu_p\mu_r}{D}$$

#4

1.



2.

$$\underline{\underline{A}} = \begin{array}{c|ccc|c} & 0 & 1 & 2 & \\ \hline & -\mu_b & 0 & \mu_b & 0 \\ & \lambda_f+\lambda_c & -(\lambda_f+\lambda_c+\mu) & \mu & 1 \\ & \lambda_c & 2\lambda_h & -(\lambda_c+2\lambda_h) & 2 \end{array}$$

3. $\underline{\pi}\cdot\underline{\underline{A}} = \underline{0} \Longrightarrow$

$$\begin{cases} \pi_2 = \dfrac{\mu_b(\lambda_c+\lambda_f+\mu)}{(\lambda_c+\mu_b)(\lambda_c+\lambda_f+\mu)+2\lambda_h(\lambda_c+\lambda_f+\mu)} = 0.99476 \\ \pi_1 = \dfrac{2\lambda_h}{\lambda_c+\lambda_f+\mu}\pi_2 = 0.00445 \\ \pi_0 = 1-\pi_1-\pi_2 = 0.00079 \end{cases}$$

4. [99.476 %, 0.445 %, 0.079 %] = percentage of time being in [2, 1, 0]

5. we use the frequencies of departures:

$$f = \pi_1 v_{12} + \pi_0 v_{02}\cdot 2 = \pi_1\mu + 2\pi_0\mu_b = 3.6\cdot 10^{-4}\ h^{-1}$$

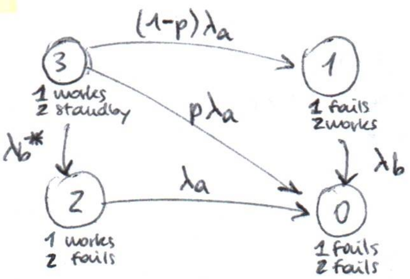
In 5 years $\rightarrow$ $5\cdot 365\cdot 24\cdot f = 15.8$ expected number of repairs in 5 years

6. again, frequencies of arrivals (this time):

$$f = \pi_1 v_{10} + \pi_2 v_{20} = \pi_1(\lambda_f+\lambda_c) + \pi_2(\lambda_c) = 3.15338\cdot 10^{-5}$$

In 5 years $\Longrightarrow$ $5\cdot 365\cdot 24\cdot f = 1.38$

#5



$$\begin{array}{cccc|c} 0 & 1 & 2 & 3 & \\ \hline 0 & 0 & 0 & 0 & 0 \\ \lambda_b & -\lambda_b & 0 & 0 & 1 \\ \lambda_a & 0 & -\lambda_a & 0 & 2 \\ p\lambda_a & (1-p)\lambda_a & \lambda_b^* & -(\lambda_a+\lambda_b^*) & 3 \end{array}$$

3. Solve for reliability $\rightarrow$ MTTF $= \tilde{R}(0) = \Sigma_{j\in S}\tilde{P}_j(0)$

$$\dot{\underline{P}}(t) = \underline{P}(t)\cdot\underline{\underline{A}} \rightarrow s\tilde{\underline{P}}(s) - \underline{P}(0) = \tilde{\underline{P}}(s)\cdot\underline{\underline{A}} \rightarrow [\tilde{P}_0(s), \tilde{P}_1(s), \tilde{P}_2(s), \tilde{P}_3(s)]$$

$$\rightarrow \tilde{R}(0) = \tilde{P}_1(0) + \tilde{P}_2(0) + \tilde{P}_3(0) = \frac{(1-p)\lambda_a}{\lambda_b(\lambda_a+\lambda_b)} + \frac{\lambda_b^*}{\lambda_a(\lambda_a+\lambda_b)} + \frac{1}{\lambda_a+\lambda_b^*} = \frac{\lambda_a^2(1-p)+\lambda_b\lambda_b^*+\lambda_a\lambda_b}{\lambda_a\lambda_b(\lambda_a+\lambda_b^*)}$$

4. $$\text{MTTF} = \tilde{R}(0) = \{\lambda = \lambda_a = \lambda_b = \lambda_{b^*}\} = \frac{3\lambda^2 - \lambda^2 p}{\lambda^2(2\lambda)} = \frac{3-p}{2\lambda}$$