lasar

laboratory of signal and risk analysis

ENERGY DEPARTMENT

POLITECNICO DI MILANO

Exercise Session 01.03.2021
Probabilistic Models

MEI CHEN
mei.chen@polimi.it

Exercise 1

The reliability engineer of a nuclear power plant is not sure that the installed alarm system, composed by a single alarm, is enough reliable. If there is something wrong with the reactor, the probability that the alarm triggers is 0.99. Assume also that if nothing is wrong with the reactor, the probability of the alarm to not trigger is still 0.99.

Suppose that something is wrong with the reactor only one day out of 100.

1. What is the probability that something is wrong if the alarm triggers? Comment the result.
2. If we add a second alarm (equal to the first one), what is the probability that something is wrong if also the second alarm triggers?

Exercise 2

Ten compressors, each one with a failure probability of 0.1, are tested independently.

1. What is the expected number of compressors that are found failed?
2. What is the variance of the number of compressors that are found failed?
3. What is the probability that none will fail?
4. What is the probability that two or more will fail?

Exercise 3

models (distr.)

Consider the occurrence of misprints in a book, and suppose that they occur at the rate of 2 per page.

1. What is the probability that the first misprint will not occur in the first page?
2. What is the expected number of pages until the first misprint appears?
3. Comment on the applicability of the Poisson assumption (independence, homogeneity, fixed period) in this case

Exercise 4

An aircraft flight panel is fitted with two types of artificial horizon indicators. The times to failure of each indicator from the start of a flight follow and exponential distribution with a mean value of 15 hours for one and 30 hours for the other. A flight lasts for a period of 3 hours.

1. What is the probability that the pilot will be without an artificial horizon indication by the end of a flight?
2. What is the mean time to this event, if the flight is of a long duration?

Exercise 5

A machine has been observed to survive a period of 100 hours without failure with probability 0.5. Assume that the machine has a constant failure rate λ.

1. Determine the failure rate λ.
2. Find the probability that the machine will survive 500 hours without failure.
3. Determine the probability that the machine fails within 1000 hour, assuming that the machine has been observed to be functioning at 500 hours.

Exercise 6

In considering the safety of a building, the total force acting on the columns of the building must be examined. This would include the effect of the dead load D (due to the weight of the structure), the live load L (due to the human occupancy, movable furniture, and the like) and the wind load W. Assume that the load effects on the individual columns are statistically independent Gaussian variates with:

$$\mu_D = 4.2\ kips \qquad \sigma_D = 0.3\ kips$$
$$\mu_L = 6.5\ kips \qquad \sigma_L = 0.8\ kips$$
$$\mu_W = 3.4\ kips \qquad \sigma_W = 0.7\ kips$$

1. Determine the mean and standard deviation of the total load acting on a column.
2. If the strength R of the column is also Gaussian with a mean equal to 1.5 times the total mean force, what is the probability of failure of the column? Assume that the coefficient of variation of the strength δ_R is 15% and the strength and load effects are statistically independent.

Exercise 7

The following relationship arises in the study of the earthquake-resistant design:

$$Y = ce^X$$

where Y is the ground motion intensity at the building site, X is the magnitude of an earthquake and c is related to the distance between the site and center of the earthquake.

If X is exponentially distributed,

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

find the cumulative distribution function of Y, $F_Y(y)$.

#1 (#2)

A = alarm triggers
W = something wrong

$\mathbb{P}(A|W) = 0.99$

$\mathbb{P}(\bar{A}|\bar{W}) = 0.99$

$\mathbb{P}(W) = 0.01$

1. $\mathbb{P}(W|A) = \dfrac{\mathbb{P}(A|W)\,\mathbb{P}(W)}{\mathbb{P}(A)} = \dfrac{\mathbb{P}(A|W)\,\mathbb{P}(W)}{\mathbb{P}(A|W)\,\mathbb{P}(W) + \mathbb{P}(A|\bar{W})\,\mathbb{P}(\bar{W})} = \dfrac{0.99\cdot 0.01}{0.99\cdot 0.01 + (1-0.99)(1-0.01)} = 0.5$

2. $\mathbb{P}(W|A,A) = \dfrac{\mathbb{P}(A,A,W)}{\mathbb{P}(A,A)} = \dfrac{\mathbb{P}(A,A|W)\,\mathbb{P}(W)}{\mathbb{P}(A,A|W)\,\mathbb{P}(W) + \mathbb{P}(A,A|\bar{W})\,\mathbb{P}(\bar{W})}$

$$= \frac{\mathbb{P}(A|W)\,\mathbb{P}(A|W)\,\mathbb{P}(W)}{\mathbb{P}(A|W)\,\mathbb{P}(A|W)\,\mathbb{P}(W) + \mathbb{P}(A|\bar{W})\,\mathbb{P}(A|\bar{W})\,\mathbb{P}(\bar{W})} = \frac{(0.99)^2\cdot 0.01}{(0.99)^2\cdot 0.01 + (0.01)^2\cdot 0.99} = 0.99$$

this is **not** $\mathbb{P}(A)^2$, they're not $\perp\!\!\!\perp$, they're conditionally $\perp\!\!\!\perp$

#2

X = # failed compressors
$X \sim \text{Binomial}(n,p) = Bi(10, 0.1)$

1. $\mathbb{E}[X] = p\cdot n = 10\cdot 0.1 = 1$
2. $Var(X) = np(1-p) = 10\cdot 0.1\,(1-0.1) = 0.9$
3. $\mathbb{P}(X=0) = \binom{n}{0} p\,(1-p)^{n-0} \overset{|}{=} \frac{n!}{n!\,0!}\,1\cdot(1-0.1)^{10} = 0.3487$
4. $\mathbb{P}(X\geq 2) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) = 1 - 0.3487 - \binom{10}{1}(0.1)^1(1-0.1)^9 = 0.2639$

#3

X = # misprints per page
Occurrence of events in a continuous time of period $\longrightarrow$ $X \sim P(\lambda) = \mathbb{P}(2)$

1. $\mathbb{P}(X=0) = \dfrac{2^0}{0!} e^{-2} = e^{-2} = 0.1353$

! 2. Number of pages modeled through a geometric distribution (number of pages up until the first misprint)

$T \sim \mathcal{G}(p) = \mathcal{G}(\mathbb{P}(X>0)) = \mathcal{G}(1 - 0.1353)$

$\mathbb{E}[T] = \dfrac{1}{p} = \dfrac{1}{1-0.1353} = 1.16$

T = # trials between two successive occurrences of success (where the success is to have the errorprint)

#4

$X_1 \sim \mathcal{E}$ $\quad \mathbb{E}[X_1] = \frac{1}{\lambda_1} = 15 \quad \Longrightarrow \quad \lambda_1 = \frac{1}{15} = 0.0\bar{6}$

$X_2 \sim \mathcal{E}$ $\quad \mathbb{E}[X_2] = \frac{1}{\lambda_2} = 30 \quad \Longrightarrow \quad \lambda_2 = \frac{1}{30} = 0.1\bar{3}$

$\boxed{MTTF = \int_0^\infty t f(t)\,dt = \int_0^\infty R(t)\,dt}$

ALTERNATIVE

1. $\mathbb{P}(X_1<3, X_2<3) = \mathbb{P}(X_1<3)\,\mathbb{P}(X_2<3) = (1-e^{-1/15\cdot 3})(1-e^{-1/30\cdot 3}) = 0.01725$
2. $MTTF = \int_0^{+\infty} t\, f_{12}(t)\,dt = \int_0^\infty R_{AB}(t)\,dt = \int_0^\infty (1-F_{AB}(t))\,dt = \int_0^\infty (1-F_A(t)F_B(t))\,dt$

$f_{12}(t) = \frac{d}{dt}F_{12}(t) = \frac{d}{dt}\left(F_1(t)F_2(t)\right) = f_1(t)F_2(t) + F_1(t)f_2(t)$! $\neq f_1(t)f_2(t)$

$$MTTF = \int_0^{+\infty} t\left[(1-e^{-\lambda_1 t})(\lambda_2 e^{-\lambda_2 t}) + (1-e^{-\lambda_2 t})(\lambda_1 e^{-\lambda_1 t})\right]dt$$

$$\overset{|}{=} \int_0^{+\infty} t\left[(\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}) - e^{-(\lambda_1+\lambda_2)t}(\lambda_1+\lambda_2)\right]dt$$

$$\overset{|}{=} \int_0^{+\infty} xe^{-x}\frac{1}{\lambda_1}dx + \int_0^{+\infty} xe^{-x}\frac{1}{\lambda_2}dx - \int_0^{+\infty} xe^{-x}\frac{1}{\lambda_1+\lambda_2}dx = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2} = 35$$

$\left(\int_0^{+\infty} x^{\alpha-1}e^{-x}dx = \Gamma(\alpha)\right)$

#5

T = time of failure $\sim \mathcal{E}(\lambda)$

$\mathbb{P}(T > 100) = 0.5$

1. $\mathbb{P}(T>100) = 1 - \mathbb{P}(T \leq 100) = e^{-\lambda 100} = \frac{1}{2} \longrightarrow \lambda = -\frac{1}{100}\ln\left(\frac{1}{2}\right) = 0.00693$

2. $\mathbb{P}(T > 500) = 1 - \mathbb{P}(T \leq 500) = e^{-\lambda 500} = 0.03127$

3. $\mathbb{P}(T < 1000 \mid T > 500) = \dfrac{\mathbb{P}(500 < T < 1000)}{\mathbb{P}(T > 500)} = \dfrac{\mathbb{P}(T<1000) - \mathbb{P}(T \leq 500)}{\mathbb{P}(T > 500)} = 0.9688201$

! simpler! $\mathbb{P}(T < 1000 \mid T > 500) = 1 - e^{-\lambda(1000-500)} = 0.96872693$

#6

$D \sim N(4.2,\ 0.3)$ $\qquad N(\mu, \sigma)$

$L \sim N(6.5,\ 0.8)$

$W \sim N(3.4,\ 0.7)$

1. $Z = D + L + W \sim N\left(\mu_D + \mu_L + \mu_W,\ \sqrt{\sigma_D^2 + \sigma_L^2 + \sigma_W^2}\right)$

   mean $= \mu_D + \mu_L + \mu_W = 14.1 \qquad = \mu_Z$

   std dev. $= \sqrt{\sigma_D^2 + \sigma_L^2 + \sigma_W^2} = 1.1045 \qquad = \sigma_Z$

2. $R \sim N(1.5\,\mu_Z,\ \sigma_R)$

   $\sigma_R = \delta_R \cdot \mu_Z \cdot 1.5$!

   $\mathbb{P}(\text{failure}) = \mathbb{P}(R < Z) = \mathbb{P}(R - Z < 0) := \mathbb{P}(T < 0) \quad : \quad T \sim N\left(\mu_R - \mu_Z,\ \sqrt{\sigma_R^2 + \sigma_Z^2}\right)$

   $\mu_T = 7.05$

   $\sigma_T = 3.35$

   $\mathbb{P}(T<0) = \mathbb{P}\left(\frac{T-\mu}{\sigma} < \frac{0-\mu}{\sigma}\right) = \Phi\left(\frac{-7.05}{3.35}\right) = 1 - \Phi\left(\frac{7.05}{3.35}\right) = 1 - \Phi(2.102) = 0.018$

#7

$X \sim \mathcal{E}(\lambda)$, $\quad Y = ce^X$

$\mathbb{P}(Y \leq t) = \mathbb{P}(ce^X \leq t) = \mathbb{P}\left(X \leq \ln\left(\frac{t}{c}\right)\right) = 1 - e^{-\lambda \ln\left(\frac{t}{c}\right)} = 1 - \left(\frac{t}{c}\right)^{-\lambda} = 1 - \left(\frac{c}{t}\right)^{\lambda}$