OPTIMIZATION

joint course with

"Discrete Optimization" and "Nonlinear Optimization"

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Chapter 1: Introduction

Optimization is an active and successful branch of applied mathematics with a very wide range of relevant applications.

General problem: Given a set $X \subset \mathbb{R}^n$ and a function $f: X \rightarrow \mathbb{R}$ to be minimized, find an optimal solution $\underline{x}^* \in X$, i.e., such that

$$f(\underline{x}^*) \leq f(\underline{x}) \qquad \forall \underline{x} \in X.$$

Course's aim: Present the main concepts and methods of discrete and nonlinear (continuous) optimization, covering also modeling aspects.

Prerequisites: linear programming (simplex algorithm), graph optimization (minimum spanning tree, maximum flow), basics of integer linear programming (Branch and Bound, Gomory fractional cuts), AMPL modeling language.

Many decision-making problems cannot be appropriately formulated or approximated in terms of linear models due to **intrinsic nonlinearity**.

Examples

1) Production planning

Determine the production levels so as to maximize the total profit while respecting the resource availability constraints.

- Since prices are "elastic" unit profit of a good decreases when the amount produced increases.

- Due to "economy of scale", the unit cost often decreases when the amount produced increases.

2) Discrete decisions modeled with binary/integer variables. Very special type of nonlinearity: $x \in \mathbb{Z}$ can be expressed as $sin(\pi x) = 0$.

Examples of optimization problems and models

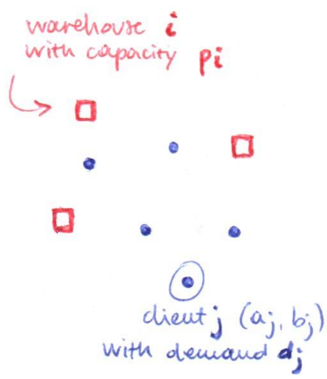
1) Location and transportation

Given

- m warehouses, indexed with $i = 1 \dots m$, with capacity p_i and an area $A_i \subseteq \mathbb{R}^2$ in which it can be located,
- n clients with known coordinates (a_j, b_j) and demand d_j, where $j = 1 \dots n$,

decide where to locate the warehouses and how to serve the clients so as to minimize the transportation costs (proportional to distance and amount of product) while respecting warehouse capacities and client demands.

Assumptions: single type of product and $\sum_{i=1}^{m} p_i \geq \sum_{j=1}^{n} d_j$



Decision variables:

(x_i, y_i)	coordinates of the i-th warehouse, $1 \le i \le m$
w_{ij}	amount of product transported from warehouse i to client j, $1 \le i \le m$ and $1 \le j \le n$
t_{ij}	distance between warehouse i and client j, $1 \le i \le m$ and $1 \le j \le n$

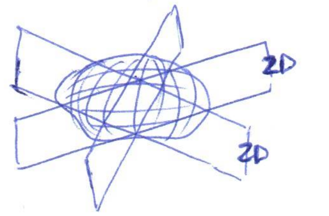
Optimization model:

objective function (not linear)

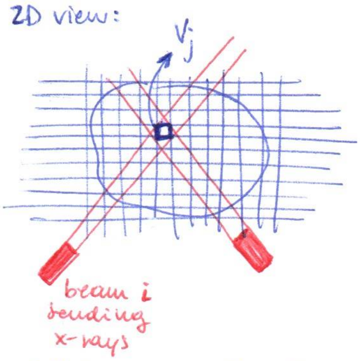
$$
\begin{array}{rrll}
\min & \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} t_{ij} & & \\
s.t. & \sum_{j=1}^{n} w_{ij} \le p_i & \forall i & \text{capacity constraint} \\
 & \sum_{i=1}^{m} w_{ij} \ge d_j & \forall j & \text{demand constraint} \\
 & t_{ij} = \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} & \forall i, j & (1) \\
 & (x_i, y_i) \in A_i \subseteq \mathbb{R}^2 & \forall i & \\
 & w_{ij} \ge 0 & \forall i, j & \\
 & t_{ij} \ge 0 & \forall i, j &
\end{array}
$$

N.B.: t_{ij} are not necessary, use equations (1) to substitute t_{ij} in the objective function.

We measure 2D slices and then we recompose it into 3D:



2D view:



we're reconstructing the density of the volume from the images of the slices

2) Image reconstruction (Computer Tomography)

Part of the body $V \subseteq \mathbb{R}^3$ subdivided into n small cubes V_j called "voxels".

Assumption: matter density is constant within each voxel (e.g., pixels).

Problem: Given the measurements provided by m beams, reconstruct a 3-D image of V, that is, determine the density x_j for each V_j.

i-the beam intersects a subset of voxels indexed by $J_i \subseteq \{1, \dots, n\}$.
For every $j \in J_i$, let a_{ij} denote the path length of i-th beam within voxel V_j.

i-th beam attenuation depends on amount of matter on the way: $\sum_{j \in J_i} a_{ij} x_j$.

Let b_i be the measurement of the i-th beam at the exit point.

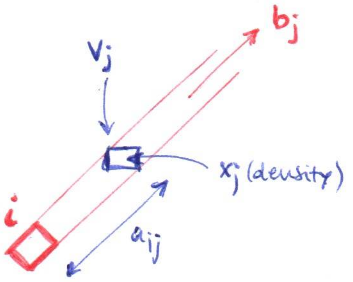
Given m beams with prescribed directions, the linear system

$$
\begin{array}{ll}
\sum_{j \in J_i} a_{ij} x_j = b_i & i = 1, \dots, m \\
x_j \ge 0 & j = 1, \dots, n
\end{array}
$$

is usually infeasible due to measurement errors, non uniformity of the V_js,...

Possibile formulation

$$
\begin{array}{ll}
\min & \sum_{i=1}^{m} (b_i - \sum_{j \in J_i} a_{ij} x_j)^2 \\
s.t. & x_j \ge 0 \quad j = 1, \dots, n.
\end{array}
$$



Since $n >> m$, to avoid alternative optimal solutions we may minimize:

$$
f(\underline{x}) = \sum_{i=1}^{m} (b_i - \sum_{j \in J_i} a_{ij} x_j)^2 + \delta \sum_{j=1}^{n} x_j \qquad \text{with } \delta > 0
$$

favors solutions with small densities

The objective function $f(\underline{x})$ may also involve

- nonlinear terms accounting for the properties of matter/image
- stochastic model of attenuation and appropriate maximum likelihood estimator.

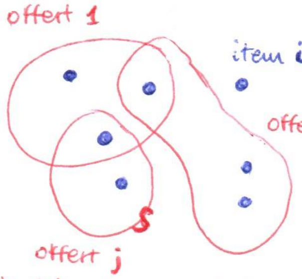
The number and directions of beams may also be optimized. (beams have costs)

Current development: 4-D optimization to account for respiratory motion.

3) Combinatorial auctions

Their popularity has been growing with internet explosion.

Participants (bidders) (offerenti) can place bids on combinations of discrete items.



$b_j(S)$ = how much the bidder j is willing to pay for S

Examples: airport time slots, wireless bandwidth, delivery routes, railroad segments, rare stamps or coins,...

Given

- a set of bidders N,
- a discrete set M of m distinct items being auctioned,
- for every subset of items $S \subseteq M$, $b_j(S)$ is the bid that bidder $j \in N$ is willing to pay for subset S.

Natural assumption: if $S \cap T = \emptyset$ then $b_j(S) + b_j(T) \le b_j(S \cup T)$

The bidder is willing to pay more for $S \cup T$ than for S and T individually (e.g., complete collection of rare stamps).

Key problem: Determine the winner of each item so as to maximize the total revenue.

Let $b(S) = \max_{j \in N} b_j(S)$. = as much as we can get from S

For every $S \subseteq M$, consider $x_S = 1$ if the highest bid on subset S is accepted, and $x_S = 0$ otherwise.

Formulation:

$$\begin{array}{lll} \max & \sum_{S \subseteq M} b(S) x_S & \\ s.t. & \sum_{S \subseteq M \,:\, i \in S} x_S \leq 1 & \forall i \in M \\ & x_S \in \{0,1\} & \forall S \subset M. \end{array}$$

involving $2^{|M|}$ variables (exponential in the number of items).

N.B.: when $x_S = 1$, the subset of items S is given to a bidder willing to pay the largest amount.

General optimization problem

$$\boxed{\begin{array}{lll} \min & f(\underline{x}) & \\ s.t. & \boxed{g_i(\underline{x}) \leq 0} & 1 \leq i \leq m \\ & \underline{x} \in S \subseteq \mathbb{R}^n & \end{array}}$$

always ≤ 0, if it's ≥ 0 then we multiply by (-1)

- the algebraic and set constraints define the **feasible region**

$$X = S \cap \{\underline{x} \in \mathbb{R}^n \;:\; g_i(\underline{x}) \leq 0, 1 \leq i \leq m\},$$

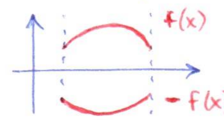
- **objective function** $f(\underline{x})$ must be defined at least on X, namely $f : X \to \mathbb{R}$,

- constraint functions $g_i(\underline{x})$ must be defined at least on S, namely $g_i : S \to \mathbb{R}$ for $i = 1, \dots, m$.

We just need to consider minimization problems since

$$\boxed{\max\{f(\underline{x}) \;:\; \underline{x} \in X\} = -\min\{-f(\underline{x}) \;:\; \underline{x} \in X\}.}$$

→ it's always a MINIMIZATION problem



max f(x) = - min(-f(x))

w.l.o.g. we can also assume that all algebraic constraints are inequality constraints since

$$\boxed{g(\underline{x}) = 0 \qquad \equiv \qquad \left\{ \begin{array}{l} g(\underline{x}) \leq 0 \\ g(\underline{x}) \geq 0. \end{array} \right.}$$

Definition

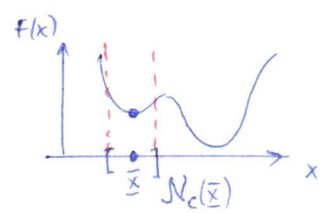
i) A feasible solution $\underline{x}^* \in X$ is a **global optimum** if

$$f(\underline{x}^*) \leq f(\underline{x}) \qquad \forall \underline{x} \in X.$$

ii) A feasible solution $\overline{\underline{x}} \in X$ is a **local optimum** if $\exists\, \epsilon > 0$ such that

$$f(\overline{\underline{x}}) \leq f(\underline{x}) \qquad \forall \underline{x} \in X \cap \mathcal{N}_\epsilon(\overline{\underline{x}})$$

where $\mathcal{N}_\epsilon(\overline{\underline{x}}) = \{\underline{x} \in X : \|\underline{x} - \overline{\underline{x}}\| \leq \epsilon\}$.



For difficult problems, we have to settle for finding a good local optimum within a reasonable computing time.

Main classes of optimization problems

Terminology: programming $\equiv$ optimization

f	g_i	S	problem type
linear	linear	$S = \mathbb{R}^n$	Linear Programming (LP)
linear	linear	$S \subseteq \mathbb{Z}^n$	Integer L. P. (ILP)
linear	linear	$S \subseteq \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ with $n = n_1 + n_2$	Mixed Integer L. P. (MILP)
at least one nonlinear		$S \subseteq \mathbb{R}^n$	Nonlinear Programming (NLP)
at least one nonlinear		$S \subseteq \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ with $n = n_1 + n_2$	Mixed Integer NLP (MINLP)

Some important special cases:

Quadratic programming: $f(\underline{x}) = \underline{x}^T Q \underline{x} + \underline{c}^T \underline{x}$ with linear constraints

Convex programming: f, the g_i functions and S are, respectively, convex functions and convex set.

Some fields of application

- health care planning and management (treatment planning, workforce scheduling, operating theater scheduling,...)
- logistics (location of plants and services, transportation, routing) and supply chain design and management
- data mining and machine learning: classification, clustering, approximation,..
- optimal control (determine the trajectory of a robot arm, airplane, shuttle)
- computational biology (determine the 3-D structure of proteins,...)
- economics (risk management, portfolio optimization, combinatorial auctions, equilibria of games,...)
- network planning and management (wired and wireless telecommunications, electric networks,...)
- production planning and inventory management (manufacturing, chemical processes, energy generation,...)

Some fields of application

- management of environmental and territorial resources (water, forest,...)
- design of experiments (for chemical and pharmaceutical companies)
- signal and image processing (2-D and 3-D reconstruction)
- statistics (e.g., nonlinear regression, estimation of distribution parameters)
- agriculture and agri-food industry
- dimensioning and optimization of structures (bridge, aircraft profile,...)
- ...