Minimum cost flow

k_{ij} = capacity, c_{ij} = cost, b_i = demand

$$\min \sum_{(i,j)} x_{ij} c_{ij}$$

$$\sum_{(h,i)\in\delta^-(i)} x_{hi} - \sum_{(i,j)\in\delta^+(i)} x_{ij} = b_i \qquad \forall i$$

$$x_{ij} \leq k_{ij} \qquad \forall (i,j)$$

$$x_{ij} \geq 0 \qquad \forall (i,j)$$

Shortest path

$$\min \sum_{(i,j)} x_{ij} c_{ij}$$

$$\sum_{(h,i)} x_{hi} - \sum_{(i,j)} x_{ij} = \begin{cases} -1 & t \\ 1 & s \\ 0 & \end{cases}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} \leq 1$$

$$x_{ij} \in \{0,1\}$$

Max flow

$$\max \sum_{(i,j)} x_{ij}$$

$$\sum_{(h,i)} x_{hi} = \sum_{(i,j)} x_{ij}$$

$$x_{ij} \leq k_{ij}$$

$$x_{ij} \geq 0$$

$\forall i \in V \setminus \{s,t\}$

$$\min\{\underline{c}^T\underline{x} : A\underline{x} \geq b,\ D\underline{x} \geq \underline{d},\ \underline{x} \in \mathbb{Z}^n\}$$

$$X = \{A\underline{x} \geq \underline{b},\ \underline{x} \in \mathbb{Z}^n\}$$

$$w^* = \min\{\underline{c}^T\underline{x} : D\underline{x} \geq \underline{d},\ \underline{x} \in \text{conv}(X)\}$$

$\Longrightarrow$ since $\text{conv}(X) \subseteq \{\underline{x} \in \mathbb{R}^n : A\underline{x} \geq \underline{b}\}$

$$z_{LP} \leq w^* \leq z^*$$

The Lagrangian duality solution is at least as good as the LP relaxation solution

If $\text{conv}(X) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} \geq \underline{b}\} \Longrightarrow z_{LP} = w^*$