Stochastic dynamical models, Mathematical Engineering 2019/2020 - SDM 1

✗ **Exercise 1**. Let X_0 be a random variable with values in a countable set I. Let $Z_0, Z_1, Z_2 \dots$ be a sequence of independent, identically distributed (iid) random variables with values in a set E. Assume that X_0 and the sequence $(Z_n)_{n\geq 0}$ are independent.

1 . Suppose we are given a function

$$F : I \times E \to I$$

and define inductively

$$X_{n+1} = F(X_n, Z_n), \quad n = 0, 1, 2, \dots$$

i) Show that $(X_n)_{n\geq 0}$ is a homogeneous Markov chain.

ii) Find its transition matrix.

2 . Suppose now that $Z_1, Z_2, \dots$ are independent, identically distributed random variables, such that $Z_i = 1$ with probability p and $Z_i = 0$ with probability $1-p$ ($i.e. Z_i \simeq B_e(p)$). Set $S_0 = 0,\ S_n = Z_1 + Z_2 + \dots + Z_n$. In each of the following cases determine whether $(X_n)_{n\geq 0}$ is a Markov chain:

a) $X_n = Z_n$

b) $X_n = S_n$

d) $X_n = S_0 + S_1 + \dots + S_n$

e) $X_n = (S_n, S_0 + S_1 + \dots + S_n)$

In the case where $(X_n)_{n\geq 0}$ is a Markov chain find its state space and transition matrix, and in the case where it is not a Markov chain find an example where $P(X_{n+1} = j | X_n = i, X_{n-1} = k)$ is not independent of k.

✗ **Exercise 2**. Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $I = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

Suppose that the law of X_0 is uniform on I, i.e. $\mathcal{L}(X_0) = \frac{1}{3}(1, 1, 1)$.

a) Compute $P(X_{100} = 2)$.

b) Compute $P(X_0 = 1, X_2 = 3)$.

c) Compute $P(X_0 + X_2 = 4)$.

d) Compute $P(X_0 = 1, X_2 = 3 | X_0 + X_2 = 4)$.

e) Determine the joint law of (X_2, X_3) and compute $E(X_1 X_2)$.

✗ **Exercise 3**. A particle is placed uniformly at one of the 9 points in a 3×3 square grid. The particle then performs a random walk such that at each step one of the adjacent points (to the right or left, upwards or downwards) is chosen with equal probabilities. This means that the particle never remains in a point or moves diagonally.

a) Describe the random walk of the particle with a Markov chain.

b) Determine the state space and the transition matrix P.

c) Find the probability that the particle after 3 steps is at the central point.

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(X) **Exercise 4.** Let $(Y_k)_{k\geq 1}$ be a sequence of iid random variables taking values in $\mathbb{N}^*$ and let $S_k = \sum_{j=1}^{k} Y_j$. Put

$$N_0 = 0, \quad N_n = \sum_{k\geq 1} 1_{\{S_k \leq n\}}, \; n \geq 1.$$

Prove that the following statements are equivalent:

a) $(N_n)_{n\geq 1}$ is a homogeneous Markov chain.

b) The random variableY_k has a geometric law.

#1 (#1)

X_0 random variable in I
$Z_0, Z_1, Z_2, \dots$ random variables in E $(\perp\!\!\!\perp, \text{iid})$ $\Big\}$ $\perp\!\!\!\perp$ $(X_0 \perp\!\!\!\perp (Z_n)_{n\geq 0})$

1. $F: I \times E \to I$; $X_{n+1} = F(X_n, Z_n)$

$$\mathbb{P}(X_{n+1}=j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(F(X_n, Z_n) = j \mid F(X_{n-1}, Z_{n-1}) = i, \dots, X_0 = i_0)$$

$\perp\!\!\!\perp$ all var. $(Z_n)_{n\geq 0}$ and $\perp\!\!\!\perp$ X_0

$$= \mathbb{P}(F(i, Z_n) = j \mid F(i_{n-1}, Z_{n-1}) = i, \dots, F(i_0, Z_0) = i_1, X_0 = i_0)$$
$$= \mathbb{P}(F(i, Z_n) = j \mid F(i_{n-1}, Z_{n-1}) = i)$$
$$= \mathbb{P}(X_n = j \mid X_{n-1} = i) = p_{ij}$$

Homogeneous?

$$\mathbb{P}(X_3 = j \mid X_2 = i) = \mathbb{P}(F(i, Z_2) = j \mid F(i_1, Z_1) = i)$$
$$\mathbb{P}(X_2 = j \mid X_1 = i) = \mathbb{P}(F(i, Z_1) = j \mid F(i_0, Z_0) = i)$$

Since Z_0, Z_1, Z_2 are iid and independent from $X_0 = i_0$ without loss of generality we can confuse (Z_2, Z_1) with (Z_1, Z_0)

$\Longrightarrow$ the probability above are equal

$\Longrightarrow$ the MC is homogeneous

2. $Z_1, Z_2, \dots$ iid : $Z_i \sim Be(p)$

$S_0 = 0$, $S_n = Z_1 + Z_2 + \dots + Z_n$

a. $(X_n)_{n\geq 0} = (Z_n)_{n\geq 0}$ $\quad Z_n \sim$ iid $Be(p)$

they're all iid $(Z_0, Z_1, \dots)$

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, \dots, X = i_1) = \mathbb{P}(Z_{n+1} = j \mid Z_n = i, \dots, Z_1 = i_1) = \mathbb{P}(Z_{n+1} = j \mid Z_n = i) = \mathbb{P}(Z_{n+1} = j)$$

$$p_{ij} = \begin{cases} p & j = 1 \\ 1-p & j = 0 \end{cases} \Longrightarrow P = \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix}$$

b. $(X_n)_{n\geq 0} = (S_n)_{n\geq 0} = (\sum_{i=1}^n Z_i)_{n\geq 0}$ $\quad S_n \sim Bi(p, n)$

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, \dots, X_0 = 0) = \mathbb{P}(S_{n+1} = j \mid S_n = i, \dots, S_0 = 0)$$
$$= \mathbb{P}(\textstyle\sum_{k=1}^{n+1} Z_k = j \mid S_n = i, \dots, S_0 = 0)$$
$$= \mathbb{P}(Z_{n+1} = j - i \mid S_n = i, \dots, S_0 = 0)$$

because $i_1, \dots, i_{n-1}$ do not appear in any form.

$$= \mathbb{P}(Z_{n+1} = j - i \mid S_n = i)$$

$Z_{n+1} \perp\!\!\!\perp Z_n, \dots, Z_1$ so $Z_{n+1} \perp\!\!\!\perp f(Z_n, \dots, Z_1)$

$$= \mathbb{P}(Z_{n+1} = j - i)$$
$$= p_{ij} = \begin{cases} p & j - i = 1 \\ 1-p & j - i = 0 \\ 0 & \text{otherwise} \end{cases}$$

c. $(X_n)_{n\geq 0} = (S_0 + \dots + S_n)_{n\geq 0}$

$X_{n+1} = X_n + S_{n+1} \quad \forall n$

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = 0) = \mathbb{P}(X_n + S_{n+1} = j \mid X_n = i, \dots, X_0 = 0)$$
$$= \mathbb{P}(S_{n+1} = j - i \mid X_n = i, \dots, X_0 = 0)$$
$$= \mathbb{P}(S_{n+1} = j - i \mid S_n = i - i_{n-1}, \dots, X_0 = 0) = *$$

$S_{n+1} = S_n + Z_{n+1}$

$$\Longrightarrow \quad * = \mathbb{P}(Z_{n+1} = (j-i) - (i - i_{n-1}) \mid S_n = (i - i_{n-1}), \dots, X_0 = 0)$$

$$\stackrel{!}{=} \mathbb{P}(Z_{n+1} = j + i_{n-1} - 2i \mid S_n = (i - i_{n-1}),\ X_{n-1} = i_{n-1})$$

$$\neq \mathbb{P}(Z_{n+1} = j + i_{n-1} - 2i \mid S_n = (i - i_{n-1}))$$

here we're not defying i_{n-1}, we need $X_{n-1} = i_{n-1}$

$\Longrightarrow$ <u>not</u> a MC

counter example:

$$\mathbb{P}(X_3 = 3 \mid X_2 = 1, X_1 = 0) = \mathbb{P}(Z_3 = 1) = p \in (0,1)$$

$$\mathbb{P}(X_3 = 3 \mid X_2 = 1, X_1 = 1) = 0 \qquad (\text{if } X_1 = X_2 = 1 \Rightarrow X_3 \geq 5)$$

$$X_3 = S_1 + S_2 + S_3 = Z_1 + (Z_1 + Z_2) + (Z_1 + Z_2 + Z_3)$$
$$= 3Z_1 + 2Z_2 + Z_3$$

We found an example where $\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = k)$ $\not\!\!\perp\!\!\!\perp$ k

d. $X_n = \begin{bmatrix} S_n \\ S_0 + S_1 + \dots + S_n \end{bmatrix}$

$$\mathbb{P}\left(X_{n+1} = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} \middle| X_n = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, \dots, X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) =$$

$$= \mathbb{P}\left(\begin{bmatrix} S_{n+1} \\ S_0 + \dots + S_{n+1} \end{bmatrix} = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} \middle| \begin{bmatrix} S_n \\ S_0 + \dots + S_n \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, \dots, \begin{bmatrix} S_0 \\ S_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$= \mathbb{P}(S_{n+1} = j_1,\ S_0 + \dots + S_n = j_1 - j_2 \mid S_n = i_1,\ S_0 + \dots + S_n = i_2, \dots)$$

$$= \mathbb{P}(S_{n+1} = j_1,\ i_2 = j_1 - j_2 \mid S_n = i_1,\ S_0 + \dots + S_n = i_2, \dots)$$

$$= \mathbb{P}(Z_{n+1} = S_{n+1} - S_n = j_1 - i_1,\ i_2 = j_1 - j_2 \mid S_n = i_1,\ S_0 + \dots + S_n = i_2, \dots)$$

$Z_{n+1} \perp\!\!\!\perp Z_1, \dots, Z_{n-1}$ in this case (only if $i_2 = j_1 - j_2$)

$$= \mathbb{P}(Z_{n+1} = j_1 - i_1 \mid S_n = i_1,\ S_0 + \dots + S_n = i_2)$$

$\Longrightarrow$ $(X_n)_{n \geq 0}$ is a MC:

$$p_{ij} = \begin{cases} p & j_1 - i_1 = 1, \quad i_2 = j_1 - j_2 \\ 1-p & j_1 - i_1 = 0, \quad i_2 = j_1 - j_2 \\ 0 & \text{otherwise} \end{cases}$$

#2 (#1)

$(X_n)_{n\geq 0}$ MC , $I = \{1,2,3\}$, $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$

$X_0 \sim \frac{1}{3}[1,1,1]$

a. $\mathbb{P}(X_{100}=2) = \mathbb{P}(X_0=2)\cdot P^{100} = \frac{1}{3}P^{100} = [..]$

$\pi^{(0)} = \mathcal{L}(X_0) = (\frac{1}{3},\frac{1}{3},\frac{1}{3}) = (\pi_1^{(0)}, \pi_2^{(0)}, \pi_3^{(0)})$

$\pi^{(n)} = (\pi_1^{(n)}, \pi_2^{(n)}, \pi_3^{(n)})$

$\boxed{\pi^{(n)} = \pi^{(0)} P^n}$

$P = QDQ^{-1}$

$P^n = QD^nQ^{-1}$

b. $\mathbb{P}(X_0=1, X_2=3) = \sum_{j=1}^{3} \mathbb{P}(X_0=1, X_1=j, X_2=3)$

$= \sum_{j=1}^{3} \mathbb{P}(X_2=3 \mid X_1=j, X_0=1)\ \mathbb{P}(X_1=j, X_0=1)$

Markov

$= \sum_{j=1}^{3} \mathbb{P}(X_2=3 \mid X_1=j)\ \mathbb{P}(X_1=j \mid X_0=1)\ \mathbb{P}(X_0=1)$

$= \sum_{j=1}^{3} p_{j3}\, p_{1j} \cdot \frac{1}{3}$

$= \frac{1}{3}\left[\cancel{p_{13}p_{11}} + p_{23}p_{12} + \cancel{p_{33}p_{13}}\right] = \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{4}{27}$

Alternatively:

$\mathbb{P}(X_0=1, X_2=3) = \mathbb{P}(X_2=3 \mid X_0=1)\,\mathbb{P}(X_0=1) = \frac{1}{3}p_{13}^{(2)} = \frac{1}{3}(P^2)_{13}$

$P^2 = \frac{1}{9}\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 4 \\ 4 & 4 & 1 \\ 1 & 4 & 4 \end{bmatrix}\cdot\frac{1}{9}$

c. $\mathbb{P}(X_0+X_2=4) = \sum_{k=1}^{3} \mathbb{P}(X_0+X_2=4, X_0=k)$

$= \sum_{k=1}^{3} \mathbb{P}(X_0+X_2=4 \mid X_0=k)\ \mathbb{P}(X_0=k)$

$= \frac{1}{3}\left(\mathbb{P}(X_2=3 \mid X_0=1) + \mathbb{P}(X_2=2 \mid X_0=2) + \mathbb{P}(X_2=1 \mid X_0=3)\right)$

$= \frac{1}{3}\left(p_{13}^{(2)} + p_{22}^{(2)} + p_{31}^{(2)}\right) = [..] = \frac{1}{3}$

d. $\mathbb{P}(X_0=1, X_2=3 \mid X_0+X_2=4) = \dfrac{\mathbb{P}(X_0=1, X_2=3, X_0+X_2=4)}{\mathbb{P}(X_0+X_2=4)}$

$= \dfrac{\mathbb{P}(X_0+X_2=4 \mid X_0=1, X_2=3)\quad \mathbb{P}(X_2=3, X_0=1)}{1/3}$

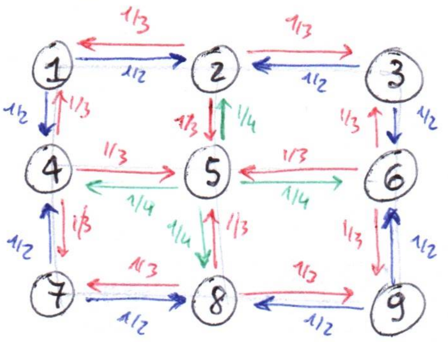
$= \dfrac{\mathbb{P}(X_2=3 \mid X_0=1)\ \mathbb{P}(X_0=1)}{1/3} = p_{13}^{(2)} = \frac{4}{9}$

e. $\mathbb{P}(X_3=i, X_2=j) \overset{?}{=} \mathbb{P}(X_1=i, X_0=j) = \mathbb{P}(X_1=i \mid X_0=j)\ \mathbb{P}(X_0=j) = \frac{1}{3}p_{ij}$

$\mathbb{E}[X_1\cdot X_2] = \sum_{(i,j)\in I\times I} i\cdot j\cdot \mathbb{P}(X_1=i, X_2=j) = \sum_{(i,j)\in I\times I} i\cdot j\ \frac{1}{3}p_{ij} = \frac{11}{3}$

$\frac{2}{3}\left(\frac{2}{3}+\frac{1}{3}\right) + \frac{3}{3}\left(\frac{1}{3}+\frac{2}{3}\right) + \frac{6}{3}\left(\frac{2}{3}+\frac{1}{3}\right)$

3



a. $(X_n)_{n\geq 0}$ stochastic process describing the position of the particle in the 3×3 grid

b. $E = \{1,2,3,4,5,6,7,8,9\}$

We denote E_i the set of adjacent points to i belonging to the set E (adiacent: left/right, upwards/downwards)

$$\Longrightarrow \quad p_{ij} = \frac{1}{\text{card}(E_i)} \quad \forall j \in E \quad (\forall i \in E)$$

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

c. $\mathbb{P}(X_3 = 5) = \sum_{k=1}^{9} \mathbb{P}(X_3 = 5 \mid X_0 = k)\, \mathbb{P}(X_0 = k)$

$\mathbb{P}(X_3 = 5 \mid X_0 = k) \neq 0 \iff k = \{2,4,6,8\}$

Since: $\underset{3^{rd}\text{ step}}{5} \leftarrow \underset{2^{nd}\text{ step}}{\{2,4,6,8\}} \leftarrow \underset{1^{st}\text{ step}}{\{1,3,5,7,9\}} \longleftarrow \underset{X_0}{\{2,4,6,8\}}$

We consider $k=4$: (the prob. is = for every $k \in \{2,4,6,8\}$)

possible paths:

3^{rd}	2^{nd}	1^{st}	acessible from 4?
5	2	1	
		3	x
		5	
	4	1	
		5	
		7	
	6	3	x
		5	
		9	x
	8	5	
		7	
		9	x

$\Longrightarrow$ Possible paths:

$$4 \rightarrow \begin{cases} 1 \to 2 \to 5 \\ 5 \to 2 \to 5 \\ 1 \to 4 \to 5 \\ 5 \to 4 \to 5 \\ 7 \to 4 \to 5 \\ 5 \to 6 \to 5 \\ 5 \to 8 \to 5 \\ 7 \to 8 \to 5 \end{cases}$$

$$\begin{aligned} \mathbb{P}(X_3 = 5 \mid X_0 = 4) &= \mathbb{P}(X_3 = 5, X_2 = 2, X_1 = 1 \mid X_0 = 4) + \cdots \\ &= \mathbb{P}(X_3 = 5 \mid X_2 = 2, X_1 = 1, X_0 = 4)\, \mathbb{P}(X_2 = 2 \mid X_1 = 1, X_0 = 4)\, \mathbb{P}(X_1 = 1 \mid X_0 = 4) \ldots \\ &= \mathbb{P}(X_3 = 5 \mid X_2 = 2)\, \mathbb{P}(X_2 = 2 \mid X_1 = 1)\, \mathbb{P}(X_1 = 1 \mid X_0 = 4) + \cdots \\ &= p_{25}\, p_{12}\, p_{41} + \cdots = [\ldots] = \frac{1}{3} = \mathbb{P}(X_3 = 5 \mid X_0 = m) \quad m = \{2,6,8\} \end{aligned}$$

$$= p_{41}(p_{12}p_{25} + p_{14}p_{45}) + p_{47}(p_{78}p_{85} + p_{74}p_{45}) + p_{45}(p_{54}p_{45} + p_{52}p_{25} + p_{56}p_{65} + p_{58}p_{85})$$

#4 (#1)

$(Y_k)_{k\geq 1}$ iid (values in $\mathbb{N}^*$), $\quad S_k := \sum_{j=1}^{k} Y_j$

$N_0 = 0$

$N_n = \sum_{k\geq 1} \mathbb{1}\{S_k \leq n\} \qquad n \geq 1$

$(N_n)_{n\geq 1}$ homogeneous MC $\iff Y_k \sim \mathcal{G}(\cdot)$

$(\Rightarrow)$ $(N_n)_{n\geq 1}$ homogeneous MC

$S_k = S_{k-1} + Y_k \longrightarrow S_k > S_{k-1}$ $\left((S_k)_{k\geq 1} \text{ increasing sequence} \right)$

$Y_1 = n+1 \iff S_1 = n+1 \iff \begin{array}{l} N_{n+1} = \mathbb{1}\{S_1 \leq n+1\} = 1 \\ N_n = \mathbb{1}\{S_1 \leq n\} = 0 \end{array}$

$\Longrightarrow \mathbb{P}(Y_1 = n+1) = \mathbb{P}(N_{n+1} = 1,\ N_n = 0)$

$$\Longrightarrow \frac{\mathbb{P}(Y_1 = n+1)}{\mathbb{P}(Y_1 = n)} = \frac{\mathbb{P}(N_{n+1} = 1,\ N_n = 0)}{\mathbb{P}(N_n = 1,\ N_{n-1} = 0)} = \frac{\mathbb{P}(N_{n+1} = 1 \mid N_n = 0)\ \mathbb{P}(N_n = 0)}{\mathbb{P}(N_n = 1 \mid N_{n-1} = 0)\ \mathbb{P}(N_{n-1} = 0)}$$

$$= \frac{p_{ij}}{p_{ij}} \cdot \frac{\mathbb{P}(N_n = 0)}{\mathbb{P}(N_{n-1} = 0)}$$

- $n = 1$: $\dfrac{\mathbb{P}(N_1 = 0)}{\mathbb{P}(N_0 = 0)} = \dfrac{\mathbb{P}(Y_1 > 1)}{1} = 1 - \mathbb{P}(Y_1 = 1) = 1 - p$

  $\dfrac{\mathbb{P}(N_1 = 0)}{\mathbb{P}(N_0 = 0)} = \dfrac{\mathbb{P}(Y_1 = 2)}{\mathbb{P}(Y_1 = 1)} = \dfrac{\mathbb{P}(Y_1 = 2)}{p}$

  $\Longrightarrow \mathbb{P}(Y_1 = 2) = p(1-p)$

- n generic: $\mathbb{P}(Y_1 = n+1) = p(1-p)^{n+1}$
  (induction)