## Sign test (Hp. data are iid)

- One sample (2 sided): $X_1, \dots, X_n$

$$\begin{cases} H_0: \mathbb{P}(X > c_0) = q \\ H_1: \mathbb{P}(X > c_0) \neq q \end{cases} \quad \left(\text{if } q = 0.5: \begin{cases} H_0: MED(X) = c_0 \\ H_1: MED(X) \neq c_0 \end{cases}\right)$$

if the distribution is simmetric this is also a test for the mean (if $\exists$)

Test statistic: $W = \sum_{i=1}^n \mathbb{1}_{\{X_i > c_0\}} \in \{0, \dots, n\}$

Under $H_0$: $W \sim Bi(n, q)$
Under $H_1$: $W \sim Bi(n, p)$ $p \neq q$

$$\Longrightarrow \begin{cases} H_0: W \sim Bi(n,q) \\ H_1: W \sim Bi(n,p),\ p \neq q \end{cases}$$

this is the test that we perform

Test statistic (another formulation): $W = \dfrac{n + \sum_{i=1}^n sign(x_i - c_0)}{2}$

- Two samples - paired $\rightarrow$ $(X_1, Y_1), \dots, (X_n, Y_n)$ $\quad Z_1, \dots, Z_n \rightarrow$ one sample

$$\begin{cases} H_0: \mathbb{P}(X > Y) = q \\ H_1: \mathbb{P}(X > Y) \neq q \end{cases} = \begin{cases} H_0: \mathbb{P}(Z > 0) = q \\ H_1: \mathbb{P}(Z > 0) \neq q \end{cases}$$

we're testing the Hp. that in a random pair $(X,Y)$, $X$ has the same probability of being larger or smaller than $Y$

## Rank test: Mann-Whitney U-test (Hp. $X_i$ iid, $Y_i$ iid, $\{X_i\}_i \perp\!\!\!\perp \{Y_i\}_i$)

- Two samples: $X_1, \dots, X_n \overset{iid}{\sim} X$, $Y_1, \dots, Y_m \overset{iid}{\sim} Y$:

$$\begin{cases} H_0: \mathbb{P}(X > Y) = \frac{1}{2} \\ H_1: \mathbb{P}(X > Y) \neq \frac{1}{2} \end{cases}$$

Test statistics: $U_1 = \sum_{i=1}^n \sum_{j=1}^m \mathbb{1}_{\{X_i > Y_j\}} = n \cdot m \cdot \hat{\mathbb{P}}(X > Y)$, $U_2 = \sum_{i=1}^n \sum_{j=1}^m \mathbb{1}_{\{Y_j > X_i\}} = n \cdot m \cdot \hat{\mathbb{P}}(Y > X)$

under $H_0$ they have the same distribution

Computationally:

- $\mathbb{X} = (X_1, \dots, X_n, Y_1, \dots, Y_m)$, $\dim(\mathbb{X}) = n+m$
- $\mathbb{R} = (r(X_1), \dots, r(X_n), r(Y_1), \dots, r(Y_m))$

$r(X_i) = \sum_{elem \in \mathbb{X}} \mathbb{1}_{\{X_i \geq elem\}}$ = # wins/ties of $X_i$ vs. $\mathbb{X}$

$r(Y_j) = \sum_{elem \in \mathbb{X}} \mathbb{1}_{\{Y_j \geq elem\}}$ = # wins/ties of $Y_j$ vs. $\mathbb{X}$

if $r(\text{element}) = 1$ (the element's rank is 1) then the element is the worst player (we look at it as a contest of team $X$ vs. team $Y$)

- $R_1 = \sum_{i=1}^n r(X_i)$ = $\Sigma$ ranks of $X$
  $R_2 = \sum_{j=1}^m r(Y_j)$ = $\Sigma$ ranks of $Y$
- $U_1 = R_1 - \dfrac{n(n+1)}{2}$, $U_2 = R_2 - \dfrac{m(m+1)}{2}$

$R_1$: # ties/wins of the first sample ($X$) against the pooled sample ($\mathbb{X}$)

$\frac{n(n+1)}{2}$: # wins/ties of the first sample against itself

## Signed-Rank test: Wilcoxon (Hp. data are iid)

- One sample: $X_1, \dots, X_n$

$$\begin{cases} H_0: \mathbb{P}(X > c_0) = q \\ H_1: \mathbb{P}(X > c_0) \neq q \end{cases}$$

Test statistic: $W^+ = \sum_{i=1}^n \mathbb{1}_{\{X_i > c_0\}} R(|X_i - c_0|)$

Under $H_0$: $[\mathbb{1}_{\{X_i > c_0\}} \sim Be(q)] \perp\!\!\!\perp [R(|X_i - c_0|) \sim \mathcal{U}(\{1, 2, \dots, n\})]$

Under $H_1$ $\mathbb{1}_{\{X_i > c_0\}} \sim Be(p)$ with $p \neq q$ and $\mathbb{1}_{\{X_i > c_0\}}$ and $R(|X_i - c_0|)$ are not $\perp\!\!\!\perp$

- Two samples - paired $\rightarrow$ $(X_1, Y_1), \dots, (X_n, Y_n) \rightarrow Z_i = X_i - Y_i$

$$\begin{cases} H_0: \mathbb{P}(X > Y) = q \\ H_1: \mathbb{P}(X > Y) \neq q \end{cases} = \begin{cases} H_0: \mathbb{P}(Z > 0) = q \\ H_1: \mathbb{P}(Z > 0) \neq q \end{cases}$$

Test statistic (↑): $W^+ = \sum_{i=1}^n \mathbb{1}_{\{X_i > Y_i\}} R(|X_i - Y_i|)$

## Permutation test : 2 ⫫ samples

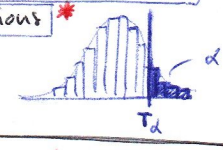
(Hp. $X_{1i}$ iid, $X_{2i}$ iid, $\{X_{1i}\}_i \perp\!\!\!\perp \{X_{2i}\}_i$)

$X_{11}, X_{12}, \dots, X_{1n_1} \overset{iid}{\sim} X_1$
$X_{21}, X_{22}, \dots, X_{2n_2} \overset{iid}{\sim} X_2$

$\begin{cases} H_0 : X_1 \overset{d}{=} X_2 \\ H_1 : X_1 \overset{d}{\neq} X_2 \end{cases}$

Test statistic : $T_0 = |\bar{X}_1 - \bar{X}_2|$ $\longrightarrow$

- Compute $T_0$ of the original sample
- $\mathbb{X}_{pooled} = (X_{11}, \dots, X_{1n_1}, X_{21}, \dots, X_{2n_2})$
  $\pi(\mathbb{X}_{pooled})$ = permutation of $\mathbb{X}_{pooled}$
- Compute $T = T(\pi(\mathbb{X}_{pooled}))$ ∀permutation $\pi$ *
- p-value = $\#\{T \geq T_0\} / \#$ permutations *
- critical region = $\{T_0 : T_0 \geq T_\alpha\}$
  $T_\alpha$ = superior $\alpha$ quantile




* if considering all the perm.s is too expensive ⟹ we consider only B random permutations

## Permutation test : One-Way ANOVA

(Hp. $X_{ji}$ iid ~ group j, ⫫ among groups)

$X_{j1}, \dots, X_{jn_j} \overset{iid}{\sim} X_j \quad j = 1, \dots, G$

$\begin{cases} H_0 : X_1 \overset{d}{=} X_2 \overset{d}{=} \dots \overset{d}{=} X_G \\ H_1 : \exists i,j \text{ s.t. } X_i \overset{d}{\neq} X_j \end{cases}$

Test statistic (from the param. case) : $T_0 = \dfrac{\sum_{j=1}^{G} n_j (\bar{x}_j - x_j)^2}{\sum_{j=1}^{G} \sum_{i=1}^{n_j} (x_{ji} - \bar{x}_j)^2}$

(There are other choices, this one is sensitive w.r.t. shifts)

## Permutation test : simple regression

(Hp. 1. The conditional mean of Y increases/decreases linearly in the value of x
2. The error is an additive error
3. The errors are iid )

$Y_i | X_i = x_i = \beta_0 + \beta_1 x_i + \varepsilon_i \qquad \varepsilon_i \overset{iid}{\sim} \varepsilon$

$\begin{cases} H_0 : \beta_1 = 0 \\ H_1 : \beta_1 \neq 0 \end{cases} = \begin{cases} H_0 : Y_1, \dots, Y_n \overset{iid}{\sim} Y = \beta_0 + \varepsilon \\ H_1 : Y_1, \dots, Y_n \not\sim \text{iid} : Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \end{cases}$

Test statistic : $T_0 = \left( \dfrac{\sum_{i=1}^{n} (x_i - \bar{x}_n)(y_i - \bar{y}_n)}{\sum_{i=1}^{n} (x_i - \bar{x}_n)^2} \right)^2$

Idea : under $H_0$ the new combination of $\vec{Y}$ - responses will have the same probability of being observed as the original. This is not true under $H_1$.

## Permutation test : center of simmetry

(Hp. data are iid)

$X_1, \dots, X_n \overset{iid}{\sim} X$ s.t. $\exists c : f(c+h) = f(c-h) \quad \forall h \in \mathbb{R}$

$\begin{cases} H_0 : c = c_0 \\ H_1 : c \neq c_0 \end{cases}$

Test statistic : $T_0 = |\bar{X}_n - c_0|$

Transformations : $\pi(X_1, \dots, X_n)$ = every element can be taken as it is or flipped on the other side of the simmetry (≠ permutation)

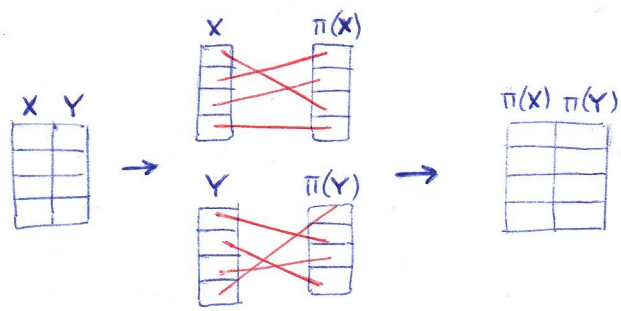
Idea : under $H_0$, since we have simmetry, a datum taken on one side of the simmetry or the other (with the same distance w.r.t. the simmetry) has the same impact on the likelihood

## Permutation test : independence

$(X_1, Y_1), \dots, (X_n, Y_n) \overset{iid}{\sim} (X, Y)$

$\begin{cases} H_0 : X \perp\!\!\!\perp Y \\ H_1 : X \not\perp\!\!\!\perp Y \end{cases}$

**Idea** : under $H_0$ the new sample $(\pi(X), \pi(Y))$ will have the same probability of being obtained than the original one.



## Permutation test: 2 ⫫ samples multivariate

(Hp. $X_{1i}$ iid, $X_{2i}$ iid, $\{X_{1i}\}_i \perp\!\!\!\perp \{X_{2i}\}_i$)

$X_{11}, X_{12}, \dots, X_{1n_1} \overset{iid}{\sim} X_1$

$X_{21}, X_{22}, \dots, X_{2n_2} \overset{iid}{\sim} X_2$

$\begin{cases} H_0: X_1 \overset{d}{=} X_2 \\ H_1: X_1 \overset{d}{\neq} X_2 \end{cases}$

Test statistic: $\| \bar{X}_1 - \bar{X}_2 \|^2_{\mathbb{R}^p} = T_0$

## Permutation test: center of simmetry multivariate

(Hp. data are iid)

$X_1, \dots, X_n \overset{iid}{\sim} X \in \mathbb{R}^p$ s.t. $\exists c \in \mathbb{R}^p : f(c+h) = f(c-h) \ \forall h \in \mathbb{R}^p$

$\begin{cases} H_0: c = c_0 \\ H_1: c \neq c_0 \end{cases}$

Test statistic: $T_0 = \| \bar{X}_n - c_0 \|^2_{\mathbb{R}^p}$

Transformations: $\pi(X_1, \dots, X_n) =$ each element is taken as it is or flipped (the whole vector is flipped)

## Permutation test: regression - multiple

(Hp. linearity of the conditional mean w.r.t. the regressors, additive action on the response of the error term, errors iid)

$Y_i \mid X_{1i} = x_{1i}, X_{2i} = x_{2i} = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i \qquad \varepsilon_i \overset{iid}{\sim} \varepsilon$

- Global test: (F-test)

$\begin{cases} H_0: (\beta_1, \beta_2) = (0,0) \\ H_1: (\beta_1, \beta_2) \neq (0,0) \end{cases} = \begin{cases} H_0: Y_1, \dots, Y_n \overset{iid}{\sim} Y = \beta_0 + \varepsilon \\ H_1: Y_1, \dots, Y_n \not\sim \text{iid} : Y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i \end{cases}$

Transformations: permutations of residuals *

Test statistic: F-statistic (it considers also the covariances)

- Partial test: (t-test)

$\begin{cases} H_0: \beta_1 = 0 \\ H_1: \beta_1 \neq 0 \end{cases} = \begin{cases} H_0: Y_i - (\beta_0 + \beta_2 x_{2i}) = \varepsilon_i \overset{iid}{\sim} \varepsilon \\ H_1: Y_i - (\beta_0 + \beta_2 x_{2i}) = \beta_1 x_{1i} + \varepsilon_i \not\sim \text{iid} \end{cases}$

Transformations: permutations of residuals *

Test statistic: t-test

* Procedure (Freedman and Lane):

- Evaluate $T_0$ (test statistic (t/F)) on the complete model
- calculate the residuals on the reduced model ($H_0$)
- permute the residuals
- new data = fitted values of the original (complete) model + permuted residuals
- Test statistic on the complete model with new data

(on loop)

- p-value = #(test_stat_perm ≥ test_stat_original) / #iterations (loop)

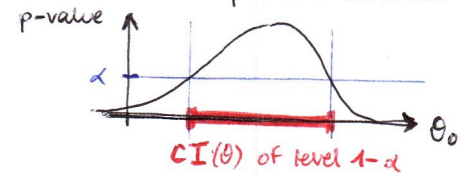
→ The null model enters for estimating the residuals

→ The complete model enters for the test statistic

## Permutational Confidence Intervals

$CI(\theta)$? → $\begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta \neq \theta_0 \end{cases}$ →

- Find the right test from the aboves,
- select a grid of values for $\theta_0$,
- perform the test choosen with all the values of $\theta_0$ in the grid finding the p-values (one p-value for each $\theta_0$),
- consider the p-value function:




$CI(\theta)$ of level $1-\alpha$

- $CI_{1-\alpha}(\theta) = \{\theta_0 : \text{p-value}(\theta_0) \geq L\alpha\rfloor\}$

This means to go to the first reachable level $\alpha$ from below (it's a discrete function the p-val's)

# Bootstrap

$X_1, .., X_n$ : $\mathbb{X}_0 = [X_1 | X_2 | .. | X_n] \longrightarrow \mathbb{X}_1, \mathbb{X}_2, .., \mathbb{X}_B$ : $[\mathbb{X}_i]_j$ = sampled from $\mathbb{X}_0$ (each element has 1/n prob. to be sampled)

$\mathbb{X}_1, .., \mathbb{X}_B \longrightarrow \hat{\theta}^*_1, .., \hat{\theta}^*_B \longrightarrow$ we can build the cumulative distribution

($\hat{\theta}$ = parameter of interest)

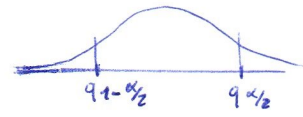
- **Confidence Intervals: (Reverse-Percentile CI)** → bootstrapping of $\hat{\theta} - \theta$

  Once we have the distribution of the estimator (bootstrap distribution), we compute the superior quantile $\alpha/2$ and the inferior quantile $1-\alpha/2$ :

  $$CI_{1-\alpha}(\theta): \quad 2\hat{\theta} - \hat{\theta}^*_{\alpha/2} < \theta < 2\hat{\theta} - \hat{\theta}^*_{1-\alpha/2}$$

  only asymptotically exact (if n is small ⇒ they have a real coverage which is smaller than the nominal)

  Notations:

  $\hat{\theta}$ = parameter of interest's estimate with $\mathbb{X}_0$

  $\hat{\theta}^*_{\alpha/2}, \hat{\theta}^*_{1-\alpha/2}$ = quantiles of the bootstrap distribution

- **Confidence Intervals: (T intervals)** → bootstrapping of $\frac{\hat{\theta} - \theta}{\hat{\sigma}_{\hat{\theta}}}$

  $$CI_{1-\alpha}(\theta): \quad \hat{\theta} - t^*_{\alpha/2}\,\hat{\sigma}_{\hat{\theta}} < \theta < \hat{\theta} - t^*_{1-\alpha/2}\,\hat{\sigma}_{\hat{\theta}}$$

  $\hat{\theta}$: estimate associated to the original sample ($\mathbb{X}_0$)

  $t^*_{\alpha/2}, t^*_{1-\alpha/2}$: quantiles of the bootstrap distribution of: $\frac{\hat{\theta}^* - \hat{\theta}}{\hat{\sigma}_{\hat{\theta}^*}}$

  $\hat{\sigma}_{\hat{\theta}^*}$: * estimate of the standard deviation of the estimator computed on the bootstrap sample

  Procedure:
  - Evaluate $\hat{\theta}$ on $\mathbb{X}_0$
  - Create $\mathbb{X}_i$ sampling from $\mathbb{X}_0$ (with rep.)
  - Evaluate $\hat{\theta}^*$ on $\mathbb{X}_i$ (same formula, diff. sample)
  - Evaluate $(\hat{\theta}^* - \hat{\theta}) / \hat{\sigma}_{\hat{\theta}^*} = T^*$
  - Given the $T^*$ for every bootstrapped sample, find the quantiles $\alpha/2$ and $1-\alpha/2$ (on loop (B))
  - $CI_{1-\alpha}(\theta)$: (↑) where $\hat{\sigma}_{\hat{\theta}} \neq \hat{\sigma}_{\hat{\theta}^*}$ (in CI we use the one from the original sample)

  We need $\hat{\sigma}_{\hat{\theta}}$ / $\hat{\sigma}_{\hat{\theta}^*}$: in the case of proportion (data which are prob.) the standard dev. is given by: $\sqrt{\frac{p(1-p)}{n}}$

- **Approaches:**
  - NAIVE (↑): resampling from the same sample
  - SMOOTH: we smooth the empirical distribution
  - PARAMETRIC: we know the family but not the parameters

- **Regression:** (CI for regression)

  $Y_i | X_i = x_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ , $\varepsilon_i \overset{iid}{\sim} \varepsilon$

  Procedure:
  - fit the regression model under $H_1$ * and obtain $T_0$.
  - compute the residuals (values − fitted values) ← residuals from the $H_1$ model
  - new data = fitted values from * + sampled residuals (from residuals)
  - fit with new data ($H_1$) and obtain T
  - Given T for every bootstrap case → bootstrap distr. and CI (↑)

- **2/more ⫫ samples**

  We independently bootstrap the different samples.

- **Test & p-values**

  $\begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta \neq \theta_0 \end{cases}$ → Through the previous procedures we obtain $CI_{1-\alpha}(\theta)$ → Varying $\alpha$ we may contain or not $\theta_0$ : the p-value of the test is the smallest $\alpha$ for which $CI_{1-\alpha}$ does not contain $\theta_0$.

  (we do a grid for $\alpha$ that we want consider, we evaluate $CI_{1-\alpha}$ for every $\alpha$ and we check if $\theta_0$ is inside. The smallest $\alpha$ for which $\theta_0$ is not in the interval → p-value)