Stochastic dynamical models

June 17^{th}, 2020

- Pocket calculators without wifi connection function are allowed. Pocket calculators with wifi connection function are NOT allowed, even if set in airplane mode.
- The teacher's or your own lecture notes are allowed, other notes are not.
- You can not ask questions. If you think that there is an error or something ambiguous, explain why and write your solution accordingly.
- Solutions can be written either in English or Italian (also French and Spanish).

Exercises

Exercise 1. Let $(Z_k)_{k\geq 1}$ be a sequence of independent identically distributed random variables with $Z_k \sim \mathcal{U}(\{0,1,2\})$ (uniform density on the set $\{0,1,2\}$) and let $(X_n)_{n\geq 0}$ be the sequence of random variables defined by

$$X_n = \begin{cases} 0 & \text{if } n = 0 \\ \sum_{k=1}^{n} Z_k & \text{if } n \geq 1 \end{cases}.$$

(1) Show that the process $(X_n)_{n\geq 0}$ is an homogeneous Markov chain and write the transition matrix.

(2) Compute the probability of arrival in the state 4 without any passage from the state 2.

(3) For all $n \geq 0$, let M_n be the random variable

$$M_n = X_n - n\,.$$

Show that the process $(M_n)_{n\geq 0}$ is a martingale with respect to the natural filtration $(\mathcal{F}_n)_{n\geq 0}$ of the Markov chain.

(4) Is the process $(M_n)_{n\geq 0}$ a Markov chain? If yes write the transition matrix.

(5) If yes establish if it is recurrent or transient.

X **Exercise 2.** **Caution!** In the following p is arbitrary with $0 < p < 1$. However, if the last digit of your "Codice persona" is:

- $0, 1, 2, 3, 4$ then $\lambda = 1$, $\mu = 2$, $\alpha = 3$,
- $5, 6, 7, 8, 9$ then $\lambda = 1$, $\mu = 3$, $\alpha = 2$.

Customers arrive randomly at an ATM cash machine. The time between two consecutive arrivals is an exponential random variable with parameter λ (customers per hour). The ATM is in a small space that can hold only two people (according to covid regulations level 3), one using the ATM and the other waiting and the other for her/his turn. If another customer comes and finds two people in the small space he leaves and looks for another ATM.
When a customer accesses the machine, he does some financial operation that takes a random time exponentially distributed with parameter μ (operations per hour). A fraction p of customers does another financial operation (in another exponential time with parameter α), the remaining fraction $1 - p$ does only one operation and no customer does 3 or more operations.

(1) Construct a Markov chain model and write the transition rate matrix of the Markov chain $(X_t)_{t\geq 0}$. [*Hint: distinguish the cases where the customer at the machine is doing his first operation or his second operation otherwise the process will not be a Markov chain.*]
(2) Is the Markov chain irreducible? Does it admit a unique invariant density?
(3) Find all invariant densities.

Thinking now on the long time behaviour of the system answer the following.

(4) What is, on average, the fraction of customers that go to the small space, enter to the small space and eventually succeed using the machine?
(5) What is the probability that a customer who succeeds using the machine gets in the small space when there is someone at the machine doing his second operation?
(6) If you enter in the small space and you find another customer at the machine doing his second operation, what is your average waiting time before you can access the machine?
(7) How about the case in which the customer at the machine is doing his first operation?
(8) How much time do you wait, on average ? (Without knowing if the customer at the machine is doing his first or second operation?
(9) Suppose that the bank that owes the ATM charges b € for each operation (so that a customer realizing 2 operations is charged $2b$ €) and operation costs of the ATM machine amount to c € per unit time. What relationship must fulfill the parameters of problem for the cashier to be self-financing?

WTF

Remember :
when the queue is limited
then we have to control to
be in!

waiting time? It must be
conditioned to the fact that
we're in the queue

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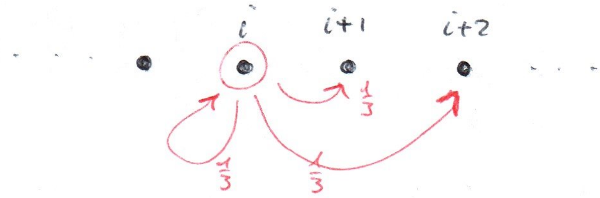
1

- $Z_k \sim U(\{0,1,2\})$
- $X_n = \begin{cases} 0 & n=0 \\ \sum_{k=1}^n Z_k & n \geq 1 \end{cases}$



1. $\mathbb{P}(X_{n+1}=j \mid X_n = i, \dots, X_0 = i_0) = \mathbb{P}\left(\sum_{k=1}^{n+1} Z_k = j \mid \sum_{k=1}^{n} Z_k = i, \dots, X_0 = i_0\right)$

by $\perp\!\!\!\perp$ Z_{n+1} from all the others

$= \mathbb{P}(Z_{n+1} = j-i)$

$= \mathbb{P}\left(Z_{n+1} = j-i \mid \sum_{k=1}^n Z_k = i\right)$ same reason

$= \begin{cases} \frac{1}{3} & j-i = 0,1,2 \\ 0 & j-i = \text{other} \end{cases} \qquad j = i, i+1, i+2$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ & & \cdots & & & & \end{bmatrix}$$

Moreover:
$\mathbb{P}(X_{n+1}=j \mid X_n = i) = \mathbb{P}(X_1 = j \mid X_0 = i)$
since Z_k are iid

2. let $(h_i)_{i=0,1,3}$ be the probability of arriving in 4 without passing through 2. $(h_2 = 0)$

if we're in 0 we may go to 0 or 1 or 2:

$h_0 = \frac{1}{3}h_0 + \frac{1}{3}h_1 + \cancel{\frac{1}{3}h_2} = \frac{1}{3}h_0 + \frac{1}{3}h_1$

if we're in 1 $\longrightarrow$ 1, 2, 3:

$h_1 = \frac{1}{3}h_1 + \cancel{\frac{1}{3}h_2} + \frac{1}{3}h_3 = \frac{1}{3}h_1 + \frac{1}{3}h_3$

if we're in 3 $\to$ 3, 4:

$h_3 = \frac{1}{3}h_3 + \frac{1}{3}$

$\Longrightarrow [h_0, h_1, h_2, h_3] = [\frac{1}{8}, \frac{1}{4}, 0, \frac{1}{2}]$

3. $M_n = X_n - n$

$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_{n+1} - (n+1) \mid \mathcal{F}_n] = \mathbb{E}\left[\sum_{k=1}^{n+1} Z_k \mid \mathcal{F}_n\right] - (n+1)$

$= \sum_{k=1}^n Z_k - (n+1) + \mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]$

$= X_n - n + \mathbb{E}[Z_{n+1}] - 1 = X_n - n = M_n$

and so by the projective property of the conditional exp.:

$\mathbb{E}[M_n \mid \mathcal{F}_m] = M_m \qquad \forall n \geq m$

4. $\mathbb{P}(M_{n+1}=j \mid M_n=i, \dots, M_0=i_0) = \mathbb{P}(X_{n+1}-(n+1)=j \mid X_n-n=i, \dots, X_0=i_0)$

$= \mathbb{P}(X_{n+1}=j+(n+1) \mid X_n=i+n, \dots)$

$= \mathbb{P}(X_{n+1}=j+(n+1) \mid X_n=i+n)$

$= \mathbb{P}(X_{n+1}-(n+1)=j \mid X_n-n=i)$

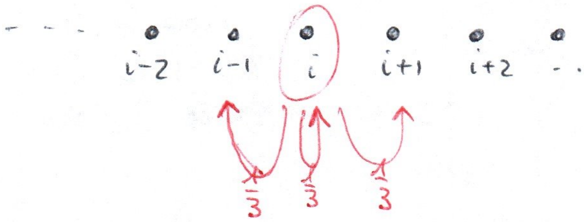
$= \mathbb{P}(M_{n+1}=j \mid M_n=i)$

$$\mathbb{P}(X_{n+1}=j \mid X_n=i) = \begin{cases} \frac{1}{3} & j=i, i+1, i+2 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \mathbb{P}(M_{n+1}=j \mid M_n=i) = \begin{cases} \frac{1}{3} & j \cancel{+n+1} = i+\cancel{n}, i+\cancel{n}+1, i+\cancel{n}+2 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1/3 & j=i-1, i, i+1 \\ 0 & \text{else} \end{cases}$$

5.



It's an irreducible MC (all states communicate).

$y_j = \frac{1}{3} y_{j-1} + \frac{1}{3} y_j + \frac{1}{3} y_{j+1}$

$y_{j-1} - 2y_j + y_{j+1} = 0$

$x^2 - 2x + 1 = 0 \qquad x_{1/2} = 1$

$(x-1)^2$

$y_j = A + Bj \Rightarrow \nexists A, B$ s.t. y_j is bounded non const $\Rightarrow$ recurrent

#2

- time between arrivals $\sim \mathcal{E}(\lambda)$
- there can be only two people : one using ATM, the other waiting
- permanence $\sim \mathcal{E}(\mu)$ (of everyone)

  someone do another operation $\sim \mathcal{E}(\alpha)$

$1-p =$ frac

$p =$ frac

States : 0 1 2

but we need to diversify :

$\{0, 1_F, 1_S, 2_F, 2_S\}$

one customer and he's doing his first operation

one customer and he's doing his second op.

two customers, one is doing his first op.

two customers, one is doing his second op.

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2 (2)

1. $0 \to 0, 1_f$: $q_{01_f} = \lambda$, $q_{00} = -\lambda$

$1_f \to 0, 1_s, 2_f, 1_f$: $q_{1_f 2_f} = \lambda$, $q_{1_f 1_s} = p\mu$, $q_{1_f 0} = (1-p)\mu$

$q_{1_f 1_f} = -(\lambda+\mu)$

- $1_f \to 0$ with probability $1-p$, in an exp time $\sim \mathcal{E}(\mu)$
- $1_f \to 1_s$ with probability p, in an exp time $\sim \mathcal{E}(\mu)$
- $1_f \to 2_f$ after exp time $\sim \mathcal{E}(\lambda)$

$1_s \to 0, 2_s, 1_s$: $q_{1_s 0} = \alpha$, $q_{1_s 2_s} = \lambda$, $q_{1_s 1_s} = -(\alpha+\lambda)$

$2_f \to 2_s, 1_f, 2_f$: $q_{2_f 2_s} = \mu p$, $q_{2_f 1_f} = \mu(1-p)$, $q_{2_f 2_f} = -\mu$

$2_s \to 1_f, 2_s$: $q_{2_s 1_f} = \alpha$, $q_{2_s 2_s} = -\alpha$

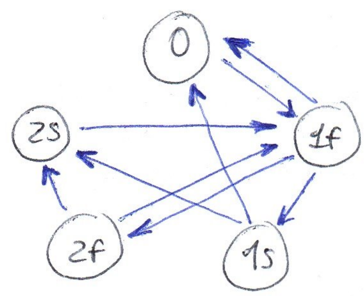
$Q =$

	0	1f	1s	2f	2s
0	$-\lambda$	λ	0	0	0
1f	$\mu(1-p)$	$-(\lambda+\mu)$	$p\mu$	λ	0
1s	α	0	$-(\alpha+\lambda)$	0	λ
2f	0	$\mu(1-p)$	0	$-\mu$	μp
2s	0	α	0	0	$-\alpha$

2. Discrete skeleton:

$P =$

	0	1f	1s	2f	2s
0	0	1	0	0	0
1f	$\frac{\mu(1-p)}{\lambda+\mu}$	0	$\frac{p\mu}{\lambda+\mu}$	$\frac{\lambda}{\lambda+\mu}$	0
1s	$\frac{\alpha}{\alpha+\lambda}$	0	0	0	$\frac{\lambda}{\alpha+\lambda}$
2f	0	$1-p$	0	0	p
2s	0	1	0	0	0



irreducible $\Longrightarrow \exists!$ invariant density

3. $\pi Q = 0$:
$$\begin{cases} \pi_0(-\lambda) + \pi_{1f}(\mu(1-p)) + \alpha\pi_{1s} = 0 \\ \lambda\pi_0 - (\lambda+\mu)\pi_{1f} + \mu(1-p)\pi_{2f} + \alpha\pi_{2s} = 0 \\ p\mu\pi_{1f} - (\alpha+\lambda)\pi_{1s} = 0 \\ \lambda\pi_{1f} - \mu\pi_{2f} = 0 \\ \lambda\pi_{1s} + \mu p\,\pi_{2f} - \alpha\pi_{2s} = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \pi_2 = \dfrac{p\mu}{\alpha+\lambda}\pi_1 \\ \pi_3 = \dfrac{\lambda}{\mu}\pi_1 \\ \pi_4 = \left(\dfrac{\lambda}{\alpha}\dfrac{p\mu}{\alpha+\lambda} + \dfrac{\mu p}{\alpha}\dfrac{\lambda}{\mu}\right)\pi_1 \\ \pi_0 = \left(\dfrac{\mu(1-p)}{\lambda} + \dfrac{\alpha}{\lambda}\dfrac{p\mu}{\alpha+\lambda}\right)\pi_1 \\ \pi_0+\pi_1+\pi_2+\pi_3+\pi_4 = 1 \end{cases}$$

[..]

4. Fraction of customers that uses that ATM?
For using the service there must be at least one space

$$\Longrightarrow \pi_0 + \pi_{1f} + \pi_{1s}$$

alternatively: 1 − fraction of people who don't
$1 - \mathbb{P}(X_n = 2f) - \mathbb{P}(X_n = 2s) =$
$1 - \pi_{2f} - \pi_{2s}$

5. $\mathbb{P}$(a customer who succeeds using ATM comes where there is someone using the machine for the 2nd operation)?

$$\Longrightarrow \frac{\pi_{1s}}{\pi_0 + \pi_{1f} + \pi_{1s}}$$

$\mathbb{P}$(1 person doing 2nd | succeed) =
$= \dfrac{\mathbb{P}(\text{1 person doing 2nd, succeed})}{\mathbb{P}(\text{succeed})}$
$= \dfrac{\mathbb{P}(\text{succeed} \mid \text{1 person doing 2nd})\ \mathbb{P}(\text{1 person doing 2nd})}{\mathbb{P}(\text{succeed})}$

6. Average waiting time and the customer is doing his second operation
Since the service is $\sim \mathcal{E}(\alpha) \Longrightarrow$ waiting time (average) $= \dfrac{1}{\alpha}$

7. Average waiting if he's doing his first?

$$\frac{1}{\mu} + \frac{1}{\alpha}$$

8.
$$\underbrace{\frac{\pi_{1f}}{\pi_0+\pi_{1f}+\pi_{1s}}}_{\substack{\text{probability of}\\ \text{succeed using the ATM}\\ \text{and coming when there is}\\ \text{already a customer}}}\left(\frac{1}{\alpha}+\frac{1}{\mu}\right) + \frac{\pi_{1s}}{\pi_0+\pi_{1f}+\pi_{1s}}\left(\frac{1}{\alpha}\right)$$

9. $\mathbb{E}[\text{return per unit of time}] = \mathbb{E}[\text{return} \mid 1^{st}\text{ op.}]\ \mathbb{P}(1^{st}) + \mathbb{E}[\text{return} \mid 2]\ \mathbb{P}(2^{nd})$

remember: $\mathbb{P}$(doing 1^{st} operation) is not π_{1f} but

$$\frac{\pi_{1f}}{\pi_0+\pi_{1f}+\pi_{1s}} = \mathbb{P}(\text{doing } 1^{st} \text{ op} \mid \text{succeeding})$$

$$\Longrightarrow b\,\frac{\pi_{1f}}{\pi_0+\pi_{1f}+\pi_{1s}} + 2b\,\frac{\pi_{1s}}{\pi_0+\pi_{1f}+\pi_{1s}} \geq c$$