General problem:

$$\min f(\underline{x}) \quad \text{s.t.} \quad g_i(\underline{x}) \le 0 \quad i \in I = \{1,\dots,m\} \quad := (P)$$
$$\underline{x} \in X \subseteq \mathbb{R}^n$$
$$h_\ell(\underline{x}) = 0 \quad \ell \in L = \{1,\dots,p\}$$

Suff. conditions for CQ:

- all g_i are linear (Karlin) } $\longrightarrow$ CQ $\forall \underline{x} \in S$
- all g_i are convex and $\exists \underline{x}^*: g_i(\underline{x}^*) < 0 \ \forall i$ (Slater) } $\longrightarrow$ CQ $\forall \underline{x} \in S$
- $\nabla g_i(\bar{\underline{x}})$ are linearly ⫫ $\forall i \in I(\bar{\underline{x}})$ } $\longrightarrow$ CQ in $\bar{\underline{x}} \in S$

First order (necessary) optimality conditions (KKT):

$f \in C^1$, $g_i \in C^1$, CQ holds at $\bar{\underline{x}}$.

if $\bar{\underline{x}}$ is a local minimum of (P) $\Longrightarrow$ $\exists u_i \ge 0 \ \forall i \in I(\bar{\underline{x}})$; $v_\ell \in \mathbb{R}$:

- $\nabla f(\bar{\underline{x}}) + \sum_{i \in I} u_i \nabla g_i(\bar{\underline{x}}) = \underline{0}$ — $\nabla f(\bar{\underline{x}}) + \sum_{i \in I} u_i \nabla g_i(\bar{\underline{x}}) + \sum_{\ell \in L} v_\ell \nabla h_\ell(\bar{\underline{x}}) = \underline{0}$
- $u_i g_i(\bar{\underline{x}}) = 0 \quad \forall i \in I$ — $h_\ell(\bar{\underline{x}}) = 0$
- $g_i(\bar{\underline{x}}) \le 0 \quad \forall i \in I$

- If the problem is convex $\Longrightarrow$ the condition is necessary and sufficient
- If f and g_i are convex and $\exists \underline{x}^*: g_i(\underline{x}^*) < 0 \ \forall i$ $\Longrightarrow$ the condition is necessary and sufficient

Lagrangian function associated: $L(\underline{x}, \underline{u}) = f(\underline{x}) + \sum_{i \in I} u_i g_i(\underline{x}) \quad \forall \underline{x} \in X, \ \underline{u} \ge \underline{0}$

Dual function: $w(\underline{u}) = \min_{\underline{x} \in X} L(\underline{x}, \underline{u})$

Dual problem: $\max_{\underline{u} \ge \underline{0}} w(\underline{u})$

Properties:

- $\forall \underline{x} \in X$ ($\underline{x}$ feasible solution) and $\underline{u} \ge \underline{0}$: $w(\underline{u}) \le f(\underline{x})$ (weak duality)
- If $\exists \bar{\underline{u}}, \bar{\underline{x}}$ s.t. $w(\bar{\underline{u}}) = f(\bar{\underline{x}})$ we have strong duality: $\bar{\underline{u}}$ is optimal for the dual and $\bar{\underline{x}}$ is optimal for the original problem (P). Moreove $(\bar{\underline{x}}, \bar{\underline{u}})$ is a saddle point of $L(\underline{x}, \underline{u})$.
- Generally we can have a duality gap: $\max_{\underline{u} \ge \underline{0}} w(\underline{u}) < \min_{\underline{x} \in S} f(\underline{x})$

  but if the problem is convex (or it holds the strong duality) the problem has a finite optimal solution and we have no duality gap.

How can we solve the dual:

If $X \subseteq \mathbb{Z}^n$ then $w(\underline{u})$ is piecewise concave and we can find the global optimal solutions with the subgradient method (since $w(\underline{u})$ may not be everywhere continuously differentiable we use the gradient if the point is cont. diff., the subgradient if it's not).

$w(\underline{u})$ is always concave, if $X \subseteq \mathbb{Z}^n$ then it's piecewise concave