$$w_i = 1 + \sum_{j\in T} p_{ij} w_j$$

+ Coupons
+ Gambler (N)

- $\mathbb{P}_i(V > n) = \ldots = U_i^{(n)}$
- $U_i^{(n+1)} \to U_i^{(n+2)} \to U_i^{(n+m)} \leq \sum_{j\in T} p_{ij}^{(m)} U_j^{(n)}$
- $i,j$ transient: $p_{ij}^{(n)} \to 0$

  $\Longrightarrow M = \max_i \left\{ \sum_{j\in T} p_{ij}^{(m)} \right\} < 1$
- $U_i^{(n)} \leq \ldots \leq M \max_j U_j^{(n-m)}$
- iterating: $\max U_i^{(n)} \leq \ldots \leq M^{\frac{n}{m}} \max U_j^{(n-\frac{n}{m}m)}$
- $\mathbb{P}_i(V > n) \leq \max U_i^{(n)} \leq \ldots \leq M^{-1}(M^{\frac{1}{m}})^n$
- $\mathbb{E}_i[V] = \sum_n \mathbb{P}_i(V>n) \leq \ldots \leq (M(1-M^{1/m}))^{-1} < \infty$

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- $\mathbb{E}_i[V] = \sum_n n \cdot \mathbb{P}_i(V = n) = [(\cdot)|X_1 \in T] + [(\cdot)|X_1 \in C] = \ldots$

$E$ = set of all states (finite)

$T$ = set of transient states

$C$ = recurrent class ($\exists!$)

$w_i = \mathbb{E}_i[V]$ = mean abs. time

$\Longrightarrow (w_i)_i$ finite and:

$$w_i = 1 + \sum_{j \in T} p_{ij} w_j$$

$$V_i = \sum_{j \in C} p_{ij} + \sum_{j \in T} p_{ij} V_j$$

+ Gambler

vs a bank

- $V_i^{(n)}$
- $V_i^{(1)}, \quad V_i^{(n)} = \cdots = \sum_{j \in T} V_j^{(n-1)} p_{ij}$
- event "absorption"
- $V_i = \mathbb{P}_i(\text{event absorption}) = \cdots = (\uparrow)$

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- $X_i$ smaller solution
- $X_i > V_i^{(1)}$
- $X_i \geq \sum_{k=1}^{n} V_i^{(k)} \Rightarrow X_i \geq \sum_{k=1}^{n+1} V_i^{(k+1)}$

$\Longrightarrow$ lim $\lightning$

$T$ = set of transient states

$C$ = recurrent class

$V_i = \mathbb{P}_i(\text{being absorbed in } C)$

$\Longrightarrow (V_i)_i$ smaller $[0,1]$-sol. of:

$$V_i = \sum_{j \in C} p_{ij} + \sum_{j \in T} p_{ij} V_j$$

$U_i = \sum_{j \in T} p_{ij} U_j$

+ Gambler vs a bank

- $U_i^{(n)}$
- $U_i = \lim_{n \to \infty} U_i^{(n)}$
- $U_i^{(n+1)} = \dots = \sum_{j \in T} p_{ij} U_j^{(n)}$
- $\lim_{n \to \infty}$

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- $V_i$ bigger solution
- $U_i^{(1)} \geq V_i$
- $U_i^{(n)} \geq V_i \implies U_i^{(n+1)} \geq V_i \implies \lim$ ↯

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- $U_i = \sum_{k \in T} p_{ik}^{(2)} U_k = \dots = \sum_{k \in T} p_{ik}^{(n)} U_k$
- $U_i \leq \sum_{j \in T} p_{ij}^{(n)} \to 0 \implies U_i = 0$

$\mathbf{T}$ = set of transient states

$$U_i = \mathbb{P}_i\left(\bigcap_{n \geq 1} \{X_n \in \mathbf{T}\}\right) \qquad i \in T$$

$\Longrightarrow$ $(U_i)_i$ is the biggest $[0,1]$ - sol. of

$$U_i = \sum_{j \in T} p_{ij} U_j$$

Remark : $\mathbf{T}$ finite $\Longrightarrow$ $U_i \equiv 0$

## Number of visits of a recurrent state is $\infty$.

- $N_i = \sum_{n \geq 1} \mathbb{1}_{\{X_n = i\}}$
- $T_i^{(k+1)} = \inf\{n > T_i^{(k)} : X_n = i\}$
- $i$ recurrent $\iff \mathbb{P}_i(T_i^{(1)} < \infty) = 1$

  $\iff \mathbb{P}_i(N_i \geq 1) = 1$

- $T_i^{(2)}$ is the first entrance in $i$ for the restarted $\implies \mathbb{P}_i(T_i^{(2)} < \infty) = 1$
- $T_i^{(k)}$ is the ↑ for the restarted

  $\implies \mathbb{P}_i(T_i^{(k)} < \infty) = 1 \iff \mathbb{P}_i(N_i \geq k) = 1$

$$m_i(z) = \begin{cases} 1 & i \text{ recurrent} \\ z \sum_j p_{ij} m_j(z) & i \in S \\ \sum_j p_{ij} m_j(z) & i \in T \setminus S \end{cases}$$

+ Gambler (3)

- $i$ recurrent $\Rightarrow T = 0$
- $i \in T$ : $T_S = \mathbb{1}\{X_0 \in S\} + \sum_{n \geq 1} \mathbb{1}\{X_n \in S\} = \mathbb{1} + \tilde{T}$
- $\mathbb{E}_i[z^T] = \sum_j \mathbb{E}_i[z^T \mid X_1 = j]\, \mathbb{P}(X_1 = j)$

  $= z^{\mathbb{1}\{X_0 \in S\}} \sum_j \mathbb{E}_j[z^T]\, p_{ij}$

$m_i(z) = \sum_{k \geq 0} z^k \, \mathbb{P}_i(T = k)$

$m_i = \mathbb{E}_i[z^T] =$ moment generating fun.

## Transience: $\sum_{k \in E} p_{jk} y_k = y_j$ + Queue

($\Longrightarrow$)
- $e$ unique state for which doesn't hold
- $\tilde{p}_{ij}$
- $\hat{V}_i = \mathbb{P}_i(T_e < \infty) < 1$ otherwise ..
- $\tilde{V}_i < 1$, $\tilde{V}_e = 1$ absorption probs. in $e$

$\Longrightarrow \tilde{V}_i = \tilde{p}_{ie} + \sum_{k \neq e} \tilde{p}_{ik} \hat{V}_k \Longrightarrow$ ..

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($\Longleftarrow$)
- formula $\forall i \neq e \rightarrow \tilde{p}_{ij}$ : formula $\forall i$
- iterating $\rightarrow \sum_{k \in E} p_{ik}^{(n)} y_k = y_i$
- Recurrent $\Rightarrow \lim p_{je}^{(n)} = 1$

$\Longrightarrow \forall j \neq e$ : $|y_j - y_e| = .. \leq \sup |y_k| \lim (1 - p_{je}^{(n)})$

$= 0 \Longrightarrow y_i$ constant ↯

$(X_n)_{n\geq 0}$ irreducible

$(X_n)_{n\geq 0}$ transient $\iff$ $\exists$ bounded non-const solution of:

$$\sum_{k\in E} p_{jk}\, y_k = y_j$$