emergency service location

combinatorial auctions

airline crew scheduling

Binary Knapsack problem

- ILP formulation
- Multidimensional Binary Knapsack problem:
  - ILP formulation
  - Suroggate relaxation
  - Lagrangian relaxation
- Cutting plane approach: cover inequality and separation of cover inequalities
- Facet defining inequalities: lifting (strenghtening) procedure for cover inequalities

State the definition and explain why is valid

How can it be solved?

Set Covering/Packing/Partitioning problems

- ILP formulations

Assignment problem

we have n projects and n persons: which one to who?

- ILP (ideal) formulation
- The matrix of the constraints is TU

! Uncapacitated Facility Location (UFL)

m clients have demand for depots. We can open some depots in n candidate sites (fixed costs of opening + transport costs).

which is better? why?

- MILP formulation
- MILP formulation variant: every depot has limited capacity
- MILP alternative formulation (pag. 4)
- Heuristic for primal bounds: local search methods
- Lagrangian relaxation (pag. 20)

STSP: quale relazione esiste tra i rilassamenti lineari delle due formulazioni?

Traveling Salesman problem (TSP)

we want to visit exactly once each node (Hamiltonian circuit tour) at min cost

with a direction

- Asymmetric (ATSP): ILP formulation with cut set inequalities
- Asymmetric (ATSP): ILP formulation with subtour elimination inequalities
- Cutting plane approach: separation of cut-set inequalities

undirected

- Symmetric (STSP): ILP formulation with cut set inequalities
- Symmetric (STSP): ILP formulation with subtour elimination inequalities

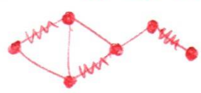
How many constraints do they have? (ordine di grandezza)

- Facet defining inequalities: strenghtening of subtour elimination inequalities
- Lagrangian relaxation based on the 1-tree (recall the definition, explain which ILP formulation we start from and which group of constraints is relaxed)

+ explain how we can solve the lagrangian subproblems and the lagrangian dual pb.

Perfect matching problem (PM)

Subset of edges without common nodes but incident to all the nodes:



- ILP (ideal) formulation
- Maximum Matching problem:
  - ILP formulation
  - Chvàtal-Gomory inequality

Uncapacitated Lot-Sizing problem (ULS)

Plan the production of a single type of product for the next n periods

- MILP formulation
- MILP extended formulation

Fixed charge network flow problem (FCNF)

determine a feasible flow of minimum cost which satisfies all demand and capacity constraints

- MILP formulation

Transportation problem

single type of product: determine a transp. plan so as to minimize total transp. costs while satisfying clients demands and all the plant capacity constraints

- ILP formulation
- The matrix of the constraints is TU

Minimum cost flow problem

determine a feasible flow of minimum total cost satisfying all demands

- ILP formulation
- The matrix of the constraints is TU
- Shortest path and maximum flow are special cases of the min cost flow

---

Methods

- Cutting plane methods
  - Mayer's theorem
  - Valid inequality
    - Chvàtal-Gomory procedure
    - Mixed-integer rounding (MIR) aggregated inequality
    - Gomory mixed integer (GMI) inequality
  - Cutting plane and separation problem
  - Cutting plane algorithm
- Branch and Cut: state of art
- Lagrangian relaxation
  - Lagrangian subproblem
  - Lagrangian dual
- Column generation method

  + describe the method for the 1-D cutting stock problem (CSP)

DISCRETE OPTIMIZATION

- MILP

$$\min \underline{c}_1^T \underline{x} + \underline{c}_2^T \underline{y} \quad : \quad A_1 \underline{x} + A_2 \underline{y} \geq \underline{b}, \quad \underline{x} \geq \underline{0} \text{ integer}, \quad \underline{y} \geq \underline{0}$$

$A_1 \in \mathbb{Z}^{m \times n_1}$, $A_2 \in \mathbb{Z}^{m \times n_2}$, $\underline{c}_1 \in \mathbb{Z}^{n_1}$, $\underline{c}_2 \in \mathbb{Z}^{n_2}$, $\underline{b} \in \mathbb{Z}^m$

- ILP

$$\min \underline{c}^T \underline{x} \quad : \quad A\underline{x} \geq \underline{b}, \quad \underline{x} \geq \underline{0} \text{ integer}$$

if $x_i \in \{0,1\}$ $\forall i$ then the problem is an 0-1-ILP

KNAPSACK PROBLEM (Binary choice)

- n objects
- $\forall i: 1 \leq i \leq n$: p_i = profit, a_i = weig
- b capacity

Goal: decide which objects to take to maximize the profit while respecting capacity constr.

ILP formulation:

$$\max \sum_{i=1}^n p_i x_i \quad \text{s.t.} \quad \sum_{i=1}^n a_i x_i \leq b, \quad x_i \in \{0,1\}$$

$$x_i = \begin{cases} 1 & i \text{ selected} \\ 0 & i \text{ not selected} \end{cases} \quad 1 \leq i \leq n$$

COVERING / PACKING / PARTITIONING PROBLEM (Binary choice)

- $M = \{1, \dots, m\}$ ground set
- $\{M_1, \dots, M_n\}$ collection of subsets indexed by $N = \{1, \dots, n\}$ $(M_j \subseteq M \ \forall j \in N)$
- c_j cost for M_j, $j \in N$

Covering goal: find a cover of M (i.e. $F \subseteq N : \bigcup_{j \in F} M_j = M$) of minimum cost

ILP formulation:

$$\min \sum_{j=1}^n c_j x_j \quad \text{s.t.} \quad \sum_{j \in N} a_{ij} x_j \geq 1 \ \forall i, \quad x_j \in \{0,1\} \ \forall j$$

$$a_{ij} = \begin{cases} 1 & \text{if } i \in M_j \\ 0 & \text{if } i \notin M_j \end{cases} \qquad x_j = \begin{cases} 1 & \text{if } M_j \text{ is selected} \\ 0 & \text{if } M_j \text{ is not selected} \end{cases}$$

$$= \min \sum_{j=1}^n c_j x_j \quad \text{s.t.} \quad \boxed{A\underline{x} \geq \underline{1}}, \quad \underline{x} \in \{0,1\}^n$$

each element is covered at least once

Packing goal: c_j represents the profits. Find a packing of M (each i is covered at most 1) that maximizes the profit

ILP formulation:

$$\max \sum_{j=1}^n c_j x_j \quad \text{s.t.} \quad \boxed{A\underline{x} \leq \underline{1}}, \quad \underline{x} \in \{0,1\}^n$$

each element is covered at most once

Partitioning goal: find a partition (each i is covered exactly 1) that minimize the costs

ILP formulation:

$$\min \sum_{j=1}^n c_j x_j \quad \text{s.t.} \quad \boxed{A\underline{x} = \underline{1}}, \quad \underline{x} \in \{0,1\}^n$$

each element is covered exactly once

ASSIGNMENT PROBLEM (Association between entities)

The LP relaxation is an **IDEAL** formulation

- n projects, n persons
- c_{ij} cost for assignment project i to person j

Goal: decide the combination to minimize costs

ILP formulation:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad 1 \le ij \le n$$
$$\text{s.t.} \quad \sum_{i=1}^{n} x_{ij} = 1 \qquad \forall j \quad (j \in \text{persons})$$
$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \quad (i \in \text{projects})$$
$$x_{ij} \in \{0,1\}$$

UNCAPACITATED FACILITY LOCATION (UFL) (forcing constraints)

- $M = \{1,\dots,m\}$ clients, $i \in M$
- $N = \{1,\dots,n\}$ candidate sites for depots (depositi), $j \in N$
- f_j cost for opening a depot in j
- c_{ij} transportation cost if the whole demand of i is served from j

Goal: decide where to open depots and how to serve clients to min costs

MILP formulation:

$$\min \sum_{i\in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$
$$\text{s.t.} \quad \sum_{j \in N} x_{ij} = 1 \qquad \forall i \in M$$
$$\boxed{\sum_{i \in M} x_{ij} \le m y_j \qquad \forall j \in M}$$
$$y_j \in \{0,1\}$$
$$0 \le x_{ij} \le 1$$

x_{ij} = fraction of demand of client i served by depot j

$y_j = \begin{cases} 1 & \text{depot } j \text{ is open} \\ 0 & \text{otherwise} \end{cases}$

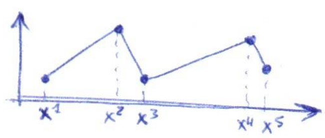
ALTERNATIVE formulation:
$x_{ij} \le y_j \quad \forall i \in M \ \forall j \in N$
(stronger)

x_{ij} and y_j are linked:
$\begin{cases} x_{ij} > 0 \Rightarrow y_j = 1 \\ y_j = 0 \Rightarrow x_{ij} = 0 \end{cases}$

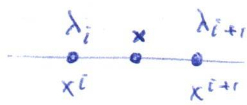
if: d_i = demand of i
k_j = capacity of j

$$\sum_{i \in M} d_i x_{ij} \le k_j y_j \qquad \forall j$$

PIECEWISE LINEAR COST FUNCTION



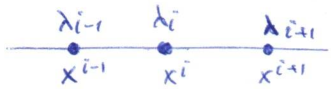
$$x = \lambda_i x^i + \lambda_{i+1} x^{i+1}$$
$$y_i = \mathbb{1}\{x \in [x^i, x^{i+1}]\}$$



Goal: minimize f over $[x^1, x^k]$

MILP formulation:

$$\min \sum_{i=1}^{k} \lambda_i f(x^i)$$
$$\text{s.t.} \quad \sum_{i=1}^{k} \lambda_i = 1$$
$$\sum_{i=1}^{k-1} y_i = 1 \qquad \longleftarrow \quad \forall \text{ point belongs to just one subinterval}$$
$$\left.\begin{array}{l} \lambda_1 \le y_1 \\ \lambda_k \le y_{k-1} \\ \lambda_i \le y_{i-1} + y_i \end{array}\right\} \longleftarrow$$
$$\lambda_i \ge 0, \ y_i \in \{0,1\}$$



if $\lambda_i \neq 0 \Rightarrow$ either we're in $[x^{i-1}, x^i]$ or in $[x^i, x^{i+1}]$ $\Rightarrow \lambda_i \le y_i + y_{i-1}$

D.O. (2)

ASYMMETRIC TRAVELING SALESMAN PROBLEM (ATSP)

- $G = (V, A)$ complete directed graph, $n = |V|$ (V nodes, A arcs)
- $c_{ij} \in \mathbb{R}$ cost for the arc $(i,j) \in A$

Goal: determine an Hamiltonian circuit (visit exactly once each node) of minimum cost

ILP formulation (1):

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij} \qquad x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is included} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{s.t.} \quad \sum_{j \in V: j \neq i} x_{ij} = 1 \quad \forall i$$

← $\forall i$ we have 1 out going arc

$$\sum_{i \in V: i \neq j} x_{ij} = 1 \quad \forall j$$

← $\forall j$ we have 1 ingoing arc

$$* \boxed{\sum_{(i,j)\in \delta^+(S)} x_{ij} \geq 1 \qquad \forall S \subset V,\ S \neq \emptyset}$$

CUT SET INEQUALITIES

$\delta^+(S) = \{(i,j) \in A : \ i \in S,\ j \in V \setminus S\}$

$$x_{ij} \in \{0,1\} \quad \forall (i,j) \in A$$

ILP formulation (2):

...

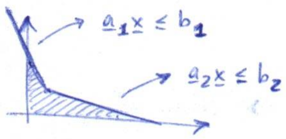
$$* \boxed{\sum_{(i,j)\in E(S)} x_{ij} \leq |S| - 1 \qquad \forall S \subseteq V,\ 2 \leq |S| \leq n-1}$$

SUBTOUR ELIMINATION INEQ.

$E(S) = \{(i,j) \in A : \ i \in S,\ j \in S\}$

If $S \neq V$, the number of arcs fully contained in S must be strictly smaller than the number of the nodes in S

DISJUNCTIVE CONSTRAINTS



either: $\begin{cases} a_1 x \leq b_1 \\ a_2 x \leq b_2 \end{cases}$ s.t. $\underline{0} \leq \underline{x} \leq \underline{u}$

MILP formulation:

$$\underline{a}_i \underline{x} - b_i \leq M(1 - y_i) \qquad i = 1,2$$
$$y_1 + y_2 = 1$$
$$y_i \in \{0,1\} \qquad i = 1,2$$
$$\underline{0} \leq \underline{x} \leq \underline{u}$$

$M \geq \max_{1 \leq i \leq 2} \{\underline{a}_i \underline{x} - b_i,\ \underline{0} \leq \underline{x} \leq \underline{u}\}$

$y_i = \begin{cases} 1 & \text{if } i \text{ is satisfied} \\ 0 & \text{otherwise} \end{cases}$

LINEARIZATION OF PRODUCT VARIABLES

- two binary variables:

$z = y_1 \cdot y_2$, $y_1, y_2 \in \{0,1\}$, $z \in \{0,1\}$:

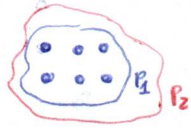
$$z \leq y_1$$
$$z \leq y_2$$
$$z \geq y_1 + y_2 - 1$$

- binary variable · bounded continuous variable

$z = x \cdot y$, $x \in [0,u]$, $y \in \{0,1\}$, $z \in [0,u]$:

$$0 \leq z \leq uy$$
$$z \leq x$$
$$z \geq x - (1-y)u$$

- A (M)ILP has ∞ formulations.
  $P = \{(\underline{x},\underline{y}) \in \mathbb{R}^{n_1+n_2} : A_1\underline{x} + A_2\underline{y} \geq \underline{b},\ \underline{x} \geq \underline{0},\ \underline{y} \geq \underline{0}\} \subseteq \mathbb{R}^{n_1+n_2}$ is a formulation for a mixed integer set $X \subseteq \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \iff X = P \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2})$
- P_1, P_2 formulations for a (M)ILP : X. P_1 stronger than P_2 if $P_1 \subset P_2$.



SYMMETRIC TSP (STSP)

- $G=(V,E)$ undirected graph
- c_e cost for every $e=\{i,j\} \in E$

Goal: determine an Hamiltonian cycle

ILP formulation (1):

$$\min \sum_{e \in E} c_e x_e$$
$$\text{s.t. } \sum_{e \in \delta(i)} x_e = 2 \qquad i \in V$$
$$\boxed{\sum_{e \in \delta(S)} x_e \geq 2 \qquad S \subset V,\ S \neq \emptyset}$$
$$x_e \in \{0,1\}$$

CUT SET INEQ.
$\delta(S) = \{\{i,j\} \in E : i \in S, j \in E \setminus S\}$

ILP formulation (2): (equally strong)

...

$$\boxed{\sum_{e \in E(S)} x_e \leq |S| - 1 \qquad S \subset V,\ S \neq \emptyset}$$

SUBTOUR ELIMINATION INEQ.
$E(S) = \{\{i,j\} \in S : i \in S, j \in S\}$

- Mayer's theorem: $X \subseteq \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}$ feasible set of a MILP $\Rightarrow$ $conv(X)$ is a rational polyhedron and the extreme points of $conv(X)$ belong to X.
  ($\min\{\underline{c}^T\underline{x} : \underline{x} \in X\} = \min\{\underline{c}^T\underline{x} : conv(X)\}$)

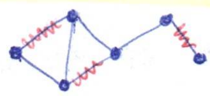
A formulation P is an ideal formulation for X if $P \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$: $P = conv(X)$.
We look for formulations close to $conv(X)$.

PERFECT MATCHING PROBLEM (PM)

Edmond's theorem:

The LP relaxation is an IDEAL formulation

- $G=(V,E)$ undirected graph, $n = |V|$
- c_e cost for every $e=\{i,j\} \in E$



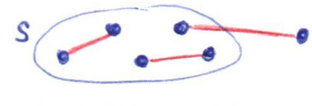
Goal: determine the perfect matching (subset of edges without common nodes but incident to all the nodes) of minimum cost

ILP formulation:

$$\min \sum_{e \in E} c_e x_e$$
$$\text{s.t. } \sum_{e \in \delta(i)} x_e = 1 \qquad \forall i \in V$$
$$x_e \in \{0,1\} \qquad \forall e \in E$$
$$\sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall S \subset V,\ |S| \text{ odd}$$

for every i we select exactly 1 edge

If $|S|$ is odd only one node must be not completely included in S:



D.O. (3)

- The formulations including additional variables are <u>extended formulations.</u>

UNCAPACITATED LOT-SIZING (ULS)

- f_t fixed cost for production during t
- p_t unit cost for production during t
- h_t unit storage cost in period t
- d_t demand in period t

<u>Goal:</u> determine a production plan (single type of product) for the next n periodts that minimize the total cost (production + storage) while sat. all the demand. <u>Hp.</u> stock empty at the beginning and at the en

<u>MILP formulation:</u>

$$\min \sum_{t=1}^{n} p_t x_t + \sum_{t=1}^{n} h_t s_t + \sum_{t=1}^{n} f_t y_t$$
$$\text{s.t.} \quad \left.\begin{aligned} & s_t = s_{t-1} + x_t - d_t \\ & x_t \le M y_t \\ & s_0 = s_n = 0 \\ & s_t, x_t \ge 0 \\ & y_t \in \{0,1\} \end{aligned}\right\} \forall t$$

x_t = amount produced in t

$y_t = \begin{cases} 1 & \text{production in period } t \\ 0 \end{cases}$

s_t = amount in stock at the end of period t

$M = \sum_{t=1}^{n} d_t + s_n - s_0$ (for instance)

since $s_t = \sum_{i=1}^{t} x_i + s_0 - \sum_{i=1}^{t} d_i \implies s_t$ can be eliminated

<u>MILP extended formulation:</u> (<u>stronger</u>) ⟵ $\boxed{\text{the LP relaxation is } \textbf{IDEAL}}$

$$\min \sum_{i=1}^{n} \sum_{t=1}^{n} c_{it} w_{it} + \sum_{t=1}^{n} f_t y_t$$
$$\begin{aligned} \text{s.t.} \quad & \sum_{i=1}^{t} w_{it} = d_t && \forall t,\ 1 \le t \le n \\ & \sum_{i=1}^{n} w_{i,n+1} = 0 \\ & w_{it} \le d_t y_t && i \le t \quad (\forall) \\ & w_{it} \ge 0 && i \le t \quad (\forall) \\ & y_t \in \{0,1\} && \forall t \end{aligned}$$

w_{it} = amount produced in i to satisfy the period t $(1 \le i \le t \le n+1)$

$y_t = \begin{cases} 1 & \text{production is in period } t \\ 0 \end{cases}$

$c_{it} = p_i + h_i + \dots + h_{t-1}$

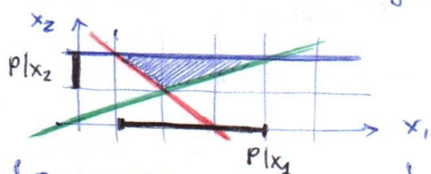
= produce something in i and take it stored till $t-1$

- Comparison betw. formulations with different variables: orthogonal projection.

Fourier-Motzkin <u>elimination method:</u>

at each iteration one variable x_i is eliminated (an equivalent linear system without x_i is delivered). The process ends when the resulting system contains a single variable.

Ex.
$$-x_2 \ge -2$$
$$x_1 + x_2 \ge 3$$
$$-\tfrac{1}{2} x_1 + x_2 \ge 0$$



- eliminate x_2: $\begin{cases} 3 - x_1 \le x_2 \\ \tfrac{1}{2} x_1 \le x_2 \\ x_2 \le 2 \end{cases} \longrightarrow \begin{cases} 3 - x_1 \le 2 \\ \tfrac{1}{2} x_1 \le 2 \end{cases} \longrightarrow 1 \le x_1 \le 4$

- eliminate x_1: $\begin{cases} 3 - x_2 \le x_1 \\ x_1 \le 2x_2 \\ x_2 \le 2 \end{cases} \longrightarrow \begin{cases} 3 - x_2 \le 2x_2 \\ x_2 \le 2 \end{cases} \longrightarrow 1 \le x_2 \le 2$

- To <u>strength a formulation</u> we look for an extended one (which better approx. conv(X)).

FIXED CHARGE NETWORK FLOW PROBLEM (**FCNF**)

minimum flow cost problem with fixed costs (for the min cost flow problem we neglect —) The LP relaxation of the min cost flow is IDEAL (TU matrix)

- $G = (V, A)$ directed graph
- for each $(i,j) \in A$:
  - f_{ij} fixed cost
  - c_{ij} unit cost
  - u_{ij} capacity
- b_i demand $\forall i \in V$

Goal: feasible flow of minimum total cost which sat. demand & constr.

MILP formulation:

$$\min \sum_{(i,j)\in A} (c_{ij} x_{ij} - f_{ij} y_{ij})$$

$$\text{s.t. } \sum_{(h,i)\in \delta^-(i)} x_{hi} - \sum_{(i,j)\in \delta^+(i)} x_{ij} = b_i \quad \forall i \in V$$

$$0 \le x_{ij} \le u_{ij} y_{ij} \quad \forall (i,j) \in A$$

$$y_{ij} \in \{0,1\} \quad \forall (i,j) \in A$$

$$(\delta^+(i) = \{(i,j)\in A : j \in V\},\ \delta^-(i) = \{(h,i) \in A : h \in V\})$$

x_{ij} = amount of flow throug (i,j)

$y_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is used} \\ 0 & \text{if not} \end{cases}$

- Using the concept of TU matrices we can figure out if a formulation is IDEAL for a ILP problem (not MILP).

A matrix $m \times n$ is **TU** if every squared submatrix has determinant $\in \{-1, 0, +1\}$.

- $P(\underline{b}) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b},\ \underline{x} \ge \underline{0}\} \ne \emptyset$

  if A is TU $\Longrightarrow$ all the extreme points of $P(\underline{b})$ are integers
- $P(\underline{b}) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} \ge \underline{b},\ \underline{x} \ge \underline{0}\} \ne \emptyset$

  if A is TU $\Longrightarrow$ all the extreme points (vertices) of $P(\underline{b})$ are integers
- A $m \times n$ is TU if:

(sufficient)

  1. $a_{ij} \in \{-1, 0, 1\}$
  2. $\forall$ column contains at most 2 non-zero coefficients
  3. the rows can be partitioned in two groups such that: for each column j with 2 non-zero coefficients: $\sum_{i \in group 1} a_{ij} - \sum_{i \in group 2} a_{ij} = 0$
- A $m \times n$ is TU iff: each subset of the rows can be partitioned into two subsets I_1 and I_2 s.t. $(\sum_{i\in I_1} a_{ij} - \sum_{i \in I_2} a_{ij}) \in \{0, 1, -1\}$ $\forall$ column j

TRANSPORTATION PROBLEM

special case of **UFL**

- single product, **m** production plants, p_i max amount producible in i
- n clients, d_j demand of client j
- c_{ij} = transportation cost of 1 unit of product from plant i to client j
- q_{ij} = maximum amount transportable

Goal: transport plan to minimize costs

ILP formulation:

$$\min \sum_{i=1}^{m} \sum_{i=1}^{n} c_{ij} x_{ij}$$

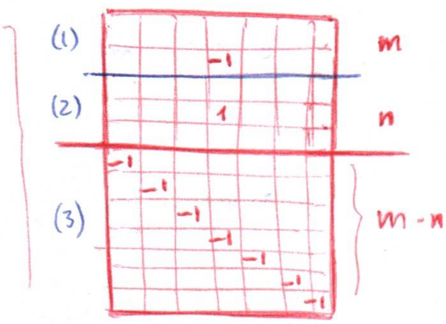
$$\text{s.t. } \sum_{j=1}^{n} x_{ij} \le p_i \quad \forall i \quad (1) \quad = -\sum x_{ij} \ge p_i$$

$$\sum_{i=1}^{m} x_{ij} \ge d_j \quad \forall j \quad (2)$$

$$x_{ij} \le q_{ij} \quad \forall i \forall j \quad (3) \quad = -x_{ij} \ge -q_{ij}$$

$$x_{ij} \ge 0 \text{ integer } \forall i \forall j$$

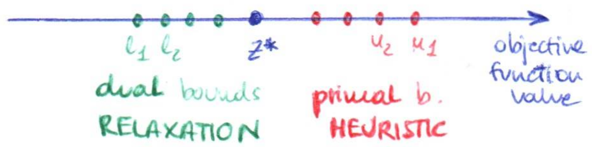
x_{ij} = amount of product transported from i to j



TU $\Longrightarrow$ the LP relax. is **IDEAL**

D.O. (4)

- For a generic problem $z^* = \min\{c(\underline{x}) : \underline{x} \in X\}$ we look for primal and dual bounds:



(for a max problem: the dual provides upper bounds, the primal lower bounds)

- $z^* := \min\{c(\underline{x}) : \underline{x} \in X\}$, $\tilde{z} := \min\{\tilde{c}(\underline{x}) : \underline{x} \in \tilde{X}\}$.
  $\tilde{z}$ is a relaxation if $X \subseteq \tilde{X}$ and $\tilde{c}(\underline{x}) \le c(\underline{x}) \quad \forall \underline{x} \in X$.
- RELAXATIONS:

MULTI-DIMENSIONAL BINARY KNAPSACK PROBLEM

- m knapsacks of capacity W_i
- n items of weight w_j
- p_j profit of the item j

Goal: fit the knapsacks to maximize the total profit

MILP formulation:

$$\max \sum_{i=1}^m \sum_{j=1}^n p_j x_{ij}$$

m constraints

$$\text{s.t. } \boxed{\sum_{j=1}^n w_j x_{ij} \le W_i \qquad \forall i \in \{1,..,m\}}$$
$$\boxed{\sum_{i=1}^m x_{ij} \le 1 \qquad \forall j \in \{1,..,n\}}$$
$$x_{ij} \in \{0,1\} \qquad \forall i, \forall j$$

$x_{ij} = 1$ if the j-th item is inserted in the i-th pocket

Surrogate relaxation:

...

1 constraint

$$\boxed{\sum_{i=1}^m \lambda_i \sum_{j=1}^n w_j x_{ij} \le \sum_{i=1}^m \lambda_i W_i}$$

We replace a subset of constraints with their linear combination with multipliers $\lambda_i \ge 0$.
Here every item j has m copies and can be selected at most one copy.

Every $\{\lambda_i\}_i$ creates a different relaxation.
Since we want the tightest upper bound
we look for: $\min_{\underline{\lambda}} z_{S(\underline{\lambda})}$ (surrogate dual)
where $z_{S(\underline{\lambda})}$ is the solution of the surrogate.

Lagrangian relaxation:

Eliminate the "difficult" constraints and add a penalty term in the obj. funct. (with a multiplier $\underline{u}$) that penalizes the violation of the constraints.

...

$$\boxed{\max \sum_{i=1}^m \sum_{j=1}^n p_j x_{ij} + \sum_{j=1}^n u_j \left(1 - \sum_{i=1}^m x_{ij}\right) \qquad \underline{u} \ge \underline{0}}$$

We had $1 - \sum_{i=1}^m x_{ij} \ge 0$ and now we added it to the obj. function. We added a penalty term in the obj. function that penalizes the violation of the constraint.
"max → +", "min → −"

Again, $\forall \underline{u}$ we have a relaxation, so:
$\min_{\underline{u}} z_{L(\underline{u})}$ (Lagrangian dual)

By elimination: we eliminate some constraints (very weak).

- "by elimination" is dominated by Lagrangian and surrogate.
- Surrogate dominates the Lagrangian but the Lagrangian dual is easier to solve.
  (In the Lagrangian we can get rid of the linking constraints, which are the most difficult)
- HEURISTIC: greedy method
  Construct a feasible solution piece by piece from scratch. At each step we chose the option that generates the best local profit without reconsidering previous steps.
- HEURISTIC: local search method
  Consider a generic $\min c(\underline{x})$ and try to improve iteratively a current feasible solution.
  Define for any feasible solution $\underline{x}$ a neighborhood $N(\underline{x})$:= subset of nearby feasible solutions.
  At the next step the solution $\underline{x}'$ will be the best solution of the set $N(\underline{x})$. To avoid local min we can allow moves (in the neighborhood) even with worse obj. function.

- For a generic ILP $\exists$ an ideal formulation (Mayer's). However it might be difficult to determine $\Longrightarrow$ improve a initial formulation by adding valid inequalities.
(A better formulation is a better approximation of $conv(X)$).

- $\underline{\pi}^T \underline{x} \le \pi_0$ is a <u>valid inequality</u> for X if $\underline{\pi}^T \underline{x} \le \pi_0 \quad \forall \underline{x} \in X$.
(It's a valid inequality if is satisfied by all the points in $\mathbf{X}$)

CUTTING PLANE METHOD = add valid inequalities only if needed

ILP: $\min\{\underline{c}^T \underline{x} : \underline{x} \in X = P \cap \mathbb{Z}^n\}$, $P = \{\underline{x} \in \mathbb{R}^n_+ : A\underline{x} \le \underline{b}\}$ LP relaxation.

Given $\underline{x}' \in P$ $(\underline{x}' \notin X)$ a <u>cutting plane</u> is an $\underline{\pi}^T \underline{x} \le \pi_0$ s.t.:

- $\underline{\pi}^T \underline{x} \le \pi_0$ valid for $X = P \cap \mathbb{Z}^n$
- $\underline{\pi}^T \underline{x}' > \pi_0$