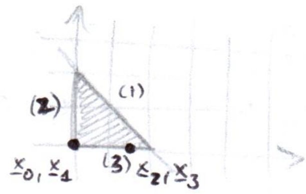
19/06/2020

Consider the quadratic problem:

$$\min \; x_1^2 + x_2^2 - x_1 x_2 - 3x_1$$
$$\text{s.t.} \quad x_1 + x_2 \le 2 \quad (1)$$
$$x_1 \ge 0 \quad (2)$$
$$x_2 \ge 0 \quad (3)$$



Describe the first 3 iterations starting from $\underline{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Problem:
$$\min \; x_1^2 + x_2^2 - x_1 x_2 - 3x_1$$
$$\text{s.t.} \quad x_1 + x_2 - 2 \le 0$$
$$-x_1 \le 0$$
$$-x_2 \le 0$$

$g_1(\underline{x}) = x_1 + x_2 - 2$
$g_2(\underline{x}) = -x_1$
$g_3(\underline{x}) = -x_2$

1. $\underline{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $W_0 = \{2, 3\}$

We look for $\underline{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$: $\min\{ q(\underline{x}_0 + \underline{d}) : \underline{a}_2^T \underline{d} = 0, \; \underline{a}_3^T \underline{d} = 0 \}$

$$\left.\begin{array}{l} \underline{a}_2^T \underline{d} = [-1 \;\; 0] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \Rightarrow d_1 = 0 \\ \underline{a}_3^T \underline{d} = [0 \;\; -1] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \Rightarrow d_2 = 0 \end{array}\right\} \rightarrow \underline{d} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{x}_1 = \underline{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

KKT conditions: $\nabla f(\underline{x}_k) + \sum_{i \in W_k} u_i^k \nabla g_i(\underline{x}_k) = \underline{0}$

$$\begin{bmatrix} -3 \\ 0 \end{bmatrix} + u_2^0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_3^0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 \Rightarrow \begin{cases} -3 - u_2^0 = 0 \\ -u_3^0 = 0 \end{cases} \quad \begin{cases} u_2^0 = -3 \\ u_3^0 = 0 \end{cases}$$

$$\Longrightarrow W_1 = W_0 \setminus \{2\} = \{3\}$$

2. $\underline{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $W_1 = \{3\}$

We look for $\underline{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$: $\min \{ q(\underline{x}_1 + \underline{d}) : \underline{a}_3^T \underline{d} = 0 \}$

$$\underline{a}_3^T \underline{d} = [0 \;\; -1] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \Rightarrow d_2 = 0$$

$$q(\underline{x}_1 + \underline{d}) = q\left(\begin{bmatrix} d_1 \\ 0 \end{bmatrix}\right) = d_1^2 - 3d_1$$

$$\nabla_{d_1} q = 2d_1 - 3 = 0 \Rightarrow d_1 = \frac{3}{2} \Rightarrow \underline{d}_1 = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$$

α_1?

$i \notin W$:

- $i = 1$: $\underline{a}_1^T \underline{d}_1 = [1 \;\; 1] \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} = \frac{3}{2} > 0$ (✓)

$$\frac{b_1 - \underline{a}_1^T \underline{x}_1}{\underline{a}_1^T \underline{d}_1} = \frac{2 - 0}{3/2} = \frac{4}{3}$$

- $i = 2$: $\underline{a}_2^T \underline{d}_1 = [-1 \;\; 0] \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} = -\frac{3}{2} < 0$ (×)

$$\alpha_1 = \min\left\{1, \frac{4}{3}\right\} = 1$$

$$\Longrightarrow \underline{x}_2 = \underline{x}_1 + \underline{d}_1 = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$$

$W_2 = W_1$

3. $\underline{x}_2 = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$, $W_2 = \{3\}$

We look for $\underline{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$: $\min\{ q(\underline{x}_2 + \underline{d}) : \underline{a}_3^T \underline{d}_2 = 0 \}$

$\underline{a}_3^T \underline{d} = [0 \ -1] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \rightarrow d_2 = 0$

$$q(\underline{x}_2 + \underline{d}) = q\left(\begin{bmatrix} 3/2 + d_1 \\ 0 \end{bmatrix}\right) = \left(\frac{3}{2} + d_1\right)^2 - 3\left(\frac{3}{2} + d_1\right)$$
$$= \frac{9}{4} + d_1^2 + \cancel{3d_1} - \frac{9}{2} - \cancel{3d_1}$$
$$= \frac{9}{4} + d_1^2 - \frac{9}{2}$$

$\nabla_{d_1} q(\underline{x}_2 + \underline{d}) = 2d_1 = 0 \Rightarrow d_1 = 0 \Rightarrow \underline{d}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

KKT conditions : $\nabla f(\underline{x}_k) + \sum_{i \in W_k} u_i^k \nabla g_i(\underline{x}_k) = \underline{0}$

$$\begin{bmatrix} 0 \\ -3/2 \end{bmatrix} + u_3^{(2)} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow -\frac{3}{2} - u_3^{(2)} = 0 \Longrightarrow u_3^{(2)} = -\frac{3}{2}$$

$\Longrightarrow W_3 = W_2 \setminus \{3\} = \emptyset$