		$\mathbb{E}[X]$	$Var(X)$
Bernoulli(p)	$f(x) = p^x(1-p)^{1-x}\,\mathbb{1}_{\{0,1\}}(x)$	p	$p(1-p)$
Binomiale(n,p)	$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Poisson(λ)	$p(k) = \frac{e^{-\lambda}\lambda^k}{k!}$	λ	λ
Uniform(a,b)	$f(x) = \frac{1}{b-a}\mathbb{1}_{[a,b]}(x)$, $F(x) = \frac{x-a}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential(λ)	$f(x) = \lambda e^{-\lambda x}\mathbb{1}_{[0,\infty)}(x)$, $F(x) = (1-e^{-\lambda x})\mathbb{1}_{[0,\infty)}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal(μ,σ^2)	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Gamma(α,β) (shape α, rate β), $\alpha>0, \beta>0$	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\mathbb{1}_{[0,\infty)}(x)$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Inv-Gamma(α,β), $\alpha>0, \beta>0$	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{x}\right)^{\alpha+1} e^{-\frac{\beta}{x}}\mathbb{1}_{[0,\infty)}(x)$	$\frac{\beta}{\alpha-1}$ $\alpha>1$	$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ $\alpha>2$
Beta(α,β)	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$
Weibull(λ,k), $\lambda>0, k>0$ (λ → scale, k → shape)	$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\mathbb{1}_{(0,+\infty)}(x)$	$\frac{\lambda}{k}\Gamma\left(\frac{1}{k}\right)$	$\frac{\lambda^2}{k^2}\left[2k\Gamma\left(\frac{2}{k}\right) - \Gamma^2\left(\frac{1}{k}\right)\right]$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\,dt \implies \begin{cases} \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \\ \Gamma(n+1) = n! \\ \Gamma(1/2) = \sqrt{\pi} \\ \Gamma(1) = 1 \end{cases}$$

- $X_i \sim \mathcal{E}(\beta) = Gamma(1,\beta) \implies \sum X_i \sim Gamma(n,\beta)$, $\bar{X}_n \sim Gamma(n, n\beta)$
- $Gamma\left(\frac{n}{2}, \frac{1}{2}\right) = \chi^2(n)$: $\mathbb{E}[X \sim \chi^2(n)] = n$
- $X \sim Gamma(\alpha,\beta) \implies kX \sim Gamma\left(\alpha, \frac{\beta}{k}\right)$
- $X \sim Gamma(n,\beta) \implies 2\beta X \sim Gamma\left(\frac{2n}{2}, \frac{1}{2}\right) = \chi^2(2n)$
- $X_i \sim Gamma(\alpha_i,\beta) \implies \sum X_i \sim Gamma(\sum \alpha_i, \beta)$
- $X \sim \mathcal{G}(\alpha_1,\beta),\ Y \sim \mathcal{G}(\alpha_2,\beta) \implies (X+Y) \perp\!\!\!\perp \frac{X}{X+Y} \perp\!\!\!\perp \frac{Y}{X+Y}$
- $X \sim \mathcal{G}(\alpha_1,1),\ Y \sim \mathcal{G}(\alpha_2,1) \implies \frac{X}{X+Y} \sim Beta(\alpha_1,\alpha_2)$
- $X \sim Beta(\alpha,\beta) \implies 1-X \sim Beta(\alpha,\beta)$
- $X \sim Beta(1,1) \implies X \sim \mathcal{U}([0,1])$

$f(x|\theta,\lambda) \longrightarrow m(x) = \int_{\Theta,\Lambda} f(x|\theta,\lambda)\,\pi(\theta,\lambda)\,d\theta\,d\lambda$ (se una è discreta $\Rightarrow \Sigma$)

$\mathbb{E}[X]$? $\mathbb{E}[X] = \int x \cdot m(x)\,dx = \mathbb{E}\left[\mathbb{E}[X|\theta,\lambda]\right] = \mathbb{E}\left[\int_X f(x|\theta,\lambda)\cdot x\,dx\right]$ *

*da qui: $\mathbb{E}[h(\theta,\lambda)] = \mathbb{E}\left[\mathbb{E}[h(\theta,\lambda)|\theta]\right] = \mathbb{E}[h_1(\theta)]\,\mathbb{E}[h_2(\lambda)]$

Posterior (exact) : **BAYES** $\pi(\theta|x) = \frac{f(x|\theta)\,\pi(\theta)}{m(x)}$, $m(x) = \int_\Theta f(x|\theta)\,\pi(\theta)\,d\theta$

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marg. post : $\pi(\theta,\lambda|\underline{x}) \implies$ marg. post of θ : $\pi(\theta|\underline{x}) = \int_\Lambda \pi(\theta,\lambda|\underline{x})\,d\lambda$

Multinomial – Dirichlet → e.g. draw $n = N_1 + .. + N_k$ balls from a urn with k colors balls

$N_1, .., N_k \mid \underline{\lambda} \sim$ multinomial $(\lambda_1, .., \lambda_k)$

$\lambda_1, .., \lambda_k \sim$ Dirichlet $(\alpha_1, .., \alpha_k)$

likelihood: $p(N_1, .., N_k \mid \lambda_1, .., \lambda_k) = \frac{n!}{N_1! \cdots N_k!} \lambda_1^{N_1} .. \lambda_k^{N_k}$ $\quad (N_j = X_j)$

Prior: $\pi(\underline{\lambda}) = \frac{1}{B(\underline{\alpha})} \left(\prod_{j=1}^{k} \lambda_j^{\alpha_j - 1} \right) \mathbb{1}_{\Delta_{k-1}}(\underline{\lambda})$

Posterior: $\pi(\underline{\lambda} \mid \underline{N}) = \left(\prod_{j=1}^{k} \lambda_j^{\alpha_j + N_j - 1} \right) \mathbb{1}_{\Delta_{k-1}}(\underline{\lambda}) \Rightarrow$ Dirichlet$(\alpha_1 + N_1, .., \alpha_k + N_k)$

Suppose to have a Gibb sampler but one of the full conditionals is not known.
Two parameters: θ, λ:

$$\pi(\theta, \lambda \mid \underline{x}) \propto L(\theta, \lambda\ \underline{x})\ \pi(\theta, \lambda) \Rightarrow \begin{cases} \pi(\theta \mid -) \propto \ ..\ \checkmark \\ \pi(\lambda \mid -) \propto (\text{something}_1\ \ \text{something}_2) \leftarrow \text{not known} \end{cases}$$

If something_1 is known ⇒ METROPOLIS – HASTINGS:

- proposal density $= \text{something}_1(\theta, \lambda)$
- $(\theta^{(j)}, \lambda^{(j)})$ current state of the chain:
  - $\theta = \theta^{(j)}$
  - $\lambda' \sim \text{something}_1(\theta)$
  - $\lambda^{(j+1)} = \lambda'$ with prob. $\min\left(1, \frac{\pi(\lambda' \mid \theta, \underline{x})}{\pi(\lambda^{(j)} \mid \theta, \underline{x})} \cdot \frac{\text{something}_1(\theta, \lambda^{(j)})}{\text{something}_1(\theta, \lambda')}\right)$

ATTENZIONE ! $\mathbb{E}[X] = \int x\, f(x)\, dx \neq \mathbb{E}[X \mid \theta] = \int x\, f(x \mid \theta)\, dx$

$$\mathbb{E}[X] = \int x \int f(x \mid \theta)\, \pi(\theta)\, d\theta\, dx$$

$V_1, .., V_{N-1} \sim \text{Beta}(1, a)$, $\quad V_N = 1$, $\quad P_1 = V_1$, $\quad P_k = V_k \prod_{j=1}^{k-1} (1 - V_j)$

1. $\prod_{j=1}^{k} (1 - V_j) = (1 - V_k) \prod_{j=1}^{k-1} (1 - V_j) = \prod_{j=1}^{k-1} (1 - V_j) - V_k \prod_{j=1}^{k-1} (1 - V_j) = \prod_{j=1}^{k-1} (1 - V_j) - P_k$
2. $\sum_{k=2}^{N} P_k = \sum_{k=2}^{N} \left[\prod_{j=1}^{k-1} (1 - V_j) - \prod_{j=1}^{k} (1 - V_j) \right] = \left[(1 - V_1) - \cancel{\prod_{j=1}^{2} (1 - V_j)} \right] + .. + \left[\cancel{\prod_{j=1}^{N-1} (1 - V_j)} - \underbrace{\prod_{j=1}^{N} (1 - V_j)}_{= 0 \text{ because } V_N = 1} \right] = (1 - V_1) = 1 - P_1$
   $(\sum_{k=1}^{N} P_k = 1)$
3. $\mathbb{E}[P_1] = \mathbb{E}[V_1] = \frac{1}{\alpha + 1}$ (Beta$(1, a)$)
   $\mathbb{E}[P_k] = \mathbb{E}[V_k \prod_{j=1}^{k-1} (1 - V_j)] = \mathbb{E}[V_k] \prod_{j=1}^{k-1} (1 - \mathbb{E}[V_j]) = \frac{1}{\alpha + 1} \prod_{j=1}^{k-1} \left(1 - \frac{1}{\alpha + 1}\right) = \frac{1}{\alpha + 1} \left(\frac{\alpha}{\alpha + 1}\right)^{k-1}$
   $= \frac{\alpha}{\alpha + 1} \mathbb{E}[P_{k-1}] \Rightarrow \mathbb{E}[P_k] = \eta(\alpha)\, \mathbb{E}[P_{k-1}]$, $\quad \eta(\alpha) = \frac{\alpha}{\alpha + 1}$

Autoregressive models: $X_t = \rho X_{t-1} + \varepsilon_t$, $\quad \varepsilon_t \sim N(0, \sigma^2)$

Conditional distr. of $(X_1, .., X_T)$ given (ρ, σ^2): $\chi(X_{1:T} \mid \rho, \sigma^2) = \prod_{t=1}^{T} \chi(X_t \mid X_{t-1}, \rho, \sigma^2)$

likelihood: $L(\rho, \sigma^2 \mid x_{1:T}) = \prod_{t=1}^{T} \chi(X_t \mid X_{t-1}, \rho, \sigma^2) = \prod_{t=1}^{T} N(X_t \mid \rho X_{t-1}, \sigma^2)$

$$= \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_t - \rho x_{t-1})^2}$$

Model that we've assuming for the data? = marginal of the data

$$m(\underline{x}) = \int_{\Theta} L(\theta, \underline{x})\, \pi(d\theta) \qquad \left(= \int_{\Theta} L(\theta, \underline{x})\, \pi(\theta)\, d\theta \right)$$

Predictive:

- $\mathbb{P}(X_{n+1} \leq k \mid \underline{X}) = \int_{\Theta} \mathbb{P}(X_{n+1} \leq k \mid \theta)\, \pi(\theta \mid \underline{X})\, d\theta = \int_{\Theta} F_{X\mid\theta}(k)\, \pi(\theta \mid \underline{X})\, d\theta$
- density : $p(x_{n+1} \mid x_1, \dots, x_n) = \dfrac{m(\underline{x}, x_{n+1})}{m(\underline{x})}$ This usually updates the parameters !

$\underbrace{\phantom{m(\underline{x}, x_{n+1})}}$ lasciamotutto con "X_{n+1}" come r.v. se poi c'è $\mathbb{P}(X_{n+1} = k \mid \underline{X})$ allora sostituiamo in (↑)

$$m_{X_{n+1}\mid\underline{X}}(x_{n+1} \mid \underline{x}) = p(X_{n+1} \mid X_1, \dots, X_n) = \int_{\Theta} p(X_{n+1} \mid \theta)\, \pi(\theta \mid \underline{X})\, d\theta$$

Posterior point estimate under the ~~quadratic~~ 0-1 loss function ~~⟹ post. mean (expect.)~~ ⟹ post. mode

Posterior point estimate under the quadratic loss function ⟹ post. mean (expect.)

TEST HYPOTHESIS:

$$\begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1 \end{cases} \Longrightarrow BF_{01} = \frac{\mathbb{P}(\Theta_0 \mid \underline{X})}{\mathbb{P}(\Theta_1 \mid \underline{X})} \frac{\mathbb{P}(\Theta_1)}{\mathbb{P}(\Theta_0)}$$

$$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta \neq \theta_0 \end{cases} \Longrightarrow BF_{01} = \frac{\prod_{i=1}^{n} f(x_i; \theta = \theta_0)}{m_1(\underline{x})} = \frac{\prod_{i=1}^{n} f(x_i; \theta = \theta_0)}{\int_{\Theta} f(\underline{x} \mid \theta)\, \pi_1(\theta)\, d\theta}$$

$$\begin{cases} H_0 : \text{model } 1 \\ H_1 : \text{model } 2 \end{cases} \Longrightarrow BF_{12} = \frac{m(\underline{x}, \theta_1)}{m(\underline{x}, \theta_2)}$$

	in favor of H_1	weak for H_0	in favor of H_0	strong for H_0	very strong for H_0
BF01	< 1	1-3	3-12	12-150	> 150
$2 \log BF_{01}$	< 0	0-2	2-5	5-10	> 10

DISCRETE

- Prior on θ :

θ	$\pi(\theta)$
⋮	⋮

⟹ posterior:

θ	$\pi(\theta)$	$L(\theta; \underline{x})\pi(\theta)$	$\pi(\theta \mid \underline{x})$
⋮	⋮	⋮	⋮

- $\mathbb{E}[\theta \mid \underline{x}] = \sum_{i=1}^{n} \theta \cdot \pi(\theta \mid \underline{x})$

SURVIVAL :

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{S(t)}$$

$$F(t) = 1 - e^{-\int_0^t h(u)\, du}$$

$$S(t) = 1 - F(t) \qquad (S_i(t) \,!)$$

$$L(\underline{x}, \theta) \propto \left[\prod_{i=1}^{n} \frac{1}{x_i} \right] \cdots \Longrightarrow L(\underline{x}, \theta) \propto \left[e^{-\sum_{i=1}^{n} \log(x_i)} \right] \cdots$$

Test Hp. $\pi(\theta \mid \underline{x}) \Longrightarrow \approx N(\mathbb{E}[\theta \mid \underline{X}], var(\theta \mid X))$

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likelihood

Bernoulli (Binomial)	⟹	conj: $Beta(\alpha, \beta)$
Normal (σ^2 known)	⟹	$\mu \sim N(\mu_0, \tau^2)$, $\mu \mid - \sim N(\mu_n, \tau_n^2)$
Normal (σ^2 not known)	⟹	$\mu \mid \sigma^2 \sim N(\cdot, \cdot)$, $\sigma^2 \sim inv\Gamma(\cdot, \cdot)$
Poisson	⟹	$\lambda \sim \Gamma(\cdot, \cdot)$
Multinomial	⟹	Dirichlet
$(N_1, \dots, N_K) \sim multinomial(\lambda_1, \dots, \lambda_K)$	→	$(\lambda_1, \dots, \lambda_K) \sim Dir(\alpha_1, \dots, \alpha_K)$ | $D(\alpha_1 + N_1, \dots, \alpha_K + N_K)$

Gibbs sampler:

$X_1, .., X_n | \theta_1, .., \theta_n \sim \; .. (\theta_1, .., \theta_n)$

Prior: $\pi(\theta_1), .., \pi(\theta_n)$

Posterior: $\pi(\theta_1, .., \theta_n | \underline{X}) \propto \underbrace{L(\theta_1, .., \theta_n | \underline{X}) \; \pi(\theta_1) .. \pi(\theta_n)}$

to keep it simple discard the quantities $\perp\!\!\!\perp \underline{\theta}$

Prepare for the Gibbs sampler (full conditionals):

$\pi(\theta_1 | -) \propto \; .. \; [\text{tutto quello che ha } \theta_1]$

$\vdots$

$\pi(\theta_n | -) \propto \; .. \; [\text{tutto quello che ha } \theta_n]$

Likelihood: $L(\underline{x}, \text{param}) = \prod_{i=1}^n f_i(x_i, \text{param})$

$L(\theta, ..) \propto \theta^n e^{..\theta} \Longrightarrow$ Gamma

$\propto \frac{1}{\theta^n} e^{..\theta} \Longrightarrow$ Inverse-Gamma

Gamma(α, β): $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{(0,+\infty)}(x)$

$\mathbb{E}[X] = \frac{\alpha}{\beta}$, $Var(X) = \frac{\alpha}{\beta^2}$

Inv-Gamma(α, β): $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{x}\right)^{\alpha+1} e^{-\frac{\beta}{x}} \mathbb{1}_{(0,+\infty)}(x)$

$\mathbb{E}[X] = \frac{\beta}{\alpha-1}$ $(\alpha > 1)$, $Var(X) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ $(\alpha > 2)$

$\Gamma(n+1) = n!$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$X_i \sim \mathcal{E}(\lambda)$

$\sum_{i=1}^n X_i \sim$ Gamma(n, λ)

$\bar{X}_n \sim$ Gamma$(n, n\lambda)$

Gamma$(1, \beta) = \mathcal{E}(\beta)$

Gamma$(\frac{n}{2}, \frac{1}{2}) = \chi^2(n)$

$U \sim \mathcal{G}(n, \beta) \Rightarrow 2\beta U \sim \chi^2(2n)$

$2\beta U \sim \mathcal{G}(\frac{2n}{2}, \frac{1}{2})$

EQUIVALENT SAMPLE PRINCIPLE

- MLE of θ: $\frac{\partial}{\partial\theta} \log L(\theta, ..) = 0 \Rightarrow \hat{\theta}_{MLE}$
- $\mathbb{E}[\theta | \underline{X}] = (..) \, \mathbb{E}[\theta] + (..) \, \hat{\theta}_{MLE}$
- We compare the red:
  - param $-^1$ plays the role of "n"
  - param $-^2$ plays the role of "$g(\underline{x})$"
- Elicitation: $\underset{\text{dim} = m}{\underline{x}^{old}} \rightarrow$
  - $-^1 = m$
  - $-^2 = g(\underline{x}^{old})$

INDICATOR FUNCTIONS (SUPPORT)

$\boxed{\text{CLT}}$ X_i: $\mathbb{E}[X_i]$, $Var(X_i) \longrightarrow \sum_{i=1}^n X_i \approx N(n \cdot \mathbb{E}[X_i], n \cdot Var(X_i))$

$\bar{X}_n \approx N(\mathbb{E}[X_i], \frac{1}{n} Var(X_i))$

HPD

$CI_{1-\alpha} = \left[\bar{x}_n - z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \; \bar{x}_n + z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right]$

JEFFREYS' PRIOR

$\pi_J(\theta) \propto \sqrt{|I(\theta)|}$: $I(\theta) = \mathbb{E}\left[\left(\frac{\partial}{\partial\theta} \log f(x, \theta)\right)^2 \middle| \theta\right]$ $(x, \cancel{\underline{X}})$

$= -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2} \log f(x, \theta) \middle| \theta\right]$

Then:

$\pi(\theta | \underline{x}) \propto L(\underline{x}, \theta) \, \pi_J(\theta)$