Stochastic dynamical models, Mathematical Engineering 2019/2020 - SDM 5

Exercise 1.
Let P_t $(t \geq 0)$ be the 4×4 matrix

$$P_t = \begin{bmatrix} \frac{1}{6}+\frac{1}{3}\mathrm{e}^{-3t}+\frac{1-\alpha}{2}\mathrm{e}^{-4t} & \frac{1}{3}-\frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{3}-\frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{6}+\frac{1}{3}\mathrm{e}^{-3t}-\frac{1-\alpha}{2}\mathrm{e}^{-4t} \\ \frac{1}{6}+\frac{1}{3}\mathrm{e}^{-3t}-\frac{1}{2}\mathrm{e}^{-4t} & \frac{1}{3}+\frac{2}{3}\mathrm{e}^{-3t} & \frac{1}{3}-\frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{6}-\frac{2}{3}\mathrm{e}^{-3t}+\frac{1}{2}\mathrm{e}^{-4t} \\ \frac{1}{6}-\frac{2}{3}\mathrm{e}^{-3t}+\frac{1}{2}\mathrm{e}^{-4t} & \frac{1}{3}-\frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{3}+\frac{2}{3}\mathrm{e}^{-3t} & \frac{1}{6}+\frac{1}{3}\mathrm{e}^{-3t}-\frac{1}{2}\mathrm{e}^{-4t} \\ \frac{1}{6}+\frac{1}{3}\mathrm{e}^{-3t}-\frac{1}{2}\mathrm{e}^{-4t} & \frac{1}{3}-\frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{3}-\frac{1}{3}\mathrm{e}^{-3t} & \frac{1}{6}+\frac{1}{3}\mathrm{e}^{-3t}+\frac{1}{2}\mathrm{e}^{-4t} \end{bmatrix}$$

1) Determine the only value of the parameter $\alpha \in \mathbb{R}$ for which $(P_t)_{t \geq 0}$ is the transition semigroup of a continuous time Markov chain $(X_t)_{t \geq 0}$ with states $\{1, 2, 3, 4\}$.

2) For the above α determine the matrix Q of transition rates.

3) Is the Markov chain recurrent or transient? In the first case determine all the invariant densities.

R **Exercise 2.** Blood donors arrive at a center according to a Poisson process with parameter λ and all of them donate 1 liter. An ambulance comes to the center from time to time to withdraw the collected bottle. Suppose that the time elapsed between the arrival of an ambulance and the next one is an exponential random variable with parameter μ and that the inter-arrival times are independent random variables.

1) Construct a Markov chain representing the number X_t of blood bottles available at time t at the center.

2) Classify the states and determine the invariant probability distributions.

R **Exercise 3.** In a gas station there are 2 fuel pumps. Since the station is very small, each costumer, who arrives and finds the pompes both, can not stop there if there is an other customer waiting for the service. Suppose that the refuelling times of different customers are independent and with exponential distribution of mean 2 minutes, and that the arrivals to the gas station are described by a Poisson process with parameter $1/10$. Assume moreover that the inter-arrival times are independent of the refuelling times.
Let $(Q_t)_{t \geq 0}$ the Markov chain counting the number of customers in the gas station at the time t.

1) Write the rates transition matrix of the chain. Is it irreducible?

2) Compute the probability that in 15 minutes arrive 2 customers, knowing that in the first 10 minutes 1 customer has arrived.

3) Find, if they exist, all invariant distributions.

4) Write the random variable W_t describing the time that a customer, arriving at the time t, spends at the gas station and compute its expectation in stationary conditions.

5) Compute the mean number of customers in the system (in stazionary conditions) at the time t. Does the Little Law hold?

Exercise 4. In a small call center there are 3 telephone operators. Suppose that the number of the incoming calls is described by a Poisson process with parameter $1/5$ (one call each 5 minutes), and the duration of each call is independent of the others and has an exponential distribution with mean 3 minutes. The calls arriving when the 3 lines are busy are put on hold, and then redirected to the first free line, according to the order of arrival. Suppose moreover that it is possible to put on hold at most 2 calls and that the following ones are lost.
Denote with $(Q_t)_t$ the number of calls at the time t (calls on hold+calls with the operators), with $Q_0 = 0$.

$\mathbb{P}$(no call till min 1 , between min 1 and 3 $\boxed{\text{at least}}$ 2 calls) !

$\mathbb{P}$(first call arrives in 6 min) + $\boxed{\mathbb{P}(\text{no call arrives in 6 min})}$!

1) Determine the transition rates matrix of the chain $(Q_t)_t$.

2) What is the probability that in the first 15 minutes arrive at most 2 calls?

3) What is the probability that the second call arrives after 6 minutes? and the probability that the first call arrives after 1 minute and the second one before of 3 minutes?

4) Does exist an invariant distribution π? If yes, determine it.

5) Compute in stationary conditions the mean permanence time in the system of a call arriving at the time t.

6) In stationary conditions what is the probability that the calls are lost?

Exercise 5. The queue in a post office is modeled as follows:

- there are 2 servers for the customers;
- the service times and the arrival times of each costumer are independent;
- the service time is an exponential random variable with parameter 1;
- the time between the arrival of two consecutive customers is an exponential random variable with parameter 2;
- the number of the customers in the system at any given time is less than or equal to 3: the customer, who arrives and finds the 2 servers busy and a third customer in the waiting line, leaves the office.

We denote the states of the system as: 0 = no customers in the queue; 1 = one served customer; 2 = two served customers; 3 = two served customers and one customer in the waiting line. (The queue size includes those customers who are currently being served).

1) Write the transition rates matrix of the Markov chain $(X_t)_{t\geq 0}$ describing the number of the customers in the system and find the invariant distribution.

2) If at time $t = 0$ there are no customers in the system, how long on average does it take until the queue reaches its maximum length?

#1 (#5)

1. $\boxed{P_0 = I} \Rightarrow \quad p_{11}(0) = 1 = \frac{1}{6} + \frac{1}{3} + \frac{1}{2} - \frac{\alpha}{2} \quad \Longrightarrow \quad \alpha = 0$

(also: $p_{14}(0) = 0 \Longrightarrow \frac{1}{6} + \frac{1}{3} - \frac{1-\alpha}{2} = 0$ if $\alpha = 0$)

2. Q?

$$\boxed{Q = \left.\frac{dP_0}{dt}\right|_{t=0}} = \begin{bmatrix} -1-2 & 1 & 1 & -1+2 \\ -1+2 & -2 & 1 & 2-2 \\ 2-2 & 1 & -2 & -1+2 \\ -1+2 & 1 & 1 & -1-2 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

3. Irreducible continuous time MC:

$\forall i,j \in I \quad \exists n \geq 1$ and $\exists k_1, .., k_n$ s.t. $i \neq k_1, k_1 \neq k_2, .., k_n \neq j$ and $q_{ik_1} q_{k_1 k_2} \cdots q_{k_n j} > 0$

This MC is irreducible and it's also finite $\Longrightarrow$ it must be recurrent.

$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$: $\quad 0 = \pi Q$:

$$\begin{cases} -3\pi_1 + \pi_2 + \pi_4 = 0 \\ \pi_1 - 2\pi_2 + \pi_3 + \pi_4 = 0 \\ \pi_1 + \pi_2 - 2\pi_3 + \pi_4 = 0 \\ \pi_1 + \pi_3 - 3\pi_4 = 0 \end{cases} \Longrightarrow [..] \Rightarrow \quad \pi = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right)$$

#2

Donors arrive $\sim \mathcal{E}(\lambda)$ and donate 1ℓ.
Time between two ambulances $\sim \mathcal{E}(\mu)$ $\Big\} \perp\!\!\!\perp$

X_t = # bottles at time t

we exit from i :
- because a donor comes
- because an ambulance comes

$B(i)$ = time for a donor to come $\sim \mathcal{E}(\lambda)$
$D(i)$ = time for an ambulance to come $\sim \mathcal{E}(\mu)$

1. • starting from 0:
   possible transition : $0 \to 1$
   first exit time from 0: $\sim \mathcal{E}(\lambda)$: $\quad q_{01} = \lambda, \quad q_{00} = -\lambda \quad q_{0j} = 0 \quad \forall j \neq 0/1$

   • starting from 1:
   possible transition : $1 \to 0, \quad 1 \to 2$
   first exit time from 0 $=$ $\min\{D(1), B(1)\} \sim \mathcal{E}(\mu + \lambda)$
   $q_{10} = \mu, \quad q_{12} = \lambda, \quad q_{11} = -(\mu+\lambda), \quad q_{1j} = 0$ otherwise

   • $i \geq 2$: * $\quad q_{i(i-1)} = \mu, \quad q_{i(i+1)} = \lambda, \quad q_{ii} = -(\mu+\lambda), \quad q_{ij} = 0$ otherw.

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ & \cdots & & & & & \end{bmatrix}$$

(*) How to compute $q_{i(i+1)}$?

$$p_{i(i+1)} = \mathbb{P}(\min\{B(i), D(i)\} = B(i)) = \mathbb{P}(B(i) < D(i))$$
$$= \mathbb{P}((B(i), D(i)) \in \{(x,y) : x \leq y\})$$

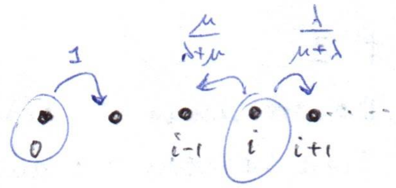
$$X = B(i) \sim \mathcal{E}(\lambda)$$
$$Y = D(i) \sim \mathcal{E}(\mu)$$

$$\mathbb{P}(X<Y) = \mathbb{P}((X,Y) \in (0,Y) \times \mathbb{R}^+) = \int_0^{+\infty}\int_0^Y f_{XY}(x,y)\,dx\,dy$$
$$\overset{\perp\!\!\!\perp}{=} \int_0^{+\infty} f_Y(y) \int_0^y f_X(x)\,dx\,dy$$
$$= \int_0^{+\infty} \mu e^{-\mu y} \int_0^y \lambda e^{-\lambda x}\,dx\,dy = [\ldots] = \frac{\lambda}{\lambda+\mu} = \frac{q_{i(i+1)}}{-q_{ii}}$$

2. i recurrent for cont. MC $\iff$ i recurrent for discrete skeleton

$$p_{ij} = \begin{cases} \frac{q_{ij}}{-q_{ii}} & i \neq j \quad q_{ii} \neq 0 \\ 0 & (i \neq j, q_{ii} = 0) \vee (i = j, q_{ii} \neq 0) \\ 1 & i = j \quad q_{ii} = 0 \end{cases}$$

$$\Longrightarrow \quad P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \cdots \\ 0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \cdots \\ & \cdots & & & & \end{bmatrix}$$



This transition matrix P corresponds to a MC $:= (Y_n)_{n\geq 0}$.
$(Y_n)_{n\geq 0}$ is an irreducible MC ($\exists$ only 1 class state)

$$j \geq 1: \quad y_j = \frac{\mu}{\lambda+\mu} y_{j-1} + \frac{\lambda}{\lambda+\mu} y_{j+1} \Longrightarrow \lambda y_{j+1} - (\lambda+\mu) y_j + \mu y_{j-1} = 0$$

$$\lambda x^2 - (\lambda+\mu)x + \mu = 0 \Longrightarrow x_{1/2} = 1, \frac{\mu}{\lambda}$$

$$\Longrightarrow y_j = A + \left(\frac{\mu}{\lambda}\right)^j B$$

$\Longrightarrow$
- $\mu \geq \lambda \Longrightarrow \nexists$ bounded non-const. solution $\Longrightarrow$ recurrent
- $\mu < \lambda \Longrightarrow \exists$ bounded non-const. solution $\Longrightarrow$ transient

π invariant exists if $\mu \geq \lambda$:

$$0 = \pi Q \iff \begin{cases} -\lambda \pi_0 + \mu \pi_1 = 0 \\ \lambda \pi_{i-1} - (\lambda+\mu)\pi_i + \mu \pi_{i+1} = 0 \\ \boxed{\sum \pi_i = 1} \; ! \end{cases}$$

$$\mu x^2 - (\lambda+\mu)x + \lambda = 0 \Rightarrow x_{1/2} = 1, \frac{\lambda}{\mu} \longrightarrow \pi_j = A + \left(\frac{\lambda}{\mu}\right)^j B$$

$$\left(\pi_1 = \frac{\lambda}{\mu}\pi_0\right) \wedge \left(\sum_{i\geq 0} \pi_i = 1\right) \Longrightarrow A = 0, \; B = \pi_0$$

#2 (#5)

$$\sum_{i\geq 0} \pi_0 \left(\frac{\lambda}{\mu}\right)^i = \pi_0 \frac{1}{1-\frac{\lambda}{\mu}} = \pi_0 \frac{\mu}{\mu-\lambda} = 1 \longrightarrow \begin{cases} \pi_0 = \frac{\mu-\lambda}{\mu} \\ \pi_j = \frac{\mu-\lambda}{\mu}\left(\frac{\lambda}{\mu}\right)^j \quad j\geq 1 \end{cases}$$

#3

- 2 fuel pumps
- refuelling time $\sim \mathcal{E}(\frac{1}{2})$
- arrivals $\sim \mathcal{E}(\frac{1}{10})$ $\Big\}$ $\perp\!\!\!\perp$
- $(Q_t)_{t\geq 0}$: Q_t = # customers at time t

1. $Q_t \in \{0,1,2,3\}$
   - starting from 0:
     $0 \to 1$
     first exit from 0: $q_{01} = \frac{1}{10}$, $q_{00} = -\frac{1}{10}$, $q_{0j} = 0$ $\quad j = 2,3$
   - starting from 1:
     $1 \to 2$, $1 \to 0$
     first exit from 1: $q_{10} = \frac{1}{2}$, $q_{12} = \frac{1}{10}$, $q_{11} = -\frac{3}{5}$, $q_{13} = 0$
   - starting from 2:
     $2 \to 1$, $2 \to 3$
     first exit from 2: $q_{21} = 1$, $q_{23} = \frac{1}{10}$, $q_{22} = -\frac{11}{10}$, $q_{20} = 0$
     it can leave either one or the other customer
     $1/2 \lor 1/2 \Rightarrow \frac{1}{2} + \frac{1}{2} = 1$
   - starting from 3:
     $3 \to 2$
     first exit from 3: $q_{32} = 1$, $q_{33} = -1$, $q_{31} = q_{30} = 0$

$$Q = \begin{bmatrix} -\frac{1}{10} & \frac{1}{10} & 0 & 0 \\ \frac{1}{2} & -\frac{3}{5} & \frac{1}{10} & 0 \\ 0 & 1 & -\frac{11}{10} & \frac{1}{10} \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

2. $X_t = X(t)$ = number of customers arrived up to time t
   $\mathbb{P}(X(15) - X(10) = \quad | X(10) - X(0) = 1) = \mathbb{P}(X(15) - X(10) = 1)$
   $\boxed{X(t+s) - X(s) \sim \mathcal{E}(t\lambda)}$
   $\mathbb{P}(X(15) - X(10) = 1) = \frac{1}{2} e^{-\frac{1}{2}}$

3. Invariant distribution:

MC finite and irreducible $\Longrightarrow$ MC recurrent

$$0 = \pi Q \quad \Longleftrightarrow \quad \begin{cases} -\frac{\pi_0}{10} + \frac{\pi_1}{2} = 0 \\ \frac{\pi_0}{10} - \frac{3\pi_1}{5} + \pi_2 = 0 \\ \frac{\pi_1}{10} - \frac{11\pi_2}{10} + \pi_3 = 0 \\ \frac{\pi_2}{10} - \pi_3 = 0 \\ \sum \pi_i = \end{cases}$$

$\Longrightarrow \quad \pi = \left(\frac{500}{611}, \frac{100}{611}, \frac{10}{611}, \frac{1}{611} \right)$ invariant distribution

4. W_t = time spent at the gas station, (for a client arrived at time t) $\quad \mathbb{E}[W_t] = ?$

$$W_t = \underset{\substack{\text{service}\\ \text{time}}}{S_t} + \underbrace{Y_t \mathbb{1}_{\{Q_t \geq 2\}}}_{\text{waiting time } (Y_t)}$$

Under stationarity conditions $Q_t \sim \pi$

$$\Longrightarrow \mathbb{E}_\pi[W_t] = \mathbb{E}_\pi[S_t] + \mathbb{E}_\pi[Y_t \mathbb{1}_{\{Q_t \geq 2\}}]$$

$$= 2 + \mathbb{E}_\pi[f(Y_t, Q_t)]$$

$$= 2 + \sum_{ij} f(i,j)\, \mathbb{P}(Y_t = i, Q_t = j)$$

$$\sum_{ij} i\, \mathbb{1}_{\{j=2\}} \mathbb{P}(Y_t = i, Q_t = j)$$

$$\sum_i i\, \mathbb{P}(Y_t = i, Q_t = 2)$$

$$\sum_i i\, \mathbb{P}(Y_t = i \mid Q_t = 2)\, \mathbb{P}(Q_t = 2)$$

$$= 2 + \mathbb{E}[Y_t \mid Q_t = 2]\, \mathbb{P}(Q_t = 2)$$

if $Q_t = 2$
Y_t = waiting time
$Y_t \sim \mathcal{E}(\frac{1}{2} + \frac{1}{2})$

$\mathbb{P}_\pi(Q_t = 2) = \pi_2 = \frac{10}{611}$

$$= 2 + \frac{10}{611} = \frac{1232}{611}$$

5. $\mathbb{E}_\pi[Q_t]$?

Stationary conditions $\Longrightarrow Q_t \sim \pi \Longrightarrow \mathbb{E}_\pi[Q_t] = \sum_i i\, \mathbb{P}(Q_t = i)$

$$\Longrightarrow \mathbb{E}_\pi[Q_t] = \sum_i i\, \mathbb{P}(Q_t = i) = 0 \cdot \pi_0 + 1\pi_1 + 2\pi_2 + 3\pi_3 = \frac{123}{611}$$

$$\boxed{\text{Little law: } \mathbb{E}_\pi[Q_t] = \underset{\substack{\text{arrival}\\ \text{param.}}}{\lambda}\, \mathbb{E}_\pi[W_t]}$$

but $\left.\begin{array}{l} \mathbb{E}_\pi[Q_t] = \frac{123}{611} \\ \mathbb{E}_\pi[W_t] \cdot \frac{1}{10} = \frac{1232}{6110} \end{array}\right\} \neq$

#4 (#5)

- 3 phone operators
- incoming calls $\sim \mathcal{E}(\frac{1}{5})$
- call duration $\sim \mathcal{E}(\frac{1}{3})$
- at most 2 on hold

$Q_t \in \{0,1,2,3,4,5\}$

!!!

Remember
when there are 2 (or+) occupied operators, the prob. that 1 gets free must consider both (or+) getting free

$(Q_t)_{t \geq 0}$ $\quad Q_t = \#$ calls at time t $\quad$ (calls on hold + calls w/ operators) $\quad , \quad Q_0 = 0$

1.
- starting from 0:
  $0 \to 1: \quad q_{01} = \frac{1}{5}, \quad q_{00} = -\frac{1}{5}, \quad q_{0j} = 0 \quad j \in \{2,3,4,5\}$
- starting from 1:
  $1 \to 0, \ 1 \to 2: \quad [q_{10}, q_{11}, q_{12}, q_{13}, q_{14}, q_{15}] = [\frac{1}{3}, -\frac{8}{15}, \frac{1}{5}, 0, 0, 0]$

if two are busy either one or the other can get free $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

- starting from 2:
  $2 \to 1, \ 2 \to 3: \quad [q_{20}, q_{21}, q_{22}, q_{23}, q_{24}, q_{25}] = [0, \frac{2}{3}, -\frac{13}{15}, \frac{1}{5}, 0, 0]$
- starting from 3:
  $3 \to 2, \ 3 \to 4: \quad [q_{30}, q_{31}, q_{32}, q_{33}, q_{34}, q_{35}] = [0, 0, 1, -\frac{6}{5}, \frac{1}{5}, 0]$
- starting from 4:
  $4 \to 3, \ 4 \to 5: \quad [q_{40}, q_{41}, q_{42}, q_{43}, q_{44}, q_{45}] = [0, 0, 0, 1, -\frac{6}{5}, \frac{1}{5}]$
- starting from 5:
  $5 \to 4: \quad [q_{50}, q_{51}, q_{52}, q_{53}, q_{54}, q_{55}] = [0, 0, 0, 0, \frac{1}{3}, -\frac{1}{3}]$

either one of the 3 busy operators got free: $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$

2. $\mathbb{P}$(in 15 minutes number of calls ≤ 2) ?

$X(t) = \#$ calls up to t

$X(t+s) - X(s) \sim \mathcal{E}(t\frac{1}{5})$

$$\mathbb{P}(X(15) - X(0) \leq 2) = \mathbb{P}(X(15)-X(0)=0) + \mathbb{P}(X(15)-X(0)=1) + \mathbb{P}(X(15)-X(0)=2)$$

$X(15) - X(0) \sim \mathcal{P}(15 \cdot \frac{1}{5}) = \mathcal{P}(3) \qquad \mathbb{P}(\cdot = k) = \frac{e^{-3} 3^k}{k!}$

$$= e^{-3} + 3e^{-3} + \frac{1}{2} 9 e^{-3} = \frac{17}{2} e^{-3}$$

3. $\mathbb{P}$(second calls arrives after 6 minutes) :

$= \mathbb{P}$(first call in 6 minutes) $+ \ \mathbb{P}$(0 calls in 6 minutes)

$= \mathbb{P}(X(6) - X(0) = 1) + \mathbb{P}(X(6) - X(0) = 0)$

$= \frac{6}{5} e^{-6/5} + e^{-6/5} = \frac{11}{5} e^{-6/5}$

$$\mathbb{P}(X(1) - X(0) = 0, \ X(3) - X(1) \geq 2) = \mathbb{P}(X(1) - X(0) = 0)\ \mathbb{P}(X(3) - X(1) \geq 2)$$

$$= \mathbb{P}(X(1) - X(0) = 0)\left(1 - \mathbb{P}(X(3) - X(1) = 0) - \mathbb{P}(X(3) - X(0) = 1)\right)$$

$$= (e^{-1/5})\left(1 - e^{-2/5} - \frac{2}{5} e^{-2/5}\right) = [\ldots] = e^{-3/5}\left(e^{2/5} - \frac{7}{5}\right) \simeq 0.05$$

4. Invariant distribution:

$$0 = \pi Q \implies \begin{cases} -\frac{1}{5}\pi_0 + \frac{1}{3}\pi_1 = 0 \\ \frac{1}{5}\pi_0 - \frac{8}{15}\pi_1 + \frac{2}{3}\pi_2 = 0 \\ \frac{1}{5}\pi_1 - \frac{13}{15}\pi_2 + \pi_3 = 0 \\ \frac{1}{5}\pi_2 - \frac{6}{5}\pi_3 + \pi_4 = 0 \\ \frac{1}{5}\pi_3 - \frac{6}{5}\pi_4 + \pi_5 = 0 \\ \frac{1}{5}\pi_4 - \pi_5 = 0 \\ \sum \pi_i = 1 \end{cases}$$

$$\pi = \frac{1}{11404}\begin{array}{cccccc} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \\ \end{array}$$
$$\left[6250, 3750, 1125, 225, 45, 9\right]$$

5. $$P_t = \underset{\text{duration}}{D_t} + \underbrace{W_t \mathbb{1}_{\{Q_t \geq 3\}}}_{\text{permanance in the queue}}$$

$$D_t \sim \mathcal{E}\left(\frac{1}{3}\right), \quad Q_t = 3 \implies W_t \sim \mathcal{E}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) = \mathcal{E}(1)$$
$$Q_t = 4 \implies W_t \sim \underbrace{\mathcal{E}(1) + \mathcal{E}(1)}_{\substack{\text{we have to wait} \\ \text{tha 2 operators} \\ \text{get free} \\ (\text{one get free} \sim \mathcal{E}(1))}} = \Gamma(2,1)$$

standard formula

$$\mathbb{E}_\pi[P_t] = \mathbb{E}_\pi[D_t] + \mathbb{E}_\pi[W_t \mid Q_t = 3]\, \mathbb{P}(Q_t = 3) + \mathbb{E}_\pi[W_t \mid Q_t = 4]\, \mathbb{P}(Q_t = 4)$$
$$= 3 + 1 \cdot \pi_3 + 2 \cdot \pi_4$$
$$= 3 + \frac{225}{11404} + \frac{90}{11404}$$
$$= \frac{34527}{11404} \simeq 3.0276$$

6. $$\mathbb{P}_\pi(\text{call lost}) = \mathbb{P}(Q_t = 5) = \pi_5 = \frac{9}{11404} \simeq 0.000789$$