11. 5 def: $\pi(x)$, $Q(x,dy)$, $\alpha(x,y)$, $p(x,y)$, $P(x,A)$.

Reversebility: $p(x,y)\,\pi(x) = p(y,x)\,\pi(y)$

invariancy: $\int_E P(x,A)\,\pi(x)\,dx = \pi(A)$

12.

- $(y_i, \underline{x}_i)$, y_i continuous: $\mathbb{E}[Y_i]$, $\underline{x}_i$?
- Model
- likelihood: s^2, $\hat{\beta}_{MLE} = (X^TX)^{-1}X^Ty$, $(y-X\beta)^T(y-X\beta)^T = \dots$
- σ^2 known: $(\tau X^TX + B_0^{-1})^{-1}$
- σ^2 unknown: $(X^TX + B_0^{-1})^{-1} = B_n$ —— $\begin{cases} \beta\mid\sigma^2 \sim N_p(b_0, \sigma^2 B_0) \\ \beta\mid\sigma^2, y \sim N_p(b_n, \sigma^2 B_n) \end{cases}$
- Jeffreys': $\sigma^2(X^TX)^{-1}$

13.

- $\mathbb{P}(m=j)$, M_j: likelihood, prior
- $\pi(\theta_j \mid y, M_j)$
- $\mathbb{P}(m=j \mid y)$

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14. CPO$_i$ = CONDITIONAL PREDICTIVE ORDINATE i $\quad (\mathcal{L}(Y_i \mid \underline{Y}_{-i}))$

LPML = LOG-PSEUDO-MARGINAL LIKELIHOOD

15.

- $Z_i \Leftrightarrow Y_i$: $(y_i, \underline{x}_i)$: $y_i \in \{0,1\}$
- $[\beta \mid \underline{z}, y] \propto \mathcal{L}(\underline{z}\mid\beta)\,\pi(\beta)$ $\quad \beta\mid - \sim N_p\left((X^TX+B_0^{-1})^{-1}(X^TX\hat{\beta}_{MLE} + B_0^{-1}b_0),\ (X^TX+B_0^{-1})^{-1}\right)$
- $[\underline{z}\mid\beta, y] \propto \mathcal{L}(\underline{Y}\mid\underline{z})\,\mathcal{L}(\underline{z}\mid\beta)$ $\quad \begin{cases} z_i \mid \beta, y_i = 1 & - \\ z_i \mid \beta, y_i = 0 & - \end{cases}$

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16. $L(\theta \mid D) = \prod_{i=1}^n \left[(f_i(y_i\mid\theta))^{\delta_i} (S_i(y_i\mid\theta))^{1-\delta_i} \right]$ $\quad (y_i, \delta_i)$ $\quad y_i = \min\{T_i, C_i\}$, $\delta_i = \mathbb{1}_{\{T_i \le C_i\}}$

conditions: 1. $T_i \perp\!\!\!\perp C_i$ 2. $C_i \perp\!\!\!\perp \theta$ 3. all the subjects in the sample are $\perp\!\!\!\perp \mid$ params

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17.

$$\mathbb{E}[P(A)\,P(B)] = \frac{\alpha(A\cap B) - \alpha(A)\alpha(B)}{a(a+1)}$$

$$\mathrm{Cov}(P(A), P(B)) = \frac{\alpha_0(A\cap B) - \alpha_0(A)\alpha_0(B)}{a+1}$$

$$\mathbb{E}[P(A)] = \alpha_0(A)$$

$$\mathrm{Var}(P(A)) = \frac{\alpha_0(A)\,\alpha_0(A^c)}{a+1}$$

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18. $\mathcal{L}(X_1, \dots, X_n) = \mathcal{L}(X_1)\mathcal{L}(X_2\mid X_1)\dots \Rightarrow \mathcal{L}(X_1) = \alpha_0 \Rightarrow \mathcal{L}(X_2\mid X_1) = \dots \Rightarrow \mathcal{L}(X_i \mid X_{i-1}, \dots, X_1) = \dots$

$\Rightarrow \mathcal{L}(X_1, \dots, X_n) = \alpha_0 \prod_{j=2}^n \left(\frac{a}{a+j-1}\alpha_0 + \frac{1}{a+j-1}\sum_{i=1}^{j-1}\delta_{x_i} \right)$

$\Rightarrow \mathcal{L}(X_{n+1}\mid -) = \frac{a}{a+n}\alpha_0 + \frac{1}{a+n}\sum_{j=1}^n \delta_{x_j}$

19.

- Exchangeability
- Polyà urn with 2 colors : $X_i \sim Be(\theta)$
- $\mathbb{P}(X_1, \dots, X_n)$
- De Finetti : $\lim \bar{X}_n \to Beta(w, B)$

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20. $\begin{cases} X_1, \dots, X_n \mid P \sim f(x)(w) = \int_\Theta k(x;\theta)\,P(d\theta) \\ P \sim \mathcal{D}_\alpha \end{cases}$ $\quad\Big\|\quad$ $\begin{matrix} X_i\mid\theta_i \sim k(\cdot;\theta_i) \\ \theta_1,\dots,\theta_n \mid P \sim P \\ P \sim \mathcal{D}_\alpha \end{matrix}$ $\Bigg\}$ $\underline{X}\mid\underline{\theta} \perp\!\!\!\perp P\mid\underline{\theta}$

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21. $P\mid\underline{X} \sim \int \mathcal{D}_{\alpha + \sum_{j=1}^n \delta_{\theta_j}} H(d\underline{\theta}\mid\underline{X})$: $H(d\underline{\theta}\mid\underline{X}) \propto \left[\prod_{j=1}^n k(x_j;\theta_j)\right]\left[\alpha_0(d\theta_1)\prod_{i=2}^n \left[\alpha(d\theta_i) + \sum_{j=1}^{i-1}\delta_{\theta_j}(d\theta_i)\right]\right]$

22./23. $\rho_n^* = \arg\min_{\hat{\rho}_n} \mathbb{E}[\ell(\rho, \hat{\rho}_n) \mid X_1, \dots, X_n]$

Binder: $\ell(\rho, \hat{\rho}_n) = \sum_{i,j}\left[C_1 \mathbb{1}\{s_i = s_j\}\mathbb{1}\{\hat{s}_i \neq \hat{s}_j\} + C_2 \mathbb{1}\{s_i \neq s_j\}\mathbb{1}\{\hat{s}_i = \hat{s}_j\}\right]$

$M_{ij} = \mathbb{P}(s_i = s_j \mid X_1, \dots, X_n)$, $\quad s^* = \arg\min_{\hat{s}} \sum_{i,j} \left| \mathbb{1}\{\hat{s}_i = \hat{s}_j\} - M_{ij} \right|$

3. $X_1, \dots, X_n | \theta \sim P_\theta$, $\theta \sim \pi(\theta)$ $\Rightarrow$ $\mathbb{P}(\theta \in B | \underline{X}_n = \underline{x}) = \dfrac{\int_B f(\underline{x}|\theta)\pi(d\theta)}{\int_\Theta f(\underline{x}|\theta)\pi(d\theta)}$

proof.

- I $\gamma(A \times B) = \int_B P_\theta(A)\pi(d\theta) = \dots = \int_A \int_B f(\underline{x}|\theta)\,\pi(d\theta)\,\lambda^{(n)}(d\underline{x})$
- $\mu_n(A) = \gamma(A \times \Theta) = \dots = \int_A m_n(\underline{x})\,\lambda^{(n)}(d\underline{x})$
- II $\gamma(\bullet \times B) \ll \mu_n(\bullet) \Rightarrow$ RN: $\gamma(A \times B) = \int_A \pi(\underline{x}, B)\,\mu_n(d\underline{x}) = \dots = \int_A \int_\Theta \pi(\underline{x}, B) f(\underline{x}|\theta)\pi(d\theta)\lambda^{(n)}$
- $\gamma(A \times B) = \mathbb{P}(\underline{X} \in A, \theta \in B) = \int \dots$ $\Rightarrow$ $\pi(\underline{x}, B) = \pi(\theta \in B | \underline{X}_n = \underline{x})$
- $\mathrm{I} = \mathrm{II}$

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5. $\mathbb{P}(|err(T)| > c) = \mathbb{P}\left(\dfrac{|err(T)|}{\sqrt{\sigma^2/T}} > \dfrac{c}{\sqrt{\sigma^2/T}}\right) = 2\left(1 - \phi\left(\dfrac{c}{\sqrt{\sigma^2/T}}\right)\right) \approx 2\left(1 - \phi\left(\dfrac{c}{\sqrt{\sigma^2(T)/T}}\right)\right)$