**Stochastic dynamical models, Mathematical Engineering 2019/2020** - SDM 2

X **Exercise 1**. In My pocket there are 2 coins: one is fair and the other is not fair; namely if you toss the unfair coin the probability of the outcome Head is $p$ $(0 < p < 1,\ p \neq 1/2)$. Put $q = 1 - p$. I choose the coin randomly (with probability 1/2 for each coin) and I toss it many times. Let $X_n$ be the number of Heads after the $n - th$ toss ($X_0 = 0$).

1 If $i_1, i_2, \dots, i_n \in \mathbb{N}$, where $i_1 \in \{0,1\}$ and $i_{k+1} - i_k \in \{0,1\}$ for all $k = 1, 2, \dots, n-1$, compute the probability

$$\mathbb{P}(X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_1 = i_1).$$

Show that $(X_n)_{n \geq 1}$ is a Markov chain.

2) Determine all the values of $p \in (0,1)$ such that $(X_n)_{n\geq 1}$ is a homogeneous Markov chain.

X **Exercise 2**. Let $(X_n)_{n\geq 0}$ be a Markov with state space $I = \{1,2,3,4,5,6,7\}$, initial state $X_0 = 2$ and transition matrix:

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 1/2 & 0 & 1/4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/4 & 1/4 & 1/8 & 1/4 & 1/8 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

1) Classify the states of the chain.

2) Determine (if they exist) all invariant distributions of the chain.

X **Exercise 3**. Let $(X_n)_{n\geq 0}$ be a Markov chain on $I = \{1,2,3,4\}$ with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1) Classify the states of the chain.

2) Compute the probability starting from the state 3, the Markov chain hits $\{1,2\}$.

3) Starting from 3, find the law of the hitting time of $\{1,2\}$.

X **Exercise 4**. Consider the chain on $\{1,2,3,4\}$ with the following transition matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1) Starting from 2 what is the probability of absorption in 4?

2) Starting from 2, how long does it take until the chain is absorbed in 1 or 4?

mean absorption time

X **Exercise 5**. The queue in a public office can be modeled as follows:
- There is a single counter that serves customers according to their order of arrival in the office;
- At every moment $n = 0, 1, 2, 3, \dots$ $A_n$ new customers arrive, where $A_n \sim \mathcal{U}(\{0,1,2\})$ ($\mathcal{U}(\{0,1,2\})$ is the uniform law on $\{0,1,2\}$);

- At every moment $n = 0, 1, 2, 3, \dots$ the served customers $V_n$ leave the office, where $V_n \sim B(1, 3/7)$ if in the office there is at least one person, otherwise $V_n = 0$.
- All the random variables $(A_n)_{n \geq 0}$ and $(V_n)_{n \geq 0}$ are independent;
- At time $n = 0$ the office is empty.

Let $(X_n)_{n \geq 0}$ the process:
$X_n$ =the number of customers in the office at the time $n$.

1) Write the transition matrix of the Markov chain $(X_n)_{n \geq 0}$.

2) Classify all the states of the Markov chain.

3) Determine (if they exist) all invariant distributions of the chain.

**Exercise 6**. Consider a Markov chain on $\mathbb{N}$ with transition matrix $P$ such that

$$p_{i,0} = 1 - p, \quad p_{i,i+1} = p, \quad p_{ij} = 0; \quad p \in (0, 1)$$

1) Show that the Markov chain is irreducible and aperiodic.

2) Let $T_0$ the time of the first return to 0

$$T_0 = \inf\{n \geq 1 | X_n = 0\}$$

Prove that for all $i$, starting from $i$, the law of $T_0$ is geometric.

3) Is the Markov chain recurrent?

! 4) Prove that the geometric law with parameter $1 - p$ is the unique invariant distribution for the Markov chain.

# #1 (#2)

2 coins : fair (F) , not fair (NF) : $\mathbb{P}(H) = \begin{cases} 1/2 & F \\ p & NF \end{cases}$

$X_n = \#$ heads after $n$ toss $(X_0 = 0)$

1. $\mathbb{P}(X_n = i_n, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n, \dots, X_1 = i_1 \mid F)\, \mathbb{P}(F) + \mathbb{P}(X_n = i_n, \dots, X_1 = i_1 \mid NF)\, \mathbb{P}(NF)$

$$= \mathbb{P}(X_n - X_{n-1} = i_n - i_{n-1}, \dots, X_2 - X_1 = i_2 - i_1, X_1 = i_1 \mid F)\, \tfrac{1}{2} \; +$$
$$\mathbb{P}(X_n - X_{n-1} = i_n - i_{n-1}, \dots, X_1 = i_1 \mid NF) \cdot \tfrac{1}{2}$$

Conditioning to F : $Z_k := X_k - X_{k-1}$ $\quad Z_k \sim Be(\tfrac{1}{2})$

NF : $Z_k := X_k - X_{k-1}$ $\quad Z_k \sim Be(p)$

$$\mathbb{P}(X_n = i_n, \dots, X_1 = i_1) = \tfrac{1}{2}\left[ \left( \left(\tfrac{1}{2}\right)^{i_n - i_{n-1}} \left(\tfrac{1}{2}\right)^{1-(i_n - i_{n-1})} \right) \cdot \dots \left( \left(\tfrac{1}{2}\right)^{i_1} \left(\tfrac{1}{2}\right)^{1-i_1} \right) \right] +$$
$$+ \tfrac{1}{2}\left[ \left( p^{i_n - i_{n-1}} q^{1-(i_n - i_{n-1})} \right) \cdot \dots \cdot \left( p^{i_1} q^{1-i_1} \right) \right]$$
$$= \tfrac{1}{2}\left[ \left(\tfrac{1}{2}\right)^n + p^{i_n} q^{n-i_n} \right]$$

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1) = \frac{\mathbb{P}(X_{n+1} = j, X_n = i, \dots, X_1 = i_1)}{\mathbb{P}(X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1)}$$
$$= \frac{\cancel{1/2}\left[ \left(\tfrac{1}{2}\right)^{n+1} + p^j q^{n+1-j} \right]}{\cancel{1/2}\left[ \left(\tfrac{1}{2}\right)^{n} + p^i q^{n-i} \right]}$$

This probability depends only on $i$ and $j$, the information up to time $n-1$ is useless : $\mathbb{P}(X_{n+1} = j \mid X_n = i, \dots, X_1 = i_1) = \mathbb{P}(X_{n+1} = j \mid X_n = i) \quad \forall n$

$\Longrightarrow$ MC.

2. Homogeneity : for instance : $\mathbb{P}(X_2 = 0 \mid X_1 = 0) = \mathbb{P}(X_1 = 0 \mid X_0 = 0)$

$$\Longrightarrow \frac{\left(\tfrac{1}{2}\right)^2 + q^2}{\tfrac{1}{2} + q} = \frac{\tfrac{1}{2} + q}{1 + 1} \Longrightarrow q^2 - q + \tfrac{1}{4} = 0 \Longrightarrow q = p = \tfrac{1}{2}$$

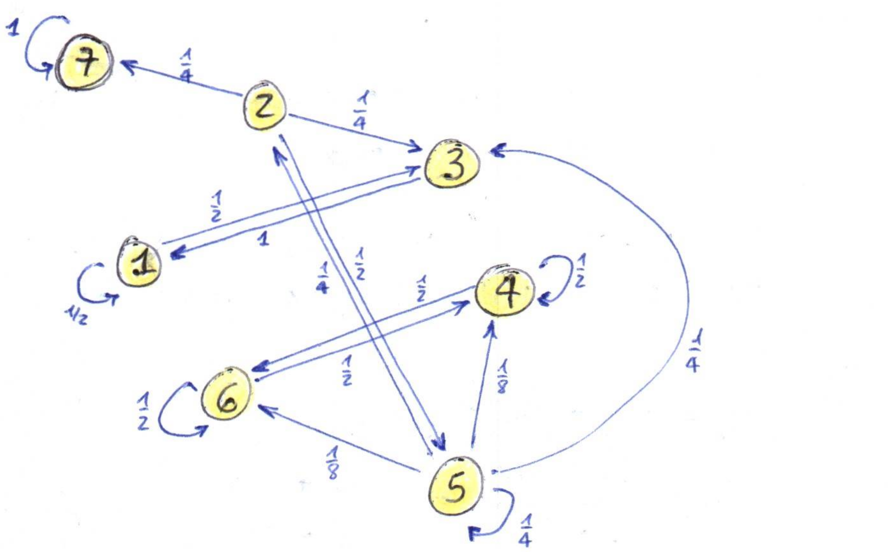
The MC is homogeneous $\Longleftrightarrow$ both coins are fair

# #2

$(X_n)_{n \geq 0}$ MC

$X_0 = 2$

$$P = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[ \begin{array}{ccccccc} 1/2 & & 1/2 & & & & \\ & & 1/4 & & 1/2 & & 1/4 \\ 1 & & & & & & \\ & & & 1/2 & & 1/2 & \\ & 1/4 & 1/4 & 1/8 & 1/4 & 1/8 & \\ & & & 1/2 & & 1/2 & \\ & & & & & & 1 \end{array} \right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}$$



1. Def. CLOSED CLASS: A class $C$ is said to be closed if :

$$\begin{cases} i \in C \\ j \text{ accessible from } i \end{cases} \longrightarrow j \in C$$

States classification:

classes : $\{7\}$ recurrent, closed
$\{1,3\}$ recurrent, closed
$\{4,6\}$ recurrent, closed
$\{2,5\}$ transient, not closed ($2 \to 3$, $2 \in \{2,5\}$, $3 \notin \{2,5\}$)

2. Invariant distributions:

$\pi = (\pi_1, \dots, \pi_7)$ inv. $\Longleftrightarrow$ $\pi P = \pi$

$$[\pi_1\ \pi_2\ \pi_3\ \dots\ \pi_7]\ P = [\pi_1\ \pi_2\ \dots\ \pi_7]$$

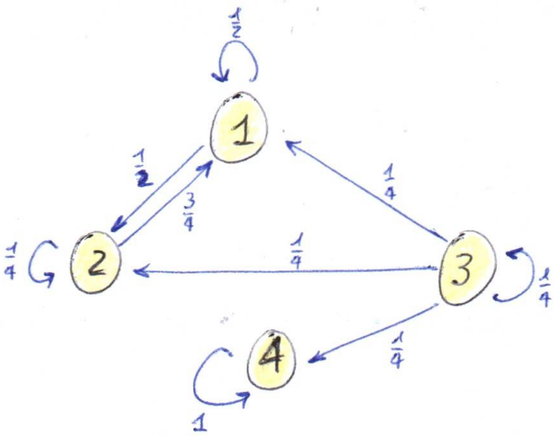
$$\begin{cases} \pi_1 = \frac{1}{2}\pi_1 + \pi_3 \\ \pi_2 = \frac{1}{4}\pi_5 \\ \pi_3 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{4}\pi_5 \\ \pi_4 = \frac{1}{2}\pi_4 + \frac{1}{8}\pi_5 + \frac{1}{2}\pi_6 \\ \pi_5 = \frac{1}{2}\pi_2 + \frac{1}{4}\pi_5 \\ \pi_6 = \frac{1}{2}\pi_4 + \frac{1}{8}\pi_5 + \frac{1}{2}\pi_6 \\ \pi_7 = \frac{1}{4}\pi_2 + \pi_7 \\ \sum_{i=1}^{7} \pi_i = 1 \end{cases} \Longrightarrow \begin{cases} \pi_1 = 2a \\ \pi_2 = 0 \\ \pi_3 = a \\ \pi_4 = b \\ \pi_5 = 0 \\ \pi_6 = b \\ \pi_7 = 1 - 3a - b \end{cases}$$

$a \in 0, \frac{1}{2}$
$b \in [0,1]$
$3a + 2b \in [0,1]$

# #3

$(X_n)_{n \geq 0}$ MC

$$P = \begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \\ \hline & 1/2 & 1/2 & & & 1 \\ & 3/4 & 1/4 & & & 2 \\ & 1/4 & 1/4 & 1/4 & 1/4 & 3 \\ & & & & 1 & 4 \end{array}$$




1. $\{1,2\}$ recurrent, closed $\{3\}$ transient, not closed
$\{4\}$ recurrent, closed

2. Hitting probability of $\{1,2\}$ when $X_0 = 3$ :
equivalent to the absorption probabilities :

$$\boxed{(V_i)_{i \in I} : \quad V_i = \sum_{j \in \{1,2\}} p_{ij} + \sum_{j \in T} p_{ij} V_j \qquad T = \{3\}}$$

$$V_3 = p_{31} + p_{32} + p_{33} V_3$$

$$V_3 = \frac{p_{31} + p_{32}}{1 - p_{33}} = \frac{1/4 + 1/4}{1 - 1/4} = \frac{2/4}{3/4} = \frac{2}{3}$$

# #3 (#2)

3. $X_0 = 3$, $T_{\{1,2\}} = \inf\{n \geq 1 : X_n = 1 \vee X_n = 2\} \sim ?$

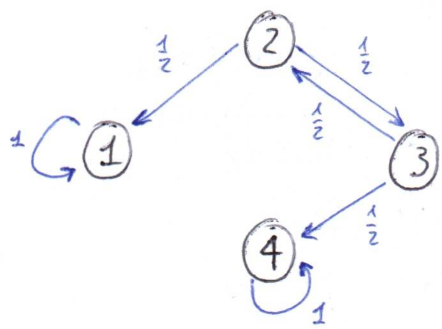
$$\begin{aligned}
\mathbb{P}(T_{\{1,2\}} = n) &= \mathbb{P}(X_n \in \{1,2\}, X_{n-1} \notin \{1,2\}, \dots \mid X_0 = 3) \\
&= \mathbb{P}(X_n = 1, X_{n-1} \in \{3,4\}, \dots \mid X_0 = 3) + \\
&\quad + \mathbb{P}(X_n = 2, X_{n-1} \in \{3,4\}, \dots \mid X_0 = 3) \\
&= \sum_{j \in \{1,2\}} \sum_{i_{n-1}, \dots, i_1 \in \{3,4\}} \mathbb{P}(X_n = j, X_{n-1} = i_{n-1}, \dots \mid X_0 = 3) \\
&= \sum_j \sum_i \frac{\mathbb{P}(X_n = j \mid X_{n-1} = i_{n-1})\, \mathbb{P}(X_{n-1} = i_{n-1} \mid X_{n-2} = i_{n-2}) \dots \mathbb{P}(X_1 = i_1 \mid X_0 = 3)}{\cancel{\mathbb{P}(X_0 = 3)}} \quad {=}3) \\
&= \sum_j \mathbb{P}(X_n = j \mid X_{n-1} = 3)\, \underbrace{\mathbb{P}(X_{n-1} = 3 \mid X_{n-2} = 3) \dots \mathbb{P}(X_1 = 3 \mid X_0 = 3)}_{n-1} \\
&= (p_{31} + p_{32})(p_{33})^{n-1} \\
&= \frac{1}{2} \cdot \left(\frac{1}{4}\right)^{n-1}
\end{aligned}$$

we can't go to 4 since it's a recurrence class ($\{1,2\} \not\subset \{4\}$)

$$\mathbb{P}(T_{\{1,2\}} = +\infty) = 1 - \mathbb{P}(T_{\{1,2\}} < +\infty) = 1 - \sum_{n \geq 1}\left(\frac{1}{2} \cdot \left(\frac{1}{4}\right)^{n-1}\right) = \frac{1}{3} \quad (?)$$

# #4

$$P = \begin{array}{c} \\ \\ \end{array}
\begin{array}{cccc|c}
1 & 2 & 3 & 4 & \\
\hline
1 & & & & 1 \\
1/2 & & 1/2 & & 2 \\
& 1/2 & & 1/2 & 3 \\
& & & 1 & 4
\end{array}$$




Classes $\{1\}$ (R), $\{4\}$ (R), $\{2,3\}$ (T)

**1.** $(V_i)_{i \in I}$: $V_i = \sum_{j \in C} p_{ij} + \sum_{j \in T} p_{ij} V_j$ $\Longrightarrow$ $\begin{cases} V_2 = p_{24} + p_{22} V_2 + p_{23} V_3 \\ V_3 = p_{34} + p_{33} V_3 + p_{32} V_2 \end{cases}$

absorp. in 4 starting from 2 $\qquad C = \{4\}$ $\qquad T = \{2,3\}$

$$\Longrightarrow \begin{cases} V_2 = \frac{1}{2} V_3 \\ V_3 = \frac{1}{2} + \frac{1}{2} V_2 \end{cases} \Longrightarrow V_2 = \frac{1}{2} \quad \left(V_3 = \frac{2}{3}\right)$$

2. Absorption time in $\{1\}$ / $\{4\}$ : $w_i = 1 + \sum_{j \in T} p_{ij} w_j$

$$\begin{cases} w_2 = 1 + p_{22} w_2 + p_{23} w_3 \\ w_3 = 1 + p_{32} w_2 + p_{33} w_3 \end{cases} \Longrightarrow \begin{cases} w_2 = 1 + \frac{1}{2} w_3 \\ w_3 = 1 + \frac{1}{2} w_2 \end{cases} \longrightarrow w_2 = 2 \quad (w_3 = 2)$$

# #5

Queue:
- single counter
- at time $n$ arrive $A_n$ people: $A_n \sim U(\{0,1,2\})$
- at time $n$ $V_n$ people leave: $V_n \sim Bi(1, \frac{3}{7})$ $X_n \geq 1$; $V_n = 0$ $X_n = 0$

If there's a person, this person leaves with prob. = 3/7

- $(A_n)_{n\geq 0}, (V_n)_{n \geq 0}$ $\perp\!\!\!\perp$
- at time $n=0$ the office is empty
- $X_n :=$ # customers at time $n$

1. $E = \{0,1,2,\dots\}$

$$X_{n+1} = \begin{cases} (X_n - V_n) + A_{n+1} & X_n \geq 1 \\ A_{n+1} & X_n = 0 \end{cases}$$

otherwise we can check that:
$\mathbb{P}(i \to i-1) = \mathbb{P}(A_i = 0, V_{i-1} = 1) = \mathbb{P}(A_i = 0)\mathbb{P}(V_{i-1} = 1) = \frac{1}{3}\cdot\frac{3}{7} = \frac{1}{7}$
$\mathbb{P}(i \to i+1) = \mathbb{P}(A_i = 1, V_{i-1} = 0) + \mathbb{P}(A_i = 2, V_{i-1} = 1) \dots$
$\mathbb{P}(i \to i+2) = \dots$
$\mathbb{P}(i \to i) = \dots$

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \mathbb{P}(X_{n+1} = j \mid X_n = 0) + \mathbb{P}(X_{n+1} = j \mid X_n \geq 1)$$

$$\mathbb{P}(X_{n+1} = j \mid X_n = 0) = \mathbb{P}(A_{n+1} = j \mid X_n = 0) = \mathbb{P}(A_{n+1} = j) = \frac{1}{3} \qquad j \in \{0,1,2\}$$

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \mathbb{P}(X_n - V_n + A_{n+1} = j \mid X_n = i)$$
$(i \geq 1)$
$$= \mathbb{P}(A_{n+1} - V_n = j - i \mid X_n = i)$$
$$= \mathbb{P}(A_{n+1} - V_n = j - i)$$

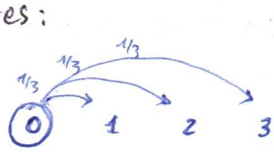
since in $\mathbb{P}(\bullet \mid X_n)$, $\bullet$ is a function only of $A_n$ and $V_n$

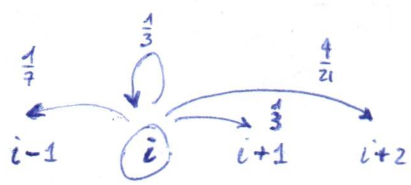
$$= \underbrace{\mathbb{P}(A_{n+1} = j - i \mid V_n = 0)}_{\frac{1}{3}} \underbrace{\mathbb{P}(V_n = 0)}_{\frac{4}{7}} + \underbrace{\mathbb{P}(A_{n+1} = j - i + 1 \mid V_n = 1)}_{\frac{1}{3}} \underbrace{\mathbb{P}(V_n = 1)}_{\frac{3}{7}}$$

$j - i \in \{0,1,2\}$ $\quad$ $1 + j - i \in \{0,1,2\}$
$j \in \{i, 1+i, 2+i\}$ $\quad$ $j \in \{i-1, i, i+1\}$

$$p_{ij} = \begin{cases} \frac{1}{3} & i = 0,\ j \in \{1,2,0\} \\ \frac{1}{7} & i > 0 \quad j = i-1 \\ \frac{4}{21} & i > 0 \quad j = i+2 \\ \frac{1}{3} & i > 0 \quad j = i, i+1 \end{cases}$$

2. States:




The MC is irreducible (all the states communicate).
We consider $T_0 = \inf\{n \geq 1 : X_n = 0\}$ (hitting time of $\{0\}$)

Generally: 
$$\boxed{\begin{aligned} v_i &= P_i(T_0 < +\infty) = \begin{cases} 1 & i \text{ recurrent} \\ <1 & i \text{ transient} \end{cases} \\ v_j &= p_{ji} + \sum_{k \in E \setminus \{i\}} p_{jk} v_k \end{aligned}}$$

$$v_0 = p_{00} + \sum_{k \in E \setminus \{0\}} p_{0k} v_k \implies v_0 = p_{00} + p_{01} v_1 + p_{02} v_2$$

# #5 (#2)

2. $i=1$: $V_1 = p_{10} + p_{11}V_1 + p_{12}V_2 + p_{13}V_3$

$$= \frac{1}{7} + \frac{1}{3}V_1 + \frac{1}{3}V_2 + \frac{4}{21}V_3$$

$i \geq 2$: $V_i = p_{i0} + p_{i(i-1)}V_{i-1} + p_{ii}V_i + p_{i(i+1)}V_{i+1} + p_{i(i+2)}V_{i+2}$

$$= \frac{1}{7}V_{i-1} + \frac{1}{3}V_i + \frac{1}{3}V_{i+1} + \frac{4}{21}V_{i+2}$$

$$\Longrightarrow \frac{4}{21}V_{i+2} + \frac{1}{3}V_{i+1} - \frac{2}{3}V_i + \frac{1}{7}V_{i-1} = 0$$

$$\Longrightarrow \frac{4}{21}x^3 + \frac{1}{3}x^2 - \frac{2}{3}x + \frac{1}{7} = 0$$

$$\Longrightarrow 4x^3 + 7x^2 - 14x + 3 = 0 \longrightarrow \begin{cases} x_1 = 1 \\ x_2 = -3 \\ x_3 = 1/4 \end{cases}$$

$$\Longrightarrow V_i = A + B\left(\frac{1}{4}\right)^i + C(-3)^i$$

Constraints: $0 \leq V_i \leq 1$:

$C = 0$

$0 \leq A + B\left(\frac{1}{4}\right)^i \leq 1 \longrightarrow 0 \leq A \leq 1$

since we want it to be minimal: $A = 0$

$V_i = B\left(\frac{1}{4}\right)^i \quad i \geq 1$

$i=1$: $B\frac{1}{4} = \frac{1}{7} + \frac{1}{3}B\frac{1}{4} + \frac{1}{3}B\frac{1}{16} + \frac{4}{21}B\frac{1}{64} \Longrightarrow B = 1$

$V_i = \left(\frac{1}{4}\right)^i \quad i \geq 1$

$$V_0 = \frac{1}{3} + \frac{1}{3}V_1 + \frac{1}{3}V_2 = \frac{1}{3}\left(1 + \frac{1}{4} + \frac{1}{16}\right) = \frac{7}{16} < 1 \Longrightarrow i \text{ transient}$$

! 3. invariant distribution

$\pi = (\pi_j)_{j\in E}$ invariant $\Longleftrightarrow \pi P = \pi \Longleftrightarrow \pi_j = \sum_{i\geq 0} p_{ij}\pi_i$

$j$ transient $\longrightarrow \lim_{n\to\infty} p_{ij}^{(n)} = 0 \quad \forall i \in E$

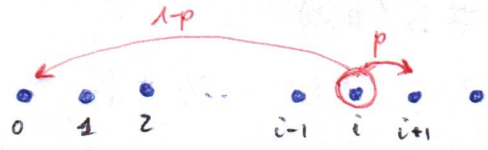
$\boxed{\pi P = \pi \Longleftrightarrow \pi P^n = \pi} \Longleftrightarrow \pi_j = \sum_{i\geq 0} p_{ij}^{(n)}\pi_i$

as $n \to \infty$ $\quad \pi_j \longrightarrow 0$ but $\sum_{j\in E} \pi_j = 1$, contraddiction

$\longrightarrow \nexists (\pi_j)_j$ invariant

# 6

$E = \mathbb{N}$ $\quad p_{ij} = \begin{cases} 1-p & j=0 \\ p & j=i+1 \\ 0 & \text{otherwise} \end{cases}$




1. $i$ and $j$ communicate $\forall i,j$?

$i<j$ : $i,j$ communicate with $(j-i)$ 1-step-ahead jumps

$i>j$ : $i,j$ communicate with a jump to zero and $(j-i)$ 1-step jumps

$\Longrightarrow$ Irreducible aperiodic MC (*)

2. $T_0 = \inf\{n \geq 1 : X_n = 0\}$ $\quad T_0 \sim ?$

$$\mathbb{P}_i(T_0=n) = \mathbb{P}(X_n=0, X_{n-1}\neq 0, \dots, X_1 \neq 0 \mid X_0=i)$$

$$= \frac{\mathbb{P}(X_n=0, X_{n-1}\neq 0, \dots, X_1\neq 0, X_0=i)}{\mathbb{P}(X_0=i)}$$

$$= \mathbb{P}(X_n=0 \mid X_{n-1}\neq 0, \dots)\, \mathbb{P}(X_{n-1}\neq 0 \mid X_{n-2}\neq 0, \dots) \dots \mathbb{P}(X_1\neq 0 \mid X_0=i) \frac{\cancel{\mathbb{P}(X_0=i)}}{\cancel{\mathbb{P}(X_0=i)}}$$

$$= (1-p)\,p^{n-1} \quad \Longrightarrow \quad T_0 \sim \mathcal{G}(1-p) \quad \forall i \geq 0$$

3. Recurrent? Is irreducible $\Rightarrow$ we analyze only 0:

$$\mathbb{P}_0(T_0 < +\infty) = \sum_{n=1}^{\infty} \mathbb{P}_0(T_0=n) = \sum_{n=1}^{\infty}(1-p)p^{n-1} = (1-p)\sum_{n=0}^{\infty} p^n = \frac{(1-p)}{(1-p)} = 1$$

$\Longrightarrow$ 0 is recurrent

$\Longrightarrow$ the MC is recurrent

4. $\pi \sim \mathcal{G}(1-p)$: $\quad \pi = (\pi_1, \pi_2, \dots) \quad : \quad \pi P = \pi$

$\pi = (1-p, (1-p)p, (1-p)p^2, \dots)$

$$\mathbb{E}_0[T_0] = \sum_{n\geq 1} \mathbb{P}_0(T_0=n)\cdot n = \sum_{n\geq 1} n\,p^{n-1}(1-p)$$

$$= \left(\sum_{n\geq 1} n p^{n-1}\right)(1-p) \qquad \sum_n n p^{n-1} = \left(\sum p^n\right)' = \left(\frac{1}{1-p}\right)' = \frac{1}{(1-p)^2} \quad !$$

$$= \frac{1}{(1-p)^2}(1-p) = \frac{1}{1-p} < +\infty \longrightarrow 0 \text{ is positive recurrent}$$

(*) The chain is irreducible $\longrightarrow$ $\exists!$ class of states, $i$ is positive recurrent $\forall i$

$\Longrightarrow$ $\exists!$ invariant distr.

$$\pi_0 = \lim_{n\to\infty} p_{00}^{(n)} = \frac{1}{\mathbb{E}_0[T_0]} = (1-p)$$

$$\pi_j = \sum_{i\in E} \pi_i p_{ij} = \pi_{j-1}p = \dots = \pi_0 p^j = (1-p)p^j$$

$\longrightarrow (1-p, (1-p)p, \dots) \quad (\exists!)$