Stochastic dynamical models, Mathematical Engineering 2019/2020 - SDM 8

Exercise 1.

1) Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. Prove that the process $M_t = N_t - \lambda t$ is a martingale with respect to the natural filtration of $(N_t)_{t\geq 0}$.

2) Let $B = (B_t)_{t\geq 0}$ be a Brownian motion. Prove that $X_t = B_t^2 - t$ is a martingale with respect to the natural filtration of B.

(?) **Exercise 2.** Let $I = \mathbb{N}$ and $(X_n)_{n\geq 0}$ be a Markov chain with state space E and let $P = (p_{jk})_{j,k\in E}$ its transition matrix. Suppose that $\lambda \in (0,1]$ is an eigenvalue of P with eigenvector a bounded function $f :\to \mathbb{R}$, i.e.

$$(Pf)(j) = \sum_{k\in E} p_{jk} f(k) = \lambda f(j), \quad \forall j \in E$$

Prove that the process

$$M_n = \lambda^{-n} f(X_n)$$

is a martingale with respect to the natural filtration $(\mathcal{F}_n)_{n\geq 0}$ of the Markov chain $(X_n)_{n\geq 0}$ ($\mathcal{F}_n = \sigma\{X_j | j \leq n\}$.

Exercise 3. A car washing system does the job in two stages. When a car enters in the system, it first goes through stage 1, then stage 2, and finally it goes away. Service times in both stages are independent (and independent of the car) with exponential distribution with parameters a and b respectively. Potential customers arrive according to a Poisson process with parameter λ (independent of all other exponential random variables) but new cars enter in the system only if there are no other cars (at stage 1 and at stage 2).

1) Write the transition rate matrix Q of a three-state 0,1,2 continuous time Markov chain model.

2) Find the frequency of visit in each state in the stationary regime.

Suppose now that $\lambda = 1/24$ (i.e. the interarrival time of two cars is an exponential random variable with average 24 minutes , to fix the ideas), $a = 1/4$, $b = 1/2$ and denote by $(X_t)_{t\geq 0}$ the three-state Markov chain.

3) Let $f : \{0,1,2\} \to \mathbb{R}$ be the function $f(0) = 24,\ f(1) = 0,\ f(2) = 16$. Show that the stochastic process $(M_t)_{t\geq 0}$ defined by

$$M_t = f(X_t) - \int_0^t (4\,\mathbb{1}_{\{X_s>0\}} - \mathbb{1}_{\{X_s=0\}})ds$$

is a martingale with respect to the natural filtration of $(X_t)_{t\geq 0}$.

4) Find the average service time (stage 1 + stage 2) applying the stopping theorem.

5) Find the matrix P_t of the transition probabilities at time t of the Markov chain model

$$P_t = \begin{bmatrix} \frac{4}{5} + e^{-\frac{3}{8}t} - \frac{4}{5}e^{-\frac{5}{12}t} & \frac{2}{15} - \frac{1}{3}e^{-\frac{3}{8}t} + \frac{1}{5}e^{-\frac{5}{12}t} & \frac{1}{15} - \frac{2}{3}e^{-\frac{3}{8}t} + \frac{3}{5}e^{-\frac{5}{12}t} \\ \frac{4}{5} - 8e^{-\frac{3}{8}t} + \frac{36}{5}e^{-\frac{5}{12}t} & \frac{2}{15} + \frac{8}{3}e^{-\frac{3}{8}t} - \frac{9}{5}e^{-\frac{5}{12}t} & \frac{1}{15} + \frac{16}{3}e^{-\frac{3}{8}t} - \frac{27}{5}e^{-\frac{5}{12}t} \\ \star & \star & \star \end{bmatrix}$$

Exercise 4. A family has three children. When one of them catches a cold he remains infectious, that is he can transmit the disease to another child, for an exponentially distributed random time with mean 6 (days). Luckily, his illness heals in an exponentially distributed random time with mean 3 (days).

1) Construct a Markov chain representing the number N_t of infected children at time t.

2) Classify states and determine invariant probability distributions.

3) Let $f : \{0, 1, 2, 3\} \to \mathbb{R}$ be the function $f(0) = 0,\ f(1) = 4,\ f(2) = 6,\ f(3) = 7$. Show that the stochastic process $(M_t)_{t \geq 0}$ defined by

$$M_t = f(N_t) + \int_0^t \mathbb{1}_{\{N_s > 0\}} ds$$

is a martingale.

4) Applying the martingale stopping theorem, compute the mean time for the extinction of the epidemic starting from $N_0 = 1$.

5) Compute the probability that, starting from $N_0 = 1$, the ill child recovers before infecting other children.

#1 (#8)

1. $(N_t)_{t\geq 0}$ Poisson process with param. λ

$M_t := N_t - t\lambda$, $(M_t)_{t\geq 0}$ martingale with natural filtr. $(N_t)_{t\geq 0}$?

- $\mathbb{E}[|M_t|] < \infty \quad \forall t$
- M_t is $\sigma(N_t)$-meas.
- $\forall s<t$: $\mathbb{E}[M_t \mid \sigma(N_s)] = \mathbb{E}[N_t - t\lambda \mid \sigma(N_s)]$

$$= \mathbb{E}[N_t - N_s \mid \sigma(N_s)] + \mathbb{E}[N_s \mid \sigma(N_s)] - t\lambda$$
$$= \mathbb{E}[N_t - N_s] + N_s - t\lambda$$
$$= (t-s)\lambda + N_s - t\lambda = N_s - \lambda s = M_s$$

2.

$(B_t)_{t\geq 0} = B$ Brownian motion

$X_t := B_t^2 - t$, $(X_t)_{t\geq 0}$ martingale w.r.t. $(B_t)_{t\geq 0}$

- $\mathbb{E}[|B_t^2 - t|] < \infty \quad \forall t$
- X_t is $\sigma(B_t)$-meas.
- $\forall s<t$: $\mathbb{E}[X_t \mid \sigma(B_s)] = \mathbb{E}[B_t^2 - t \mid \sigma(B_s)]$

$$= \mathbb{E}[(B_t - B_s + B_s)^2 - t \mid \sigma(B_s)]$$
$$= \mathbb{E}[(B_t - B_s)^2 + (B_s)^2 + 2B_s(B_t - B_s) - t \mid \sigma(B_s)]$$
$$= \mathbb{E}[(B_t - B_s)^2] + (B_s)^2 + 2B_s\,\mathbb{E}[B_t - B_s] - t$$
$$= (t-s) + B_s^2 - t = B_s^2 - s = X_s$$

#2

- $\mathbb{E}[|M_n|] < +\infty \quad \forall n$
- M_n is $\sigma(X_j \mid j\leq n)$-meas.
- $\mathbb{E}[M_{n+1} \mid \sigma(X_j \mid j \leq n)] \overset{?}{=} M_n$

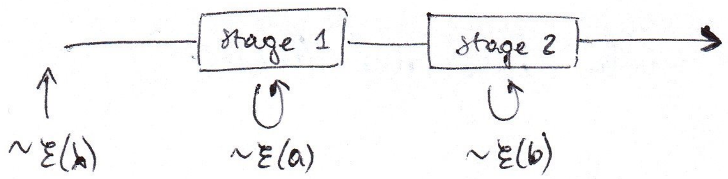
$\mathcal{D} := \sigma(X_j \mid j\leq n)$: if $\mathbb{E}[ZW] = \mathbb{E}[XW] \; \forall W$ $\mathcal{D}$-meas $\Rightarrow Z = \mathbb{E}[X \mid \mathcal{D}]$
Z, X $\mathcal{D}$-meas.

Here: $X = M_{n+1}$, $Z = M_n$, $W = g(X_0, \dots, X_n)$

$$\mathbb{E}[M_{n+1}\, g(X_0,\dots,X_n)] = \mathbb{E}[\lambda^{-(n+1)} f(X_{n+1})\, g(X_0,\dots,X_n)]$$
$$= \sum_{\delta_0,\dots,\delta_{n+1}\in I} \lambda^{-(n+1)} f(\delta_{n+1})\, g(\delta_0,\dots,\delta_n)\, \mathbb{P}(X_0=\delta_0,\dots,X_{n+1}=\delta_{n+1})$$
$$= \sum_{\delta_0,\dots,\delta_n\in I} \lambda^{-(n+1)} \Big(\sum_{\delta_{n+1}\in I} f(\delta_{n+1})\, p_{\delta_n\delta_{n+1}}\Big)\, \mathbb{P}(X_0=\delta_0,\dots,X_n=\delta_n)\, g(\delta_0,\dots,\delta_n)$$
$$= \sum_{\delta_0,\dots,\delta_n\in I} \lambda^{-(n+1)} (\lambda f(\delta_n))\, \mathbb{P}(X_0=\delta_0,\dots,X_n=\delta_n)\, g(\delta_0,\dots,\delta_n) = \mathbb{E}[M_n\, g(X_0,\dots,X_n)]$$

$\Rightarrow M_n = \mathbb{E}[M_{n+1} \mid \sigma(X_j \mid j\leq n)]$

#3



1. • starting from 0: — first time exit from 0 $\sim \mathcal{E}(\lambda)$

$0 \to 1: \quad [q_{00}, q_{01}, q_{02}] = [-\lambda, \lambda, 0]$

• starting from 1: — first time exit from 1 $\sim \mathcal{E}(a)$

$1 \to 2: \quad [q_{10}, q_{11}, q_{12}] = [0, -a, a]$

• starting from 2: — first time exit from 2 $\sim \mathcal{E}(b)$

$2 \to 0: \quad [q_{20}, q_{21}, q_{22}] = [b, 0, -b]$

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -a & a \\ b & 0 & -b \end{bmatrix}$$

2. $\boxed{\text{Frequency of visit at stationarity} \Rightarrow \pi}$

$$0 = \pi Q \implies \begin{cases} \cancel{\lambda}\pi_0 + b\pi_2 = 0 \\ \lambda\pi_0 - a\pi_1 = 0 \\ a\pi_1 - b\pi_2 = 0 \\ \sum \pi_i = 1 \end{cases} \implies \pi = \left(\frac{1}{1 + \frac{\lambda}{a} + \frac{\lambda}{b}}, \frac{\lambda}{a}\pi_0, \frac{\lambda}{b}\pi_0 \right)$$

Suppose: $\lambda = \frac{1}{24}$, $a = \frac{1}{4}$, $b = \frac{1}{2}$

3. $f : \{0,1,2\} \to \mathbb{R}$: $\quad f(0) = 24, \quad f(1) = 0, \quad f(2) = 16$

$$M_t := f(X_t) - \int_0^t 4\mathbb{1}_{\{X_s > 0\}} - \mathbb{1}_{\{X_s = 0\}} \, ds$$

$$Qf\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1/24 & 1/24 & 0 \\ 0 & -1/4 & 1/4 \\ 1/2 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 24 \\ 0 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix} = 4\mathbb{1}_{\{X_s > 0\}} - \mathbb{1}_{\{X_s = 0\}}$$

$\implies M_t = f(X_t) - \int_0^t (Qf)(X_s)\, ds$ is a Martingale because of the thm.

4. Mean service time: starting from 1 to 0: $\mathbb{E}_1[T_0]$ $\quad T_0 = \inf(t > 0 : X_t = 0)$

$$\boxed{\mathbb{E}_1[M_0] = \mathbb{E}_1[M_{T \wedge t}] = \mathbb{E}_1[M_t] \quad \forall t \quad \text{(stopping theorem)}}$$

$$\mathbb{E}_1[M_0] = \mathbb{E}_1\left[f(X_0) - \int_0^0 .. \, ds\right] = \mathbb{E}_1[f(X_0)] = \mathbb{E}_1[f(1)] = 0$$

$$\mathbb{E}_1[M_{T \wedge t}] = \mathbb{E}_1\left[f(X_{T \wedge t}) - \int_0^{T \wedge t} 4\mathbb{1}_{\{X_s > 0\}} - \mathbb{1}_{\{X_s = 0\}} \, ds\right] =$$

$$= \mathbb{E}_1\left[f(X_{T \wedge t}) - \int_0^{T \wedge t} 4\mathbb{1}_{\{X_s > 0\}} \, ds\right]$$

#3 (#8)

4. Monotone convergence thm. : $\mathbb{E}_1[t\wedge T] \xrightarrow{t\to\infty} \mathbb{E}[T]$

Fatou's lemma : $\mathbb{E}_1[M_{t\wedge T}] \xrightarrow{t\to\infty} \mathbb{E}[M_T]$ if $\mathbb{P}(T<\infty)=1$

$\Longrightarrow$ $X_{T\wedge t} \xrightarrow{t\to\infty} X_T = 0$, $T\wedge t \to T$

$\longrightarrow$ by Lebesgue's thm : $\mathbb{E}_1[f(X_{T\wedge t})] \xrightarrow{t\to\infty} \mathbb{E}_1[f(X_T)]$

$\Longrightarrow$ $0 = \mathbb{E}_1\left[f(0) - \int_0^T 4\mathbb{1}_{\{X_s>0\}}\,ds\right] = \mathbb{E}_1[24-4T] \Longrightarrow \mathbb{E}_1[T_0] = 6$

$\mathbb{E}_1\left[f(X_T) - \int_0^T 4\mathbb{1}_{\{X_s>0\}}\,ds\right]$

!

5. $p_{20}(t)$? $p_{21}(t)$? $p_{22}(t)$?

FKE : $p_{ij}'(t) = \sum_{k\in E} p_{ik}(t)\, q_{kj}$

BKE : $p_{ij}'(t) = \sum_{k\in E} q_{ik}\, p_{kj}(t)$

- BKE : $p_{20}'(t) = q_{20}p_{00}(t) + q_{21}p_{10}(t) + q_{22}p_{20}(t)$
  $\overset{!}{=} \frac{2}{5} + \frac{1}{2}e^{-\frac{3}{8}t} - \frac{2}{5}e^{-\frac{5}{12}t} - \frac{1}{2}p_{20}(t)$

Homogeneous : $p_{20}(t) = ke^{-\frac{1}{2}t}$

Complete : $p_{20}(t) = ke^{-\frac{1}{2}t} + Ae^{-\frac{3}{8}t} + Be^{-\frac{5}{12}t} + C$

Conditions :

$p_{20}'(t) = \frac{2}{5} + \frac{1}{2}e^{-\frac{3}{8}t} - \frac{2}{5}e^{-\frac{5}{12}t} - \frac{1}{2}p_{20}(t)$, $p_{20}(0) = 0$

$\Rightarrow -\frac{1}{2}ke^{-\frac{1}{2}t} - \frac{3}{8}Ae^{-\frac{3}{8}t} - \frac{5}{12}Be^{-\frac{5}{12}t} = \frac{2}{5} + \frac{1}{2}e^{-3/8t} - \frac{2}{3}e^{-5/12t} - \frac{1}{2}Ke^{-1/2t} - \frac{1}{2}Ae^{-3/8t} - \frac{1}{2}Be^{-5/12t} - \frac{1}{2}C$

$$\left.\begin{aligned} &\tfrac{2}{5} - \tfrac{1}{2}C = 0 \\ &-\tfrac{3}{8}A = \tfrac{1}{2} - \tfrac{1}{2}A \\ &-\tfrac{5}{12}B = -\tfrac{2}{3} - \tfrac{1}{2}B \\ &k + 4 - \tfrac{24}{5} + \tfrac{4}{5} = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 4 \\ B &= -24/5 \\ C &= 4/5 \\ k &= 0 \end{aligned}$$

$$\boxed{p_{20}(t) = \frac{4}{5} + 4e^{-\frac{3}{8}t} - \frac{24}{5}e^{-\frac{5}{12}t}}$$

- BKE : $p_{21}'(t) = q_{20}p_{01}(t) + q_{21}p_{11}(t) + q_{22}p_{21}(t)$
  $\overset{!}{=} \frac{1}{2}p_{01}(t) - \frac{1}{2}p_{21}(t)$
  $\overset{!}{=} \frac{1}{15} - \frac{1}{6}e^{-3/8t} + \frac{1}{10}e^{-5/12t} - \frac{1}{2}p_{21}(t)$

$p_{21}(t) = ke^{-\frac{1}{2}t} + Ae^{-\frac{3}{8}t} + Be^{-\frac{5}{12}t} + C \Rightarrow \cdots \Rightarrow$ $A = -4/3$, $B = 6/5$, $C = 2/15$, $k = 0$

$$\boxed{p_{21}(t) = \frac{2}{15} - \frac{4}{3}e^{-\frac{3}{8}t} + \frac{18}{5}e^{-\frac{5}{12}t}}$$

$$p_{22}(t) = 1 - p_{20}(t) - p_{21}(t) \quad = \frac{1}{15} - \frac{8}{3}e^{-\frac{3}{8}t} + \frac{18}{5}e^{-\frac{5}{12}t}$$

#4

transmission $\sim \mathcal{E}(\frac{1}{6})$
healing $\sim \mathcal{E}(\frac{1}{3})$

1. starting from 0: $[q_{00}, q_{01}, q_{02}, q_{03}] = [0,0,0,0]$
starting from 1: $1 \to 0,2 \quad = [\frac{1}{3}, -\frac{1}{2}, \frac{1}{6}, 0]$
2: $2 \to 3,1 \quad = [0, \frac{2}{3}, -1, \frac{1}{3}]$
3: $3 \to 2 \quad = [0,0,1,-1]$

not only one of the two can heal but also one of the two can transmiss

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/3 & -1/2 & 1/6 & 0 \\ 0 & 2/3 & -1 & 1/3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad (N_t)_{t\geq 0} \qquad N_t = \#\text{infected at time } t$$

2. Discrete skeleton:

$$p_{ij} = \begin{cases} \frac{q_{ij}}{-q_{ii}} & q_{ii} \neq 0,\ i \neq j \\ 0 & (q_{ii}=0,\ i\neq j) \vee (q_{ii}\neq 0,\ i=j) \\ 1 & q_{ii}=0,\ i=j \end{cases} \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\{0\}$ recurrent absorbing
$\{1,2,3\}$ transient

3. $f : \{0,1,2,3\} \to \mathbb{R}$: $\quad f\left(\begin{bmatrix}0\\1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}0\\4\\2\\7\end{bmatrix}$

$$(Qf)\begin{bmatrix}0\\1\\2\\3\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/3 & -1/2 & 1/6 & 0 \\ 0 & 2/3 & -1 & 1/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix}0\\4\\6\\7\end{bmatrix} = \begin{bmatrix}0\\-1\\-1\\-1\end{bmatrix} = \mathbb{1}_{\{N_s>0\}}$$

$$\Longrightarrow \quad M_t = f(N_t) - \int_0^t (Qf)(N_s)\,ds \quad \text{is a Martingale (thm.)}$$

4. $\mathbb{E}_1[T_0]$? $\quad T_0 = \inf\{t\geq 0 : N_t = 0\}$

Stopping theorem: $\mathbb{E}_1[M_T] = \mathbb{E}_1[M_0] = \mathbb{E}_1[M_{T\wedge t}]$		(1)
$T\wedge t \xrightarrow[t\to\infty]{} T$ $\quad\Longrightarrow \mathbb{E}_1[T\wedge t] = \mathbb{E}_1[T]$	monotone conv. thm.	(2)
$\Longrightarrow \mathbb{E}_1[f(N_{T\wedge t})] \to \mathbb{E}_1[f(N_T)]$	Lebesgue's thm.	(3)

$(1) \Rightarrow \mathbb{E}_1[M_0] = \mathbb{E}[f(N_0)] = \mathbb{E}[f(1)] = 4$

$$4 = \mathbb{E}_1[M_0] = \mathbb{E}_1[M_T] = \mathbb{E}_1\left[f(0) + \int_0^T \mathbb{1}_{\{X_s>0\}}\,ds\right] = \mathbb{E}_1[+T] = \mathbb{E}_1[T]$$

$$\Longrightarrow \quad \mathbb{E}_1[T] = 4$$

#4 (#8)

5. healing $\sim \mathcal{E}\left(\frac{1}{3}\right)$ $\quad \mathbb{P}_1(T_{healing} < T_{transm.})$

transmission $\sim \mathcal{E}\left(\frac{1}{6}\right)$

$$\min(\text{Healing}, \text{Trans}) \sim \mathcal{E}\left(\frac{1}{2}\right)$$

$$\mathbb{P}(\min(\text{Heal}, \text{Trans}) = \text{Heal}) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

$$= p_{10} = \frac{q_{10}}{-q_{11}} = \frac{1/3}{1/2} = \frac{2}{3}$$

$$= \underbrace{\mathbb{P}(N_{T_i} = j)}_{\text{probability of going to } j \text{ after leaving } i} = \frac{q_{ij}}{-q_{ii}}$$