ESSENTIAL SUPREMUM

The essential supremum of a function is the smallest value that is larger or equal than the function values everywhere when allowing for ignoring what the function does at a set of points of measure zero.

Eg.

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\sup_{\mathbb{R}} f(x) = 1$$

$$\operatorname{ess\,sup}_{\mathbb{R}} f(x) = 0$$

because we are allowed to ignore what the function does at $x=0$.

ESSENTIALLY BOUNDED FUNCTION

- f essentially bounded if :

$$\operatorname{ess\,sup}_{X} f < \infty$$

- f essentially bounded if :

$$\exists M \geq 0 \; : \; \mu(\{x \in X : |f(x)| > M\}) = 0$$

- f essentially bounded if :

$$f = g \quad \text{a.e.} \qquad g \text{ bounded function}$$

$$\mathcal{L}^{\infty}(X, \mathcal{A}, \mu) := \{ f : X \to \overline{\mathbb{R}} : f \in \mathcal{M}(X, \mathcal{A}), \; f \text{ essentially bounded} \}$$

$$\|f\|_{\infty} := \operatorname{ess\,sup}_{X} |f|$$

$$= \inf \{ M \geq 0 : \mu(\{x \in X : f(x) > M\}) = 0 \}$$

$f : X \to \mathbb{R} \iff$

$f : (X, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

measurable if :

$$f^{-1}(E) \in \mathcal{A} \qquad \forall E \in \mathcal{B}(\mathbb{R})$$

DCT in L^1

$\{f_n\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$

$f \in \mathcal{M}_+(X, \mathcal{A})$ s.t. $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X.

If $\exists g \in L^1(X, \mathcal{A}, \mu)$ s.t. $|f_n| \leq g \quad \forall n \in \mathbb{N}$

$$\Longrightarrow \begin{cases} 1. & f_n, f \in L^1(X, \mathcal{A}, \mu) \quad \forall n \in \mathbb{N} \\ 2. & \int_X |f_n - f| \, d\mu \xrightarrow[n\to\infty]{} 0 \\ 3. & \int_X f_n \, d\mu \xrightarrow[n\to\infty]{} \int_X f \, d\mu \end{cases}$$

DCT in L^p

$\{f_n\}_n \subseteq \mathcal{M}_+(X, \mathcal{A})$

$f \in \mathcal{M}_+(X, \mathcal{A})$ s.t. $f_n \xrightarrow[n\to\infty]{} f$ a.e. in X.

If $\exists g \in L^p(X, \mathcal{A}, \mu)$ s.t. $|f_n| \leq g \quad \forall n \in \mathbb{N}$

or if $\exists g \in L^1(X, \mathcal{A}, \mu)$ s.t. $|f_n|^p \leq g \quad \forall n \in \mathbb{N}$

$$\Longrightarrow \int_X |f_n - f|^p \, d\mu \xrightarrow[n\to\infty]{} 0$$

$$\left(\Longleftrightarrow \|f_n - f\|_{L^p} \xrightarrow[n\to\infty]{} 0 \Longleftrightarrow f_n \xrightarrow[n\to\infty]{} f \text{ in } L^p\right)$$