Plain gradient based direct frequency estimation using second-order constrained adaptive IIR notch filter

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An investigation of the problem of direct frequency estimation with a constrained second-order adaptive IIR notch filter using the plain gradient algorithm is reported. The closed form expressions for the bias and mean square error of the proposed algorithm are derived. Simulations show that the proposed algorithm yields a faster convergence speed than the conventional indirect frequency estimation algorithms.

Introduction: Let $d(n)$ be a measurable signal of the form

$$d(n) = A\cos(\omega_0 n + \theta) + v(n) \tag{1}$$

where $\{A \neq 0, \omega_0, \theta\}$ are the amplitude, (angular) frequency and phase parameter of the sinusoid and $v(n)$ is a statistically independent additive broadband noise. To estimate the frequency ω_0 with the only available signal $d(n)$, the following second-order constrained adaptive IIR notch filter can be used (see, e.g. [1, 2]):

$$H(z^{-1}) = \frac{1 + az^{-1} + z^{-2}}{1 + \rho a z^{-1} + \rho^2 z^{-2}} \tag{2}$$

where $0 < \rho < 1$ is a constant close to 1, controlling the notch bandwidth. It is easy to see that, when $a = -2\cos\omega_0 \equiv a_0$, $\hat{v}(n)$, the output of $H(z^{-1})$ excited with $d(n)$, is almost equal to $v(n)$. The unknown frequency ω_0 can be estimated using online processing. It is well known that the PG algorithm is popularly used because of its simplicity and its well-balanced complexity and performance. The following PG algorithm can be used to estimate a_0:

$$a(n+1) = a(n) - \mu\hat{v}(n)s_a(n) \tag{3}$$

where $s_a(n)$ is the gradient of $\hat{v}(n)$ with respect to a at $a = a(n)$. It is in fact the output of the gradient transfer function, denoted as $H_a(z^{-1})$, excited by $d(n)$. In [2], $H_a(z^{-1})$ is approximated by:

$$H_a(z^{-1}) \simeq (1-\rho)z^{-1}\frac{1 - \rho z^{-2}}{1 + \rho a z^{-1} + \rho^2 z^{-2}} \tag{4}$$

and the closed form expressions for estimation bias and MSE of the adaptive IIR notch filter were developed. It should be pointed out that this is an indirect frequency estimation. In many applications, the frequency, denoted as $\hat{\omega}_0(n)$, is needed and hence has to be computed from $a(n) = -2\cos\hat{\omega}_0(n)$ at each iteration if (3) is used. This conversion is computationally costly. Furthermore, it is possible to have such a situation that $|a(n)| > 2$, and hence no frequency $\hat{\omega}_0(n)$ can be found. More seriously, the filter may become unstable [1]. In [3], a lattice based algorithm was proposed, where the adaptive parameter $k(n)$ is defined as $k(n) = -\cos\hat{\omega}_0(n)$. This is, however, still an indirect frequency estimation and the stability problem remains.

Proposed algorithm and performance analysis: We propose a direct frequency estimation algorithm. The basic idea is to replace a in (2) with $-2\cos\hat{\omega}_0$. In that case, the notch filter is parameterised with the frequency parameter $\hat{\omega}_0$ rather than a:

$$H(z^{-1}) = \frac{1 - 2\cos\hat{\omega}_0 z^{-1} + z^{-2}}{1 - 2\rho\cos\hat{\omega}_0 z^{-1} + \rho^2 z^{-2}} \equiv H(z^{-1}, \hat{\omega}_0) \tag{5}$$

It is easy to see that the true frequency ω_0 can be estimated with the following PG algorithm:

$$\hat{\omega}_0(n+1) = \hat{\omega}_0(n) - \mu\hat{v}(n)s_\omega(n) \tag{6}$$

where $s_\omega(n)$ is the gradient of $\hat{v}(n)$ with respect to $\hat{\omega}_0$ at $\hat{\omega}_0 = \hat{\omega}_0(n)$, which is the output of the corresponding gradient transfer function, denoted as $H_\omega(z^{-1})$. Using the same idea as in [2], $H_\omega(z^{-1})$ can be approximated as:

$$H_\omega(z^{-1}) \simeq 2\sin\hat{\omega}_0\left[z^{-1} - \rho z^{-1}H(z^{-1}, \hat{\omega}_0)\right] \tag{7}$$

Let us now study the performance of the proposed adaptive direct frequency estimation algorithm governed by (5)–(7). Note $H(e^{-j\omega}, \hat{\omega}_0)$ is a function of $\hat{\omega}_0$ (for a given ω) and $H(e^{-j\omega}, \omega) = 0$. Without giving the details, we mention that, using the Taylor series expansion, $H(e^{-j\omega}, \hat{\omega}_0)$ in the vicinity of $\hat{\omega}_0 = \omega$ can be approximated as:

$$H(e^{-j\omega}, \hat{\omega}_0) \simeq [B_1(\omega)\delta(\omega) + B_2(\omega)\delta^2(\omega)]e^{-j\phi(\omega)} + B_3(\omega)\delta^2(\omega)e^{-j2\phi(\omega)} \tag{8}$$

where $\delta(\omega) \equiv \hat{\omega}_0 - \omega$, and

$$\begin{aligned} B_1(\omega) &\equiv \frac{2\sin\omega}{(1-\rho)\sqrt{(1+\rho)^2 - 4\rho\cos^2\omega}} \\ B_2(\omega) &\equiv \frac{\cos\omega}{2\sin\omega}B_1(\omega), \quad B_3(\omega) \equiv -\rho B_1^2(\omega) \\ \phi(\omega) &\equiv \begin{cases} \tan^{-1}\dfrac{(1+\rho)\sin\omega}{(1-\rho)\cos\omega}, & \omega \le \dfrac{\pi}{2} \\ \pi + \tan^{-1}\dfrac{(1+\rho)\sin\omega}{(1-\rho)\cos\omega}, & \omega > \dfrac{\pi}{2} \end{cases} \end{aligned} \tag{9}$$

Using the same approach as that used in [2], the output of the notch filter $\hat{v}(n)$ can be expressed as:

$$\begin{aligned} \hat{v}(n) = {}& A[B_1(\omega_0)\delta(\omega_0) + B_2(\omega_0)\delta^2(\omega_0)]\cos[\psi(n) - \phi] \\ &+ AB_3(\omega_0)\delta^2(\omega_0)\cos[\psi(n) - 2\phi] + v_1(n) \end{aligned} \tag{10}$$

where $\psi(n) \equiv \omega_0 n + \theta$, $\phi = \phi(\omega_0)$ and $v_1(n)$ is the output of the notch filter excited with $v(n)$. Similarly, the gradient signal can be expressed as:

$$\begin{aligned} s_\omega(n) = {}& \{-2\rho\sin\omega_0 AB_1(\omega_0)\cos[\psi^*(n) - \phi] \\ &+ 2A\cos\omega_0\cos[\psi^*(n)]\}\delta(\omega_0) + \rho A\cos[\psi^*(n) - \phi] \\ &\times \{-\cos\omega_0 B_1(\omega_0) - 2B_1(\omega_0)\cos\omega_0\}\delta^2(\omega_0) \\ &+ \{-A\sin\omega_0\cos[\psi^*(n)] \\ &\quad -2\rho A\sin\omega_0 B_3(\omega_0)\cos[\psi^*(n) - 2\phi]\}\delta^2(\omega_0) \\ &\quad + 2\sin\omega_0 A\cos[\psi^*(n)] + v_2(n) \end{aligned} \tag{11}$$

where $\psi^*(n) \equiv \psi(n) - \omega_0$ and $v_2(n)$ is the output of the gradient filter $H_\omega(z^{-1})$ excited by $v(n)$.

It follows from (6) that

$$\delta_\omega(n+1) = \delta_\omega(n) - \mu\hat{v}(n)s_\omega(n) \tag{12}$$

where $\delta_\omega(n)$ is actually equal to $\delta(\omega_0)$ with $\hat{\omega}_0$ replaced with the frequency $\hat{\omega}_0(n)$ updated with the PG (6), and $\hat{v}(n)$, $s_\omega(n)$ are given by (10) and (11), respectively, but with $\delta(\omega_0)$ replaced by $\delta_\omega(n)$.

It can be shown without giving the details that the instantaneous bias and MSE can be computed by

$$\begin{aligned} E[\delta_\omega(n+1)] &= (1 - \mu\xi_{11})E[\delta_\omega(n)] + \mu\xi_{12}E[\delta_\omega^2(n)] - \mu R_{1,2} \\ E[\delta_\omega^2(n+1)] &= (1 - 2\mu\xi_{11} + \mu^2\xi_{22})E[\delta_\omega^2(n)] \\ &\quad - (2\mu R_{1,2} + \mu^2\xi_{21})E[\delta_\omega(n)] + \mu^2\xi_{23} \end{aligned} \tag{13}$$

where

$$\begin{aligned} \sigma_1^2 &= \frac{\sigma_v^2}{\rho^2} - \frac{1-\rho}{1+\rho}\frac{(1+\rho^2)(1+\rho)^2 - 8\rho^2\cos^2\omega_0}{\rho^2(\rho^4 - 2\rho^2\cos 2\omega_0 + 1)}\sigma_v^2 \\ \sigma_2^2 &= 4\sigma_v^2\left[\frac{(1-\rho)^3(1+\rho^2)}{(1+\rho)(\rho^4 - 2\rho^2\cos 2\omega_0 + 1)} + \frac{2\rho(1-\rho)}{1+\rho}\right]\sin^2\omega_0 \\ R_{1,2} &= 2\sigma_v^2\left[-\frac{\rho(1-\rho)\sin 2\omega_0}{1+\rho} - \frac{1}{1+\rho}\frac{(1-\rho)^3\sin 2\omega_0}{\rho^4 - 2\rho^2\cos 2\omega_0 + 1}\right] \end{aligned} \tag{14}$$

with $\sigma_v^2 = E[v^2(n)]$ and

$$\begin{aligned}
\xi_{11} &= A^2 B_1 \sin\omega_0 \cos(\omega_0 - \phi) \\
\xi_{12} &= -0.5\xi_{11} \tan\omega_0 - A^2 B_1 \cos(\omega_0 - \phi)\left(\cos\omega_0 - 2\rho B_1 \sin\omega_0 \cos\phi\right) \\
\xi_{21} &= 4A^2 \sin\omega_0\left[\rho\sigma_1^2 B_1 \sin\omega_0 \cos\phi - \cos\omega_0\sigma_1^2 - B_1 \cos\left(\omega_0 - \phi\right)R_{1,2}\right] \\
\xi_{22} &= 2A^2\sigma_1^2(\cos 2\omega_0 + \rho^2 B_1^2 \sin^2\omega_0) \\
&\quad + A^4 B_1^2 \sin^2\omega_0\left[0.5 + \cos^2\left(\omega_0 - \phi\right)\right] \\
&\quad - 2\rho A^2\sigma_1^2\left[B_1 \sin 2\omega_0 \cos\phi + 2\sin^2\omega_0\left(B_2\cos\phi + B_3\cos 2\phi\right)\right] \\
&\quad + 4A^2 \sin\omega_0 R_{1,2}\left[B_2\cos\left(\omega_0 - \phi\right) + B_3\cos\left(\omega_0 - 2\phi\right)\right] \\
&\quad + 2A^2 B_1 R_{1,2}\left[2\cos\omega_0\cos\left(\omega_0 - \phi\right) - \rho B_1 \sin 2\omega_0\right] + 0.5A^2 B_1^2\sigma_2^2 \\
\xi_{23} &= 2A^2\sigma_1^2 \sin^2\omega_0 + \sigma_1^2\sigma_2^2 + 2R_{1,2}^2
\end{aligned} \tag{15}$$

with $B_1 = B_1(\omega_0)$.

Examples: We compare the performance of the two indirect frequency estimation algorithms, i.e. (3) and the lattice based one proposed in [3], with the one proposed in this Letter.

In the first example, $d(n)$ is generated from (1), $\omega_0 = 1.1$, $A = 2$ and $\sigma_v^2 = 0.2$. We run the two algorithms with the same $\hat{\omega}_0(0) = 1.416$ for (6) (equivalently, $a(0) = -0.3084$ for (3), and $k(0) = -0.1542$ for the lattice one), $\mu = 0.0002$ and $\rho = 0.95$. Fig. 1 shows a typical evolution of the estimated frequency $\hat{\omega}_0(n)$ for each algorithm. Clearly, our proposed algorithm has a much faster convergence speed than the other two.

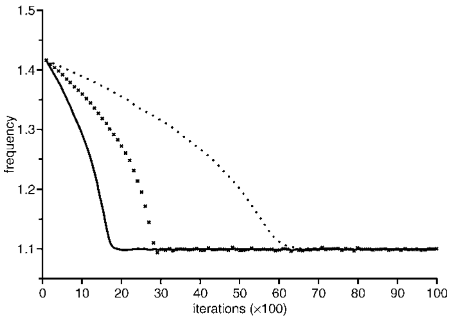

Fig. 1 *Evolution of estimated frequency*

— proposed direct frequency estimation algorithm, $\hat{\omega}_0(n)$
• indirect frequency estimation algorithm using (3), $\cos^{-1}[-a(n)/2]$
× lattice based indirect frequency estimation algorithm, $\cos^{-1}[-k(n)]$

In the second example, the conditions are exactly the same as the first example except (i) $\rho = 0.85$, and (ii) 50 sets of data are generated for $d(n)$. Fig. 2 shows the theoretical MSE of the direct frequency estimation (6) obtained using (13), the equivalent frequency MSE obtained by converting the theoretical MSE of the indirect frequency estimation (3), and the simulated MSE of the algorithms (3) and (6), with 50 independent runs. We observe that, for the proposed algorithm, the theoretical MSE is close to the simulated one, and by comparing the two theoretical MSE curves, one can see that the proposed algorithm converges faster than the indirect algorithm.

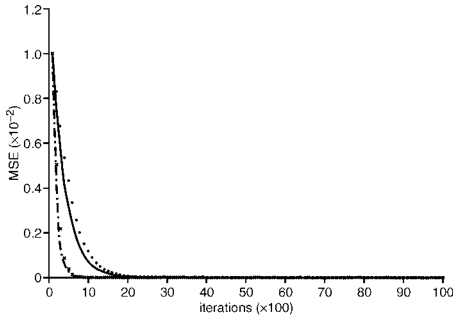

Fig. 2 *Comparison of MSE between algorithms (3) and (6)*

–·–· theoretical MSE from (13)
—— equivalent frequency theoretical MSE given in [2]
× × simulated MSE of algorithm (6)
• • • simulated MSE of algorithm (3)

Conclusions: We have proposed an adaptive algorithm for direct frequency estimation. The advantages of this algorithm include avoiding frequency conversion that exists in the conventional algorithms and good stability properties. The theoretical MSE is confirmed with simulations which also show that the proposed algorithm usually yields a faster convergence speed than the traditional ones. This is particularly true when the frequency to be estimated is large.

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