Adaptive algorithm for direct frequency estimation

H.C. So and P.C. Ching

Abstract: Based on the linear prediction property of sinusoidal signals, a new adaptive method is proposed for frequency estimation of a real tone in white noise. Using the least mean square approach, the estimator is computationally efficient and it provides unbiased and direct frequency measurements on a sample-by-sample basis. Convergence behaviour of the estimated frequency is analysed and its variance in white Gaussian noise is derived. Computer simulations are included to corroborate the theoretical analysis and to show its comparative performance with two adaptive frequency estimators in non-stationary environments.

1 Introduction

Estimating the frequency of sinusoidal signals in noise has applications in many areas [1–3] such as carrier and clock synchronisation, angle of arrival estimation, demodulation of frequency-shift keying (FSK) signals, and Doppler estimation of radar and sonar wave returns. In this work, we consider single real tone frequency estimation in white noise. The discrete-time noisy sinusoid is modelled as

$$x_n = \alpha \cos(\omega n + \phi) + q_n \stackrel{\Delta}{=} s_n + q_n \tag{1}$$

where the noise q_n is assumed to be a white zero-mean random process with unknown noise power σ_q^2 while α, ω and $\phi \in [0, 2\pi)$ which represent the tone amplitude, frequency and phase of the sinusoid, respectively, are unknown. Without loss of generality, the sampling period is assigned to be 1 s. The task here is to find $\omega \in (0, \pi)$ from x_n.

If the sinusoidal parameters are constant in time, classical batch techniques [2, 3] include maximum likelihood estimation [4] and eigenanalysis algorithms such as Pisarenko's harmonic retrieval method [5] and MUSIC [6], can be employed to achieve accurate frequency estimation. On the other hand, when the environment is non-stationary, such as the frequency is an abruptly changing function of time and the amplitude/phase is time-varying, tracking of ω is necessary. Griffiths [7] was the first to formulate the adaptive frequency estimation problem and Thompson [8] was the first to propose a constrained least mean square (LMS) algorithm [9] to obtain an unbiased estimate of the sinusoidal frequency, which can be considered as an online implementation of Pisarenko's method. The key idea of the non-stationary frequency estimation method suggested by Etter and Hush [10] is to maximise the mean square difference between x_n and its delayed version using an adaptive time delay estimator (ATDE) [11] and the frequency estimate is given by π over the estimated delay. Since the delay of the ATDE is restricted to be an integral multiple of the sampling interval, the algorithm cannot give accurate frequency estimation, particularly for large ω. An improvement to [10] was made by providing fractional sample delays in the ATDE with the use of Lagrange interpolation [12]. However, the frequency estimate of the modified method is still biased because the Lagrange interpolator cannot perfectly model subsample delays for sinusoidal signals. Generally, finite length fractional delay filters are never ideal for non-integer delays [13, 14]. Other recent adaptive frequency estimators include constrained pole–zero notch filtering [15], Pisarenko's method combined with Kamen's pole factorisation [16] and adaptive IIR-BPF [17], which is an LMS-style linear prediction algorithm with an IIR band-pass filter for noise reduction.

In this paper, a new adaptive frequency estimator in white noise is proposed [18] based on linear prediction of sinusoidal signals. The main scientific advance of the work is to minimize the mean square value of a modified linear prediction error function, which is characterised by the estimated frequency only and its minimum exactly corresponds to the sinusoidal frequency. As a result, direct and unbiased frequency estimation is achieved. The proposed approach can be considered as a specific application of the unbiased impulse response estimation algorithm [19, 20] derived from minimising the mean square value of the equation error under a constant norm constraint. Starting from the property that a pure sinusoid is predictable from its past two sampled values, the modified linear prediction error function is developed. The LMS algorithm is then applied to minimise the cost function and the frequency estimate is updated explicitly on a sample-by-sample basis. Performance measures of the adaptive estimator, namely convergence behaviour and variance of the estimated frequency, are also analysed. It is noteworthy that the proposed frequency estimation framework has been extended to least squares type realisations, which provide higher estimation accuracy at the expense of larger computational requirement, and interested readers are referred to [21–24]. Simulation results are presented to corroborate the theoretical analyses and to illustrate the superiority of the proposed frequency estimation algorithm over the adaptive Pisarenko's algorithm [8] and adaptive IIR-BPF [17].

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2 Direct frequency estimator (DFE)

It is easy to verify that s_n obeys the following simple recurrence relation [25]:

$$s_n = 2\cos(\omega)s_{n-1} - s_{n-2} \tag{2}$$

With the measurement $\{x_n\}$, we can predict s_n using

$$\hat{s}_n = 2\cos(\hat{\omega})x_{n-1} - x_{n-2} \tag{3}$$

where $\hat{\omega}$ represents an estimate of ω. Defining the linear prediction error function as

$$e_n \triangleq x_n - \hat{s}_n \tag{4}$$

It can be shown that the mean square error function $E\{e_n^2\}$ is

$$E\{e_n^2\} = 4[\cos(\hat{\omega}) - \cos(\omega)]^2\sigma_s^2 + 2[2+\cos(2\hat{\omega})]\sigma_q^2 \tag{5}$$

where $\sigma_s^2 = \alpha^2/2$ denotes the tone power. Apparently, minimising $E\{e_n^2\}$ with respect to $\hat{\omega}$ will not give the desired solution because of the noise component. When the value of σ_q^2 is available, then unbiased frequency estimation [26, 27] can be attained with the use of $E\{e_n^2\}$. On the other hand, without knowing the noise power, unbiased frequency estimates can still be obtained via minimising (5) subject to the constraint $2+\cos(2\hat{\omega})$ is a constant. This constrained optimisation problem is in fact equivalent to the unconstrained minimisation of a scaled version of $E\{e_n^2\}$ [20], denoted by $E\{\zeta_n^2\}$, which has the form

$$E\{\zeta_n^2\} = \frac{E\{e_n^2\}}{2[2+\cos(2\hat{\omega})]} = \frac{2[\cos(\hat{\omega}) - \cos(\omega)]^2\sigma_s^2}{2+\cos(2\hat{\omega})} + \sigma_q^2 \tag{6}$$

It is worth noting that (6) can be considered as an alternative form of the modified mean square error suggested in [28] but there was no theoretical analysis of their frequency estimator. The advantages of using (6) are that we can obtain direct frequency measurements and derive the estimator performance in a simpler way. Investigating the first and second derivatives of (6) shows that for $\omega \in (0, \pi)$, the performance surface $E\{\zeta_n^2\}$ has a unique minimum at $\hat{\omega} = \omega$ with the value of σ_q^2, but it also has a maximum when $\hat{\omega} < \pi/3$ or $\hat{\omega} > 2\pi/3$. This suggests that minimisation of $E\{\zeta_n^2\}$ can be achieved via gradient search methods if the initial value of $\hat{\omega}$ is chosen between $\pi/3$ and $2\pi/3$. In this work, the computationally attractive LMS algorithm is utilised to estimate ω iteratively. From (6), the instantaneous value of $E\{\zeta_n^2\}$, ζ_n^2, is

$$\zeta_n^2 = \frac{e_n^2}{2[2+\cos(2\hat{\omega}_n)]} \tag{7}$$

where $\hat{\omega}_n$ denotes the estimate of ω at time n. Note that ζ_n^2 is in fact an estimate of σ_q^2 as $\hat{\omega} \to \omega$. The stochastic gradient estimate is computed by differentiating ζ_n^2 with respect to $\hat{\omega}_n$ and is given by

$$\frac{\partial \zeta_n^2}{\partial \hat{\omega}_n} = \frac{2\sin(\hat{\omega}_n)}{[2+\cos(2\hat{\omega}_n)]^2} e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}] \tag{8}$$

Since the term $2\sin(\hat{\omega}_n)/[2+\cos(2\hat{\omega}_n)]^2$ is positive for $\hat{\omega}_n \in (0, \pi)$, it does not affect the sign of the gradient estimate. As a result, the LMS updating equation for the direct frequency estimator (DFE) can be simplified as

$$\hat{\omega}_{n+1} = \hat{\omega}_n - \mu e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}] \tag{9}$$

where μ is the step size of the adaptive algorithm. To reduce computation, the value of the cosine function is retrieved from a pre-stored cosine vector of the form $[1\ \cos(\pi/L)\cdots\cos(\pi(L-1)/L)]$ where L is the vector length. Notice that when L increases, the frequency resolution increases but a larger memory will be needed. As a result, the method requires only five multiplications, five additions and one look-up operation for each sampling interval. Comparing with its recursive least squares (RLS) realisation [23] which involves eight additions, nine multiplications, one division, one square root and one arccosine operation per iteration, the LMS implementation is more computationally simple, but at the expense of larger variance for the frequency estimate. It is noteworthy that the RLS algorithm can also be derived from a total least squares minimisation framework [24], and its frequency variance decreases linearly and quadratically with the length of $\{x_n\}$ at low and high signal-to-noise ratio (SNR) conditions, respectively.

Prior to deriving the convergence behaviour of the frequency estimate, we evaluate the expected value of the learning increment of (9):

$$\begin{aligned}
&E\{e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\} \\
&\quad = E\{[s_n - 2s_{n-1}\cos(\hat{\omega}_n) + s_{n-2} + q_n \\
&\qquad - 2\cos(\hat{\omega}_n)q_{n-1} + q_{n-2}] \\
&\qquad \cdot [(s_n + s_{n-2} + q_n + q_{n-2})\cos(\hat{\omega}_n) + s_{n-1} + q_{n-1}]\} \\
&\quad = \sigma_s^2[(1+\cos(2\omega)) + \cos(\hat{\omega}_n) + \cos(\omega) - (2\cos(\omega) \\
&\qquad + 2\cos(\omega))\cos^2(\hat{\omega}_n) - 2\cos(\hat{\omega}_n) \\
&\qquad + (\cos(2\omega)+1)\cos(\hat{\omega}_n) + \cos(\omega)] + \sigma_q^2[\cos(\hat{\omega}_n) \\
&\qquad - 2\cos(\hat{\omega}_n) + \cos(\hat{\omega}_n)] \\
&\quad = 2\sigma_s^2[\cos(\hat{\omega}_n)\cos(2\omega) - \cos(\omega)\cos(2\hat{\omega}_n)]
\end{aligned} \tag{10}$$

Obviously, $\hat{\omega}_n = \omega$ is a stationary point of (10). Moreover, the derivative of (10) with respect to $\hat{\omega}_n$ at $\hat{\omega}_n = \omega$ is easily shown to be

$$\begin{aligned}
&\left.\frac{\partial 2\sigma_s^2[\cos(\hat{\omega}_n)\cos(2\omega) - \cos(\omega)\cos(2\hat{\omega}_n)]}{\partial \hat{\omega}_n}\right|_{\hat{\omega}_n=\omega} \\
&\quad = 2\sigma_s^2\sin(\omega)(2\cos^2(\omega)+1)
\end{aligned} \tag{11}$$

which is always positive for $\omega \in (0, \pi)$. As a result, the local stability of the algorithm is proved [29]. Substituting (10) into (9), we obtain the mean convergence trajectory of the frequency estimate as

$$\begin{aligned}
\hat{\omega}_{n+1} &= \hat{\omega}_n - 2\mu\sigma_s^2[\cos(\hat{\omega}_n)\cos(2\omega) - \cos(\omega)\cos(2\hat{\omega}_n)] \\
&= \hat{\omega}_n - 2\mu\sigma_s^2\left[\sin\left(\frac{\hat{\omega}_n - \omega}{2}\right)\sin\left(\frac{3(\hat{\omega}_n + \omega)}{2}\right)\right. \\
&\quad \left. + \sin\left(\frac{\hat{\omega}_n + \omega}{2}\right)\sin\left(\frac{3(\hat{\omega}_n - \omega)}{2}\right)\right]
\end{aligned} \tag{12}$$

Considering local convergence when $\hat{\omega}_n$ approaches ω, (12) can be approximated as

$$\begin{aligned}
\hat{\omega}_{n+1} &\approx \hat{\omega}_n - \mu\sigma_s^2(\hat{\omega}_n - \omega) \\
&\qquad \times \left[\sin\left(\frac{3(\hat{\omega}_n + \omega)}{2}\right) + 3\sin\left(\frac{\hat{\omega}_n + \omega}{2}\right)\right] \\
&= \hat{\omega}_n(1 - \mu\sigma_s^2 g(\hat{\omega}_n)) + \mu\sigma_s^2\omega g(\hat{\omega}_n)
\end{aligned} \tag{13}$$

where

$$g(\hat{\omega}_n) \triangleq \sin\left(\frac{3(\hat{\omega}_n + \omega)}{2}\right) + 3\sin\left(\frac{\hat{\omega}_n + \omega}{2}\right) \tag{14}$$

A closed form expression for $\hat{\omega}_n$ is not available because the geometric ratio $(1 - \mu\sigma_s^2 g(\hat{\omega}_n))$ is changing at each iteration, but the convergence trajectory can be easily acquired using (13) by brute force. Nevertheless, some observation can be made from (13). First, the mean convergence rate of $\hat{\omega}_n$ is independent of the noise level. To ensure convergence and stability, μ should be chosen so that $|1 - \mu\sigma_s^2 g(\hat{\omega}_n)| < 1$ is satisfied. Since $0 < g(\hat{\omega}_n) < 4$, the bound for μ can thus be computed from $|1 - 4\mu\sigma_s^2| < 1$, which gives $0 < \mu < 1/(2\sigma_s^2)$. In addition, the algorithm has a time-varying time constant of $1/(2\mu\sigma_s^2 g(\hat{\omega}_n))$. Considering when $\hat{\omega}_n \to \omega$, it is found that $g(\hat{\omega}_n)$ will approach zero at $\omega = 0$ or $\omega = \pi$, which implies that the steady state learning rate of $\hat{\omega}_n$ is fairly slow if the frequency is close to one of these extreme values.

Assuming that q_n is of Gaussian distribution and using (9) again, the steady state frequency variance of the DFE algorithm, denoted by $\mathrm{var}(\hat{\omega})$, is derived as (see the Appendix)

$$\mathrm{var}(\hat{\omega}) \triangleq \lim_{n\to\infty} E\{(\hat{\omega}_n - \omega)^2\}$$

$$\approx \frac{\mu\sigma_q^2}{2\mathrm{SNR}\sin(\omega)}\left(\frac{\cos(4\omega)}{2+\cos(2\omega)} + 1\right) \tag{15}$$

where $\mathrm{SNR} = \sigma_s^2/\sigma_q^2$. It can be seen that $\mathrm{var}(\hat{\omega})$ is proportional to μ and σ_q^2 and inversely proportional to SNR. Investigating the term $(\cos(4\omega)/(2+\cos(2\omega)) + 1)/\sin(\omega)$ reveals that the frequency variance approaches its minimum value of $0.32\mu\sigma_q^2/\mathrm{SNR}$ if $\omega \approx 0.28\pi$ or $\omega \approx 0.72\pi$ while it has a large value when ω is close to 0 or π. At $\omega = 0$ and $\omega = \pi$, $\mathrm{var}(\hat{\omega}) \to \infty$, while the variance of $\hat{\omega}_n$ has zero value in the absence of noise. From (13) and (15), the choice of μ should be a trade-off between a fast convergence rate and a small variance, as in the standard LMS algorithm [9]. We also note that the estimation performance of the DFE is relatively poor when ω is close to 0 or π because both the convergence time and variance are large.

On the other hand, the steady state variance of the noise power estimate using (7) is given by

$$\mathrm{var}(\zeta^2) \triangleq \lim_{n\to\infty} E\{(\zeta_n^2 - \sigma_q^2)^2\} \approx 2\sigma_q^4 \tag{16}$$

It is interesting to note that $\mathrm{var}(\zeta^2)$ does not depend on μ and ω.

3 Simulation results

Computer simulations had been conducted to evaluate the sinusoidal frequency estimation performance of the DFE in the presence of white Gaussian noise for non-stationary conditions. Comparisons with two LMS-style frequency estimators which are claimed to provide unbiased estimation, namely, the adaptive Pisarenko's algorithm [8] and adaptive IIR-BPF [17] were also made. For each iteration, the former requires ten multiplications, four divisions, five additions, one look-up and one square root operation while the latter needs six multiplications, six additions and one look-up operation. The signal power was unity, which corresponded to $\alpha = \sqrt{2}$ and we scaled the noise sequence to obtain different SNRs. The length of the cosine vector L was chosen to be 1000 and this provided a frequency resolution of $\pi/1000$ rad/s. The initial frequency estimates of all three methods were set to be 0.5π rad/s and the 3 dB bandwidth coefficient in [17] was $\beta = 0.5$. All the results were based on 100 independent runs.

In the first example, ω was a piecewise constant function and the SNR was 10 dB. The actual frequency had a value of 0.95π rad/s during the first 4000 iterations and then changed instantaneously to 0.55π rad/s and to 0.3π rad/s, at the 4000th and the 8000th iteration, respectively. The step size parameters of the DFE and adaptive Pisarenko's algorithm were chosen to be 0.002 while that of the adaptive IIR-BPF was 0.005. Figure 1 shows the trajectories for the frequency estimates of the three algorithms in tracking this time-varying frequency. It can be seen that $\hat{\omega}_n$ converged to the desired values at approximately the 2000th, 5300th and 9000th iterations. The convergence time for $\omega = 0.95\pi$ rad/s almost doubled that of $\omega = 0.3\pi$ rad/s because upon convergence, the term $\sin(3(\hat{\omega}_n + \omega)/2) + 3\sin((\hat{\omega}_n + \omega)/2)$ approached 0.16 and 3.4 in the former and the latter case, respectively. In addition, we observe that (13) had predicted the learning behaviour of the frequency estimate accurately. On the other hand, the adaptive Pisarenko's algorithm also estimated the step-changing frequency accurately but with different convergence behaviors, while the adaptive IIR-BPF was incapable of tracking the true frequency after the 4000th iteration. Figure 2 plots a time evolution of x_n to quantify visually the difficulty of accurate frequency estimation.

As mentioned in Section 2, the modified cost function $E\{\zeta_n^2\}$ has two maxima in $(0, \pi/3)$ and $(2\pi/3, \pi)$, which

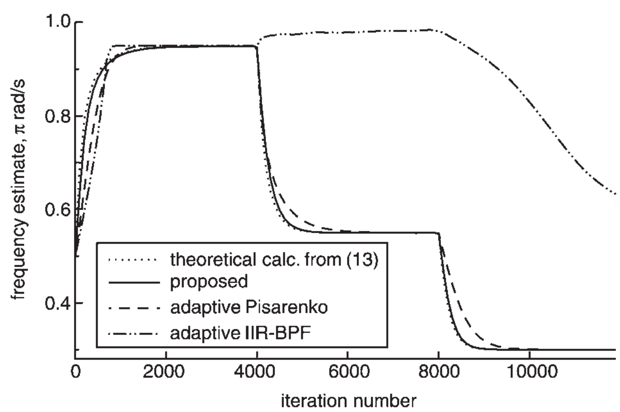

Fig. 1 *Frequency estimates for step changes in frequency at SNR = 10 dB*

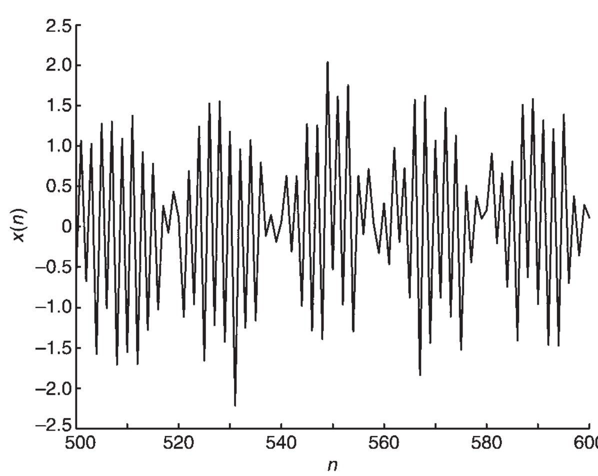

Fig. 2 *Time segment of x(n)*

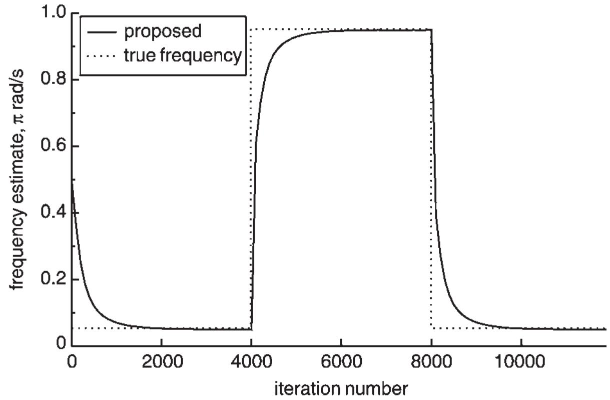

Fig. 3 *Frequency estimate of DFE for very large step changes in frequency at SNR = 10 dB*

implies that the algorithm cannot achieve global convergence when there is a very large change in frequency abruptly. Nevertheless, in this case the frequency estimate will converge to a value outside its admissible range of $(0, \pi)$. As a result, a simple solution is to set $\omega_n = \pi/2$, or more generally any value in $(\pi/3, 2\pi/3)$, whenever $\omega_n < 0$ or $\omega_n > \pi$. Figure 3 shows the tracking performance of the DFE for very large step changes in frequency at SNR = 10 dB with the use of the above suggestion, namely, ω_n was set to $\pi/2$ whenever its value was outside $(0, \pi)$. The actual frequency had a value of 0.05π rad/s during the first 4000 iterations and then changed instantaneously to 0.95π rad/s and back to 0.05π rad/s, at the 4000th and the 8000th iteration, respectively. We observe that the proposed algorithm tracked the step-changing frequency accurately via the slight modification.

A comprehensive test was then performed for a wide range of $\omega \in [0.05\pi, 0.95\pi]$ at SNR = 10 dB, and the steady state mean square frequency errors (MSFEs) of the three algorithms were measured and plotted in Fig. 4. In order to provide a fair comparison, μ was fixed at 0.002 while we adjusted the step sizes of [8] and [17] such that their convergence times were approximately identical for each tested frequency. It is seen that the measured MSFEs of the DFE agreed with their theoretical values, particularly when ω was close to 0.5π rad/s. Furthermore, the value of $\mathrm{var}(\hat{\omega})$ was bounded by 2.2×10^{-5} rad^2/s^2 for $\omega \in [0.2\pi, 0.8\pi]$ rad/s. Interestingly, the frequency dependence of the MSFEs was similar to that of the adaptive Pisarenko's method but the DFE had smaller variances for all cases. Although [17] gave the best performance for $\omega < 0.2\pi$ and $\omega > 0.8\pi$, it had much larger MSFEs for other frequencies,

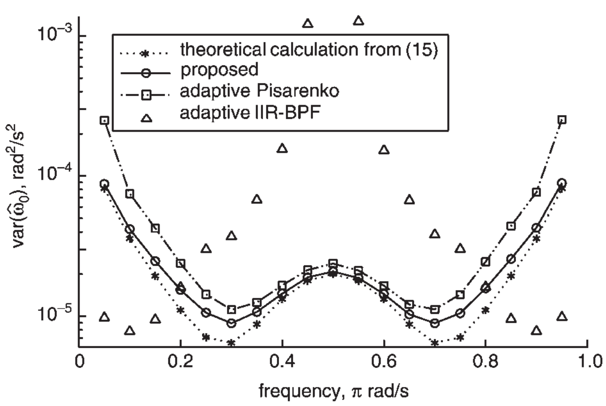

Fig. 4 *Frequency variances at SNR = 10 dB*

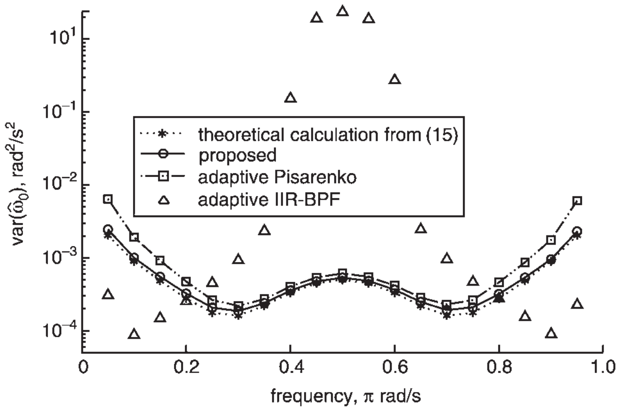

Fig. 5 *Frequency variances at SNR = 3 dB*

particularly when ω was close to 0.5π rad/s, and it failed to work at this frequency. This test was repeated for a lower SNR condition of 3 dB and the results are plotted in Fig. 5, which gave similar observations.

Figures 6 and 7 show the estimated noise power using (7) and its variance for different frequencies, respectively, at SNR = 10 dB. Along the frequency axis, the estimated noise power and variance fluctuated around their nominal values, with minimum and maximum values of 9.90×10^{-2} and 1.01×10^{-1}, and 1.88×10^{-2} and 2.08×10^{-1}, respectively. This implies that for all frequencies, (7) estimated σ_q^2 accurately while the mean square errors of the noise power estimates agreed with (16), and as expected, their frequency dependence was negligible.

Figure 8 demonstrates the carrier frequency estimation performance for a noisy binary phase-shift keying (BPSK)

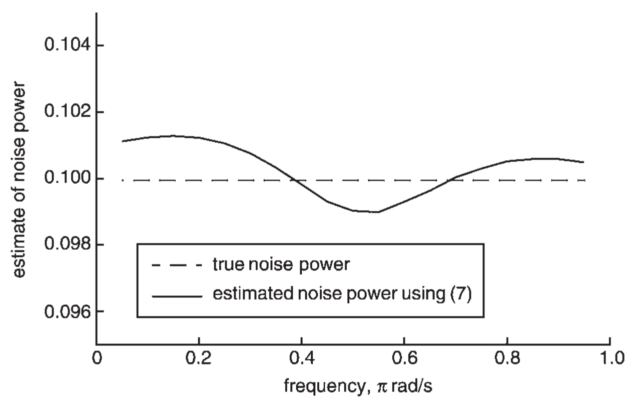

Fig. 6 *Estimate of noise power at SNR = 10 dB*

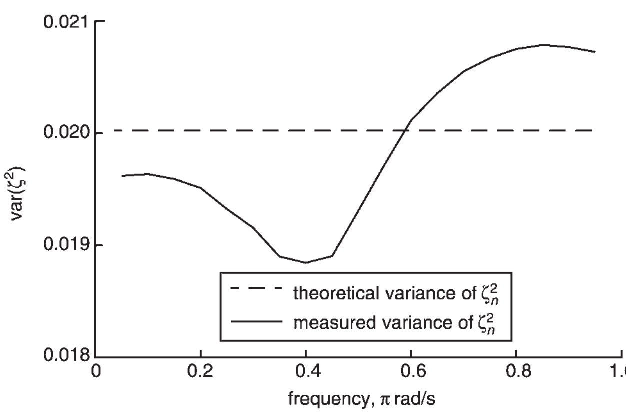

Fig. 7 *Estimate of var(ζ^2) at SNR = 10 dB*

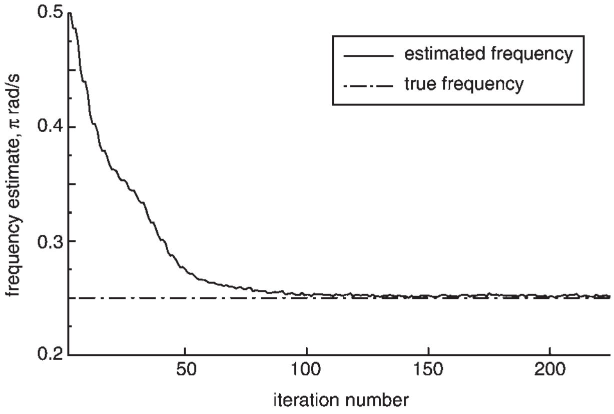

Fig. 8 *Frequency estimate of DFE for an BPSK signal at SNR = 10 dB*

signal where its amplitude as well as phase were nonstationary, at SNR $= 10\,\text{dB}$. The baud rate and the carrier frequency of the BPSK signal was selected as $0.05\pi\,\text{rad/s}$ and $0.25\pi\,\text{rad/s}$, respectively, and thus there were 40 samples for each symbol. In this example, $\mu = 0.02$ was used to achieve fast convergence at the expense of a larger variance. We can see that the DFE algorithm converged at approximately the 100th iteration and an accurate estimate of the carrier frequency was obtained.

4 Conclusions

A computationally attractive algorithm, called the DFE, has been proposed for tracking the frequency of a real sinusoid embedded in white noise. Using an LMS-style method, the frequency estimate is adjusted directly on a sample-by-sample basis. Learning behaviour and mean square error of the estimated frequency in white Gaussian noise are derived and verified by computer simulations. It is shown that the DFE gives unbiased frequency estimates in several non-stationary conditions and has high frequency estimation accuracy when the frequency is neither close to 0 nor π. In addition, the DFE outperforms two existing LMS-style frequency estimators in terms of estimation accuracy, computational complexity and/or tracking capability. It is noteworthy that the proposed LMS algorithm will give biased frequency estimation for a linearly varying frequency because in this case, the recurrence of (2) will not hold exactly. Moreover, the algorithm development assumes white noise, and if the noise is coloured with known spectrum, one possible solution is to filter the noisy sinusoid by a linear whitening filter which makes the noise component at the filter output white. Accurate frequency estimation in the scenarios of linear variation of signal frequency and/or unknown coloured noise will be our future research directions.

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7 Appendix

The steady state mean square error of $\hat{\omega}_n$ is derived as follows. Subtracting ω from both sides of (9), squaring both sides, taking expectation and then considering $n \to \infty$ yields

$$
\begin{aligned}
&2 \lim_{n\to\infty} E\{(\hat{\omega}_n - \omega)e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\} \\
&\quad = \mu \lim_{n\to\infty} E\{e_n^2[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]^2\} \qquad (17)
\end{aligned}
$$

Suppose μ is chosen sufficiently small such that $\hat{\omega}_n \to \omega$ upon convergence. The component which involves both signal and noise in the RHS of (17) is approximated as

$$\begin{aligned}
&\mu E\Big\{[q_n - 2\cos(\omega)q_{n-1} + q_{n-2}]^2[\alpha\cos(\omega n + \phi)\cos(\omega)\\
&\quad + \alpha\cos(\omega(n-2) + \phi))\cos(\omega)\\
&\quad + \alpha\cos(\omega(n-1) + \phi)]^2\Big\}\\
&= \mu E\Big\{\big[q_n^2 + 4\cos^2(\omega)q_{n-1}^2 + q_{n-2}^2\big]\cdot\alpha^2\\
&\quad \cdot[\cos^2(\omega n + \phi)\cos^2(\omega)\\
&\quad + \cos^2(\omega(n-2) + \phi)\cos^2(\omega) + \cos^2(\omega(n-1) + \phi)\\
&\quad + 2\cos(\omega n + \phi)\cos(\omega(n-2) + \phi)\cos^2(\omega)\\
&\quad + 2\cos(\omega n + \phi)\cos(\omega(n-1) + \phi)\cos(\omega)\\
&\quad + 2\cos(\omega(n-1) + \phi)\cos(\omega(n-2) + \phi)\cos(\omega)]\Big\}\\
&= 2\mu\sigma_s^2\sigma_q^2(\cos(2\omega) + 2)^3 \qquad (18)
\end{aligned}$$

Furthermore, the component due to noise only in the RHS of (17) can be estimated as

$$\begin{aligned}
&\mu E\{[q_n - 2\cos(\omega)q_{n-1} + q_{n-2}]^2\\
&\quad \times[\cos(\omega)q_n + \cos(\omega)q_{n-2} + q_{n-1}]^2\}\\
&= \mu E\big\{\big(q_n^2 + 4\cos^2(\omega)q_{n-1}^2 + q_{n-2}^2 - 4\cos(\omega)q_nq_{n-1}\\
&\quad + 2q_nq_{n-2} - 4\cos(\omega)q_{n-1}q_{n-2}\big)\big(\cos^2(\omega)q_n^2\\
&\quad + \cos^2(\omega)q_{n-2}^2 + q_{n-1}^2 + 2\cos^2(\omega)q_nq_{n-2}\\
&\quad + 2\cos(\omega)q_nq_{n-1} + 2\cos(\omega)q_{n-1}q_{n-2}\big)\big\}\\
&= 2\mu\sigma_q^2(\cos(2\omega) + 2)^2 \qquad (19)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty} E\{(\hat{\omega}_n - \omega)e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\}\\
&\quad = \lim_{n\to\infty} E\{(\hat{\omega}_{n-2} - \omega)e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\}\\
&\quad\quad - \mu\sum_{i=1}^{2}\lim_{n\to\infty} E\{e_{n-i}[(x_{n-i} + x_{n-i-2})\\
&\quad\quad \times\cos(\hat{\omega}(n-i)) + x_{n-i-1}]\\
&\quad\quad \cdot e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\}\\
&\quad \approx \lim_{n\to\infty} E\{(\hat{\omega}_n - \omega)e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\}\\
&\quad\quad - \mu\sum_{i=1}^{2}\lim_{n\to\infty} E\{e_{n-i}[(x_{n-i} + x_{n-i-2})\cos(\hat{\omega}_n) + x_{n-i-1}]\\
&\quad\quad \cdot e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\} \qquad (20)
\end{aligned}$$

With the use of (13), the first term of (20) can be evaluated as

$$\begin{aligned}
&\lim_{n\to\infty} E\{(\hat{\omega}_n - \omega)e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\}\\
&= 2\sigma_s^2\lim_{n\to\infty} E\Big\{(\hat{\omega}_n - \omega)\Big[\sin\Big(\frac{\hat{\omega}_n - \omega}{2}\Big)\sin\Big(\frac{3(\hat{\omega}_n + \omega)}{2}\Big)\\
&\quad + \sin\Big(\frac{\hat{\omega}_n + \omega}{2}\Big)\sin\Big(\frac{3(\hat{\omega}_n - \omega)}{2}\Big)\Big]\Big\}\\
&\approx \sigma_s^2\lim_{n\to\infty} E\Big\{(\hat{\omega}_n - \omega)^2\Big[\sin\Big(\frac{3(\hat{\omega}_n + \omega)}{2}\Big) + 3\sin\Big(\frac{\hat{\omega}_n + \omega}{2}\Big)\Big]\Big\}\\
&\approx \sigma_s^2\mathrm{var}(\hat{\omega})(\sin(3\omega) + 3\sin(\omega))
\end{aligned} \qquad (21)$$

It can also be shown that

$$\begin{aligned}
&\lim_{n\to\infty} E\{e_{n-1}[(x_{n-1} + x_{n-3})\cos(\hat{\omega}_n) + x_{n-2}]\\
&\quad \cdot e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\}\\
&\approx E\{[q_{n-1} - 2\cos(\omega)q_{n-2} + q_{n-3}]\\
&\quad \times[(x_{n-1} + x_{n-3})\cos(\omega) + x_{n-2}]\\
&\quad \cdot[q_n - 2\cos(\omega)q_{n-1} + q_{n-2}]\\
&\quad \times[(x_n + x_{n-2})\cos(\omega) + x_{n-1}]\}\\
&= -4\sigma_s^2\sigma_q^2\cos^2(\omega)(\cos(2\omega) + 2)^2\\
&\quad + \sigma_q^4(4\cos^4(\omega) - 12\cos^2(\omega) + 1) \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} E\{e_{n-2}[(x_{n-2} + x_{n-4})\cos(\hat{\omega}_n) + x_{n-3}]\\
&\quad \cdot e_n[(x_n + x_{n-2})\cos(\hat{\omega}_n) + x_{n-1}]\}\\
&\approx E\{(q_{n-2} - 2\cos(\omega)q_{n-3} + q_{n-4})\\
&\quad \times[(x_{n-2} + x_{n-4})\cos(\omega) + x_{n-3}]\\
&\quad \cdot(q_n - 2\cos(\omega)q_{n-1} + q_{n-2})[(x_n + x_{n-2})\cos(\omega) + x_{n-1}]\}\\
&= \sigma_s^2\sigma_q^2(2\cos^2(\omega) - 1)(\cos(2\omega) + 2)^2 + 2\sigma_q^4\cos^2(\omega)
\end{aligned} \qquad (23)$$

Substituting (18)–(23) into (17) and after simplification, we obtain (15).