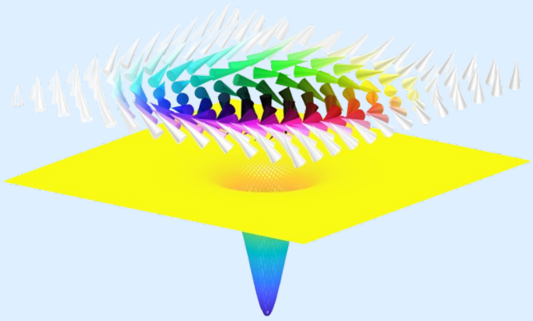

Composite topological excitations in ferromagnetic superconductors:
Magnetic skyrmion-superconducting vortex pairs

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Motivation

Motivation

- Possibility of coexisting ferromagnetic (FM) and superconducting (SC) states was proposed by Greenside, Blount & Varma[1]
- Ferromagnetic superconductor properties modeled by combining Ginzburg–Landau theory with a mean field theoretic model of the ferromagnetic subsystem[2]
- Magnetic order is associated with local moments, while the conduction electrons carry superconductivity
- There exists a stable temperature range below $T_m < T_C$ such that the magnetization $\vec{m} \in S^2$

$\Rightarrow$ Topological magnetic spin textures coexisting with superconducting states

- Composite topological excitations: **magnetic skyrmion-superconducting vortex pair** (SVP)
- SVPs already observed experimentally in chiral magnet-superconductor (CMSC) heterostructures[3]

[1] H.S. Greenside, E.I. Blount and C.M. Varma, Phys. Rev. Lett. 46 (1981) 49
[2] E. I. Blount and C. M. Varma, Phys. Rev. Lett. 42, 1079 (1979)
[3] EY.-J. Xie, A. Qian, B. He, Y.-B. Wu, S. Wang, B. Xu et al., Phys. Rev. Lett. 133 (2024) 166706

Motivation

- In CMSC heterostructures, vortices usually approximated by thin film Pearl vortex (no back-reaction)[4,5]

$\Rightarrow$ Chiral magnetic system with external **inhomogeneous** applied magnetic field

- SC vortex interactions[6] and FM skyrmion interactions[7] independently well understood
- Interactions of composite SVPs poorly understood
- Intertype superconductivity (vortex clustering) predicted using ill-defined perturbative expansion[8]

$\Rightarrow$ We want to understand long-range interactions of SVPs

$\Rightarrow$ Can type 1.5 superconductivity occur in this single superconducting OP model?

[4] S.S. Apostoloff, E.S. Andriyakhina, P.A. Vorobyev, O.A. Tretiakov and I.S. Burmistrov, Phys. Rev. B 107 (2023) L220409

[5] S.S. Apostoloff, E.S. Andriyakhina and I.S. Burmistrov, Phys. Rev. B 109 (2024) 104406

[6] N.S. Manton and J.M. Speight, Commun. Math. Phys. 236 (2003) 535

[7] B.M.A.G. Piette, B.J. Schroers and W.J. Zakrzewski, Z. Phys. C 65 (1995) 165

[8] A. Vagov, T.T. Saraiva, A.A. Shanenko, A.S. Vasenko, J.A. Aguiar, V.S. Stolyarov et al., Commun. Phys. 6 (2023) 284

Ferromagnetic superconductor model

Model setup and parameters

- Superconducting order parameter $\psi \in \mathbb{C}$
- $|\psi|^2$ is a measure of local density of Cooper pairs
- Electromagnetic gauge field $\vec{A} = (A_1, A_2, A_3)$
- Associated magnetic field $\vec{B} = \vec{\nabla} \times \vec{A} = (\partial_2 A_3, -\partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)$
- Gauge covariant derivative $\vec{D}\psi = \vec{\nabla}\psi + iq\vec{A}\psi$
- Cooper pair: effective charge $q \sim 2e$
- Fixed length magnetization $\vec{m} \in S^2 \subset \mathbb{R}^3$
- The total Gibbs free energy functional of the system consists of three parts

$$F[\psi, \vec{A}, \vec{m}] = F_{\mathrm{sc}}[\psi, \vec{A}] + F_{\mathrm{mag}}[\vec{m}] + F_{\mathrm{int}}[\psi, \vec{A}, \vec{m}] \tag{1}$$

Ferromagnetic superconductor model

- In the exchange approximation, the free energy of an isotropic ferromagnet in the absence of an applied magnetic field is given by

$$F_{\text{mag}}[\vec{m}] = \int_{\mathbb{R}^2} \mathrm{d}^2x \left\{ \frac{\alpha(T)}{2}|\vec{m}|^2 + \frac{\beta}{4}|\vec{m}|^4 + \frac{1}{2}|\nabla\vec{m}|^2 \right\}, \quad \alpha(T) = \alpha_0 \frac{(T - T_m)}{T_m} \tag{2}$$

- The superconducting order parameter is described by the Ginzburg–Landau free energy

$$F_{\text{sc}}[\psi, \vec{A}] = \int_{\mathbb{R}^2} \mathrm{d}^2x \left\{ \frac{a(T)}{2}|\psi|^2 + \frac{b}{4}|\psi|^4 + \frac{1}{2}|\vec{D}\psi|^2 + \frac{1}{2}|\vec{\nabla} \times \vec{A}|^2 \right\}, \quad a(T) = a_0 \frac{(T - T_c)}{T_c} \tag{3}$$

- Two main interactions of the superconducting OP $\psi \in \mathbb{C}$ with the magnetization $\vec{m} \in S^2$

$\Rightarrow$ Spin-flip scattering (direct) and the Zeeman interaction (indirect)

Interactions

- One is via the direct effects of **spin-flip scattering** of conduction electrons with the magnetic moments and conduction-electron polarization[9],

$$F_{\text{spin-flip}}[\psi, \vec{m}] = \int_{\mathbb{R}^2} \mathrm{d}^2x \left\{ \left(\eta_1 |\vec{m}|^2 + \eta_2 |\nabla \vec{m}|^2 \right) |\psi|^2 \right\} \tag{4}$$

- The second is an indirect interaction which arises from the **Zeeman interaction**[10]

$$F_{\text{zeeman}}[\vec{A}, \vec{m}] = -\int_{\mathbb{R}^2} \mathrm{d}^2x \, (\vec{\nabla} \times \vec{A}) \cdot \vec{m} \tag{5}$$

- We will consider the effect of only the Zeeman interaction in this talk

[9] E.I. Blount and C.M. Varma, Phys. Rev. Lett. 42, 1079 (1979)
[10] S.-Z. Lin, L.N. Bulaevskii and C.D. Batista, Phys. Rev. B 86 (2012) 180506

Ground state configurations

- The potential energy is given by $\mathcal{F}_p = \frac{a}{2}|\psi|^2 + \frac{b}{4}|\psi|^4 + \frac{\alpha}{2}|\vec{m}|^2 + \frac{\beta}{4}|\vec{m}|^4$
- The associated uniform ground state configurations are found to by solving the system of equations

$$\left.\frac{\delta \mathcal{F}_p}{\delta|\psi|}\right|_{(u,m_0)} = au + bu^3 = 0, \quad \left.\frac{\delta \mathcal{F}_p}{\delta|\vec{m}|}\right|_{(u,m_0)} = \alpha m_0 + \beta m_0^3 = 0 \tag{6}$$

- This gives us the ground state

$$u^2 = -\frac{a}{b}, \quad m_0^2 = -\frac{\alpha}{\beta} \tag{7}$$

- The corresponding ground state free energy is determined to be

$$\mathcal{F}_p^* = -\frac{a^2}{4b} - \frac{\alpha^2}{4\beta} \tag{8}$$

Field equations

- Superconducting vortices and magnetic skyrmions are solutions of the Euler-Lagrange field equations

$$\frac{\delta F}{\delta \psi^*} = -\frac{1}{2}\vec{D}\cdot\vec{D}\psi - \frac{b}{2}\left(u^2 - |\psi|^2\right)\psi = 0, \tag{9}$$

$$\frac{\delta F}{\delta \vec{A}} = q^2|\psi|^2\vec{A} + \frac{iq}{2}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right) + \vec{J} - \vec{J}_m = \vec{0}, \tag{10}$$

$$\frac{\delta F}{\delta \vec{m}} = -\Delta\vec{m} - \vec{\nabla}\times\vec{A} = \vec{0}. \tag{11}$$

- From the gauge field equation (10), we get the supercurrent

$$\vec{J} = \vec{\nabla}\times\vec{B} = \vec{J}_m - q^2|\psi|^2\vec{A} - \frac{iq}{2}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right) \tag{12}$$

and the magnetization current

$$\vec{J}_m = \vec{\nabla}\times\vec{m} \tag{13}$$

Superconducting (Meissner) state

- In the SC phase, with $T_m < T < T_c$, the SC OP is uniform $|\psi| = u$
- Magnetic field is expelled from the bulk $\vec{B} = \vec{0}$ and magnetization absent $|\vec{m}| = 0$
- The free energy of the superconducting phase, for $T < T_c$, is simply

$$\mathcal{F}_{\mathrm{SC}} = -\frac{a^2}{4b} \tag{14}$$

Ferromagnetic phase

- Characterized by suppression of superconductivity and vanishing of Cooper pairs, i.e. $|\psi| = 0$ everywhere
- Magnetic field in FM phase is given by $\vec{B} = \vec{m}$
- Uniform ground state configuration is

$$\left.\frac{\delta F}{\delta \vec{m}}\right|_{|\vec{m}|=m_0} = \left.\left(\alpha \vec{m} + \beta |\vec{m}|^2 \vec{m} - \vec{\nabla} \times \vec{A}\right)\right|_{|\vec{m}|=m_0} = \vec{0} \quad \Rightarrow \quad m_0^2 = \frac{1-\alpha}{\beta} \tag{15}$$

- Corresponding free energy density in FM phase is[11]

$$\mathcal{F}_{\mathrm{FM}} = -\frac{(\alpha(T)-1)^2}{4\beta} \text{ for } T < T_m^0 \tag{16}$$

- Critical temperature T_m^0 at which $\mathcal{F}_{\mathrm{FM}} = 0$ is found by solving $\alpha(T_m^0) = 1$, which gives

$$T_m^0 = \left(1 + \frac{1}{\alpha_0}\right) T_m > T_m \leftarrow \text{Curie temperature} \tag{17}$$

[11] H.S. Greenside, E.I. Blount and C.M. Varma, Phys. Rev. Lett. 46 (1981) 49

Superconducting ferromagnetic phase

- There also exists the possibility of a mixed superconducting and ferromagnetic phase, in some range $T_t < T < T_m$, where $|\psi| = u$, $\vec{m} = \vec{m}_0$ and the magnetic field is expelled from the bulk $\vec{B} = \vec{0}$

$\Rightarrow$ Screening currents restricted to surface of SC to compensate the external field in the bulk[12]

- In this superconducting ferromagnetic phase, the free energy density is found to be

$$\mathcal{F}_{\mathrm{SCFM}} = -\frac{a^2}{4b} - \frac{\alpha^2}{4\beta} \text{ for } T < T_m \tag{18}$$

- At $T = T_m$ there is a phase transition from the SC phase to the SCFM phase
- For $T < T_m$, this mixed phase is energetically favorable over the superconducting phase
- Another phase transition at some $T = T_t$ from the SCFM state to the FM state
- For this phase transition to be physical, we require $0 < T_t < T_m$

[12] Z. Devizorova, S. Mironov and A. Buzdin, Phys. Rev. Lett. 122 (2019) 117002

Superconducting ferromagnetic phase

- If this is the case, then SCFM phase is stable over the range $T_t < T < T_m$
- The transition temperature T_t is determined by solving

$$\mathcal{F}_{\mathrm{SCFM}} = \mathcal{F}_{\mathrm{FM}} \quad \Rightarrow \quad \frac{a_0^2(T_t - T_c)^2}{bT_c^2} = \frac{1}{\beta} - \frac{2\alpha_0(T_t - T_m)}{\beta T_m}. \tag{19}$$

- Solutions of this are found to be given by

$$T_t^{\pm} = T_c \left\{ \left(1 - \frac{\alpha_0 b T_c}{a_0^2 \beta T_m} \right) \pm \frac{1}{2} \sqrt{ \left(\frac{2\alpha_0 b T_c}{a_0^2 \beta T_m} - 2 \right)^2 - 4 \left(1 - \frac{b}{a_0^2 \beta} - \frac{2\alpha_0 b}{a_0^2 \beta} \right) } \right\} \tag{20}$$

- For particular parameters, the SCFM phase is found to exist in finite temperature

Superconducting ferromagnets

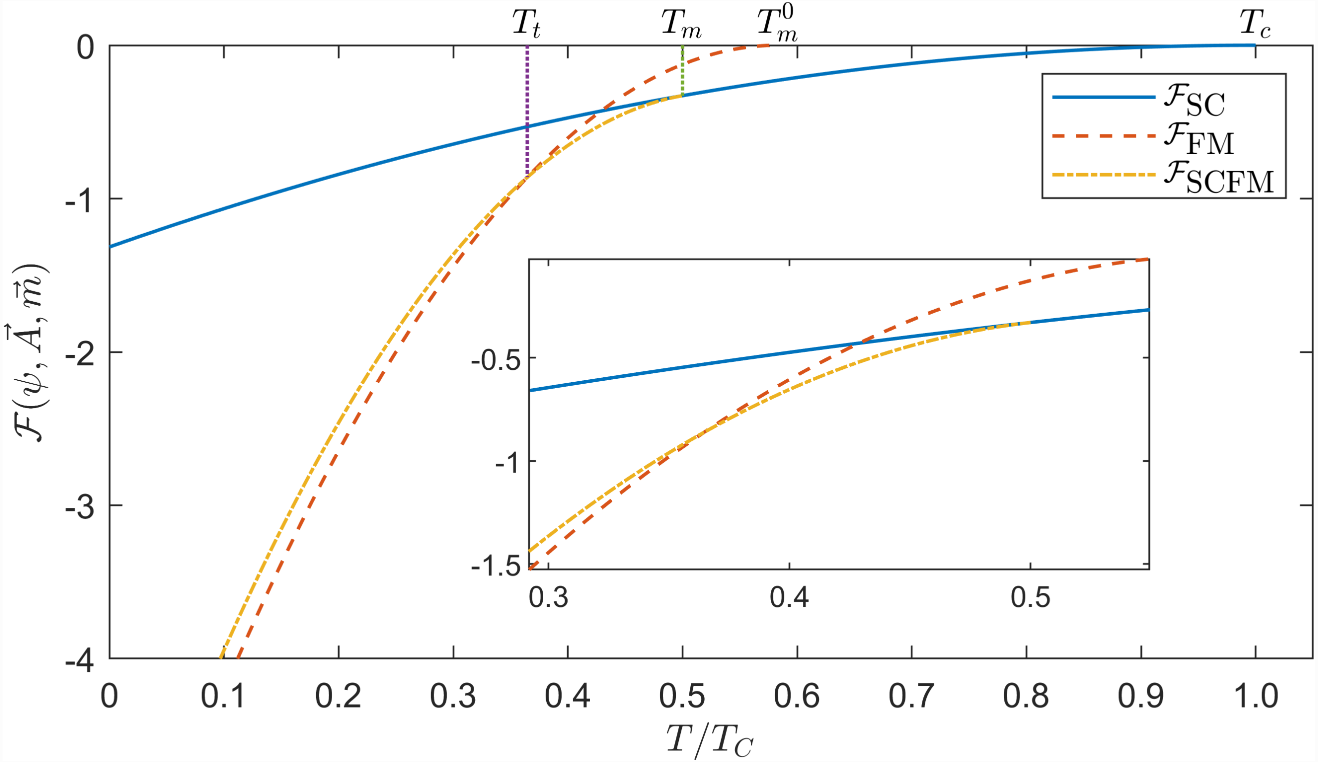
- Coexistence of superconductivity and ferromagnetism observed in Uranium based heavy-fermion superconductors UGe_2, URhGe and UCoGe[13]
- These ferromagnetic superconductors have an orthorhombic structure
- They exhibit superconductivity well below their Curie temperature, $T_m \gg T_C$
- Coexisting superconductivity and ferromagnetism also found in hole-doped $RbEuFe_4As_4$[14] and hole-doped $EuFe_2As_2$[15]
- Curie temperature in these materials is about $T_m \sim T_c/2$
- We consider such ferromagnetic superconductors with $T_m < T_C$

[13] A.D. Huxley, Physica C 514 (2015) 368

[14] Y. Liu, Y.-B. Liu, Z.-T. Tang, H. Jiang, Z.-C. Wang, A. Ablimit et al., Phys. Rev. B 93 (2016) 214503

[15] S. Nandi, W.T. Jin, Y. Xiao, Y. Su, S. Price, D.K. Shukla et al., Phys. Rev. B 89 (2014) 014512

Ferromagnetic superconductor model



Composite skyrmion-vortex pair

Composite magnetic skyrmion-superconducting vortex pair

- Extended Nielsen-Olesen multi-vortex ansatz[16]

$$\psi = \sigma(r)e^{iN\theta}, \quad \vec{A} = \left(-\frac{a(r)}{r}\sin\theta, \frac{a(r)}{r}\cos\theta, g(r)\right), \quad N \in \mathbb{Z} \tag{21}$$

- Profile functions satisfy BCs $\sigma(0) = 0, \sigma(\infty) = u$, $a(0) = 0, a(\infty) = N/q$ and $g'(0) = g(\infty) = 0$
- By Stoke's theorem, the total magnetic flux through the xy-plane is thus

$$\Phi = \int_{\mathbb{R}^2} \mathrm{d}^2x\, B_3 = 2\pi \int_0^\infty \mathrm{d}r \frac{\mathrm{d}a}{\mathrm{d}r} = \frac{2\pi N}{q} \equiv N\Phi_0 \quad \leftarrow \quad \text{flux quantum } \Phi_0 \tag{22}$$

and

$$\int_{\mathbb{R}^2} \mathrm{d}^2x\, (B_1, B_2) = \int_0^\infty \mathrm{d}r\, r\frac{\mathrm{d}g}{\mathrm{d}r} \int_0^{2\pi} \mathrm{d}\theta\, (\sin\theta, -\cos\theta) = (0,0) \tag{23}$$

[16] H.B. Nielsen and P. Olesen, Nucl. Phys. B 61 (1973) 45

Composite magnetic skyrmion-superconducting vortex pair

- For the magnetization field, the axially symmetric ansatz[17]

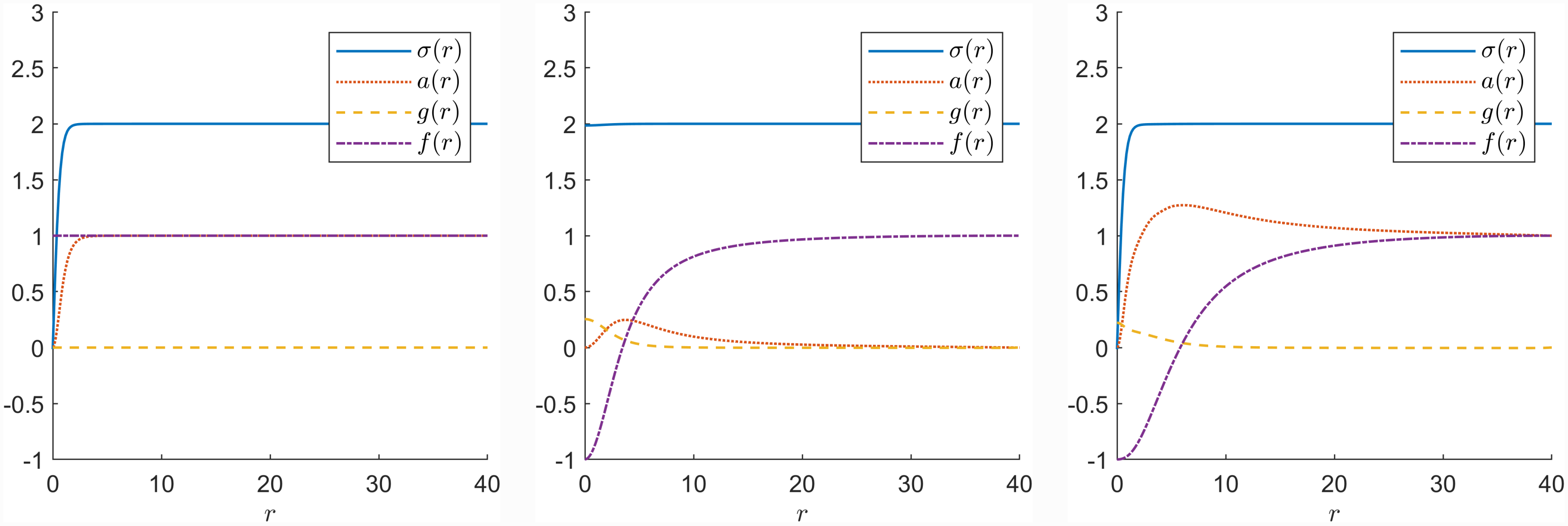
$$\vec{m} = \begin{pmatrix} \sqrt{1-f(r)^2}\cos(\phi) \\ \sqrt{1-f(r)^2}\sin(\phi) \\ f(r) \end{pmatrix} \tag{24}$$

- Monotonically increasing profile function with BCs $f(0) = -1$ and $f(\infty) = 1$
- Spin down $\vec{m}_\downarrow$ states at $r = 0$, spin up $\vec{m}_\uparrow$ states as $r \to \infty$.
- Energy minimized for **Bloch** skyrmion ($\phi = \theta + \pi/2$)
- The topological degree of the magnetization field is given by

$$n = \frac{1}{4\pi}\int_{\mathbb{R}^2} \mathrm{d}^2x \left[\vec{m}\cdot\left(\partial_1\vec{m}\times\partial_2\vec{m}\right)\right] = \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\theta\int_0^\infty \mathrm{d}r\,\frac{\mathrm{d}f}{\mathrm{d}r}\sin f(r) = -1 \tag{25}$$

[17] A.N. Bogdanov and A. Hubert, J. Magn. Magn. Mater. 138 (1994) 255.

Composite magnetic skyrmion-superconducting vortex pair



(a) Vortex (b) Bloch skyrmion (c) Composite skyrmion-vortex pair

Asymptotic form of skyrmion-vortex pairs

Linearization of the Gibbs free energy

- Recall that the Gibbs free energy density is

$$\mathcal{F} = \frac{1}{2}(D_i\psi)^*(D_i\psi) + \frac{1}{4}F_{ij}F_{ij} + \frac{b}{4}\left(u^2 - |\psi|^2\right)^2 + \frac{1}{2}\partial_j m_i \partial_j m_i - \epsilon_{ijk} m_i \partial_j A_k \tag{26}$$

- Let us linearize about the ground state in ferromagnetic superconducting phase

$$\psi = u + \phi, \quad \vec{A} = 0 + \vec{\alpha}, \quad \vec{m} = \vec{m}_0 + \vec{n} \tag{27}$$

- To determine the form of the perturbation $\vec{n}$, consider the expansion[18]

$$\vec{m} = \sqrt{1 - \vec{n}\cdot\vec{n}}\,\vec{m}_0 + \vec{n} \approx \vec{m}_0 + \vec{n} + O(\vec{n}\cdot\vec{n}) \tag{28}$$

- The magnitude of this is

$$\begin{aligned} \vec{m}\cdot\vec{m} &= (1 - \vec{n}\cdot\vec{n})(\vec{m}_0\cdot\vec{m}_0) + \vec{n}\cdot\vec{n} + 2\sqrt{1 - \vec{n}\cdot\vec{n}}\,(\vec{m}_0\cdot\vec{n}) \\ &= 1 + 2\sqrt{1 - \vec{n}\cdot\vec{n}}\,(\vec{m}_0\cdot\vec{n}) \\ &\overset{!}{=} 1 \quad \Rightarrow \quad \vec{m}_0\cdot\vec{n} = 0 \quad \Rightarrow \quad \vec{n} \in T_{\vec{m}_0}S^2 \end{aligned} \tag{29}$$

[18] B.M.A.G. Piette, B.J. Schroers and W.J. Zakrzewski, Z. Phys. C 65 (1995) 165

Linearization of the Gibbs free energy

- Linearized energy is

$$\mathcal{F}_{\text{lin}} = \frac{1}{2}|\vec{\nabla}\phi|^2 + bu^2\phi^2 + \frac{1}{2}|\vec{\nabla}\times\vec{\alpha}|^2 + \frac{1}{2}q^2u^2|\vec{\alpha}|^2 + \frac{1}{2}|\nabla\vec{n}|^2 - \vec{n}\cdot(\vec{\nabla}\times\vec{\alpha}) \tag{30}$$

- Superconducting OP is described by a **Klein-Gordon** equation

$$\frac{\delta\mathcal{F}_{\text{lin}}}{\delta\phi} = \left(-\Delta + 2bu^2\right)\phi = 0 \tag{31}$$

- Gauge field by a **Proca** equation with source generated by the (curl of the) magnetization

$$\frac{\delta\mathcal{F}_{\text{lin}}}{\delta\vec{\alpha}} = -\Delta\vec{\alpha} + \vec{\nabla}(\vec{\nabla}\cdot\vec{\alpha}) + q^2u^2\vec{\alpha} - \vec{\nabla}\times\vec{n} = 0 \tag{32}$$

- Magnetization by a vector **Poisson** equation, where the magnetic field provides the source

$$\frac{\delta\mathcal{F}_{\text{lin}}}{\delta\vec{n}} = -\Delta\vec{n} - \vec{\nabla}\times\vec{\alpha} = 0 \tag{33}$$

Asymptotic form of the composite state

- Linearized field equation for OP reduces to Bessel's modified equation of zeroth order,

$$r^2\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}+r\frac{\mathrm{d}\phi}{\mathrm{d}r}-2bu^2r^2\phi=0 \quad \Rightarrow \quad \phi(r)=c_\psi K_0\left(\sqrt{2bu^2}r\right) \tag{34}$$

- Superconducting OP asymptotically behaves as $\psi(r)\sim u+c_\psi K_0\left(\sqrt{2bu^2}r\right)$
- Linearized field equation for in-plane gauge field $\vec{\alpha}_{r\theta}=\alpha(r)\vec{e}_\theta$ becomes modified Bessel equation of first order

$$r^2\frac{\mathrm{d}^2\alpha}{\mathrm{d}r^2}+r\frac{\mathrm{d}\alpha}{\mathrm{d}r}-\left(q^2u^2r^2+1\right)\alpha=0 \quad \Rightarrow \quad \alpha(r)=c_A K_1(qur) \tag{35}$$

- In-plane gauge field has the asymptotic behaviour $\vec{A}_{r\theta}(r)\sim c_A K_1(qur)\vec{e}_\theta$
- Identical to single-band GL vortex asymptotics[19]

[19] J.M. Speight, Phys. Rev. D 55 (1997) 3830

Asymptotic form of the composite state

- Multiple choices for magnetization ansatz
- Bloch skyrmion lowest energy skyrmion numerically $\rightarrow$ Bloch perturbations $\vec{n} = f(r)\vec{e}_\theta$

$\Rightarrow$ Coupled system of ODEs:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{1}{r^2}f - \frac{\mathrm{d}\alpha_z}{\mathrm{d}r} = 0 \qquad (36)$$

$$\frac{\mathrm{d}^2 \alpha_z}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\alpha_z}{\mathrm{d}r} - q^2u^2\alpha_z + \frac{\mathrm{d}f}{\mathrm{d}r} + \frac{f}{r} = 0 \qquad (37)$$

- General solution for the asymptotic out-of-plane gauge field is

$$\alpha_z(r) = -\frac{c_m}{\sqrt{q^2u^2-1}}K_0\left(\sqrt{q^2u^2-1}\,r\right) \qquad (38)$$

- Magnetization asymptotically is found to be given by

$$\vec{n}(r) = -\frac{1}{2}\frac{c_m}{\sqrt{q^2u^2-1}}rK_0\left(\sqrt{q^2u^2-1}\,r\right)\vec{e}_\theta \qquad (39)$$

Asymptotics and length scales

- Summary of asymptotics:

$$\phi(r) = c_\psi K_0\left(\frac{r}{\xi_s}\right), \qquad \xi_s = \frac{1}{\sqrt{2bu^2}} = \frac{1}{\sqrt{-2a}} \tag{40}$$

$$\vec{\alpha}(r) = c_A K_1\left(\frac{r}{\lambda}\right)\vec{e}_\theta - c_m \xi_m K_0\left(\frac{r}{\xi_m}\right)\vec{e}_z, \qquad \lambda = \frac{1}{qu}, \lambda_z = \xi_m \tag{41}$$

$$\vec{n}(r) = -\frac{1}{2}c_m \xi_m r K_0\left(\frac{r}{\xi_m}\right)\vec{e}_\theta, \qquad \xi_m = \frac{1}{\sqrt{q^2u^2-1}} \tag{42}$$

- The coherence lengths are $\xi_{s,m}$ and magnetic penetration depths are λ
- Magnetization coherence length ξ_m is real for $qu > 1$

Long-range interactions of skyrmion-vortex pairs

Long-range interaction energy of composite states

- We want to construct a linearized field theory such that its solutions are identical to asymptotics of the SVP

$\Rightarrow$ Introduce an external source into our energy

$$\mathcal{F} = \mathcal{F}_{\text{lin}} + \mathcal{F}_{\text{source}}, \quad \mathcal{F}_{\text{source}} = -\rho\phi - j_i\alpha_i - \sigma_i n_i \tag{43}$$

- This gives us the modified system of coupled ODEs

$$\left(-\Delta + 2bu^2\right)\phi = \rho, \tag{44}$$

$$\left(-\Delta + q^2u^2\right)\vec{\alpha} = \vec{j} + \vec{\nabla} \times \vec{n} - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{\alpha}\right), \tag{45}$$

$$-\Delta\vec{n} = \vec{\sigma} + \vec{\nabla} \times \vec{\alpha}. \tag{46}$$

- Need to solve this system using our already determined asymptotic forms

External sources

- Static Klein-Gordon equation in 2D has Green's function K_0, that is

$$\left(-\Delta + \lambda^2\right) K_0(\lambda r) = 2\pi\delta(r) \tag{47}$$

- Substituting $\phi(r) = c_\psi K_0(r/\xi_s)$ into modified field eqn yields

$$\rho(r) = \left(-\Delta + \frac{1}{\xi_s^2}\right) c_\psi K_0\left(\frac{r}{\xi_s}\right) = c_\psi 2\pi\delta(r) \tag{48}$$

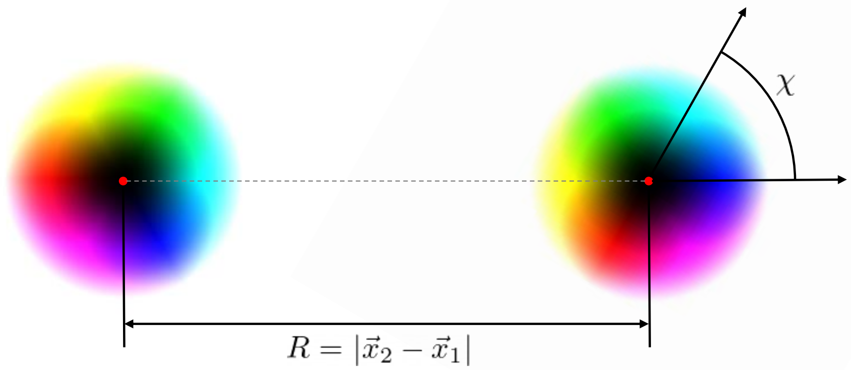
- Similar approach allows us to determine the other sources

$$\vec{j}(r) = -2\pi\lambda c_A[\vec{e}_z \times \vec{\nabla}\delta(r)] - c_m\left[2\pi\xi_m\delta(r) + \frac{1}{2}rK_1\left(\frac{r}{\xi_m}\right)\right]\vec{e}_z, \tag{49}$$

$$\vec{\sigma}(r) = \frac{1}{2}\frac{c_m}{\xi_m} rK_0\left(\frac{r}{\xi_m}\right)\vec{e}_\theta \tag{50}$$

Long-range interaction energy setup

- Can now compute asymptotic interaction energy of well-separated SVPs
- Consider a pair at $\vec{x}_1$ and label that pairs as SVP$^{(1)}$, and another pair SVP$^{(2)}$ at $\vec{x}_2$
- Allow a relative $\mathrm{SO}(2)_{\mathrm{iso}}$ iso-rotation of the separated skyrmions
- Parameterize this by a rotation angle $\chi \in [0, 2\pi)$ that acts on in-plane magnetization (n_r, n_θ) components of, say, SVP$^{(1)}$



Long-range interaction energy of composite states

- Interaction energy between well-separated SVPs comes from cross-terms in the linearization,

$$E_{\text{int}}(\vec{x}_1, \vec{x}_2) = -\int_{\mathbb{R}^2} \mathrm{d}^2\vec{x} \left(\rho^{(1)}\phi^{(2)} + \vec{j}^{(1)} \cdot \vec{\alpha}^{(2)} + \vec{n}^{(1)} \cdot \vec{\sigma}^{(2)} + \vec{n}^{(1)} \cdot (\vec{\nabla} \times \vec{\alpha})^{(2)} \right) \tag{51}$$

- After a bit of work we arrive at the interaction energy in terms of SVP separation $R = |\vec{x}_2 - \vec{x}_1|$ and relative skyrmion orientation χ:

$$\begin{aligned} E_{\text{int}}(R, \chi) = {} & \underbrace{\left[-\frac{\pi^2}{2} c_m^2 \xi_m^4 K_1\left(\frac{R}{\xi_m}\right) + \frac{\pi^2}{4} c_m^2 \xi_m^3 R K_0\left(\frac{R}{\xi_m}\right) \right]}_{\text{Zeeman + skyrmion-skyrmion interactions}} \cos(\chi) - \underbrace{\frac{2\pi c_m^2 \xi_m^4}{\lambda^2} K_0\left(\frac{R}{\xi_m}\right)}_{\text{OOP magnetic attraction}} \\ & + \underbrace{2\pi \left\{ c_A^2 K_0\left(\frac{R}{\lambda}\right) - c_\psi^2 K_0\left(\frac{R}{\xi_s}\right) \right\}}_{\text{IP magnetic repulsion + core-core attraction}} . \end{aligned} \tag{52}$$

Vortex-vortex contribution to interaction energy

- Interaction energy to leading order can be expressed as

$$E_{\text{int}}(R,\chi) \approx \frac{\pi}{\sqrt{R}} e^{-R/\xi_m} \left\{ c_m^2 \xi_m^3 \sqrt{\frac{\pi \xi_m}{2}} \left(\frac{\pi}{2} \left[\frac{R}{2} - \frac{17\xi_m}{16} \right] \cos(\chi) - \frac{2\xi_m}{\lambda^2} \right) \right.$$
$$\left. + 2c_A^2 \sqrt{\frac{\pi\lambda}{2}} e^{-R(\xi_m-\lambda)/(\lambda\xi_m)} - 2c_\psi^2 \sqrt{\frac{\pi\xi_s}{2}} e^{-R(\xi_m-\xi_s)/(\xi_s\xi_m)} \right\} \tag{53}$$

- Two terms contributing to the vortex-vortex interaction: scalar core-core attraction and magnetic repulsion
- These are proportional to

$$U_V(R) = c_A^2 \sqrt{\lambda} e^{-R(\xi_m-\lambda)/(\lambda\xi_m)} - c_\psi^2 \sqrt{\xi_s} e^{-R(\xi_m-\xi_s)/(\xi_s\xi_m)} \tag{54}$$

- First term originates from the gauge field, it repels vortices due to circulating currents

Intertype (1.5) superconductivity

- When core-core interaction dominates, the force $-U_V'(R)$ between vortices is attractive and the vortex cores (zeroes of the order parameter ψ) coincide
- This occurs when

$$\frac{\xi_m - \xi_s}{\xi_s \xi_m} < \frac{\xi_m - \lambda}{\lambda \xi_m} \quad \Rightarrow \quad \lambda < \xi_s \tag{55}$$

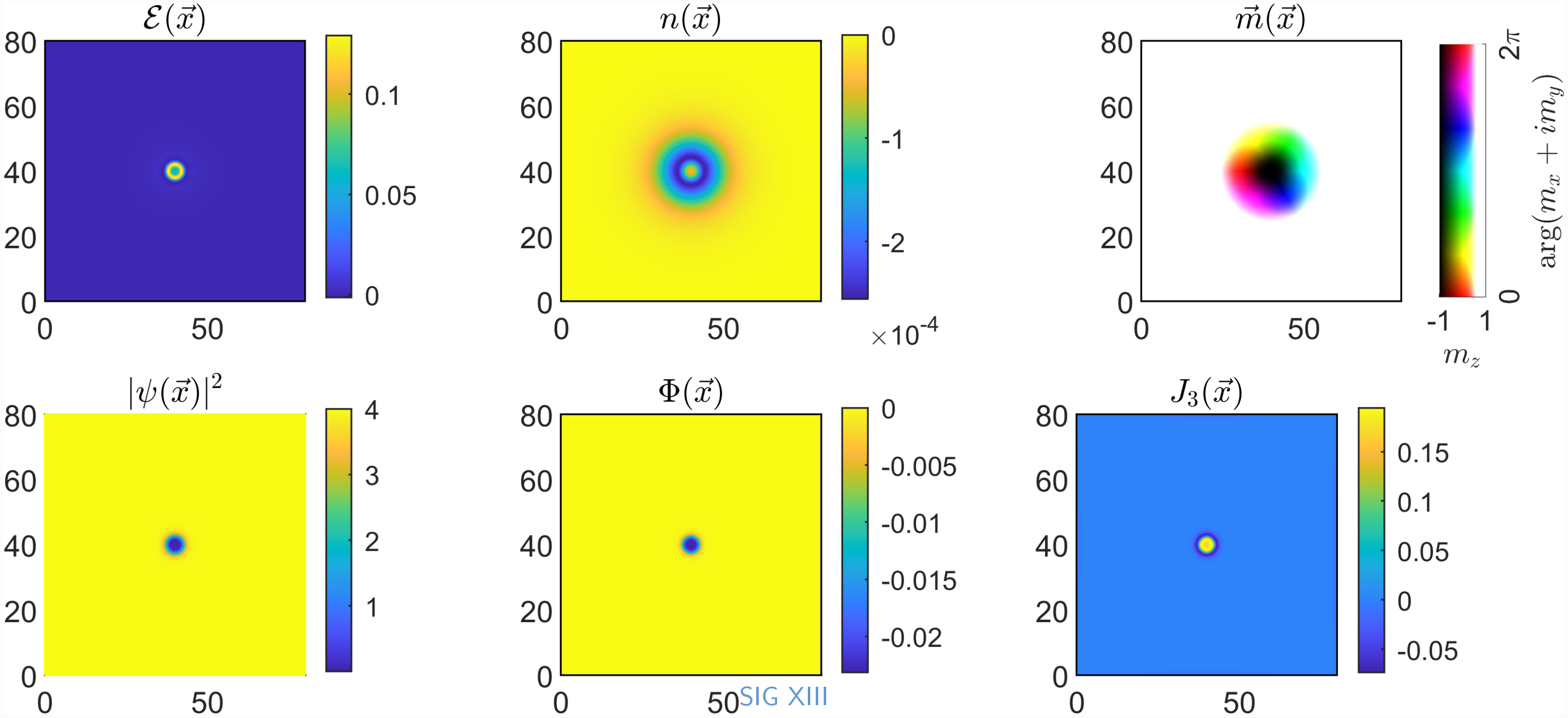
- On the other hand, the magnetic repulsion dominates and force between vortices is repulsive when

$$\frac{\xi_m - \xi_s}{\xi_s \xi_m} > \frac{\xi_m - \lambda}{\lambda \xi_m} \quad \Rightarrow \quad \lambda > \xi_s \tag{56}$$

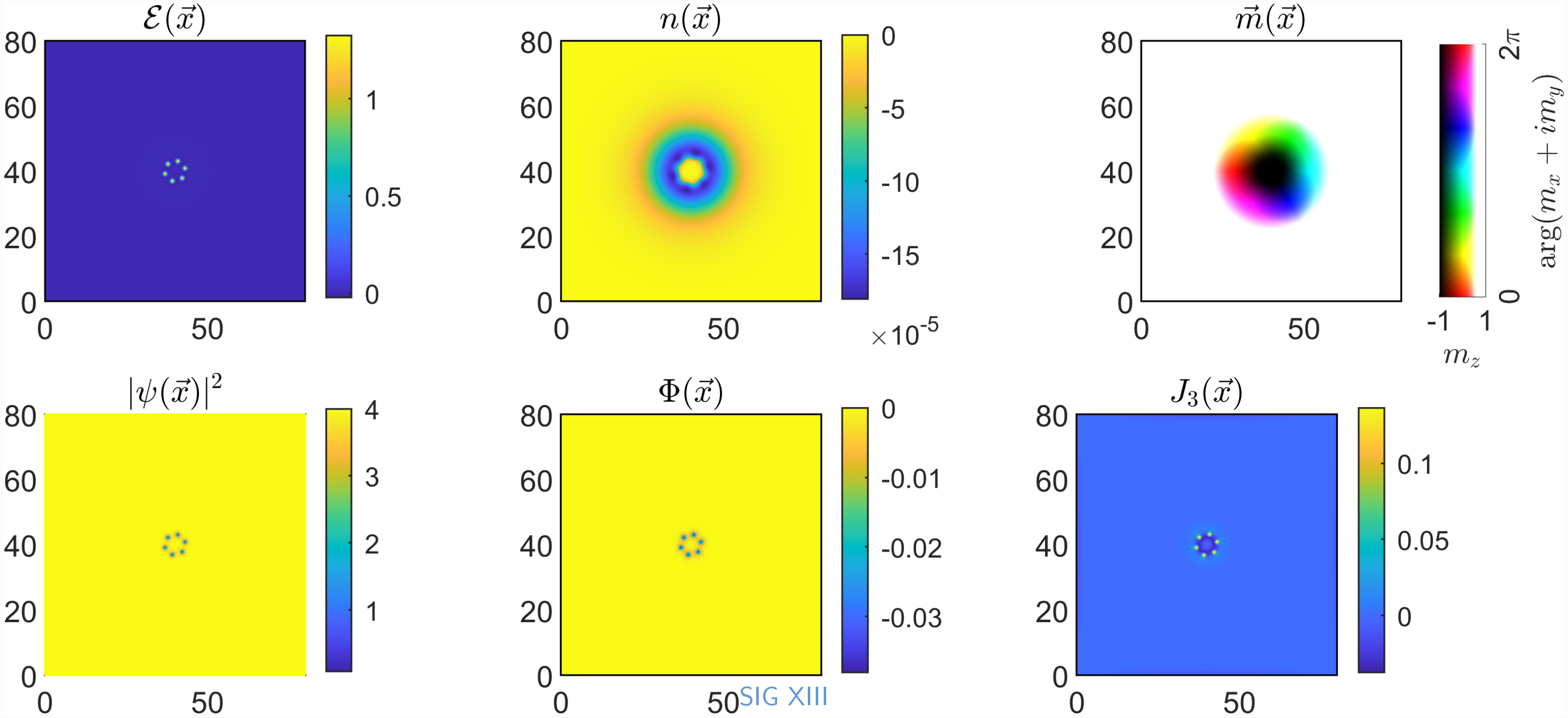
- For **vortex clustering**, we need $\xi_s < \lambda < \xi_m$
- For $qu > 1$, it is always true that $\lambda < \xi_m$

$\Rightarrow$ For type 1.5 superconductivity we only need $\lambda > \xi_s$, which amounts to choosing $b > \frac{1}{2}q^2$

$b < \frac{1}{2}q^2 \quad \Rightarrow \quad$ Type I (6-vortex,1-skyrmion)



$b > \frac{1}{2}q^2 \quad \Rightarrow \quad$ Type 1.5 (6-vortex,1-skyrmion)



Skyrmion contribution to interaction energy

- Impact of skyrmion iso-rotation angle $\chi \in [0, 2\pi)$ on interaction energy,

$$\frac{\partial E_{\text{int}}}{\partial \chi} = -\frac{\pi}{2} c_m^2 \xi_m^3 \sqrt{\frac{\pi \xi_m}{2}} \left[\frac{R}{2} - \frac{17\xi_m}{16}\right] \sin(\chi), \quad \frac{\partial^2 E_{\text{int}}}{\partial \chi^2} = -\frac{\pi}{2} c_m^2 \xi_m^3 \sqrt{\frac{\pi \xi_m}{2}} \left[\frac{R}{2} - \frac{17\xi_m}{16}\right] \cos(\chi) \tag{57}$$

$\Rightarrow$ Extremized for the choice $\chi = k\pi$ with $k \in \{0, 1\}$

- For large R, we have

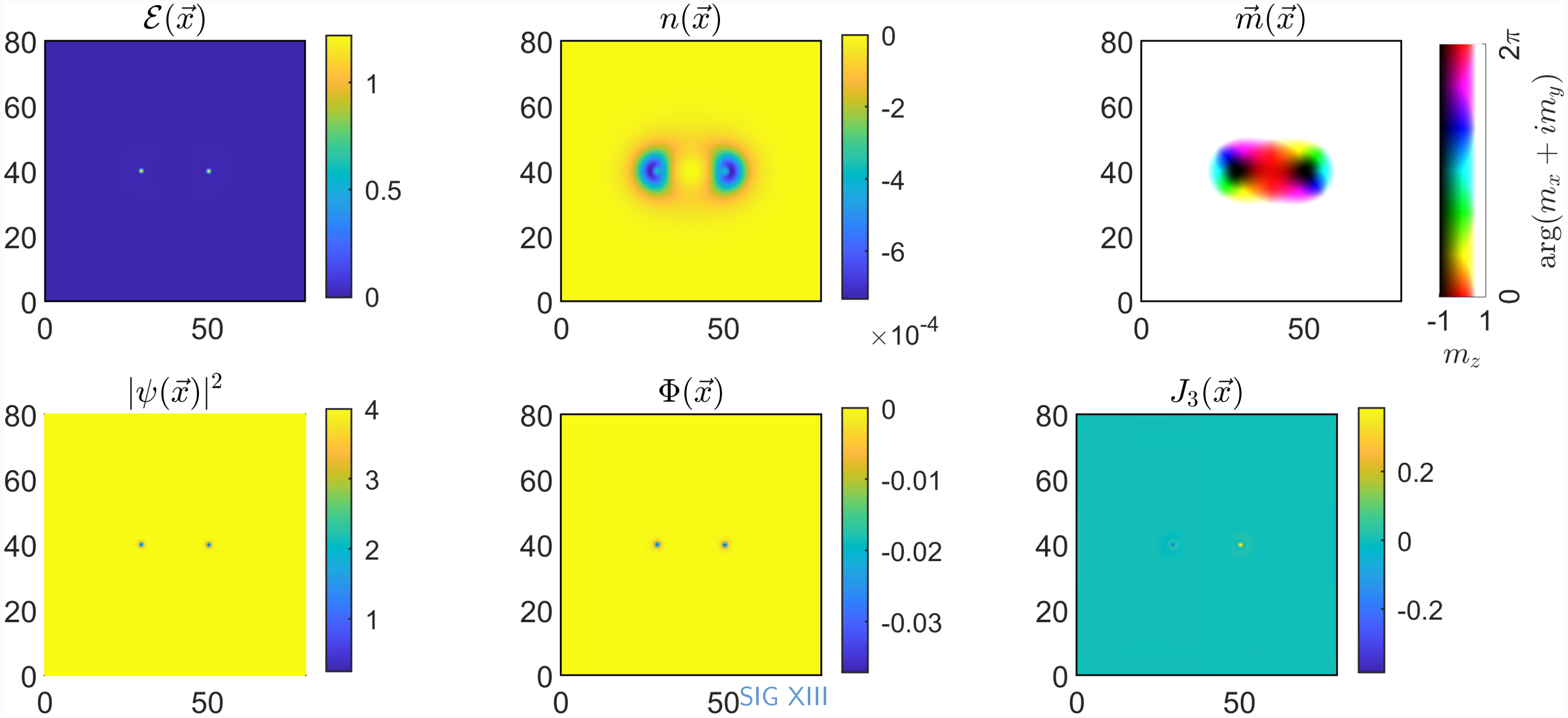
$$R \gg \xi_m \quad \Rightarrow \quad -\frac{\pi}{2} c_m^2 \xi_m^3 \sqrt{\frac{\pi \xi_m}{2}} \left[\frac{R}{2} - \frac{17\xi_m}{16}\right] < 0 \tag{58}$$

- Hessian $\partial^2 E_{\text{int}}/\partial \chi^2$ is positive definite if $k = 1$, i.e. $\chi = \pi$ minimizes E_{int}

$\Rightarrow$ E_{int} is minimized if **skyrmions are anti-aligned**

- Then they experience **short range repulsion** and **long-range attraction**

$b > \frac{1}{2}q^2 \quad \Rightarrow \quad$ Type 1.5 SVP-SVP bound state



Conclusion and further work

Conclusion

- Shown that superconducting vortices can coexist with magnetic skyrmions
- They form skyrmion-vortex pairs due to Zeeman interaction $-\vec{B}\cdot\vec{m}$
- Skyrmions prefer to be anti-aligned, similar to baby Skyrme model
- Vortices exhibit intertype superconductivity with clustering
- SVPs form bound states with other SVPs
- We considered fixed length magnetization

$\Rightarrow$ **Hybridisation of modes** if we let the magnetization length vary (via spin-flip scattering terms)

Further work

- Can extend to unconventional superconductors with equal spin triplet pairing[20]
- Describes the Uranium compounds UGe_2, URhGe and UCoGe

$\Rightarrow$ Multi-component GL model + magnetism[21]:

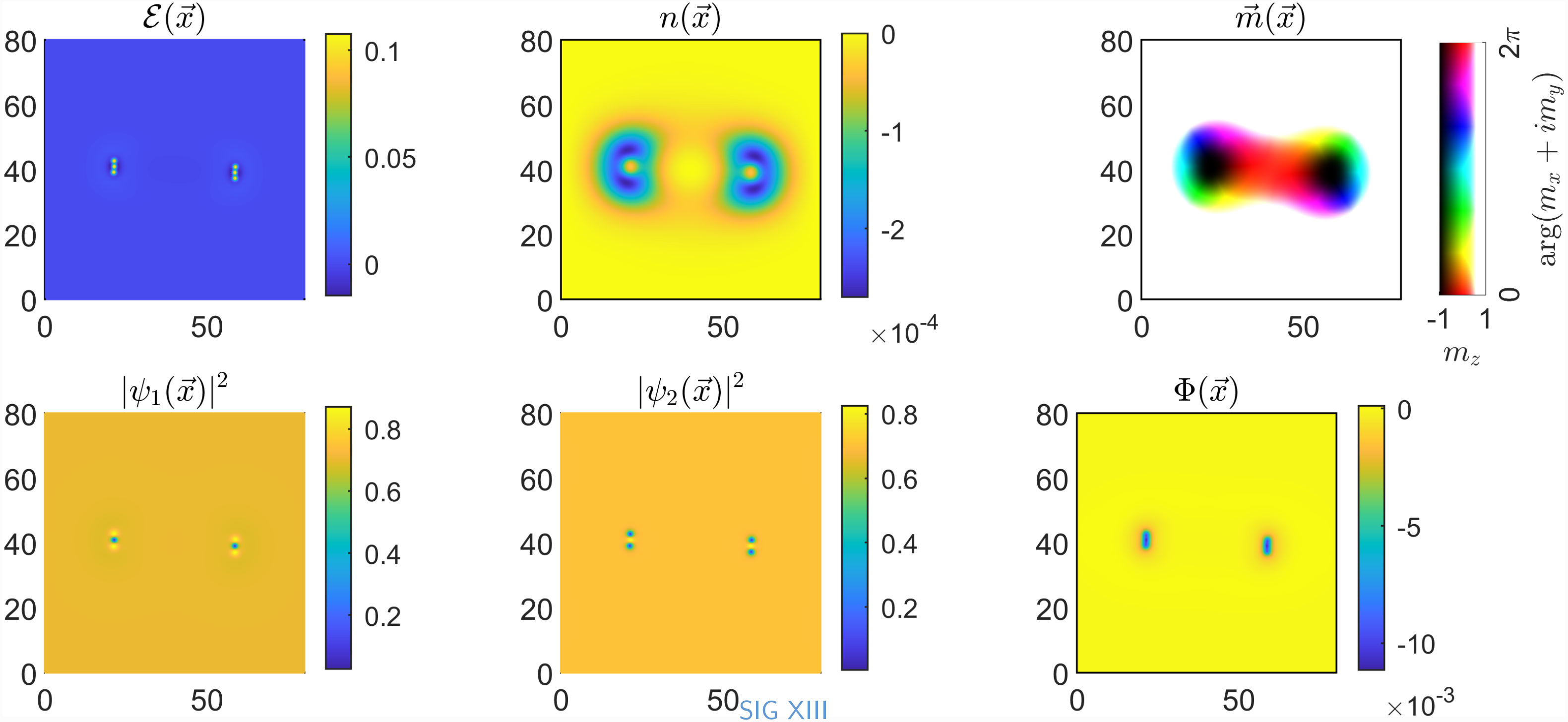
$$F = \int_{\mathbb{R}^2} \mathrm{d}^2x \left\{ \frac{a(T)}{2}|\psi_\alpha|^2 + \frac{b_1}{4}|\psi_\alpha|^4 + b_2|\psi_1|^2|\psi_2|^2 + c\left(\psi_1\psi_2^* + \psi_1^*\psi_2\right) + \frac{1}{2}|\vec{D}\psi_\alpha|^2 + \frac{1}{2}|\vec{\nabla}\times\vec{A}|^2 \right.$$
$$\left. + \frac{\alpha(T)}{2}|\vec{m}|^2 + \frac{\beta}{4}|\vec{m}|^4 + \frac{1}{2}|\nabla\vec{m}|^2 - (\vec{\nabla}\times\vec{A})\cdot\vec{m} \right\}, \tag{59}$$

- Bands can carry different flux quanta $\psi_\alpha = u_\alpha\sigma_\alpha(r)e^{iN_\alpha\theta}$
- Can lead to SVPs carrying **fractional** quantum flux $N = \frac{u_1^2N_1+u_2^2N_2}{u_1^2+u_2^2} = (N_1+N_2)/2$

[20] V.P. Mineev, Phys. Rev. B 95 (2017) 104501

[21] V.P. Mineev, Low Temp. Phys. 44 (2018) 510

$N_1 = 2, N_2 = 4 \Rightarrow N = 3 \quad \rightarrow \quad \Phi_{\mathrm{SVP}} = 1.5\Phi_0$ (No bilinear Josephson)



$N_1 = 1, N_2 = 2 \Rightarrow N = 1.5 \quad \rightarrow \quad \Phi_{\mathrm{SVP}} = 1.5\Phi_0$ (No bilinear Josephson)