Neutron stars from skyrmion branes

Paul Leask

School of Mathematics, University of Leeds, Leeds, LS2 9JT, England, UK

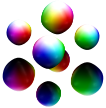
SKCM2, Hiroshima University, November 2023

Table of Contents

Motivation

Motivation

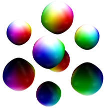
- Main aim: Describe baryonic matter on all scales from **finite atomic nuclei** to **dense infinite nuclear matter**



Motivation

- Main aim: Describe baryonic matter on all scales from **finite atomic nuclei** to **dense infinite nuclear matter**
- The Skyrme model can be used to model **neutron crystals**, which exist under high pressure inside neutron stars [*Nucl. Phys. B* **262** 133–143 (1985)]

Motivation



- Main aim: Describe baryonic matter on all scales from **finite atomic nuclei** to **dense infinite nuclear matter**
- The Skyrme model can be used to model **neutron crystals**, which exist under high pressure inside neutron stars [*Nucl. Phys. B* **262** 133–143 (1985)]
- Within the Skyrme framework for various crystals, the neutron stars so far have been generically **crustless**

Motivation

- Main aim: Describe baryonic matter on all scales from **finite atomic nuclei** to **dense infinite nuclear matter**
- The Skyrme model can be used to model **neutron crystals**, which exist under high pressure inside neutron stars [*Nucl. Phys. B* **262** 133–143 (1985)]
- Within the Skyrme framework for various crystals, the neutron stars so far have been generically **crustless**
- Neutron stars with crusts previously obtained by **interpolating** between **high density Skyrme EoS** and **low density nuclear EoS** [*Phys. Lett. B* **811** 135928 (2020)]

Motivation

- Main aim: Describe baryonic matter on all scales from **finite atomic nuclei** to **dense infinite nuclear matter**
- The Skyrme model can be used to model **neutron crystals**, which exist under high pressure inside neutron stars [*Nucl. Phys. B* **262** 133–143 (1985)]
- Within the Skyrme framework for various crystals, the neutron stars so far have been generically **crustless**
- Neutron stars with crusts previously obtained by **interpolating** between **high density Skyrme EoS** and **low density nuclear EoS** [*Phys. Lett. B* **811** 135928 (2020)]
- Can we obtain a **single EoS** that yields neutron stars with crusts?

Motivation

- Main aim: Describe baryonic matter on all scales from **finite atomic nuclei** to **dense infinite nuclear matter**
- The Skyrme model can be used to model **neutron crystals**, which exist under high pressure inside neutron stars [*Nucl. Phys. B* **262** 133–143 (1985)]
- Within the Skyrme framework for various crystals, the neutron stars so far have been generically **crustless**
- Neutron stars with crusts previously obtained by **interpolating** between **high density Skyrme EoS** and **low density nuclear EoS** [*Phys. Lett. B* **811** 135928 (2020)]
- Can we obtain a **single EoS** that yields neutron stars with crusts?
- Can these neutron stars have sufficient maximal masses?

Skyrme model

Skyrme model

- It was initially believed that descriptions of the low energy regime of QCD must contain explicit quarks

Skyrme model

- It was initially believed that descriptions of the low energy regime of QCD must contain explicit quarks
- Witten showed quarks can be integrated away [*Nucl. Phys. B* **160** 57–115 (1979)]

Skyrme model

- It was initially believed that descriptions of the low energy regime of QCD must contain explicit quarks
- Witten showed quarks can be integrated away [*Nucl. Phys. B* **160** 57–115 (1979)]

$\Longrightarrow$ Degrees of freedom are **hadrons** (mesons and baryons)

Skyrme model

- It was initially believed that descriptions of the low energy regime of QCD must contain explicit quarks
- Witten showed quarks can be integrated away [*Nucl. Phys. B* **160** 57–115 (1979)]

$\Longrightarrow$ Degrees of freedom are **hadrons** (mesons and baryons)

- In the large N_c-limit, QCD can be reduced to an effective field theory of mesons

Skyrme model

- It was initially believed that descriptions of the low energy regime of QCD must contain explicit quarks
- Witten showed quarks can be integrated away [*Nucl. Phys. B* **160** 57–115 (1979)]

$\Longrightarrow$ Degrees of freedom are **hadrons** (mesons and baryons)

- In the large N_c-limit, QCD can be reduced to an effective field theory of mesons
- Skyrme's idea [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]: effective mesonic Lagrangian involving only **pions**, with **baryons emerging as stable topological solitons**

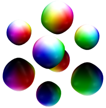
Skyrme model

- The theory has $N_f = 2$ flavours of quarks (u,d) that make up the pion fields

Skyrme model

- The theory has $N_f = 2$ flavours of quarks (u,d) that make up the pion fields
- These are encoded in the Skyrme field

$$\varphi = \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi^1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix} \in \mathrm{SU}(2)$$



Skyrme model

- The theory has $N_f = 2$ flavours of quarks (u,d) that make up the pion fields
- These are encoded in the Skyrme field

$$\varphi = \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi^1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix} \in \mathrm{SU}(2)$$

- This is a map $\varphi : \mathbb{R}^3 \to \mathrm{SU}(2)$ with the constraint $\sigma^2 + \vec{\pi} \cdot \vec{\pi} = 1$

Skyrme model

- The theory has $N_f = 2$ flavours of quarks (u,d) that make up the pion fields
- These are encoded in the Skyrme field

$$\varphi = \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi^1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix} \in \mathrm{SU}(2)$$

- This is a map $\varphi : \mathbb{R}^3 \to \mathrm{SU}(2)$ with the constraint $\sigma^2 + \vec{\pi} \cdot \vec{\pi} = 1$
- We impose the vacuum B.C. $\varphi(\vec{x} \to \infty) = \mathbb{I}_2$

Skyrme model

- The theory has $N_f = 2$ flavours of quarks (u,d) that make up the pion fields
- These are encoded in the Skyrme field

$$\varphi = \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi^1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix} \in \mathrm{SU}(2)$$

- This is a map $\varphi : \mathbb{R}^3 \to \mathrm{SU}(2)$ with the constraint $\sigma^2 + \vec{\pi} \cdot \vec{\pi} = 1$
- We impose the vacuum B.C. $\varphi(\vec{x} \to \infty) = \mathbb{I}_2$
- One-point compactification of space $\mathbb{R}^3 \cup \{\infty\} \cong S^3$

Skyrme model

- The theory has $N_f = 2$ flavours of quarks (u,d) that make up the pion fields
- These are encoded in the Skyrme field

$$\varphi = \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi^1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix} \in \mathrm{SU}(2)$$

- This is a map $\varphi : \mathbb{R}^3 \to \mathrm{SU}(2)$ with the constraint $\sigma^2 + \vec{\pi} \cdot \vec{\pi} = 1$
- We impose the vacuum B.C. $\varphi(\vec{x} \to \infty) = \mathbb{I}_2$
- One-point compactification of space $\mathbb{R}^3 \cup \{\infty\} \cong S^3$

$\Longrightarrow$ Skyrme field is now a map $\varphi : S^3 \to \mathrm{SU}(2)$

Skyrme model

- The theory has $N_f = 2$ flavours of quarks (u,d) that make up the pion fields
- These are encoded in the Skyrme field

$$\mathrm{SU}(2) \ni \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi^1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix} \leftrightarrow (\sigma, \pi_1, \pi_2, \pi_3) \in S^3$$

- This is a map $\varphi : \mathbb{R}^3 \to \mathrm{SU}(2)$ with the constraint $\sigma^2 + \vec{\pi} \cdot \vec{\pi} = 1$
- We impose the vacuum B.C. $\varphi(\vec{x} \to \infty) = \mathbb{I}_2$
- One-point compactification of space $\mathbb{R}^3 \cup \{\infty\} \cong S^3$

$\Longrightarrow$ Skyrme field is now a map $\varphi : S^3 \to \mathrm{SU}(2) \cong S^3$

Skyrme model

- The theory has $N_f = 2$ flavours of quarks (u,d) that make up the pion fields
- These are encoded in the Skyrme field

$$\varphi = \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi^1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix} \cong (\sigma, \pi_1, \pi_2, \pi_3)$$

- This is a map $\varphi : \mathbb{R}^3 \to \mathrm{SU}(2)$ with the constraint $\sigma^2 + \vec{\pi} \cdot \vec{\pi} = 1$
- We impose the vacuum B.C. $\varphi(\vec{x} \to \infty) = \mathbb{I}_2$
- One-point compactification of space $\mathbb{R}^3 \cup \{\infty\} \cong S^3$

$\implies$ Skyrme field is now a map $\varphi : S^3 \to \mathrm{SU}(2) \cong S^3$

- Disjoint homotopy classes labelled by $B \in \pi_3(S^3) = \mathbb{Z}$

$\implies$ Fields are **topologically stable** and B is identified with the **baryon number**

Generalized Skyrme Lagrangian

- Skyrme's original model:

$$\mathcal{L}_{24} = \frac{F_\pi^2}{16\hbar} g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2} g^{\mu\alpha} g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right)$$

$$L_\mu = \varphi^\dagger \partial_\mu \varphi \in \mathfrak{su}(2)$$

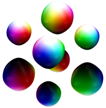
Generalized Skyrme Lagrangian

- Skyrme's original model:

$$\mathscr{L}_{24} = \frac{F_\pi^2}{16\hbar} g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2} g^{\mu\alpha} g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right)$$

$$L_\mu = \varphi^\dagger \partial_\mu \varphi \in \mathfrak{su}(2)$$

- This is $(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2 \cong \mathrm{SO}(4)$ invariant and the **pions are massless**



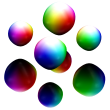
Generalized Skyrme Lagrangian

- Skyrme's original model:

$$\mathscr{L}_{24} = \frac{F_\pi^2}{16\hbar} g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2} g^{\mu\alpha} g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right)$$

$$L_\mu = \varphi^\dagger \partial_\mu \varphi \in \mathfrak{su}(2)$$

- This is $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2 \cong \mathrm{SO}(4)$ invariant and the **pions are massless**
- Boundary condition $\varphi(\vec{x} \to \infty) = \mathbb{I}_2$ spontaneously breaks chiral $\mathrm{SO}(4)$ symmetry to an isospin $\mathrm{SO}(3)$ symmetry, which acts on $\vec{\pi}$



Generalized Skyrme Lagrangian

- The standard massive Skyrme model includes the pion mass potential:

$$\mathscr{L}_{024} = -\frac{F_\pi^2 m_\pi^2}{8\hbar^3}\,\mathrm{Tr}\left(\mathbb{I}_2 - \varphi\right) + \frac{F_\pi^2}{16\hbar} g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2} g^{\mu\alpha} g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right)$$

Generalized Skyrme Lagrangian

- Generalized Skyrme model includes a sextic term [*Phys. Lett. B* **154**, 101–106 (1985)], which is related to the ω-Skyrme model [*Phys. Lett. B* **137**, 251–256 (1984)]:

$$\mathcal{L}_{0246} = -\frac{F_\pi^2 m_\pi^2}{8\hbar^3}\mathrm{Tr}\left(\mathbb{I}_2 - \varphi\right) + \frac{F_\pi^2}{16\hbar}g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2}g^{\mu\alpha}g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right)$$
$$-\pi^4\lambda^2 g^{\mu\nu}\mathcal{B}_\mu\mathcal{B}_\nu, \qquad \lambda^2 = g_\omega^2/(2\pi^4 m_\omega^2)$$

Generalized Skyrme Lagrangian

- Generalized Skyrme model includes a sextic term [*Phys. Lett. B* **154**, 101–106 (1985)], which is related to the ω-Skyrme model [*Phys. Lett. B* **137**, 251–256 (1984)]:

$$\mathscr{L}_{0246} = -\frac{F_\pi^2 m_\pi^2}{8\hbar^3}\operatorname{Tr}\left(\mathbb{I}_2 - \varphi\right) + \frac{F_\pi^2}{16\hbar} g^{\mu\nu}\operatorname{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2} g^{\mu\alpha} g^{\nu\beta}\operatorname{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right) \\ - \pi^4\lambda^2 g^{\mu\nu}\mathscr{B}_\mu\mathscr{B}_\nu$$

- Exhibits short range ω-meson-like repulsion while still describing scalar meson effects

Generalized Skyrme Lagrangian

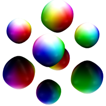
- Generalized Skyrme model includes a sextic term [*Phys. Lett. B* **154**, 101–106 (1985)], which is related to the ω-Skyrme model [*Phys. Lett. B* **137**, 251–256 (1984)]:

$$\mathcal{L}_{0246} = -\frac{F_\pi^2 m_\pi^2}{8\hbar^3}\mathrm{Tr}\left(\mathbb{I}_2 - \varphi\right) + \frac{F_\pi^2}{16\hbar}g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2}g^{\mu\alpha}g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right) \\ - \pi^4\lambda^2 g^{\mu\nu}\mathcal{B}_\mu\mathcal{B}_\nu$$

- Exhibits short range ω-meson-like repulsion while still describing scalar meson effects
- Baryon d.o.f. not explicitly visible $\longrightarrow$ topology: Homotopy invariant $\leftrightarrow$ Baryon number

$$\pi_3(\mathrm{SU}(2)) = \mathbb{Z} \ni B = \int_{\mathbb{R}^3} \mathrm{d}^3x\sqrt{-g}\,\mathcal{B}^0, \quad \mathcal{B}^\mu = \frac{1}{24\pi^2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(L_\nu L_\rho L_\sigma)$$

Generalized Skyrme Lagrangian



- Generalized Skyrme model includes a sextic term [*Phys. Lett. B* **154**, 101–106 (1985)], which is related to the ω-Skyrme model [*Phys. Lett. B* **137**, 251–256 (1984)]:

$$\mathscr{L}_{0246} = -\frac{F_\pi^2 m_\pi^2}{8\hbar^3}\mathrm{Tr}\left(\mathbb{I}_2 - \varphi\right) + \frac{F_\pi^2}{16\hbar} g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2} g^{\mu\alpha} g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right) \\ - \pi^4\lambda^2 g^{\mu\nu}\mathscr{B}_\mu\mathscr{B}_\nu$$

- Exhibits short range ω-meson-like repulsion while still describing scalar meson effects
- Baryon d.o.f. not explicitly visible $\longrightarrow$ topology: Homotopy invariant $\longleftrightarrow$ Baryon number

$$\pi_3(\mathrm{SU}(2)) = \mathbb{Z} \ni B = \int_{\mathbb{R}^3} \mathrm{d}^3x\sqrt{-g}\,\mathscr{B}^0, \quad \mathscr{B}^\mu = \frac{1}{24\pi^2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(L_\nu L_\rho L_\sigma)$$

- Baryons realized as non-perturbative excitations of the pions $\Longrightarrow$ solutions of the Euler–Lagrange field equations - topological solitons (**skyrmions**)

Generalized Skyrme model

- We are interested in **static** solutions and adopt the usual Skyrme units of length $\tilde{L} = 2\hbar/eF_\pi$ and energy $\tilde{E} = F_\pi/4e$

Generalized Skyrme model

- We are interested in **static** solutions and adopt the usual Skyrme units of length $\tilde{L} = 2\hbar/eF_\pi$ and energy $\tilde{E} = F_\pi/4e$
- In Skyrme units the energy-momentum tensor is

$$\frac{\pi^4\lambda^2 e^4 F_\pi^2}{2\hbar^3} = c_6 \boxtimes$$

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}\mathscr{L}_{0246}\right)}{\partial g^{\mu\nu}} \\ &= -\operatorname{Tr}(L_\mu L_\nu) - \frac{1}{4}g^{\alpha\beta}\operatorname{Tr}([L_\mu, L_\alpha][L_\nu, L_\beta]) + 2c_6\mathscr{B}_\mu\mathscr{B}_\nu + g_{\mu\nu}\mathscr{L}_{0246} \end{aligned}$$

Generalized Skyrme model

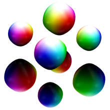
- We are interested in **static** solutions and adopt the usual Skyrme units of length $\tilde{L} = 2\hbar/eF_\pi$ and energy $\tilde{E} = F_\pi/4e$
- In Skyrme units the energy-momentum tensor is

$$T_{\mu\nu} = -\operatorname{Tr}(L_\mu L_\nu) - \frac{1}{4}g^{\alpha\beta}\operatorname{Tr}([L_\mu, L_\alpha][L_\nu, L_\beta]) + 2c_6\mathcal{B}_\mu\mathcal{B}_\nu + g_{\mu\nu}\mathcal{L}_{0246}$$

- The adimensional static energy is thus ($T_{00} = \mathcal{E}_{\text{stat}} + \mathcal{E}_{\text{kin}}$)

$$\begin{aligned} M_B(\varphi, g) &= \int_{\mathbb{R}^3} \mathrm{d}^3x\sqrt{-g}\,\mathcal{E}_{\text{stat}} \\ &= \int_M \mathrm{d}^3x\sqrt{-g}\left\{-\frac{1}{2}g^{ij}\operatorname{Tr}(L_iL_j) - \frac{1}{16}g^{ik}g^{jl}\operatorname{Tr}\left([L_i, L_j][L_k, L_l]\right)\right. \\ m = \frac{2m_\pi}{F_\pi e} \rightarrow \quad & \left. +m^2\operatorname{Tr}\left(\mathbb{I}_2 - \varphi\right) + c_6\frac{\epsilon^{ijk}\epsilon^{abc}}{(24\pi^2\sqrt{-g})^2}\operatorname{Tr}(L_iL_jL_k)\operatorname{Tr}(L_aL_bL_c)\right\} \end{aligned}$$

Generalized Skyrme model



- We are interested in **static** solutions and adopt the usual Skyrme units of length $\tilde{L} = 2\hbar/eF_\pi$ and energy $\tilde{E} = F_\pi/4e$
- In Skyrme units the energy-momentum tensor is

$$T_{\mu\nu} = -\operatorname{Tr}(L_\mu L_\nu) - \frac{1}{4} g^{\alpha\beta} \operatorname{Tr}([L_\mu, L_\alpha][L_\nu, L_\beta]) + 2c_6 \mathcal{B}_\mu \mathcal{B}_\nu + g_{\mu\nu} \mathcal{L}_{0246}$$

- The adimensional <u>static</u> energy is thus

$$M_B(\varphi, g) = \int_{\mathbb{R}^3} \mathrm{d}^3x \sqrt{-g} \left\{ -\frac{1}{2} g^{ij} \operatorname{Tr}(L_i L_j) - \frac{1}{16} g^{ik} g^{jl} \operatorname{Tr}\left([L_i, L_j][L_k, L_l] \right) \right.$$
$$\left. + m^2 \operatorname{Tr}\left(\mathbb{I}_2 - \varphi \right) + c_6 \frac{\epsilon^{ijk} \epsilon^{abc}}{(24\pi^2 \sqrt{-g})^2} \operatorname{Tr}(L_i L_j L_k) \operatorname{Tr}(L_a L_b L_c) \right\}$$

- Skyrmions are **energy minimizing** static solutions of the Euler–Lagrange equations associated to M_B

Linking in the Skyrme model

Baryon number as the linking number of vortices

- The Skyrme field can be written in terms of two vortices $\psi_1, \psi_2 \in \mathbb{C}$ as

$$\varphi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

Baryon number as the linking number of vortices

- The Skyrme field can be written in terms of two vortices $\psi_1, \psi_2 \in \mathbb{C}$ as

$$\varphi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

- Consider the Hopf map $H : S^3 \to S^2$ due to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$

Baryon number as the linking number of vortices

- The Skyrme field can be written in terms of two vortices $\psi_1, \psi_2 \in \mathbb{C}$ as

$$\varphi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

- Consider the Hopf map $H : S^3 \to S^2$ due to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$
- The map $\varphi : \mathbb{R}^3 \cup \{\infty\} \cong S^3 \to S^3$ of degree B has Hopf charge $Q = B$ under the Hopf map $H : S^3 \to S^2$ [*Phys. Rev. D* **101**, 065011 (2020)]

Baryon number as the linking number of vortices

- The Skyrme field can be written in terms of two vortices $\psi_1, \psi_2 \in \mathbb{C}$ as

$$\varphi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

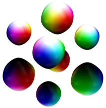
- Consider the Hopf map $H : S^3 \to S^2$ due to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$
- The map $\varphi : \mathbb{R}^3 \cup \{\infty\} \cong S^3 \to S^3$ of degree B has Hopf charge $Q = B$ under the Hopf map $H : S^3 \to S^2$ [*Phys. Rev. D* **101**, 065011 (2020)]
- Distinct regular points on S^2 under $H \circ \varphi : \mathbb{R}^3 \cup \{\infty\} \to S^2$ have preimages on $\mathbb{R}^3 \cup \{\infty\}$ that are linked $Q = B$ times

Baryon number as the linking number of vortices

- The Skyrme field can be written in terms of two vortices $\psi_1, \psi_2 \in \mathbb{C}$ as

$$\varphi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

- Consider the Hopf map $H : S^3 \to S^2$ due to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$
- The map $\varphi : \mathbb{R}^3 \cup \{\infty\} \cong S^3 \to S^3$ of degree B has Hopf charge $Q = B$ under the Hopf map $H : S^3 \to S^2$ [*Phys. Rev. D* **101**, 065011 (2020)]
- Distinct regular points on S^2 under $H \circ \varphi : \mathbb{R}^3 \cup \{\infty\} \to S^2$ have preimages on $\mathbb{R}^3 \cup \{\infty\}$ that are linked $Q = B$ times
- Antipodal points on S^2 identified as **vortex lines** (zeros) and are linked $Q = B$ times



Skyrmion solutions

$B = 1$ 🦔 skyrmion [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]

- The hedgehog field is $\varphi(\vec{x}) = \exp\left(i f(r) \vec{x} \cdot \vec{\tau}\right)$

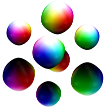
$B = 1$ 🦔 skyrmion [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]

- The hedgehog field is $\varphi(\vec{x}) = \exp\left(i f(r)\vec{x}\cdot\vec{\tau}\right)$
- Known as hedgehogs because the pion fields point radially outwards

$B = 1$ skyrmion [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]

- The hedgehog field is $\varphi(\vec{x}) = \exp\left(if(r)\vec{x} \cdot \vec{\tau}\right)$
- Known as hedgehogs because the pion fields point radially outwards
- Profile function $f(r)$ must satisfy the B.C.s $f(\infty) = 0$ and $f(0) = \pi$

$B = 1$  skyrmion [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]



- The hedgehog field is $\varphi(\vec{x}) = \exp\left(i f(r)\vec{x}\cdot\vec{\tau}\right)$
- Known as hedgehogs because the pion fields point radially outwards
- Profile function $f(r)$ must satisfy the B.C.s $f(\infty) = 0$ and $f(0) = \pi$
- The hedgehog solution has baryon number $B = 1$ since

$$B = -\frac{1}{2\pi^2}\int_0^{\infty}\frac{\sin^2 f}{r^2}\frac{\mathrm{d}f}{\mathrm{d}r}4\pi r^2 dr = \frac{1}{\pi}f(0) = 1$$

$B = 1$ 🦔 skyrmion [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]

- The hedgehog field is $\varphi(\vec{x}) = \exp\left(i f(r) \vec{x} \cdot \vec{\tau}\right)$
- Known as hedgehogs because the pion fields point radially outwards
- Profile function $f(r)$ must satisfy the B.C.s $f(\infty) = 0$ and $f(0) = \pi$
- The hedgehog solution has baryon number $B = 1$ since

$$B = -\frac{1}{2\pi^2} \int_0^\infty \frac{\sin^2 f}{r^2} \frac{\mathrm{d}f}{\mathrm{d}r} 4\pi r^2 dr = \frac{1}{\pi} f(0) = 1$$

- For the hedgehog ansatz, the (massless) static energy is

$$M_1 = 4\pi \int_0^\infty \left[r^2 \left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 + 2\sin^2 f \left(1 + \left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 \right) + \frac{\sin^4 f}{r^2} \right] \mathrm{d}r$$

$B = 1$ skyrmion [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]

- The hedgehog field is $\varphi(\vec{x}) = \exp\left(i f(r)\vec{x}\cdot\vec{\tau}\right)$
- Known as hedgehogs because the pion fields point radially outwards
- Profile function $f(r)$ must satisfy the B.C.s $f(\infty) = 0$ and $f(0) = \pi$
- The hedgehog solution has baryon number $B = 1$ since

$$B = -\frac{1}{2\pi^2}\int_0^\infty \frac{\sin^2 f}{r^2}\frac{\mathrm{d}f}{\mathrm{d}r}4\pi r^2 dr = \frac{1}{\pi}f(0) = 1$$

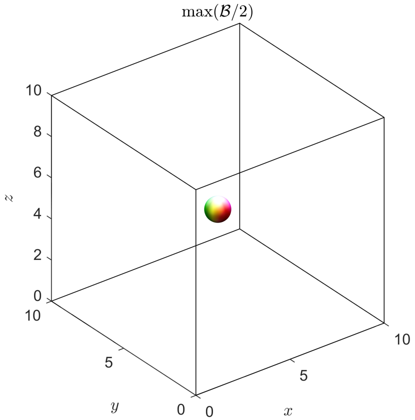
- For the hedgehog ansatz, the (massless) static energy is

$$M_1 = 4\pi\int_0^\infty \left[r^2\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + 2\sin^2 f\left(1 + \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2\right) + \frac{\sin^4 f}{r^2}\right]\mathrm{d}r$$

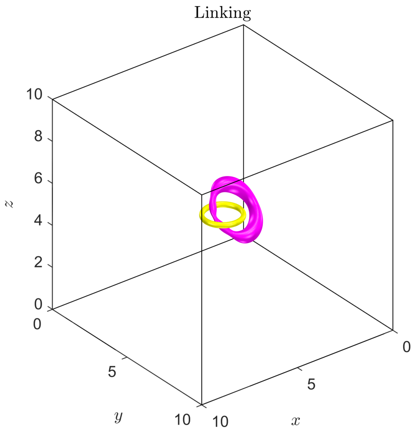
- E–L equations reduce to a 2nd order non-linear ODE that can only be solved numerically,

$$\left(r^2 + 2\sin^2 f\right)\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}f}{\mathrm{d}r} + \sin 2f\left[\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 - 1 - \frac{\sin^2 f}{r^2}\right] = 0$$

$B = 1$ 🦔 skyrmion [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]



(a) Isosurface plot of the baryon density $\mathcal{B}_0$

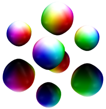


(b) Linking of two preimages

How to construct larger Skyrmions

- Asymptotic interactions of two $B = 1$ skyrmions have preferred orientation (**attractive channel**) [*Commun. Math. Phys.* **245** 123–147 (2004)]

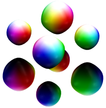
How to construct larger Skyrmions



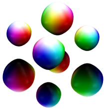
- Asymptotic interactions of two $B = 1$ skyrmions have preferred orientation (**attractive channel**) [*Commun. Math. Phys.* **245** 123–147 (2004)]
- Can place $B = 1$ Skyrmions in the attractive channel on a subcluster of a bravais lattice, most favourable being a FCC lattice [*Phys. Lett. B* **208** 491–494 (1988)]

How to construct larger Skyrmions

- Asymptotic interactions of two $B = 1$ skyrmions have preferred orientation (**attractive channel**) [*Commun. Math. Phys.* **245** 123–147 (2004)]
- Can place $B = 1$ Skyrmions in the attractive channel on a subcluster of a bravais lattice, most favourable being a FCC lattice [*Phys. Lett. B* **208** 491–494 (1988)]
- Skyrme fields can be constructed using a rational map approximation (RMA) [*Nucl. Phys. B* **510**, 507–537 (1998)]

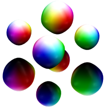


How to construct larger Skyrmions



- Asymptotic interactions of two $B = 1$ skyrmions have preferred orientation (**attractive channel**) [*Commun. Math. Phys.* **245** 123–147 (2004)]
- Can place $B = 1$ Skyrmions in the attractive channel on a subcluster of a bravais lattice, most favourable being a FCC lattice [*Phys. Lett. B* **208** 491–494 (1988)]
- Skyrme fields can be constructed using a rational map approximation (RMA) [*Nucl. Phys. B* **510**, 507–537 (1998)]
- One could also relate Skyrmions to instantons via the Atiyah–Manton approximation [*Phys. Lett. B* **222** 438–442 (1989)] or using ADHM data [*Nonlinearity* **35**, 3944–3990 (2022)]

How to construct larger Skyrmions



- Asymptotic interactions of two $B = 1$ skyrmions have preferred orientation (**attractive channel**) [*Commun. Math. Phys.* **245** 123–147 (2004)]
- Can place $B = 1$ Skyrmions in the attractive channel on a subcluster of a bravais lattice, most favourable being a FCC lattice [*Phys. Lett. B* **208** 491–494 (1988)]
- Skyrme fields can be constructed using a rational map approximation (RMA) [*Nucl. Phys. B* **510**, 507–537 (1998)]
- One could also relate Skyrmions to instantons via the Atiyah–Manton approximation [*Phys. Lett. B* **222** 438–442 (1989)] or using ADHM data [*Nonlinearity* **35**, 3944–3990 (2022)]
- Quite large skyrmions (up to $B = 108$) have been constructed by gluing α-particles together [*Proc. R. Soc. A* **463** 261–279 (2007)] or using a multi-layer RMA based on the Skyrme crystal [*Phys. Rev. D* **87** 0850834 (2013)]

Skyrmions from rational maps [*Nucl. Phys. B* **510**, 507–537 (1998)]

- Rational maps are functions from $S^2 \to S^2$, whereas as $\varphi : \mathbb{R}^3 \to S^3$.

Skyrmions from rational maps [*Nucl. Phys. B* **510**, 507–537 (1998)]

- Rational maps are functions from $S^2 \to S^2$, whereas as $\varphi : \mathbb{R}^3 \to S^3$.
- Identify RM target S^2 with spheres of constant latitude on S^3, and RM domain S^2 with spheres in $\mathbb{R}^3$ of radius r.

Skyrmions from rational maps [*Nucl. Phys. B* **510**, 507–537 (1998)]

- Rational maps are functions from $S^2 \to S^2$, whereas as $\varphi : \mathbb{R}^3 \to S^3$.
- Identify RM target S^2 with spheres of constant latitude on S^3, and RM domain S^2 with spheres in $\mathbb{R}^3$ of radius r.
- Using polar coords for $\mathbb{R}^3$, $z = \tan(\theta/2)\exp(i\varphi)$, with radius r, the RM ansatz is

$$\varphi(r, z) = \exp\left[\frac{if(r)}{1+|R|^2}\begin{pmatrix}1-|R|^2 & 2\bar{R} \\ 2R & |R|^2-1\end{pmatrix}\right],$$

Skyrmions from rational maps [*Nucl. Phys. B* **510**, 507–537 (1998)]

- Rational maps are functions from $S^2 \to S^2$, whereas as $\varphi : \mathbb{R}^3 \to S^3$.
- Identify RM target S^2 with spheres of constant latitude on S^3, and RM domain S^2 with spheres in $\mathbb{R}^3$ of radius r.
- Using polar coords for $\mathbb{R}^3$, $z = \tan(\theta/2)\exp(i\varphi)$, with radius r, the RM ansatz is

$$\varphi(r, z) = \exp\left[\frac{if(r)}{1+|R|^2}\begin{pmatrix} 1-|R|^2 & 2\bar{R} \\ 2R & |R|^2-1 \end{pmatrix}\right],$$

- $f(r)$ is a radial profile function with B.C.s $f(0) = \pi$ and $f(\infty) = 0$

Skyrmions from rational maps [*Nucl. Phys. B* **510**, 507–537 (1998)]

- Rational maps are functions from $S^2 \to S^2$, whereas as $\varphi : \mathbb{R}^3 \to S^3$.
- Identify RM target S^2 with spheres of constant latitude on S^3, and RM domain S^2 with spheres in $\mathbb{R}^3$ of radius r.
- Using polar coords for $\mathbb{R}^3$, $z = \tan(\theta/2)\exp(i\varphi)$, with radius r, the RM ansatz is

$$\varphi(r, z) = \exp\left[\frac{if(r)}{1+|R|^2}\begin{pmatrix} 1-|R|^2 & 2\bar{R} \\ 2R & |R|^2-1 \end{pmatrix}\right],$$

- $f(r)$ is a radial profile function with B.C.s $f(0) = \pi$ and $f(\infty) = 0$
- $R(z) = p(z)/q(z)$ is a rational map of degree $B = \max(\deg p, \deg q)$.

Skyrmions from rational maps [*Nucl. Phys. B* **510**, 507–537 (1998)]

- Rational maps are functions from $S^2 \to S^2$, whereas as $\varphi : \mathbb{R}^3 \to S^3$.
- Identify RM target S^2 with spheres of constant latitude on S^3, and RM domain S^2 with spheres in $\mathbb{R}^3$ of radius r.
- Using polar coords for $\mathbb{R}^3$, $z = \tan(\theta/2)\exp(i\varphi)$, with radius r, the RM ansatz is

$$\varphi(r, z) = \exp\left[\frac{i f(r)}{1 + |R|^2}\begin{pmatrix} 1 - |R|^2 & 2\bar{R} \\ 2R & |R|^2 - 1 \end{pmatrix}\right],$$

- $f(r)$ is a radial profile function with B.C.s $f(0) = \pi$ and $f(\infty) = 0$
- $R(z) = p(z)/q(z)$ is a rational map of degree $B = \max(\deg p, \deg q)$.
- Rational maps for $B = 1, \ldots, 4$:

B	$R(z)$	Symmetry
1	z	$O(3)$
2	z^2	$O(2) \times \mathbb{Z}$
3	$\frac{z^3 - \sqrt{3} i z}{\sqrt{3} i z^2 - 1}$	T_d
4	$\frac{z^4 + 2\sqrt{3} i z^2 + 1}{z^4 - 2\sqrt{3} i z^2 + 1}$	O_h

Skyrmions from rational maps [*Nucl. Phys. B* **510**, 507–537 (1998)]

- Substituting the RMA into the massless static energy functional yields

$$M_B = 4\pi \int_0^\infty r^2 \left\{ \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + 2B\sin^2 f \left(\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + 1 \right) + \mathcal{I}\frac{\sin^4 f}{r^2} \right\} \mathrm{d}r,$$

Skyrmions from rational maps [*Nucl. Phys. B* **510**, 507–537 (1998)]

- Substituting the RMA into the massless static energy functional yields

$$M_B = 4\pi \int_0^\infty r^2 \left\{ \left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 + 2B \sin^2 f \left(\left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 + 1 \right) + \mathcal{I} \frac{\sin^4 f}{r^2} \right\} \mathrm{d}r,$$

- $\mathcal{I}$ is the purely angular integral to be minimised for choice of rational map R:

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{\mathrm{d}R}{\mathrm{d}z} \right| \right)^4 \frac{2i \mathrm{d}z \mathrm{d}\bar{z}}{(1 + |z|^2)^2}.$$

Skyrmions from rational maps [*Nucl. Phys. B* **510**, 507–537 (1998)]

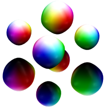
- Substituting the RMA into the massless static energy functional yields

$$M_B = 4\pi \int_0^\infty r^2 \left\{ \left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 + 2B \sin^2 f \left(\left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 + 1 \right) + \mathcal{I} \frac{\sin^4 f}{r^2} \right\} \mathrm{d}r,$$

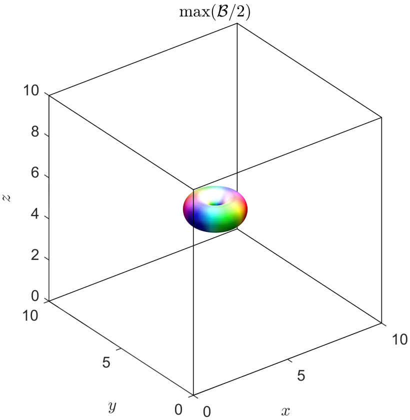
- $\mathcal{I}$ is the purely angular integral to be minimised for choice of rational map R:

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{\mathrm{d}R}{\mathrm{d}z} \right| \right)^4 \frac{2i\,\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}.$$

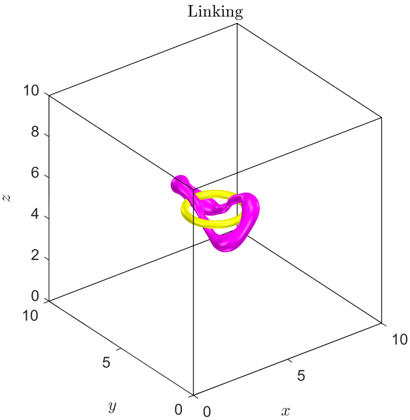
- Optimising $\mathcal{I}$ and the profile function $f(r)$ gives approximate Skyrmions, but further numerical relaxation is required to find true Skyrmions.



$B = 2$ skyrmion [*Phys. Lett. B* **195**, 235–239 (1987)]

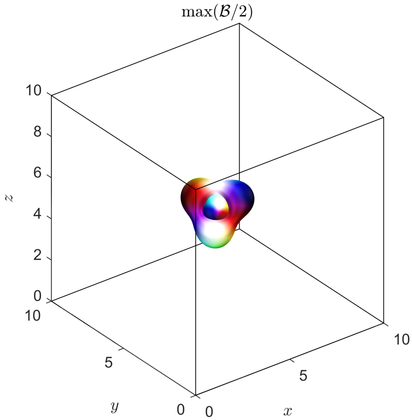


(a) Isosurface plot of the baryon density $\mathscr{B}_0$

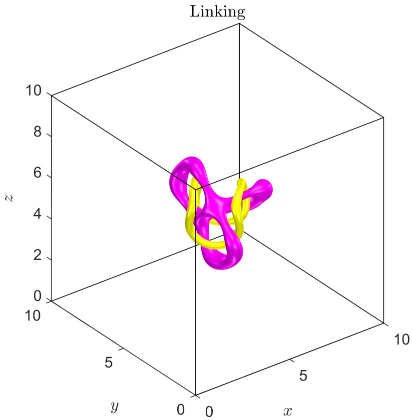


(b) Linking of two preimages

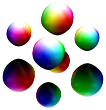
$B = 3$ skyrmion [*Phys. Lett. B* **235**, 147–152 (1990)]



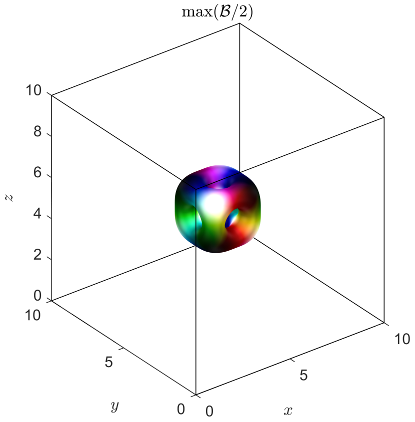
(a) Isosurface plot of the baryon density $\mathscr{B}_0$



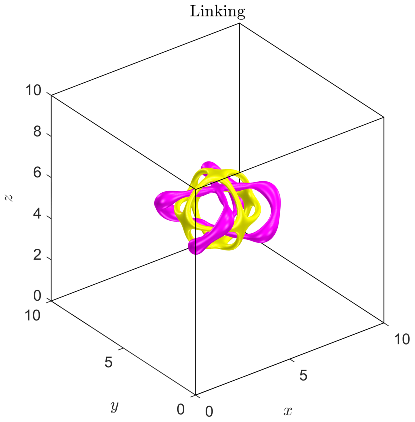
(b) Linking of two preimages



$B = 4$ skyrmion [*Phys. Lett. B* **235**, 147–152 (1990)]



(a) Isosurface plot of the baryon density $\mathscr{B}_0$



(b) Linking of two preimages

Skyrmion crystals and phases of skyrmion matter

Motivation of Skyrme crystals

- Aim: construct an **equation of state** (EoS) to model neutron stars

Motivation of Skyrme crystals

- Aim: construct an **equation of state** (EoS) to model neutron stars

$\Longrightarrow$ We need to understand **phases** and **phase transitions** of nuclear matter

Motivation of Skyrme crystals

- Aim: construct an **equation of state** (EoS) to model neutron stars

$\Longrightarrow$ We need to understand **phases** and **phase transitions** of nuclear matter

- Ground state of dense nuclear matter has a **crystalline** structure in the classical approximation

Motivation of Skyrme crystals

- Aim: construct an **equation of state** (EoS) to model neutron stars

$\Longrightarrow$ We need to understand **phases** and **phase transitions** of nuclear matter

- Ground state of dense nuclear matter has a **crystalline** structure in the classical approximation
- In order to determine skyrmion crystals, we first need to define what a crystal really is!

Skyrmion crystals

- Skyrme crystals are energy minimizing maps

$$\varphi : \mathbb{R}^3/\Lambda_\diamond \to SU(2), \quad \Lambda_\diamond = \left\{ n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z} \right\}$$

Skyrmion crystals

- Skyrme crystals are energy minimizing maps

$$\varphi : \mathbb{R}^3/\Lambda_\circ \to SU(2), \quad \Lambda_\circ = \left\{ n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z} \right\}$$

- They are critical and stable w.r.t. variations of the lattice Λ about $\Lambda_\circ$

Skyrmion crystals

- Skyrme crystals are energy minimizing maps

$$\varphi : \mathbb{R}^3/\Lambda_\circ \to SU(2), \quad \Lambda_\circ = \left\{ n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z} \right\}$$

- They are critical and stable w.r.t. variations of the lattice Λ about $\Lambda_\circ$
- Key idea [*Comm. Math. Phys.* **332** 355-377 (2014)]: Identify all 3-tori via diffeomorphism (with $\mathbb{T}^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$)

$$F : (\mathbb{T}^3, g) \to (\mathbb{R}^3/\Lambda, d), \quad (x^1, x^2, x^3) \mapsto x^1\vec{X}_1 + x^2\vec{X}_2 + x^3\vec{X}_3$$

Skyrmion crystals

- Skyrme crystals are energy minimizing maps

$$\varphi : \mathbb{R}^3/\Lambda_\circ \to \mathrm{SU}(2), \quad \Lambda_\circ = \left\{ n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z} \right\}$$

- They are critical and stable w.r.t. variations of the lattice Λ about $\Lambda_\circ$
- Key idea [*Comm. Math. Phys.* **332** 355-377 (2014)]: Identify all 3-tori via diffeomorphism (with $\mathbb{T}^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$)

$$F : (\mathbb{T}^3, g) \to (\mathbb{R}^3/\Lambda, d), \quad (x^1, x^2, x^3) \mapsto x^1\vec{X}_1 + x^2\vec{X}_2 + x^3\vec{X}_3$$

- The metric on $\mathbb{T}^3$ is the pullback $g = F^*d = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, $g_{ij} = \vec{X}_i \cdot \vec{X}_j$

Skyrmion crystals

- Skyrme crystals are energy minimizing maps

$$\varphi : \mathbb{R}^3/\Lambda_\circ \to \mathrm{SU}(2), \quad \Lambda_\circ = \left\{ n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z} \right\}$$

- They are critical and stable w.r.t. variations of the lattice Λ about $\Lambda_\circ$
- Key idea [*Comm. Math. Phys.* **332** 355-377 (2014)]: Identify all 3-tori via diffeomorphism (with $\mathbb{T}^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$)

$$F : (\mathbb{T}^3, g) \to (\mathbb{R}^3/\Lambda, d), \quad (x^1, x^2, x^3) \mapsto x^1\vec{X}_1 + x^2\vec{X}_2 + x^3\vec{X}_3$$

- The metric on $\mathbb{T}^3$ is the pullback $g = F^* d = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, $g_{ij} = \vec{X}_i \cdot \vec{X}_j$
- Fix Skyrme field to be the map $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$

Skyrmion crystals

- Skyrme crystals are energy minimizing maps

$$\varphi : \mathbb{R}^3/\Lambda_\circ \to \mathrm{SU}(2), \quad \Lambda_\circ = \left\{ n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z} \right\}$$

- They are critical and stable w.r.t. variations of the lattice Λ about $\Lambda_\circ$
- Key idea [*Comm. Math. Phys.* **332** 355-377 (2014)]: Identify all 3-tori via diffeomorphism (with $\mathbb{T}^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$)

$$F : (\mathbb{T}^3, g) \to (\mathbb{R}^3/\Lambda, d), \quad (x^1, x^2, x^3) \mapsto x^1\vec{X}_1 + x^2\vec{X}_2 + x^3\vec{X}_3$$

- The metric on $\mathbb{T}^3$ is the pullback $g = F^*d = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, $g_{ij} = \vec{X}_i \cdot \vec{X}_j$
- Fix Skyrme field to be the map $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$
- Vary metric g_s with $g_0 = F^*d \Longleftrightarrow$ vary lattice Λ_s with $\Lambda_0 = \Lambda$

Skyrmion crystals

- Skyrme crystals are energy minimizing maps

$$\varphi : \mathbb{R}^3/\Lambda_\circ \to \mathrm{SU}(2), \quad \Lambda_\circ = \left\{ n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z} \right\}$$

- They are critical and stable w.r.t. variations of the lattice Λ about $\Lambda_\circ$
- Key idea [*Comm. Math. Phys.* **332** 355-377 (2014)]: Identify all 3-tori via diffeomorphism (with $\mathbb{T}^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$)

$$F : (\mathbb{T}^3, g) \to (\mathbb{R}^3/\Lambda, d), \quad (x^1, x^2, x^3) \mapsto x^1\vec{X}_1 + x^2\vec{X}_2 + x^3\vec{X}_3$$

- The metric on $\mathbb{T}^3$ is the pullback $g = F^*d = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, $g_{ij} = \vec{X}_i \cdot \vec{X}_j$
- Fix Skyrme field to be the map $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$
- Vary metric g_s with $g_0 = F^*d \iff$ vary lattice Λ_s with $\Lambda_0 = \Lambda$
- Energy minimized over variations of $g \iff$ energy minimizing period lattice $\Lambda_\circ$

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- For fixed $\mathscr{L}_{024}$-field φ, there always **exists** a critical point of $M_B(\varphi, g)$ w.r.t. variations of g and it is in fact a **unique** c.p. (generalizes to $\mathscr{L}_{0246}$-model)

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- For fixed $\mathscr{L}_{024}$-field φ, there always **exists** a critical point of $M_B(\varphi, g)$ w.r.t. variations of g and it is in fact a **unique** c.p. (generalizes to $\mathscr{L}_{0246}$-model)
- Four crystal solutions were found for unit cells with charge $B_{\text{cell}} = 4$

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- For fixed $\mathscr{L}_{024}$-field φ, there always **exists** a critical point of $M_B(\varphi, g)$ w.r.t. variations of g and it is in fact a **unique** c.p. (generalizes to $\mathscr{L}_{0246}$-model)
- Four crystal solutions were found for unit cells with charge $B_{\text{cell}} = 4$
- These are the φ_{FCC}, φ_α, φ_{string} and φ_{brane} crystals

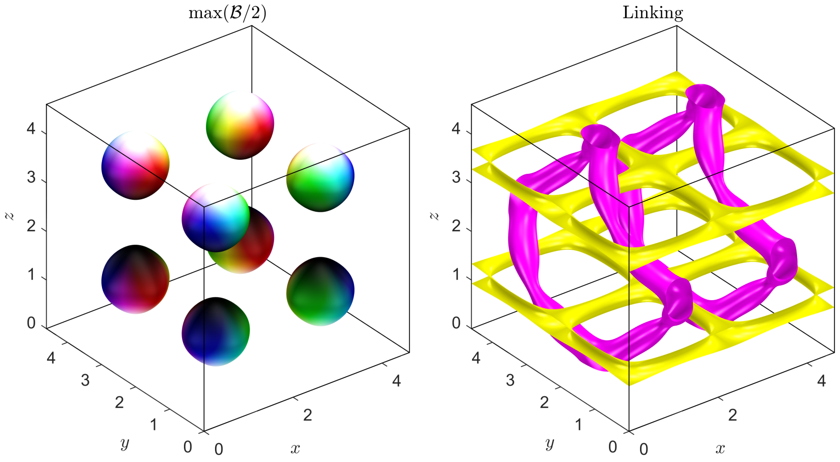
Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- For fixed $\mathscr{L}_{024}$-field φ, there always **exists** a critical point of $M_B(\varphi, g)$ w.r.t. variations of g and it is in fact a **unique** c.p. (generalizes to $\mathscr{L}_{0246}$-model)
- Four crystal solutions were found for unit cells with charge $B_{\text{cell}} = 4$
- These are the φ_{FCC}, φ_α, φ_{string} and φ_{brane} crystals
- The φ_{FCC}-crystal [*Phys. Lett. B* **208** 491–494 (1988)] can be obtained from a Fourier series-like expansion as an initial configuration [*Nucl. Phys. A* **501** 801–812 (1989)],

$$\varphi^0 = -c_1 c_2 c_3, \quad \varphi^1 = s_1 \sqrt{1 - \frac{s_2^2}{2} - \frac{s_3^2}{2} + \frac{s_2^2 s_3^2}{3}}, \quad \text{and cyclic,}$$

where $s_i = \sin(2\pi x^i / L)$ and $c_i = \cos(2\pi x^i / L)$, with initial metric $g = L^2 \mathbb{I}_3$.

Half-skyrmion (FCC) crystal [*Phys. Lett. B* **208** 491–494 (1988)]



(a) Isosurface plot of the baryon density $\mathscr{B}_0$

(b) Linking of two preimages

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- From φ_{FCC}, the other three crystals can be constructed by applying a chiral $\mathrm{SO}(4)$ transformation $Q \in \mathrm{SO}(4)$, such that $\varphi = Q\varphi_{\mathrm{FCC}}$, and minimizing M_B w.r.t. variations of φ and g

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- From $\varphi_{\rm FCC}$, the other three crystals can be constructed by applying a chiral $\mathrm{SO}(4)$ transformation $Q \in \mathrm{SO}(4)$, such that $\varphi = Q\varphi_{\rm FCC}$, and minimizing M_B w.r.t. variations of φ and g
- These are

$$Q \in \left\{ \mathbb{I}_4, \underbrace{\begin{pmatrix} (0,-1,1,1)/\sqrt{3} \\ * \end{pmatrix}}_{Q_\alpha}, \underbrace{\begin{pmatrix} (0,0,0,1) \\ * \end{pmatrix}}_{Q_{\rm brane}}, \underbrace{\begin{pmatrix} (0,0,1,1)/\sqrt{2} \\ * \end{pmatrix}}_{Q_{\rm string}} \right\}.$$

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- From $\varphi_{\rm FCC}$, the other three crystals can be constructed by applying a chiral $\mathrm{SO}(4)$ transformation $Q \in \mathrm{SO}(4)$, such that $\varphi = Q\varphi_{\rm FCC}$, and minimizing M_B w.r.t. variations of φ and g
- These are

$$Q \in \left\{ \mathbb{I}_4, \underbrace{\begin{pmatrix} (0,-1,1,1)/\sqrt{3} \\ * \end{pmatrix}}_{Q_\alpha}, \underbrace{\begin{pmatrix} (0,0,0,1) \\ * \end{pmatrix}}_{Q_{\rm brane}}, \underbrace{\begin{pmatrix} (0,0,1,1)/\sqrt{2} \\ * \end{pmatrix}}_{Q_{\rm string}} \right\}.$$

- The $\varphi_{\rm brane}$-crystal is the lowest energy solution at all baryon densities $n_B = B_{\rm cell}/V_{\rm cell}$

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- From φ_{FCC}, the other three crystals can be constructed by applying a chiral $\mathrm{SO}(4)$ transformation $Q \in \mathrm{SO}(4)$, such that $\varphi = Q\varphi_{\mathrm{FCC}}$, and minimizing M_B w.r.t. variations of φ and g
- These are

$$Q \in \left\{ \mathbb{I}_4, \underbrace{\begin{pmatrix} (0,-1,1,1)/\sqrt{3} \\ * \end{pmatrix}}_{Q_\alpha}, \underbrace{\begin{pmatrix} (0,0,0,1) \\ * \end{pmatrix}}_{Q_{\mathrm{brane}}}, \underbrace{\begin{pmatrix} (0,0,1,1)/\sqrt{2} \\ * \end{pmatrix}}_{Q_{\mathrm{string}}} \right\}.$$

- The φ_{brane}-crystal is the lowest energy solution at all baryon densities $n_B = B_{\mathrm{cell}}/V_{\mathrm{cell}}$
- $\Delta E = E_{\mathrm{isolated}} - E_{\mathrm{min}}$ is minimized for the choice of crystal φ_{brane}

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- From φ_{FCC}, the other three crystals can be constructed by applying a chiral $\text{SO}(4)$ transformation $Q \in \text{SO}(4)$, such that $\varphi = Q\varphi_{\text{FCC}}$, and minimizing M_B w.r.t. variations of φ and g
- These are

$$Q \in \left\{ \mathbb{I}_4, \underbrace{\begin{pmatrix} (0,-1,1,1)/\sqrt{3} \\ * \end{pmatrix}}_{Q_\alpha}, \underbrace{\begin{pmatrix} (0,0,0,1) \\ * \end{pmatrix}}_{Q_{\text{brane}}}, \underbrace{\begin{pmatrix} (0,0,1,1)/\sqrt{2} \\ * \end{pmatrix}}_{Q_{\text{string}}} \right\}.$$

- The φ_{brane}-crystal is the lowest energy solution at all baryon densities $n_B = B_{\text{cell}}/V_{\text{cell}}$
- $\Delta E = E_{\text{isolated}} - E_{\text{min}}$ is minimized for the choice of crystal φ_{brane}

$\Longrightarrow$ Should yield a **lower compression modulus** than previous studies

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- From φ_{FCC}, the other three crystals can be constructed by applying a chiral $\mathrm{SO}(4)$ transformation $Q \in \mathrm{SO}(4)$, such that $\varphi = Q\varphi_{\mathrm{FCC}}$, and minimizing M_B w.r.t. variations of φ and g
- These are

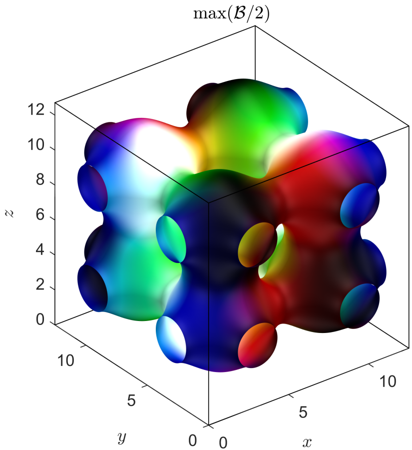
$$Q \in \left\{ \mathbb{I}_4, \underbrace{\begin{pmatrix} (0,-1,1,1)/\sqrt{3} \\ * \end{pmatrix}}_{Q_\alpha}, \underbrace{\begin{pmatrix} (0,0,0,1) \\ * \end{pmatrix}}_{Q_{\mathrm{brane}}}, \underbrace{\begin{pmatrix} (0,0,1,1)/\sqrt{2} \\ * \end{pmatrix}}_{Q_{\mathrm{string}}} \right\}.$$

- The φ_{brane}-crystal is the lowest energy solution at all baryon densities $n_B = B_{\mathrm{cell}}/V_{\mathrm{cell}}$
- $\Delta E = E_{\mathrm{isolated}} - E_{\mathrm{min}}$ is minimized for the choice of crystal φ_{brane}

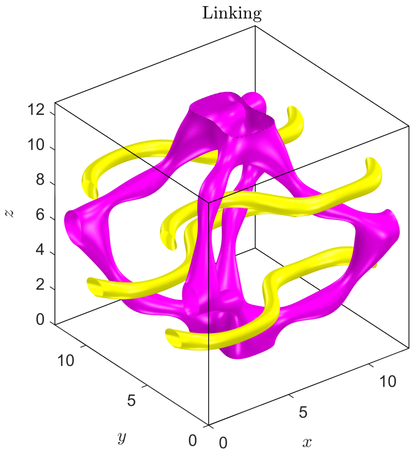
$\Longrightarrow$ Should yield a **lower compression modulus** than previous studies

$\Longrightarrow$ **Brane crystal** is an ideal candidate for **dense nuclear matter**

Brane or domain wall crystal [*J. Math. Phys.* **64** 103503 (2023)]



(a) Isosurface plot of the baryon density $\mathcal{B}_0$



(b) Linking of two preimages

Varying the metric on $\mathbb{T}^3$

- Let g_s be a smooth one-parameter family of metrics on $\mathbb{T}^3$ with $g_0 = F^* d$

Varying the metric on $\mathbb{T}^3$

- Let g_s be a smooth one-parameter family of metrics on $\mathbb{T}^3$ with $g_0 = F^* d$
- Set $\delta g = \partial_s g_s|_{s=0} \in \Gamma(\odot^2 T^*\mathbb{T}^3)$ (symmetric 2-covariant tensor field on $\mathbb{T}^3$)

Varying the metric on $\mathbb{T}^3$

- Let g_s be a smooth one-parameter family of metrics on $\mathbb{T}^3$ with $g_0 = F^*d$
- Set $\delta g = \partial_s g_s|_{s=0} \in \Gamma(\odot^2 T^*\mathbb{T}^3)$ (symmetric 2-covariant tensor field on $\mathbb{T}^3$)
- Inner product on the space of 2-covariant tensor fields $\langle A, B \rangle_g = A_{ij} g^{jk} B_{kl} g^{li}$

Varying the metric on $\mathbb{T}^3$

- Let g_s be a smooth one-parameter family of metrics on $\mathbb{T}^3$ with $g_0 = F^* d$
- Set $\delta g = \partial_s g_s|_{s=0} \in \Gamma(\odot^2 T^* \mathbb{T}^3)$ (symmetric 2-covariant tensor field on $\mathbb{T}^3$)
- Inner product on the space of 2-covariant tensor fields $\langle A, B \rangle_g = A_{ij} g^{jk} B_{kl} g^{li}$
- First variation of M_B w.r.t. g_s is

$$\left.\frac{\mathrm{d}M_B(\varphi, g_s)}{\mathrm{d}s}\right|_{s=0} = \int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g}\, \langle S(\varphi, g), \delta g \rangle_g, \quad S(\varphi, g) \in \Gamma(\odot^2 T^* \mathbb{T}^3)$$

Varying the metric on $\mathbb{T}^3$

- Let g_s be a smooth one-parameter family of metrics on $\mathbb{T}^3$ with $g_0 = F^* d$
- Set $\delta g = \partial_s g_s|_{s=0} \in \Gamma(\odot^2 T^* \mathbb{T}^3)$ (symmetric 2-covariant tensor field on $\mathbb{T}^3$)
- Inner product on the space of 2-covariant tensor fields $\langle A, B \rangle_g = A_{ij} g^{jk} B_{kl} g^{li}$
- First variation of M_B w.r.t. g_s is

$$\left.\frac{\mathrm{d}M_B(\varphi, g_s)}{\mathrm{d}s}\right|_{s=0} = \int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g}\, \langle S(\varphi, g), \delta g \rangle_g, \qquad S(\varphi, g) \in \Gamma(\odot^2 T^* \mathbb{T}^3)$$

- $S(\varphi, g)$ is the **stress-energy tensor**:

$$S_{ij} = \frac{1}{2}\left[m^2 \operatorname{Tr}(\mathrm{Id} - \varphi) - \frac{1}{2} g^{kl} \operatorname{Tr}(L_k L_l) - \frac{1}{16} g^{km} g^{ln} \operatorname{Tr}(\Omega_{kl}\Omega_{mn}) - c_6 (B_0)^2\right] g_{ij}$$
$$+ \frac{1}{2} \operatorname{Tr}(L_i L_j) + \frac{1}{8} g^{kl} \operatorname{Tr}(\Omega_{ik}\Omega_{jl}).$$

Numerical minimization of the field and lattice

- Fix $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$ and think of the energy as a map $E_\varphi : \mathrm{SPD}_3 \to \mathbb{R}$ such that $E_\varphi := M_B(\varphi|_{\text{fixed}}, g)$

Numerical minimization of the field and lattice

- Fix $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$ and think of the energy as a map $E_\varphi : \mathrm{SPD}_3 \to \mathbb{R}$ such that $E_\varphi := M_B(\varphi|_{\text{fixed}}, g)$
- We use arrested Newton flow on SPD_3 to minimize E_φ w.r.t. g

Numerical minimization of the field and lattice

- Fix $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$ and think of the energy as a map $E_\varphi : \mathrm{SPD}_3 \to \mathbb{R}$ such that $E_\varphi := M_B(\varphi|_{\text{fixed}}, g)$
- We use arrested Newton flow on SPD_3 to minimize E_φ w.r.t. g
- Explicitly, we are solving the system of 2nd order ODEs

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} (g_{ij})_s = -\frac{\partial E_\varphi}{\partial g_{ij}} = -\int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g}\, S_\varphi^{ij}, \quad (g_{ij})_0 = \vec{X}_i \cdot \vec{X}_j$$

where $S_\varphi \equiv S(\varphi|_{\text{fixed}}, g)$ is the fixed field stress tensor

Numerical minimization of the field and lattice

- Fix $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$ and think of the energy as a map $E_\varphi : \mathrm{SPD}_3 \to \mathbb{R}$ such that $E_\varphi := M_B(\varphi|_{\text{fixed}}, g)$
- We use arrested Newton flow on SPD_3 to minimize E_φ w.r.t. g
- Explicitly, we are solving the system of 2nd order ODEs

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} (g_{ij})_s = -\frac{\partial E_\varphi}{\partial g_{ij}} = -\int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g}\, S_\varphi^{ij}, \quad (g_{ij})_0 = \vec{X}_i \cdot \vec{X}_j$$

where $S_\varphi \equiv S(\varphi|_{\text{fixed}}, g)$ is the fixed field stress tensor

- In conjunction, we minimize $M_B(\varphi, g|_{\text{fixed}})$ w.r.t. φ for some initial field φ_0

Numerical minimization of the field and lattice

- Fix $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$ and think of the energy as a map $E_\varphi : \mathrm{SPD}_3 \to \mathbb{R}$ such that $E_\varphi := M_B(\varphi|_{\text{fixed}}, g)$
- We use arrested Newton flow on SPD_3 to minimize E_φ w.r.t. g
- Explicitly, we are solving the system of 2nd order ODEs

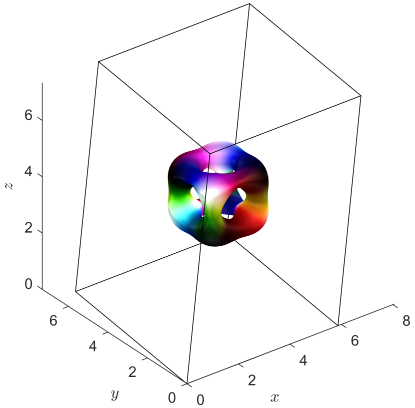
$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} (g_{ij})_s = -\frac{\partial E_\varphi}{\partial g_{ij}} = -\int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g}\, S_\varphi^{ij}, \quad (g_{ij})_0 = \vec{X}_i \cdot \vec{X}_j$$

where $S_\varphi \equiv S(\varphi|_{\text{fixed}}, g)$ is the fixed field stress tensor

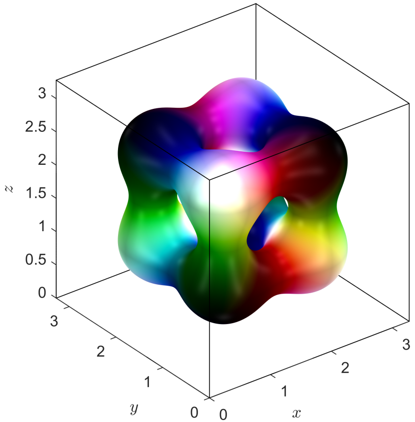
- In conjunction, we minimize $M_B(\varphi, g|_{\text{fixed}})$ w.r.t. φ for some initial field φ_0

$\Longrightarrow$ Laddering of minimizations

An example: the α-particle



(a) Initial configuration of a $B = 4$ RMA in a non-cubic lattice Λ



(b) Relaxed final solution of the cubic α-particle crystal

Phases of skyrmion matter

- Consider fixed baryon density n_B variations of $M_B(\varphi, g)$ w.r.t. g

Phases of skyrmion matter

- Consider fixed baryon density n_B variations of $M_B(\varphi, g)$ w.r.t. g
- vol_g is required to be invariant under variations g_s of the metric:

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g_s} = \frac{1}{2} \int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g}\, g^{ij} \delta g_{ij} = 0$$

Phases of skyrmion matter

- Consider fixed baryon density n_B variations of $M_B(\varphi, g)$ w.r.t. g
- vol_g is required to be invariant under variations g_s of the metric:

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g_s} = \frac{1}{2} \int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g} g^{ij} \delta g_{ij} = 0$$

$\implies$ δg is trace-free, i.e. $\mathrm{Tr}_g(\delta g) = 0$

Phases of skyrmion matter

- Consider fixed baryon density n_B variations of $M_B(\varphi, g)$ w.r.t. g
- vol_g is required to be invariant under variations g_s of the metric:

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g_s} = \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}^3x \sqrt{g} g^{ij} \delta g_{ij} = 0$$

$\implies$ δg is trace-free, i.e. $\mathrm{Tr}_g(\delta g) = 0$

- Leads to modifying the (fixed φ field) stress-energy tensor via the mapping

$$S_\varphi \mapsto \tilde{S}_\varphi = S_\varphi - \frac{1}{3}\mathrm{Tr}_g(S_\varphi)\, g$$

Phases of skyrmion matter

- Consider fixed baryon density n_B variations of $M_B(\varphi, g)$ w.r.t. g
- vol_g is required to be invariant under variations g_s of the metric:

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \int_{\mathbb{T}^3} \mathrm{d}^3x\sqrt{g_s} = \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}^3x\sqrt{g}g^{ij}\delta g_{ij} = 0$$

$\implies$ δg is trace-free, i.e. $\mathrm{Tr}_g(\delta g) = 0$

- Leads to modifying the (fixed φ field) stress-energy tensor via the mapping

$$S_\varphi \mapsto \tilde{S}_\varphi = S_\varphi - \frac{1}{3}\mathrm{Tr}_g(S_\varphi)\, g$$

- Convergence criterion becomes $\max(\tilde{S}_\varphi) < \mathrm{tol}$

Phases of skyrmion matter

- Consider fixed baryon density n_B variations of $M_B(\varphi, g)$ w.r.t. g
- vol_g is required to be invariant under variations g_s of the metric:

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\int_{\mathbb{T}^3}\mathrm{d}^3x\sqrt{g_s} = \frac{1}{2}\int_{\mathbb{T}^3}\mathrm{d}^3x\sqrt{g}g^{ij}\delta g_{ij} = 0$$

$\implies$ δg is trace-free, i.e. $\mathrm{Tr}_g(\delta g) = 0$

- Leads to modifying the (fixed φ field) stress-energy tensor via the mapping

$$S_\varphi \mapsto \tilde{S}_\varphi = S_\varphi - \frac{1}{3}\mathrm{Tr}_g(S_\varphi)\, g$$

- Convergence criterion becomes $\max(\tilde{S}_\varphi) < \mathrm{tol}$
- This process enables us to determine an **energy-density** curve

Phases of skyrmion matter

- Consider fixed baryon density n_B variations of $M_B(\varphi, g)$ w.r.t. g
- vol_g is required to be invariant under variations g_s of the metric:

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \int_{\mathbb{T}^3} \mathrm{d}^3x\sqrt{g_s} = \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}^3x\sqrt{g}g^{ij}\delta g_{ij} = 0$$

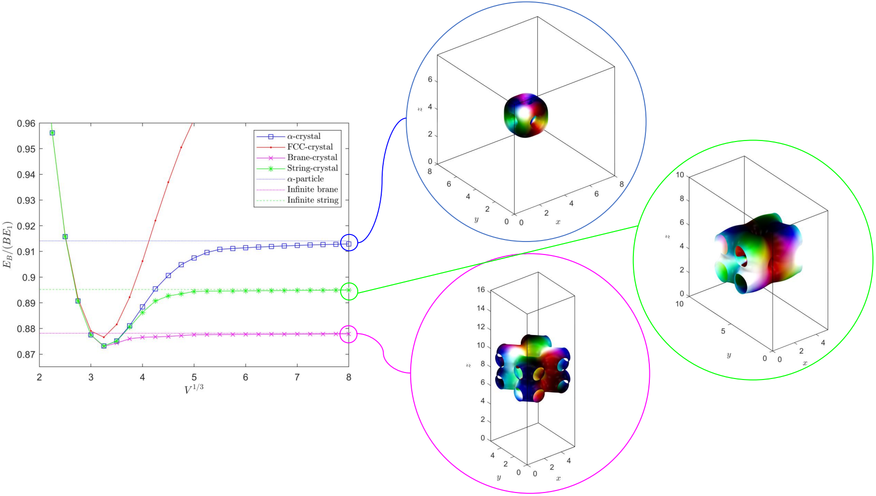
$\implies$ δg is trace-free, i.e. $\mathrm{Tr}_g(\delta g) = 0$

- Leads to modifying the (fixed φ field) stress-energy tensor via the mapping

$$S_\varphi \mapsto \tilde{S}_\varphi = S_\varphi - \frac{1}{3}\mathrm{Tr}_g(S_\varphi)\, g$$

- Convergence criterion becomes $\max(\tilde{S}_\varphi) < \mathrm{tol}$
- This process enables us to determine an **energy-density** curve
- This is key to obtaining an **equation of state** within our framework

Phases of skyrmion matter



Quantum skyrmion crystals and the symmetry energy

Isospin quantization

- Non-renormalizable theory $\Longrightarrow$ isospin asymmetry is included by semi-classically quantizing isospin collective coordinates: $\varphi(x) \mapsto \hat{\varphi}(x,t) = A(t)\varphi(x)A^{\dagger}(t)$ [*Nucl. Phys. B* **262** 133–143 (1985)]

Isospin quantization

- Non-renormalizable theory $\implies$ isospin asymmetry is included by semi-classically quantizing isospin collective coordinates: $\varphi(x) \mapsto \hat{\varphi}(x,t) = A(t)\varphi(x)A^{\dagger}(t)$ [*Nucl. Phys. B* **262** 133–143 (1985)]
- Can use a mean-field approximation of a large chunk ($B = N_{\text{cell}}B_{\text{cell}}$) in a generic quantum state with fixed eigenvalue [*Phys. Rev. D* **106** 114031 (2022)]

$$I_3 = \frac{(Z-N)}{2} = -\frac{(1-2\gamma_p)}{2}N_{\text{cell}}B_{\text{cell}}, \quad \gamma_p \text{ is the proton fraction}$$

Isospin quantization

- Non-renormalizable theory $\implies$ isospin asymmetry is included by semi-classically quantizing isospin collective coordinates: $\varphi(x) \mapsto \hat{\varphi}(x,t) = A(t)\varphi(x)A^\dagger(t)$ [*Nucl. Phys. B* **262** 133–143 (1985)]
- Can use a mean-field approximation of a large chunk ($B = N_{\text{cell}}B_{\text{cell}}$) in a generic quantum state with fixed eigenvalue [*Phys. Rev. D* **106** 114031 (2022)]

$$I_3 = \frac{(Z-N)}{2} = -\frac{(1-2\gamma_p)}{2}N_{\text{cell}}B_{\text{cell}}, \quad \gamma_p \text{ is the proton fraction}$$

- $I = I_3$ minimizes the isospin energy since by definition $I^2 \geq I_3^2$

Isospin quantization

- Non-renormalizable theory $\Longrightarrow$ isospin asymmetry is included by semi-classically quantizing isospin collective coordinates: $\varphi(x) \mapsto \hat{\varphi}(x,t) = A(t)\varphi(x)A^\dagger(t)$ [*Nucl. Phys. B* **262** 133–143 (1985)]
- Can use a mean-field approximation of a large chunk ($B = N_{\text{cell}}B_{\text{cell}}$) in a generic quantum state with fixed eigenvalue [*Phys. Rev. D* **106** 114031 (2022)]

$$I_3 = \frac{(Z-N)}{2} = -\frac{(1-2\gamma_p)}{2}N_{\text{cell}}B_{\text{cell}}, \quad \gamma_p \text{ is the proton fraction}$$

- $I = I_3$ minimizes the isospin energy since by definition $I^2 \geq I_3^2$
- The isospin correction (per unit cell) to the energy of the crystal is found to be $E_{\text{iso}} = \frac{\hbar^2}{8U_{33}}B_{\text{cell}}^2\delta^2$

Symmetry energy

- The asymmetry of matter is determined by the isospin asymmetry parameter $\delta = (N - Z)/(N + Z) = 1 - 2\gamma_p$

Symmetry energy

- The asymmetry of matter is determined by the isospin asymmetry parameter $\delta = (N - Z)/(N + Z) = 1 - 2\gamma_p$
- Binding energy per baryon number of asymmetric nuclear matter is given by

$$\frac{E}{B}(n_B, \delta) = E_N(n_B) + S_N(n_B)\delta^2 + O(\delta^3), \quad \begin{aligned} n_0 &= 0.160\,\mathrm{fm}^{-3} \\ E_N(n_0) &= 923\,\mathrm{MeV} \\ S_N(n_0) &\approx 30\,\mathrm{MeV} \end{aligned}$$

Symmetry energy

- The asymmetry of matter is determined by the isospin asymmetry parameter $\delta = (N - Z)/(N + Z) = 1 - 2\gamma_p$
- Binding energy per baryon number of asymmetric nuclear matter is given by

$$\frac{E}{B}(n_B, \delta) = E_N(n_B) + S_N(n_B)\delta^2 + \mathrm{O}(\delta^3), \quad \begin{aligned} n_0 &= 0.160\,\mathrm{fm}^{-3} \\ E_N(n_0) &= 923\,\mathrm{MeV} \\ S_N(n_0) &\approx 30\,\mathrm{MeV} \end{aligned}$$

- The isospin symmetric binding energy is defined by $E_N = M_B/B$

Symmetry energy

- The asymmetry of matter is determined by the isospin asymmetry parameter $\delta = (N - Z)/(N + Z) = 1 - 2\gamma_p$
- Binding energy per baryon number of asymmetric nuclear matter is given by

$$\frac{E}{B}(n_B, \delta) = E_N(n_B) + S_N(n_B)\delta^2 + O(\delta^3), \qquad \begin{aligned} n_0 &= 0.160\,\mathrm{fm}^{-3} \\ E_N(n_0) &= 923\,\mathrm{MeV} \\ S_N(n_0) &\approx 30\,\mathrm{MeV} \end{aligned}$$

- The isospin symmetric binding energy is defined by $E_N = M_B/B$
- The symmetry energy S_N dictates how the binding energy changes going from symmetric $(\delta = 0)$ to asymmetric $(\delta \neq 0)$ nuclear matter

Symmetry energy

- The asymmetry of matter is determined by the isospin asymmetry parameter $\delta = (N - Z)/(N + Z) = 1 - 2\gamma_p$
- Binding energy per baryon number of asymmetric nuclear matter is given by

$$\frac{E}{B}(n_B, \delta) = E_N(n_B) + S_N(n_B)\delta^2 + \mathrm{O}(\delta^3), \quad \begin{aligned} n_0 &= 0.160\,\mathrm{fm}^{-3} \\ E_N(n_0) &= 923\,\mathrm{MeV} \\ S_N(n_0) &\approx 30\,\mathrm{MeV} \end{aligned}$$

- The isospin symmetric binding energy is defined by $E_N = M_B/B$
- The symmetry energy S_N dictates how the binding energy changes going from symmetric $(\delta = 0)$ to asymmetric $(\delta \neq 0)$ nuclear matter
- It is obtained from the quantum isospin energy $S_N(n_B) = \frac{E_{\mathrm{iso}}}{B_{\mathrm{cell}}\delta^2} = \frac{\hbar^2}{8U_{33}} V_{\mathrm{cell}} n_B$

Symmetry energy

- The asymmetry of matter is determined by the isospin asymmetry parameter $\delta = (N - Z)/(N + Z) = 1 - 2\gamma_p$
- Binding energy per baryon number of asymmetric nuclear matter is given by

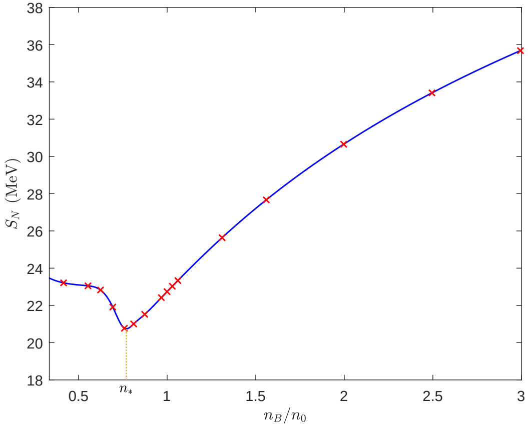
$$\frac{E}{B}(n_B, \delta) = E_N(n_B) + S_N(n_B)\delta^2 + \mathrm{O}(\delta^3),$$

$$\boxed{\begin{aligned} n_0 &= 0.160\,\mathrm{fm}^{-3} \\ E_N(n_0) &= 923\,\mathrm{MeV} \\ S_N(n_0) &\approx 30\,\mathrm{MeV} \end{aligned}}$$

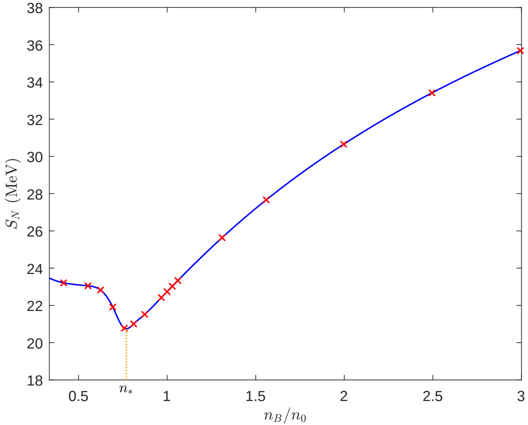
- The isospin symmetric binding energy is defined by $E_N = M_B/B$
- The symmetry energy S_N dictates how the binding energy changes going from symmetric $(\delta = 0)$ to asymmetric $(\delta \neq 0)$ nuclear matter
- It is obtained from the quantum isospin energy $S_N(n_B) = \frac{E_{\mathrm{iso}}}{B_{\mathrm{cell}}\delta^2} = \frac{\hbar^2}{8U_{33}} V_{\mathrm{cell}} n_B$
- In [arXiv:2306.04533 (2023)], at saturation we find

$$\boxed{n_0 = 0.160\,\mathrm{fm}^{-3},\ E_N(n_0) = 912\,\mathrm{MeV} \text{ and } S_N(n_0) = 22.7\,\mathrm{MeV}}$$

Symmetry energy and the cusp structure

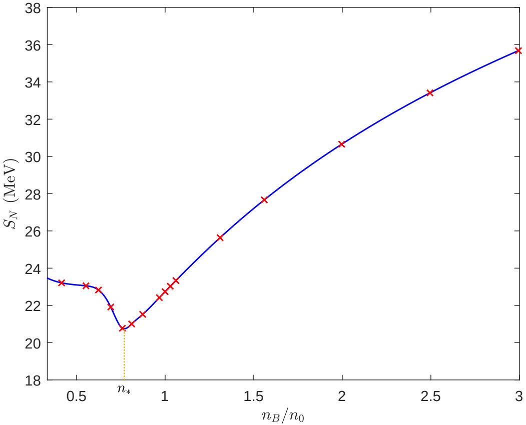


Symmetry energy and the cusp structure



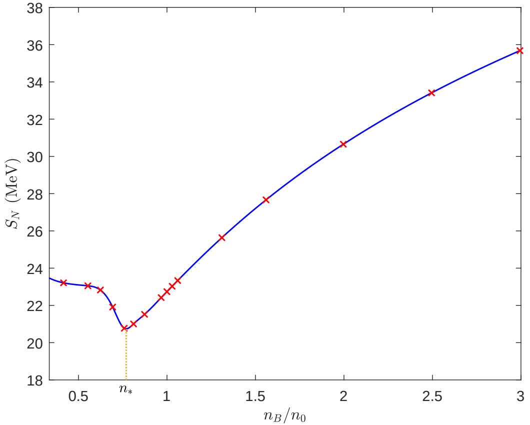
- **Cusp** below saturation at $n_* \sim 3n_0/4$

Symmetry energy and the cusp structure



- **Cusp** below saturation at $n_* \sim 3n_0/4$
- Symmetry energy at zero density $S_N(0) = 23.77\,\mathrm{MeV}$ (finite symmetric nucl. mat.)

Symmetry energy and the cusp structure

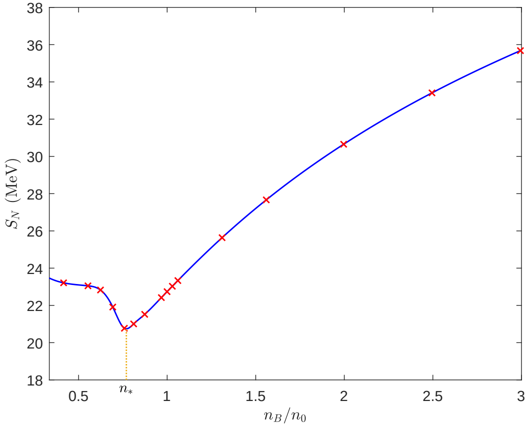


- **Cusp** below saturation at $n_* \sim 3n_0/4$
- Symmetry energy at zero density $S_N(0) = 23.77\,\text{MeV}$ (finite symmetric nucl. mat.)
- Bethe–Weizsäcker SEMF asymmetry energy $E_A = a_A\delta^2 B$

Neutron stars from skyrmion branes

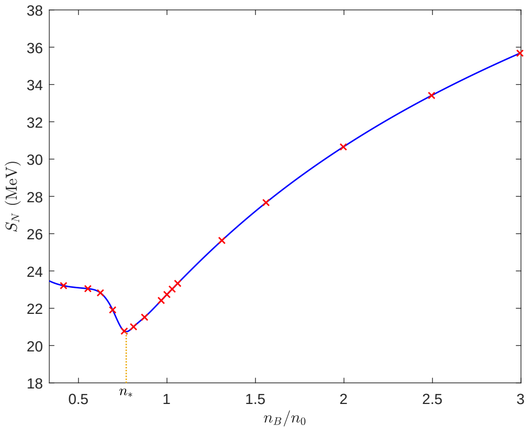
Paul Leask

Symmetry energy and the cusp structure

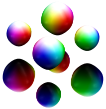


- **Cusp** below saturation at $n_* \sim 3n_0/4$
- Symmetry energy at zero density $\boxed{S_N(0) = 23.77\,\text{MeV}}$ (finite symmetric nucl. mat.)
- Bethe–Weizsäcker SEMF asymmetry energy $E_A = a_A \delta^2 B$
- Can identify $\boxed{S_N(0) \sim a_A = 23.7\,\text{MeV}}$

Symmetry energy and the cusp structure



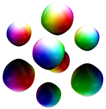
- **Cusp** below saturation at $n_* \sim 3n_0/4$
- Symmetry energy at zero density $S_N(0) = 23.77\,\mathrm{MeV}$ (finite symmetric nucl. mat.)
- Bethe–Weizsäcker SEMF asymmetry energy $E_A = a_A \delta^2 B$
- Can identify $S_N(0) \sim a_A = 23.7\,\mathrm{MeV}$
- Cusp origin: **phase transition** between **infinite isospin asymmetric nuclear matter** and somewhat **isolated finite nuclear matter** [arXiv:2306.04533 (2023)]



Particle fractions of $npe\mu$ matter in β-equilibrium

- Global **charge neutrality** by including background of charged leptons $n_p = n_e + n_\mu$

Particle fractions of $npe\mu$ matter in β-equilibrium



- Global **charge neutrality** by including background of charged leptons $n_p = n_e + n_\mu$
- Lepton-nucleon exchange described by **simultaneous** processes [N. K. Glendenning, *Compact Stars* (1997)]:
  - Electron capture: $p + l \longrightarrow n + \nu_l$
  - β-decay: $n \longrightarrow p + l + \bar{\nu}_l$

Particle fractions of $npe\mu$ matter in β-equilibrium

- Global **charge neutrality** by including background of charged leptons $n_p = n_e + n_\mu$
- Lepton-nucleon exchange described by **simultaneous** processes [N. K. Glendenning, *Compact Stars* (1997)]:
  - Electron capture: $p + l \longrightarrow n + \nu_l$
  - β-decay: $n \longrightarrow p + l + \bar{\nu}_l$
- As n_B increases then so too does n_p and $n_e \rightarrow \mu_e \geq m_\mu = 105.66\,\mathrm{MeV}$

Particle fractions of $npe\mu$ matter in β-equilibrium

- Global **charge neutrality** by including background of charged leptons $n_p = n_e + n_\mu$
- Lepton-nucleon exchange described by **simultaneous** processes [N. K. Glendenning, *Compact Stars* (1997)]:
  - Electron capture: $p + l \to n + \nu_l$
  - β-decay: $n \to p + l + \bar{\nu}_l$
- As n_B increases then so too does n_p and $n_e \to \mu_e \geq m_\mu = 105.66\,\mathrm{MeV}$

$\implies$ Energetically favourable for muons to appear

Neutron stars from skyrmion branes
Paul Leask
Motivation
Skyrme model
Linking in the Skyrme model
Skyrmion solutions
Skyrmion crystals and phases of skyrmion matter
Quantum skyrmion crystals and the symmetry energy
Neutron stars
Final remarks

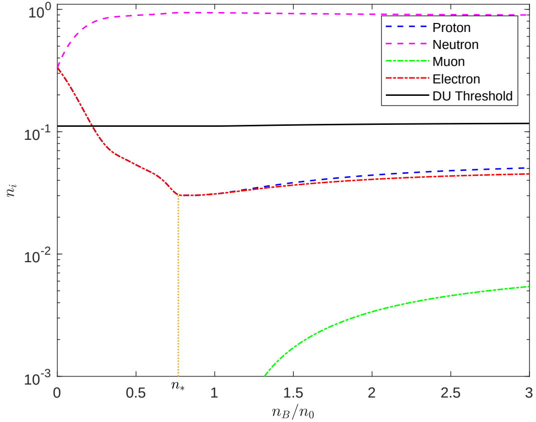
Particle fractions of $npe\mu$ matter in β-equilibrium

- Global **charge neutrality** by including background of charged leptons $n_p = n_e + n_\mu$
- Lepton-nucleon exchange described by **simultaneous** processes [N. K. Glendenning, *Compact Stars* (1997)]:
  - Electron capture: $p + l \longrightarrow n + \nu_l$
  - β-decay: $n \longrightarrow p + l + \bar{\nu}_l$
- As n_B increases then so too does n_p and $n_e \rightarrow \mu_e \geq m_\mu = 105.66\,\mathrm{MeV}$

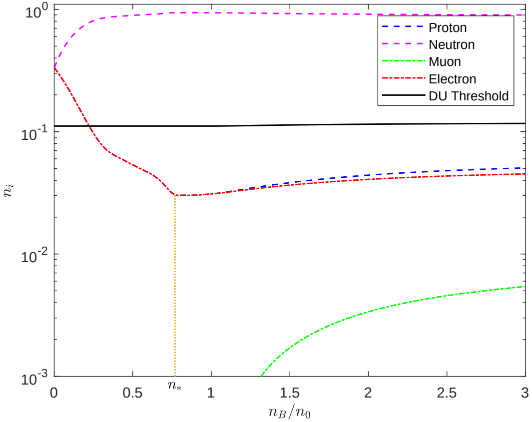
$\Longrightarrow$ Energetically favourable for muons to appear

- The simultaneous β-decay and electron capture processes allow the calculation of the proton fraction γ_p at a prescribed density n_B [*Phys. Rev. D* **106** 114031 (2022)]

Particle fractions of $npe\mu$ matter in β-equilibrium

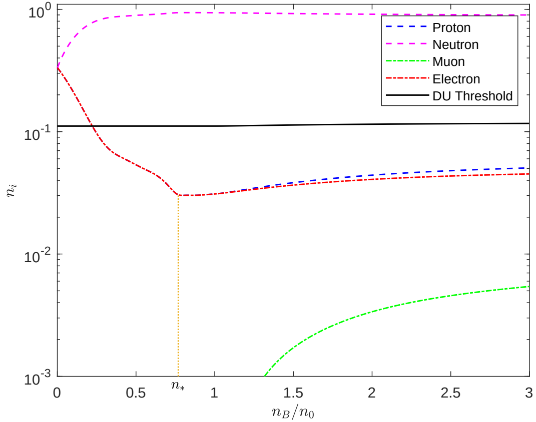


Particle fractions of $npe\mu$ matter in β-equilibrium



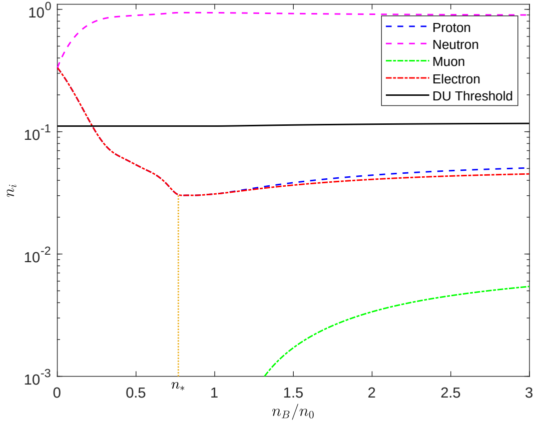
- **Cusp** also present at n_*

Particle fractions of $npe\mu$ matter in β-equilibrium



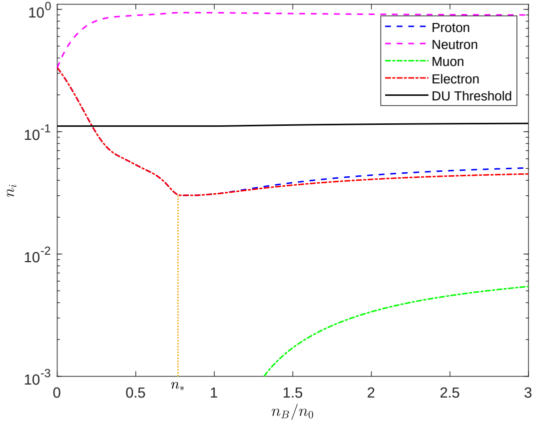
- **Cusp** also present at n_*
- Reinforces the proposition that the **cusp** indicates the start of a **phase transition** between **infinite asym matter** and **finite sym matter**

Particle fractions of $npe\mu$ matter in β-equilibrium



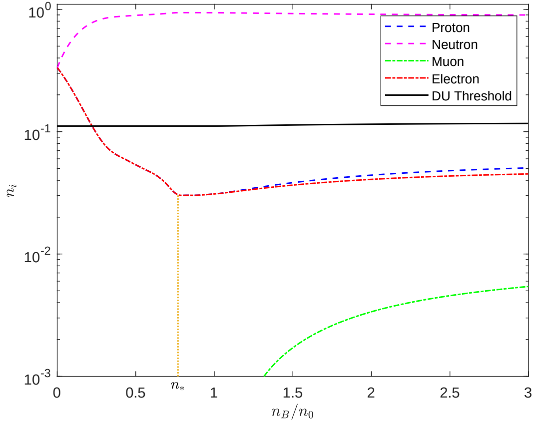
- **Cusp** also present at n_*
- Reinforces the proposition that the **cusp** indicates the start of a **phase transition** between **infinite asym matter** and **finite sym matter**
- The crust of NS is iron rich with $\gamma_p = 0.46$ for $^{56}\mathrm{Fe}$

Particle fractions of $npe\mu$ matter in β-equilibrium



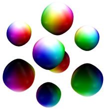
- **Cusp** also present at n_*
- Reinforces the proposition that the **cusp** indicates the start of a **phase transition** between **infinite asym matter** and **finite sym matter**
- The crust of NS is iron rich with $\gamma_p = 0.46$ for $^{56}\mathrm{Fe}$
- We find as $n_B \to 0$ then $\gamma_p = 0.5$

Particle fractions of $npe\mu$ matter in β-equilibrium



- **Cusp** also present at n_*
- Reinforces the proposition that the **cusp** indicates the start of a **phase transition** between **infinite asym matter** and **finite sym matter**
- The crust of NS is iron rich with $\gamma_p = 0.46$ for $^{56}\mathrm{Fe}$
- We find as $n_B \to 0$ then $\gamma_p = 0.5$

$\Longrightarrow$ These correspond quite well



Particle fractions of $npe\mu$ matter in β-equilibrium

- Global **charge neutrality** by including background of charged leptons $n_p = n_e + n_\mu$
- Lepton-nucleon exchange described by **simultaneous** processes [N. K. Glendenning, *Compact Stars* (1997)]:
  - Electron capture: $p + l \longrightarrow n + \nu_l$
  - β-decay: $n \longrightarrow p + l + \bar{\nu}_l$
- As n_B increases then so too does n_p and $n_e \rightarrow \mu_e \geq m_\mu = 105.66\,\mathrm{MeV}$

$\Longrightarrow$ Energetically favourable for muons to appear

- The simultaneous β-decay and electron capture processes allow the calculation of the proton fraction γ_p at a prescribed density n_B [*Phys. Rev. D* **106** 114031 (2022)]
- Energy of a relativistic Fermi gas at zero temperature (lepton energy)

$$E_l(n_B) = \frac{B_{\mathrm{cell}}}{n_B \hbar^3 \pi^2} \int_0^{\hbar k_F} k^2 \sqrt{k^2 + m_l^2}\, \mathrm{d}k, \quad k_F = (3\pi^2 n_l)^{1/3}, \quad n_l = \gamma_l n_B$$

Particle fractions of $npe\mu$ matter in β-equilibrium

- Global **charge neutrality** by including background of charged leptons $n_p = n_e + n_\mu$
- Lepton-nucleon exchange described by **simultaneous** processes [N. K. Glendenning, *Compact Stars* (1997)]:
  - Electron capture: $p + l \longrightarrow n + \nu_l$
  - β-decay: $n \longrightarrow p + l + \bar{\nu}_l$
- As n_B increases then so too does n_p and $n_e \rightarrow \mu_e \geq m_\mu = 105.66\,\mathrm{MeV}$

$\Longrightarrow$ Energetically favourable for muons to appear

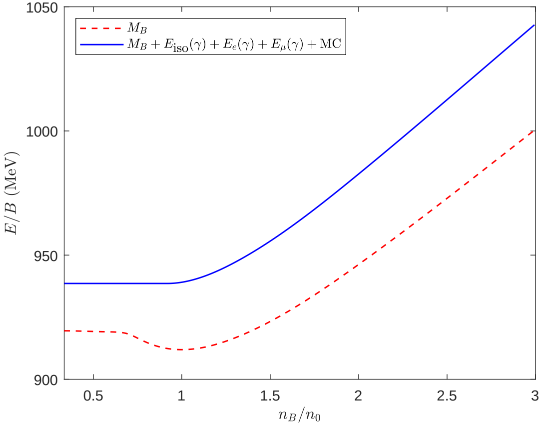
- The simultaneous β-decay and electron capture processes allow the calculation of the proton fraction γ_p at a prescribed density n_B [*Phys. Rev. D* **106** 114031 (2022)]
- Energy of a relativistic Fermi gas at zero temperature (lepton energy)

$$E_l(n_B) = \frac{B_{\text{cell}}}{n_B \hbar^3 \pi^2} \int_0^{\hbar k_F} k^2 \sqrt{k^2 + m_l^2}\, \mathrm{d}k, \quad k_F = (3\pi^2 n_l)^{1/3}, \quad n_l = \gamma_l n_B$$

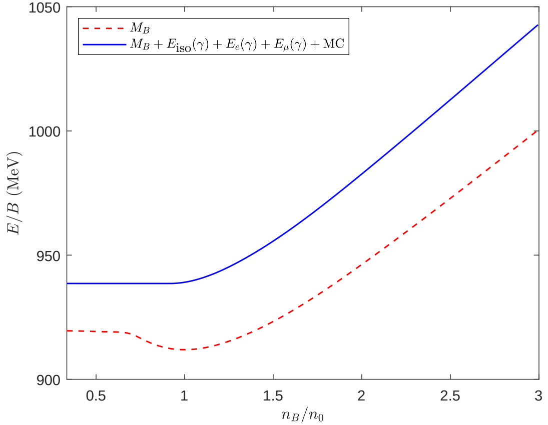
- Energy per unit cell of β-equilibrated matter

$$E_{\text{cell}}(n_B) = M_B(n_B) + E_{\text{iso}}(n_B) + E_e(n_B) + E_\mu(n_B)$$

Isospin asymmetric equation of state



Isospin asymmetric equation of state

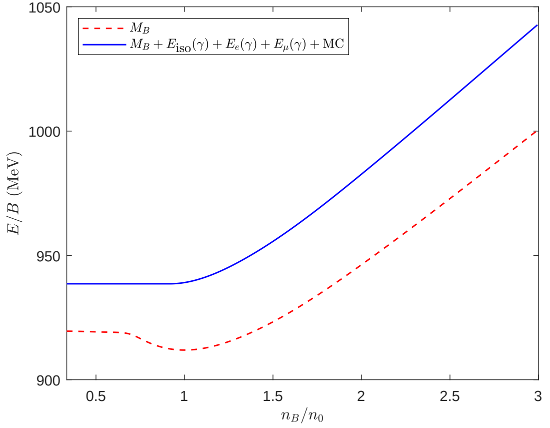


- Can obtain the pressure p and energy density ρ from the $E(n_B)$ curve, with

$$\rho = \frac{E}{V} = \frac{n_B}{B} E_{\text{cell}}$$

$$p = -\frac{\partial E}{\partial V} = \frac{n_B^2}{B}\frac{\partial E_{\text{cell}}}{\partial n_B}$$

Isospin asymmetric equation of state



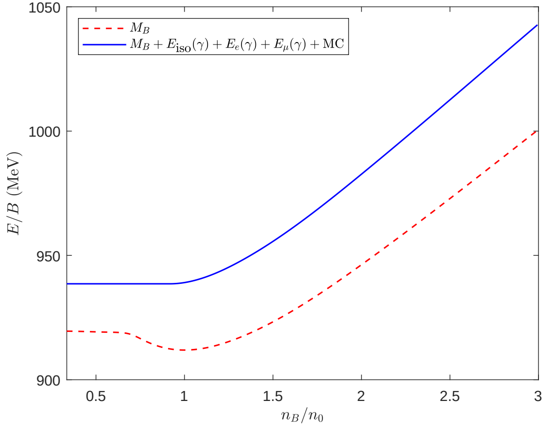
- Can obtain the pressure p and energy density ρ from the $E(n_B)$ curve, with

$$\rho = \frac{E}{V} = \frac{n_B}{B} E_{\rm cell}$$

$$p = -\frac{\partial E}{\partial V} = \frac{n_B^2}{B} \frac{\partial E_{\rm cell}}{\partial n_B}$$

$\Longrightarrow$ Isospin asymmetric nuclear matter EoS $\rho_{\rm brane} = \rho_{\rm brane}(p)$

Isospin asymmetric equation of state



- Can obtain the pressure p and energy density ρ from the $E(n_B)$ curve, with

$$\rho = \frac{E}{V} = \frac{n_B}{B} E_{\text{cell}}$$

$$p = -\frac{\partial E}{\partial V} = \frac{n_B^2}{B} \frac{\partial E_{\text{cell}}}{\partial n_B}$$

$\implies$ Isospin asymmetric nuclear matter EoS $\rho_{\text{brane}} = \rho_{\text{brane}}(p)$

- We will use this EoS to obtain NS within the Skyrme model

Neutron stars

Coupling to gravity

- In order to describe neutrons stars within the Skyrme framework, we need to couple the generalized Skyrme model to gravity

Coupling to gravity

- In order to describe neutrons stars within the Skyrme framework, we need to couple the generalized Skyrme model to gravity
- Introduce the Einstein–Hilbert–Skyrme action

$$S = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^4x\sqrt{-g}R + S_{\text{matter}}$$

Coupling to gravity

- In order to describe neutrons stars within the Skyrme framework, we need to couple the generalized Skyrme model to gravity
- Introduce the Einstein–Hilbert–Skyrme action

$$S = \frac{1}{16\pi G}\int_{\Sigma} \mathrm{d}^4x\sqrt{-g}R + S_{\text{matter}}$$

- S_{matter} describes matter inside NS

Coupling to gravity

- In order to describe neutrons stars within the Skyrme framework, we need to couple the generalized Skyrme model to gravity
- Introduce the Einstein–Hilbert–Skyrme action

$$S = \frac{1}{16\pi G}\int_{\Sigma} \mathrm{d}^4x\sqrt{-g}R + S_{\text{matter}}$$

- S_{matter} describes matter inside NS
- NS Interior well described by **perfect fluid** of nearly free neutrons & degenerate gas of electrons:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} = (\rho(p) + p)u_\mu u_\nu + p g_{\mu\nu}$$

Coupling to gravity

- In order to describe neutrons stars within the Skyrme framework, we need to couple the generalized Skyrme model to gravity
- Introduce the Einstein–Hilbert–Skyrme action

$$S = \frac{1}{16\pi G}\int_{\Sigma} d^4x\sqrt{-g}R + S_{\mathrm{matter}}$$

- S_{matter} describes matter inside NS
- NS Interior well described by **perfect fluid** of nearly free neutrons & degenerate gas of electrons:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{matter}}}{\delta g^{\mu\nu}} = (\rho(p)+p)u_\mu u_\nu + p g_{\mu\nu}$$

- The energy density ρ and the pressure p are related by the (Brane) crystal EoS $\rho(p) = \rho_{\mathrm{brane}}(p)$ [*Phys. Lett. B* **811** 135928 (2020)]

The Tolman–Oppenheimer–Volkoff system

- Our aim is to calculate M_{max} and R_{max} for a NS described by our system

The Tolman–Oppenheimer–Volkoff system

- Our aim is to calculate $M_{\max}$ and $R_{\max}$ for a NS described by our system
- Need to solve the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ for some particular choice of $g_{\mu\nu}$

The Tolman–Oppenheimer–Volkoff system

- Our aim is to calculate M_{max} and R_{max} for a NS described by our system
- Need to solve the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ for some particular choice of $g_{\mu\nu}$
- Simplest case: **static** & **non-rotating** neutron star

The Tolman–Oppenheimer–Volkoff system

- Our aim is to calculate $M_{\max}$ and $R_{\max}$ for a NS described by our system
- Need to solve the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ for some particular choice of $g_{\mu\nu}$
- Simplest case: **static** & **non-rotating** neutron star
- Spherically symmetric ansatz of the spacetime metric

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

The Tolman–Oppenheimer–Volkoff system

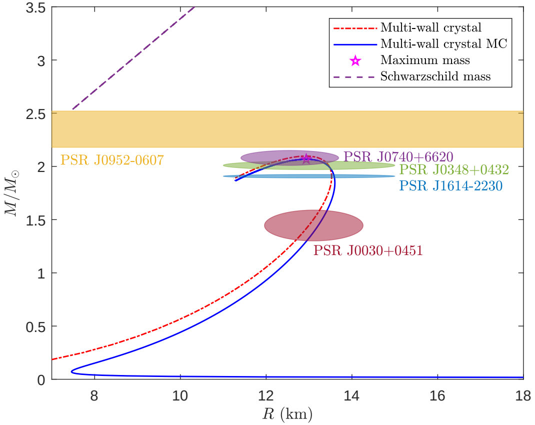
- Our aim is to calculate M_{max} and R_{max} for a NS described by our system
- Need to solve the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ for some particular choice of $g_{\mu\nu}$
- Simplest case: **static** & **non-rotating** neutron star
- Spherically symmetric ansatz of the spacetime metric

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

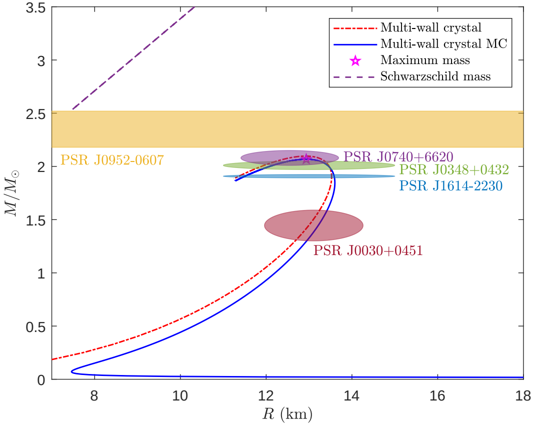
- Substituting this into the Einstein equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$ yields the TOV system

$$\frac{\mathrm{d}A}{\mathrm{d}r} = A(r)r\left(8\pi G B(r)p(r) - \frac{1-B(r)}{r^2}\right)$$
$$\frac{\mathrm{d}B}{\mathrm{d}r} = B(r)r\left(8\pi G B(r)\rho(p(r)) + \frac{1-B(r)}{r^2}\right)$$
$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{p(r)+\rho(p(r))}{2A(r)}\frac{\mathrm{d}A}{\mathrm{d}r}$$

Neutron star properties and the mass-radius curve



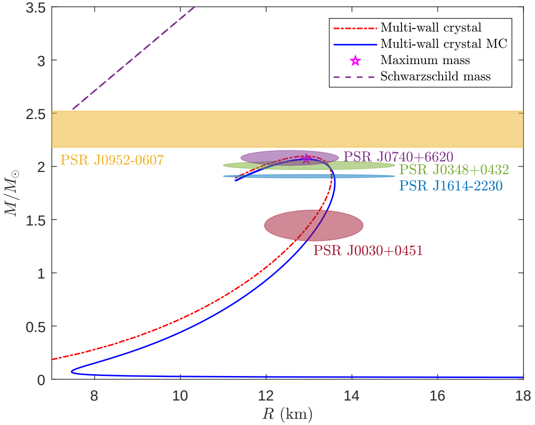
Neutron star properties and the mass-radius curve



- Mass M obtained from Schwarzschild metric definition outside the star

$$B(R_{\mathrm{NS}}) = \frac{1}{1 - \frac{2MG}{R_{\mathrm{NS}}}}$$

Neutron star properties and the mass-radius curve

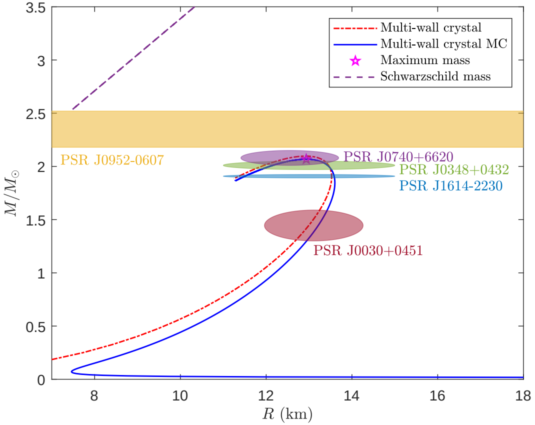


- Mass M obtained from Schwarzschild metric definition outside the star

$$B(R_{\mathrm{NS}}) = \frac{1}{1 - \frac{2MG}{R_{\mathrm{NS}}}}$$

- $M_{\mathrm{max}} = 2.0971 M_\odot$, occurring for a neutron star of radius $R_{\mathrm{NS}} = 13.12\,\mathrm{km}$.

Neutron star properties and the mass-radius curve



- Mass M obtained from Schwarzschild metric definition outside the star

$$B(R_{NS}) = \frac{1}{1 - \frac{2MG}{R_{NS}}}$$

- $M_{max} = 2.0971 M_\odot$, occurring for a neutron star of radius $R_{NS} = 13.12\,\text{km}$.

$\Longrightarrow$ Resulting neutron stars agree well with recent NICER/LIGO observational data

Final remarks

Final remarks

- Cusp structure in the symmetry energy observed in the hidden-local-symmetric (HLS) Skyrme model [*Phys. Rev. C* **83** 025206 (2011)]

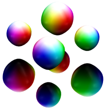
Final remarks

- Cusp structure in the symmetry energy observed in the hidden-local-symmetric (HLS) Skyrme model [*Phys. Rev. C* **83** 025206 (2011)]
- Attributed to the behavior of the chiral condensates combined with the dilaton condensate near saturation n_0 [*Mod. Phys. Lett. A* **37** 2230003 (2022)]

Final remarks

- Cusp structure in the symmetry energy observed in the hidden-local-symmetric (HLS) Skyrme model [*Phys. Rev. C* **83** 025206 (2011)]
- Attributed to the behavior of the chiral condensates combined with the dilaton condensate near saturation n_0 [*Mod. Phys. Lett. A* **37** 2230003 (2022)]
- There is a topological phase transition where the FCC lattice of hedgehog skyrmions fractionalize into half-skyrmions (FCC crystal)

Final remarks



- Cusp structure in the symmetry energy observed in the hidden-local-symmetric (HLS) Skyrme model [*Phys. Rev. C* **83** 025206 (2011)]
- Attributed to the behavior of the chiral condensates combined with the dilaton condensate near saturation n_0 [*Mod. Phys. Lett. A* **37** 2230003 (2022)]
- There is a topological phase transition where the FCC lattice of hedgehog skyrmions fractionalize into half-skyrmions (FCC crystal)
- Analogous to “pseudo-gap” phenomenon in condensed matter physics

Open problems

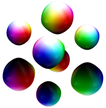
- Brane solution improves on compressibility at saturation

Open problems

- Brane solution improves on compressibility at saturation
- However, the **compression modulus** is still **too high**, $K_0 \sim 4K_{\text{exp}}$

Open problems

- Brane solution improves on compressibility at saturation
- However, the **compression modulus** is still **too high**, $K_0 \sim 4K_{\text{exp}}$
- Inhomogeneous solutions are not enough alone



Open problems

- Brane solution improves on compressibility at saturation
- However, the **compression modulus** is still **too high**, $K_0 \sim 4K_{\text{exp}}$
- Inhomogeneous solutions are not enough alone

$\implies$ Inclusion of other d.o.f. such as **vector mesons** necessary?

Open problems

- Brane solution improves on compressibility at saturation
- However, the **compression modulus** is still **too high**, $K_0 \sim 4K_{\text{exp}}$
- Inhomogeneous solutions are not enough alone

$\Longrightarrow$ Inclusion of other d.o.f. such as **vector mesons** necessary?

- Charged pion condensation normally indicates that the state is superconducting

Open problems

- Brane solution improves on compressibility at saturation
- However, the **compression modulus** is still **too high**, $K_0 \sim 4K_{\text{exp}}$
- Inhomogeneous solutions are not enough alone

$\Longrightarrow$ Inclusion of other d.o.f. such as **vector mesons** necessary?

- Charged pion condensation normally indicates that the state is superconducting

$\Longrightarrow$ Couple skyrmion matter to electromagnetism: $\partial_\mu \varphi \mapsto D_\mu \varphi = \partial_\mu \varphi - ieA_\mu [Q, \varphi]$

- Estimation of **SEMF coefficients** a_V, a_S, a_C, a_A

Open problems

- Brane solution improves on compressibility at saturation
- However, the **compression modulus** is still **too high**, $K_0 \sim 4K_{\text{exp}}$
- Inhomogeneous solutions are not enough alone

$\Longrightarrow$ Inclusion of other d.o.f. such as **vector mesons** necessary?

- Charged pion condensation normally indicates that the state is superconducting

$\Longrightarrow$ Couple skyrmion matter to electromagnetism: $\partial_\mu \varphi \mapsto D_\mu \varphi = \partial_\mu \varphi - ieA_\mu [Q, \varphi]$

- Estimation of **SEMF coefficients** a_V, a_S, a_C, a_A

$\Longrightarrow$ Reducing binding energies and using the α-particle approximation [*Phys. Rev. C* **99** 044312 (2019)] should be able to estimate the coefficients