Atividade 2

Cadeias de Markov

Paulo Ricardo Seganfredo Campana

25 de julho de 2023

1) Considere o exemplo da cadeia de Markov com dois estados. Se $X_n, n \geqslant 0$, é tal cadeia, calcule
a) $\mathrm{P}(X_1 = 0 \mid X_0 = 0, X_2 = 0)$;
b) $\mathrm{P}(X_1 \neq X_2)$.

$$\mathrm{P}(X_1 = 0 \mid X_0 = 0, X_2 = 0) =$$

2) Seja $X_n, n \geqslant 0$, uma cadeia de Markov com função de transição P.
a) Mostre que se π_n denota a distribuição da variável aleatória X_n, então, para $m \geqslant 1$, temos que
$\mathrm{P}(X_n = x_n, X_{n+1} = x_{n+1}, \cdots, X_{n+m} = x_{n+m}) = \pi_n(x_n)\mathrm{P}(x_n, x_{n+1}) \cdots \mathrm{P}(x_{n+m-1}, x_{n+m})$.

$$\mathrm{P}(X_n = x_n, X_{n+1} = x_{n+1}, \cdots, X_{n+m} = x_{n+m}) = \mathrm{P}\left(\bigcap_{j=n}^{n+m} X_j = x_j\right) =$$

$$\mathrm{P}\left(X_{n+m} = x_{n+m} \cap \bigcap_{j=n}^{n+m-1} X_j = x_j\right) =$$

$$\mathrm{P}\left(X_{n+m} = x_{n+m} \mid \bigcap_{j=n}^{n+m-1} X_j = x_j\right) \cdot \mathrm{P}\left(\bigcap_{j=n}^{n+m-1} X_j = x_j\right) =$$

$$\mathrm{P}(X_{n+m} = x_{n+m} \mid X_{n+m-1} = x_{n+m-1}) \cdot \mathrm{P}\left(\bigcap_{j=n}^{n+m-1} X_j = x_j\right) =$$

$$\mathrm{P}(x_{n+m-1}, x_{n+m}) \cdot \mathrm{P}\left(\bigcap_{j=n}^{n+m-1} X_j = x_j\right) =$$

$$\mathrm{P}(x_{n+m-1}, x_{n+m}) \cdot \mathrm{P}(x_{n+m-2}, x_{n+m-1}) \cdot \mathrm{P}\left(\bigcap_{j=n}^{n+m-2} X_j = x_j\right) =$$

$$\prod_{i=1}^{m} \mathrm{P}(x_{n+m-i}, x_{n+m-i+1}) \cdot \mathrm{P}\left(\bigcap_{j=n}^{n+m-m} X_j = x_j\right) =$$

$$\prod_{i=1}^{m} \mathrm{P}(x_{n+m-i}, x_{n+m-i+1}) \cdot \mathrm{P}(X_n = x_n) =$$

$$\pi_n(x_n) \prod_{k=1}^{m} \mathrm{P}(x_{n+k-1}, x_{n+k}) =$$

$$\pi_n(x_n)\mathrm{P}(x_n, x_{n+1})\mathrm{P}(x_{n+1}, x_{n+2}) \cdots \mathrm{P}(x_{n+m-1}, x_{n+m})$$

b) Use a parte a) para mostrar que
$P(X_1 = x_1 \mid X_2 = x_2, \cdots, X_n = x_n) = P(X_1 = x_1 \mid X_2 = x_2)$.

$$P(X_1 = x_1 \mid X_2 = x_2, \cdots, X_n = x_n) = \frac{P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)}{P(X_2 = x_2, \cdots, X_n = x_n)} =$$

$$\frac{\pi_1(x_1)P(x_1, x_2)\cdots P(x_{n-1}, x_n)}{\pi_2(x_2)P(x_2, x_3)\cdots P(x_{n-1}, x_n)} = \frac{\pi_1(x_1)P(x_1, x_2)}{\pi_2(x_2)} =$$

$$\frac{P(X_1 = x_1)P(X_2 = x_2 \mid X_1 = x_1)}{P(X_2 = x_2)} = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_2 = x_2)} = P(X_1 = x_1 \mid X_2 = x_2)$$

3) Seja $X_n, n \geqslant 0$, uma cadeia de Markov sobre o espaço de estados $\mathcal{S}$. Para $y \in \mathcal{S}$ fixo, expresse $P(X_1 = y)$ em termos da distribuição inicial $\pi_0(x) = P(X_0 = x)$ e a

função de transição $P(x, y) = P(X_1 = y \mid X_0 = x)$.

$$P(X_1 = y) = P([X_1 = y] \cap \Omega) = P\left([X_1 = y] \cap \bigcup_{x \in \mathcal{S}} [X_0 = x]\right) =$$

$$P\left(\bigcup_{x \in \mathcal{S}} [X_1 = y \cap X_0 = x]\right) = \sum_{x \in \mathcal{S}} P([X_1 = y \cap X_0 = x]) =$$

$$\sum_{x \in \mathcal{S}} P(X_1 = y \mid X_0 = x) P(X_0 = x) = \sum_{x \in \mathcal{S}} P(x, y)\pi_0(x)$$