UNIVERSIDADE FEDERAL DA PARAÍBA
Introdução à Álgebra Linear
Terceira Prova
Paulo Ricardo Seganfredo Campana - 20210044220

Questão 1.

i)

Produto interno é uma função de $V \longrightarrow \mathbb{R}$ que obedece 4 propriedades:

$$p1 : \langle v,\ v \rangle \geq 0 \quad \forall v \text{ e somente 0 quando } v = 0$$
$$p2 : \langle u + v,\ w \rangle = \langle u,\ w \rangle + \langle v,\ w \rangle$$
$$p3 : \langle u,\ v \rangle = \langle v,\ u \rangle$$
$$p4 : \langle \alpha u,\ v \rangle = \alpha \langle u,\ v \rangle \quad \forall \alpha \in \mathbb{R}$$

ii)

$$\langle u,\ v \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$$
$$p1 : \langle u,\ u \rangle = x_1^2 - x_1x_2 - x_2x_1 + 3x_2^2 = x_1^2 - 2x_2x_1 + 3x_2^2 = (x_1 - x_2)^2 + 2x_2^2 \geq 0$$
$$\text{pois são números reais ao quadrado e somente 0 quando } u = (0, 0)$$
$$p2 : \langle u + v,\ w \rangle = (x_1 + y_1)z_1 - (x_1 + y_1)z_2 - (x_2 + y_2)z_1 + 3(x_2 + y_2)z_2 =$$
$$= x_1z_1 + y_1z_1 - x_1z_2 - y_1z_2 - x_2z_1 - y_2z_1 + 3x_2z_2 + 3y_2z_2 =$$
$$= x_1z_1 - x_1z_2 - x_2z_1 + 3x_2z_2 + y_1z_1 - y_1z_2 - y_2z_1 + 3y_2z_2 = \langle u,\ w \rangle + \langle v,\ w \rangle$$
$$p3 : \langle u,\ v \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2 = y_1x_1 - y_1x_2 - y_2x_1 + 3y_2x_2 = \langle v,\ u \rangle$$
$$p4 : \langle \alpha u,\ v \rangle = \alpha x_1y_1 - \alpha x_1y_2 - \alpha x_2y_1 + 3\alpha x_2y_2 =$$
$$\alpha(x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2) = \alpha \langle u,\ v \rangle$$

iii)

$$\|(3, 4) - (1, 5)\| = \|(2, -1)\| = \sqrt{\langle (2, -1),\ (2, -1) \rangle}$$

Com o produto interno usual:

$$\sqrt{\langle (2, -1),\ (2, -1) \rangle} = \sqrt{4 + 1} = \sqrt{5}$$

Com o produto interno $x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$

$$\sqrt{\langle (2, -1),\ (2, -1) \rangle} = \sqrt{4 + 4 + 3} = \sqrt{11}$$

São diferentes pois usam produtos interno diferentes.

Questão 2.

$$\|u+v\|^2 - \|u-v\|^2$$
$$\langle u+v,\ u+v\rangle - \langle u-v,\ u-v\rangle$$
$$\langle u,\ u\rangle + 2\langle u,\ v\rangle + \langle v,\ v\rangle - \langle u,\ u\rangle + 2\langle u,\ v\rangle - \langle v,\ v\rangle)$$
$$4\langle u,\ v\rangle$$
$$4\langle u,\ v\rangle = \|u+v\|^2 - \|u-v\|^2$$

Questão 3.

$$\langle p,\ q\rangle = a_0b_0 + a_1b_1 + a_2b_2$$

i)

$$\left\langle x^2-2x+3,\ 3x-4\right\rangle = 1\cdot 0 + (-2)\cdot 3 + 3\cdot(-4) = -18$$

ii)

$$\|x^2-2x+3\| = \sqrt{\langle x^2-2x+3,\ x^2-2x+3\rangle} = \sqrt{1\cdot 1+(-2)\cdot(-2)+3\cdot 3} = \sqrt{14}$$
$$\|3x-4\| = \sqrt{\langle 3x-4,\ 3x-4\rangle} = \sqrt{3\cdot 3+(-4)\cdot(-4)} = \sqrt{25} = 5$$

iii)

$$\|x^2-2x+3+3x-4\| = \|x^2+x-1\| = \sqrt{\langle x^2+x-1,\ x^2+x-1\rangle} =$$
$$= \sqrt{1\cdot 1+1\cdot 1+(-1)\cdot(-1)} = \sqrt{3}$$

iv)

$$\langle u,\ v\rangle = \cos\theta\|u\|\|v\|$$
$$\left\langle x^2-2x+3,\ 3x-4\right\rangle = \cos\theta\|x^2-2x+3\|\|3x-4\|$$
$$-18 = \cos\theta\sqrt{14}\cdot 5$$
$$\cos\theta = \frac{-18}{5\sqrt{14}}$$
$$\theta = \arccos\left(\frac{-18}{5\sqrt{14}}\right)$$

v)

$$W = [3x-4] \qquad W^\perp = \{(a_2x^2+a_1x+a_0)\in\mathbb{P}_2; \left\langle a_2x^2+a_1x+a_0,\ 3x-4\right\rangle = 0\}$$
$$\left\langle a_2x^2+a_1x+a_0,\ 3x-4\right\rangle = 3a_1 - 4a_0 = 0$$
$$3a_1 = 4a_0 \longrightarrow \frac{3}{4}a_1 = a_0 \text{ e } a_2 \text{ livre}$$
$$p = \left(a_2x^2+a_1x+\frac{3}{4}a_1\right) = a_2(x^2) + a_1\left(x+\frac{3}{4}\right)$$
$$W^\perp = \left[x^2, \left(x+\frac{3}{4}\right)\right]$$

Questão 4.

i)

$$v_1' = (1,1,1,1) \quad u_1 = \frac{(1,1,1,1)}{\|(1,1,1,1)\|} = \frac{(1,1,1,1)}{2} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$v_2' = (1,1,2,4) - \frac{\langle (1,1,2,4),\ (1,1,1,1)\rangle}{\langle (1,1,1,1),\ (1,1,1,1)\rangle}(1,1,1,1) = (1,1,2,4) - \frac{8}{4}(1,1,1,1) = (-1,-1,0,2)$$

$$u_2 = \frac{(-1,-1,0,2)}{\|(-1,-1,0,2)\|} = \frac{(-1,-1,0,2)}{\sqrt{6}} = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}\right)$$

$$v_3' = (1,2,-4,-3) - \frac{\langle (1,2,-4,-3),\ (-1,-1,0,2)\rangle}{\langle (-1,-1,0,2),\ (-1,-1,0,2)\rangle}(-1,-1,0,2) - \frac{\langle (1,2,-4,-3),\ (1,1,1,1)\rangle}{\langle (1,1,1,1),\ (1,1,1,1)\rangle}(1,1,1,1)$$

$$(1,2,-4,-3) - \frac{-9}{6}(-1,-1,0,2) - \frac{-4}{4}(1,1,1,1)$$

$$(2,3,-3,-2) + \left(-\frac{3}{2}, -\frac{3}{2}, 0, 3\right) = \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)$$

$$u_3 = \frac{\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)}{\left\|\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)\right\|} = \frac{\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)}{\sqrt{\frac{25}{2}}} = \frac{\sqrt{2}}{5}\left(\frac{1}{2}, \frac{3}{2}, -3, 1\right) = \left(\frac{\sqrt{2}}{10}, \frac{3\sqrt{2}}{10}, \frac{-3\sqrt{2}}{5}, \frac{\sqrt{2}}{5}\right)$$

$$\beta = \left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}\right), \left(\frac{\sqrt{2}}{10}, \frac{3\sqrt{2}}{10}, \frac{-3\sqrt{2}}{5}, \frac{\sqrt{2}}{5}\right)\right\}$$

ii)

$$\omega = (1,2,-6,\pi) = a\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + b\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}\right) + c\left(\frac{\sqrt{2}}{10}, \frac{3\sqrt{2}}{10}, \frac{-3\sqrt{2}}{5}, \frac{\sqrt{2}}{5}\right)$$

$$\begin{cases} \frac{a}{2} - \frac{b}{\sqrt{6}} + \frac{c\sqrt{2}}{10} = 1 \\ \frac{a}{2} - \frac{b}{\sqrt{6}} + \frac{3c\sqrt{2}}{10} = 2 \\ \frac{a}{2} + 0 + \frac{-3c\sqrt{2}}{5} = -6 \\ \frac{a}{2} + \frac{2b}{\sqrt{6}} + \frac{c\sqrt{2}}{5} = \pi \end{cases} \longrightarrow \begin{cases} 0 - 0 + \frac{2c\sqrt{2}}{10} = 1 \longrightarrow c = \frac{5}{\sqrt{2}} \\ \frac{a}{2} + 0 - 3 = -6 \longrightarrow \frac{a}{2} = -3 \longrightarrow a = -6 \\ -3 - \frac{b}{\sqrt{6}} + \frac{1}{2} = 1 \longrightarrow \frac{b}{\sqrt{6}} = -\frac{7}{2} \longrightarrow b = -\frac{7\sqrt{6}}{2} \\ \frac{a}{2} + \frac{2b}{\sqrt{6}} + \frac{c\sqrt{2}}{5} = \pi \end{cases}$$

Porém o conjunto $\left\{-6, -\frac{7\sqrt{6}}{2}, \frac{5}{\sqrt{2}}\right\}$ não é solução para a quarta equação $\frac{a}{2} + \frac{2b}{\sqrt{6}} + \frac{c\sqrt{2}}{5} = \pi$

O sistema não tem solução, portanto $\omega = (1,2,-6,\pi)$ não pode ser escrito como combinação linear de β.

iii)

$W = [(1,1,1,1),(1,1,2,4),(1,2,-4,-3)] \qquad W^{\perp} = \{(x,y,z,t) \in \mathbb{R}^4; \langle (x,y,z,t),\ w\rangle = 0\}$

$\langle (x,y,z,t),\ (1,1,1,1)\rangle = x + y + z + t = 0$

$\langle (x,y,z,t),\ (1,1,2,4)\rangle = x + y + 2z + 4t = 0$

$\langle (x,y,z,t),\ (1,2,-4,-3\rangle = x + 2y + -4z - 3t = 0$

$$\begin{cases} x + y + z + t = 0 \\ x + y + 2z + 4t = 0 \longrightarrow z + 3t = 0 \longrightarrow z = -3t \\ x + 2y - 4z - 3t = 0 \longrightarrow y - 5z - 4t = 0 \longrightarrow y + 11t = 0 \longrightarrow y = -11t \end{cases}$$

$(x,y,z,t) = (13t, -11t, -3t, t) = t(13,-11,-3,1)$

$W^{\perp} = [(13,-11,-3,1)]$