UNIVERSIDADE FEDERAL DA PARAÍBA
Probabilidade II
Atividade 4
Paulo Ricardo Seganfredo Campana

Questão 1 e 2.

a) $X \sim \text{Bernoulli}(p) \sim p^x(1-p)^{1-x} \qquad x = \{0, 1\}$

$$M_X(t) = E(e^{Xt}) = \sum_{x=0}^{1} e^{xt}p^x(1-p)^{1-x} = e^0p^0(1-p)^1 + e^tp(1-p)^0 = (1-p) + e^tp = pe^t - p + 1$$

$$M'_X(t) = pe^t \qquad M'_X(0) = E(X) = pe^0 = p$$

$$M''_X(t) = pe^t \qquad M''_X(0) = E(X^2) = pe^0 = p$$

$$\text{Var}(X) = p - p^2 = p(1-p)$$

b) $X \sim \text{Geo}(p) \sim p(1-p)^x \qquad x = 0, 1, 2, 3, \cdots$

$$M_X(t) = E(e^{Xt}) = \sum_{x=0}^{\infty} e^{xt}p(1-p)^x = p\sum_{x=0}^{\infty}(e^t(1-p))^x = \frac{p}{1-e^t(1-p)} = \frac{p}{pe^t - e^t + 1}$$

$\vdots$

$$M'_X(t) = -\frac{(p-1)pe^t}{(pe^t - e^t + 1)^2} \qquad M'_X(0) = E(X) = -\frac{p(p-1)}{(p-1+1)^2} = \frac{1-p}{p}$$

$\vdots$

$$M''_X(t) = \frac{pe^t(p-1)(pe^t - e^t - 1)}{(pe^t - e^t + 1)^3} \qquad M''_X(0) = E(X^2) = \frac{p(p-1)(p-2)}{p^3} = \frac{p^2 - 3p + 2}{p^2}$$

$$\text{Var}(X) = \frac{p^2 - 3p + 2}{p^2} - \frac{p^2 - 2p + 1}{p^2} = \frac{1-p}{p^2}$$

c) $X \sim \text{Gama}(\alpha, \beta) \sim \dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\mathbb{I}_{(0,\infty)}(x)$

$$M_X(t) = E(e^{Xt}) = \int_0^\infty e^{xt}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{(t-\beta)x}dx \qquad \begin{cases} y = -(t-\beta)x \\ dy = -(t-\beta)dx \end{cases}$$

$$\frac{\beta^\alpha}{\Gamma(\alpha)(\beta-t)}\int_0^\infty \left(\frac{y}{\beta-t}\right)^{\alpha-1} e^{-y}dy = \frac{\beta^\alpha}{\Gamma(\alpha)(\beta-t)^\alpha}\int_0^\infty y^{\alpha-1}e^{-y}dy = \frac{\beta^\alpha\Gamma(\alpha)}{\Gamma(\alpha)(\beta-t)^\alpha} = \frac{\beta^\alpha}{(\beta-t)^\alpha}$$

$M'_X(t) = \dfrac{\alpha\beta^\alpha}{(\beta - t)^{\alpha+1}}$ $\qquad$ $M'_X(0) = E(X) = \dfrac{\alpha\beta^\alpha}{\beta^{\alpha+1}} = \dfrac{\alpha}{\beta}$

$M''_X(t) = \dfrac{\alpha(\alpha+1)\beta^\alpha}{(\beta - t)^{\alpha+2}}$ $\qquad$ $M''_X(0) = E(X^2) = \dfrac{\alpha(\alpha+1)\beta^\alpha}{\beta^{\alpha+2}} = \dfrac{\alpha(\alpha+1)}{\beta^2}$

$$\mathrm{Var}(X) = \frac{\alpha(\alpha+1)}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

Questão 3.

a) $M_X(0) = E(e^{X0}) = E(1) = 1$

b) $M_Y(t) = E(e^{Yt}) = E(e^{(aX+b)t}) = E(e^{aXt+bt}) = E(e^{aXt}e^{bt}) = e^{bt}E(e^{aXt}) = e^{bt}M_X(at)$

Questão 4.

$M_Y(t) = M_X(2t) = E(e^{2Xt}) = \displaystyle\int_0^\infty e^{2xt}\lambda e^{-\lambda x}dx = \lambda\int_0^\infty e^{(2t-\lambda)x}dx$ $\qquad$ $\begin{cases} y = -(2t-\lambda)x \\ dy = -(2t-\lambda)dx \end{cases}$

$$\frac{\lambda}{\lambda - 2t}\int_0^\infty e^{-y}dy = \frac{\lambda}{\lambda - 2t}(-e^{-y})\Big|_0^\infty = \frac{\lambda}{\lambda - 2t}$$

$M'_Y(t) = \dfrac{2\lambda}{(\lambda - 2t)^2}$ $\qquad$ $M'_Y(0) = E(X) = \dfrac{2\lambda}{\lambda^2} = \dfrac{2}{\lambda}$

$M''_Y(t) = \dfrac{8\lambda}{(\lambda - 2t)^3}$ $\qquad$ $M''_Y(0) = E(X^2) = \dfrac{8\lambda}{\lambda^3} = \dfrac{8}{\lambda^2}$

$$\mathrm{Var}(Y) = \frac{8}{\lambda^2} - \frac{4}{\lambda^2} = \frac{4}{\lambda^2}$$

Questão 5.

$X \sim \text{Poisson}(\lambda)$ $\quad$ Sabemos que $\mu = E(X) = \mathrm{Var}(X) = \lambda$

$P(|X - \lambda| \geq \alpha) = P(X - \lambda \geq \alpha) + P(X - \lambda \leq -\alpha)$ $\qquad$ Pois são eventos disjuntos

$P(X \geq \lambda + \alpha) + P(X \leq \lambda - \alpha)$ Tomamos $\alpha = \dfrac{\lambda}{2}$

$P\left(X \geq \dfrac{3\lambda}{2}\right) + P\left(X \leq \dfrac{\lambda}{2}\right) \geq P\left(X \leq \dfrac{\lambda}{2}\right)$ Pois $P\left(X \geq \dfrac{3\lambda}{2}\right) \geq 0$

$$P\left(X \leq \frac{\lambda}{2}\right) \leq P\left(|X - \lambda| \geq \frac{\lambda}{2}\right) \leq \left(\frac{2}{\lambda}\right)^2 \lambda = \frac{4}{\lambda}$$

Questão 6.

A partir da desigualdade clássica de Chebyshev:

$P(|X - \mu| \geq \alpha) \leq \dfrac{1}{\alpha^2}\mathrm{Var}(X)$ Tomamos $\alpha = k\sigma$ e $\mathrm{Var}(X) = \sigma^2$

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2\sigma^2}\sigma^2$$

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$