UNIVERSIDADE FEDERAL DA PARAÍBA
Probabilidade II
Atividade 4
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Questão 1.

$$X \sim Exp\left(\lambda = \frac{1}{10}\right) \qquad F_X(x) = 1 - e^{-\dfrac{x}{10}}$$

a) $P(X > 10) = 1 - P(X \leqslant 10) = 1 - F_X(10)$

$$1 - (1 - e^{-1}) = 1 - 1 + e^{-1} = \frac{1}{e}$$

b) $P(10 \leqslant X \leqslant 20) = F_X(20) - F_X(10)$

$$(1 - e^{-2}) - (1 - e^{-1}) = e^{-1} - e^{-2} = \frac{1}{e} - \frac{1}{e^2}$$

Questão 2.

a) $S_Y = Y \circ S_x = h(X) \circ (0,1) = (0, \infty)$

b) $X \sim U(0,1) \longrightarrow F_X(x) = x$

$$F_Y(y) = P(Y \leqslant y) = P(-\ln(1-X) \leqslant y) = P(\ln(1-X) \geqslant -y) = P(1 - X \geqslant e^{-y})$$

$$P(-X \geqslant e^{-y} - 1) = P(X \leqslant 1 - e^{-y}) = F_X(1 - e^{-y}) = 1 - e^{-y}$$

$$F_Y(y) = \begin{cases} 0, & \text{se } y < 0, \\ 1 - e^{-y}, & \text{se } y \geqslant 0. \end{cases}$$

c) $F'_Y(y) = e^{-y}$

$$f_Y(y) = \begin{cases} 0, & \text{se } y < 0, \\ e^{-y}, & \text{se } y \geqslant 0. \end{cases} \longrightarrow Y \sim Exp(\lambda = 1)$$

Questão 3.

$$M_X(t) = E(e^{xt}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \int_0^\infty \lambda e^{-(\lambda - t)x} dx \qquad u = (\lambda - t)x \quad du = (\lambda - t)dx$$

$$\frac{\lambda}{\lambda - t} \int_0^\infty e^{-u} du = \frac{\lambda}{\lambda - t}$$

$$M'_X(t) = \frac{\lambda}{(\lambda - t)^2} \longrightarrow M'_X(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$M''_X(t) = \frac{2\lambda}{(\lambda - t)^3} \longrightarrow M''_X(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2} \qquad \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$E(X) = \frac{1}{\lambda}, \qquad Var(X) = \frac{1}{\lambda^2}$$

Questão 4.

$$\int_{-2}^{2} cdx+\int_{2}^{4} 2cdx = 1 \longrightarrow cx\Big|_{-2}^{2}+2cx\Big|_{2}^{4} = 1 \longrightarrow (2c+2c)+(8c-4c) = 1 \longrightarrow 8c = 1 \longrightarrow c = \frac{1}{8}$$

$$E(X) = \frac{1}{8}\left(\int_{-2}^{2} xdx + \int_{4}^{2} 2xdx\right) = \frac{1}{8}\left(\frac{x^2}{2}\Big|_{-2}^{2} + x^2\Big|_{2}^{4}\right) = \frac{1}{8}(2-2+16-4) = \frac{12}{8} = \frac{3}{2}$$

$$E(X^2) = \frac{1}{8}\left(\int_{-2}^{2} x^2dx + \int_{4}^{2} 2x^2dx\right) = \frac{1}{8}\left(\frac{x^3}{3}\Big|_{-2}^{2} + \frac{2x^3}{3}\Big|_{2}^{4}\right) = \frac{1}{24}(8+8+128-16) = \frac{16}{3}$$

$$E(X) = \frac{3}{2}, \qquad Var(X) = \frac{16}{3} - \left(\frac{3}{2}\right)^2 = \frac{37}{12}$$

Questão 5.

a) $E(Y) = E(-\ln X) \qquad f_Y = 1 \cdot \mathbb{I}_{(0,1)}(x)$

$$\int_{-\infty}^{\infty} -\ln x \cdot \mathbb{I}_{(0,1)}(x)dx = \int_{0}^{1} -\ln xdx = \int_{1}^{0} \ln xdx$$

$$x\ln x - x\Big|_{1}^{0} = (0\cdot(-\infty)-0)-(1\cdot\ln 1 - 1) = 1$$

b) $X \sim U(0,1) \longrightarrow F_X(x) = x$

$$F_Y(y) = P(Y \leqslant y) = P(-\ln X \leqslant y) = P(\ln X \geqslant -y) = P(X \geqslant e^{-y})$$

$$1 - P(X \leqslant e^{-y}) = 1 - F_X(e^{-y}) = 1 - e^{-y}$$

$$f_Y(y) = F_Y'(y) = e^{-y} \longrightarrow Y \sim Exp(\lambda = 1)$$

c) $S_Y = (0,\infty)$

$$E(Y) = \int_{0}^{\infty} ye^{-y}dy \qquad \begin{cases} u = y, & du = dy \\ v = -e^{-y}, & dv = e^{-y} \end{cases}$$

$$-ye^{-y}\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-y}dy \longrightarrow -\frac{y}{e^y}\Big|_{0}^{\infty} + 1 \longrightarrow E(Y) = 1$$

Alternativamente:

$$E\left(Y \sim Exp\left(\lambda\right)\right) = \frac{1}{\lambda} \longrightarrow \lambda = 1 \longrightarrow E(Y) = 1 \quad \checkmark$$