UNIVERSIDADE FEDERAL DA PARAÍBA
Probabilidade II
Segunda Prova
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Questão 1.

$$X \sim U\left(0, \frac{1}{2}\right) \qquad Y = e^{2X}$$

$$S_X = \left(0, \frac{1}{2}\right) \qquad S_Y = (e^{2\cdot 0}, e^{2\cdot\frac{1}{2}}) = (1, e)$$

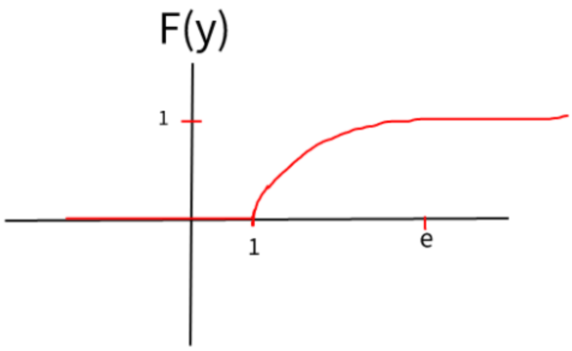
$$F_X(x) = \frac{x-a}{b-a} = \frac{x}{\frac{1}{2}} = 2x \qquad a \le x \le b$$

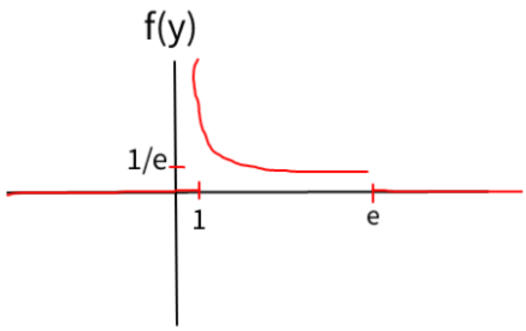
a)

$$F_Y(y) = P(Y \le y) = P(e^{2X} \le y) = P(2X \le \ln y) = P\left(X \le \frac{\ln y}{2}\right) = F_X\left(\frac{\ln y}{2}\right)$$

$$F_X\left(\frac{\ln y}{2}\right) = 2\left(\frac{\ln y}{2}\right) = \ln y \longrightarrow F_Y(y) = \begin{cases} 0 & \text{se} \quad y \le 1, \\ \ln y & \text{se } 1 \le y \le e, \\ 1 & \text{se } e \le y. \end{cases}$$

$$f_Y(y) = F'_Y(y) = (\ln y)' = \frac{1}{y} \longrightarrow f_Y(y) = \begin{cases} 0 & \text{se} \quad y \le 1, \\ \dfrac{1}{y} & \text{se } 1 \le y \le e, \\ 0 & \text{se } e \le y. \end{cases}$$





b)

$$E(Y) = \int_1^e y\frac{1}{y}dy = \int_1^e 1dy = y\Big|_1^e = e - 1$$

$$E(Y^2) = \int_1^e y^2\frac{1}{y}dy = \int_1^e ydy = \frac{y^2}{2}\Big|_1^e = \frac{e^2-1}{2}$$

$$Var(Y) = \frac{e^2-1}{2} - (e-1^2) = \frac{e^2-1}{2} - \frac{2e^2-4e+2}{2} = \frac{-e^2+4e-3}{2} = -\frac{1}{2}(e-1)(e-3)$$

Questão 2.

$$X \sim Exp(\lambda) \qquad Y = aX + b$$

$$E(X) = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2} \qquad M_X(t) = \frac{\lambda}{\lambda - t}$$

a)

$$M_Y(t) = e^{bt}M_X(at) = e^{bt}\frac{\lambda}{\lambda - at} = \lambda(e^{bt}(\lambda - at)^{-1})$$

b)

$$M'_Y(t) = \lambda(e^{bt}(a)(\lambda - at)^{-2} + (\lambda - at)^{-1}be^{bt})$$

$$M'_Y(0) = \lambda(a\lambda^{-2} + \lambda^{-1}b) = \frac{a}{\lambda} + b = \frac{a + b\lambda}{\lambda}$$

$$M''_Y(t) = \lambda(a(e^{bt}(2a)(\lambda - at)^{-3}) + be^{bt}(\lambda - at)^{-2} + (\lambda - at)^{-1}b^2e^{bt} + (a)(\lambda - at)^{-2}be^{bt})$$

$$M''_Y(0) = \lambda(2a^2\lambda^{-3} + ab\lambda^{-2} + \lambda^{-1}b^2 + a\lambda^{-2}b)$$

$$= \frac{2a^2 + 2ab\lambda + b^2\lambda^2}{\lambda^2} = \frac{a^2 + (a + b\lambda)^2}{\lambda^2}$$

$$Var(Y) = \frac{a^2 + (a + b\lambda)^2}{\lambda^2} - \frac{(a + b\lambda)^2}{\lambda^2} = \frac{a^2}{\lambda^2}$$

c)

$$E(Y) = E(aX + b) = b + aE(X) = b + a\frac{1}{\lambda} = \frac{a + b\lambda}{\lambda}$$

$$Var(Y) = Var(aX + b) = a^2Var(X) = a^2\frac{1}{\lambda^2} = \frac{a^2}{\lambda^2}$$

Questão 3.

$$X \sim Exp(\lambda) \qquad \mu = E(X) = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

Pela desigualdade clássica de Chebyshev:

$$P(|X - \mu| \geq \alpha) \leq \frac{1}{\alpha^2}Var(X)$$

$$P\left(\left|X - \frac{1}{\lambda}\right| \geq \alpha\right) \leq \frac{1}{\alpha^2\lambda^2}$$

Por serem eventos disjuntos, podemos separar $P(|X - \lambda| \geq \alpha)$ em duas probabilidades:

$$P\left(\left|X - \frac{1}{\lambda}\right| \geq \alpha\right) = P\left(X - \frac{1}{\lambda} \geq \alpha\right) + P\left(X - \frac{1}{\lambda} \leq -\alpha\right) =$$

$$P\left(X \geq \frac{1}{\lambda} + \alpha\right) + P\left(X \leq \frac{1}{\lambda} - \alpha\right)$$

$P\left(X \leq \frac{1}{\lambda} - \alpha\right)$ é positivo por ser uma probabilidade, portanto:

$$P\left(X \geq \frac{1}{\lambda} + \alpha\right) \leq P\left(\left|X - \frac{1}{\lambda}\right| \geq \alpha\right) \leq \frac{1}{\alpha^2\lambda^2}$$

Escolha $\alpha = \frac{2}{\lambda}$

$$P\left(X \geq \frac{3}{\lambda}\right) \leq P\left(\left|X - \frac{1}{\lambda}\right| \geq \alpha\right) \leq \frac{1}{4}$$

$$P\left(X \geq \frac{3}{\lambda}\right) \leq \frac{1}{4}$$