Estimation of AoA, based on the division of subspaces

Far field and near field

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Steering vector modeling for ULA in far field

Due to the path difference, the signal received at the m-th antenna $s_m(t)$ arrives with a time delay τ_m given by:

$$\tau_m = \frac{\Delta d}{c}$$

where Δd is the relative distance between d_k^m and d_k. The phase difference (γ_m) at the m-th antenna is given by:

$$\gamma_m = 2\pi f \tau_m = \frac{2\pi(d_k^m - d_k)}{\lambda}$$

where, from the Fourier transform (2), we can observe that the time shift causes a phase shift in the frequency domain:

$$\mathcal{F}\{s_m(t-\tau_m)\} = \int_{-\infty}^{\infty} s(t-\tau_m)e^{-j2\pi f t}dt = S(f)e^{-j2\pi f \tau_m}$$

The spectral magnitude $|S(f)|$ remains unchanged, meaning that only the continuous phase spectrum ($\angle S(f)$) is affected.

Steering vector modeling for ULA in far field

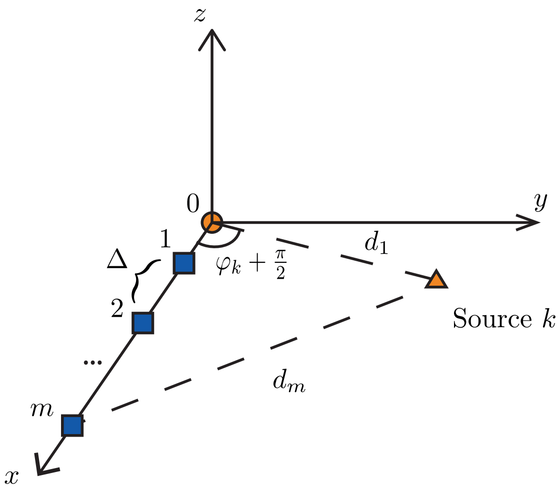

Figura: Geometry of the relative distances between m antennas and the source.

Steering vector modeling for ULA in far field

The relative distance between the m-th antenna and the source (d_k^m) can be obtained using the law of cosines. Prior knowledge of the array's spatial configuration is required for modeling. Assuming the array structure shown in Figure 1:

$$[d_k^m]^2 = d_k^2 + \left((m-1)\Delta\right)^2 + 2d_k \cdot (m-1)\Delta \cdot \left(\cos(\varphi) + \frac{\pi}{2}\right)$$

$$d_k^m = \sqrt{d_k^2 + \left((m-1)\Delta\right)^2 + 2d_k \cdot (m-1)\Delta \cdot \sin(\varphi)}$$

Steering vector modeling for ULA in far field

In the far-field regime $d_k \gg (m-1)\Delta$, we can approximate $\tilde{d}_k^m$ using the Taylor expansion. Factoring:

$$d_k^m = d_k \cdot \sqrt{1 + \frac{\left((m-1)\Delta\right)^2}{d_k^2} + \frac{2(m-1)\Delta}{d_k}\sin(\varphi)}$$

Using the Taylor expansion for $\sqrt{1+x}$ when $|x| \ll 1$:

$$\sqrt{1+x} \approx 1 + \frac{x}{2}$$

We obtain:

$$\begin{aligned}
\tilde{d}_k^m &\approx d_k \left[1 + \frac{1}{2}\left(\frac{\left((m-1)\Delta\right)^2}{d_k^2} + \frac{2(m-1)\Delta}{d_k}\sin(\varphi)\right)\right] \\
&\approx d_k + \frac{d_k}{2} \cdot \frac{\left((m-1)\Delta\right)^2}{d_k^2} + d_k \cdot \frac{(m-1)\Delta}{d_k}\sin(\varphi) \\
&\approx d_k + (m-1)\Delta \cdot \sin(\varphi).
\end{aligned}$$

Steering vector modeling for ULA in far field

Substituting d_k^m:

$$(\tilde{d}_k^m - d_k) = d_k + ((m-1)\Delta \cdot \sin(\varphi) - d_k)$$

$$\therefore (\tilde{d}_k^m - d_k) = (m-1)\Delta \cdot \sin(\varphi).$$

The corresponding phase difference is:

$$\gamma_m = 2\pi \frac{\tilde{d}_k^m - d_k}{\lambda} = \frac{2\pi}{\lambda}(m-1)\Delta \cdot \sin(\varphi).$$

Note: The phase difference for each element of the steering vector is independent of the distance between the transmitter and the receiver.

Steering vector modeling for ULA in far field

The Fraunhofer distance (d_F) provides the minimum distance at which it is possible to approximate the curvature of a wave as planar. Thus, the steering vector model has limitations for distances shorter than d_{FA}.

$$d_{FA} = \frac{2D^2}{\lambda}$$

It follows that the steering vector model for **received** signals in the far-field regime is given by:

$$\mathbf{a}(\theta_m) = \left[1, e^{-j\frac{2\pi\Delta}{\lambda}\sin(\theta)}, e^{-j\frac{4\pi\Delta}{\lambda}\sin(\theta)}, \cdots, e^{-j\frac{2\pi\Delta}{\lambda}(m-1)\sin(\theta)}\right]$$

Path loss model

Path loss can be decomposed into three terms, describing the phenomena of path loss, shadowing, and fading. Each term introduces different variations in the signal amplitude due to the area over which each term is averaged. The area mean is the long-term path loss term, characterizing a slow variation, and measures the average attenuation of the signal amplitude relative to distance, being deterministic.

Parametric channel gain model

Modeling a LoS scenario, the free-space model with $\varphi = 2$:

$$g = \frac{A_{\text{iso}}}{A_{\text{sphere}}} = \frac{\frac{\lambda}{4\pi}}{4\pi d^2} = \frac{\text{3GHz}}{(4\pi)^2 f}\left(\frac{1}{d}\right)^2$$

The average propagation can be described by a parametric channel gain model, where g is the large-scale fading coefficient, $g = \frac{1}{\text{PL}}$, adjustable for different scenarios. Defining $\Upsilon = (\frac{\lambda}{4\pi})^\alpha$, observe which Υ is correlated with the system frequency:

$$g = \Upsilon\left(\frac{1}{d}\right)^\alpha$$

Modeling the channel

At a given instant t, we can model the signals incident on the array:

$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix}}_{\mathbf{y}\in\mathbb{C}^m} = [\mathbf{g}^\top]^{\frac{1}{2}} \underbrace{\begin{bmatrix} a_1(\varphi_1) & a_1(\varphi_2) & \dots & a_1(\varphi_k) \\ a_2(\varphi_1) & a_2(\varphi_2) & \dots & a_2(\varphi_k) \\ \vdots & \vdots & \ddots & \vdots \\ a_m(\varphi_1) & a_m(\varphi_2) & \dots & a_m(\varphi_k) \end{bmatrix}}_{\mathbf{A}\in\mathbb{C}^{m\times k}} \underbrace{\begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_k(t) \end{bmatrix}}_{\mathbf{s}\in\mathbb{C}^k} + \underbrace{\begin{bmatrix} n_1(t) \\ n_2(t) \\ \vdots \\ n_m(t) \end{bmatrix}}_{\mathbf{n}\in\mathbb{C}^m}.$$

where $\mathbf{g}^\top = \left[g_1, g_2, \cdots, g_k\right] \in \mathbb{R}^{1\times k}$. $y_m \in \mathbb{C}$ is the linear combination of k transmitted signals, with the effects of the channel, additive noise at the receivers, and phase shifts due to the array response:

$$y_m(t) = \sum_{k=1}^{K} \sqrt{g_k} s_k(t) e^{-j\gamma_2(m-1)\sin(\varphi_k)} + n(t)$$

where $n(t) \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$.

Modeling the channel

We construct the matrix $\mathbf{Y} \in \mathbb{C}^{m \times n}$ with the sample functions of the process, where each coefficient $y_m(t)$ is the combination of signals received by an antenna in the array at a given instant t.

$$\mathbf{Y} = \sqrt{\boldsymbol{G}}\mathbf{A} \cdot \mathbf{S} + \mathbf{N}, \quad \mathbf{N} \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2 \mathbf{I}_M)$$

where $\boldsymbol{G}^{\frac{1}{2}} = \text{diag}(\sqrt{g_1}, \sqrt{g}_2, \cdots, \sqrt{g_k})$, $\mathbf{A}$ is the steering vector, $\mathbf{S}$ is the matrix of transmitted signals, and $\mathbf{N}$ is the noise matrix. In this way, we characterize the process based on its statistical averages.

Modeling the channel

We model the signals received by the array as a random process.

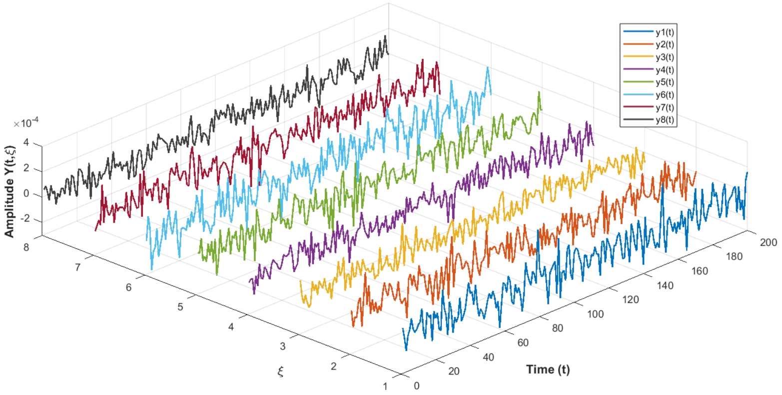

Figura: Model of signals received in the array for an 8-antenna configuration.

Modeling the channel

We perform a subjective graphical analysis of stationarity by observing that there is no apparent trend.

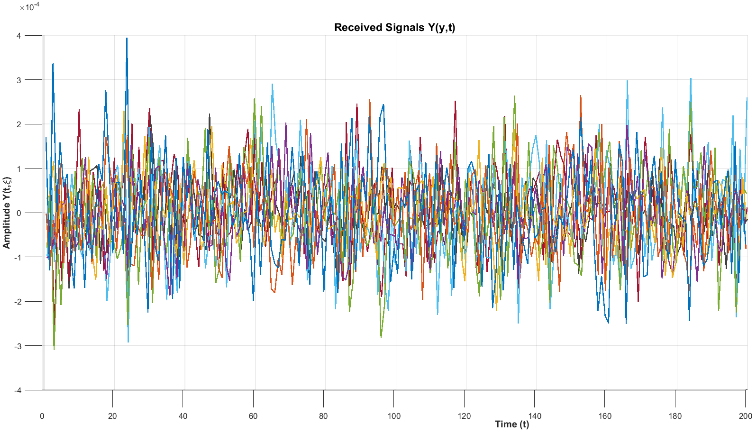

Figura: Time series of the received signal process.

Modeling the channel

We model $\mathbf{Y}$ as at least Wide-Sense Stationary (WSS), which is valid only for fixed sources. The autocorrelation of a signal y_m as a function of a time shift τ is defined as:

$$\gamma(\tau) = \mathbb{E}[y_m(t) \cdot y_m(t + \tau)]$$

Given that $\tau = t_2 - t_1$, for $\tau = 0$:

$$\gamma(0) = \mathbb{E}\left[y_m(t)^2\right]$$

Thus, the total power of the process must be equal to the average power of any individual sample function $y_m(t)$. To avoid waste and achieve maximum spectral efficiency, most modern modulations remove the DC component of signals before transmission. We can neglect the means, so the introduced error will be small compared to the effect of noise. In this case, the covariance matrix is equivalent to the autocorrelation matrix.

Noise subspace based localization

$$
\begin{aligned}
\mathbf{C}_{\mathbf{Y}} &= \mathbb{E}[(\mathbf{Y} - \mu_{\mathbf{Y}})^H(\mathbf{Y} - \mu_{\mathbf{Y}})] \\
&\approx \mathbb{E}[\mathbf{Y}^H\mathbf{Y}] = \mathbf{\Gamma}_{\mathbf{Y}} \\
&= \mathbb{E}\big[(\boldsymbol{G}^{\frac{1}{2}}\mathbf{AS} + \mathbf{N})(\boldsymbol{G}^{\frac{1}{2}}\mathbf{AS} + \mathbf{N})^H\big] \\
&= \mathbb{E}\big[[\boldsymbol{G}^{\frac{1}{2}}]^H\mathbf{SAS}^H\mathbf{A}^H\boldsymbol{G}^{\frac{1}{2}}\big] + \mathbb{E}\big[\mathbf{NN}^H\big] \\
&= [\boldsymbol{G}^{\frac{1}{2}}]^H\mathbf{A}\mathbf{\Gamma}_{\mathbf{S}}\mathbf{A}^H\boldsymbol{G}^{\frac{1}{2}} + \sigma^2\mathbf{I}.
\end{aligned}
$$

Noise subspace based localization

Through $C_{\mathbf{Y}}$, the eigendecomposition of a matrix is performed, as shown in eq. (16), where the matrix is represented in terms of its eigenvalues and eigenvectors.

$$\mathbf{R_Y} = \mathbf{V\Sigma V}^H = \mathbf{V_s\Sigma_s V_s}^H + \mathbf{V_n}\sigma^{\mathbf{2}}\mathbf{V_n}^H$$

where $\mathbf{V}$ is a vector space, which can be understood as an orthonormal matrix since each column represents an eigenvector orthogonal to the other eigenvectors of the field
Applying spectral decomposition, we have:

$$[\mathbf{V}, \mathbf{\Sigma}] = \text{eig}(\mathbf{R_Y})$$

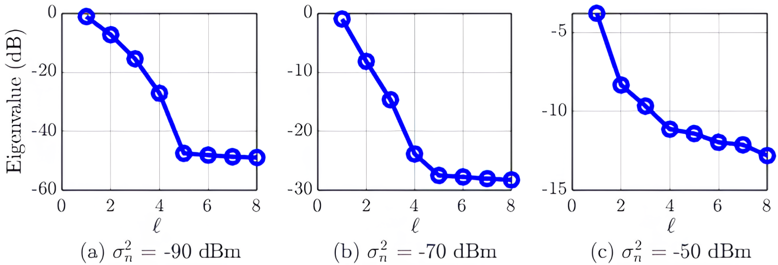

Figura: Natural separation of eigenvalues.

Source estimation

In order to minimize uncertainty in the obtained information and to reduce the cross-entropy ($\mathcal{H}$) between $\mathbf{R_Y}$ and the proposed model ($\hat{\mathbf{R}}_\mathbf{Y}$), we minimize $\mathcal{H}$ using Maximum Likelihood (ML), a method that seeks the parameters that best fit the observed data.

$$\hat{\ell}(p\,|\,\hat{\lambda}) = \ln \mathcal{L} = \sum_{i=1}^{n} \ln \mathbf{R_Y}(\hat{\lambda}_{p+1}, \ldots, \hat{\lambda}_m\,|\,p)$$

Given the true value (p_0) belonging to the parametric model $\{R_Y(\lambda_{p+1}, \ldots, \lambda_m|\,p\,),\, p \in \mathcal{P}\}$, where $\mathcal{P}$ is the parametric space, we aim to maximize the conditional probability $\hat{\ell}(\,p\,|\,\lambda)$ using the eigenvalues obtained from $\mathbf{R_Y}$. We evaluate the value of p, which is correlated with the number of noise eigenvectors used to explain the data, directly influencing the model complexity. Greater complexity increases likelihood but may overestimate the number of users.

Source estimation

To address which, the Akaike Information Criterion (AIC) is used to penalize complex models:

$$\text{AIC}(p) = -k \ln\left(\frac{\prod_{i=p+1}^{m} \hat{\lambda}_i}{[\hat{\sigma}^2]^{m-p}}\right) + p \cdot (2m - p)$$

For this criterion, the estimated variance of the noise is given by the average of the $m - p$ smallest eigenvalues of the covariance matrix:

$$\hat{\sigma}^2 = \frac{1}{m-p} \sum_{i=p+1}^{m} \hat{\lambda}_i$$

Pseudo Spectrum

$$P_{\mathsf{MUSIC}}(\varphi) = \left[\mathbf{a}(\varphi)^H \hat{\mathbf{V}}_n \hat{\mathbf{V}}_n^H \mathbf{a}(\varphi)\right]^{-1}$$

The estimated AoA is obtained as follows:

$$\hat{\theta} = \arg \text{máx}\, P_{MUSIC}(\theta)$$

Multiple Signal Classification Overview

Cuadro: Values used on simulation parameters (MUSIC)

Parameters	**Values**
Snapshots	$N = 2000$ realizations
Number of users	$K = 4$ Users
Number of antennas	$M = 8, M = 32, M = 64$
ULA spacing	$\Delta = 0,5\lambda$
Carrier Frequency	$f_c = 15{,}0$Ghz
Transmitted signal power	$P = 100$mW
Noise power	$\sigma^2 = 10$mW
Path-loss Expoent	$\alpha = 2$

Multiple Signal Classification Overview

A technique to estimate the AoA, based on the separation between the noise and signal subspaces of the received signal covariance matrix.

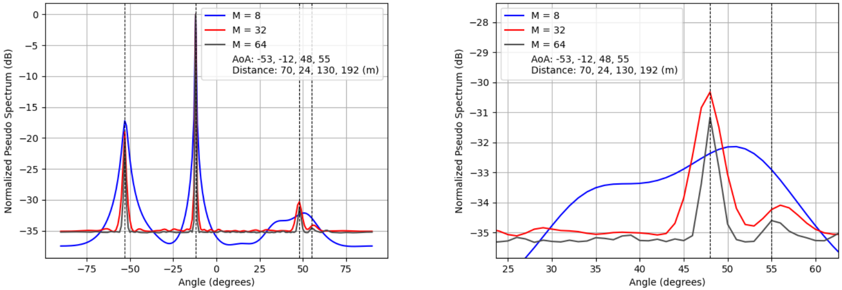

Figura: (a) Overall pseudo-spectrum representation, (b) Zoom in on the region with the most distant users.

edge effects

Assuming $\varphi > 4$, noise dominates the covariance matrix $R \approx \sigma^2\mathbf{I}$, being **N** $\sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2\mathbf{I}_M)$ we have $\Theta \sim \mathcal{U}[-\frac{\pi}{2}, \frac{\pi}{2}]$ given by:

$$E[\Theta^2] = \frac{1}{2a}\int_{-a}^{a} \Theta^2 d\Theta = \frac{1}{2a} \cdot \left[\frac{\Theta^3}{3}\right]_{-a}^{a} = \frac{a^2}{3} = 2{,}7 \cdot 10^3$$

$$\text{RMSE} = \sqrt{E[\Theta^2]} = 51{,}96^\circ$$

At the edges, the steering vectors have less relative variation with respect to small increments of θ, which reduces their distance from the noise subspace. This weakens the distinction between signal and noise, $\frac{d}{dx}sin(\varphi) = cos(\varphi)$. The nonlinearity of the steering vector can concentrate estimates close to the center. The graphs converge statistically due to the superposition of the effects of the steering vector geometry, the path loss and the uniform distribution as a basis.

edge effects

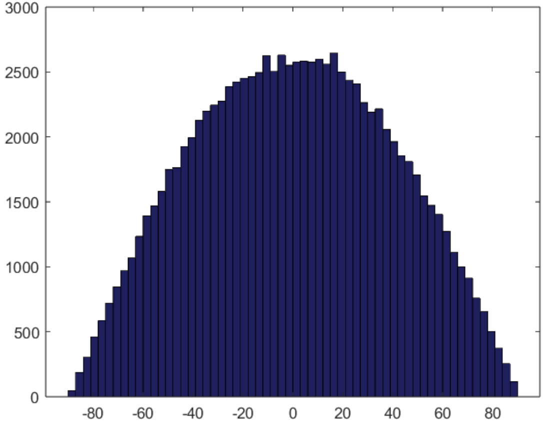

Figura: monte carlo simulation for 100.000 snapshots, $\alpha > 4$

Modeling steering vector near-field

$$\tilde{d}_k^m(\varphi) = \sqrt{d_k^2 + \left((m-1)\Delta\right)^2 + 2d_1 \cdot (m-1)\Delta \cdot \sin(\varphi)}$$

a aproximação de Taylor de segunda ordem para $|x| \ll 1$:

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$$

é obtida assumindo $d^3 \ll r_k^3$ e $d^4 \ll r_k^4$.

$$\approx r_k \left(1 + \tfrac{(m-1)^2 d^2}{2r_k^2} - \tfrac{(m-1)d}{r_k}\sin(\varphi_k) - \tfrac{(m-1)^4 d^4}{8r_k^4} + \tfrac{(m-1)^3 d^3}{2r_k^3}\sin(\varphi_k)\right)$$

$$\approx r_k \left(1 - \frac{(m-1)d}{r_k}\sin(\varphi_k) + \frac{(m-1)^2 d^2}{2r_k^2}\left(1 - \sin^2(\varphi_k)\right)\right),$$

Modeling steering vector near-field

$$\mathbf{a}(\varphi_k, d_k) = e^{j2\pi\Delta}\frac{dm - d_k}{\lambda}$$

$$\mathbf{a}(\varphi_k, d_k) = e^{\frac{j2\pi\Delta}{\lambda}(m-1)\sin(\varphi_k)+(m-1)^2-\pi\frac{\Delta^2}{\lambda d_k}\cos^2(\varphi_k)}$$

$$\mathbf{a}(\theta_k, d_k) = \left[1, e^{j(\gamma_k+\phi_k)}, e^{2(\gamma_k+2\phi_k)}, \cdots, e^{(m-1)\gamma_k+(m-1)^2\phi_k}\right]$$

$\gamma_k = \frac{2\pi\Delta}{\lambda}\sin(\varphi_k)$ represents the linear phase term and $\phi_k = \frac{-\pi\Delta^2}{\lambda d}\cos^2(\varphi_k)$ is the curvature-dependent term.

Path loss model Near Field

In near-field, each antenna is at a slightly different distance from the source, as the wavefront is no longer flat, but rather spherical.

$$g = \Upsilon\left(\frac{1}{d_k + (m-1)\frac{\Delta^2}{2d_k}}\right)^{\alpha}$$

Multiple Signal Classification Near field

$$P_{\text{MUSIC}}(\varphi, d) = \left[\mathbf{a}(\varphi, d)^H \hat{\mathbf{V}}_n \hat{\mathbf{V}}_n^H \mathbf{a}(\varphi, d)\right]^{-1}$$

Multiple Signal Classification Near field

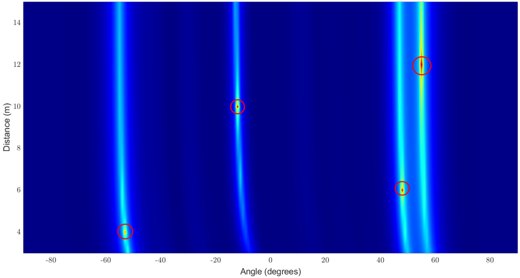

Figura: Pseudo spectrum MUSIC 2D

Multiple Signal Classification Near field

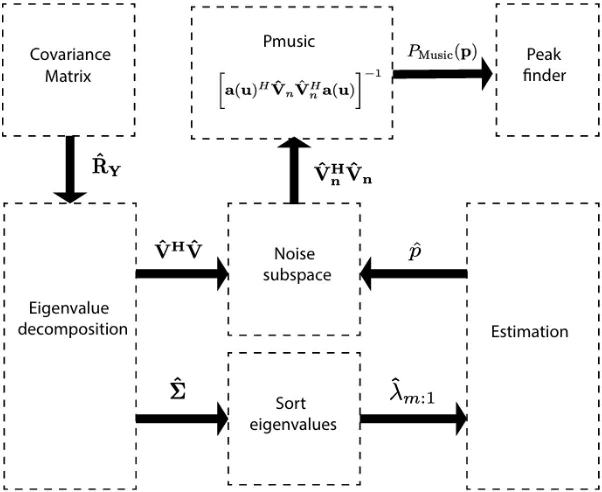

Figura: diagram MUSIC