Modeling Arrival Time

The Poisson Distribution

Model Limitations: All Distributions are Uniform

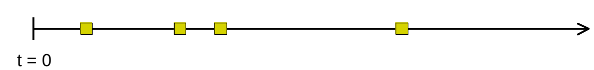
- How do we know that customers will arrive randomly with a Uniform distribution?
  - Actually they probably won’t
- How do we know that service time occurs randomly with a Uniform distribution?
  - It probably doesn’t
- So we need to model our events with distributions other than Uniform
  - But what distribution should we use?
  - and how can we implement them?
    - C only has rand() which is Uniformly distributed

Poisson Arrivals

- Poisson Distribution
  - discrete probability distribution
  - describes a random variable representing the number of events that occur in a time interval
- There is a convenient relationship between Poisson (discrete) and Exponential (continuous) distributions
  - Poisson: number of events in a time interval
  - Exponential: time between events
- We exploit these two distributions extensively
  - queueing theory: math is straightforward enough for analysis
  - simulation: “time to next arrival” variates easy to generate

Poisson Arrivals

- Assuming:
  - arrivals occur one at a time, at some rate
  - arrivals are as likely to occur at any time as at any other
  - nothing else is known about inter-arrival times
- *We can calculate distribution of inter-arrival times*



Poisson Arrivals (cont.)

- Let Δt represent some small interval of time
- Since we can consider Δt as small as we like, and arrivals are as likely to occur at any instant:
  1. probability of more than one arrival in Δt is 0
  2. probability of one arrival in Δt is proportional to Δt
     - i.e. $\lambda\Delta t$, where λ is a constant reflecting overall arrival rate
  3. arrivals in one time interval are independent of those in another disjoint one
- Let $p(k,t)$ be the probability of k arrivals occurring in the interval $[0, t]$. Consider the interval $[t, t + \Delta t]$:
  - probability of arrival in $[t, t + \Delta t] = \lambda\Delta t$
  - probability of no arrival in $[t, t + \Delta t] = 1 - \lambda\Delta t$

Poisson Arrivals (cont.)

So:

$p(k, t + \Delta t)$ = prob. k arrivals in $[0, t]$ and zero in $[t, t + \Delta t]$
+ prob. $(k - 1)$ arrivals in $[0,t]$ and 1 in $[t, t + \Delta t]$
$= p(k, t)\cdot(1 - \lambda\Delta t) + p(k - 1, t)\cdot\lambda\Delta t$

- Rearranging:

$$\frac{p(k,t+\Delta t)-p(k,t)}{\Delta t}=\lambda[p(k-1,t)-p(k,t)]$$

Poisson Arrivals (cont.)

- As $\Delta t \to 0$, the LHS of this becomes a derivative:

$$\frac{d}{dt}p(k,t)=-\lambda[p(k,t)-p(k-1,t)]$$

  - this differential equation has initial conditions:
    - $p(k, 0) = 0$ for all $k > 0$
    - $p(0, 0) = 1$
- Solving for this equation and its boundary conditions (which is non-trivial) yields:

$$p(k,t)=\frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

  - probability of k arrivals occuring in $[0, t]$

Poisson Arrivals (cont.)

- Probability mass function (i.e. $t = 1$):

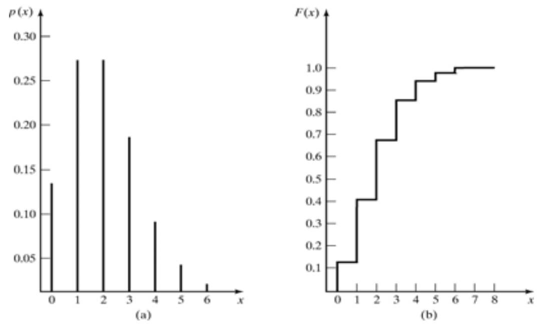
$$p(x)=\begin{cases}\dfrac{e^{-\lambda}\lambda^x}{x!}, & x=0,1,2,\ldots\\ 0, & otherwise\end{cases}$$

- Example (text)
  - computer repair person paged each time there is a service call -- - number of beeps per hour is known to agree with a Poisson distribution with mean $\lambda = 2$
    - probability of 3 beeps in next hour?

$$p(3)=\frac{e^{-2}2^3}{3!}=\frac{(0.135)(8)}{6}=0.18$$

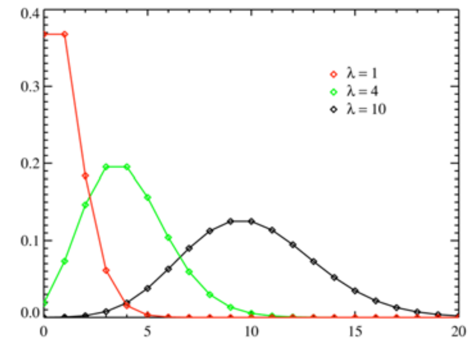
    - probability of 2 or more beeps?

Poisson Arrivals (cont.)

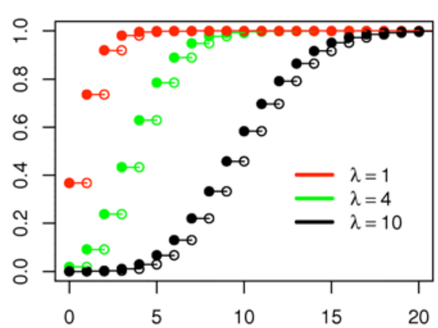


Poisson PMF $\lambda = 2$ Poisson CDF

Poisson Arrivals (cont.)



Poisson Arrivals (cont.)



Inter-arrival Time

- Let X be the random variable equal to the length of time between arrivals: $F_X(t) = P(X \le t)$
- Note: the following two events are complementary (1 and only 1 occurs):
  - waiting time $\le t$ (i.e. at least one arrival in $[0, t]$)
  - no arrivals in $[0, t]$
- Thus: $F_X(t) + e^{-\lambda t} = 1$

$$F_X(t) = 1 - e^{-\lambda t}$$

$$f_X(t) = \frac{d}{dt} F_X(t) = \lambda e^{\lambda t}$$

distribution of inter-arrival times is an exponential distribution

Application to Simulation

- "Poisson Arrivals" commonly used in modelling
  - e.g. particles from a radioactive source have Poisson arrivals
  - in the absence of any information, it is common to assume arrival events are Poisson

- Benefit:
  - inter-arrival time is exponential
    - easily integrated with time-based/event driven simulations
  - relatively simple distribution
    - simple situations can be modelled mathematically when necessary

- Aside
  - service times are often assumed to be exponential, however it is debatable whether this is justified