Master Thesis: Climate change economic effect on health through temperature

Paulo Gugelmo Cavalheiro Dias

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Contents

1 Introduction

Temperature is a fundamental component of the physical environment, shaping human activity across multiple dimensions. As climate change alters the global distribution of temperature, there is growing concern about its broader economic consequences. While substantial attention has been given to the direct effects of temperature on labor productivity, agricultural yields, and conflict, less is understood about how temperature-driven changes in health may propagate through the economy over the long run.

Health is a critical input into individual well-being and economic performance. Variations in temperature can influence the incidence and severity of disease, the functioning of the human body, and the demand for medical services. These effects, in turn, may alter labor supply, human capital accumulation, and lifetime income trajectories. Understanding the economic cost associated with these health effects is essential for quantifying the full burden of climate change and for informing effective adaptation policy.

1.1 Related literature

This paper contributes to three main strands of literature: health economics, the economics of climate change, and the intersection of climate and health. Each of these fields provides foundational insights but leaves open important questions regarding the long-run economic consequences of temperature-induced health variation.

1.1.1 Health Economics

A large literature has documented the central role of health in shaping economic behavior over the life cycle. De Nardi et al. (2023) demonstrate that health status significantly affects labor supply, retirement decisions, and savings dynamics, establishing health as a key determinant of productivity and economic welfare. Beyond contemporaneous productivity effects, poor health imposes substantial intertemporal costs. Both De Nardi et al. (2023) and Capatina (2015) quantify the lifetime income and utility losses associated with adverse health trajectories, emphasizing the compounding nature of health shocks over the life course.

1.1.2 Climate and Economics

The economic impacts of climate change are multifaceted, with temperature emerging as a particularly influential channel. Burke et al. (2015) document substantial aggregate output losses associated with rising temperatures, particularly in countries with limited adaptive capacity. More recently, Bilal and Kanzig (2024, 2025) show that temperature shocks have asset-pricing implications, underscoring the forward-looking nature of climate risks.

At the same time, the literature has explored how extreme heat may exacerbate social and political instability. Burke, Hsiang, and Miguel (2015) provide compelling evidence that hotter temperatures increase the likelihood of conflict in developing countries, suggesting that the consequences of climate change extend well beyond output measures. Importantly, adaptation has been shown to significantly mitigate the economic burden of climate shocks. Carleton et al. (2022) estimate the global mortality consequences of warming while explicitly accounting for adaptation costs and benefits, illustrating how policy and behavioral responses shape climate damages.

From a methodological standpoint, the empirical identification of climate impacts presents persistent challenges. Deryugina and Hsiang (2017), Hsiang (2016), and Nordhaus (2019) all highlight the difficulties of isolating causal effects in the presence of spatial and temporal heterogeneity. Bilal and Kanzig (2024, 2025) further emphasize the econometric complexity involved in measuring forward-looking responses to climate risks.

1.1.3 Climate and Health

A growing body of work links temperature variation to health outcomes. Barreca et al. (2016) show that the temperature-mortality relationship in the U.S. has declined substantially over the twentieth century, pointing to significant adaptation in developed economies. Nevertheless, the potential for major health shocks remains. The IPCC (2022, Chapter 7) outlines the anticipated health risks under various warming scenarios, concluding that climate-related health burdens are likely to intensify even in high-income countries.

1.1.4 Gap in the Literature

While prior research has examined the contemporaneous effects of temperature on economic output, and others have quantified the cost of poor health on lifetime economic outcomes, little is known about how temperature-induced health variation translates into long-run income losses at the individual level. Moreover, existing work often abstracts from or aggregates over individual responses.

This Master Thesis addresses this gap by quantifying the lifetime economic cost of temperature-driven health deterioration in the context of the USA, under current levels of adaptation and within a structural life-cycle framework of individual decision-making.

1.2 Research question and strategy

The economic consequences of temperature-induced health variation reflect a fundamental tradeoff embedded in individual decision-making. Higher temperatures can deteriorate health outcomes, impairing both physical capacity and cognitive functioning, which in turn depresses labor productivity and expected longevity. These changes can have long-lasting effects on income profiles, particularly when health deteriorates early in life.

Yet even in the absence of institutional responses or targeted health investments, individuals may adjust their economic behavior in response to deteriorating health. A worsening health trajectory may lead agents to reoptimize by altering labor supply, savings, or consumption paths. These behavioral responses—though constrained—can partially absorb the economic shock. The key question is then whether such individual adjustments are sufficient to mitigate the lifetime income loss induced by temperature-related health shocks, or whether the long-run economic cost remains substantial despite endogenous reoptimization.

This tension motivates the central research question of this paper: How do temperature-induced health shocks affect individuals' lifetime income, when only individual-level behavioral responses are allowed, and what are the economic mechanisms through which these effects propagate?

To address this question, the analysis proceeds in two stages. First, an empirical investigation quantifies the causal impact of temperature variation on individual health status. Using micro-level data, the analysis estimates how both short-term and sustained exposure to high temperatures affect health outcomes across demographic groups and age cohorts.

Second, these empirical estimates are embedded into a structural, life-cycle model of individual behavior. In the model, health enters as a state variable that evolves over time and affects both survival and future health probability distribution. Individuals maximize lifetime utility by choosing labor supply and consumption paths, taking health dynamics as exogenous but responsive to temperature. Crucially, the model does not incorporate health investment or collective adaptation, isolating the role of individual optimization.

By simulating the model under alternative temperature scenarios, the analysis computes the long-run income losses attributable to temperature-induced health shocks. These results yield a quantitative assessment of the intertemporal economic cost of climate-related health deterioration, under a benchmark of minimal adaptation.

2 Setting

This section is dedicated to the presentation of the general setting in which individuals live in this model. First, the relationships between the different elements in a general framework will be presented. Then, the data used will be presented and discussed. The methods used to estimate the functional forms of the relationships will then be presented, and will be refered as the specific case of this work. Finally, results of the estimation process of these relationships will be presented and discussed.

2.1 Formal Description

At each period, an exogenous weather realization occurs, and individuals draw a health and living status.

2.1.1 History vectors

In a general framework, we can think of the health of an individual at time t as a vector $\mathcal{H}_t \in \mathbb{R}^t$ containing all the health status of the individual throughout their life. Similarily, we can think of the weather experienced by an individual at time t as a vector $\mathcal{W}_t \in \mathbb{R}^t$ containing all the weather conditions experienced by the same individual throughout their life.

2.1.2 Health Status

Let $H_t \in \Omega(H)$ be a random variable denoting the health status of an individual at time t. The functional form of its distribution f_h will depend on its sample space $\Omega(H)$. The past health history and the temperature also affect the probability distribution of health status. Generically, we can therefore write:

$$H_t \sim f_h(\mathcal{H}_{t-1}, \mathcal{W}_t) \tag{1}$$

Also, it follows that:

$$\mathcal{H}_t = (H_1, ..., H_t) \tag{2}$$

2.1.3 Living Status

Let $L_t \in \{0, 1\}$ be a random binary variable denoting the living status of an individual at time t. It is determined by a Bernoulli distribution with parameter p_t such that:

$$L_t \sim \mathcal{B}(p_t) \tag{3}$$

The probability parameter p_t depends on their health, age, and temperature. In a general approach, we can rewrite the first equation such as:

$$L_t \sim \mathcal{B}(p_t(\mathcal{H}_t, \mathcal{W}_t)) \tag{4}$$

2.2 Data

Three main datasets were used to estimate these relationships: the Health and Retirement Study (HRS) dataset for health information and survival, the Berkeley Earth dataset for annual temperature, and the Federal Reserve Bank of Saint Louis (FRED) dataset for other economic variables.

2.2.1 HRS Data

The HRS has a main survey performed every two years on a panel of individuals in the United States of America (USA). An exit survey occurs in parallel, that targets individuals identified as dead, in which questions are asked to relatives. The exit survey was used to identify dead individual, for which the living status was noted as 0 at the year of the survey. Dead individuals were kept in the final analyzed dataset if they had been observed in the immediate previous survey. To avoid mortality effects of the COVID-19, the latest year selected for this study was 2018. Due to different encoding, the 2000 and prior surveys were not taken into account.

Four variables were used in this dataset: Year of the survey, age of individuals, living status, and health status.

The age of individuals was determined based on the year of birth question, that was substracted to the year of the survey. Living status was encoded as 1 if an individuals was present in the main survey, and 0 if they were in the exit survey. The health status of dead individuals was encoded as the same as in the previous survey. The Health Proxy defined above was constitued by adding four variables, from the main survey, and obtained by asking individuals if they have had blood pressure, lung disease, hearth condition, and stroke accidents. The health status was identified through the self-reported health. This measure is less precise than alternative composites, based on weighted average of specific questions of health, but perform well enough and is more tractable.

In the HRS data, health status can take eight possible values:

- 1: Excellent
- 2: Very Good
- 3: Good
- 4: Fair
- 5: Poor
- 8, -8, or 9: Non Available, Not answered, or refused to answer[1]

[1] This last category was removed from the final analyzed dataset.

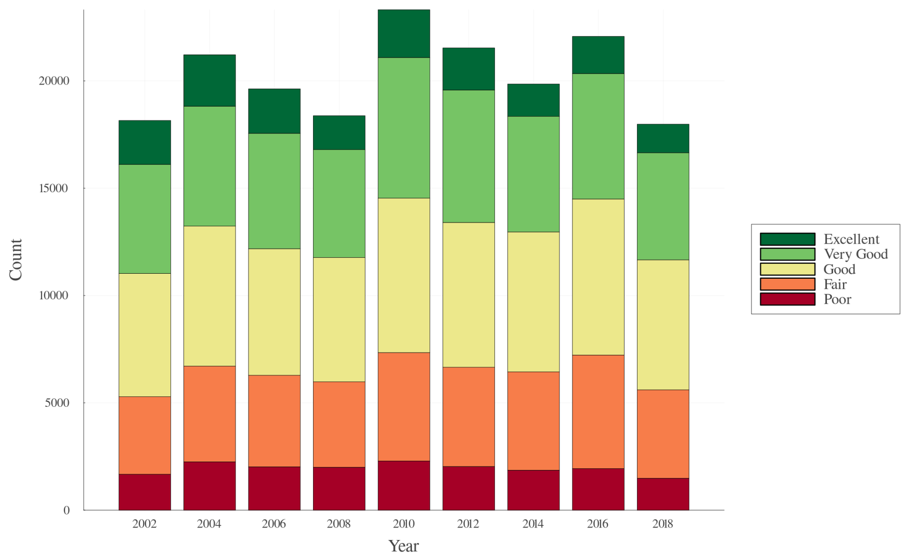


Figure 1: Health Status distribution per Year, from the HRS data

2.2.2 Climate Data

The climate data of Berkeley Earth was used to collect information on average annual or pluriannual global temperature on land. The name of the chosen dataset was the “Land Monthly Average Temperature”. For each month, from 1880 to 2022, two anomaly extrema values are given. They correspond to the 95% confidence interval for the average temperature. The average temperature is computed as a deviation from the average annual temperature computed between Januray 1951 and December 1980.

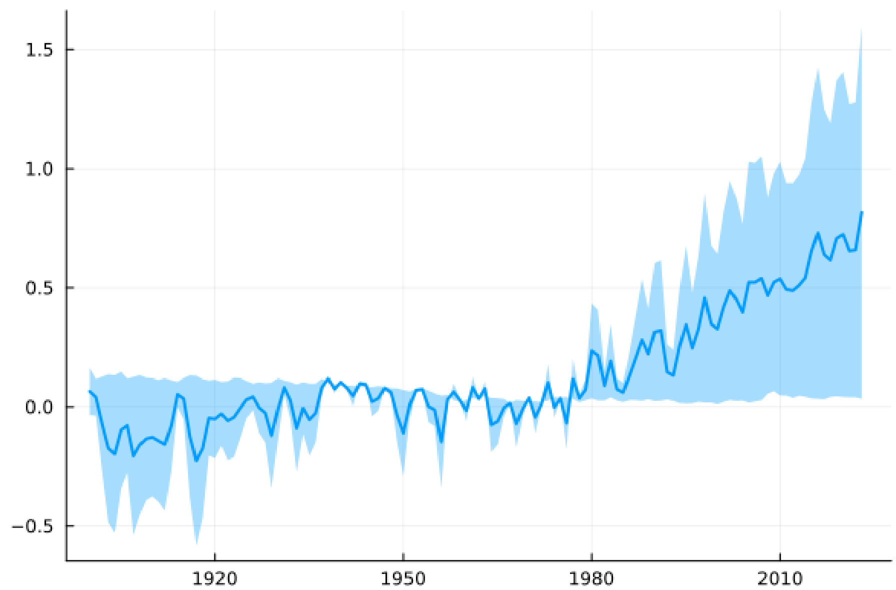


Figure 2: Average annual temperature from the Berkeley Dataset
The light blue area is delimited by the anomalies extrema of each year, and the dark blue line represents the average of the anomalies.

2.2.3 Economic Data

Finally, the FRED data was used to get two economic variables. First, the annual Gross Domestic Product (GDP) of the USA. Second, the interest rate, whose average is then used to calibrate the economic model.

3 Estimation Methods

3.1 Health Transition

The HRS dataset provides rich information regarding the health status of individuals. To make use of the five different values of the self reported health, it was therefore decided not to recode the variables into a binary health status variable $H \in \{Bad, Good\}$.

Several methods were considered to estimate the relationship between current health and past health and other covariates. Given that health is here an discrete, ordinal variable, ordered response models were chosen. More specifically, the ordered logit and ordered probit models were selected, due to their simplicity and broad usage (**Wooldridge˙2010**).

For tractability and computational reasons, it was chosen not to overload the function with history parameters, and just to focus on recent health, and on current temperature. Therefore, $f_h(\mathcal{H}_{t-1}, \mathcal{W}_t)$ is considered as $f_h(H_{t-1}, T_t)$ from now onwards.

Since $\Omega(H_t) = [\![1, 5]\!]$, we can consider f_h as a categorical distribution function[2]. It can therefore be written as:

$$f_h(H_{t-1} = j, T_t) = f_{h,j}(T_t),\ \forall j \in [\![1, 5]\!] \tag{5}$$

To estimate $f_{h,j}(T_t)$, i.e. the probabilities to go to another health state given that $H_{t-1} = j$, a first naive approach would consist in running an ordinal logistic regression with H_t as the dependent variable, and the with covariates including the age, temperature, and control variables.

There are two main issues with this regression. First, there is a large amount of omitted variables affecting the survival probability in this formulation. Second, when we try to include economic variables reflecting the progress in medicine, economic development, or other kinds of control to take into account possible omitted variables, we are faced with a colinearity issues with temperature.

Another important element to take into account to estimate $f_h(\cdot)$ are the interaction effects between the different covariates. For example, it seems plausible that age and previous health status interact: Being in "Fair" health should not have the same effect on the health transition probability for a 20 years old individual compared to a 80 years old individual.

To tackle these problems, it is possible to use a IV-based approach to try to isolate the effect of temperature on health. To do so, let us define a Health Proxy ($HP_{i,t}$) as the sum of binary variables at time t for individual i indicating if they have health issues. I retained four possible health accident, that can all be related to exposure to high levels of ambient temperature, such that:

$$X_{i,t}^h = \begin{bmatrix} \text{High Blood Pressure}_{i,t} \\ \text{Lung Disease}_{i,t} \\ \text{Hearth Condition}_{i,t} \\ \text{Stroke}_{i,t} \end{bmatrix}$$

Formally, the Health Proxy can therefore be written as:

$$HP_{i,t} = \sum_{j \in X_{i,t}^h} j \tag{6}$$

We can thus first run the following linear regression, to estimate the marginal effect of temperature on health through the above mentioned health accidents:

$$\widehat{HP}_{i,t}^{I} = \widehat{\beta_0} + \widehat{\beta_A} \cdot Age_{i,t} + \widehat{\beta_T} \cdot Temperature_t \tag{7}$$

[2]Also called generalized Bernoulli distribution, that can be represented as a Markov transition matrix.

Once we have the estimate $\widehat{HP}^I_{i,t}$, we can then run a second regression, with the transition probability as the dependent variables, and with $H_{i,t-1}$ and $\widehat{HP}^I_{i,t}$ as covariates.

3.2 Living Status

The living status variable, being binary, a simple logistic regression was possible.

To estimate $p_t(\mathcal{H}_t, \mathcal{W}_t)$, the first naive approach would have led to the same problems as previously seen. For example, running the non-IV regression with X containing only GDP, we find a negative coefficient on temperature, but also with GDP. This is due to the recent tendency in the USA in which the life expectancy stays stable or decrease slightly, while the GDP continues to raise importantly.

With $\Lambda(\cdot)$ being the logistic function, we could therefore run the following logistic regression to estimate the probability parameter $p_{i,t}$, i.e. the survival probability of i at t, with the previous estimate $\widehat{HP}^I_{i,t}$:

$$\widehat{p_{i,t}} = \Lambda\left(\widehat{\beta_0} + \widehat{\beta_H} \cdot Health_{i,t} + \widehat{\beta_{HP}} \cdot \widehat{HP}^I_{i,t}\right) \tag{8}$$

3.3 Estimation Results

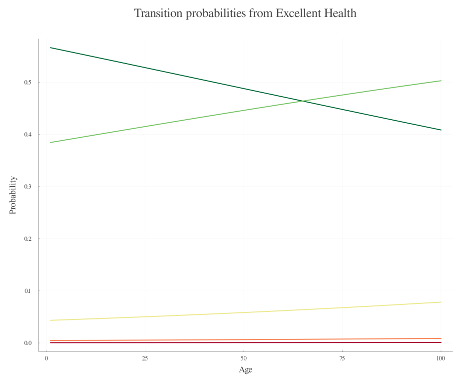
The first regression to explain the Health Proxy yields:

	HP
(Intercept)	-0.487***
	(0.068)
Age	0.020***
	(0.001)
Temperature	-0.032
	(0.123)
Age × Temperature	0.005**
	(0.002)
N	182,947
R^2	0.086

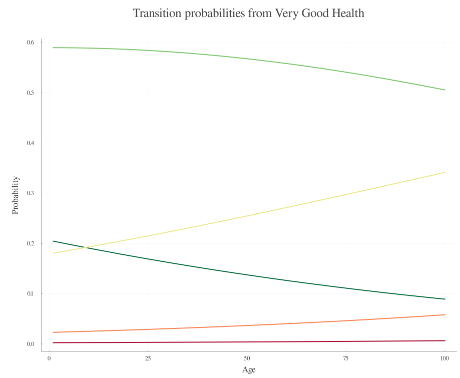
Table 1: Regression of Health Proxy on Age, Temperature, and their interaction.

3.3.1 Health transition

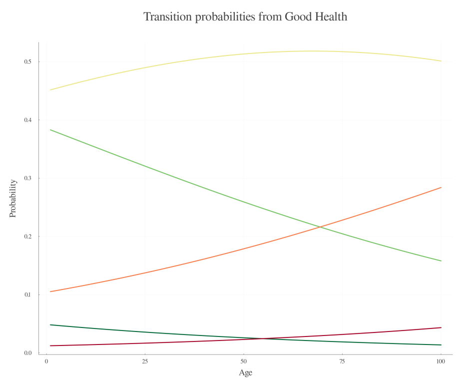
The regression to estimate the health transition probabilities yields.



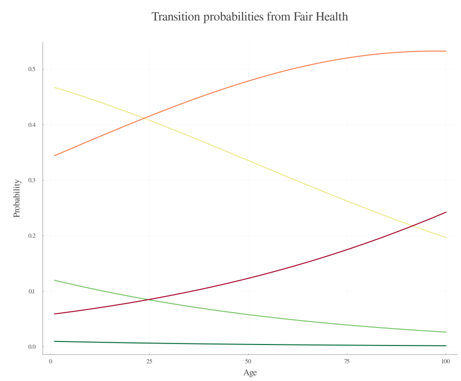
(a) Transition probabilities from Excellent Health



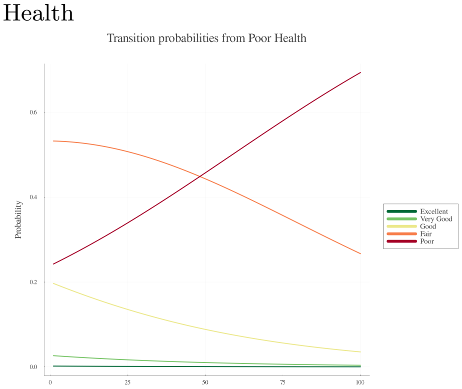
(b) Transition probabilities from Very Good Health



(c) Transition probabilities from Good Health



(d) Transition probabilities from Fair Health



(e) Transition probabilities from Poor Health

Figure 3: Transition probabilities

From these estimates, it is possible to simulate the collective health trajectory of a population.

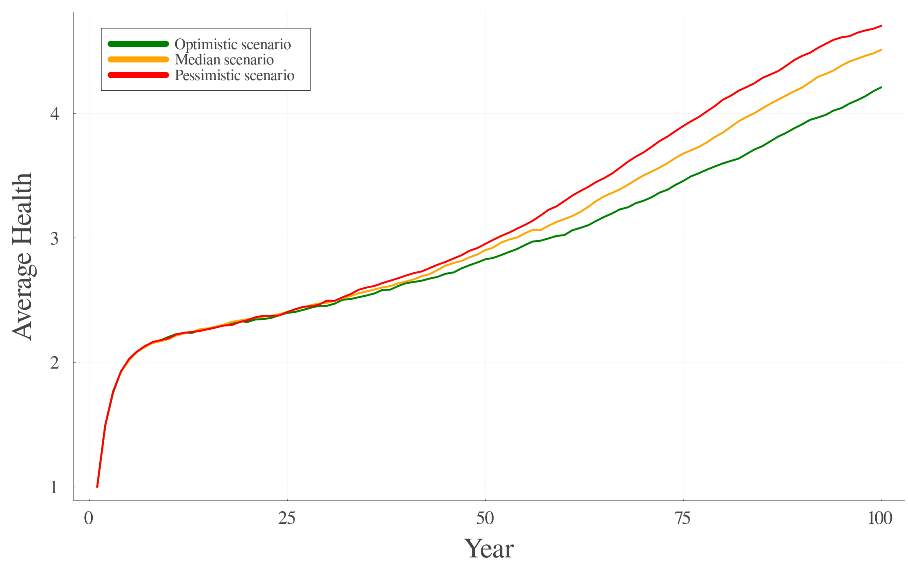


Figure 4: Annual average health status as a function of temperature scenarios.

3.3.2 Survival

When taking into account the health transition estimates and the survival probability estimates, it is then possible to visualize the demographic evolution of a population through time.

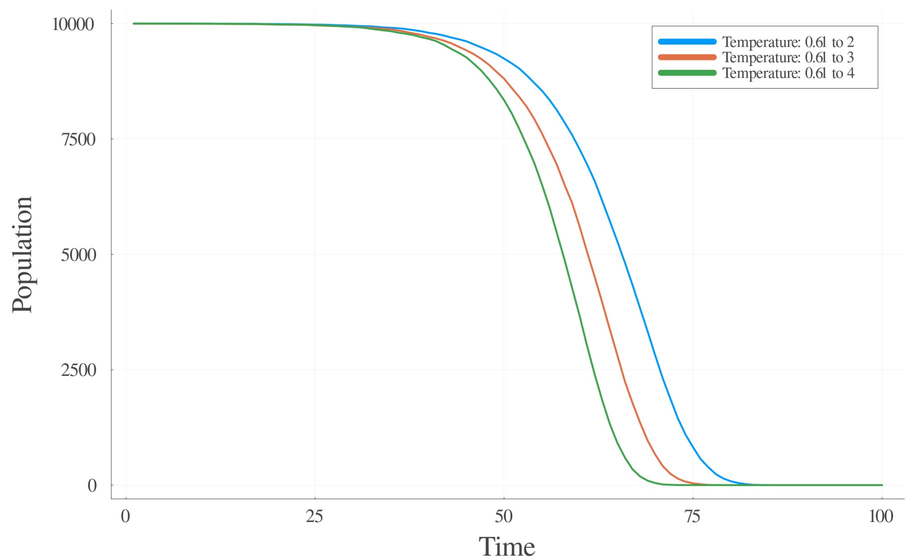


Figure 5: Annual probability of survival as a function of age and health status, obtained with N_0 =10 000

Temperature	Life expectancy
4.0	48.0
2.0	59.86
1.5	63.95
0.61	73.26
$0.5-4$	57.33
$0.5-2$	64.83
$0.0-1.5$	69.29

Table 2: Life expectancy simulations with fixed and variable temperatures.

We see that the effect of temperature on life expectancy is negative. This is due to the double effect of temperature on health transition and on survival probability.

DISCUSSION

These estimates are now going to be used in the economic model.

4 Model

This section is dedicated to the formal description of the model, as well as its analytical analysis. First, its main mechanisms will be explained, and then, the inexistence of analytical solution in most cases will be shown.

4.1 Baseline specification

The agents maximizes:

$$\max_{\{c_t, l_t, s_{t+1}\}_{t=1}^{100}} \mathbb{E}\left[\sum_{t=1}^{100} \beta^t \cdot u(c_t, l_t)\right]$$

Their utility function is:

$$u(c_t, l_t) = \frac{c_t^{1-\rho}}{1-\rho} - \xi_t \cdot \frac{l_t^{1+\varphi}}{1+\varphi}$$

With :
- c_t the consumption
- l_t the quantity of labor supply provided by the agent
- h_t the health status
- w_t the weather variable, which is here temperature
- ξ_t the labor disutility coefficient
- ρ the risk aversion coefficient
- φ the Frisch elasticity

The agent is subject to the following budget constraint:

$$c_t + s_{t+1} \leq l_t \cdot z_t + s_t \cdot (1 + r_t)$$

With:
- c_t the consumption at period t
- s_{t+1} the savings for period $t+1$
- l_t the labor supply provided by the agent at period t
- z_t the productivity at time t
- s_t the savings available at the beginning of period t
- r_t the interest rate at period t

Also, agents are subject to the following borrowing constraint, defined as:

$$s_{t+1} \geq \underline{s}, \forall t \in [\![1, T]\!]$$

We can note the First Order Conditions, such that:

$$c_t^{-\rho} \cdot z_t = \xi_t \cdot l_t^{\varphi} \iff \begin{cases} c_t = \left[\frac{\xi_t \cdot l_t^{\varphi}}{z_t}\right]^{-\frac{1}{\rho}} \\ l_t = \left[\frac{c_t^{-\rho} z_t \cdot}{\xi_t}\right]^{\frac{1}{\rho}} \end{cases} \tag{9}$$

And

$$c_t^{-\rho} = \beta \cdot \mathbb{E}\left[c_{t+1}^{-\rho} \cdot (1 + r_{t+1})\right] + \gamma_t \tag{10}$$

The first corresponds to the equalization of marginal benefit and cost of labor, and the second corresponds to the Euler equation.

The first equilbrium condition implies an within decision, driven by the labor disutility coefficient ξ and the productivity z. There is a unique mapping between consumption and labor at any period, to equalize the benefits and the costs of labor.

The second equilibrium condition implies an intertemporal decision. The marginal utility of consumption at one period must be equal to the expected marginal utility of conusmption next period, discounted by the discounting factor β and the interest rate next period $(1 + r_{t+1})$, plus the marginal benefit of violating the borrowing constraint at the current period.

It is now important to describe what the Expectation operator $\mathbb{E}$ entails here. In a generic formulation, one could expect the uncertainty to affect the interest rate at the next period, which is the reason $(1 + r_{t+1})$ is within the operator.

Another specification could exclude any uncertainty from the interest rate. The uncertainty could then come from the health and survival draw. If the uncertainty only comes from these two draws, the expectation operator can be formalized such as:

$$\mathbb{E}\left[c_{t+1}\right] \equiv p_{t+1}(\mathcal{H}_t, \mathcal{W}_t) \cdot c_{t+1}$$

In this specification, the only uncertainty is whether the agent will be alive or not in next period. The baseline model will first focus on this simplified assumption. Variations will be introduced later.

4.2 Analytical Solution Inexistence

Proposition This maximization program is impossible to solve analytically in most cases[3].

If we consider the model altogether, it is impossible to describe analytically the optimal policy functions of the three choice variables. While the entire proof is available in the appendix, a quick explanation is possible here. First, the objective functions is linear with the savings at next period s_{t+1}, making it disappear from the F.O.C.s. This term requires therefore the labor and consumption policies to be solved, and then plugged into the budget constraint, to have a solution. However, if we try to solve the two other policy functions, we end up with transcendantal equations of the form $a \cdot x^{\alpha} + b \cdot x + c = 0$, with $\alpha \notin \mathbb{N}$. This transcendantal equations can be overcome with specific combinations of parameters, but these are however absurd in our context. This inexistence of analytical solution calls henceforth for a numerical solving of the model. The next section discusses the different methods used in order to do so.

[3]The proof of this proposition is in the Appendix.

5 Numerical methods

Several ways have been considered to solve this model numerically. This section is dedicated to the presentation of the different methods used in order to do so. First, the auxiliary functions are presented. Second, the different main algorithms specifications and their performance are presented. Finally, the aggregation methods and different numerical results are discussed.

5.1 Functions

This subsection is dedicated to the description of the fundamental programmatic functions that were used to solve the model numerically.

- **Budget clearing function:** The budget clearing function computes the amount of non used income for a set of state and choice variables. Since at optimal, the budget constraint is binding, the underlying theoreical result indicates that the budget clearing function should be zero. Given the imprecision of numerical methods, the average budget clearing function was used as a measure of the precision performance of each algorithm.

  It is equal to:

  $$B(s_t, l_t, c_t, s_{t+1}) = l_t \cdot z_t + s_t \cdot (1 + r_t) - s_{t+1} - c_t$$

- **Bellman function:** The Bellman function takes as an argument the value function next period, and maximizes the current utility plus the discounted value function next period.

  It is equal to:

  $$V(s_t) = \max_{\{c_t, l_t, s_{t+1}\} \in \Gamma(s_t)} \{u(c_t, l_t) + \beta \cdot V(s_{t+1})\}$$

  With $\Gamma(s_t)$ the feasibility set given by the state variable s_t.

- **Backwards function:** The backwards function iterates the Bellman function from the last period to the first one. In the case of policy iteration, it iterates over the policy function, and not the value function. It aggregates the optimal decisions and returns a grid of optimal choices associated to each period and state variable value.

  For the pure numerical value function iteration, the backwards function is as following:

```
for t from 100 to 1

    if t is equal to 100
        Bellman next period = Vector of zeros
    end

    for s in the possible set of s
        for c,l,s' in the feasible set of the current s value

            # We compute the budget clearing:

            bc = budget_clearing(c,l,s')

            # If it does not,
            # we set the value function to a very low number.

            if bc < 0

                V[c,l,s',s] = -Inf

            # If it does, we compute the value function
            # for this combination of choice variables.

            else if bc >= 0

                V[c,l,s',s] =
                     utility(c,l,s') +
                     beta * probability of survival *
                     Bellman next period[s']
            end
        end

        # We set the value function for s and t to
        # the maximum value found.

    end

# We set the value function at next period to current one
Bellman_next_period = Value_function[index_s,t]

end
```

Figure 6: Pseudo-code of the backward function.

5.2 Algorithms

This section details the different algorithms built upon the above-mentioned fundamental functions. Indeed, if one can think of the pure numerical value function iteration to solve the model, multiple approaches exist, that vary depending on the targeted tradeoff between precision performance and speed performance[4].

5.2.1 Pure numerical value function iteration

The pure numerical value function iteration algorithm consists in verifying all possible combination of choice variables for each level of state variable to determine what is the best possible response given a certain amount of state variable.

Here, the algorithm goes through all the possible values of c, l, and s', without using any approximation obtained through the FOC mentioned above. This is quite computational-intensive, but has the advantage of not using analytical results, which can lead to approximation depending on the resolution of the ranges used.

5.2.2 F.O.C. approximated value function iteration

The FOC approximated value function iteration algorithms make use of the two expression of consumption and labor supply derived from the FOC seen in the previous section. They are faster by order of magnitudes when compared to the pure numerical value function iteration algorithm, but contain more errors, measured by the budget clearing function[5].

5.2.3 Interpolated algorithms

The interpolated algorithms use interpolation techniques to approximate the value of the next period Bellman equation. This interpolation can be implemented in the pure numerical algorithm, and in the FOC-approximated ones.

They allow for a smoother shape of policy function, and have graphical results that are more easily interpretable. However, their speed performance is slightly worse, and the effect of interpolation on precision performance is ambiguous.

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[4]For more information, the different steps of the algorithms and their source code are available online. The steps and comments of the present work are available here: `https://www.paulogcd.com/Master_Thesis/`, and the documented replication package, coded in Julia, is available here: `https://www.paulogcd.com/Master_Thesis_Paulogcd_2025/`.

[5]Note that it is impossible to use the second FOC, i.e. the Euler equation, containing the Lagrangien multiplier γ_t. However, the numerical solving process allows for an estimation of γ_t.

5.3 Performance

Algorithm	Error	Time (in seconds)	Memory (in Mb)
Pure Numerical Value Function Iteration	0.0179	0.6604	1033.4409
FOC approximation 1 (fixing labor supply)	0.042	0.2506	98.1834
FOC approximation 2 (fixing consumption)	0.042	0.0188	55.6981

Figure 7: Algorithms and their performance

5.4 Policy Function Results

- Consumption policy
  - Labor supply policy
  - Savings policy

6 Results

6.1 Lifetime income

7 Discussion

8 References

9 Appendix

9.1 Proof of Impossibility

This section is dedicated to the proof that the maximization program has no analytical solution in most cases.

We will show this absence of analytical solution by attempting to solve it in three different ways: First by using the Budget Constraint binding, then by using the F.O.C. and the Budget Constraint, and lastly trying to go to the last period to solve it recursively.

9.1.1 Maximization program

$$\max_{\{c_t, l_t, s_{t+1}\}_{t=1}^{100}} \mathbb{E}\left[\sum_{t=1}^{100} \beta^t \cdot \frac{c_t^{1-\rho}}{1-\rho} - \xi_t \cdot \frac{l_t^{1+\varphi}}{1+\varphi}\right]$$

Subject to budget and borrowing constraints:

$$c_t + s_{t+1} \leq l_t \cdot z_t + s_t \cdot (1 + r_t)$$

$$s_{t+1} \geq \underline{s}, \forall t \in [\![1, 100]\!]$$

9.1.2 Budget constraint binding

A first solving attempt consists in assuming that the budget constraint binds. We can then obtain the following expression for consumption:

$$c_t = l_t \cdot z_t + s_t \cdot (1 + r_t) - s_{t+1}$$

Plugging it into the maximization program, we obtain:

$$\max_{\{l_t, s_{t+1}\}_{t=1}^{100}} \mathbb{E}\left[\sum_{t=1}^{100} \beta^t \cdot \frac{(l_t \cdot z_t + s_t \cdot (1 + r_t) - s_{t+1})^{1-\rho}}{1-\rho} - \xi_t \cdot \frac{l_t^{1+\varphi}}{1+\varphi}\right]$$

The F.O.C. with respect to labor implies:

$$l_t^{\varphi} \cdot \xi_t = [l_t \cdot z_t + s_t \cdot (1 + r_t) - s_{t+1}]^{-\rho} \cdot z_t \tag{11}$$

We can develop the decomposition of consumption if and only if $\rho \in \mathbb{N}$. Indeed, this equation is of form $x = (x - \alpha)^{\beta} \cdot z$. With $\beta \notin \mathbb{N}$, is a transcendantal equation.

9.1.3 F.O.C. and Budget clearing

We can now try to compute the F.O.C. first, and then make use of the Budget Constraint. The Lagrangien function associated witht the maximization program of the agent is:

$$\begin{aligned}\mathcal{L}(c_t, l_t, s_{t+1}; \lambda_t, \gamma_t) = \mathbb{E}\Big[\sum_{t=1}^{100} \beta^t \cdot ((\frac{c_t^{1-\rho}}{1-\rho} - \xi_t \cdot \frac{l_t^{1+\varphi}}{1+\varphi}) \\ + \lambda_t \cdot (l_t \cdot z_t + s_t \cdot (1+r_t) - c_t - s_{t+1}) \\ + \gamma_t \cdot (s_{t+1} - \underline{s}))\Big]\end{aligned} \tag{12}$$

The First Order Conditions are the following:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \iff c_t^{-\rho} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = 0 \iff \lambda_t \cdot z_t = \xi_t \cdot l_t^{\varphi}$$

$$\frac{\partial \mathcal{L}}{\partial s_{t+1}} = 0 \iff \lambda_t = \beta \cdot \mathbb{E}\left[\lambda_{t+1} \cdot (1 + r_{t+1})\right] + \gamma_t$$

We first note that we must obtain a closed-form solution for c_t and l_t to obtain the optimal value of s_{t+1}. Indeed, since s_{t+1} is linear in $\mathcal{L}$, we would need to plug the closed-form solutions of c_t and l_t in the budget constraint.

Replacing the expression of λ_t in the two other equation yields:

$$c_t^{-\rho} \cdot z_t = \xi_t \cdot l_t^{\varphi} \iff \begin{cases} c_t = \left[\frac{\xi_t \cdot l_t^{\varphi}}{z_t}\right]^{-\frac{1}{\rho}} \\ l_t = \left[\frac{c_t^{-\rho} z_t \cdot}{\xi_t}\right]^{\frac{1}{\rho}} \end{cases} \tag{13}$$

And

$$c_t^{-\rho} = \beta \cdot \mathbb{E}\left[c_{t+1}^{-\rho} \cdot (1 + r_{t+1})\right] + \gamma_t \tag{14}$$

Assuming that the budget constraint binds, it becomes, as previously seen:

$$c_t + s_{t+1} = l_t \cdot z_t + s_t \cdot (1 + r_t) \iff c_t = l_t \cdot z_t + s_t \cdot (1 + r_t) - s_{t+1}$$

This leads to the following equation system:

$$\begin{cases} c_t = \left[\frac{\xi_t \cdot l_t^{\varphi}}{z_t}\right]^{-\frac{1}{\rho}} \\ c_t = l_t \cdot z_t + s_t \cdot (1 + r_t) - s_{t+1} \end{cases}$$

$$\iff$$

$$\left[\frac{\xi_t \cdot l_t^{\varphi}}{z_t}\right]^{-\frac{1}{\rho}} = l_t \cdot z_t + s_t \cdot (1 + r_t) - s_{t+1}$$

$$\Longleftrightarrow$$

$$l_t^{-\frac{\varphi}{\rho}} \cdot \left(\frac{\xi_t}{z_t}\right)^{-\frac{1}{\rho}} = l_t \cdot z_t + s_t \cdot (1 + r_t) - s_{t+1}$$

$$\Longleftrightarrow$$

$$l_t^{-\frac{\varphi}{\rho}} \cdot \left(\frac{z_t}{\xi_t}\right)^{\frac{1}{\rho}} - l_t \cdot z_t - s_t \cdot (1 + r_t) - s_{t+1} = 0$$

This is a transcendantal equation of form $x^{\alpha} \cdot b - x \cdot y - c = 0$, which admits a solution if and only if $-\frac{\varphi}{\rho} \in \mathbb{N}$. This condition seems unrealistic in our context:

- $-\varphi > 0$ implies that labor has a decreasing disutility, which makes the maximization program absurd.
- $-\rho > 0$ implies a risk-loving agent, which changes drastically the framework of our model, and would require another whole interpretation.

Note that if we set $-\rho \in \mathbb{N}$ and further develop the last equation in the budget constraint binding attempt, we end up with the same condition.

9.1.4 Backwards solving attempt

If we try to solve it backwards, we now go to the last period. At the last period, $s_{t+1} = \underline{s}$ for sure: Since there is no future, the agent will borrow as much as they can, or will at least not save anything more than what is imposed by the constraint.

For simplification, let $\underline{s}$ be fixed such that: $\underline{s} = 0$. The new optimality condition is:

$$l_t^{\varphi} \cdot \xi_t = [l_t \cdot z_t + s_t \cdot (1 + r_t)]^{-\rho} \cdot z_t \tag{15}$$

Although we simplified the term at the exponential of which we have $-\rho$, this is still a transcendantal equation due to the sum of labor income and income coming from savings of last period, and the problem remain the same.