Microeconomics Project

Chambon Lionel
Gugelmo Cavalheiro Dias Paulo
Pavlovich Karina
Soufflet Morgane

May 1, 2024

Abstract

In the following paper our aim is to use game theory to determine the optimal tax the environmentally discerning government should impose on a representative firm in order to ensure the firm invests in Research and Development and upgrades its' production technology to thus minimise the pollution per unit of product produced.

We start by building a classic Stackelberg game to find the optimal tax, we then proceed to building a cheap talk game with imperfect information under the assumption that the R&D investment might not lead to successful drop in pollution per unit of product produced to investigate the optimal strategy profile for the firm and for the government.

Contents

1 Introduction

In this work, we apply game theory framework to model an efficient taxation policy the government could implement, taking into account a strategic incentive for a firm to falsely declare the pollution per unit of good produced. We also describe the changes in the optimal taxation policy in case of persistent pollution and/or fruitless R&D.

Game theory provides a beneficial framework for contemporary researchers to analyse strategic interactions between the economic agents and to effectively trace the consequences of their decisions on each other, as well as on the environment.

The general consensus is that the emissions tax is one of the most effective policies the government can impose to increase the environmental quality, the optimal amount of the tax being determined by the marginal environmental damage of the production firm's market share (Dorothée Brécard, 2010). In case of highly polluting enterprises, Wei Yu, Ruizhu Han show that particularly heavy taxation should be implemented (Wei Yu, Ruizhu Han, 2019).

Yenipazarli (2019) further examines the effectiveness of the emissions tax within a duopolistic competition framework, taking into account heterogeneous consumer preferences for greener goods.

Drake *et al.* (2018) introduce asymmetric regulations across different markets in their models to analyze the presence of so-called carbon leakages, which occur when firms relocate to markets with less strict emissions regulations instead of adapting production in their domestic market.

Studies demonstrate that the transition to green production can be challenging for firms, since it requires a major capital investment in R&D, making final goods more costly for the consumer. Hence the firms' decision to opt for green technologies largely depends on consumers' willingness to increase their expenses. (Conrad, 2005, Krass 2013)

The government subsidies may stimulate the firms to switch to greener production and increase the market share of green products, making the green goods cheaper to produce which would result in a substantial decrease in the environmental pollution.

That being said, Ling *et al.* (2022) demonstrate that higher subsidies might not result in a successful decrease in environmental pollution, as the effectiveness of the subsidy and the amount of the subsidy largely depend on the manufacturing investment, R&D investment, and consumers' preferences.

As demonstrated so far, there is substantial body of academic research that aims to investigate how to reduce environmental pollution by focusing on unsustainable production. Based on our research, we were able to identify two factors that require further analysis.

First, it is unlikely that the government can accurately monitor the emissions on a firm-level, thus it might be difficult to differentiate the individual-firm's emissions from aggregate levels. As a consequence, the governments might find themselves making policy decisions based on the low-quality data shared with them by the firms with outstanding credibility and spotless reputation.

Several instances happened in the past when firms took advantage and falsely declared their emitted pollution levels, one notable example of such a case is the Volkswagen scandal of 2013 (Link!).

Second, many papers assume that an optimal level of R&D investment *perfectly* offsets or reduces production emissions. However, it is also possible that efforts to adapt production technologies might fail or do not reduce emissions. In that case, firms might achieve a sub-optimal outcome as they are forced to absorb the sunk R&D costs.

In this paper we analyse the effects of informational asymmetries and frictions on the optimal tax rate based on the classic Stackelberg game framework and the "cheap talk" game.

2 Baseline Model

We model the economy using the following assumptions:

- There exist two agents in the economy, namely the government and the producer (the firm), consumers are not included.
- The firms are homogeneous and therefore representable.

First, we model the interactions between the government and the firm using the Stackelberg game structure (a game with strategic interactions that implies perfect information, non-simultaneous actions and two players, namely the leader and the follower). The pollution in the past is not taken into account in the simplified model, but will be included in the extension.

2.1 Players

We assign the role of the leader to the firm and the role of a follower to the government. The firm "plays" first, by maximising its' profit. The government observes the action of the firm and then makes a move by setting an optimal tax depending on the firm's profit.

2.2 Firm

2.2.1 Technology of the firm

The firm is a monopoly that produces a single good. It's production function takes two inputs:

- R&D investment (r)
- manufacturing investment (m)

The main goal of the R&D investment is to diminish the pollution produced by the firm per unit of product produced, while the manufacturing investment is used by the firm to produce a quantity q of its' single good.

The pollution g_t generated by producing one unit of good at time t is defined as:

$$g_t = \frac{1}{r_t} \tag{1}$$

Where r_t is the R&D investment done by the firm to reduce its unit pollution level g_t.

The manufacturing investment m_t needed to produce a quantity q_t of a good is defined as:

$$m_t = q_t \cdot c_t \tag{2}$$

Where q_t is the quantity of good produced by the firm at time t and c_t is the cost of manufacturing one good by the firm at time t

Lastly, the firm is subject to a tax τ based on the pollution level emitted per unit produced.

2.2.2 Profit of the firm

Given those technology constraints, the firm maximises its profit:

$$\begin{aligned} &\max_{r_t, m_t} \pi_{F,t} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t \\ &\text{subject to : } B_t \leq r_t + m_t \end{aligned} \tag{3}$$

Where :

- q_t is the quantity of good produced and sold at time t
- p_t is the price at which one unit of good is sold
- c_t is the cost of production of a unit of good
- $\tau_t \in \mathbb{R}_+$ is the amount at which one unit of good is taxed, depending on the level of pollution produced per unit, at time t
- $g_t \in \mathbb{R}_+$ is the level of pollution emitted by producing a unit of product at time t
- B_t is the budget of the firm at time t

2.3 Government

The government maximises the social welfare, which we define as a utility function that takes the economic activities EAt and the risk of environmental disaster EQt as inputs:

$$u_{G,t} = \mathrm{EA}t^{\alpha} \cdot \mathrm{EQ}t^{1-\alpha} \tag{4}$$

Where :

- EA_t is the Economic Activity of society at time t
- EQ_t is the Environmental Quality of the world at time t

- $\alpha \in (0,1)$ is the relative importance of Economic Activity compared to the Environmental Quality of the World.

This function has two main properties :

1. Positive but decreasing returns for both Economic Activity and Environmental Quality[1].
2. Both Economic Activity and Environmental Quality are complementary goods, as the cross-derivative is positive [2].

2.4 Model solution

2.5 Firm's problem

In this simplified model, we first solve the Firm's problem for one period (our previous maximisation problem (3)) :

$$\max_{r_t, m_t} \pi_{F,t} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t$$
$$\text{subject to : } B_t \leq r_t + m_t$$

Hence the Lagrangian is:

$$\mathcal{L} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t + \lambda \left(B - r_t - m_t\right)$$

Expressing the q_t as a function of m_t, and the g_t as a function of r_t gives:

$$\mathcal{L} = \frac{m_t}{c_t} \cdot (p_t - c_t) - \tau_t \cdot \frac{m_t}{c_t} \cdot \frac{1}{r_t} + \lambda \left(B - r_t - m_t\right)$$

Applying the First Order Condition[3] gets the manufacturing investment:

$$m_t = \frac{1}{\tau_t} \left(r_t^2 (p_t - c_t) - r_t \cdot \tau_t \right) = \frac{r_t^2 \left(p_t - c_t\right)}{\tau_t} - r_t \tag{5}$$

This expression gives the optimal level of manufacturing investment for the firm. Since it has a negative relationship with the pollution tax τ_t, we can conclude that, the bigger the pollution tax, the more incentives the firm has to invest in R&D instead of simply manufacturing goods with high pollution level.

From that, we can identify the optimal level of research r for the firm[4] :

$$r_t = \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}} \tag{6}$$

Thus, the maximum profit of the firm is [5] :

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right)^2 \tag{7}$$

[1]See Appendix for proof.
[2]See Appendix for proof.
[3]See Appendix for proof.
[4]See Appendix for proof.
[5]See Appendix for proof.

2.6 Government problem

In this simplified model, we set the Economic Activity EA_t equal to the profit of the firm. The environment quality EQt at time t depends on the environmental pollution at time t. The expression for the EQt is the following:

$$EQ_t \equiv e^{-q_t \cdot g_t} \tag{8}$$

Where $q_t \cdot g_t$ is the amount of pollution produced at time t
Thus, the maximisation program of the government is:

$$\max_{\tau_t} u_{G,t} = \pi_t^{\alpha} \cdot e^{-q_t \cdot g_t \cdot (1-\alpha)}$$

Since the Stackelberg game implies perfect information, the government knows the allocation decisions of the firm. Therefore the government knows the values of r_t and m_t, and the maximisation program of the government is[5]:

$$u_{G,t} = \frac{e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha}}{c^{\alpha} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}}} \tag{9}$$

To maximise this utility function, we find first derivatives with respect to τ_t and set them equal to zero. Solving for τ_t gives the expression for the optimal pollution tax τ_t^* that maximises the objective function of the government [6] :

$$\tau_t^* = \frac{B_t\,(p_t - c_t)\,(1-\alpha)\left[\alpha \cdot 4 \cdot c_t + 1 - \alpha - \sqrt{(1-\alpha)^2 + \alpha \cdot 8 \cdot c_t(1-\alpha)}\right]}{\alpha^2 \cdot 8 \cdot c_t^2} \tag{10}$$

We can observe that the optimal pollution tax has the following properties:

- It has a negative relationship with the manufacturing cost c_t
- It has a negative relationship with the preference of the government for economic activity α
- It has a positive relationship with the budget of the firm cost B_t
- It has a positive relationship with the selling price p_t

This model allows us to say several things about the strategic interactions between the firm and the govenrment. First of all, we see that the tradeoff between economic activity and environmental quality is determinant in the choice of the government. The negative relationship between τ_t and c_t and the positive relationship between τ_t and B_t shows it. If the firm is faced with higher production costs, it is in the interest of the government to still to have highger costs of production

[5] See Appendix for proof.
[6] See Appendix for proof.

3 Extension

3.1 Motivation

So far, we have shown that under friction-less and perfect information conditions in a Stackelberg framework, there exists a government-chosen, optimal tax level that enforces a level of R&D spending which maximizes both firm profits and social welfare, by taking into account pollution and environmental risks.

Now we propose a simple "cheap talk" game between the Government and the representative Firm following Crawford and Sobel (1982).

In this extension we take into account the imperfect information about the success of the technological investment. We believe that this is a suitable construct as firms could have an incentive to be dishonest about their true level of emissions in order to tempt the government to enact a lower tax thus increasing the firm profits. We also consider the case where a firm might not under-report its emissions by too large of a margin to preserve a reasonable degree of credibility.

The following section will set up the game, present a solution, identify equilibrium conditions and discuss results.

3.2 Game Setup

3.2.1 Assumptions

As before, the players are given by a welfare-maximizing Government and a profit-maximizing Firm.

- *Nature* draws a technology shock ϵ uniformly from $U \sim [0, 2]$. Thus, positive and negative technology shocks are equally likely.
- Only the Firm observes the nature of its technology shock and its subsequent level of emissions.
- The Government knows the distribution of the technology shock and the lowest and highest possible level of emissions a firm can declare.
- Using the same procedure than in the previous section, the optimal tax τ as a function of the shock ϵ becomes:
- Similarly, the firm's optimal profit as a function of ϵ is:

3.2.2 Available Actions and Payoffs

- The firm sends a message about the nature of a technology shock $\epsilon \sim [0, 2]$. Its payoff function is:

$$U_F = -(a - \epsilon + \delta)^2 \tag{11}$$

  where

  - $a \in A$ is the action taken by the government afterwards to calculate the next tax τ
  - $\delta \in [0, \epsilon]$ represents a variable that captures misreporting by the firm regarding its true ϵ that a firm might choose to convince the government to choose a lower tax level in the next period.

- Based on its beliefs about the credibility of the firm, the government chooses a when deciding the new optimal tax, where government utility is given by

$$U_G = -(a - \epsilon)^2 \tag{12}$$

- Optimal choices for the firm and the government, respectively, are:

$$a_F = \epsilon - \delta$$

$$a_G = \epsilon$$

In this setup, the government wishes to perfectly anticipate the firm's true value of ϵ, while the firm wishes to deviate and underreport it by a factor of δ.

3.3 Solving the Game

Proposition 1. *There exists a Perfect Bayesian Equilibrium in the form of a "babbling" equilibrium in which the firm chooses a message regardless of the value of ϵ. Then, the firm's best repose is to set*

$$a(m) = \mathbb{E}[\epsilon] = 1$$

which leads to the equilibrium derived earlier.

This case on its own, however, is not particularly interesting. Therefore, we will seek to derive a two-message equilibrium to identify a "threshold of underreporting" given by δ. Suppose thus that the firm can choose a message such that $a(m_1) < a(m_2)$. That is, suppose the firm faces a level of ϵ and must now decide by *by how much* to underreport it, such that $\epsilon > m_2 > m_1$.

Plugging in the two messages into the firm's utility function yields the marginal benefit from choosing $a(m_2)$ over $a(m_1)$:

$$\Delta(\epsilon) = -(a(m_2) - \epsilon - \delta)^2 + (a(m_1) - \epsilon - \delta)^2 \tag{13}$$

This is increasing in ϵ.

Proposition 2. *There exists an ϵ such that the firm chooses m_1 if $\epsilon \in [0, \epsilon)$ and m_2 if $\epsilon \in (\epsilon^*, 2]$. Then, knowing the distribution of ϵ and by Bayes' rule :*

$$a(m_1) = \frac{\epsilon^*}{2}$$

$$a(m_2) = \frac{2 + \epsilon^*}{2}$$

These conditions now enable us to solve for ϵ^* by plugging both expressions into (1).

$$\epsilon^* - \delta - \frac{\epsilon}{2} = \frac{2 + \epsilon}{2} - \epsilon^* + \delta$$

$$\Leftrightarrow \frac{2\epsilon^* - 2}{2} = 2\delta$$

$$\epsilon^* = 2\delta + 1 \tag{14}$$

for $\delta \in [-\frac{1}{2}, 0]$ since $\epsilon \in [0, 1]$.

3.4 Discussion

In our setting, we have identified boundaries for our underreporting parameter δ. The previous result tells us that

- If the firm faces the worst possible technology shock $\epsilon = 1$, then it has no incentive to report untruthfully, i.e., $\delta = 0$.
- If the firm faces the best possible technology shock $\epsilon = 0$, it will choose $\delta = -\frac{1}{2}$.

4 Conclusion

We have shown that under frictionless and perfect information conditions in a Stackelberg framework, there exists a government-chosen, optimal tax level that enforces a level of R&D spending which maximizes both firm profits and social welfare, by taking into account pollution and environmental risks.

5 Appendix

5.1 Properties of the utility function of the government

5.1.1 Diminishing returns of economic activity and environmental quality

The Government utility function is :

$$u_{G,t} = \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}t^{1-\alpha} \tag{4}$$

For EAt the economic activity and EQt the environmental quality to yields positive but diminishing returns, the first derivatives must be positive and the second derivatives must be negatives.

We compute :

$$\frac{\partial u_{G,t}}{\partial \mathrm{EA}t} = \alpha \cdot \mathrm{EA}t^{\alpha-1} \cdot \mathrm{EQ}_t^{1-\alpha} > 0$$

$$\frac{\partial u_{G,t}}{\partial \mathrm{EQ}t} = (1-\alpha) \cdot \mathrm{EA}t^{\alpha} \cdot \mathrm{EQ}_t^{-\alpha} > 0$$

$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EA}t} = \alpha \cdot (\alpha - 1) \cdot \mathrm{EA}t^{\alpha-2} \cdot \mathrm{EQ}_t^{1-\alpha} < 0$$

$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EQ}t} = (1-\alpha) \cdot (-\alpha) \cdot \mathrm{EA}t^{\alpha-1} \cdot \mathrm{EQ}_t^{-\alpha-1} < 0$$

Thus, we indeed have positive but diminishing returns for both EAt and EQt.

5.1.2 Complementarity of economic activity and environmental quality

For EA_t the economic activity and EQ_t the environmental quality to be complements, the crosse-derivative has to be positive. We compute :

$$\frac{\partial^2 u_{G,t}}{\partial \mathrm{EA}t \partial \mathrm{EQ}t} = \alpha \cdot (1-\alpha) \cdot \mathrm{EA}t^{\alpha-1} \cdot \mathrm{EQ}t^{-\alpha} > 0$$

Thus, EA_t and EQ_t are complement. This means that in this model, the government does not prefer a world without economic activity or without some environmental quality, but it prefers a mixture of them.

5.2 First Order Conditions

Differentiating the Lagrangian with respect to the R&D investment r_t gives:

$$\frac{\partial \mathcal{L}}{\partial r_t} = 0$$

$$\Leftrightarrow \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2} - \lambda = 0$$

$$\Leftrightarrow \lambda = \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2}$$

Differentiating the Lagrangian with respect to the manufacturing investment m_t gives:

$$\frac{\partial \mathcal{L}}{\partial m_t} = 0$$
$$\Leftrightarrow \frac{p_t - c_t}{c_t} - \frac{\tau_t}{c_t \cdot r_t} - \lambda = 0$$
$$\Leftrightarrow \frac{p_t - c_t}{c_t} - \frac{\tau_t}{c_t \cdot r_t} - \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2} = 0$$
$$\Leftrightarrow r^2 t \cdot (pt - c_t) - r_t \cdot \tau_t - \tau_t \cdot m_t = 0$$

Hence the manufacturing investment is:

$$m_t = \frac{1}{\tau_t}\left(r_t^2(p_t - c_t) - r_t \cdot \tau_t\right) = \frac{r_t^2\,(p_t - c_t)}{\tau_t} - r_t \tag{5}$$

5.3 Optimal Level of Research

Starting with the Budget constraint of the firm:

$$B_t = r_t + m_t$$

Plugging into the constraint the expression for m_t we had found earlier gives:

$$B_t = r_t + \frac{r_t^2\,(p_t - c_t)}{\tau_t} - r_t$$
$$\Leftrightarrow B_t = \frac{r_t^2\,(p_t - c_t)}{\tau_t}$$

Rearranging the expression with respect to the research level r_t gives:

$$r_t = \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}} \tag{6}$$

5.4 Maximum Profit

The First Order Condition is:

$$\pi_t = \frac{m_t}{c_t} \cdot (p_t - c_t) - \tau_t \cdot \frac{m_t}{c_t} \cdot \frac{1}{r_t}$$

Factorising by $\frac{m_t}{c_t}$ gives:

$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \frac{\tau_t}{r_t}\right)$$

We now plug in the expression for r_t:

$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \frac{\tau_t}{\sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}}}\right)$$

Which is equivalent to:

$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \sqrt{\frac{\tau_t(p_t - c_t)}{B_t}} \right)$$

We now plug in the expression for m_t:

$$\pi_t = \frac{B_t - \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}}}{c_t} \cdot \left(p_t - c_t - \sqrt{\frac{\tau_t(p_t - c_t)}{B_t}} \right)$$

Factorising by B_t gives:

$$\pi_t = \frac{1}{c_t} \cdot \sqrt{B_t} \left(\sqrt{B_t} - \sqrt{\frac{\tau_t}{p_t - c_t}} \right) \cdot \left(\frac{\sqrt{B_t}(p_t - c_t) - \sqrt{\tau_t(p_t - c_t)}}{\sqrt{B_t}} \right)$$

We now simplify by $\sqrt{B_t}$:

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t} - \sqrt{\frac{\tau_t}{p_t - c_t}} \right) \cdot \left(\sqrt{B_t}(p_t - c_t) - \sqrt{\tau_t(p_t - c_t)} \right)$$

$$\Leftrightarrow \pi_t = \frac{1}{c_t} \cdot \left(\frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{p_t - c_t}} \right) \cdot \left(\sqrt{B_t}(p_t - c_t) - \sqrt{\tau_t(p_t - c_t)} \right)$$

Dividing by $\sqrt{(}p_t - c_t)$:

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right) \cdot \left(\sqrt{B_t}(\sqrt{p_t - c_t}) - \sqrt{\tau_t} \right)$$

The simplified version of the profit function is therefore

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right)^2 \tag{7}$$

5.5 Government maximisation problem

The government maximisation problem is:

$$\max_{\tau_t} \left(\frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right)^2 \right)^{\alpha} \cdot e^{-q_t \cdot g_t (1-\alpha)}$$

We know that:

$$q_t \cdot g_t = \frac{m_t}{c_t} \cdot \frac{1}{r_t} = \frac{1}{c_t} \left(B_t - \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}} \right) \left(\frac{\sqrt{p_t - c_t}}{\sqrt{B_t \cdot \tau_t}} \right)$$

$$\Leftrightarrow q_t \cdot g_t = \frac{1}{c_t} \cdot \left(\frac{\sqrt{B_t \cdot (p_t - c_t)}}{\sqrt{\tau_t}} - 1 \right) = \frac{1}{c_t} \cdot \frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{\tau_t}}$$

We can now plug the expression for $q_t \cdot g_t$ in the government's utility function:

$$u_{G,t} = \left(\frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}\right)^2\right)^\alpha \cdot e^{-\frac{1}{c_t} \cdot \frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{\tau_t}} (1-\alpha)}$$

The simplified utility function of the government is therefore:

$$u_{G,t} = \frac{1}{c_t^\alpha} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}\right)^{2\cdot\alpha} \cdot e^{-(1-\alpha)\cdot\frac{1}{c_t}\cdot\frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{\tau_t}}}$$

5.6 Optimal pollution tax

To find the optimal pollution tax, we set the derivative of the social utility function to zero :

$$u_{G,t} = \frac{e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha}}{c^\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}}} \tag{9}$$

We see that this function is of form $f(\tau_t) = \frac{u(\tau_t)}{v(\tau_t)}$, therefore its derivative is of form $f'(\tau_t) = \frac{u'(\tau)v(\tau_t) - v'(\tau_t)u(\tau_t)}{v^2(\tau_t)}$. To have its derivative equal to zero, we only need $u'(\tau_t)v(\tau_t) - v'(\tau_t)u(\tau_t) = 0$, with :

$$\begin{cases} u(\tau_t) = e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha} \\ v(\tau_t) = c^\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}} \end{cases}$$

Now computing the First Order Conditions:

$$\begin{cases} \frac{\partial u(\tau_t)}{\partial \tau_t} = e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(-\frac{1}{2}\right) \cdot 2\alpha \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \\ \frac{\partial v(\tau_t)}{\partial \tau_t} = c_t^\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}} \cdot \frac{1}{c_t}(1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \left(-\frac{1}{2}\right) \cdot \tau_t^{-\frac{3}{2}} \end{cases}$$

$$\iff$$

$$\begin{cases} \frac{\partial u(\tau_t)}{\partial \tau_t} = -\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \\ \frac{\partial v(\tau_t)}{\partial \tau_t} = -\frac{c_t^{\alpha-1}}{2} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \end{cases}$$

Thus we can compute :

$$u'(\tau_t)v(\tau_t) - v'(\tau_t)u(\tau_t) = 0$$

$$\iff$$

$$-\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \cdot c^\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}}$$
$$+ \frac{c_t^{\alpha-1}}{2} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha} = 0$$

$$\iff$$

$$\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \cdot c^\alpha =$$
$$\frac{c_t^{\alpha-1}}{2} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha}$$

We simplify by $\left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha}$:

$$\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} \cdot c^{\alpha} =$$
$$\frac{c_t^{\alpha-1}}{2} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)}$$

We divide by $e^{\frac{1}{c_t}\cdot(1-\alpha)}$ and by c_t^{α}:

$$\alpha \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} = \frac{1}{2 \cdot c_t} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}}$$
$$\Longleftrightarrow$$
$$\alpha \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} = \frac{1}{2 \cdot c_t} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-1}$$

Rearranging:

$$\tau_t \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha) \cdot \sqrt{B_t(p_t - c_t)}} = \sqrt{B(p_t - c_t)} - \sqrt{\tau_t}$$

Multiplying both sides by $\sqrt{B_t(p_t - c_t)}$ and rearranging:

$$\tau_t \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} = B_t(p_t - c_t) - \sqrt{\tau_t} \cdot \sqrt{B_t(p_t - c_t)}$$
$$\Longleftrightarrow$$
$$\tau_t \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} - B_t(p_t - c_t) = -\sqrt{\tau_t} \cdot \sqrt{B_t(p_t - c_t)}$$

We now square both sides of the expression:

$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 - 2 \cdot \tau_t \cdot B_t\left(p_t - c_t\right) \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + \left(B_t(p_t - c_t)\right)^2 = \tau_t \cdot B_t(p_t - c_t)$$
$$\Longleftrightarrow$$
$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 - 2 \cdot \tau_t \cdot B_t\left(p_t - c_t\right) \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} - \tau_t \cdot B_t(p_t - c_t) + \left(B_t(p_t - c_t)\right)^2 = 0$$

Factorizing by $-\tau_t \cdot B_t(p_t - c_t)$ gives:

$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 + \tau_t \cdot (-B_t)\left(p_t - c_t\right)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) + \left(B_t(p_t - c_t)\right)^2 = 0$$

This is a standard quadratic equation $\tau_t^2 \cdot a + \tau_t \cdot b + c = 0$ that accepts two solutions :

$$\tau^* = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

With :

- $a = \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2$

- $b = (-B_t)(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)$
- $c = (B_t(p_t - c_t))^2$

The solution $\tau^* = \frac{-b+\sqrt{b^2+4\cdot a\cdot c}}{2\cdot a}$ does not make sense in the context of this problem, so we have the optimal tax :

$$\tau_t^* = \frac{-b - \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - \sqrt{\left[(-B_t)(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)\right]^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2 \cdot (B_t(p_t - c_t))^2}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - \sqrt{(B_t(p_t - c_t))^2\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2 \cdot (B_t(p_t - c_t))^2}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - \sqrt{(B_t(p_t - c_t))^2\left[\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2\right]}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - B_t(p_t - c_t)\sqrt{\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left[\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{1-\alpha} + 1\right) - \sqrt{\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{1-\alpha} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}\right]}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left[\left(\frac{\alpha \cdot 4 \cdot c_t + 1 - \alpha}{1-\alpha}\right) - \sqrt{\left(\frac{\alpha \cdot 4 \cdot c_t + 1 - \alpha}{1-\alpha}\right)^2 - \frac{\alpha^2 \cdot 16 \cdot c_t^2}{(1-\alpha)^2}}\right]}{\frac{\alpha^2 \cdot 8 \cdot c_t^2}{(1-\alpha)^2}}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left[\left(\frac{\alpha \cdot 4 \cdot c_t + 1 - \alpha}{1-\alpha}\right) - \sqrt{\frac{1}{(1-\alpha)^2}\left((\alpha \cdot 4 \cdot c_t + 1 - \alpha)^2 - (\alpha^2 \cdot 16 \cdot c_t^2)\right)}\right]}{\frac{\alpha^2 \cdot 8 \cdot c_t^2}{(1-\alpha)^2}}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\,(p_t - c_t)\left[\left(\frac{\alpha\cdot 4\cdot c_t + 1 - \alpha}{1-\alpha}\right) - \frac{1}{(1-\alpha)}\sqrt{\alpha^2\cdot 16\cdot c_t^2 + (1-\alpha)^2 + 2\cdot\alpha\cdot 4\cdot c_t(1-\alpha) - \alpha^2\cdot 16\cdot c_t^2}\right]}{\frac{\alpha^2\cdot 8\cdot c_t^2}{(1-\alpha)^2}}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\,(p_t - c_t)\,(\frac{1}{1-\alpha})\left[\alpha\cdot 4\cdot c_t + 1 - \alpha - \sqrt{(1-\alpha)^2 + 2\cdot\alpha\cdot 4\cdot c_t(1-\alpha)}\right]}{\frac{\alpha^2\cdot 8\cdot c_t^2}{(1-\alpha)^2}}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\,(p_t - c_t)\,(1-\alpha)\left[\alpha\cdot 4\cdot c_t + 1 - \alpha - \sqrt{(1-\alpha)^2 + \alpha\cdot 8\cdot c_t(1-\alpha)}\right]}{\alpha^2\cdot 8\cdot c_t^2} \tag{10}$$