DRAFT INTRO Microeconomics Project

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Abstract

In the following paper our aim is to use game theory to determine the optimal tax the environmentally discerning government should impose on a representative firm in order to ensure the firm invests in Research and Development and upgrades its' production technology to thus minimise the pollution per unit of product produced.

We start by building a classic Stackelberg game to find the optimal tax, we then proceed to building a cheap talk game with imperfect information under the assumption that the R&D investment might not lead to successful drop in pollution per unit of product produced to investigate the optimal strategy profile for the firm and for the government.

Contents

1 Introduction

In this work we used game theory framework to model an efficient taxation policy the government should implement, knowing that there exist strategic incentives for the firm to falsely declare the pollution per unit of good produced. We also demonstrate how the optimal taxation policy differs in case of persistent pollution and unsuccessful R&D results.

We have decided to dedicate our project to solving these issues for several reasons.

First, the climate change is an undeniable threat to global welfare and the green tax is often highlighted one of the most efficient means of climate protection.

Given that in the real world the government does not have the tools to instantly verify the amount of declared pollution, several instances happened in the past when the firms that are subject to the green tax falsely declared their emitted pollution levels, one notable example of such a case is the Volkswagen scandal of 2013 (Link!).

Moreover, even if the government efficiently implements the optimal tax and the firm invests substantially in the R&D aiming to transition to greener production, the pollution still might persist. The R&D is a complex process and the success is never guaranteed.

In the literature we reviewed incentives to lie and the complexity of R&D have not been taken into account when analysing the strategic interactions between the economic agents and the effectiveness of the green tax. Hence, we have decided to make it the main focus of our project.

2 Literature review

There is a growing interest in managing the environmental pollution levels by targeting the unsustainable production. Game theory provides a good framework for the contemporary researchers to analyse the strategic interactions between the economic agents and thus to effectively trace the effects their decisions have on each other and on the environment.

We have found numerous research papers that explore how game theory can be applied to study the effectiveness of environmental policies (most often the green tax) imposed on the production firms by the government.

Vast amounts of research have been done to analyse the effectiveness of green taxes in promoting sustainable production. It has been shown that heavy taxation must be implemented to achieve a drop in pollution when the government is dealing with highly polluting enterprises (Wei Yu, Ruizhu Han, 2019). Some papers claim that the emissions tax is one of the most effective policies the government can impose to increase the environmental quality, the optimal amount of the tax being determined by the marginal environmental damage of the market share of the production firm (Dorothée Brécard, 2010).

Research has demonstrated that the producers must take full responsibility for the environmental, social, economic, and cultural ramifications of the undertakings they give (Veleva and Ellenbecker, 2001; Ülkü and Engau, 2021). However, it has been shown that the transition to the green production can be challenging for the firms, since it requires a major capital investment in R&D, thus making the final goods more costly to the consumer. Hence the firms' decision to opt for green technologies largely depends on consumers' willingness to increase their expenses. (Conrad, 2005, Krass 2013)

That being said, the government subsidy may stimulate the firms to switch to greener production and increase the market share of green products, thus making the green goods cheaper to produce which would result in a substantial decrease in the environmental pollution.

On the other hand, higher subsidies might not result in a successful drop in environmental pollution, the effectiveness of the subsidy and the amount of the subsidy largely depend on the manufacturing investment, R&D investment, and consumers' preferences. (Yantao Ling, Jing Xu, M. Ali Ülkü, 2022)

3 Baseline Model

We model the economy using the following assumptions:

- There exist two agents in the economy, namely the government and the producer (the firm), consumers are not included.
- The firms are homogeneous and therefore representable.

First, we model the interactions between the government and the firm using the Stackelberg game structure (a game with strategic interactions that implies perfect information, non-simultaneous actions and two players, namely the leader and the follower). The pollution in the past is not taken into account in the simplified model, but will be included in the extension.

3.1 Players

We assign the role of the leader to the firm and the role of a follower to the government. The firm "plays" first, by maximising its' profit. The government observes the action of the firm and then makes a move by setting an optimal tax depending on the firm's profit.

3.2 Firm

3.2.1 Technology of the firm

The firm is a monopoly that produces a single good. It's production function takes two inputs:

- R&D investment (r)
- manufacturing investment (m)

The main goal of the R&D investment is to diminish the pollution produced by the firm per unit of product produced, while the manufacturing investment is used by the firm to produce a quantity q of its' single good.

The pollution g_t generated by producing one unit of good at time t is defined as:

$$g_t = \frac{1}{r_t} \tag{1}$$

Where r_t is the R&D investment done by the firm to reduce its unit pollution level g_t.

The manufacturing investment m_t needed to produce a quantity q_t of a good is defined as:

$$m_t = q_t \cdot c_t \tag{2}$$

Where q_t is the quantity of good produced by the firm at time t and c_t is the cost of manufacturing one good by the firm at time t

Lastly, the firm is subject to a tax τ based on the pollution level emitted per unit produced.

3.2.2 Profit of the firm

Given those technology constraints, the firm maximises its profit:

$$\begin{aligned} &\max_{r_t, m_t} \pi_{F,t} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t \\ &\text{subject to : } B_t \leq r_t + m_t \end{aligned} \tag{3}$$

Where :

- q_t is the quantity of good produced and sold at time t
- p_t is the price at which one unit of good is sold
- c_t is the cost of production of a unit of good
- $\tau_t \in \mathbb{R}_+$ is the amount at which one unit of good is taxed, depending on the level of pollution produced per unit, at time t
- $g_t \in \mathbb{R}_+$ is the level of pollution emitted by producing a unit of product at time t
- B_t is the budget of the firm at time t

3.3 Government

The government maximises the social welfare, which we define as a utility function that takes the economic activities EA_t and the risk of environmental disaster EQ_t as inputs:

$$u_{G,t} = \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{1-\alpha} \tag{4}$$

Where :

- EA_t is the Economic Activity of society at time t
- EQ_t is the Environmental Quality of the world at time t
- $\alpha \in (0, 1)$ is the relative importance of Economic Activity compared to the Environmental Quality of the World.

This function has two main properties :

1. Positive but decreasing returns for both Economic Activity and Environmental Quality[1].
2. Both Economic Activity and Environmental Quality are complementary goods, as the cross-derivative is positive [2].

3.4 Model solution

3.5 Firm's problem

In this simplified model, we first solve the Firm's problem for one period :

$$\begin{aligned} &\max_{r_t, m_t} \pi_{F,t} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t \\ &\text{subject to : } B_t \leq r_t + m_t \end{aligned}$$

[1]See Appendix for proof.
[2]See Appendix for proof.

Hence the Lagrangian is:

$$\mathcal{L} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t + \lambda (B - r_t - m_t)$$

Expressing the q_t as a function of m_t, and the g_t as a function of r_t gives:

$$\mathcal{L} = \frac{m_t}{c_t} \cdot (p_t - c_t) - \tau_t \cdot \frac{m_t}{c_t} \cdot \frac{1}{r_t} + \lambda (B - r_t - m_t)$$

Applying the First Order Condition[3] gets the manufacturing investment:

$$m_t = \frac{1}{\tau_t} \left(r_t^2 (p_t - c_t) - r_t \cdot \tau_t \right) = \frac{r_t^2 (p_t - c_t)}{\tau_t} - r_t \tag{5}$$

This expression gives the optimal level of manufacturing investment for the firm. Since it has a negative relationship with the pollution tax τ_t, we can conclude that, the bigger the pollution tax, the more incentives the firm has to invest in R&D instead of simply manufacturing goods with high pollution level.

From that, we can identify the optimal level of research r for the firm [4] :

$$r_t = \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}} \tag{6}$$

Thus, the maximum profit of the firm is[5]:

$$\pi_t^* = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right)^2 \tag{7}$$

3.6 Government problem

In this simplified model, we set the Economic Activity EA_t equal to the profit of the firm. The environment quality EQ_t at time t depends on the environmental pollution at time t. The expression for the EQ_t is the following:

$$EQ_t \equiv e^{-q_t \cdot g_t} \tag{8}$$

Where $q_t \cdot g_t$ is the amount of pollution produced at time t
Thus, the maximisation program of the government is:

$$\max_{\tau_t} u_{G,t} = \pi_t^{\alpha} \cdot e^{-q_t \cdot g_t \cdot (1-\alpha)}$$

Since the Stackelberg game implies perfect information, the government knows the allocation decisions of the firm. Therefore the government knows the values of r_t and m_t, and the maximisation program of the government is [6]:

$$u_{G,t} = \frac{e^{\frac{1}{c_t} \cdot (1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t} \right)^{2\alpha}}{c^{\alpha} \cdot e^{\frac{1}{c_t} \cdot (1-\alpha) \cdot \frac{\sqrt{B_t (p_t - c_t)}}{\sqrt{\tau_t}}}} \tag{9}$$

[3]See Appendix for proof.
[4]See Appendix for proof.
[5]See Appendix for proof.
[6]See Appendix for proof.

To maximise this utility function, we find first derivatives with respect to τ_t and set them equal to zero. Solving for τ_t gives the expression for the optimal pollution tax τ_t^* that maximises the objective function of the government [7] :

$$\tau_t^* = \frac{B_t\,(p_t - c_t)\,(1-\alpha)\left[\alpha \cdot 4 \cdot c_t + 1 - \alpha - \sqrt{(1-\alpha)^2 + \alpha \cdot 8 \cdot c_t(1-\alpha)}\right]}{\alpha^2 \cdot 8 \cdot c_t^2} \tag{10}$$

We can observe that the optimal pollution tax has the following properties:

- It has a negative relationship with the manufacturing cost c_t
- It has a negative relationship with the preference of the government for economic activity α
- It has a positive relationship with the budget of the firm cost B_t
- It has a positive relationship with the selling price p_t

3.7 Interpretations and extensions

This model allows us to say several things about the strategic interactions between the firm and the government.

Regarding the maximum profit of the firm, we see that π^* has a negative relationship with the tax. Under the condition that $B_t \cdot (p_t - c_t) > \tau_t$, this conclusion seems to be realistic and in harmony with what we would expect from a rational firm.

Regarding the best response of the government : first of all, the tax depend on the preference of the government, translated by α. As defined in the utility function, $u_{G,t} = \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{1-\alpha}$, α is the relative preference of government for Economic Activity. If the government suddendly has a bigger preference for economic activity, and that we have an increase of α, then its best response would be to decrease the tax.

Secondly, we see that the tradeoff between economic activity and environmental quality is determinant in the choice of the government. The negative relationship between τ_t and c_t and the positive relationship between τ_t and B_t shows it. If the firm is faced with higher production costs, it is in the interest of the government to decrease the tax, even if it implies a highger pollution level. In the same sense, if the firm has suddenly a higher budget, its profit will be positively impacted, and the government can allow itself to tax it a bit more. The conclusions of this model seem coherent with real considerations.

To extend our baseline model and make it more applicable to real life, we are now going to introduce imperfect information

[7] See Appendix for proof.

4 Appendix

4.1 Properties of the utility function of the government

4.1.1 Diminishing returns of economic activity and environmental quality

The Government utility function is :

$$u_{G,t} = \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{1-\alpha}$$

For EA_t the economic activity and EQ_t the environmental quality to yields positive but diminishing returns, the first derivatives must be positive and the second derivatives must be negatives.

We compute :

$$\frac{\partial u_{G,t}}{\partial \mathrm{EA}_t} = \alpha \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{1-\alpha} > 0$$

$$\frac{\partial u_{G,t}}{\partial \mathrm{EQ}_t} = (1-\alpha) \cdot \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{-\alpha} > 0$$

$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EA}_t} = \alpha \cdot (\alpha - 1) \cdot \mathrm{EA}_t^{\alpha-2} \cdot \mathrm{EQ}_t^{1-\alpha} < 0$$

$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EQ}_t} = (1-\alpha) \cdot (-\alpha) \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{-\alpha-1} < 0$$

Thus, we indeed have positive but diminishing returns for both EA_t and EQ_t.

4.1.2 Complementarity of economic activity and environmental quality

For EA_t the economic activity and EQ_t the environmental quality to be complements, the crosse-derivative has to be positive. We compute :

$$\frac{\partial^2 u_{G,t}}{\partial \mathrm{EA}_t \partial \mathrm{EQ}_t} = \alpha \cdot (1-\alpha) \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{-\alpha} > 0$$

Thus, EA_t and EQ_t are complement. This means that in this model, the governement does not prefer a world without economic activity or without some environmental quality, but it prefers a mixture of them.

4.2 First Order Conditions

Differentiating the Lagrangian with respect to the R&D investment r_t gives:

$$\frac{\partial \mathcal{L}}{\partial r_t} = 0$$

$$\Leftrightarrow \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2} - \lambda = 0$$

$$\Leftrightarrow \lambda = \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2}$$

Differentiating the Lagrangian with respect to the manufacturing investment m_t gives:

$$\frac{\partial \mathcal{L}}{\partial m_t} = 0$$

$$\Leftrightarrow \frac{p_t - c_t}{c_t} - \frac{\tau_t}{c_t \cdot r_t} - \lambda = 0$$

$$\Leftrightarrow \frac{p_t - c_t}{c_t} - \frac{\tau_t}{c_t \cdot r_t} - \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2} = 0$$

$$\Leftrightarrow r_t^2 \cdot (pt - c_t) - r_t \cdot \tau_t - \tau_t \cdot m_t = 0$$

Hence the manufacturing investment is:

$$m_t = \frac{1}{\tau_t} \left(r_t^2 (p_t - c_t) - r_t \cdot \tau_t \right) = \frac{r_t^2 \left(p_t - c_t \right)}{\tau_t} - r_t \tag{11}$$

4.3 Optimal Level of Research

Starting with the Budget constraint of the firm:

$$B_t = r_t + m_t$$

Plugging into the constraint the expression for m_t we had found earlier gives:

$$B_t = r_t + \frac{r_t^2 \left(p_t - c_t \right)}{\tau_t} - r_t$$

$$\Leftrightarrow B_t = \frac{r_t^2 \left(p_t - c_t \right)}{\tau_t}$$

Rearranging the expression with respect to the research level r_t gives:

$$r_t = \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}} \tag{12}$$

4.4 Maximum Profit

The First Order Condition is:

$$\pi_t = \frac{m_t}{c_t} \cdot (p_t - c_t) - \tau_t \cdot \frac{m_t}{c_t} \cdot \frac{1}{r_t}$$

Factorising by $\frac{m_t}{c_t}$ gives:

$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \frac{\tau_t}{r_t} \right)$$

We now plug in the expression for r_t:

$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \frac{\tau_t}{\sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}}} \right)$$

Which is equivalent to:

$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \sqrt{\frac{\tau_t (p_t - c_t)}{B_t}} \right)$$

We now plug in the expression for m_t:

$$\pi_t = \frac{B_t - \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}}}{c_t} \cdot \left(p_t - c_t - \sqrt{\frac{\tau_t (p_t - c_t)}{B_t}} \right)$$

Factorising by B_t gives:

$$\pi_t = \frac{1}{c_t} \cdot \sqrt{B_t} \left(\sqrt{B_t} - \sqrt{\frac{\tau_t}{p_t - c_t}} \right) \cdot \left(\frac{\sqrt{B_t}(p_t - c_t) - \sqrt{\tau_t (p_t - c_t)}}{\sqrt{B_t}} \right)$$

We now simplify by $\sqrt{B_t}$:

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t} - \sqrt{\frac{\tau_t}{p_t - c_t}} \right) \cdot \left(\sqrt{B_t}(p_t - c_t) - \sqrt{\tau_t (p_t - c_t)} \right)$$

$$\Leftrightarrow \pi_t = \frac{1}{c_t} \cdot \left(\frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{p_t - c_t}} \right) \cdot \left(\sqrt{B_t}(p_t - c_t) - \sqrt{\tau_t (p_t - c_t)} \right)$$

Dividing by $\sqrt{(p_t - c_t)}$:

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right) \cdot \left(\sqrt{B_t}(\sqrt{p_t - c_t}) - \sqrt{\tau_t} \right)$$

The simplified version of the profit function is therefore

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right)^2 \tag{13}$$

4.5 Government maximisation problem

The government maximisation problem is:

$$\max_{\tau_t} \left(\frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right)^2 \right)^{\alpha} \cdot e^{-q_t \cdot g_t (1-\alpha)}$$

We know that:

$$q_t \cdot g_t = \frac{m_t}{c_t} \cdot \frac{1}{r_t} = \frac{1}{c_t}\left(B_t - \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}}\right)\left(\frac{\sqrt{p_t - c_t}}{\sqrt{B_t \cdot \tau_t}}\right)$$

$$\Leftrightarrow q_t \cdot g_t = \frac{1}{c_t} \cdot \left(\frac{\sqrt{B_t \cdot (p_t - c_t)}}{\sqrt{\tau_t}} - 1\right) = \frac{1}{c_t} \cdot \frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{\tau_t}}$$

We can now plug the expression for $q_t \cdot g_t$ in the government's utility function:

$$u_{G,t} = \left(\frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}\right)^2\right)^{\alpha} \cdot e^{-\frac{1}{c_t} \cdot \frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{\tau_t}}}{}^{(1-\alpha)}$$

The simplified utility function of the government is therefore:

$$u_{G,t} = \frac{1}{c_t^{\alpha}} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}\right)^{2 \cdot \alpha} \cdot e^{-(1-\alpha) \cdot \frac{1}{c_t} \cdot \frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{\tau_t}}}$$

4.6 Optimal pollution tax

To find the optimal pollution tax, we set the derivative of the social utility function to zero :

$$u_{G,t} = \frac{e^{\frac{1}{c_t} \cdot (1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha}}{c^{\alpha} \cdot e^{\frac{1}{c_t} \cdot (1-\alpha) \cdot \frac{\sqrt{B_t(p_t - c_t)}}{\sqrt{\tau_t}}}}$$

We see that this function is of form $f(\tau_t) = \frac{u(\tau_t)}{v(\tau_t)}$, therefore its derivative is of form $f'(\tau_t) = \frac{u'(\tau)v(\tau_t) - v'(\tau_t)u(\tau_t)}{v^2(\tau_t)}$. To have its derivative equal to zero, we only need $u'(\tau_t)v(\tau_t) - v'(\tau_t)u(\tau_t) = 0$, with :

$$\begin{cases} u(\tau_t) = e^{\frac{1}{c_t} \cdot (1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha} \\ v(\tau_t) = c^{\alpha} \cdot e^{\frac{1}{c_t} \cdot (1-\alpha) \cdot \frac{\sqrt{B_t(p_t - c_t)}}{\sqrt{\tau_t}}} \end{cases}$$

Now computing the First Order Conditions:

$$\begin{cases} \frac{\partial u(\tau_t)}{\partial \tau_t} = e^{\frac{1}{c_t} \cdot (1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(-\frac{1}{2}\right) \cdot 2\alpha \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha - 1} \\ \frac{\partial v(\tau_t)}{\partial \tau_t} = c_t^{\alpha} \cdot e^{\frac{1}{c_t} \cdot (1-\alpha) \cdot \frac{\sqrt{B_t(p_t - c_t)}}{\sqrt{\tau_t}}} \cdot \frac{1}{c_t}(1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \left(-\frac{1}{2}\right) \cdot \tau_t^{-\frac{3}{2}} \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases} \frac{\partial u(\tau_t)}{\partial \tau_t} = -\alpha \cdot e^{\frac{1}{c_t} \cdot (1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha - 1} \\ \frac{\partial v(\tau_t)}{\partial \tau_t} = -\frac{c_t^{\alpha - 1}}{2} \cdot e^{\frac{1}{c_t} \cdot (1-\alpha) \cdot \frac{\sqrt{B_t(p_t - c_t)}}{\sqrt{\tau_t}}} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \end{cases}$$

Thus we can compute :

$$u'(\tau_t)v(\tau_t) - v'(\tau_t)u(\tau_t) = 0$$

$$\Longleftrightarrow$$

$$- \alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \cdot c^{\alpha} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}}$$
$$+ \frac{c_t^{\alpha-1}}{2} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha} = 0$$

$$\Longleftrightarrow$$

$$\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \cdot c^{\alpha} =$$
$$\frac{c_t^{\alpha-1}}{2} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha}$$

We simplify by $\left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha}$:

$$\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} \cdot c^{\alpha} =$$
$$\frac{c_t^{\alpha-1}}{2} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)}$$

We simplify by $e^{\frac{1}{c_t}\cdot(1-\alpha)}$ and by c_t^{α}:

$$\alpha \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} = \frac{1}{2\cdot c_t} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}}$$

$$\Longleftrightarrow$$

$$\alpha \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} = \frac{1}{2\cdot c_t} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-1}$$

$$\Longleftrightarrow$$

$$\alpha = \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right) \cdot \frac{1}{2\cdot c_t} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-1}$$

$$\Longleftrightarrow$$

$$\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)\cdot\sqrt{B_t(p_t - c_t)}} = \frac{\left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)}{\tau_t}$$

$$\Longleftrightarrow$$

$$\tau_t \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)\cdot\sqrt{B_t(p_t - c_t)}} = \sqrt{B(p_t - c_t)} - \sqrt{\tau_t}$$

$$\Longleftrightarrow$$

$$\tau_t \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + \sqrt{\tau_t} \cdot \sqrt{B_t(p_t - c_t)} - B_t(p_t - c_t) = 0$$

$$\Longleftrightarrow$$

$$\tau_t \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} - B_t(p_t - c_t) = -\sqrt{\tau_t} \cdot \sqrt{B_t(p_t - c_t)}$$
$$\Longleftrightarrow$$
$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 - 2 \cdot \tau_t \cdot B_t\left(p_t - c_t\right) \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + \left(B_t(p_t - c_t)\right)^2 = \tau_t \cdot B_t(p_t - c_t)$$
$$\Longleftrightarrow$$
$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 + \tau_t \cdot (-2) \cdot B_t\left(p_t - c_t\right) \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} - \tau_t \cdot B_t(p_t - c_t) + \left(B_t(p_t - c_t)\right)^2 = 0$$
$$\Longleftrightarrow$$
$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 + \tau_t \cdot B_t\left(p_t - c_t\right)\left((-2) \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} - 1\right) + \left(B_t(p_t - c_t)\right)^2 = 0$$
$$\Longleftrightarrow$$
$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 + \tau_t \cdot (-B_t)\left(p_t - c_t\right)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) + \left(B_t(p_t - c_t)\right)^2 = 0$$

This is a standard quadratic equation $\tau_t^2 \cdot a + \tau_t \cdot b + c = 0$ that accepts two solutions :

$$\tau^* = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

With :

- $a = \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2$
- $b = (-B_t)\left(p_t - c_t\right)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)$
- $c = \left(B_t(p_t - c_t)\right)^2$

The solution $\tau^* = \frac{-b+\sqrt{b^2+4 \cdot a \cdot c}}{2 \cdot a}$ does not make sense in the context of this problem, so we have the optimal tax :

$$\tau_t^* = \frac{-b - \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$
$$\Longleftrightarrow$$
$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - \sqrt{\left[(-B_t)\left(p_t - c_t\right)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)\right]^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2 \cdot \left(B_t(p_t - c_t)\right)^2}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$
$$\Longleftrightarrow$$
$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - \sqrt{\left(B_t\left(p_t - c_t\right)\right)^2\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2 \cdot \left(B_t(p_t - c_t)\right)^2}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - \sqrt{\left(B_t\left(p_t - c_t\right)\right)^2 \left[\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2\right]}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - B_t\left(p_t - c_t\right)\sqrt{\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left[\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{1-\alpha} + 1\right) - \sqrt{\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{1-\alpha} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}\right]}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left[\left(\frac{\alpha \cdot 4 \cdot c_t + 1 - \alpha}{1-\alpha}\right) - \sqrt{\left(\frac{\alpha \cdot 4 \cdot c_t + 1 - \alpha}{1-\alpha}\right)^2 - \frac{\alpha^2 \cdot 16 \cdot c_t^2}{(1-\alpha)^2}}\right]}{\frac{\alpha^2 \cdot 8 \cdot c_t^2}{(1-\alpha)^2}}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left[\left(\frac{\alpha \cdot 4 \cdot c_t + 1 - \alpha}{1-\alpha}\right) - \sqrt{\frac{1}{(1-\alpha)^2}\left((\alpha \cdot 4 \cdot c_t + 1 - \alpha)^2 - (\alpha^2 \cdot 16 \cdot c_t^2)\right)}\right]}{\frac{\alpha^2 \cdot 8 \cdot c_t^2}{(1-\alpha)^2}}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left[\left(\frac{\alpha \cdot 4 \cdot c_t + 1 - \alpha}{1-\alpha}\right) - \frac{1}{(1-\alpha)}\sqrt{\alpha^2 \cdot 16 \cdot c_t^2 + (1-\alpha)^2 + 2 \cdot \alpha \cdot 4 \cdot c_t(1-\alpha) - \alpha^2 \cdot 16 \cdot c_t^2}\right]}{\frac{\alpha^2 \cdot 8 \cdot c_t^2}{(1-\alpha)^2}}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left(\frac{1}{1-\alpha}\right)\left[\alpha \cdot 4 \cdot c_t + 1 - \alpha - \sqrt{(1-\alpha)^2 + 2 \cdot \alpha \cdot 4 \cdot c_t(1-\alpha)}\right]}{\frac{\alpha^2 \cdot 8 \cdot c_t^2}{(1-\alpha)^2}}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t\left(p_t - c_t\right)(1-\alpha)\left[\alpha \cdot 4 \cdot c_t + 1 - \alpha - \sqrt{(1-\alpha)^2 + \alpha \cdot 8 \cdot c_t(1-\alpha)}\right]}{\alpha^2 \cdot 8 \cdot c_t^2} \tag{9}$$