Microeconomics Project

Chambon Lionel
Gugelmo Cavalheiro Dias Paulo
Pavlovich Karina
Soufflet Morgane

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Abstract

Contents

1 Baseline model

This model does not use strategic interactions, and takes place within a perfect information framework.

2 Players

This model has two agents : the Firm and the Governement.

2.1 Firm

2.1.1 Technology of the firm

The firm is a monopoly, that has two possible inputs : R&D investment (r) to diminish its pollution per unit and manufacturing investment (m) to produce a quantity q of its single good. The firm is also taxed on the pollution level they emit by a tax τ implemented by the government.

$$\begin{cases} g_t = \frac{1}{r_t} \\ m_t = q_t \cdot c_t \end{cases} \tag{1}$$

Where :

- g_t is the pollution generated by the production of one unit of good by the firm at time t
- r_t is the R&D investment done by the firm to reduce its unit pollution level g_t
- m_t is the manufacturing investment of the firm, that is needed to produce a certain quantity of good q_t
- q_t is the quantity of good produced by the firm at time t
- c_t is the cost of manufacturing one good by the firm at time t

2.1.2 Profit of the firm

Given those technology constraints, the firm maximises its profit such that :

$$\begin{aligned} &\max_{r_t, m_t} \pi_{F,t} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t \\ &\text{subject to : } B_t \leq r_t + m_t \end{aligned} \tag{2}$$

Where :

- q_t is the quantity of good produced and sold at time t
- p_t is the price at which one unit of good is sold

- c_t is the cost of production of unit of good
- $\tau_t \in (0,1)$ the share at which the one unit of good is taxed, depending on the level of pollution produced per unit, at time $_t$
- $g_t \in \mathbb{R}_+$ is the level of pollution created by the production of one unit by the firm at time t
- B_t is the budget of the firm at time t

2.2 Government

The goal of the government is to maximise the social welfare function. The social welfare function here is defined as a utility function taking into account economic activities and the risk of environmental disaster. We define it as :

$$u_{G,t} = \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{1-\alpha} \tag{3}$$

Where :

- EA_t is the Economic Activity of society at time t
- EQ_t is the Environmental Quality of the world at time t
- $\alpha \in (0,1)$ is the relative importance of Economic Activity compared to the Environmental Quality of the World.

This function has two main properties :

1. It has positive but decreasing returns for both Economic Activity and Environmental Quality :

$$\frac{\partial u_{G,t}}{\partial \mathrm{EA}_t} = \alpha \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{1-\alpha} > 0$$
$$\frac{\partial u_{G,t}}{\partial \mathrm{EQ}_t} = (1-\alpha) \cdot \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{-\alpha} > 0$$

$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EA}_t} = \alpha \cdot (\alpha - 1) \cdot \mathrm{EA}_t^{\alpha-2} \cdot \mathrm{EQ}_t^{1-\alpha} < 0$$
$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EQ}_t} = (1-\alpha) \cdot (-\alpha) \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{-\alpha-1} < 0$$

2. Both Economic Activity and Environmental Quality are complementary goods, as the cross-derivative is positive :

$$\frac{\partial^2 u_{G,t}}{\partial \mathrm{EA}_t \partial \mathrm{EQ}_t} = \alpha \cdot (1-\alpha) \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{-\alpha} > 0$$

3 Model solution

3.1 Firm's problem

In this simplified model, we first solve the Firm's problem for one period :

$$\max_{r_t, m_t} \pi_{F,t} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t$$
$$\text{subject to : } B_t \leq r_t + m_t$$

We can thus write the Lagrangien such that :

$$\mathcal{L} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t + \lambda (B - r_t - m_t)$$

And if we rewrite it with the extended expression of q_t and m_t, we get :

$$\mathcal{L} = \frac{m_t}{c_t} \cdot (p_t - c_t) - \tau_t \cdot \frac{m_t}{c_t} \cdot \frac{1}{r_t} + \lambda (B - r_t - m_t)$$

We solve the First Order Conditions.

If we differentiate the Lagrangien with respect to the R&D investment r_t :

$$\frac{\partial \mathcal{L}}{\partial r_t} = 0$$
$$\iff$$
$$\frac{\tau_t \cdot m_t}{c_t \cdot r_t^2} - \lambda = 0$$
$$\iff$$
$$\lambda = \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2}$$

If we differentiate the Lagrangien with respect to the manufacturing investment m_t :

$$\frac{\partial \mathcal{L}}{\partial m_t} = 0$$
$$\iff$$
$$\frac{p_t - c_t}{c_t} - \frac{\tau_t}{c_t \cdot r_t} - \lambda = 0$$
$$\iff$$
$$\frac{p_t - c_t}{c_t} - \frac{\tau_t}{c_t \cdot r_t} - \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2} = 0$$
$$\iff$$
$$r_t^2 \cdot (p_t - c_t) - r_t \cdot \tau_t - \tau_t \cdot m_t = 0$$
$$\iff$$
$$m_t = \frac{1}{\tau_t} \left(r_t^2 (p_t - c_t) - r_t \cdot \tau_t \right) = \frac{r_t^2 (p_t - c_t)}{\tau_t} - r_t \tag{4}$$

This equation gives us the optimal level of manufacturing investment for the firm. We can see that it has a negative relationship with τ_t the pollution tax, which seems logical : the bigger the pollution tax, the more incentives the firm has to invest in R&D instead of simply manufacturing goods with high pollution level.

From that, we can identify the optimal level of research r for the firm :

$$B_t = r_t + m_t$$
$$\Longleftrightarrow$$
$$B_t = r_t + \frac{r_t^2\,(p_t - c_t)}{\tau_t} - r_t$$
$$\Longleftrightarrow$$
$$B_t = \frac{r_t^2\,(p_t - c_t)}{\tau_t}$$
$$\Longleftrightarrow$$
$$r_t = \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}} \tag{5}$$

Thus, the maximum profit of the firm is[1] :

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}\right)^2 \tag{6}$$

3.2 Government problem

In this simplified model, we say that the Economic Activity EA_t is equivalent to the profit of the firm.

The environment quality EQ_t at time t depends of the pollution in nature at time t. We can define in a simple, one period context :

$$EQ_t \equiv e^{-q_t \cdot g_t} \tag{7}$$

Where :

- $q_t \cdot g_t$ is the quantity of pollution produced at time t

Note here that we don't take into account past pollution in this simplified model. This will be taken into account in the next sections.

Thus, the maximisation program of the government is :

$$\max_{\tau_t} u_{G,t} = \pi_t^{\alpha} \cdot e^{-q_t \cdot g_t \cdot (1-\alpha)}$$

Now, in this simple framework, the government knows the allocation decisions of the firm, and more specifically knows that the values of r_t and m_t. We can thus rewrite the maximisation program of the government as :

[1]You can find the proof of this result here.

$$\max_{\tau_t} u_{G,t} = \pi_t^{\alpha} \cdot e^{-q_t \cdot g_t (1-\alpha)}$$

$$\Longleftrightarrow$$

$$\max_{\tau_t} \left(\frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right)^2 \right)^{\alpha} \cdot e^{-q_t \cdot g_t (1-\alpha)}$$

Now :

$$q_t \cdot g_t = \frac{m_t}{c_t} \cdot \frac{1}{r_t} = \frac{1}{c_t} \left(B_t - \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}} \right) \left(\frac{\sqrt{p_t - c_t}}{\sqrt{B_t \cdot \tau_t}} \right)$$

$$\Longleftrightarrow$$

$$q_t \cdot g_t = \frac{1}{c_t} \cdot \left(\frac{\sqrt{B_t \cdot (p_t - c_t)}}{\sqrt{\tau_t}} - 1 \right) = \frac{1}{c_t} \cdot \frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{\tau_t}}$$

Thus the objective function of the government is :

$$u_{G,t} = \left(\frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right)^2 \right)^{\alpha} \cdot e^{-\frac{1}{c_t} \cdot \frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{\tau_t}} (1-\alpha)}$$

$$\Longleftrightarrow$$

$$u_{G,t} = \frac{1}{c_t^{\alpha}} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t} \right)^{2 \cdot \alpha} \cdot e^{-(1-\alpha) \cdot \frac{1}{c_t} \cdot \frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{\tau_t}}}$$

$$u_{G,t} = \frac{e^{\frac{1}{c_t} \cdot (1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t} \right)^{2\alpha}}{c^{\alpha} \cdot e^{\frac{1}{c_t} \cdot (1-\alpha) \cdot \frac{\sqrt{B_t (p_t - c_t)}}{\sqrt{\tau_t}}}} \tag{8}$$

To maximises this utility function, we set the derivative of it with respect to τ_t to zero, and we solve for τ_t, and the pollution tax τ_t^* that maximises the objective function of the government is such that[2] :

$$\tau_t^* = \frac{B_t (p_t - c_t)(1-\alpha) \left[\alpha \cdot 4 \cdot c_t + 1 - \alpha - \sqrt{(1-\alpha)^2 + \alpha \cdot 8 \cdot c_t (1-\alpha)} \right]}{\alpha^2 \cdot 8 \cdot c_t^2} \tag{9}$$

After analysis of this term, it appears that :

- It has a negative relationship with the manufacturing cost c_t
- It has a negative relationship with the preference of the governement for economic activity α
- It has a positive relationship with the budget of the firm cost B_t
- It has a positive relationship with the selling price p_t

[2]You can find the proof of this result here.

3.3 Interpretations and extensions

The baseline model happens in only one period, in a simultaneous way, with perfect imperfection. In reality, information is imperfect, decisions of the government and of the firm are not simultaneous, and pollution is persistent. The following extensions will try to develop a framework in which those elements are studied.

4 Appendix

4.1 Maximum Profit

$$\pi_t = \frac{m_t}{c_t} \cdot (p_t - c_t) - \tau_t \cdot \frac{m_t}{c_t} \cdot \frac{1}{r_t}$$
$$\Longleftrightarrow$$
$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \frac{\tau_t}{r_t}\right)$$
$$\Longleftrightarrow$$
$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \frac{\tau_t}{\sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}}}\right)$$
$$\Longleftrightarrow$$
$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \sqrt{\frac{\tau_t(p_t - c_t)}{B_t}}\right)$$
$$\Longleftrightarrow$$
$$\pi_t = \frac{B_t - \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}}}{c_t} \cdot \left(p_t - c_t - \sqrt{\frac{\tau_t(p_t - c_t)}{B_t}}\right)$$
$$\Longleftrightarrow$$
$$\pi_t = \frac{1}{c_t} \cdot \sqrt{B_t}\left(\sqrt{B_t} - \sqrt{\frac{\tau_t}{p_t - c_t}}\right) \cdot \left(\frac{\sqrt{B_t}(p_t - c_t) - \sqrt{\tau_t(p_t - c_t)}}{\sqrt{B_t}}\right)$$
$$\Longleftrightarrow$$
$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t} - \sqrt{\frac{\tau_t}{p_t - c_t}}\right) \cdot \left(\sqrt{B_t}(p_t - c_t) - \sqrt{\tau_t(p_t - c_t)}\right)$$
$$\Longleftrightarrow$$
$$\pi_t = \frac{1}{c_t} \cdot \left(\frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{p_t - c_t}}\right) \cdot \left(\sqrt{B_t}(p_t - c_t) - \sqrt{\tau_t(p_t - c_t)}\right)$$
$$\Longleftrightarrow$$
$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}\right) \cdot \left(\sqrt{B_t}(\sqrt{p_t - c_t}) - \sqrt{\tau_t}\right)$$
$$\Longleftrightarrow$$
$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}\right)^2 \tag{6}$$

4.2 Optimal pollution tax

To find the optimla pollution tax, we set the derivative of the social utility function to zero :

$$u_{G,t} = \frac{e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha}}{c^\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}}}$$

We see that this function is of form $f(\tau_t) = \frac{u(\tau_t)}{v(\tau_t)}$, thus its derivative is of form $f'(\tau_t) = \frac{u'(\tau)v(\tau_t)-v'(\tau_t)u(\tau_t)}{v^2(\tau_t)}$. To have its derivative equal to zero, we thus only need $u'(\tau_t)v(\tau_t) - v'(\tau_t)u(\tau_t) = 0$, with :

$$\begin{cases} u(\tau_t) = e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha} \\ v(\tau_t) = c^\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}} \end{cases}$$

$$\iff$$

$$\begin{cases} \frac{\partial u(\tau_t)}{\partial \tau_t} = e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(-\frac{1}{2}\right) \cdot 2\alpha \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \\ \frac{\partial v(\tau_t)}{\partial \tau_t} = c_t^\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}} \cdot \frac{1}{c_t}(1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \left(-\frac{1}{2}\right) \cdot \tau_t^{-\frac{3}{2}} \end{cases}$$

$$\iff$$

$$\begin{cases} \frac{\partial u(\tau_t)}{\partial \tau_t} = -\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \\ \frac{\partial v(\tau_t)}{\partial \tau_t} = -\frac{c_t^{\alpha-1}}{2} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \end{cases}$$

Thus we can compute :

$$u'(\tau_t)v(\tau_t) - v'(\tau_t)u(\tau_t) = 0$$

$$\iff$$

$$-\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \cdot c^\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}}$$
$$+ \frac{c_t^{\alpha-1}}{2} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)\cdot\frac{\sqrt{B_t(p_t-c_t)}}{\sqrt{\tau_t}}} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha} = 0$$

$$\iff$$

$$\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha-1} \cdot c^\alpha =$$
$$\frac{c_t^{\alpha-1}}{2} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{2\alpha}$$

$$\iff$$

$$\alpha \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)} \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} \cdot c^\alpha =$$
$$\frac{c_t^{\alpha-1}}{2} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}} \cdot e^{\frac{1}{c_t}\cdot(1-\alpha)}$$

$$\iff$$

$$\alpha \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} \cdot c^{\alpha} =$$
$$\frac{c_t^{\alpha-1}}{2} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}}$$
$$\Longleftrightarrow$$
$$\alpha \cdot \tau_t^{-\frac{1}{2}} \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} = \frac{1}{2 \cdot c_t} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-\frac{3}{2}}$$
$$\Longleftrightarrow$$
$$\alpha \cdot \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)^{-1} = \frac{1}{2 \cdot c_t} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-1}$$
$$\Longleftrightarrow$$
$$\alpha = \left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right) \cdot \frac{1}{2 \cdot c_t} \cdot (1-\alpha) \cdot \sqrt{B_t(p_t - c_t)} \cdot \tau_t^{-1}$$
$$\Longleftrightarrow$$
$$\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha) \cdot \sqrt{B_t(p_t - c_t)}} = \frac{\left(\sqrt{B(p_t - c_t)} - \sqrt{\tau_t}\right)}{\tau_t}$$
$$\Longleftrightarrow$$
$$\tau_t \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha) \cdot \sqrt{B_t(p_t - c_t)}} = \sqrt{B(p_t - c_t)} - \sqrt{\tau_t}$$
$$\Longleftrightarrow$$
$$\tau_t \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + \sqrt{\tau_t} \cdot \sqrt{B_t(p_t - c_t)} - B_t(p_t - c_t) = 0$$
$$\Longleftrightarrow$$
$$\tau_t \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} - B_t(p_t - c_t) = -\sqrt{\tau_t} \cdot \sqrt{B_t(p_t - c_t)}$$
$$\Longleftrightarrow$$
$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 - 2 \cdot \tau_t \cdot B_t\left(p_t - c_t\right) \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + \left(B_t(p_t - c_t)\right)^2 = \tau_t \cdot B_t(p_t - c_t)$$
$$\Longleftrightarrow$$
$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 + \tau_t \cdot (-2) \cdot B_t\left(p_t - c_t\right) \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} - \tau_t \cdot B_t(p_t - c_t) + \left(B_t(p_t - c_t)\right)^2 = 0$$
$$\Longleftrightarrow$$
$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 + \tau_t \cdot B_t\left(p_t - c_t\right)\left((-2) \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} - 1\right) + \left(B_t(p_t - c_t)\right)^2 = 0$$
$$\Longleftrightarrow$$
$$\tau_t^2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)}\right)^2 + \tau_t \cdot (-B_t)\left(p_t - c_t\right)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) + \left(B_t(p_t - c_t)\right)^2 = 0$$

This is a standard quadratic equation $\tau_t^2 \cdot a + \tau_t \cdot b + c = 0$ that accepts two solutions :

$$\tau^* = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

With :

- $a = \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2$
- $b = (-B_t)(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)$
- $c = (B_t(p_t - c_t))^2$

The solution $\tau^* = \frac{-b+\sqrt{b^2+4 \cdot a \cdot c}}{2 \cdot a}$ does not make sense in the context of this problem, so we have the optimal tax :

$$\tau_t^* = \frac{-b - \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - \sqrt{\left[(-B_t)(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)\right]^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2 \cdot (B_t(p_t - c_t))^2}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - \sqrt{(B_t(p_t - c_t))^2\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2 \cdot (B_t(p_t - c_t))^2}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - \sqrt{(B_t(p_t - c_t))^2\left[\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2\right]}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right) - B_t(p_t - c_t)\sqrt{\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{(1-\alpha)} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left[\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{1-\alpha} + 1\right) - \sqrt{\left(2 \cdot \frac{\alpha \cdot 2 \cdot c_t}{1-\alpha} + 1\right)^2 - 4 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}\right]}{2 \cdot \left(\frac{\alpha \cdot 2 \cdot c_t}{1-\alpha}\right)^2}$$

$$\Longleftrightarrow$$

$$\tau_t^* = \frac{B_t(p_t - c_t)\left[\left(\frac{\alpha \cdot 4 \cdot c_t + 1 - \alpha}{1-\alpha}\right) - \sqrt{\left(\frac{\alpha \cdot 4 \cdot c_t + 1 - \alpha}{1-\alpha}\right)^2 - \frac{\alpha^2 \cdot 16 \cdot c_t^2}{(1-\alpha)^2}}\right]}{\frac{\alpha^2 \cdot 8 \cdot c_t^2}{(1-\alpha)^2}}$$

$$\Longleftrightarrow$$

$$
\begin{gathered}
\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left[\left(\frac{\alpha\cdot 4\cdot c_t + 1-\alpha}{1-\alpha}\right) - \sqrt{\frac{1}{(1-\alpha)^2}\left(\left(\alpha\cdot 4\cdot c_t + 1 - \alpha\right)^2 - \left(\alpha^2\cdot 16\cdot c_t^2\right)\right)}\right]}{\frac{\alpha^2\cdot 8\cdot c_t^2}{(1-\alpha)^2}} \\
\iff \\
\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left[\left(\frac{\alpha\cdot 4\cdot c_t + 1-\alpha}{1-\alpha}\right) - \frac{1}{(1-\alpha)}\sqrt{\alpha^2\cdot 16\cdot c_t^2 + (1-\alpha)^2 + 2\cdot\alpha\cdot 4\cdot c_t(1-\alpha) - \alpha^2\cdot 16\cdot c_t^2}\right]}{\frac{\alpha^2\cdot 8\cdot c_t^2}{(1-\alpha)^2}} \\
\iff \\
\tau_t^* = \frac{B_t\left(p_t - c_t\right)\left(\frac{1}{1-\alpha}\right)\left[\alpha\cdot 4\cdot c_t + 1 - \alpha - \sqrt{(1-\alpha)^2 + 2\cdot\alpha\cdot 4\cdot c_t(1-\alpha)}\right]}{\frac{\alpha^2\cdot 8\cdot c_t^2}{(1-\alpha)^2}} \\
\iff \\
\tau_t^* = \frac{B_t\left(p_t - c_t\right)(1-\alpha)\left[\alpha\cdot 4\cdot c_t + 1 - \alpha - \sqrt{(1-\alpha)^2 + \alpha\cdot 8\cdot c_t(1-\alpha)}\right]}{\alpha^2\cdot 8\cdot c_t^2}
\end{gathered}
\tag{9}
$$