# Microeconomics Project

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## 1 Baseline model

This model does not use strategic interactions, and takes place within a perfect information framework.

## 2 Players

This model has two agents : the Firm and the Governement.

### 2.1 Firm

#### 2.1.1 Technology of the firm

The firm is a monopoly, that has two possible inputs : R&D investment $(r)$ to diminish its pollution per unit and manufacturing investment $(m)$ to produce a quantity $q$ of its single good. The firm is also taxed on the pollution level they emit by a tax $\tau$ implemented by the government.

$$\begin{cases} g_t = \frac{1}{r_t} \\ m_t = q_t \cdot c_t \end{cases} \tag{1}$$

Where :

- $g_t$ is the pollution generated by the production of one unit of good by the firm at time $t$
- $r_t$ is the R&D investment done by the firm to reduce its unit pollution level $g_t$
- $m_t$ is the manufacturing investment of the firm, that is needed to produce a certain quantity of good $q_t$
- $q_t$ is the quantity of good produced by the firm at time $t$
- $c_t$ is the cost of manufacturing one good by the firm at time $t$

### 2.1.2 Profit of the firm

Given those technology constraints, the firm maximises its profit such that :

$$\begin{aligned} &\max_{r_t, m_t} \pi_{F,t} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t \\ &\text{subject to : } B_t \leq r_t + m_t \end{aligned} \tag{2}$$

Where :

- $q_t$ is the quantity of good produced and sold at time $t$
- $p_t$ is the price at which one unit of good is sold
- $c_t$ is the cost of production of unit of good
- $\tau_t \in (0,1)$ the share at which the one unit of good is taxed, depending on the level of pollution produced per unit, at time $_t$
- $g_t \in \mathbb{R}_+$ is the level of pollution created by the production of one unit by the firm at time $t$
- $B_t$ is the budget of the firm at time $t$

## 2.2 Government

The goal of the government is to maximise the social welfare function. The social welfare function here is defined as a utility function taking into account economic activities and the risk of environmental disaster. We define it as :

$$u_{G,t} = \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{1-\alpha} \tag{3}$$

Where :

- $\mathrm{EA}_t$ is the Economic Activity of society at time $t$
- $EQ_t$ is the Environmental Quality of the world at time $t$
- $\alpha \in (0,1)$ is the relative importance of Economic Activity compared to the Environmental Quality of the World.

This function has two main properties :

1. It has positive but decreasing returns for both Economic Activity and Environmental Quality :

$$\frac{\partial u_{G,t}}{\partial \mathrm{EA}_t} = \alpha \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{1-\alpha} > 0$$
$$\frac{\partial u_{G,t}}{\partial \mathrm{EQ}_t} = (1-\alpha) \cdot \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{-\alpha} > 0$$

$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EA}_t} = \alpha \cdot (\alpha - 1) \cdot \mathrm{EA}_t^{\alpha-2} \cdot \mathrm{EQ}_t^{1-\alpha} < 0$$

$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EQ}_t} = (1-\alpha) \cdot (-\alpha) \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{-\alpha-1} < 0$$

2. Both Economic Activity and Environmental Quality are complementary goods, as the cross-derivative is positive :

$$\frac{\partial^2 u_{G,t}}{\partial \mathrm{EA}_t \partial \mathrm{EQ}_t} = \alpha \cdot (1-\alpha) \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{-\alpha} > 0$$

# 3 Environment Quality

The environment quality $EQ_t$ at time $t$ depends of the pollution in nature at time $t$. We can define in a simple, one period context :

$$EQ_t := e^{-q_t \cdot g_t} \tag{4}$$

Where :

- $q_t \cdot g_t$ is the quantity of pollution produced at time $t$

Note here that we don't take into account past pollution in this simplified model. This will be taken into account in the next sections.

# 4 Model solution

## 4.1 Firm's problem

In this simplified model, we first solve the Firm's problem for one period :

$$\max_{r_t, m_t} \pi_{F,t} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t$$
$$\text{subject to : } B_t \leq r_t + m_t$$

We can thus write the Lagrangien such that :

$$\mathcal{L} = q_t \cdot (p_t - c_t) - \tau_t \cdot q_t \cdot g_t + \lambda \left(B - r_t - m_t\right)$$

And if we rewrite it with the extended expression of $q_t$ and $m_t$, we get :

$$\mathcal{L} = \frac{m_t}{c_t} \cdot (p_t - c_t) - \tau_t \cdot \frac{m_t}{c_t} \cdot \frac{1}{r_t} + \lambda \left(B - r_t - m_t\right)$$

We solve the First Order Conditions.
If we differentiate the Lagrangien with respect to the R&D investment $r_t$ :

$$\frac{\partial \mathcal{L}}{\partial r_t} = 0$$
$$\Longleftrightarrow$$
$$\frac{\tau_t \cdot m_t}{c_t \cdot r_t^2} - \lambda = 0$$
$$\Longleftrightarrow$$
$$\lambda = \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2}$$

If we differentiate the Lagrangien with respect to the manufacturing investment $m_t$ :

$$\frac{\partial \mathcal{L}}{\partial m_t} = 0$$
$$\Longleftrightarrow$$
$$\frac{p_t - c_t}{c_t} - \frac{\tau_t}{c_t \cdot r_t} - \lambda = 0$$
$$\Longleftrightarrow$$
$$\frac{p_t - c_t}{c_t} - \frac{\tau_t}{c_t \cdot r_t} - \frac{\tau_t \cdot m_t}{c_t \cdot r_t^2} = 0$$
$$\Longleftrightarrow$$
$$r_t^2 \cdot (p_t - c_t) - r_t \cdot \tau_t - \tau_t \cdot m_t = 0$$
$$\Longleftrightarrow$$
$$m_t = \frac{1}{\tau_t} \left( r_t^2 (p_t - c_t) - r_t \cdot \tau_t \right) = \frac{r_t^2 \, (p_t - c_t)}{\tau_t} - r_t \tag{5}$$

This equation gives us the optimal level of manufacturing investment for the firm. We can see that it has a negative relationship with $\tau$ the pollution tax, which seems logical : the bigger the pollution tax, the more incentives the firm has to invest in R&D instead of simply manufacturing goods with high pollution level.

From that, we can identify the optimal level of research $r$ for the firm :

$$B_t = r_t + m_t$$
$$\Longleftrightarrow$$
$$B_t = r_t + \frac{r_t^2 \, (p_t - c_t)}{\tau_t} - r_t$$
$$\Longleftrightarrow$$
$$B_t = \frac{r_t^2 \, (p_t - c_t)}{\tau_t}$$
$$\Longleftrightarrow$$
$$r_t = \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}} \tag{6}$$

Thus, the maximum profit of the firm is :

$$\pi_t = \frac{m_t}{c_t} \cdot (p_t - c_t) - \tau_t \cdot \frac{m_t}{c_t} \cdot \frac{1}{r_t}$$

$$\Longleftrightarrow$$

$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \frac{\tau_t}{r_t}\right)$$

$$\Longleftrightarrow$$

$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \frac{\tau_t}{\sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}}}\right)$$

$$\Longleftrightarrow$$

$$\pi_t = \frac{m_t}{c_t} \cdot \left(p_t - c_t - \sqrt{\frac{\tau_t (p_t - c_t)}{B_t}}\right)$$

$$\Longleftrightarrow$$

$$\pi_t = \frac{B_t - \sqrt{\frac{B_t \cdot \tau_t}{p_t - c_t}}}{c_t} \cdot \left(p_t - c_t - \sqrt{\frac{\tau_t (p_t - c_t)}{B_t}}\right)$$

$$\Longleftrightarrow$$

$$\pi_t = \frac{1}{c_t} \cdot \sqrt{B_t} \left(\sqrt{B_t} - \sqrt{\frac{\tau_t}{p_t - c_t}}\right) \cdot \left(\frac{\sqrt{B_t}(p_t - c_t) - \sqrt{\tau (p_t - c_t)}}{\sqrt{B_t}}\right)$$

$$\Longleftrightarrow$$

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t} - \sqrt{\frac{\tau_t}{p_t - c_t}}\right) \cdot \left(\sqrt{B_t}(p_t - c_t) - \sqrt{\tau (p_t - c_t)}\right)$$

$$\Longleftrightarrow$$

$$\pi_t = \frac{1}{c_t} \cdot \left(\frac{\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}}{\sqrt{p_t - c_t}}\right) \cdot \left(\sqrt{B_t}(p_t - c_t) - \sqrt{\tau (p_t - c_t)}\right)$$

$$\Longleftrightarrow$$

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}\right) \cdot \left(\sqrt{B_t}(\sqrt{p_t - c_t}) - \sqrt{\tau}\right)$$

$$\Longleftrightarrow$$

$$\pi_t = \frac{1}{c_t} \cdot \left(\sqrt{B_t \cdot (p_t - c_t)} - \sqrt{\tau_t}\right)^2$$

## 4.2 Government problem

In this simplified model, we say that the Economic Activity $\mathrm{EA}_t$ is equivalent to the profit of the firm. Also, we have the following maximisation program :

$$\max_{\tau_t, g_t} u_{G,t} = \pi_{F,t}^{\alpha} \cdot (D(X_t))^{\alpha-1}$$

subject to :

$$\begin{cases} X_t = \rho \cdot X_{t-1} + q_t \cdot g_{F,t} + \varepsilon_t \\ X_0 = 0 \end{cases}$$

Thus writing all the terms we have :

$$u_{G,t} = 2$$