1 Baseline model

This model does not use strategic interactions, and takes place within a perfect information framework.

2 Players

This model has two agents : the Firm and the Governement.

2.1 Firm

The firm is a monopoly, that maximises its profit with respect to its production technology $g_{F,t}$ and its quantity of good produced q. The firm also receives a subsidy from the Government the less its good is polluting. We define its profit functions as :

$$\pi_{F,t} = q_t \cdot \big(p - c + s_t \cdot (g_t - g_{F,t}) - \beta(g_t - g_{F,t})^k\big) \tag{1}$$

Where :

- q_t is the quantity of good produced and sold at time t
- p_t is the price at which one unit of good is sold
- c is the cost of production of unit of good
- $s_t \in (0, 1)$ the share at which the one unit of good is subsidized or taxed (depending on the level of pollution produced per unit) at time $_t$
- $g_t \in \mathbb{R}_+$ is the accepted level of pollution produced by unit of good fixed by the government at time t
- $g_{F,t} \in \mathbb{R}_+$ is the level of pollution created by the production of one unit by the firm at time t
- $\beta \in \mathbb{R}_+$ is the coefficient of Research and Development (RD) cost
- $k > 1$ is the difficulty of upgrading the production technology so that it pollutes less

2.2 Government

The goal of the government is to maximise the social welfare function. The social welfare function here is defined as a utility function taking into account economic activities and the risk of environmental disaster. We define it as :

$$u_{G,t} = \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{1-\alpha} \tag{2}$$

Where :

- EA_t is the Economic Activity of society at time t
- EQ_t is the Environmental Quality of the world at time t
- $\alpha \in (0,1)$ is the relative importance of Economic Activity compared to the Environmental Quality of the World.

This function has two main properties :

1. It has positive but decreasing returns for both Economic Activity and Environmental Quality :

$$\frac{\partial u_{G,t}}{\partial \mathrm{EA}_t} = \alpha \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{1-\alpha} > 0$$
$$\frac{\partial u_{G,t}}{\partial \mathrm{EQ}_t} = (1-\alpha) \cdot \mathrm{EA}_t^{\alpha} \cdot \mathrm{EQ}_t^{-\alpha} > 0$$

$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EA}_t} = \alpha \cdot (\alpha - 1) \cdot \mathrm{EA}_t^{\alpha-2} \cdot \mathrm{EQ}_t^{1-\alpha} < 0$$
$$\frac{\partial^2 u_{G,t}}{\partial^2 \mathrm{EQ}_t} = (1-\alpha) \cdot (-\alpha) \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{-\alpha-1} < 0$$

2. Both Economic Activity and Environmental Quality are complementary goods, as the cross-derivative is positive :

$$\frac{\partial^2 u_{G,t}}{\partial \mathrm{EA}_t \partial \mathrm{EQ}_t} = \alpha \cdot (1-\alpha) \cdot \mathrm{EA}_t^{\alpha-1} \cdot \mathrm{EQ}_t^{-\alpha} > 0$$

3 Environment Quality

The environment quality EQ_t depends on the pollution in nature. We define :

$$EQ_t = \left(R(X_t) \cdot D(X_t) + d(X_t)\right)^{-1} \tag{3}$$

$$\begin{cases} X_t \sim \mathrm{AR}(1) \\ \iff \\ X_t = \rho \cdot X_{t-1} + P_{F,t} + \varepsilon_t \end{cases} \tag{4}$$

Where :

- $R(X_t) \in [0,1]$ the risk of a natural disaster happening at time t
- $D(X_t) \in \mathbb{R}_+$ the potential dammage caused by a natural disaster happening at time t
- $d(X_t) \in \mathbb{R}_+$ the normal dammage caused by pollution at time t

- $X_t \in \mathbb{R}_+$ the total quantity of pollution in nature at time t
- $\rho \in [0, 1]$ the persistence level of pollution in nature
- $P_{F,t} = q_t \cdot g_{F,t}$ the quantity of pollution produced by the firm at time t
- ε_t a white noise

4 Model solution

4.1 Firm's problem

In this simplified model, we first solve the Firm's problem for one period :

$$\max_{g_{F,t}, q_t} \pi_{F,t} = q_t \cdot (p - c + s(g_t - g_{F,t})) - \beta(g_t - g_{F,t})^k$$

We solve the First Order Conditions. If we derive with respect to the pollution level per unit produced :

$$\frac{\partial \pi_{F,t}}{\partial g_{F,t}} = 0$$
$$\Longleftrightarrow$$
$$-s_t \cdot q_t + \beta \cdot k \cdot (g_t - g_{F,t})^{k-1} = 0$$
$$\Longleftrightarrow$$
$$\beta \cdot k(g_t - g_{F,t})^{k-1} = s_t \cdot q_t$$
$$\Longleftrightarrow$$
$$\left(\frac{s_t \cdot q_t}{\beta \cdot k}^{\frac{1}{k-1}} \right) = g_t - g_{F,t}$$
$$\Longleftrightarrow$$
$$g_{F,t} = g_t - \left(\frac{s_t \cdot q_t}{\beta \cdot k} \right)^{\frac{1}{k-1}} \tag{5}$$

Then, if we derive with respect to the quantity produced :

$$\frac{\partial \pi_{F,t}}{\partial q_t} = 0$$
$$\Longleftrightarrow$$
$$p - c + s_t \cdot (g_t - g_{F,t}) - \beta \cdot (g_t - g_{F,t})^k = 0$$
$$\Longleftrightarrow$$
$$p = c - s_t \cdot (g_t - g_{F,t}) + \beta \cdot (g_t - g_{F,t})^k \tag{6}$$

Now if we plug equation (5) into (6) we get :

$$p = c - s_t \cdot \left(g_t - g_t - \left(\frac{s}{\beta \cdot k} \right)^{\frac{1}{k-1}} \right) + \beta \cdot \left(g_t - g_t - \left(\frac{s}{\beta \cdot k} \right)^{\frac{1}{k-1}} \right)^k$$

$$\Longleftrightarrow$$

$$p = c + s_t \cdot \left(\frac{s_t}{\beta \cdot k} \right)^{\frac{1}{k-1}} - \beta \left(\left(\frac{s}{\beta \cdot k} \right)^{\frac{1}{k-1}} \right)^k$$

$$\Longleftrightarrow$$

$$p = c + s_t \cdot \left(\frac{s_t}{\beta \cdot k} \right)^{\frac{1}{k-1}} - \beta \left(\frac{s}{\beta \cdot k} \right)^{\frac{k}{k-1}}$$

$$\Longleftrightarrow$$

$$p = c + s_t^{\frac{k-1}{k-1} + \frac{1}{k-1}} \cdot (\beta \cdot k)^{-\frac{1}{k-1}} - \beta^{\frac{k-1}{k-1} - \frac{k}{k-1}} \cdot \left(\frac{s}{k} \right)^{\frac{k}{k-1}}$$

$$\Longleftrightarrow$$

$$p = c + s_t^{\frac{k}{k-1}} \cdot (\beta \cdot k)^{-\frac{1}{k-1}} - \beta^{-\frac{1}{k-1}} \cdot \left(\frac{s}{k} \right)^{\frac{k}{k-1}}$$

$$\Longleftrightarrow$$

$$p = c + s_t^{\frac{k}{k-1}} \cdot \left[(\beta \cdot k)^{-\frac{1}{k-1}} - \beta^{-\frac{1}{k-1}} \cdot \left(\frac{1}{k} \right)^{\frac{k}{k-1}} \right]$$

$$\Longleftrightarrow$$

$$p = c + s_t^{\frac{k}{k-1}} \cdot \beta^{-\frac{1}{k-1}} \cdot \left(k^{\frac{1}{k-1}} - \left(\frac{1}{k} \right)^{\frac{k}{k-1}} \right)$$

$$\Longleftrightarrow$$

$$p = c + \left(\frac{s_t^k}{\beta} \right)^{\frac{1}{k-1}} \cdot \left(k^{\frac{1}{k-1}} - k^{-\frac{k}{k-1}} \right)$$

$$\Longleftrightarrow$$

$$p = c + \left(\frac{s_t^k}{\beta} \right)^{\frac{1}{k-1}} \cdot k^{\frac{1}{k-1}} \cdot \left(1 - k^{-k} \right)$$

$$\Longleftrightarrow$$

$$p = c + \left(\frac{s_t^k \cdot k}{\beta} \right)^{\frac{1}{k-1}} \cdot \left(1 - k^{-k} \right) \tag{7}$$

Therefore, the maximised profit of the firm will be :

$$\pi_{F,t} = q_t \cdot \left(p - c + s_t \cdot (g_t - g_{F,t}) - \beta \left(g_t - g_{F,t} \right)^k \right)$$

$$\Longleftrightarrow$$

$$
\begin{aligned}
\pi_{F,t} =& q_t \cdot (c + \left(\frac{s_t^k \cdot k}{\beta}\right)^{\frac{1}{k-1}} \cdot \left(1 - k^{-k}\right) - c + s_t \left[g_t - g_t - \left(\frac{s_t}{\beta \cdot k}\right)^{\frac{1}{k-1}}\right] \\
& - \beta \left[g_t - g_t - \left(\frac{s_t}{\beta \cdot k}\right)^{\frac{1}{k-1}}\right]^k)
\end{aligned}
$$

$$\Longleftrightarrow$$

$$
\pi_{F,t} = q_t \cdot \left(\left(\frac{s_t^k \cdot k}{\beta}\right)^{\frac{1}{k-1}} \cdot \left(1 - k^{-k}\right) - s_t \left[\left(\frac{s_t}{\beta \cdot k}\right)^{\frac{1}{k-1}}\right] + \beta \left[\left(\frac{s_t}{\beta \cdot k}\right)^{\frac{1}{k-1}}\right]^k\right)
$$

$$\Longleftrightarrow$$

$$
\pi_{F,t} = q_t \cdot \left(\left(\frac{s_t^k \cdot k}{\beta}\right)^{\frac{1}{k-1}} \cdot \left(1 - k^{-k}\right) - \left(\frac{s_t^k \cdot k}{\beta}\right)^{\frac{1}{k-1}} \cdot \left(1 - k^{-k}\right)\right)
$$

$$\Longleftrightarrow$$

$$
\pi_{F,t} = q_t \cdot 0 \iff \pi_{F,t} = 0 \tag{8}
$$

At the optimal, the profit of the firm will be null.

4.2 Government problem

In this simplified model, we say that the Economic Activity EA_t is equivalent to the profit of the firm. Also, we have the following maximisation program :

$$
\max_{s_t, g_t} u_{G,t} = \pi_{F,t}^{\alpha} \cdot (R(X_t) \cdot D(X_t) + d(X_t))^{\alpha - 1}
$$

subject to :

$$
\begin{cases}
s_t \left(g_t - g_{F,t}\right) q_t \leq \pi_{F,t} \\
X_t = X_{t-1} + q_t \cdot g_{F,t} + \varepsilon_t \\
X_0 = 0
\end{cases}
$$