Chapter 3

Alon-Boppana Theorem

3.1 Statement and Consequences

Let X be a d-regular graph. From chapter 1, we know

$$\frac{d - \lambda_1(X)}{2} \leq h(X) \leq \sqrt{2d(d - \lambda_1(X))}$$

Hence, to construct good communication network, our goal is to find graphs with small λ_1. The main result of this chapter places a constraint on how small λ_1 can be.

Proposition 3.1. If $\{X_n\}$ is a sequence of connected d-regular graphs with $|X_n| \to \infty$ as $n \to \infty$, then

$$\liminf_{n \to \infty} \lambda_1(X_n) \geq 2\sqrt{d-1}$$

That is, for every $\epsilon \geq 0$, there exists an $N > 0$ s.t. $\lambda_1(X_n) > 2\sqrt{d-1} - \epsilon$ for all $n > N$

Remark
$\lambda_1(X_n)$ is at best a little bit smaller than $2\sqrt{d-1}$. Once we know the very first few terms of $\{\lambda_1(X)\}$, we know the (best possible) quality of the graph.

Definition 3.2. Suppose that X is a d-regular graph with n vertices.

- If X is not bipartite, **trivial eigenvalues** is $\lambda_0(X) = d$.
- If X is bipartite, **trivial eigenvalues** are $\lambda_0(X) = d$ and $\lambda_{n-1}(X) = -d$.

Definition 3.3. Suppose that X is a d-regular graph with n vertices.

- If X is not bipartite, define $\lambda(X) = \max\{|\lambda_1(X)|, |\lambda_{n-1}(X)|\}$.
- If X is bipartite, define $\lambda_(X) = \max\{|\lambda_1(X)|, |\lambda_{n-2}(X)|\}$.

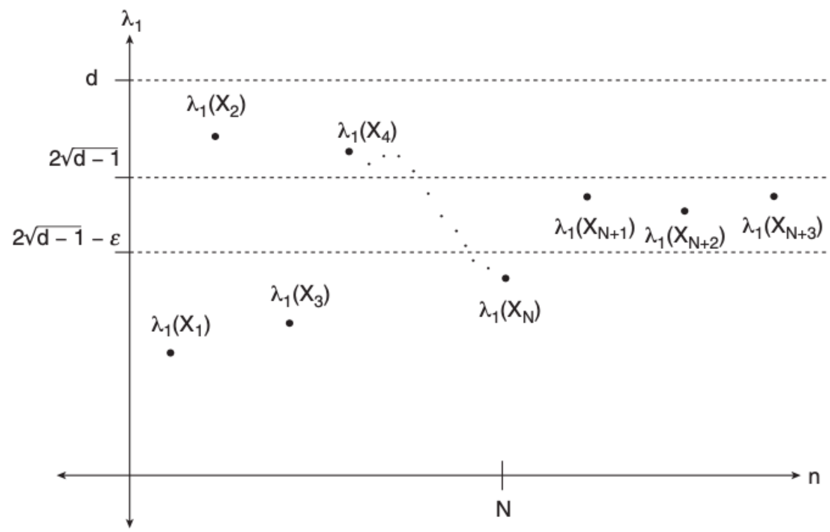

Figure 3.1:

Remark

- If X is disconnected, then $\lambda(X) = d$, since in disconnected graphs $\lambda_1(X) = \lambda_0(X)$. If X is connected,

$$\lambda(X) = \max\{|\lambda| : \lambda \text{ is a nontrivial eigenvalue of} X\}$$

- $\lambda(X) \geq \lambda_1(X)$. Hence bounding $\lambda_1(X)$ below is bounding $\lambda(X)$ below.
- $d - \lambda(X_n) \leq d - \lambda_1(X_n)$, thus $\{X_n\}$ is an expander family if $d - \lambda(X_n)$ is bounded below by some positive constant.

Proposition 3.4. (Alon-Boppana)
$\{X_n\}$ is a sequence of connected d-regular graphs with $|X_n| \to \infty$ as $n \to \infty$, then

$$\liminf_{n \to \infty} \lambda(X_n) \geq 2\sqrt{d-1}$$

Remark
The best (possible) upper bound for $\lambda(X)$ is $2\sqrt{d-1}$, which corresponds to the best (possible) lower bound for $d - \lambda(X_n)$.

Definition 3.5. d-regular graph X is **Ramanujan** if $\lambda(X) \leq 2\sqrt{d-1}$

Remark

- For $d \geq 3$, Ramanujan graph must be connected, otherwise $\lambda(X) = d > 2\sqrt{d-1}$.
- Suppose $\{X_n\}$ is a sequence of d-regular Ramanujan graphs. Then $\lambda_1(X) \leq \lambda(X) \leq 2\sqrt{d-1}$. With $d \geq 3$,

$$h(X_n) \geq \frac{d - \lambda_1(X_n)}{2} \geq \frac{d - 2\sqrt{d-1}}{2} > 0$$

  If $d \geq 3$, then any sequence of d-regular Ramanujan graphs is an expander family.

Proposition 3.6. Let X be a non-bipartite, d-regular graph with vertex set V, A be the adjacency operator of X. Then

$$\lambda(X) = \max_{\substack{f \in L_0^2(V) \\ ||f||_2 = 1}} |\langle Af, f\rangle_2| = \max_{f \in L_0^2(V)} \frac{|\langle Af, f\rangle_2|}{\langle f, f\rangle_2}$$

Proof

- Let $n = |V|$. By spectrum theorem, there is an orthonormal basis $\{f_0, f_1, \ldots, f_{n-1}\}$ s.t. f_i is associated with eigenvalue $\lambda_i = \lambda_i(X)$. Also, f_0 is constant on V.
- Let $f \in L_0^2(V)$ s.t. $||f||_2 = 1$. As in the eigenvalue version of Rayleigh-Ritz proposition, $f = c_1 f_1 + \cdots + c_{n-1} f_{n-1}$ for some $c_i \in \mathbb{C}$.
- Hence

$$\begin{aligned}
|\langle Af, f\rangle_2| &= |\langle \sum_{i=1}^{n-1} c_i \lambda_i f_i, \sum_{j=1}^{n-1} c_j f_j \langle_2| \\
&\leq \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_i \overline{c_j} |\lambda_i \langle f_i, f_j\rangle_2| \\
&= \sum_{i=1}^{n-1} c_i \overline{c_i} |\lambda_i| \\
&\leq \lambda(X) \sum_{i=1}^{n-1} c_i \overline{c_i} = \lambda(X) ||f||_2^2 = \lambda(X)
\end{aligned}$$

  .

- So

$$\lambda(X) \geq \max_{\substack{f \in L_0^2(V) \\ ||f||_2 = 1}} |\langle Af, f\rangle_2|$$

- Now we show the upper bound can be achieved. If $\lambda(X) = \lambda_1(X)$, let $f = f_1$. Otherwise, let $f = f_{n-1}$.

Remark

- It can be shown:
$$\lambda(X) = \max_{f \in L_0^2(V,\mathbb{R})} \frac{|\langle Af, f\rangle_2|}{\langle f, f\rangle_2}$$
which corresponds to the restricted version of Rayleigh-Ritz.

- Let X be a bipartite d-regular graph with vertex set V. Then
$$\lambda(X) \leq \max_{\substack{f \in L_0^2(V) \\ ||f||_2=1}} |\langle Af, f\rangle_2| = d$$
with equality iff X is disconnected.

3.2 First Proof: The Rayleigh-Ritz Method

A proof for Proposition 3.1.

Proposition 3.7. Let X be a connected d-regular graph. If $diam(X) \geq 4$, then
$$\lambda_1(X) > 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2} diam(X) - 1 \rfloor}$$

Proof

- We first show Proposition 3.7. implies Proposition 3.1.
- Fix a vertex v of X. The number of walks of length 1 starting at v is d. Length 2 is d^2
- In general, the number of walks of length a starting at v is d^a. Note that a walk of length a contains at most $a+1$ vertices.
- We can cover the entire graph by taking all walks of length $diam(X)$ from our fixed vertex v. There are $d^{diam(X)}$ such walks,each containing at most $diam(X)+1$ vertices.
- Hence
$$|X| \leq (diam(X)+1)d^{diam(X)}$$
- Let $\{X_n\}$ be a sequence of connected, d-regular graphs s.t. $|X_n| \to \infty$ as $n \to \infty$, so
$$\frac{2\sqrt{d-1}-1}{\lfloor \frac{1}{2} diam(X) - 1 \rfloor}$$
approaches 0 as $n \to \infty$. Proposition 3.1 follows.

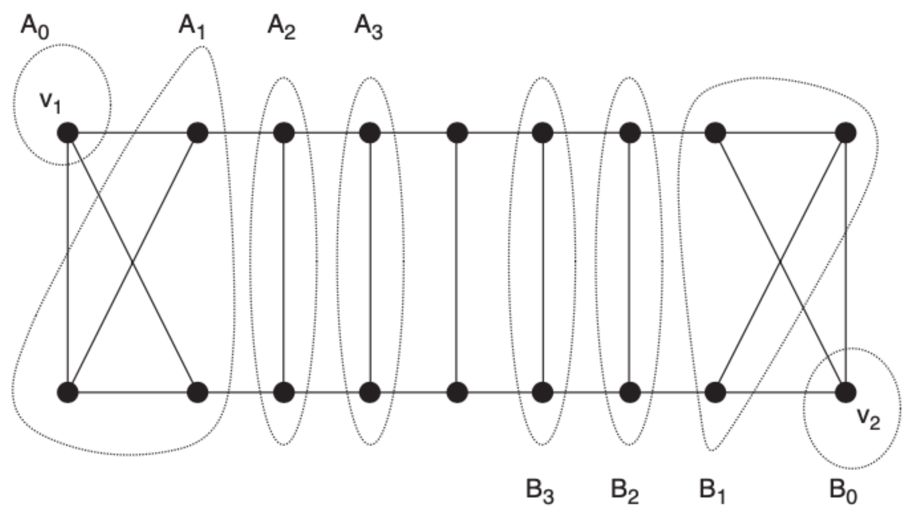

Figure 3.2:

Proof
For Proposition 3.7.

- Proof idea: Firstly, we pick two vertices v_1, v_2 that are as far apart as possible in the graph. Secondly, construct $f \in L_0^2(V, \mathbb{R})$ that has local maxima at v_1, v_2 and decreases rapidly as we move away from v_1, v_2. By Rayleigh-Ritz, $\lambda_1(X) \geq d - \frac{\langle \Delta f, f \rangle_2}{\langle f, f \rangle_2}$. Thirdly, we calculate $\langle f, f \rangle_2$. Lastly, we give an upper bound on $\langle \Delta f, f \rangle_2$, thus having a lower bound for $\lambda_1(X)$

- Throughout the proof, let $b = \lfloor \frac{1}{2} diam(X) - 1 \rfloor$, $q = d - 1$, edge multiset E, vertex set V.

- Step 1: Construct f.
  - Let v_1, v_2 be two vertices s.t. $dist(v_1, v_2) \geq 2b + 2$. Define the sets

$$A_i = \{v \in V \mid dist(v, v_1) = i\}$$
$$B_i = \{v \in V \mid dist(v, v_2) = i\}$$

    for $i = 0, 1, \cdots, b$
  - Figure 3.2 provides an example for this construction in a 3-regular graph.
  - $A_0, A_1, \cdots, A_b, B_1, \cdots, B_b$ are disjoint sets.
  - Suppose not, then $x \in A_i \cap B_j$ for some $0 \leq i, j \leq b$ ($A_i \cap A_j$ is empty by definition, same for B). Then, $dist(v_1, v_2) \leq dist(v_1, x) + dist(x, v_2) \leq 2b < 2b + 2$, contradiction.

- Observation (useful for step 3): If $x \in A_i$ where $1 \leq i \leq b-1$, then there is at least one vertex from A_{i-1} that is adjacent to x, and at most q vertices from A_{i+1} that are adjacent to x.
- Let
$$A = \bigcup_{i=0}^{b} A_i \quad \text{and} \quad B = \bigcup_{i=0}^{b} B_i$$
No vertex of A is adjacent to a vertex of B.
- Suppose $x \in A, y \in B$ are adjacent. Then $\mathrm{dist}(v_1, v_2) \leq \mathrm{dist}(v_1, x) + \mathrm{dist}(x, y) + \mathrm{dist}(y, v_2) \leq 2b+1 < 2b+2$, contradiction.
- Define $f \in L_0^2(V, \mathbb{R})$ as
$$f(x) = \begin{cases} \alpha & x \in A_0 \\ \alpha q^{-(i-1)/2} & x \in A_i for i \geq 1 \\ 1 & x \in B_0 \\ q^{-(i-1)/2} & x \in B_i for i \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$
where $\alpha \in \mathbb{R}$ will be chosen next.
- Let f_0 be the function constant on 1 on all vertices, then
$$\begin{aligned} \langle f, f_0 \rangle_2 &= \sum_{x \in V} f(x) f_0(x) \\ &= \alpha \left(|A_0| + \sum_{i=1}^{b} q^{-(i-1)/2} |A_i| \right) + \left(|B_0| + \sum_{i=1}^{b} q^{-(i-1)/2} |B_i| \right) \\ &= \alpha c_0 + c_1 \end{aligned}$$
for some real numbers $c_0, c_1 > 0$. Let $\alpha = -c_1/c_0$, then we have $\langle f, f_0 \rangle_2 = 0$

- Step 2: Compute $\langle f, f \rangle_2$.
$$\begin{aligned} \langle f, f \rangle_2 = \sum_{x \in V} f(x) \overline{f(x)} &= \sum_{i=0}^{b} \sum_{x \in A_i} |f(x)|^2 + \sum_{i=0}^{b} \sum_{x \in B_i} |f(x)|^2 \\ &= S_A + S_B \end{aligned}$$
where (by plugging in f)
$$S_A = \alpha^2 + \sum_{i=1}^{b} |A_i| \alpha^2 q^{-(i-1)}$$
$$S_B = 1 + \sum_{i=1}^{b} |B_i| q^{-(i-1)}$$
clearly S_A and S_B are positive.

- Step 3: Upper bound for $\langle \Delta f, f\rangle_2$
  - Give an orientation to the edges of X. Recall from chapter 1 that Laplacian does not depend on orientation. From Proposition 1.21, we have that

$$\langle \Delta f, f\rangle_2 = \sum_{e\in E}(f(e^+) - f(e^-))^2 = C_A + C_B$$

where

$$\begin{aligned} C_A &= \sum_{\substack{e\in E \\ e^+ \text{ or } e^- \in A}} (f(e^+) - f(e^-))^2 \\ C_B &= \sum_{\substack{e\in E \\ e^+ \text{ or } e^- \in B}} (f(e^+) - f(e^-))^2 \end{aligned}$$

  - Then for C_A

$$\begin{aligned} C_A &= \sum_{i=0}^{b-1}\sum_{x\in A_i}\sum_{y\in A_{i+1}} A_{x,y}(f(x)-f(y))^2 + \sum_{x\in A_b}\sum_{y\notin A} A_{x,y}(f(x)-0)^2 \\ &\le \sum_{i-1}^{b-1} q|A_i|(q^{-(i-1)/2} - q^{-i/2})^2\alpha^2 + q|A_b|q^{-(b-1)}\alpha^2 \\ &= \alpha^2\sum_{i-1}^{b-1} q|A_i|(q^{1/2}-1)^2 q^{-i} + \alpha^2((q^{1/2}-1)^2 + 2q^{1/2} - 1)|A_b|q^{-(b-1)} \\ &= \alpha^2(q^{1/2}-1)^2(\sum_{i=1}^{b}|A_i|q^{-(i-1)}) + \alpha^2(2q^{1/2}-1)|A_b|q^{-(b-1)} \\ &= (q^{1/2}-1)^2(S_A - \alpha^2) + \alpha^2(\frac{2\sqrt{q}-1}{b})b|A_b|q^{-(b-1)} \\ &\le (q^{1/2}-1)^2(S_A-\alpha^2) + (\frac{2\sqrt{q}-1}{b})(S_A-\alpha^2) \\ &= (q+1-2\sqrt{q} + \frac{2\sqrt{q}-1}{b})(S_A-\alpha^2) \\ &< (q+1-2\sqrt{q} + \frac{2\sqrt{q}-1}{b})S_A \end{aligned}$$

0:1 Adjacency relationship comes from definition of A_i

1:2 Plug in f. When $i=0$, $f(x)-f(y)=0$. Also, by observation, for $x\in A_i$, at most q adjacent vertices in A_{i+1}. Similar for $x\in A_b$, at most q.

2:3 $(q^{-(i-1)/2} - q^{-i/2})^2 = (q^{1/2}-1)^2 q^{-i}$, and $q = (q^{1/2}-1)^2 + 2q^{1/2} - 1$

3:4 Combine part of latter term into the former, so the former is the sum from 1 to b (instead of b-1).

4:5 Note $S_A - \alpha^2 = \alpha^2 \sum_{i=1}^{b} |A_i| q^{-(i-1)}$.

5:6 Again by observation, for $x \in A_i$, at most q vertices from A_{i+1} are adjacent to x. Since every vertex of A_{i+1} must be adjacent to some vertex in A_i, we have $|A_{i+1}| \leq q|A_i|$ for $1 \leq i \leq b-1$. Same argument works for B_i as well. So

$$|A_1| \geq q^{-1}|A_2| \geq \cdots \geq q^{-(b-1)}|A_b|$$

so $q^{-(b-1)}|A_b|$ is the minimum among all such forms, thus,

$$\alpha^2 b |A_b| q^{-(b-1)} = \alpha^2 \sum_{i=1}^{b} |A_b| q^{-(b-1)} \leq \alpha^2 \sum_{i=1}^{b} |A_i| q^{-(i-1)} = S_A - \alpha^2$$

6:7 Just computation. Also, since X is connected (thus $\mathrm{diam}(X)$ finite) and $\mathrm{diam}(X) \geq 4$, we have $d \geq 2$ (only connected $d = 1$ regular graph has only one edge). Then, $(2\sqrt{q} - 1)/b > 0$ and $(q^{1/2} - 1)^2 > 0$. Hence the coefficient is positive.

7:8 Positive coefficient and positive $S_A - \alpha^2$

– Similarly,

$$C_B < (q + 1 - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b}) S_B$$

– Hence

$$\langle \Delta f, f \rangle_2 = C_A + C_B < (q + 1 - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b})(S_A + S_B)$$

- Step 4: Rayleigh-Ritz, put $\langle \Delta f, f \rangle_2$ and $\langle f, f \rangle_2$ together.

$$\begin{aligned} d - \lambda_1(X) &= \min_{\substack{g \in L_0^2(V) \\ \|g\|_2 = 1}} \langle \Delta g, g \rangle_2 \\ &\leq \frac{\langle \Delta f, f \rangle_2}{\langle f, f \rangle_2} \\ &= \frac{C_A + C_B}{S_A + S_B} \\ &< (q + 1 - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b}) \ \text{(by final result of Step 3)} \\ &= d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{b} \end{aligned}$$

Hence

$$\lambda_1(X) > 2\sqrt{d-1} - \frac{2\sqrt{d-1} - 1}{\lfloor \frac{1}{2} diam(X) - 1 \rfloor}$$