Chapter 1 Graph Eigenvalues and the Isoperimetric Constant

1. Definitions from Graph Theory

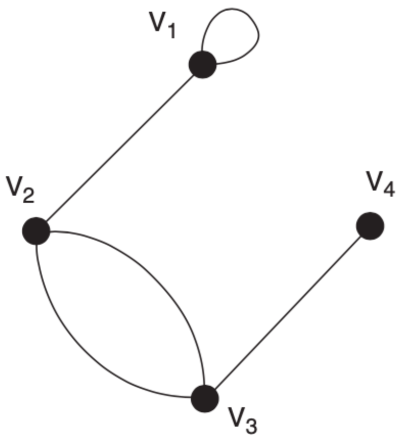
> **Definition 1.1 Multiset**
> A collection of objects where objects may appear more than once.
> Multiplicity: number of times appear for that object

Example
S = {a, a, 4, a, -1, -1, x, 15}
Multiplicity of a is 3, of -1 is 2. Size of S, denoted as |S|, is 8.

> **Definition 1.2 Graph**
>
> 1. A graph is composed of a vertex set V and an edge multiset E.
> 2. The vertex set V can be any set.
> 3. The edge multiset E is a multiset whose elements are sets of the form {v, w} or {v} where v and w are distinct vertices.
> 4. An edge of the form {v} is called a loop.
> 5. If {v, w} ∈ E, then v and w are adjacent or neighbours, and edge {v, w} is incident to v and w. Similarly, {v} is adjacent to itself and {v} is incident to v.
> 6. Degree of vertex v, deg(v) = number of edges e ∈ E s.t. v is incident to e.
> 7. For loops, we specify that one loop contribute 1 to deg(v)
> 8. Order of a graph X, |X| = number of vertices in the graph.

Remark:
When we specify loop as only contributing one to deg(v), we have cancelled Euler's Theorem, which states that in connected graphs: Eulerian <=> degree of each vertex is even.

Example:



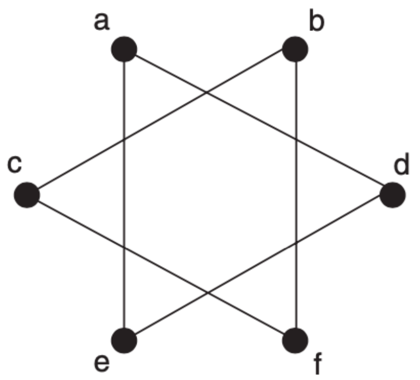
> **Definition 1.3**
>
> 1. Multigraph: graph has multiple edges (i.e. two distinct edges connecting same pair of vetrices)
> 2. Digraph: graphs in which edges have directions
> 3. d-regular graph: every vertex has degree d

> **Definition 1.4 Walk and Connectedness**
>
> 1. Walk: finite alternating sequence of vertices and edges, of the form below, where $v_i \in$ V and $e_i \in$ E, v_i is adjacent to v_{i+1} with edge e_i
> $$w = (v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n)$$
> 2. Connected: A graph X with vertex set V is connected if for every x, y ∈ V there exists a walk from x to y. Otherwise, disconnected.

> **Definition 1.5 Bipartite Graph**
> A graph X with vertex set V is bipartite if there exist V_1, $V_2 \subset$ V such that
>
> 1. V = $V_1 \cup V_2$,
> 2. $V_1 \cap V_2$ = ∅,
> 3. every edge of X is incident to a vertex in V_1 and a vertex in V_2.
> In this case, (V_1, V_2) is a bipartition of V.

Remark:
Bipartite graph <=> vertices of X can be coloured with two colours s.t. no adjacent vertices have the same colour.

Example



> **Definition 1.6 Distance**
>
> 1. Distance between x and y, dist(x, y) = minimal length of walk between x and y
> 2. Diameter of X, diam(X) = $\max_{x,y\in V} dist(x,y)$

Remark:

1. If x and y are not connected, dist(x, y) is infinity. Same for diam(X) in disconnected graphs.
2. dist defines a metric space on V

2. Cayley Graphs

Properties of Cayley graphs are connected with properties of the group.

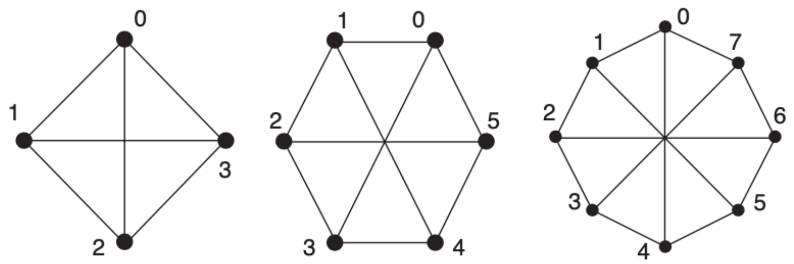
> **Definition 1.7 Symmetric multiset of a group**
> Let G be a group and Γ be a multi-subset of G. We say that Γ is symmetric if whenever y is an element of Γ with multiplicity n, then y^{-1} is an element of Γ of multiplicity n. (handwrite the notation)

> **Definition 1.8 Cayley Graph**
> Group G and symmetric multiset Γ. The Cayley graph of G with respect to Γ, $Cay(G,\Gamma)$, is defined as follows:
>
> 1. Vertices: Elements of G.
> 2. Edges: x, $y \in G$ are adjacent iff exists $\gamma \in \Gamma$ such that $x = y\gamma$. (In other words, $y^{-1}x = \gamma$.)
> 3. Multiplicity of the edge: multiplicity of $y^{-1}x$ in Γ.

Remark:

1. It's necessary for Γ to be symmetric. Due to transitivity of adjacency, $x = y\gamma$ leads to $y = x\gamma'$, thus $\gamma = \gamma'$
2. If we relax Γ to any multiset, then we get a directed Cayley graph.

Example



Cay($\mathbb{Z}_4$,{1,2,3}) Cay($\mathbb{Z}_6$,{1,3,5}) Cay($\mathbb{Z}_8$,{1,4,7})

> **Proposition 1.9**
> Group G and symmetric multiset Γ. Then:
>
> 1. $Cay(G,\Gamma)$ is $|\Gamma|$-regular.
> 2. $Cay(G,\Gamma)$ is connected <=> Γ generates G as a group.

Proof ideas:

- for any g in G, g multiply each element of Γ generates a different edge.
- Γ generates G
  <=> any g in G, $g = \gamma_1 \dots \gamma_k = 1_G\gamma_1 \dots \gamma_k$
  <=> exists a walk in X from 1_G to g
  <=> X is connected
  Remark:
  If we count a loop twice in Definition 1.2, then 1) would fail.

3. Adjacency Operator

> **Definition 1.10 Complex Vector Space $L^2(S)$**
> Finite set S. Define $L^2(S)$ as

(vector space of functions, mapping from S to complex numbers)

$$L^2(S) = \{f : S \to \mathbb{C}\}$$

$f, g \in L^2(S)$, $\alpha \in \mathbb{C}$
Sum: $(f+g)(x) = f(x) + g(x)$
Scalar multiplication: $(\alpha f)(x) = \alpha f(x)$
Standard inner product and therefore norm

$$\langle f, g\rangle_2 = \sum_{x \in S} f(x)\overline{g(x)} \quad ||f||_2 = \sqrt{\langle f, f\rangle_2} = \sqrt{\sum_{x \in S} |f(x)|^2}$$

Remark:
For simplicity, we often write $L^2(V)$, in which V is the vertex set of graph X, as $L^2(X)$. When doing so, we are viewing an element (function) f of $L^2(S)$ as:

1. a function (original definition)
2. a picture, where we label f(v) for every vertex of X
3. a (vector) value, with a particular ordering for vertices $v_1, \ldots, v_n$, as $(f(v_1), \ldots, f(v_n))^T$. Written in this way, inner product is the same as complex inner product of vectors.
   We also consider $L^2(E)$, where E is an edge multiset of X

Remark: Standard Basis
Consider a set $S = \{x_1, x_2, \ldots, x_n\}$. Let $\beta = \{\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_n}\} \in L^2(S)$, where $\delta_{x_i}(x_j) = 1$ if $i = j$, and $\delta_{x_i}(x_j) = 0$ if $i \neq j$.
If $f \in L^2(S)$, then

$$f(x) = f(x_1)\delta_{x_1}(x) + \cdots + f(x_n)\delta_{x_n}(x)$$

That is, β spans $L^2(S)$, and β is an orthonormal basis.

Definition 1.11 Adjacency Matrix
Graph X, vertices ordered as $v_1, v_2, \ldots, v_n$
Adjacency matrix of X is denoted as A, where $A_{i,j}$ is the number of edges that are incident to both v_i and v_j

Proposition 1.12 Facts about Adjacency Matrix

1. A is symmetric.
2. Suppose A is $n \times n$, then A has n real eigenvalues (not necessarily different)
3. For graph X, two adjacency matrices using different vertex ordering have same eigenvalues. (hence we can talk about eigenvalues of a graph without referring to a specific vertex ordering)

Proof

1. By definition.
2. Symmetric real matrix, then by spectrum theorem in linear algebra.
3. For two adjacency matrices of different ordering, A_1, A_2,
   - We can get from one to the other by permutations on rows and columns, and these permutations are the same, in the sense that if we exchange column i with column j, we also exchange row i with row j.
   - Say the permutation needed from A_1 to A_2 is π, with corresponding row permutation matrix R_π, column permutation matrix C_π, then:
     $A_2 = R_\pi C_\pi A_1$ (first permuting columns, then permuting rows)
     $A_2 = C_\pi A_1 R_\pi^{-1}$ (left multiply is permutation π, right multiply is permutation $-\pi$, and $R_{-\pi} = R_\pi^{-1}$)
     $A_2 = C_\pi A_1 C_\pi^T$ ($R_{-\pi} = R_\pi^{-1} = C_\pi$, and $P^{-1} = P^T$ for both $P = C_\pi$ and $P = R_\pi$)
     Note C_π is orthogonal, hence A_1 and A_2 are similar, with same eigenvalues.

Definition 1.13 Spectrum of Graph
Order the eigenvalues of graph X as:

$$\lambda_{n-1}(X) \leq \lambda_{n-2}(X) \leq \cdots \leq \lambda_1(X) \leq \lambda_0(X)$$

We call the multiset of eigenvalues of X the *spectrum* of X. For distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_r$ with multiplicities $m_1, m_2, \ldots, m_r$, we write:

$$Spec(X) = \begin{pmatrix} \mu_1 & \mu_2 & \ldots & \mu_r \\ m_1 & m_2 & \ldots & m_r \end{pmatrix}$$

Remark
Spectrum conveys much information about the graph, especially $\lambda_1(X)$. However, in most applications where graphs are large, we do not have spectrum. Our main task henceforth is to develop techniques to estimate λ_1.

Definition 1.14 Adjacency Operator
Graph X, vertex set V, adjacency matrix with ordering $v_1, v_2, \ldots, v_n$. Consider $f \in L^2(X)$, then

$$(Af)(v) = \sum_{w \in V} A_{v,w} f(w)$$

Hence A is a linear operator, $L^2(X) \to L^2(X)$. This is known as adjacency operator of X.

Remark

1. This operator is natural considering the close connection between square matrix and operator.
2. For Cayley graph $X = Cay(G, \Gamma)$, we have:

   $$(Af)(g) = \sum_{y \in \Gamma} f(gy)$$

   for all $g \in G$
   (since $A_{g,w} = 1 \iff w = gy$ for some $y \in \Gamma$)

4. Eigenvalues of Regular Graphs

Definition 1.15 Symmetric about 0
X is a d-regular graph. Then its spectrum is symmetric about 0 if whenever λ is an eigenvalue of X of multiplicity k, then −λ is also an eigenvalue of X with multiplicity k.

Proposition 1.16 Facts about regular graph's eigenvalues
If X is a d-regular graph with n vertices, then

1. d is an eigenvalue of X.

2. $|\lambda_i(X)| \leq d$ for $i = 0, \ldots, n-1$.
3. $\lambda_1(X) < \lambda_0(X)$ iff X is connected.
4. If X is bipartite, then the spectrum of X is symmetric about 0.
5. If −d is an eigenvalue of X , then X is bipartite.

Proof

1. Consider constant function $f_0(x) = 1$ for all vertices
$(Af_0)(x) = \sum_{y \in V} A_{x,y} f_0(y) = \sum_{y \in V} A_{x,y} = d$ (since X is d-regular)
2. For arbitrary eigenvalue λ with eigenfunction f, pick $x \in V$ s.t. $|f(x)| = max_{y \in V}|f(y)|$. Then,
$|\lambda||f(x)| = |(Af)(x)|$ (eigenvalue)
$= |\sum_{y \in V} A_{x,y} f(y)|$ (definition)
$\leq \sum_{y \in V} |A_{x,y}||f(y)|$ (triangle inequality)
$\leq |f(x)| \sum_{y \in V} |A_{x,y}|$ (by design of x)
$= d|f(x)|$ (d-regular)
3. By results from spectrum theorem, multiplicity of eigenvalue d is equal to the dimension of its associated eigenspace, i.e.
$E_d(A) = \{f \in L^2(V) | Af = d \cdot f\}$
WTS: $dim(E_d(A)) = 1 \iff$ X is connected
With this, λ_1 cannot be equal to $\lambda_0 = d$ (note eigenvalue λ's are labelled in descending order), hence $\lambda_1(X) < \lambda_0(X)$
    1. Suppose X is connected, and f is an arbitrarily chosen eigenfunction associated with d. We will show that f is constant on V, hence eigenspace of d is of dim 1.
    - Pick $x \in V$ s.t. $|f(x)| = max_{y \in V}|f(y)|$
    - WLOG, f(x) > 0, since -f is also an eigenfunction of d
    - Then, $f(x) = \frac{(Af)(x)}{d} = \sum_{y \in V} \frac{A_{x,y}}{d} f(y)$ (def of adjacency operator)
    - WTS: f(y) = f(x) for any y adjacent to x
    Suppose not. Then, f(t) < f(x) for some t adjacent to x. Then, since f(x) is the maximum, we have: $f(x) = \sum_{y \in V} \frac{A_{x,y}}{d} f(y) < \sum_{y \in V} \frac{A_{x,y}}{d} f(x) = f(x)$ (at least one <), contradiction.
    - Repeat the process for each y that is adjacent to x, eventually we reach every vertex of X since X is connected. Hence f is constant on X.
    2. Suppose X is disconnected.
    - WTS: $dim(E_d(A)) > 1$, we show by constructing two linearly independent eigenfunctions associated with d
    - Let V_1 be the set of all vertices w that there is a walk from v to w in X.
    - Let $V_2 = V \setminus V_1$
    - Define the functions:
    - $f_1(x) = 1$ if $x \in V_1$, $f_1(x) = 0$ if $x \in V_2$;
    - $f_2(x) = 0$ if $x \in V_1$, $f_2(x) = 1$ if $x \in V_2$;
    - Then it is easy to check two functions are linearly independent.
4. Suppose $V = V_1 \cup V_2$, is a bipartition, λ is an eigenvalue with multiplicity k. Then by spectrum theorem, there are orthogonal eigenfunctions $f_1, \ldots, f_k$. For $i = 1, \ldots, k$, define the functions

$$g_i(x) = \begin{cases} f_i(x) & x \in V_1 \\ -f_i(x) & x \in V_2 \end{cases}$$

WTS: each g_i is an eigenfunction with associated eigenvalue $-\lambda$. Hence $-\lambda$ is an eigenvalue with multiplicity $m \geq k$. Same argument, reversing the roles of λ and $-\lambda$ shows that $k \geq m$, so $m = k$, done.
For arbitrary $x \in V_1$, since every y adjacent to x is in V_2, we have:
$(Ag_i)(x) \quad = \sum_{y \in V_2} A_{x,y} g_i(y)$ (adjacency operator)
$= -\sum_{y \in V} A_{x,y} f_i(y)$ (for $y \in V_1$, $A_{x,y} = 0$ so don't matter)
$= -(Af_i)(x)$ (adjacency operator)
$= -\lambda f_i(x)$ (eigenvalue)
$= -\lambda g_i(x)$ (definition, since $x \in V_1$)
Similar for $x \in V_2$. Hence $-\lambda$ is an eigenvalue.
5. Assume X is connected.
    - Suppose $-d$ is an eigenvalue, and f be an associated eigenfunction.
    - Pick $x \in V$ s.t. $|f(x)| = \max_{y \in V} |f(y)|$
    - WLOG, $f(x) > 0$, since $-f$ is also an eigenfunction.
    - Following argument in part (3)(1), we can show that $f(y) = -f(x)$ for all y adjacent to x. By connectivity, eventually we have:

$$f(y) = \begin{cases} f(x) & \text{if } dist(x, y) \text{ is even} \\ -f(x) & \text{if } dist(x, y) \text{ is odd} \end{cases}$$

    - This gives a bipartition of V by the value of $f(y)$.
    The case for disconnected X is omitted.

Definition 1.17 Circulant Matrix

1. A circulant matrix C is of the form:

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}$$

2. The eigenvalues of the circulant matrix C given above are:

$$\chi_a = \sum_{j=0}^{n-1} c_j \xi^{aj}$$

where $\xi = \exp(2\pi i/n)$ and $a = 0, 1, 2, \ldots, n-1$

(Proof omitted)

Lemma 1.18
Let $n \geq 2$ and a be integers. Let $\xi = \exp(2\pi i/n)$. Then

$$\sum_{j=0}^{n-1} (\xi^a)^j = \begin{cases} 0 & \text{if } n \text{ does not divide } a \\ n & \text{otherwise} \end{cases}$$

Proof

- If n divides a, then $\xi^a = 1$, done.
- If not, then sum of geometric series.

Example: Eigenvalues for complete graph K_n

- Definition from graph theory: graph with n vertices, where vertices v and w are adjacent, via an edge of multiplicity 1, iff $v \neq w$.
- Reformulation with Cayley graph: $K_n = Cay(\mathbb{Z}_n, \{1, 2, \dots, n\})$
- Its adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$$

- This is circulant. Hence we know the formula for all its eigenvalues.
  If $a = 0$, then $\chi_0 = n - 1$ ($c_i = 0$ except for $i = 0$)
  If $0 < a \leq n - 1$, then a cannot divide n. Applying lemma 1.18, we have:

$$\chi_a = \sum_{j=0}^{n-1} c_j \xi^{aj} = \sum_{j=0}^{n-1} \xi^{aj} - 1 = 0 - 1 = -1$$

- Hence,

$$Spec(K_n) = \begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix}$$

- Formulation of circulant matrix provides a quick computation for eigenvalues of all complete graphs. However, most regular graphs do not have such nice property.

5. The Laplacian

In this section we consider Laplacian on graph, which measures the inflow and outflow of a vertex.

> **Definition 1.19**
> Graph X, vertex set V, edge set E.
>
> 1. Orientation: Give each edge e an arbitrary orientation: from endpoint e^- (origin) to the other endpoint e^+ (extremity). For loops, $e^- = e^+$.
> 2. Analogue of gradient: $d : L^2(V) \to L^2(E)$, s.t. for each $f \in L^2(V)$
>
> $$(df)(e) = f(e^+) - f(e^-)$$
>
> 3. Analogue of divergence: $d^* : L^2(E) \to L^2(V)$, s.t. for each $f \in L^2(E)$
>
> $$(d^* f)(v) = \sum_{\substack{e \in E \\ v = e^+}} f(e) - \sum_{\substack{e \in E \\ v = e^-}} f(e)$$
>
> 4. Laplacian operator: $\Delta = d^* d$

Remark:

- $(df)(e)$ measures the change of f along the edge e, $(d^* f)(v)$ measures the total inward flow at vertex v.
- Laplacian operator is introduced to simplify proofs in later chapters.

> **Lemma 1.20 Laplacian does not depend on the choice orientation**
> X be a k-regular graph, with vertex set V, edge multiset E, and adjacency operator A. Then, $\Delta = kI - A$

Proof
Let $f \in L^2(V)$ and $x \in V$. Then

$$\begin{aligned}
(\Delta f)(x) = (d^*(df))(x) &= \sum_{\substack{e \in E \\ v = e^+}} (df)(e) - \sum_{\substack{e \in E \\ v = e^-}} (df)(e) \\
&= \left(\sum_{\substack{e \in E \\ x = e^+}} f(x) - \sum_{\substack{e \in E \\ x = e^+ \\ y = e^-}} f(y) \right) - \left(\sum_{\substack{e \in E \\ x = e^- \\ y = e^+}} f(y) - \sum_{\substack{e \in E \\ x = e^-}} f(x) \right) \\
&= \sum_{\substack{e \in E \\ x = e^+ \text{ or } x = e^-}} f(x) - \sum_{\substack{e \in E \\ x = e^+ \text{and } y = e^- \text{ or } x = e^- \text{and } y = e^+}} f(y) \\
&= kf(x) - \sum_{y \in V} A_{x,y} f(y) \\
&= ((kI - A)f)(x)
\end{aligned}$$

First two lines by definition, third line by rearrange summation.

Remark:

- The proof relies on the definition that loops are counted only once.
- Since $\Delta = kI - A$, we have a linear transformation $\Delta : L^2(V) \to L^2(V)$
- $\Delta(\alpha f) = \alpha \Delta f$ for $\alpha \in \mathbb{C}$ (obvious from proof above)

> **Proposition 1.21 Eigenvalues of Δ**
> X is a k-regular graph, vertex set V, edge multiset E, $n = |V|$
>
> 1. The eigenvalues of Δ are:
>
> $$0 = k - \lambda_0(X) \leq k - \lambda_1(X) \leq \dots \leq k - \lambda_{n-1}(X)$$
>
> In particular, the eigenvalues of Δ lie in $[0, 2k]$.
> 2. $f \in L^2(V)$, $g \in L^2(E)$. Then
>
> $$\langle df, g \rangle_2 = \langle f, d^* g \rangle_2$$
>
> and

$$\langle \Delta f, f\rangle_2 = \sum_{e\in E} |f(e^+) - f(e^-)|^2$$

Proof

1. $Af = \lambda f$ iff $(kI - A)f = (k - \lambda)f$, so we get eigenvalues. Recall for k-regular graph, its eigenvalues are bounded by k, so we have bounds for Δ.
2. $$\begin{aligned}\langle df, g\rangle_2 &= \sum_{e\in E}(df)(e)\overline{g(e)} = \sum_{e\in E}[f(e^+ - f(e^-))]\overline{g(e)} \\ &= \sum_{e\in E} f(e^+)\overline{g(e)} - \sum_{e\in E} f(e^-)\overline{g(e)} \\ &= \sum_{v\in V} f(v) \sum_{\substack{e\in E\\ v=e^+}} \overline{g(e)} - \sum_{v\in V} f(v) \sum_{\substack{e\in E\\ v=e^-}} \overline{g(e)} \\ &= \sum_{v\in V} f(v)\overline{(d^*g)(v)} \\ &= \langle f, d^*g\rangle_2\end{aligned}$$

   First and last lines are by definition. Second line opens the bracket. For the third line, if there is some v s.t. v is not equal to any e^+, then that term would be zero. Same for the second bulk with e^-. So the equation is legitimate.
3. $$\begin{aligned}\langle \Delta f, f\rangle_2 &= \langle d^*df, f\rangle_2 = \overline{\langle f, d^*df\rangle_2} \\ &= \overline{\langle df, df\rangle_2} = \langle df, df\rangle_2 = ||df||_2^2 \\ &= \sum_{e\in E} |f(e^+) - f(e^-)|^2\end{aligned}$$

   From first line to second line by result (2), rest by definition.

6. The Isoperimetric Constant

We want to study the quality of graph as a communication network.
Standard for a good communication network: Each vertex has lots of neighbours (quick transmission), and the total number of edges is relatively small (low cost).

Definition 1.22 Isoperimetric constant

1. Graph X with vertex set V, $F \in V$. The $boundary$ of F, ∂F , is the set of edges connecting F to $V \setminus F$.
2. The isoperimetric constant of X is defined as

$$h(X) = \min\left\{ \frac{|\partial F|}{\min\{|F|, |V \setminus F|\}} \;\middle|\; F \subset V \right\}$$

Remark:

- Numerator is the number of neighbours and denominator is the size of the set. We ask the question: does the set have lots of neighbours relative to its size?
- The outer minimum takes the worst scenario across the graph.
- We only want to consider information flow from a smaller part of the graph to the rest. Hence, the denominator must be less than $n/2$, so we put another minimum there. Alternatively, we can restrict F with $|F| \le n/2$.
- In summary, isoperimetric constant measures neighbourhood situation.

Properties of Isoperimetric constant

1. Suppose that X is a d-regular graph. Then $0 \le h(X) \le d$.
2. $h(X) = 0$ iff X is disconnected.

Proof

1. LHS:
   - non-negative is obvious.
   - Singleton takes 0.
2. RHS:
   - $\frac{|\partial F|}{|F|} \le \frac{d|F|}{|F|}$, i.e. every vertex of F has all its edges incident with some vertex in $V \setminus F$, and this is a d-regular graph. An upper bound.
   - Take F to be a single point, upper bound achieved.
3. $h(X) = 0 \Leftrightarrow \exists F \in X\, s.t.\ \partial F = \varnothing$, hence X must be disconnected.

Example:

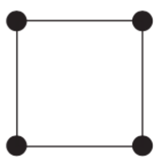


Figure 1.15 C_4

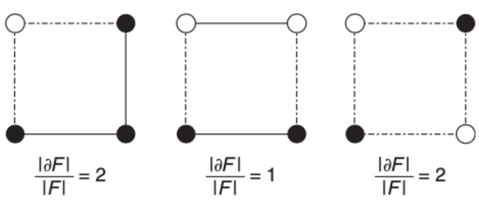


Figure 1.16

Motivation:

- $h(K_n)$ grows as at polynomial n. However, K_n has too many edges, which makes it expensive.
- We restrict the number of edges by considering regular graphs only. Then, we want to construct arbitrarily large graph with large isoperimetric constants.

> **Definition 1.23 Bounded away from zero**
> Sequence $\{a_n\}$ is bounded away from zero if there exists $\epsilon > 0$ s.t. $a_n \geq \epsilon$ for all n.

> **Definition 1.24** Expander Family
> $\{X_n\}$ is a sequence of d-regular graphs s.t. $|X_n| \to \infty$ as $n \to \infty$. We say $\{X_n\}$ is an $expander\ family$ if the sequence $\{h(X_n)\}$ is bounded away from zero.

Remark:
There are other definitions of expander family, but I postpone their introductions until they are necessary.

Example: There do not exist expander families of degree 2

- Fact: Any connected 2-regular graph must be isomorphic to C_n for some n.
- Suffice to show that $\{C_n\}$ is not an expander family.
- (book p28, omitted)

7. The Rayleigh-Ritz Theorem

Bounds for the second eigenvalue $\lambda_1(X)$

> **Definition 1.25**
> X finite set, f_0 function constant on 1 on all of X. Define
>
> $$L^2(X, \mathbb{R}) = \{f : X \to \mathbb{R}\}$$
>
> and
>
> $$\begin{aligned} L^2_0(X, \mathbb{R}) &= \{f \in L^2(X, \mathbb{R}) \mid \langle f, f_0 \rangle_2 = 0\} \\ &= \{f \in L^2(X, \mathbb{R}) \mid \sum_{x \in X} f(x) = 0\} \end{aligned}$$

Remark:
$L^2(X, \mathbb{R}) \subset L^2(X)$. However, we can also directly define $L^2_0(X)$ on entire X with $\langle f, f_0 \rangle_2 = 0$, and we still have Proposition 1.26.

> **Proposition 1.26 Rayleigh-Ritz**
> X is a d-regular graph with vertex set V. Then
>
> $$\lambda_1(X) = \max_{f \in L^2_0(V, \mathbb{R})} \frac{\langle Af, f_0 \rangle_2}{||f||_2^2} = \max_{\substack{f \in L^2_0(V, \mathbb{R}) \\ ||f||_2 = 1}} \langle Af, f_0 \rangle_2$$
>
> Equivalently,
>
> $$d - \lambda_1(X) = \min_{f \in L^2_0(V, \mathbb{R})} \frac{\langle \Delta f, f \rangle_2}{||f||_2^2} = \min_{\substack{f \in L^2_0(V, \mathbb{R}) \\ ||f||_2 = 1}} \langle \Delta f, f \rangle_2$$

Proof

1. Equivalence: Note $A = dI - \Delta$. The results follows from addition of inner product and the fact that $\langle df, f \rangle_2 = d||f||_2^2$.
2. By Spectrum Theorem, we can expand f_0 to an orthonormal basis $\{f_0, f_1, \dots, f_{n-1}\}$ of $L^2(X, \mathbb{R})$, where each f_i is an associated eigenfunction of eigenvalue $\lambda_i = \lambda_i(X)$.
   Let $f \in L^2_0(X, \mathbb{R})$ with $||f||_2 = 1$.
   Since we have a basis, $f = c_0 f_0 + c_1 f_1 + \dots + c_{n-1} f_{n-1}$ for some $c_i \in \mathbb{R}$.
   Also,

   $$\begin{aligned} 0 &= \langle f, f_0 \rangle_2 \\ &= c_0 \langle f_0, f_0 \rangle_2 + c_1 \langle f_1, f_0 \rangle_2 + \dots + c_{n-1} \langle f_{n-1}, f_0 \rangle_2 = c_0 \end{aligned}$$

   First line by definition, second line by orthonormal basis.
   So, $f = c_1 f_1 + \dots + c_{n-1} f_{n-1}$.
   Then

   $$\begin{aligned} \langle Af, f \rangle_2 &= \langle A \sum_{i=1}^{n-1} c_i f_i, \sum_{i=1}^{n-1} c_i f_i \rangle_2 = \langle \sum_{i=1}^{n-1} \lambda_i c_i f_i, \sum_{j=1}^{n-1} c_j f_j \rangle_2 \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_i c_j \lambda_i \langle f_i, f_j \rangle_2 = \sum_{i=1}^{n-1} c_i^2 \lambda_i \\ &\leq \lambda_1 \sum_{i=1}^{n-1} c_i^2 \\ &= \lambda_1 ||f||_2^2 = \lambda_1 \end{aligned}$$

   First line by expression of f, second line by the orthonormal basis (only equal to 1 for same f_i, otherwise 0), from second to third line by the fact that λ_1 is the second largest and the expression there exclude λ_0 (hence λ_1 is the largest). Last line by definition of f.
   Hence we find an upper bound. Suffice to show this upper bound can be achieved.
   Note $f_1 \in L^2_0(X, \mathbb{R})$ with $||f_1||_2 = 1$, and $\langle Af_1, f_1 \rangle_2 = \langle \lambda_1 f_1, f_1 \rangle_2 = \lambda_1$.
   Hence λ_1 is indeed the maximum.

Remark:

- Fact: $\overline{\langle Af, f \rangle_2} = \overline{\langle f, Af \rangle_2} = \langle Af, f \rangle_2$, since A as a real symmetric matrix has self-adjoint corresponding operator. Hence $\langle Af, f \rangle_2$ is real, and the proof above applies to $L^2_0(X)$ defined on entire X. That is, we have

  $$\lambda_1(X) = \max_{g \in L^2_0(X)} \frac{\langle Ag, g \rangle_2}{||g||_2^2} = \max_{\substack{g \in L^2_0(X) \\ ||g||_2^2 = 1}} \langle Af, f \rangle_2$$

- By cleverly choosing f, we can have a good lower bound for λ_1.

> **Proposition 1.27 Bounds for $h(X)$**
> X is a d-regular graph with vertex set V, edge multiset E. Then
>
> $$\frac{d - \lambda_1(X)}{2} \leq h(X) \leq \sqrt{2d(d - \lambda_1(X))}$$
>
> (Proof omitted)

Remark:
This results connect the lower bound of $\lambda_1(X)$ with the bounds of $h(X)$, hence useful for expander family. Roughly speaking, the smaller $\lambda_1(X)$ is, the larger $h(X)$ is, and the better the communication network would be.

> **Definition 1.28 Spectral gap**
>
> 1. If X is a connected d-regular graph, then $d - \lambda_1(X)$ is called the spectral gap of X.
> 2. $\{X_n\}$ is a sequence of d-regular graphs s.t. $|X_n| \to \infty$ as $n \to \infty$. We say $\{X_n\}$ is an $expander\ family$ iff $\{d - \lambda_1(X)\}$ is bounded away from zero.

Remark:
The corollary follow naturally from Proposition 1.27. Because this corollary, in the search of expander family, we focus on graph spectra.

Example:
Let $\Gamma = \{\sigma, \sigma^{-1}, \tau\} \subset S_n$ where $\sigma = (1, 2, \ldots, n)$ and $\tau = (1, 2)$. We will show that $(X_n) = (Cay(S_n, \Gamma))$ is not an expander family.

- Each X_n is 3-regular, since there are only 3 elements in Γ and decomposition of each element into xy only yields one y for fixed x.
- Suffice to show $3 - \lambda_1(X_n) \to 0$ as $n \to \infty$
- *Key: find a good f and Raleigh-Ritz*
- Define $f : S_n \to \mathbb{C}$ by $f(\alpha) = \exp(\frac{2\pi i}{n} \cdot \alpha^{-1}(1))$
- We first show that $f \in L_0^2(S_n)$, so $\lambda_1(X_n) \geq \langle Af, f\rangle_2 / \langle f, f\rangle_2$

8. Powers and Products of Adjacency Matrices

> **Definition 1.29 Digraph**
>
> 1. Define directed graph X to be a set V (vertices) and a multiset E (directed edges), where every element of E is an ordered pair (v, w), called the directed edge from v to w and visualised as an arrow from v to w, of elements of V.
> 2. Define the adjacency matrix of digraph X to be the $n \times n$ matrix whose ij entry equals the number of directed edges from v_i to v_j in X.

> **Definition 1.30** $X_1 \cdot X_2$
> X_1 and X_2 are two finite graphs (not necessarily digraphs) with same vertex set V. Define $X_1 \cdot X_2$ to be the digraph with vertex set V, where multiplicity of the directed edge from v_1 to v_2 equals the number of pairs (e_1, e_2) s.t. e_1 is a directed edge in X_1 which starts at v_1 , and e_2 is a directed edge in X_2 which points from the terminal point of e_1 to v_2.

Remark:
Essentially, we are taking a walk with length 2, first step in X_1 and second step in X_2.

Example:
From graph X and graph Y, construct $X \cdot Y$

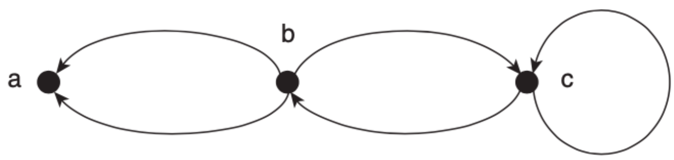


Figure 1.18 A directed graph X

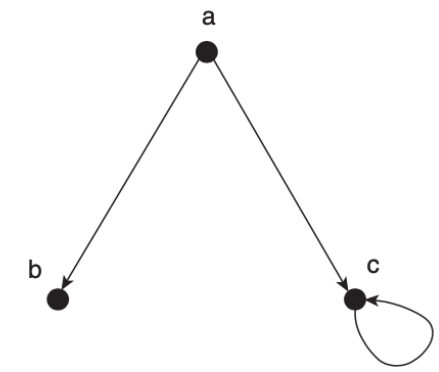


Figure 1.19 A directed graph Y

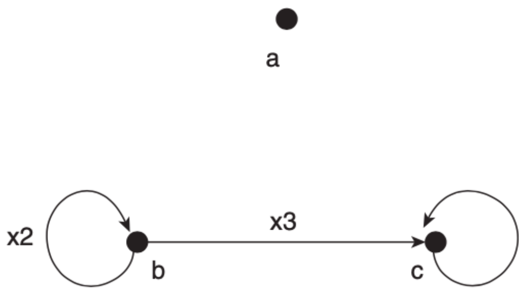


Figure 1.20 The graph $X \cdot Y$

The adjacency matrix

$$A_{X \cdot Y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

> **Definition 1.31** X^n
> X is a finite graph with vertex set V. Define X^n to be the graph with vertex set V, where the multiplicity of the edge from v_1 to v_2 equals to the number of walks of length n from v_1 to v_2. That is,
>
> $$X^n = X \cdot X \dots X$$

Remark:
Repeat $X \cdot X$ for n times, with same interpretation.

> **Proposition 1.32**
>
> 1. $A_{X_1 \cdot X_2} = A_1 \cdot A_2$ (when X_1 and X_2 has same ordering of vertices)
> 2. $A_{X^n} = A^n$

Proof:

1. Denote entries of A_1 as p_{ij}, entries of A_2 as q_{ij}, then for an arbitrary entry of $A_{X_1 \cdot X_2}$ denoted as A_{ij}, by matrix multiplication, we have:

$$A_{ij} = \sum_{k=1}^{n} p_{ik} q_{kj}$$

   which matches the definition of an edge in $X_1 \cdot X_2$ exactly.
2. Repeat (1).

> **Proposition 1.33**
> X is a finite graph with eigenvalues $\lambda_0, \dots, \lambda_{n-1}$. Then the eigenvalues of X^j are $\lambda_0^j, \dots, \lambda_{n-1}^j$.

Proof:
By spectrum theorem, we have PAP^{-1} as a matrix with diagonal entries $\lambda_0, \dots, \lambda_{n-1}$ (since A is diagonalisable), where P is unitary. Since $PA^jP^{-1} = PAP^{-1}PAP^{-1} \dots PAP^{-1}$, we have eigenvalues of X^j as $\lambda_0^j, \dots, \lambda_{n-1}^j$.

> **Definition 1.34 Directed Cayley graph**
> G is a group, and Γ is a multi-subset of G. The directed Cayley graph of G with respect to Γ, $Cay(G, \Gamma)$, is defined as follow. Vertices are the elements of G. For all $x, y \in G$, the multiplicity of the directed edge from x to y in $Cay(G, \Gamma)$ equals the number of elements γ in Γ, counted with multiplicity, s.t. $x\gamma = y$.

Note: then definition on the book is inconsistent with next proposition.

> **Proposition 1.35**
> G is a group, $\Gamma_1, \Gamma_2 \subset G$. $X_1 = Cay(G, \Gamma_1)$ and $X_2 = Cay(G, \Gamma_2)$. Let
>
> $$\Gamma_1 \Gamma_2 = \{\gamma_1 \gamma_2 \,|\, (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2\}$$
>
> Then $X_1 \cdot X_2 = Cay(G, \Gamma_1 \Gamma_2)$.

Proof: just unfold the definition