Chapter 2 Subgroups and Quotients

Negative results: under what conditions we cannot have expander families.

1. Coverings and Quotients

> **Definition 2.1 Graph Homomorphism**
>
> 1. Graph X with V, E, another graph X' with V', E'. A graph homomorphism between X and X' is a pair of maps $\phi_V : V \to V'$ and $\phi_E : E \to E'$, where whenever $e \in E$ has endpoints $a, b \in V$ and $\phi_E(e)$ has endpoints $a', b' \in V'$, then $\phi_V(\{a, b\}) = \{a', b'\}$.
> 2. Usually write ϕ for two homomorphisms.
> 3. If both ϕ_V and ϕ_E are bijective, then ϕ is an isomorphism.
> 4. For vertex v of graph X, let E_v be the set of edges incident to v. E_v is a set, we regard multiple edges as distinct elements. With homomorphism ϕ, E_v is mapped to $E_{\phi(v)}$.

> **Definition 2.2 Covering**
> Homomorphism $\phi : X \to Y$, and v is a vertex of X.
>
> 1. ϕ is bijective at v, if the map from E_v to $E_{\phi(v)}$ induced by ϕ is bijective.
> 2. ϕ is locally bijective, if ϕ is bijective at every vertex of X.
> 3. ϕ is a covering from X to Y, if ϕ is locally bijective and ϕ maps the vertex set of X surjectively onto the vertex set of Y.
> 4. We say X covers Y if such a covering ϕ exists.

Example

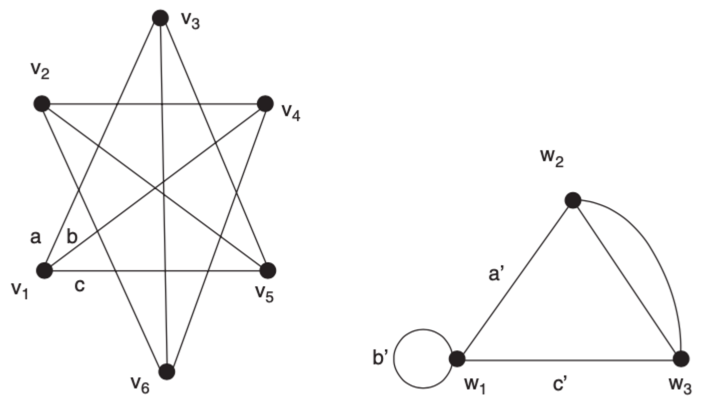

Figure 2.1 X (the left graph) covers Y (the right graph)

Covering ϕ : $\phi(v_1) = \phi(v_4) = w_1$, $\phi(v_2) = \phi(v_3) = w_2$, $\phi(v_5) = \phi(v_6) = w_3$.

> **Lemma 2.3 Properties of covering**
>
> 1. If X covers Y, then X is d-regular iff Y is d-regular.
> 2. If X covers Y and X is connected, then Y is connected.

Proof:

1. Immediately follows from bijection between E_v and $E_{\phi(v)}$.
2. Let w_1, w_2 be two vertices of Y. Since ϕ is a surjective map of vertex set from X to Y, there exists v_1, v_2 of X s.t. $\phi(v_1) = w_1, \phi(v_2) = w_2$. Since X is connected, there is a walk from v_1 to v_2 using edges $e_1, e_2, \ldots, e_n$. Then the edges $\phi(e_1), \ldots, \phi(e_n)$ is also a walk in Y from w_1 to w_2 since ϕ is a homomorphism.

> **Definition 2.4 Fibre**
> Covering $\phi : X \to Y$, w is a vertex of Y.
> Then the fibre of ϕ at w, $\phi^{-1}(w)$, is the set of all vertices v of X s.t. $\phi(v) = w$.

Remark:
i.e. the preimgae of w in X. Note ϕ is surjective for vertices.

> **Lemma 2.5 For connected graphs, all fibres have same size**
> Covering $\phi : X \to Y$.
> If Y is connected (equivalently, X is connected), then $|\phi^{-1}(w_1)| = |\phi^{-1}(w_2)|$ for all vertices w_1, w_2 of Y.

Proof:

- Induction on $dist(w_1, w_2)$, which is finite since Y is connected.
- We first show this is true for adjacent w_1, w_2.
- Let m be the number of edges between w_1, w_2. Clearly $m > 0$.
- The number of edges between a vertex in $\phi^{-1}(w_1)$ and a vertex in $\phi^{-1}(w_2)$ is $m \cdot |\phi^{-1}(w_1)|$. (Fix a vertex in $\phi^{-1}(w_1)$, then considering vertices in $\phi^{-1}(w_2)$)
- Reverse the roles of w_1 and w_2, the number is $m \cdot |\phi^{-1}(w_2)|$. (Fix a vertex in $\phi^{-1}(w_2)$, then considering vertices in $\phi^{-1}(w_1)$)
- Hence $|\phi^{-1}(w_1)| = |\phi^{-1}(w_2)|$.
- The fibre of a vertex that is j distance away has the same size as the fibre of a vertex that is $j + 1$ distance away, since we can always consider the last step in the walk connecting two vertices. Inductive steps done.

> **Lemma 2.6**
> If X and Y are finite graphs s.t X covers Y, then $h(X) \le h(Y)$.

Proof

- If Y is not connected, then by contrapositive of Lemma 2.3, so as X. All 0, done.
- Suppose Y is connected.
- Let S be a set of vertices of Y s.t. $h(Y) = \frac{|\partial S|}{|S|}$ and $|S| \le \frac{1}{2}|Y|$.

- Let

$$\phi^{-1}(S) = \{v \in V_X \mid \phi(v) \in S\}$$

- Let w be an arbitrary vertex of Y, $a = |\phi^{-1}(w)|$.
- By Lemma 2.5, $|\phi^{-1}(S)| = a|S|$, and $|X| = a|Y|$.
- Thus $|\phi^{-1}(S)| = a|S| \leq \frac{a}{2}|Y| = \frac{1}{2}|X|$, which makes a proper set to be considered for isoperimetric constant.
- Also, $|\partial[\phi^{-1}(S)]| = a|\partial S|$. Every edge in ∂S has exactly a preimage in X (a vertices collapsed into one vertex in Y, somehow similar to Lemma 2.5).
- Hence

$$h(X) \leq \frac{|\partial[\phi^{-1}(S)]|}{|\phi^{-1}(S)|} = \frac{a|\partial S|}{a|S|} = h(Y)$$

> **Definition 2.7 Coset Graph**
> Group G, Γ symmetric in G, $H \leq G$. Define the coset graph $Cos(H\backslash G, \Gamma)$ as:
>
> 1. vertex set: $H\backslash G$, set of right cosets of H in G
> 2. multiplicity between Hx and Hy: number of γ in Γ, counted with multiplicity, s.t. $Hx = Hy\gamma$.

Remark:

1. $Cos(H\backslash G, \Gamma)$ is $|\Gamma|$-regular.
2. G/H : left cosets; $G\backslash H$: right cosets.
3. If $H \triangleleft G$, then $G/H = G\backslash H$, and we have quotient group G/H. The map $G \to G/H$ by $a \mapsto aH$ is the canonical homomorphism.
4. If $H \triangleleft G$, then $Cos(H\backslash G, \Gamma) = Cay(G/H, \overline{\Gamma})$, where $\overline{\Gamma} = \{\gamma H \mid \gamma \in \Gamma\}$ is the image of Γ under the canonical homomorphism. Proof is just checking edge set and vertex set.

> **Lemma 2.8**
> Group G, $H \leq G$, Γ symmetric in G.
>
> 1. $Cay(G, \Gamma)$ covers $Cos(H\backslash G, \Gamma)$.
> 2. $h(Cay(G, \Gamma)) \leq h(Cos(H\backslash G, \Gamma))$

Proof

- For $g \in G, \gamma \in \Gamma$, denote by $e(g, \gamma)$ the edge in $Cay(G, \Gamma)$ between g and $g\gamma$ induced by γ; denote by $e(Hg, \gamma)$ the edge in $Cos(H\backslash G, \Gamma)$ between Hg and $Hg\gamma$ induced by γ.
- $\phi : Cay(G, \Gamma) \to Cos(H\backslash G, \Gamma)$, by $g \mapsto Hg$, $e(g, \gamma) \mapsto e(Hg, \gamma)$. Suffice to show ϕ is a covering.
- Surjective: trivial.
- Injective: The number of edges must be the same. (handwritten graph)
- (2) immediately follows from (1) and Lemma 2.6.

> **Definition 2.9 Sequence of Quotients**
> $\{G_n\}, \{Q_n\}$ are sequences of finite groups. We say $\{G_n\}$ admits $\{Q_n\}$ as a sequence of quotients, if for each n there exists $H_n \triangleleft G_n$ s.t. $G_n/H_n \cong Q_n$.

> **Definition 2.10 Yields an expander family**
> $\{G_n\}$ is a sequence of finite groups. We say $\{G_n\}$ yields an expander family, if for some positive integer d, there exists a sequence $\{\Gamma_n\}$ where for each n, Γ_n is symmetric in G_n with $|\Gamma_n| = d$, so the sequence $\{Cay(G_n, \Gamma_n)\}$ is an expander family.

> **Proposition 2.11 Quotients Non-expansion Principle**
> $\{G_n\}$ is a sequence of finite groups. Suppose $\{G_n\}$ admits $\{Q_n\}$ as a sequence of quotients. If $\{Q_n\}$ does not yield an expander family, then $\{G_n\}$ does not yield an expander family.

Proof

- Suppose not true. There exists a positive integer d s.t. for each n, there is a Γ_n symmetric in G_n s.t. $Cay(G_n, \Gamma_n)$ is an expander family.
- Let $\overline{\Gamma_n}$ be the image of Γ_n under the canonical homomorphism.
- Then $Cos(H_n\backslash G_n, \Gamma_n) = Cay(G_n/H_n, \overline{\Gamma_n})$ has isoperimetric constants larger than $Cay(G_n, \Gamma_n)$, hence $\{Cay(Q_n, \overline{\Gamma_n})\}$ is an expander family.

Remark:

- We can develop this principle with left cosets only. I doubt this formulation in book is designed as a comparison with next section.
- Next target is to restate the result with eigenvalues.

> **Definition 2.12 Pullback**
> Homomorphism $\phi : X \to Y$. Let $f \in L^2(Y)$.
> Define $f^* \in L^2(X)$ by $f^* = f \circ \phi$. We say f^* is the pullback of f via ϕ.

> **Lemma 2.13**
> X covers Y. Let $A, \tilde{A}$ be the adjacency operators of Y, X, respectively. Let $f \in L^2(Y)$. Then
>
> $$(Af)^* = \tilde{A}f^*$$

Proof

- v be an arbitrary vertex of X, and E_v is the set of all edges incident to v.
- If e is an edge in Y incident to v, let $e(v)$ denote the other vertex incident to e.
- Similarly, use $\tilde{e}$ for edges in X.
- Then, we can reformulate adjacency operator as

$$Af(v) = \sum_{e \in E(v)} f(e(v))$$

- Then, interpreting definitions of pullback and adjacency operator in different order

$$(Af)^*(v) = (Af)(\phi(v)) = \sum_{e \in E_{\phi(v)}} f(e(\phi(v)))$$
$$(\tilde{A}f^*)(v) = \sum_{\tilde{e} \in E_v} f^*(\tilde{e}(v)) = \sum_{\tilde{e} \in E_v} f(\phi(\tilde{e}(v)))$$

- Since ϕ is a covering, incidence is preserved, thus $e(\phi(v)) = \phi(\tilde{e}(v))$ i.e. vertices connected to $\phi(v)$ by e in Y, is the same as the map of vertices connected to v by e in X.

> **Lemma 2.14**
> X covers Y. Then every eigenvalue of Y is an eigenvalue of X.

Proof

- Let $A, \tilde{A}$ be the adjacency operators of Y, X, respectively.

- Let μ be an eigenvalue of A with corresponding eigenfunction f. Then

$$\tilde{A}f^* = (Af)^* = (\mu f)^* = \mu f^*$$

> **Proposition 2.15**
> d-regular graphs X and Y, and X covers Y. Then $\lambda_1(X) \geq \lambda_1(Y)$.

Proof

- $\lambda_0 = d$ for both X and Y.
- Possibly $\lambda_1(X)$ is not an eigenvalue of Y.

Remark

1. Idea: the quality of a graph is no better than that of any graph it covers. The covered graph is generally smaller, and it is more difficult to maintain the quality when graph becomes larger.
2. Another proof for Quotients Non-expansion Principle
    - Fact: $Cay(G_n, \Gamma_n)$ covers $Cay(G_n/H_n, \overline{\Gamma_n}) = Cay(Q_n, \overline{\Gamma_n})$.
    - Assumption: $\{Q_n\}$ not expander.
    - Then $\{d - \lambda_1(Q_n)\} \to 0$ for any d. (for any possible Γ_n, same for below)
    - Then $\{d - \lambda_1(G_n)\} \to 0$ for any d, since $\lambda_1(Q_n) \leq \lambda_1(G_n)$ for any n.
    - Then $\{G_n\}$ not expander.

2. Subgroups and Schreier Generators

> **Definition 2.16 Set of Transversals**
>
> 1. Finite group G, $H \leq G$. Let $T \subset G$ s.t. T contains exactly one element from each right coset of H in G. Then T is a set of transversals for H in G.
> 2. Denote $\overline{x}$ the unique element of T s.t. $Hx = H\overline{x}$.

> **Lemma 2.17**
> Let G, H, and T be defined as above. Then:
>
> 1. For all $x \in G$, there exists a unique $h \in H$ s.t. $x = h\overline{x}$.
> 2. For all $h \in H$, $a \in G$, we have $\overline{ha} = \overline{a}$.
> 3. For all $a, b \in G$, we have $\overline{\overline{a}b} = \overline{ab}$.
> 4. For all $t \in T$, we have $\overline{t} = t$.

Proof:

1. $h = x(\overline{x})^{-1}$, and there is only one $\overline{x}$ corresponding x, hence h is unique.
2. $H\overline{ha} = Hha = Ha = H\overline{a}$, second equality by definition of right coset.
3. $\overline{a} = h^{-1}a$ for some $h \in H$, thus $\overline{a} = h'a$ for some $h' \in H$, since $H \leq G$. Then, $\overline{\overline{a}b} = \overline{h'ab} = \overline{ab}$, last equality by (2).
4. $\overline{t} = ht$ for some $h \in H$ (same as (3)), thus $\overline{t} \in Ht$. Since $t \in Ht$, $\overline{t} = t$ since T contains only one element from each right coset of H.

Remark:

1. Following (1), we can decompose G with H and T, s.t. every $g \in G$, $g = ht$ for some $h \in H$ and $t \in T$. Also, $t = \overline{g}$, $h = g(\overline{g})^{-1}$.
2. For t, $\gamma \in G$, we introduce the notation

$$\widehat{(t, \gamma)} = t\gamma(\overline{t\gamma})^{-1}$$

   Following (1), exists a unique $h \in H$ s.t. $t\gamma = h\overline{t\gamma}$. Notice that $\widehat{(t, \gamma)} = t\gamma(\overline{t\gamma})^{-1} = h$, hence $\widehat{(t, \gamma)}$ is an element of H, and $t\gamma = \widehat{(t, \gamma)}\overline{t\gamma}$ is the unique expression of $t\gamma$ in the form of $h\overline{t\gamma}$.
3. Rigorously speaking we need redefine concepts like function, bijection, "set" multiplication etc. since we are dealing with multiset. I omit these details. Anyway the guiding principle is to pretend it is still a set and count the multiplicity (essentially not that different).

> **Definition 2.18 Schreier Generators**
> Let G, H, T, be defined as in Def 2.16. $\Gamma \in G$, define
>
> $$\hat{\Gamma} = \{\widehat{(t, \gamma)} \mid (t, \gamma) \in T \times \Gamma\}$$
>
> 1. $\hat{\Gamma}$ is the set of Schreier generators for H in G with respect to Γ.
> 2. $|\hat{\Gamma}| = [G : H] \cdot |\Gamma|$, where $[G : H]$ means the number of right cosets generated by H, and clearly $|T| = [G : H]$

> **Lemma 2.19**
> G, H, T as Def 2.16.
>
> 1. If $\Gamma \in G$, then $\hat{\Gamma} \in H$.
> 2. If Γ symmetric in G, then $\hat{\Gamma}$ symmetric in H.

Proof:

1. Let $t \in T, \gamma \in \Gamma, x = t\gamma$.
    - By Lemma 2.17, $x = h\overline{x}$ for a unique $h \in H$, so $\widehat{(t, \gamma)} = h$.
2. Define map $\phi : T \times \Gamma \to T \times \Gamma$, by $\phi(t, \gamma) = (\overline{t\gamma}, \gamma^{-1})$.
    - Last equality by Lemma 2.17, we have

$$(\phi \circ \phi)(t, \gamma) = \phi(\overline{t\gamma}, \gamma^{-1}) = (\overline{\overline{t\gamma}\gamma^{-1}}, \gamma) = (t, \gamma)$$

      Inverse of ϕ is just itself, bijective.
    - Again by Lemma 2.17

$$\left(\widehat{(t, \gamma)}\right)^{-1} = \overline{t\gamma}\gamma^{-1}t^{-1} = \overline{t\gamma}\gamma^{-1}(\overline{\overline{t\gamma}\gamma^{-1}})^{-1} = \widehat{\phi(t, \gamma)}$$

      Because ϕ is bijective, $\left(\widehat{(t, \gamma)}\right)^{-1}$ has same multiplicity in $\hat{\Gamma}$ as $\widehat{(t, \gamma)}$.

Remark:
If $Cay(G, \Gamma)$ is an undirected graph, then $Cay(H, \hat{\Gamma})$ is also an undirected graph. Following Chapter 1, we can consider an undirected graph as two (overlapping) directed graph.

Example:

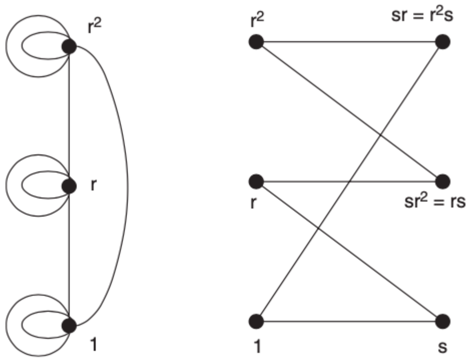

Figure 2.6 On the left, $\mathrm{Cay}(H, \hat{\Gamma})$; on the right, $\mathrm{Cay}(G, \Gamma)$

- $G = D_3 = \{1, r, r^2, s, sr, sr^2\}$, $H = \{1, r, r^2\}$, $\Gamma = \{s, sr\}$.
- Right cosets of H in G are H and $Hs = \{s, sr, sr^2\}$.
- Let $T = \{1, s\}$, and T is indeed a set of transversals for H in G.
- By collapsing edges horizontally, we get from right to left (e.g. sr collapsing to r^2). The collapsed edge (two directed edge) becomes two (directed) loops.
- This collapsing can be described by a surjective map, $G \to H$, by $ht \mapsto h$.
- Denote by $e(g, \gamma)$ each directed edge in $Cay(G, \Gamma)$ from g to $g\gamma$.

> **Lemma 2.20**
> Map [set of directed edges in $Cay(G, \Gamma)$] $\to$ [set of directed edges in $Cay(H, \hat{\Gamma})$], by
>
> $$e(ht, \gamma) \to e(h, \widehat{(t, \gamma)})$$
>
> is bijective.

Proof

- Surjective is trivial.
- Injective: same as Lemma 2.8 (how about a formal proof?)

> **Lemma 2.21**
> G, H, T as Def 2.16. Γ symmetric in G. Then
>
> $$h(Cay(G, \Gamma)) \leq \frac{h(Cay(H, \hat{\Gamma}))}{[G : H]}$$

Remark:
The actual result used in Proposition 2.24 is $h(Cay(G, \Gamma)) \leq h(Cay(H, \hat{\Gamma}))$, which is an obvious implication of Lemma 2.21.

Proof:

- Let $S \subset H$, s.t. $|S| \leq \frac{1}{2}|H|$ and $\frac{|\partial S|}{|S|} = h(Cay(H, \hat{\Gamma}))$.
- Let $\tilde{S} = \{ht \mid h \in S,\, t \in T\}$, so $|\tilde{S}| = |S| \cdot |T|$, and $|T| = [G : H]$
- Then

$$|\tilde{S}| = |S| \cdot |T| \leq \frac{1}{2}|H| \cdot |T| = \frac{1}{2}|G|$$

- Note:

$$\begin{aligned} g\gamma \in \tilde{S} &\iff (ht)\gamma \in \tilde{S} \\ &\iff ht\gamma(\overline{ht\gamma})^{-1} \in S \\ &\iff ht\gamma(\overline{t\gamma})^{-1} \in S \\ &\iff h\widehat{(t, \gamma)} \in S \end{aligned}$$

  First line: decomposition of g in G. Second line: $g\gamma = h't'$ since $g\gamma \in G$, and $h' = (g\gamma)(\overline{g\gamma})^{-1}$, which lies in S by definition of $\tilde{S}$. Third line: by (4) of Lemma 2.17. Last line: definition.
- Then:

$$\begin{aligned} |\partial\tilde{S}| &= |\{e(g, \gamma) \,:\, g \in \tilde{S}, \gamma \in \Gamma, g\gamma \notin \tilde{S}\}| \\ &= |\{e(ht, \gamma) \,:\, h \in S, t \in T, \gamma \in \Gamma, ht\gamma \notin \tilde{S}\}| \\ &= |\{e(h, \widehat{(t, \gamma)}) \,:\, h \in S, t \in T, \gamma \in \Gamma, ht\gamma \notin \tilde{S}\}| \\ &= |\{e(h, \widehat{(t, \gamma)}) \,:\, h \in S, t \in T, \gamma \in \Gamma, h\widehat{(t, \gamma)} \notin S\} \\ &= |\partial S| \end{aligned}$$

  First line is the definition of boundary set in $Cay(G, \Gamma)$, edges which has one endpoint not in $\tilde{S}$, second line decomposes g, third line by the bijection map between $e(ht, \gamma)$ and $e(h, \widehat{(t, \gamma)})$, fourth line by last point, last line is again definition of boundary in $Cay(H, \hat{\Gamma})$.
- Thus,

$$h(Cay(G, \Gamma)) \leq \frac{|\partial\tilde{S}|}{|\tilde{S}|} = \frac{|\partial S|}{|S| \cdot [G : H]} = \frac{h(Cay(H, \tilde{\Gamma}))}{[G : H]}$$

> **Corollary 2.22 Schreier Subgroup Lemma**
> If Γ generates G, then $\hat{\Gamma}$ generates H.

Proof:

- Γ generates G
- then $Cay(G, \Gamma)$ is connected (Prop 1.9)
- then $h(Cay(G, \Gamma)) > 0$, (properties of isoperimetric constant)
- then $h(Cay(H, \hat{\Gamma})) > 0$, (Lemma 2.21)
- then $Cay(H, \hat{\Gamma})$ is connected (properties of isoperimetric constant)

- then $\hat{\Gamma}$ generates H.

Remark:
The converse is false. Counterexample

> **Definition 2.23 Bounded-index sequence of subgroups**
> $\{G_n\}$ and $\{H_n\}$ are sequences of finite groups. We say that $\{G_n\}$ admits $\{H_n\}$ as a bounded-index sequence of subgroups, if $H_n \leq G_n$ for all n, and the sequence $\{[G_n : H_n]\}$ is bounded.

> **Proposition 2.24 Subgroups Nonexpansion Principle**
> $\{G_n\}$ be a sequence of finite groups. Suppose that $\{G_n\}$ admits $\{H_n\}$ as a bounded-index sequence of subgroups. If $\{H_n\}$ does not yield an expander family, so does $\{G_n\}$.

Proof:

- Suppose not true. Then, there exists a positive d and Γ_n symmetric in G_n with $|\Gamma_n| = d$ for all n, s.t. $\{Cay(G_n, \Gamma_n)\}$ is an expander family, and $h(Cay(G_n, \Gamma_n)) \geq \epsilon$ for some $\epsilon > 0$.
- Since $\{[G_n : H_n]\}$ is bounded, exist M s.t. $[G_n : H_n] \leq M$ for all n.
- Let T_n be a set of transversals for H_n in G_n for each n.
- Let

$$\Lambda_n = \widehat{\Gamma_n} \cup \{(M - [G_n : H_n])d \cdot e_n\}$$

  where $\cdot$ means the set contains $(M - [G_n : H_n])d$ copies of identity e_n of group G_n. So Λ_n is essentially the set of Schreier generators and the union is just modifying multiplicity of e_n s.t. $|\Lambda_n| = [G_n : H_n] \cdot |\Gamma_n| + (M - [G_n : H_n])d = M \cdot d$, since $|\Gamma_n| = d$.
- After modification:
  - $Cay(H_n, \Lambda_n)$ has same vertex set as $Cay(H_n, \widehat{\Gamma_n})$, and their edge sets are the same expect for additional loops in $Cay(H_n, \Lambda_n)$. So they have the same isoperimetric constant.
  - $\{Cay(H_n, \Lambda_n)\}$ is now a sequence of Md-regular graphs, since $|\Lambda_n|$ has a fixed size. We don't have this for $|\widehat{\Gamma_n}|$, so $\{Cay(H_n, \widehat{\Gamma_n})\}$ do not fulfil the precondition of expander family.
- Then by Lemma 2.21

$$h(Cay(H_n, \Lambda_n)) = h(Cay(H_n, \widehat{\Gamma_n})) \geq h((Cay(G_n, \Gamma_n))) \geq \epsilon$$

  for all n. Thus $\{Cay(H_n, \Lambda_n)\}$ is an expander family

> **Lemma 2.25**
> G, H, T as Def 2.16. Γ symmetric in G. Then
>
> $$\lambda_1(Cay(G, \Gamma)) \geq \frac{\lambda_1(Cay(H, \hat{\Gamma}))}{[G : H]}$$

Proof

- A_G, A_H are adjacency operators of $Cay(G, \Gamma), Cay(H, \hat{\Gamma})$, respectively.
- $\lambda_H = \lambda_1(Cay(H, \hat{\Gamma}))$, $f \in L_0^2(H, \mathbb{R})$, and is an eigenfunction of λ_H. Such an eigenfunction must exist, since f_0 as a constant 1 function must be an orthogonal eigenfunction (corresponding to the "constant" part of an orthogonal eigen basis), thus every other orthogonal eigenfunction must fall in $f \in L_0^2(H, \mathbb{R})$, since the definition of this space only requires functions to be orthogonal with f_0.
- Define $\tilde{f} \in L^2(G)$ by $\tilde{f}(ht) = f(h)$ for all $h \in H, t \in T$. Due to the decomposition of every $g \in G$, $\tilde{f}$ is well-defined.
- Note $\tilde{f} \in L_0^2(G, \mathbb{R})$, since

$$\sum_{g \in G} \tilde{f}(g) = \sum_{t \in T} \sum h \in H \tilde{f}(ht) = \sum_{t \in T} \sum_{h \in H} f(h) = 0$$

- By Rayleigh-Ritz

$$\lambda_1(Cay(G, \Gamma)) \geq \frac{\langle A_G \tilde{f}, \tilde{f} \rangle}{\langle \tilde{f}, \tilde{f} \rangle}$$

- Denominator:

$$\begin{aligned}
\langle \tilde{f}, \tilde{f} \rangle &= \sum_{g \in G} \tilde{f}(g)^2 \\
&= \sum_{t \in T} \sum_{h \in H} \tilde{f}(ht)^2 \\
&= \sum_{t \in T} \sum_{h \in H} f(h)^2 \\
&= \sum_{t \in T} \langle f, f \rangle \\
&= [G : H] \langle f, f \rangle
\end{aligned}$$

  All by definitions.
- Consider $ht\gamma$ as an element of G, then $ht\gamma = h't'$, and $h' = (ht\gamma)(\overline{ht\gamma})^{-1}$, $\tilde{f}(h't') = f(h')$
- Thus, $\tilde{f}(ht\gamma) = f(ht\gamma(\overline{ht\gamma})^{-1}) = f(h\widehat{(t, \gamma)})$ (last equation same as proof of Lemma 2.21)
- Numerator:

$$\begin{aligned}
\langle A_G \tilde{f}, \tilde{f} \rangle &= \sum_{g \in G} A_G \tilde{f}(g) \cdot \tilde{f}(g) \\
&= \sum_{g \in G} \sum_{\gamma \in \Gamma} \tilde{f}(g\gamma) \cdot \tilde{f}(g) \\
&= \sum_{h \in H} \sum_{t \in T} \sum_{\gamma \in \Gamma} \tilde{f}(ht\gamma) \cdot \tilde{f}(ht) \\
&= \sum_{h \in H} \sum_{t \in T} \sum_{\gamma \in \Gamma} f(h\widehat{(t, \gamma)}) \cdot f(h) \\
&= \sum_{h \in H} (A_H f)(h) \cdot \overline{f(h)} \\
&= \langle A_H f, f \rangle \\
&= \lambda_H \langle f, f \rangle
\end{aligned}$$

  Note f is a real function, thus $f = \overline{f}$. Else by definition.
- Hence

$$\lambda_1(Cay(G, \Gamma)) \geq \frac{\lambda_H \langle f, f \rangle}{[G : H] \langle f, f \rangle} = \frac{\lambda_1(Cay(H, \hat{\Gamma}))}{[G : H]}$$

Remark:

1. Another common presentation of Lemma 2.25 is with normalised adjacency operators. Normalised = largest eigenvalue is 1. In this way, the adjacency matrix is related to a random walk on X, with each entry being a transition probability.
2. Another proof for Subgroups Non-expansion Principle: