Coupled predator-prey models and their application in investing in the stock market

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Motivation

What if we can simulate the complex behavior and interaction of certain species which compete, or work together? Can we model a food chain with only simple equations? What is the relationship between foxes and rabbits in the forest, and the chaotic nature of the stock market?
When I chose this topic, I have found fascinating how mechanisms which seemingly do not have very much in common could relate to each other. In the next few pages I present a model and its extension which can describe certain aspects of our world in a new way.

Introduction

Predator-prey models are important topics not only in biology, but in other fields like plasma physics(Morel, Gürcan, and Berionni 2014), economics(Apedaille et al. 1994), or even in criminology (Sooknanan, Bhatt, and Comissiong 2016).
The first, simplest model originated from the study of fish populations of the Mediterranean after the first world war, by Lotka and Volterra. In their model there are two types of species: the prey, and the predator. They form a simple food-chain where the predator species hunts the prey species, while the prey grazes vegetation. Their behavior and the time evolution of the number of species are characterized by a simple system of two, nonlinear first order differential equations. By solving these equations, we can have insights on how the number of predators and preys evolve in time.
In this document I present the classical Lotka-Volterra (LV) equations and its modified versions. I also reproduce a financial model described in (Addison, Bhatt, and Owen 2016), which can be used by a stock market trader to eliminate some of its risks involving the trading of specific stocks by buying the shares of so-called prey companies, and sell them to a predator company. The stock market also has equilibrium, so it behaves like an ecosystem and understanding the ramifications of these mechanisms can provide the investor with insights that yield competitive advantages [4].

Project goals

In this project I first implement, evaluate and interpret the simplest of the predator-prey models, and using the results I build more complex models, which can be applied in a financial setting. The following models are going to be investigated:

- Classical LV model
- Classical LV model at equilibrium
- LV model with prey limit and predation efficiency
- LV model with one predator - two preys without interaction between the preys
- LV model with one predator - two preys, in a financial setting

My own contribution to the topic

The first three models are discussed in the (Rubin H Landau 2008) book already, but was implemented in Python and I used a sligthly modified version of the equations which was presented there. My own contribution to these tasks is to recreate them using R and its DeSolve library and to validate my further results.
The one predator - two-prey model was discussed in (Elettreby 2009) and (Kesh, Sarkar, and Roy, n.d.), but with an interaction term, either synergistic or competitive, included between the preys, so I found it interesting if I can analyze the results without the interactions.
Finally, I recreated the financial model presented in (Addison, Bhatt, and Owen 2016), and evaluated it at a certain parameter set, which gives a stable, and an unstable result.

Materials and methods

R programming language

R is a language and environment for statistical computing and graphics. It is a GNU project which is similar to the S language and environment which was developed at Bell Laboratories (formerly AT&T, now Lucent Technologies) by John Chambers and colleagues. R provides a wide variety of statistical (linear and nonlinear modelling, classical statistical tests, time-series analysis, classification, clustering, ...) and graphical techniques, and is highly extensible. I created this document using RStudio, and it was generated with the RMarkdown framework.

ggplot2 library

ggplot2 is an R library for creating graphics, based on the The Grammar of Graphics. It has been created by Hadley Wickham and is part of the tidyverse project. ("Https://Www.r-Graph-Gallery.com," n.d.)

DeSolve library

R package deSolve is the successor of R package odesolve, which is a package to solve initial value problems (IVP) of ordinary differential equations (ODE), differential algebraic equations (DAE), partial differential equations (PDE), and delay differential equations (DeDE). I used this package to solve the systems of nonlinear differential equations present in predator-prey models.
The particular function I used for the solutions is the *lsoda* function

("Https://Www.rdocumentation.org/Packages/deSolve," n.d.).

Possible source of errors

When integrating differential equations, one must always take the step-error into account. Derivatives require small differences which are prone to subtractive cancelations and round-off error accumulation. When I solve the simplest LV model, one can observe that the limit cycles do not form closed curves, which is due to the errors mentioned before.
Also I tried to set the time intervals big enough to see meaningful information on the plots and I used an appropriate resolution on these intervals to avoid further errors.

Classical LV model

This model describes the interaction of a predatory and a prey species. The predators try to catch the preys, thus increasing their population, and decreasing the prey's. In this model the prey species has unlimited resources to feed, and in this way the population of the prey species grows exponentially if no predatory species are present. We can summarize the whole model in a system of nonlinear differential equations:

$$\frac{dX}{dt} = aX - bXY$$

$$\frac{dY}{dt} = bkXY - lY$$

where x and y are the prey and predator density, respectively, a and l determines how fast the preys reproduce and the predators die of hunger. b and k describes the interaction part of the equation, where b governs the interaction rate between the two species, and k is set the efficiency of which the predators convert prey to food.
The parameters which were used in the simulations were summarized in *Table 1*.

Figure 1. Time evolution of the prey and predator densities

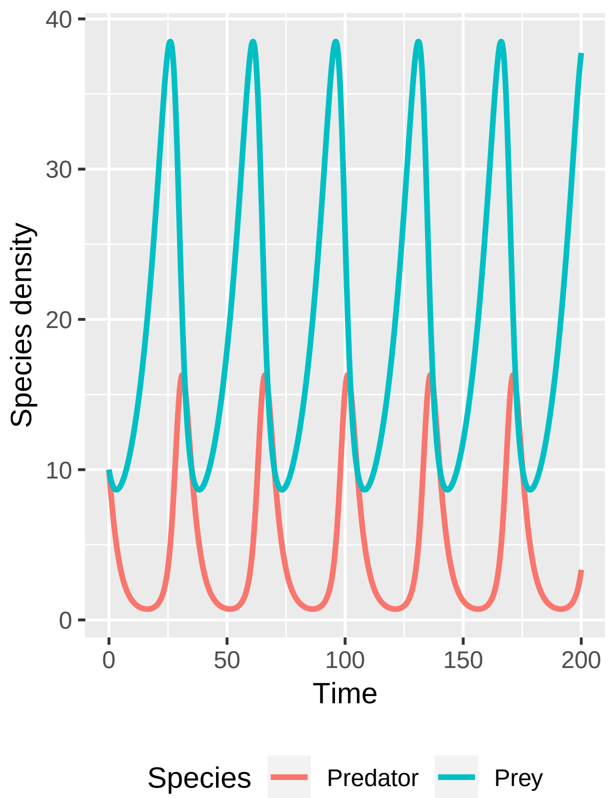


Figure 2. Phase space of prey and predator densities, with different prey initial density values

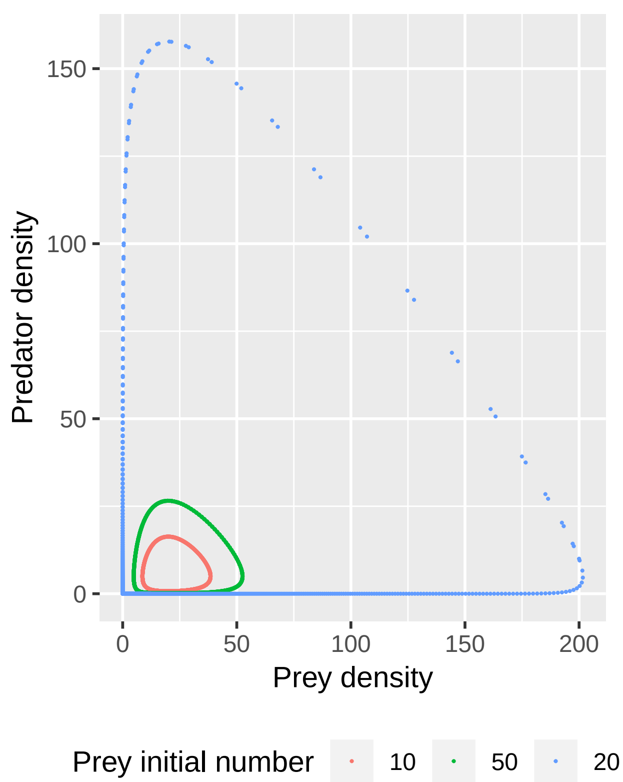


Table 1: Parameter values for the simplest LV model

parameter	value
a	0.10
b	0.20
k	0.02
l	0.40
X0	10.00
Y0	10.00

One can show that there exists a non-trivial stable solution to this model by setting the followings:

$$\dot{X} = 0$$

and

$$\dot{Y} = 0$$

gives

$$X_0 = \frac{l}{bk}, Y_0 = \frac{a}{b}$$

which results in a stable equilibrium solution. Starting the simulation with these initial values gives us a non-trivial, stable state (see *Table 2*).

Figure 3. Time evolution of the prey, and predator densities in the steady state

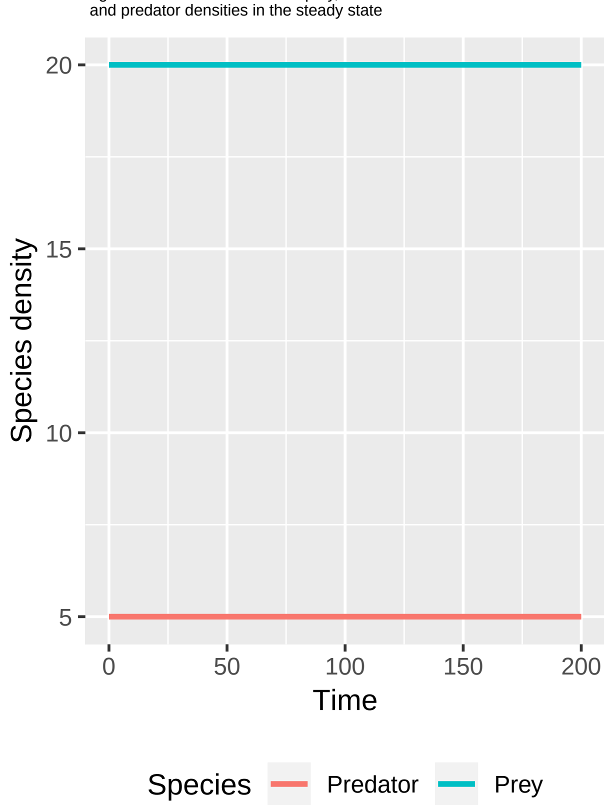


Figure 4. Phase space of prey and predator densities in the steady state

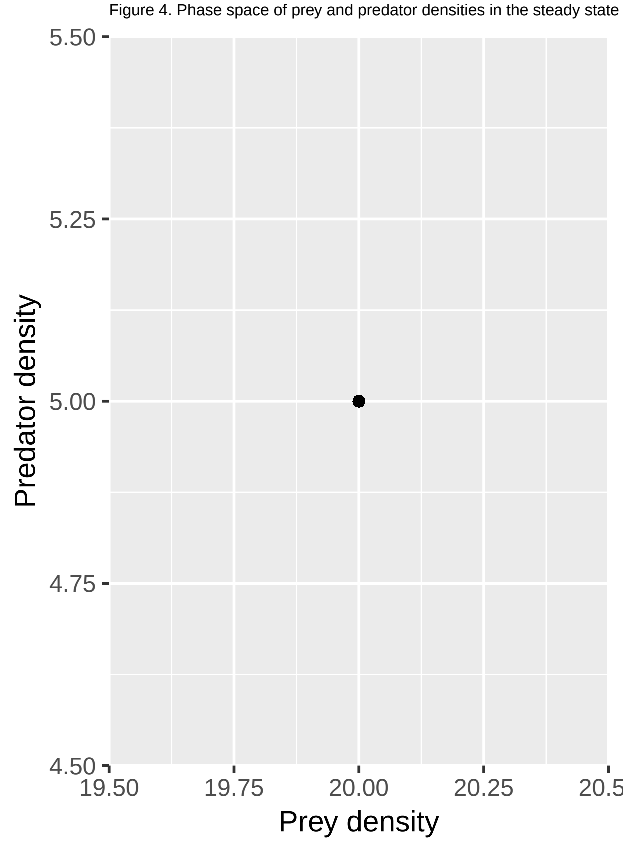


Table 2: Parameter values for the simplest LV model in equilibrium

parameter	value
a	0.10
b	0.02
k	0.02
l	0.40
X0	20.00
Y0	5.00

Discussion

On *Figure 1* one can see that the solution of this model is an oscillatory behavior, where the prey and predator density rise and fall out of phase respect to each other. Plotting the x, y phase space on *Figure 2* shows closed orbits, called limit cycles. However, the orbits are not perfectly closed due to the errors in the numerical integration of the equations (but cannot be seen in this resolution).
In *Figure 3* and *Figure 4* we can see the solution for the equilibrium state, which gives us a constant value both for the prey and predator densities, and the representation in the phase space is a single point as we expected.

LV model with prey limit and predator efficiency

This model extends the classical LV model in such a way, that it includes a so called K carrying capacity for the prey population, and a functional response palpha, as the probability a predator finds one prey. These two parameters make the model more realistic, as in reality the resources available for the preys are not unlimited, and the hunt by the predators are not always successful, and takes time. The equations describing this system is the following:

$$\frac{dX}{dt} = Xa(1 - \frac{X}{K}) - \frac{bYX}{1 + bX\tau}$$

$$\frac{dY}{dt} = bkXY - lY$$

where K is the carrying capacity of the prey population, and τ is the time the predator spends handling the prey. The initial parameters are summarized in *Table 3*.

Table 3: Parameter values for the LV model with prey limit and predatory efficiency

parameter	value
a	1e-01
b	2e-02
k	4e-01
l	2e-02
K	1e+02
tau	3e-01
X0	1e+01
Y0	1e+01

Figure 5. Time evolution of the prey and predator densities, as prey limit and predatory efficiency are included

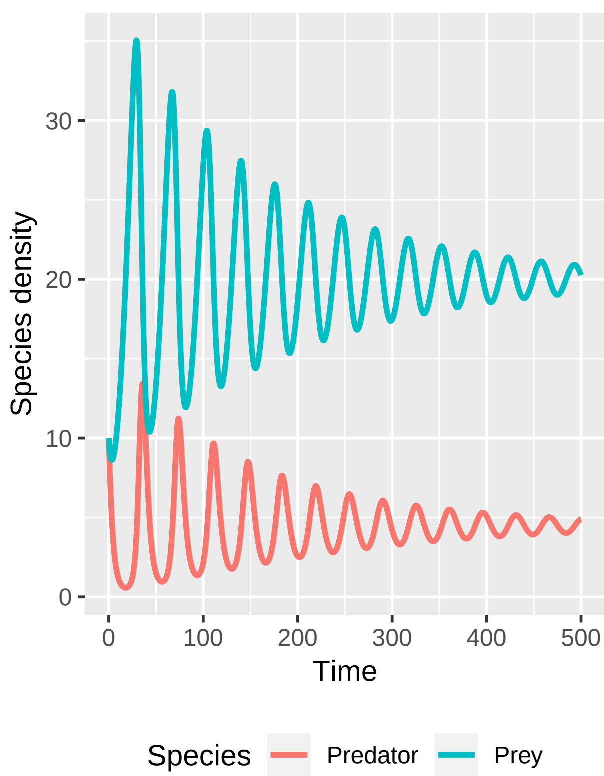
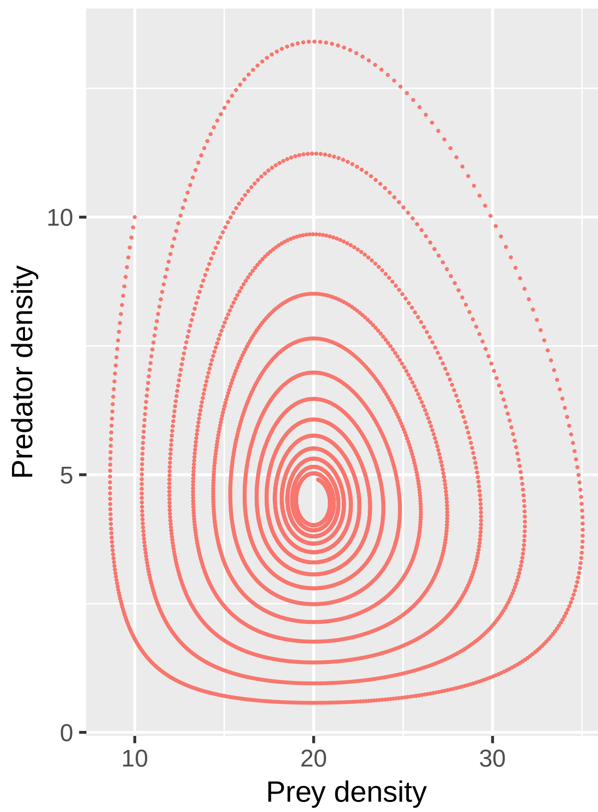


Figure 6. Phase space of prey and predator densities



Discussion

As we can see in *Figure 5* the two species density oscillates with these initial parameters, but they dampen down after some time and converges to an equilibrium. Depending on b, the result could be also an overdamped, oscillation-free function, or a limit cycle. We should note that, that the equations can be extended with predator limit as well.

Two preys, one predator LV model

This modification of the original LV model includes the extensions mentioned above, and also includes one more prey species. In reality there can be preys which can be hunted down more easily by the predator, and could have different carrying capacity. These biological information are included in the parameters of this model.

$$\frac{dX_1}{dt} = Xa_1(1 - \frac{X_1}{K_1}) - \frac{b_1 Y X_1}{1 + b_1 X_1 \tau_1}$$

$$\frac{dX_2}{dt} = Xa_2(1 - \frac{X_2}{K_2}) - \frac{b_2 Y X_2}{1 + b_2 X_2 \tau_2}$$

$$\frac{dY}{dt} = \delta_1 X_1 Y + \delta_2 X_2 Y - lY$$

where the meaning of the parameters is the same as discusses before, but indexed as for the *i-th* prey. The δ_i parameters define the gain from that the predator chooses to eat prey i.
First, I ran a simulation where the parameters for the two prey are the same, except the second prey reproduce a little faster than the first, and I observed the results (*Figure 7*). After this I was curious what happens if I set the predatory conversion rate and the predatory gain for prey 1 to a bit higher value (see *Table 5*). Biologically it would mean the predator catch that particular prey more easily and gain more from them, and also decrease the their population faster. I expect that the first prey population will decrease and converge **faster** to a value near zero, while the second prey population remains unchanged or a bit decreased compared to the unmodified system.

Figure 7. Time evolution of the prey and predator densities in the two−prey, one predator model

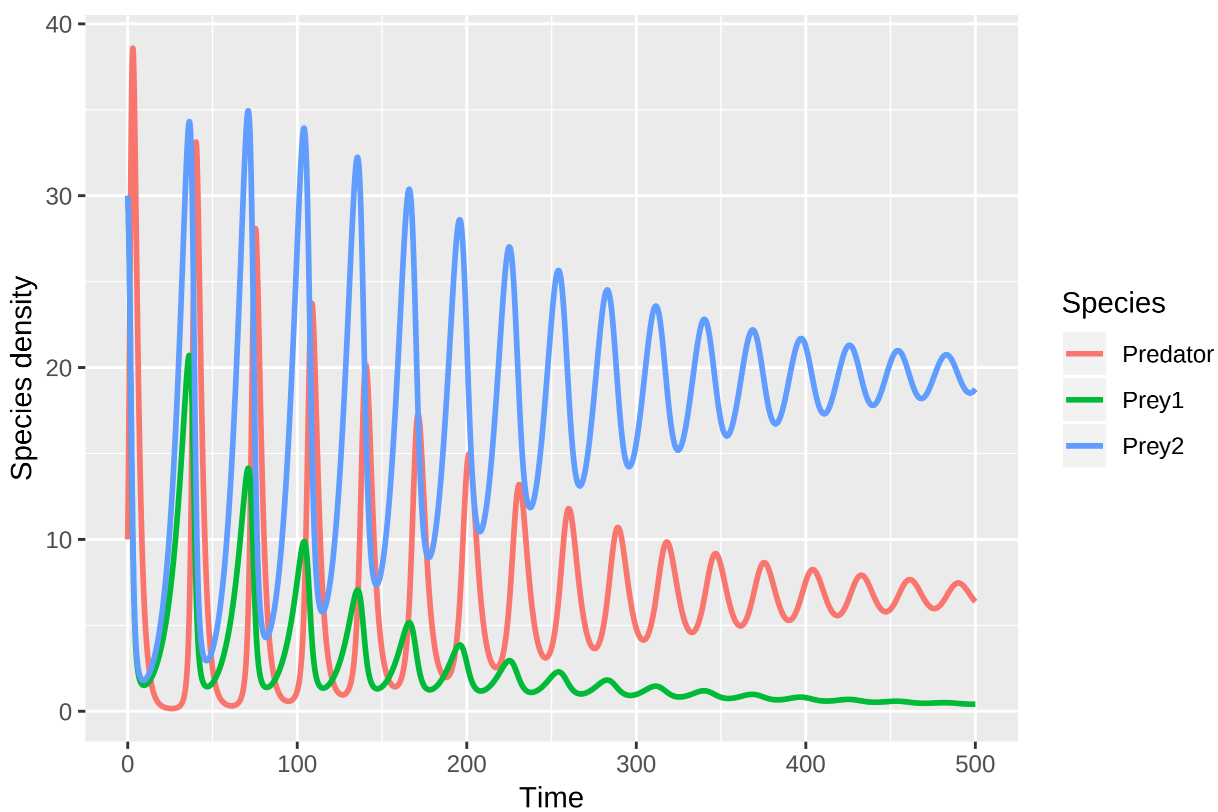


Table 4: Parameter values for the two-prey - one predator model

parameter	value
a1	0.13
a2	0.15
b1	0.02
b2	0.02
K1	100.00
K2	100.00
tau1	0.30
tau2	0.30
l	0.40
delta1	0.02
delta2	0.02
X1_0	30.00
X2_0	30.00
Y0	10.00

Figure 8. Time evolution of the prey and predator densities in the two−prey, one predator model

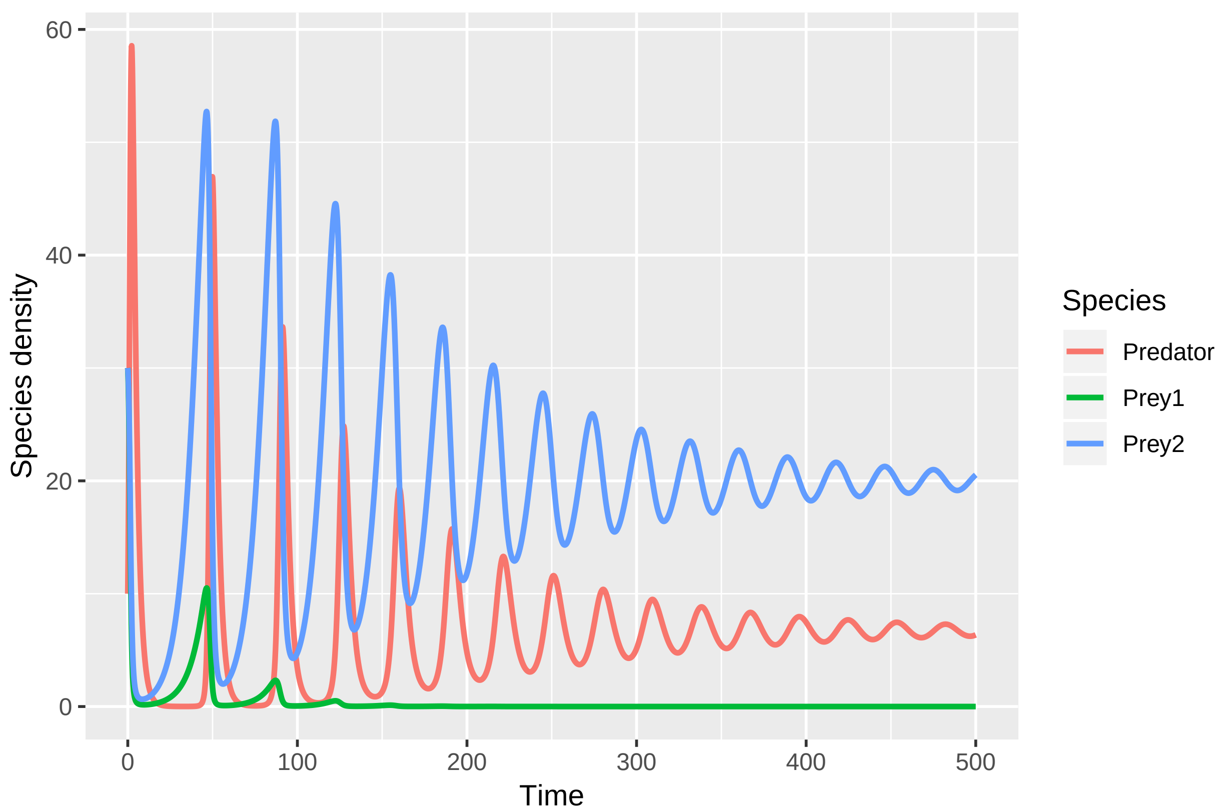


Table 5: Parameter values for the modified, two-prey - one predator model

parameter	value
a1	0.130
a2	0.150
b1	0.020
b2	0.025
K1	100.000
K2	100.000
tau1	0.300
tau2	0.300
l	0.400
delta1	0.020
delta2	0.040
X1_0	30.000
X2_0	30.000
Y0	10.000

Discussion

As one can see in *Figure 7* by letting the second prey reproduce faster means that the predator nearly kills all the preys from the first species, while maintaining a stable state of its population. Also the second prey can sustain a bigger population in long term.
In *Figure 8* we can see that the in the modified system the population of the first prey species indeed converge faster to zero as I expected, due to the fact that the predators can hunt them down more easily. Also, the predator population converges to an equilibrium a bit higher than in the unmodified system.

Stock market application

Within recent times, application of models which originate from biology have become used in finance to describe the complex behavior of the stock market. There are multiple approaches (Ghezzi 1992) which tries to apply the slightly modified version of the predator-prey models to analyze how venture capital investments exhaust the available stock of opportunities. A trader would be likely to make a more informed choice between competing stocks or shares based on comparisons among prices per share.
In the model described in this chapter the financially powerful trading company who wants to acquire smaller, financially weak companies, is the predator, while the smaller companies represent the prey. The predator makes a monetary offer for the prey shares. Once purchased, these shares are held until they are converted to predator shares.
The main objective is to simulate the system described above, and to make profit from the price changes.
In order for this to be profitable financially, the predator needs to have an estimate of the risk involved in this trade and this is based on changes in his predatory share prices.
Analyzing the simulation results can allow the trader to find the specific parameters which makes the system stable, and this would allow determine if the acquisition of prey company shares is profitable, or not.

The equations describing the problem are a modified one predator - two prey model, with a competitive interaction term between the preys, which simulates the competition on the free market between the firms. The equations can be seen below. Note that I used a slightly different notations for the parameters from the previous models to be more aligned with the original article.

$$\frac{X_1}{dt} = \alpha_1 X_1 \left(1 - \frac{X_1}{K_1}\right) - m_{12} X_1 X_2 - \beta X_1 Y r_1$$

$$\frac{X_2}{dt} = \alpha_2 X_2 \left(1 - \frac{X_2}{K_2}\right) - m_{21} X_1 X_2 - \beta X_2 Y r_2$$

$$\frac{dY}{dt} = -\mu Y + c_1 \beta_1 X_1 Y r_1 + c_2 \beta_2 X_2 Y r_2$$

$$r1 = \frac{1}{1 + \left(\frac{a_2 X_2 + b_2 Y}{X_1}\right)^n}$$

$$r2 = \frac{1}{1 + \left(\frac{a_1 X_1 + b_1 Y}{X_2}\right)^n}$$

- X_1 and X_2 are relative share prices of competing prey companies, Y is the relative price per share of the predator company. The parameters used in the model are all non-negative.
- α_1 and α_2: Intrinsic growth rate of prey share price,
- K_1 and K_2: Carrying capacity of prey shares,
- m_{12} and m_{21}: Inter-specific competition between prey market shares,
- β_1 and β_2: Predatory capture rate is the probability predator Y invests in prey X_1 or X_2,
- a_1 and a_2: Harvesting rate of prey shares,
- b_1 and b_2: Anti-predator behavior of prey shares,
- c_1 and c_2: Rate of conversion of prey shares to predator shares,
- μ: Rate of decline of predator share price,
- n : Multiplicative effect due to the predatory functional response.

I reproduced two results from the original article, where I ran simulations with parameter values which gave a stable (*Figure 9*), and unstable (*Figure 10*) results. Then I tried to modify the β_2 parameter for the second company in such a way, that the predator company invest more often to this second company. If the system is not so robust for this change, then I expect that it is going to destabilize the whole ecosystem, and drives the companies into bankruptcy (*Figure 11*). The parameters for the three simulations are summarized in *Table 6*.

Figure 9. Time evolution of the relative share price of the prey and predator companies in equilibrium

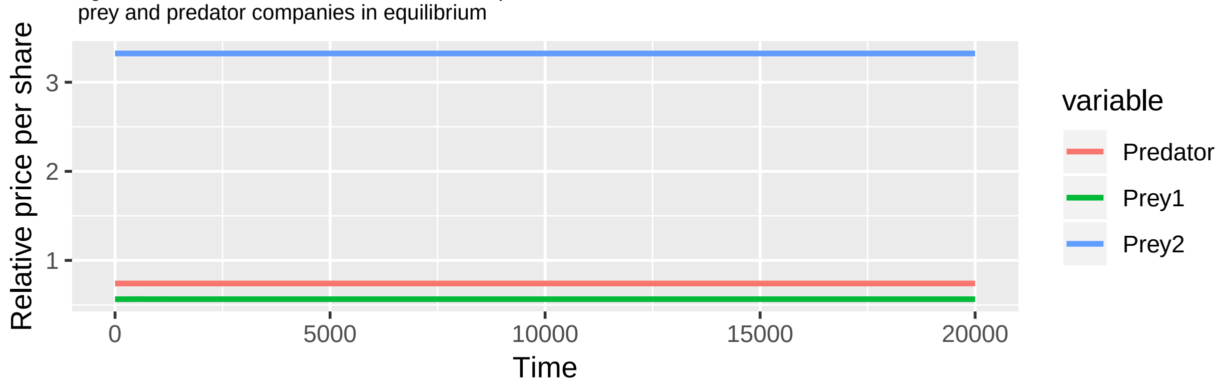


Figure 10. Time evolution of the relative share price of the prey and predator companies in an unstable state

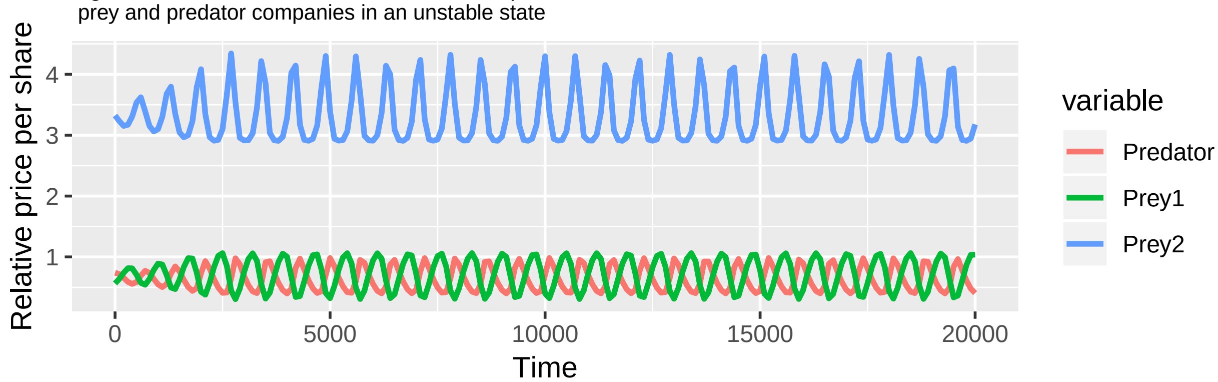
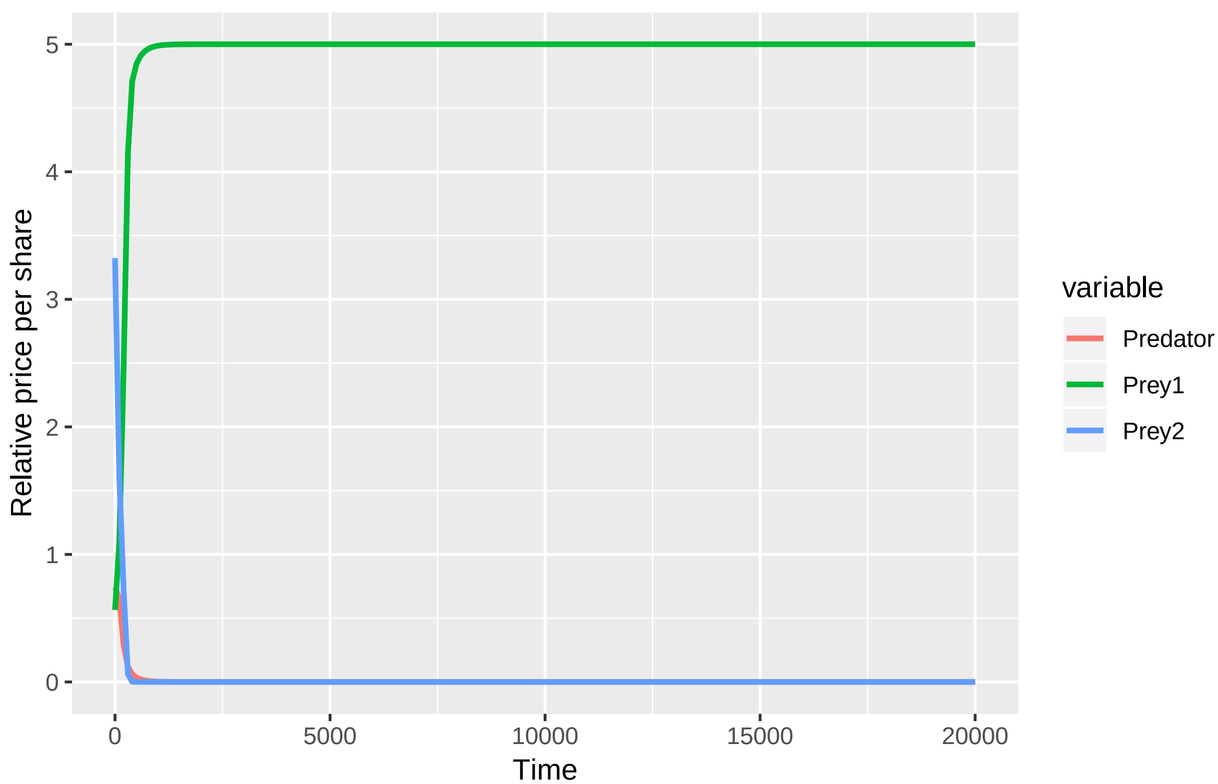


Table 6: Parameter values for the modified, two-prey - one predator model

parameter	value
a1	0.0200
a2	1.5000
b1	0.0100
b2	0.0300
alpha1	0.0400
alpha2	0.0500
beta1	0.0300
beta2	0.0300
K1	5.0000
K2	10.0000
m12	0.0100
m21	0.0200
mu	0.0200
n	1.0000
c1	0.1000
c2	0.2000
X1__0	0.5644
X2__0	3.3233
Y0	0.7406

Figure 10. Time evolution of the relative share price of the prey and predator companies in an unstable state



Discussion

In *Figure 9-10* can be seen that I successfully reproduced the results presented in the article. In *Figure 11* one can see that by changing the β_2 parameter, it caused the second prey and the predatory company to go nearly bankrupt, but not the first one. This is not what I expected, but can be explained. I think by letting the predator company to buy too much of the shares of the second prey company, it is causing a destabilization regarding them, and the first prey company can reach its prey limit capacity independently from them. Nevertheless, it is an interesting result, and need to be explore further.

Meaning of the parameters and the stability in the financial world

The *a1*, *a2*, *b1*, *b2* parameters represents the predatory functions, and in more detail the harvesting rate and the anti-predatory behavior of the preys. These represent the harvesting rate of prey and anti-predator behavior of prey respectively. In the financial world, these are taken to be the equity risk premium (excess returns based on investment in the stock market) and price volatility index (a measure of risk in terms of 'investor fear' in the stock market) respectively from the two prey stocks. The parameters *c1* and *c2*, which naturally represents in biology the conversion rates of the i-th prey to predator are represented financially as the price to earnings ratio of the prey stock respectively.

Therefore, a predator company who wishes to invest in either of these companies can be advised that if the stock parameters of those companies are kept within the stability intervals, their investment would be stable and profitable. Otherwise, parameters outside of these intervals would cause the model to become unstable and therefore it would not be wise to invest.

Summary

In this project I simulated multiple predator-prey models, and discussed their results. I validated two of my hypotheses about the outcome of the simulations if I modify their initial parameters. Last, but not least I recreated the results of an article, which applies a specific predator-prey model to a financial problem.

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