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A Markov Renewal Model for Rainfall Occurrences

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A probabilistic model for the temporal description of daily rainfall occurrences at a single location is presented. By defining an event as a day with measurable precipitation the model is cast into the discrete-time point process framework. In the proposed model the sequence of times between events is formed by sampling from two geometric distributions, according to transition probabilities specified by a Markov chain. The model belongs to the class of Markov renewal processes and exhibits clustering relative to the independent Bernoulli process. As a special case, it reduces to a renewal model with a mixture distribution for the interarrival times. The rainfall occurrence model coupled with a mixed exponential distribution for the nonzero daily rainfall amounts was applied to the daily rainfall series for Snoqualmie Falls, Washington, and was successful in preserving the short-term structure of the occurrence process, as well as the distributional properties of the seasonal rainfall amounts.

1. INTRODUCTION

Point process theory has been widely used to model the stochastic structure of short-term rainfall [cf. *Kavvas and Delleur*, 1981; *Gupta and Waymire*, 1979; *Waymire and Gupta*, 1981*a*, *b*; *Waymire et al.*, 1984; *Smith and Karr*, 1983, 1985]. Rainfall data are usually available in the form of cumulative amounts over disjoint equispaced time intervals. In adapting the continuous-time point process theory [cf. *Çinlar*, 1975; *Cox and Lewis*, 1978; *Cox and Isham*, 1980] to modeling short-term rainfall, two approaches can be followed. The first is to define an "event" as a day with measurable precipitation and develop discrete-time point process models to describe the probabilistic structure of the sequence of rainy and dry days. Such a probabilistic model is proposed in this paper. The second approach is to assume the existence of an underlying continuous-time rainfall occurrence process whose outcome is only observed as the integral of the continuous process over the given sampling interval. Under the second approach, one tries to infer the properties of the underlying continuous-time process from the observed discrete data. Results in this direction have been reported by *Rodriguez-Iturbe et al.* [1984], *Valdes et al.* [1985], and *Foufoula-Georgiou and Guttorp* [1986]. The main conclusion of these studies is that the inferred description of the underlying process depends on the time scale at which the fitting of the model is made. This poses limitations on the model in terms of inability to extrapolate at other time scales and inability to infer properties of the underlying rainfall-generating mechanism based on the sampled realizations. In addition, estimation problems arise when observations over relatively long sampling intervals, such as days, are used to estimate the parameters of continuous-time models [*Foufoula-Georgiou and Guttorp*, 1986].

The daily rainfall occurrence process has been extensively studied over the past two decades. The only discrete-time models investigated to date are Markov chains (see, for example, *Gabriel and Neumann* [1962]), the discrete autoregressive moving average models (DARMA) [*Chang et al.*, 1984], and a discrete-time alternating renewal model of [*Galloy et al.*, 1981]. Markov chains have been found, in general, to be inadequate to model the clustering dependencies present in daily rainfall occurrences. Models from the DARMA family [*Jacobs and Lewis*, 1978] were used by *Chang et al.* [1984], who reported satisfactory results in modeling daily rainfall occurrences in Indiana. In our view the main disadvantage of the DARMA models is the lack of physical motivation for the model structure and the discontinuous memory they exhibit [cf. *Keenan*, 1980]. On the other hand, point process theory permits more elegant mathematical formulations of intuitively appealing dependence properties of the process, such as the conditional intensity function or the index of dispersion, which also provide measures of clustering. Recently, *Smith* [1987] has proposed a new family of discrete point process models for daily rainfall occurrences. Theoretical and empirical comparisons of those models (termed Markov Bernoulli models) with the class of models proposed herein would be worth investigating.

Daily rainfall occurrences are the result of the interaction of several rainfall-generating mechanisms. For example, the first rainy day in a wet period may be the result of a frontal storm passing over a region, whereas subsequent rainy days in the same wet period may be considered secondary events. In that sense, times between events may come from different probability distributions, for instance, one with a small mean and coefficient of variation for the secondary events and one with a large mean and coefficient of variation for the primary events. The sequence of event types is governed by transition probabilities with higher probabilities of having secondary events after a primary event or after a small number of secondary events. In the model proposed in this paper the times between daily rainfall occurrences are sampled from two different probability distributions (this is called a two-state process) which we assume to be geometric. The transition from one interarrival type (or event type) to the other is governed by a Markov chain. This model belongs to a class known as Markov renewal models [cf. *Çinlar*, 1975; *Cox and Lewis*, 1978]. Markov renewal processes are, in general, nonrenewal, a nomenclature

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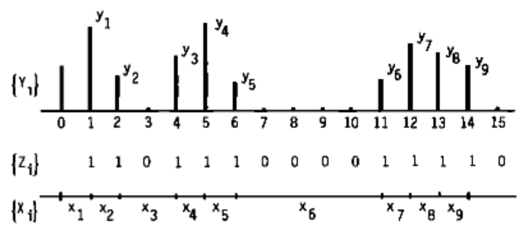

Fig. 1. Schematic representation of a daily rainfall process and definition of the $\{Y_i\}$, $\{Z_i\}$, and $\{X_i\}$ series. Note that the process starts at an arbitrary event. $\{Y_i\}$ is the series of the nonzero daily rainfall amounts, $\{Z_i\}$ is the binary series of rain-no rain, and $\{X_i\}$ is the series of interarrival times.

that may at first seem contradictory. The term Markov renewal refers to the conditional dependence of the present state (interarrival type) on the previous state only and not on states before that. By contrast, for a renewal process (which results as a special case of the Markov renewal process) the present state is independent of all previous states.

Apart from the intuitive appeal that discrete rainfall amounts represent the combined effect of several underlying mechanisms, a justification for using a mixture model may be provided by the form of the cumulative distribution function $F(x)$ of the interarrival times. For a geometric distribution with parameter p the log-survivor function ($\ln(1 - F(x))$) of the intervals is a straight line with slope $\ln(1 - p)$. Log-survivor functions of times between events at several stations located throughout the United States [*Foufoula-Georgiou and Lettenmaier*, 1986] suggest that the interarrival times come from two different geometric distributions. It should be noted, however, that graphical identification of mixtures is extremely difficult, in general, [cf. *Leytham*, 1984] and becomes even harder for discrete data.

The general class of Markov renewal processes, to which the proposed discrete-time point process model proposed belongs, were introduced by *Smith* [1955] and were later studied by *Pyke* [1961*a*, *b*] and *Cox* [1963]. An extensive bibliography of theoretical developments and applications of Markov renewal processes is given by *Teugels* [1976]. Markov renewal processes have a flexible dependence structure. It will be seen later that Markov chains, Markov processes, renewal processes, and alternating renewal processes [cf. *Çinlar*, 1975] are all special cases of the general Markov renewal process. In order to illustrate how our model differs from a Markov chain we briefly note that the probability of having a rainy day does not depend on the condition (rain-no rain) of the previous day but rather on the number of days since the last rain. Within a rainy period (consecutive rainy days), however, our process behaves as a Markov chain.

The emphasis of the work presented in this paper is on the modeling of the occurrence process. Others [e.g., *Woolhiser and Roldan*, 1982] have addressed the modeling of event scale rainfall amounts. Although this paper is concluded with an example in which both the occurrences and amounts are modeled, the amounts modeling part closely follows the work of others. It is included primarily for completeness and to allow assessment of the performance of the occurrence model as it affects the modeling of cumulative (seasonal) amounts.

2. Rainfall Occurrence Model

Consider the daily rainfall process schematically presented in Figure 1. Let $\{Y_i\}$ denote the series of nonzero daily rainfall amounts, $\{X_i\}$ denote the series of times between events (interarrival times), and $\{Z_i\}$ denote the binary series of zeros and ones, zeros for dry days and ones for wet days. Rainfall modeling at the event scale is best performed in two steps: the occurrence of rainfall is modeled first, followed by the modeling of the amounts; finally, the two models are superimposed.

Our rainfall occurrence model describes the sequence of interarrival times $\{X_i\}$. Useful properties of the binary series $\{Z_i\}$, such as transition probabilities between rainy and dry days, are subsequently derived. The model of interarrival times is a discrete-time Markov renewal model. A formal definition of a Markov renewal process in continuous time is given by *Çinlar* [1975, p. 313]:

Definition. For each $n \in N$, let a random variable S_n take on values in a countable set of states $E = \{1, 2, \cdots\}$ and a random variable T_n take on values in $R_+ = [0, +\infty)$ such that $0 = T_0 \le T_1 \le T_2 \le \cdots$. The stochastic process $(S, T) = \{S_n, T_n\}$, $n \in N$ is said to be a Markov renewal process with state space E provided that

$$P\{S_{n+1} = j, T_{n+1} - T_n \le t | S_0, \cdots, S_n; T_0, \cdots, T_n\} = P\{S_{n+1} = j, T_{n+1} - T_n \le | S_n\} \qquad (1)$$

for all $n \in N, j \in E$, and $t \in R_+$.

For the rainfall occurrence model the random variable S_n is given the interpretation of the "type" (or "state") of an interarrival time and takes on values from the binary set $E = \{1, 2\}$. This is a two-state Markov renewal model where the two types of interarrival times (type 1 and type 2) are sampled according to a Markov chain with state space E. Let $\langle X_i \rangle$ denote the type of the ith interarrival time, that is, $\langle X_i \rangle = 1, 2$ for type 1 and type 2, respectively. The transition probability matrix of the Markov chain is

$$P = \begin{bmatrix} a_1 & 1 - a_1 \\ 1 - a_2 & a_2 \end{bmatrix} \qquad (2)$$

where

$$a_j = P\{\langle X_i \rangle = j | \langle X_{i-1} \rangle = j\} \qquad j = 1, 2 \qquad (3)$$

For example, given that the interarrival time X_{i-1} is of the type 1, the probability that X_i will also be of type 1 is a_1. Associated with the Markov chain are the limit or equilibrium probabilities

$$e_j = \lim_{i \to \infty} P\{\langle X_i \rangle = j\} \qquad j = 1, 2 \qquad (4)$$

which are the unconditional probabilities of any interval X_i being of type 1 or type 2. Note that $e_2 = 1 - e_1$. From the theory of Markov chains [cf. *Cox and Miller*, 1965] it is known that

$$e_1 = \frac{1 - a_2}{2 - a_1 - a_2} \qquad (5)$$

Note that if the conditional probabilities a_1 and a_2 are equal to the unconditional probabilities e_1 and e_2 (in that case, $a_1 + a_2 = 1$), the process of the types of interarrival times reduces to a renewal process.

To complete the model description, we need to specify what the type 1 and type 2 interarrival times mean. An interarrival time X_i is said to be of type 1 (or type 2) if it is sampled from a probability distribution $f_1(x)$ (or $f_2(x)$). For the rainfall occurrence model these distributions have been assumed geometric with parameters p_1 and p_2, respectively. One can write therefore that

$$f_j(x_i) = P\{X_i = x_i | \langle X_i \rangle = j\} = p_j(1-p_j)^{x_i - 1} \qquad j = 1, 2 \tag{6}$$

The assumption of geometric distributions is supported by the data analysis presented later in this paper and by *Foufoula-Georgiou* [1985]. In the rest of this section the statistical properties of intervals and counts for the proposed two-state Markov renewal model are derived.

2.1. *Interval Properties*

The moment-generating function of the interarrival times of a two-state Markov renewal model is given as

$$\psi(z) = e_1\psi_1(z) + e_2\psi_2(z) \tag{7}$$

where $\psi_j(z)$, $j = 1, 2$ is the moment generating function of the probability distribution of the type j intervals. For a geometric distribution with parameter p,

$$\psi(z) = \frac{pz}{1 - (1-p)z} \tag{8}$$

[cf. *Parzen*, 1960]. Moments of the interarrival times are then obtained from

$$E(X^k) = (-1)^k \left.\frac{d^k\psi(z)}{dz^k}\right|_{z=1} \tag{9}$$

For instance, the mean, variance, and survivor function of the interarrival times are given by

$$E(X) = e_1/p_1 + e_2/p_2 \tag{10a}$$

$$\operatorname{Var}(X) = e_1(1-p_1)/p_1{}^2 + e_2(1-p_2)/p_2{}^2 + e_1e_2(1/p_1 - 1/p_2)^2 \tag{10b}$$

$$R(x) = e_1(1-p_1)^x + e_2(1-p_2)^x \tag{10c}$$

It is important to note here that the proposed model admits coefficients of variation of interarrival times with values less or greater than one. In contrast, both the continuous-time Neyman-Scott [e.g., *Kavvas and Delleur*, 1981] and the doubly stochastic Poisson [*Smith and Karr*, 1983] processes have coefficients of variation always greater than one. In addition, it can be shown after some algebra that the coefficient of variation of the proposed Markov renewal model is always greater than $1 - m$, where $m = 1/E(X)$ is the rate of occurrence of the process. This observation suggests that the proposed process is always overdispersed (more clustered) than an independent Bernoulli process with the same rate of occurrence, which would have a coefficient of variation equal to $1 - m$. This is a desirable property since an analysis of several rainfall series [*Foufoula-Georgiou and Lettenmaier*, 1986] suggests that most daily rainfall occurrence series are overdispersed relative to the Bernoulli process.

The autocorrelation function of the interarrival times of the two-state Markov renewal process takes the form [cf. *Cox and Lewis*, 1978, p. 196]

$$\rho_k = c\beta^k \tag{11}$$

where

$$c = \frac{e_1e_2(1/p_1 - 1/p_2)^2}{e_1(1-p_1)/p_1{}^2 + e_2(1-p_2)/p_2{}^2 + e_1e_2(1/p_1 - 1/p_2)^2} \tag{12}$$

$$\beta = a_1 + a_2 - 1 \tag{13}$$

Consequently, the spectral density function of the intervals is

$$f_+(\omega) = \frac{1}{\pi}\left(1 + 2c\,\frac{\beta\cos\omega - \beta^2}{1 + \beta^2 - 2\beta\cos\omega}\right) \qquad 0 \le \omega \le \pi \tag{14}$$

Note that the autocorrelation function of the intervals becomes zero (renewal process) for $a_1 + a_2 = 1$, in which case the Markov chain of the type of intervals has conditional probabilities of occurrence equal to the unconditional ones, that is, it reduces to a Bernoulli process. Without loss of generality, we can assume that the type 1 interarrival times are sampled from the geometric distribution with the smaller mean. Then, for persistent structures (clustering) the conditional probability of being in state 1 is greater than the unconditional probability of (5), resulting in $a_1 + a_2 > 1$. In this case, the interarrival times have a positive autocorrelation function decaying with a rate $(a_1 + a_2 - 1)$.

2.2. *Count Properties*

Let

$$Z_k = 1(Y_k > \varepsilon) \qquad k \ge 0 \tag{15}$$

be the binary series of zeros and ones, where $1(E)$ is an index function taking the value of 1 if E occurs and zero otherwise, and where zeros (ones) correspond to days with cumulative rainfall less (greater) than ε. The small quantity ε (consistent with the previous definition of an event) has been taken equal to 0.01 inches. In this section we compute the statistical properties of the Z_k series in terms of the four parameters a_1, a_2, p_1, and p_2 of the Markov renewal model. The rate of occurrence of the Z_k process is

$$m = P\{\cdots\} = P\{Z_k = 1\} = \frac{p_1p_2(2 - a_1 - a_2)}{p_1(1-a_1) + p_2(1-a_2)} \tag{16}$$

One of the most descriptive statistical properties of a continuous-time stationary point process is its conditional intensity function [cf. *Cox and Lewis*, 1978, p. 73]. An analogous property for a discrete-time point process may be defined as

$$h_k = P\{Z_{t+k} = 1 | Z_t = 1\} = P\{Z_k = 1 | Z_0 = 1\} \qquad k = 1, 2, \cdots \tag{17}$$

which essentially defines a sequence of conditional probabilities of occurrence (note that the last equality is due to stationarity). The interpretation of h_k with respect to clustering remains the same as in the continuous case; values of h_k greater than the constant (unconditional) probability of occurrence m imply that the chance of having an event at time $t + k$ due to an event at time t is greater than the chance of having an event at any arbitrary time. Below we give the expression for h_k.

Proposition. The conditional probability of occurrence h_k of the discrete-time two-state Markov renewal process described above takes the form

$$h_k = m + AW^{k-1} \qquad k = 1, 2, \cdots \tag{18}$$

where

$$A = e_1p_1 + e_2p_2 - m \tag{19}$$

$$W = 1 - p_1(1-a_1) - p_2(1-a_2) \tag{20}$$

An outline of the proof is given in Appendix A.

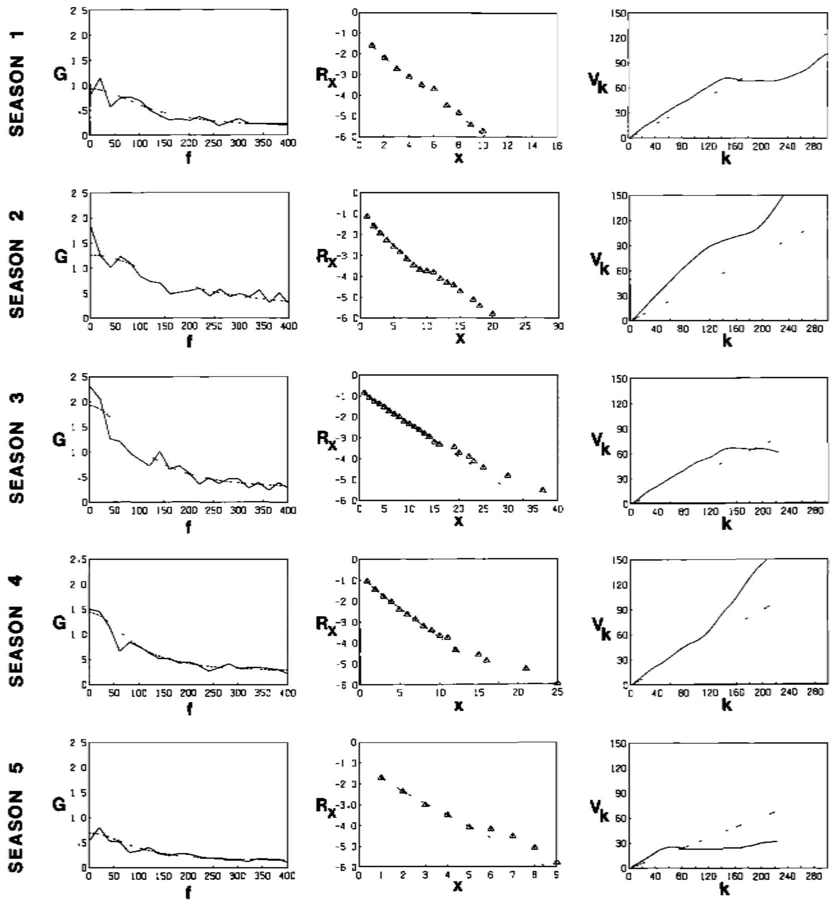

Fig. 2. Comparison of empirical (solid lines and open triangles) and theoretical (dashed lines) functions of the Markov renewal model fitted to daily rainfall occurrences at Snoqualmie Falls, Washington. G versus f is the normalized spectrum of counts $g_+'(\omega)$ versus frequency factor $f = \omega T/2\pi$, R_x versus x is the log-survivor function $\ln R(x)$ versus interarrival time x (days), and V_k versus k is the variance of counts V_k versus interval length k (days). For more details on these functions see text.

It is particularly important to note in (18) that since A is positive and $0 < W < 1$, the conditional intensity function decreases geometrically to the constant intensity m of the process. This implies that the Markov renewal process exhibits clustering. The shape of the conditional intensity function is only indicative of the presence, but not the type, of clustering. However, the fact that the coefficient of variation of the interarrival times is always greater than the coefficient of variation of the Bernoulli process suggests that the form of clustering is overdispersion relative to the Bernoulli process. This is the type of clustering found in most daily rainfall occurrence series [e.g., *Foufoula-Georgiou*, 1985].

Having an expression for the conditional intensity function, all the other properties of the counting process can be readily obtained. The expected number of events within a period of k time units (for example, days) after the occurrence of an event is given as

$$H_k = mk + A\sum_{i=1}^{k} W^i = mk + A\frac{W^{k+1}-1}{W-1} \qquad k = 1, 2, \cdots \tag{21}$$

Notice that as $k \to \infty$, $H_k - mk \to 0$, where mk is the expected number of events in any one period of length k time units. The variance of counts, that is, the variance of the number of events in a period of k time units after the occurrence of an event, is

$$V_k = mk - m^2k^2 + 2m\sum_{i=1}^{k-1}(k-i)h_i \tag{22}$$

where h_i is given by (18). Finally, the index of dispersion is $I_k = V_k/mk$. Note that the above formulae for V_k and I_k apply to any discrete-time point process with conditional intensity function h_k (see also *Guttorp* [1986]).

The spectrum of counts, $g_+(\omega)$, of the two-state Markov renewal process is

$$g_+(\omega) = \frac{m}{\pi}\left(1 - m - 2A\,\frac{W - \cos\omega}{1 - 2W\cos\omega + W^2}\right) \qquad 0 \le \omega \le \pi \tag{23}$$

and is computed by simply taking the Fourier transform of the covariance of counts $c_k = E(Z_l Z_{l+k}) = m(h_k - m) = mAW^{k-1}$. The normalized spectrum of counts is defined as $g_+'(\omega) = \pi g_+(\omega)/m$ and is usually plotted (see Figure 2) versus a frequency factor $f = \omega T/2\pi$, where T is the total length of observation.

3. METHODS FOR FITTING THE MARKOV RENEWAL MODEL

The discrete-time Markov renewal model developed in the previous section has four parameters: a_1, the transition probability from type 1 to type 1 interval; a_2, the transition probability from type 2 to type 2 interval; p_1, the parameter of the geometric distribution of the type 1 intervals; and p_2, the parameter of the geometric distribution of the type 2 intervals. Note that the interarrival times cannot be classified directly as belonging to type 1 or type 2 by observation of the series of daily rainfall events. Only probabilistic classification is possible. Thus the transition probabilities a_1 and a_2 must be estimated together with the parameters of the two geometric distributions p_1 and p_2. In the following section, maximum likelihood and method of moments estimators for the parameters a_1, a_2, p_1, and p_2 are studied.

3.1. *Maximum Likelihood Estimation*

The observations to which the Markov renewal model is fitted are the interarrival times X_i, that is, the sequence of lengths of dry periods between consecutive rainy days. Let $\boldsymbol{\theta} = (a_1, a_2, p_1, p_2)$ denote the vector of unknown parameters and $(x_1, x_2, \cdots, x_n)$ the sampled sequence of n interarrival times.

Proposition. The likelihood function of the two-state Markov renewal model takes the form

$$L(\boldsymbol{\theta}|x_1, \cdots, x_n) = EB_1PB_2P, \cdots, B_n' \tag{24}$$

where the matrices $E, B_1, \cdots, B_n'$ are functions of the four parameters of the model and the known sampled data:

$$E = (e_1 1 - e_1) = \left(\frac{1 - a_2}{2 - a_1 - a_2} \; \frac{1 - a_1}{2 - a_1 - a_2}\right) \tag{25a}$$

$$B_i = \begin{pmatrix} p_1(1 - p_1)^{x_i - 1} & 0 \\ 0 & p_2(1 - p_2)^{x_i - 1} \end{pmatrix} \quad i = 1, \cdots, n \tag{25b}$$

$$B_n' = B_n\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{25c}$$

and P is the transition probability matrix of the Markov chain as defined in (2). The proof of the above proposition is sketched in Appendix B.

Note that for independent interarrival times the process reduces to a renewal process with a mixed geometric distribution for the interarrival times. In this case, the log likelihood function is simply

$$L'(\boldsymbol{\theta}|x_1, \cdots, x_n) = \sum_{i=1}^{n} \ln \left[e_1 p_1(1 - p_1)^{x_i - 1} + (1 - e_1)p_2(1 - p_2)^{x_i - 1}\right] \tag{26}$$

3.2. *Method of Moments Estimation*

The method of moments (MOM) uses sample estimates of the first three moments and the lag 1 covariance of the interarrival times and solves for the four parameters by using the theoretical relationships between the population moments and the parameters. This method has a major drawback in that the estimate of the third moment is highly variable, so the resulting parameter estimates can be unstable. A modified method of moments estimation which uses the median instead of the third moment was also tested [*Foufoula-Georgiou*, 1985]. Due to the discreteness of the data, however, the median has poor sampling properties, which lead to unsatisfactory performance of this method. Therefore it was dropped from further consideration.

It should be noted that the moments of the interarrival times involve only the equilibrium unconditional probabilities e_1 and $e_2 = 1 - e_1$. The transition probabilities are introduced only in the second product moments, as, for example, in the autocorrelation coefficient r_1. Therefore it is possible to use the first three moments for estimation of e_1, p_1, and p_2 and then use the first autocorrelation coefficient $r_1 = c(a_1 + a_2 - 1)$, where c is given in (12), together with e_1 of (5) to solve for a_1 and a_2:

$$a_1 = (1 - e_1)(r_1/c + 1) + 2e_1 - 1 \tag{27a}$$

$$a_2 = e_1(r_1/c + 1) - 2e_1 + 1 \tag{27b}$$

From the above two equations one can see that for acceptable parameter estimates, that is, $0 < a_1, a_2 < 1$, the following inequality must hold

$$-\min (e_1/e_2, e_2/e_1) < r_1/c < 1 \tag{28}$$

Note that the value $\min (e_1/e_2, e_2/e_1)$ corresponds to the ratio of the smallest to the largest equilibrium probability, a value always less than 1. Therefore inequality (28) is consistent with the requirement that the autocorrelation function of the process, given as $r_k = c(a_1 + a_1 - 1)^k$, is less than 1 in absolute value.

4. STATISTICAL PROPERTIES OF THE ESTIMATORS

The two methods discussed in the previous section were tested for consistency (bias) and efficiency (variability) using Monte Carlo simulation. Several sets of population parameters were selected to represent a range of underlying processes consistent with the data analysis reported by *Foufoula-Georgiou and Lettenmaier* [1986]. Two kinds of dependencies were considered in selecting population parameters: dependency in the intervals (a measure of which is the autocorrelation function r_k) and dependency in the counts or clustering (a measure of which is the conditional intensity function h_k). The type of clustering (overdispersion and underdispersion relative to the Bernoulli process) is further inferred by the variance time curve and index of dispersion. It should be emphasized that independence in intervals does not imply or result from independence in counts. For instance, a renewal process may well be clustered as, for example, the renewal Cox process with Markovian intensity [*Smith and Karr*, 1983] and the renewal form of the Markov renewal process discussed herein. In the discussion that follows the dependencies in both the intervals and counts are used to characterize the underlying process. Recall that, for a Markov renewal process, these dependencies take the form $r_k = c(a_1 + a_2 - 1)^k$ and $h_k = m + AW^{k-1}$, with A and W defined in (19) and (20).

The first set of parameters tested was $\{a_1 = 0.4, a_2 = 0.3, p_1 = 0.8, p_2 = 0.2\}$. These parameter values correspond to an occurrence process with a mean interarrival time of 2.98 days, a standard deviation of 3.59 days (coefficient of variation $c_v = 1.2$), a skewness coefficient $c_s = 3.01$, and a first autocorrelation coefficient $r_1 = 0.08$. The conditional intensity function is $h_k = 0.335 + 0.186(0.38)^{k-1}$, which indicates a clustering of counts. Five hundred synthetic sequences of 50, 100, 200, 500, and 800 events (corresponding to approximately 150, 300, 600, 1500, and 2400 days of observation, as inferred by the rate of occurrence $m = 0.34$ events per day) were generated from a Markov renewal model with the above parameters. ML and MOM parameter estimates were computed for all synthetic sequences. (For the maximization of the likelihood