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Periodic imperfect preventive maintenance with two categories of competing failure modes

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Abstract

A maintenance policy is studied for a system with two types of failure modes: maintainable and non-maintainable. The quality of maintenance actions is modelled by its effect on the system failure rate. Preventive maintenance actions restore the system to a condition between as good as new and as bad as immediately before the maintenance action. The model presented permits to study the equipment condition improvement (improvement factor) as a function of the time of the preventive maintenance action. The determination of the maintenance policy, which minimizes the cost rate for an infinite time span, is examined. Conditions are given under which a unique optimal policy exists.

Keywords: Preventive maintenance; Minimal repair; Improvement factor; Maintainable failure modes; Non-maintainable failure modes

1. Introduction

The optimal determination of maintenance policies is an important issue in reliability engineering. Preventive maintenance actions may increase equipment lifetime and decrease in-service breakdowns frequency. Usually, it is assumed that preventive maintenance actions restore the system to a good as new condition. Nevertheless, this assumption does not hold always in practice. For example when only some of the components of a complex system are replaced the condition of the whole system can be considered to be between as good as new and as bad as immediately before the maintenance action [20].

One way of modelling imperfect maintenance actions is to consider that the component condition after a maintenance action is the same as immediately before the maintenance action (minimal repair) with probability p and as good as new (replacement) with probability $1-p$ [2]. Alternatively, the effect of the maintenance action can be modelled by using the system effective age or the failure rate function [10,18]. Malik [10] introduced improvement factors to model the effect of maintenance actions. Lie and Chun [8] and Nakagawa [14] considered improvement factors in failure rate function and effective age. Nakagawa [16] introduced improvement factors in hazard rate and age for a sequential preventive maintenance policy and analyzed two corresponding imperfect preventive maintenance models.

In this paper, the concept of failure mode is used. A definition of failure mode is given by Mosleh et al. [11]. They define a failure mode (page 110) as 'A description of component failure in terms of the component function that was actually or potentially unavailable'. Høyland and Rausand [6] (page 10) state that 'All technical items are designed to fulfill one or more functions. A failure mode is thus defined as non-fulfillment of one of these functions'. Accordingly, it will be written 'failure with respect to a given failure mode of the system' if the corresponding function is unavailable.

Usually maintenance actions such as oiling and cleaning or partial system replacement only restore equipment to a good as new condition with respect to some failure modes while increased failure-proneness of other failure modes due to wear, for example, is not eliminated. Lin et al. [9] modelled this phenomenon by introducing the concept of two categories of failure modes, maintainable failure modes

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Nomenclature

c_{m}	cost of a minimal repair
c_{p}	cost of a preventive maintenance action
c_{r}	cost of system replacement
$\Gamma(s)$	Gamma function, i.e. $\Gamma(s)=\int_0^\infty y^{s-1}e^{-y}\mathrm{d}y$,
t	time
T	decision variable: fixed period between preventive maintenance actions, $T>0$
T^*	optimal value of T
N	decision variable: after $N-1$, $N=2,3,\ldots$, imperfect preventive maintenance actions the next preventive maintenance action at time NT, $N=1,2,\ldots$, restores the system to a good as new condition
N^*	optimal value of N
$C(T,N)$	cost rate as a function of T and N
$C_{\mathrm{C}}(T,N)$	expected cost in a cycle as a function of T and N
δ	parameter used to specify δ_0
δ_0, p_0	parameters used to specify $p(t)$ in the form $p(t)=p_0+\delta_0\lambda(t)$, $0\leq t<T$
$\gamma_k(T)$	improvement factor of the kth preventive maintenance action in a cycle for a given value of T, $k=1,2,\ldots,N-1$, $N=2,3,\ldots$
$h(t)$	hazard rate of non-maintainable failure modes
$H(t)$	cumulative hazard rate of non-maintainable failure modes
$\lambda(t)$	hazard rate of maintainable failure modes if $p_k(t)=0$, $t\geq 0$
$\Lambda(t)$	cumulative hazard rate of maintainable failure modes if $p_k(t)=0$, $t\geq 0$, $\Pi(k,T)=\int_{(k-1)T}^{kT} r_{k,T}(t)\mathrm{d}t$, $k=1,2,\ldots,N$, $N=1,2,\ldots$
$p_k(t)$	function which models the dependence between maintainable and non-maintainable models for t in the interval $((k-1)T, kT]$, $k=1,2,\ldots,N$, $N=1,2,\ldots$. When $p_k(t)$ does not depend on k it is written simply $p(t)$.
$r_{k,T}(t)$	maintained system failure rate for t in the interval $((k-1)T, kT]$, $k=1,2,\ldots,N$, $N=1,2,\ldots$

The maintained system failure rate is denoted as $r_{k,T}(t)$, $(k-1)T\leq t<kT$, to stress its dependence on the decision variable T. To simplify the exposition the maintained system failure rate will be referred to as the system failure rate.

A cycle is defined as the period of time between two subsequent system replacements.

For a function $g(t)$, $g'(t)$ and $g''(t)$ will denote its first and second derivatives, respectively.

Throughout the paper, the terms increasing and decreasing will mean strictly increasing and strictly decreasing, respectively.

If it is not otherwise stated indexes k and N are understood to take values $k=1,2,\ldots,N$ and $N=1,2,\ldots$, respectively.

and non-maintainable failure modes, into the modelling of imperfect preventive maintenance activities. Similarly to Lin et al. [9], the system failure modes are divided into maintainable and non-maintainable failure modes. That is, it will be assumed that there are system functions (maintainable failure mode) for which the system degradation leading to its unavailability can be removed by preventive maintenance actions. Removing degradation related to other system functions (non-maintainable failure mode) is only possible by making a complete overhaul which restore the whole system to a good as new condition. Further it will be supposed that a failure rate function can be related with each failure mode. The approach presented in this paper differs from the one used by Lin et al. [9]: while they use adjustment factors in effective age and hazard rate, it is modelled explicitly the effect of preventive maintenance actions by the reduction of the failure rate of the maintainable failure mode.

The model presented in this paper is related to competing risks models. Maintainable and non-maintainable failure modes compete to provoke the system failure. Further three types of maintenance actions are considered: preventive (imperfect) maintenance, minimal repairs and replacements. For references on competing risks models see [4,19] and [22]. The dependence scheme obtained for competing risks for failures through the model of this paper can be considered as 'positive' in the sense that shorter times to failure with respect to non-maintainable failure models tend to occur together with shorter times to failure with respect to maintainable failure modes [1]. Modeling stochastic dependence is an ample subject. One useful tool in this regard is the copula [17].

The model presented in this paper can be applied to multi-component series systems in which some components are replaced frequently while others are replaced with a smaller frequency. Maintainable failure modes would correspond to frequently replaced components. Non-maintainable failure modes would correspond to less frequently replaced components. Even if a component is replaced frequently its degradation may depend on the degree of degradation of less frequently replaced components because of, for example, physical interactions like vibration or high temperature. Since the behavior of the failure rate can be used to characterize the system degradation, the dependence between maintainable and non-maintainable failure modes can be stated in terms of failure rates. In this paper, this approach is used, i.e. it is considered that the failure rate of maintainable failure modes depends on the failure rate of non-maintainable failure modes. Possible practical application can be made on systems like electric truck motors

[12]. Electric truck motors are complex series systems with interactions between components like vibration and high temperatures, which may provoke their degradation. It can be expected that failure patterns of coupled or close components like the armature and the gears differ to the failure patterns encountered when those components work without any interaction. The use of the model presented in this paper would permit to take into account possible interactions between those components.

The main contribution of this paper is the development of a model for the optimization of the imperfect maintenance policy of a system considering that: (1) competing failures modes are dependent and (2) the system failure rate improvement factors of preventive maintenance actions depend on the time at which these actions are executed. The approach presented in this paper offers an alternative to the use of the system effective age [3], adjustment factors [5,16] or different failure processes before and after system replacements [24] for modeling effects of maintenance actions on the system failure rate.

The structure of this paper is as follows. In Section 2, the model is described. Improvement factors corresponding to preventive maintenance actions are studied and it is examined how preventive maintenance actions influence the system failure rate. In Section 3 sufficient conditions for the existence of an optimal maintenance policy are studied. In Section 4, a form for the function used to model the dependence between maintainable and non-maintainable failure modes is justified. The system failure rate and cost function for Weibull failure rates are presented in Section 5. After, the model is illustrated with numerical examples. In Section 7, issues deserving further research are presented.

To facilitate the exposition the terms type I failure and type II failure will be used interchangeably for failures with respect to non-maintainable and maintainable failure modes, respectively.

2. The model

2.1. Description of the maintenance policy

Upon a failure the system is restored to an operating condition by a minimal repair action. A minimal repair action returns the system to a condition as bad as immediately before the failure.

At times kT of a cycle, $k=1,2,\ldots,N-1$, $N=2,3,\ldots$, the system is preventively maintained. Preventive maintenance actions return the system condition with respect to maintainable failure modes to a good as new condition.

At time NT of a cycle the system is restored to a good as new condition (i.e. the system is replaced).

Assumptions.

1. Times for preventive maintenance actions, minimal repairs and replacements are negligible.
2. Maintainable and non-maintainable failure modes are competing to cause system failure.
3. The hazard rate of the maintainable failure mode is $\lambda(t-(k-1)T+p_k(t)h(t)$, $(k-1)T\leq t<kT$, $p_k(t)\geq 0$, $t\geq 0$ (see explication below).
4. $\lambda(t)$ and $h(t)$ are continuous and increasing functions of time t.
5. $h'(t)$, $h''(t)$, $\lambda'(t)$ and $p'(t)$ exist.
6. c_m, c_p and c_r have positive values with c_m, c_p, $c_r<\infty$. Further it is assumed that $c_r>c_r$. That is, system replacements are more expensive than preventive maintenance actions.

2.2. System failure rate

The hazard rate of maintainable failure modes at time $t-(k-1)T$ after the $(k-1)$th maintenance action in a cycle, $k=2,3,\ldots,N-1$, or after the system replacement $(k=1)$ is equal to the sum of the hazard rate $\lambda(t-(k-1)T)$ plus a positive value $p_k(t)h(t)$. $\lambda(t-(k-1)T)$ corresponds to the maintainable failure rate of a system replaced at time $(k-1)T$ if maintainable and non-maintainable failure modes are independent. The term $p_k(t)h(t)$ arises from the dependence between failure modes and can be interpreted as the increase in the hazard rate of maintainable failure modes as a consequence of the degradation of non-maintainable failure modes.

When $p_k(t)=0$, $t\geq 0$, maintainable and non-maintainable failure modes are independent. In that case the model presented in this paper can be useful when it is worthy carrying out simultaneous maintenance actions, which restore the system to a good as new condition with respect to maintainable and non-maintainable failure modes. This would correspond to an opportunistic maintenance action [7].

Recall that $h(t)\Delta x$ is approximately the probability that the system has a type I failure in the interval $(t,t+\Delta t]$ given that the system has not undergone a type I failure before t. This approximation is better for small values of Δt. Hence if $p_k(t)$ is assumed to be a probability then $p_k(t)$ can be interpreted as the failure rate of type II failures provoked by type I failures. That is, if the system has a type I failure at time t then it will undergo also a type II failure with probability $p_k(t)$ or will not have a type II failure with probability $1-p_k(t)$. Models considering this type of failure interactions have been proposed [13,23].

Finally, the system failure rate is given by

$$r_{k,T}(t) = h(t) + \lambda(t-(k-1)T) + p_k(t)h(t),\quad (k-1)T\leq t<kT. \tag{1}$$

2.3. Improvement factors of preventive maintenance actions

For the form (1) of the failure rate the system improvement factor in the kth preventive maintenance action in a cycle, $k = 1,2,\ldots,N-1$, $N = 2,3,\ldots$, is defined as:

$$\gamma_k(T) = \frac{r_{k,T}(kT) - r_{k+1,T}(kT)}{r_{k,T}(kT)} = \frac{\lambda(T) - \lambda(0) + h(kT)[p_k(kT) - p_{k+1}(kT)]}{h(kT) + \lambda(T) + p_k(kT)h(kT)}. \qquad (2)$$

$h(t)$ characterizes the system failure proneness because of type I failures. Therefore, since the effect of the degradation of non-maintainable failure modes on maintainable failure modes is modelled by the term $p_k(t)h(t)$ hence $p_k(t)$ can be interpreted as a measure of the vulnerability of maintainable failure modes with respect to type I failures. Besides, preventive maintenance actions render the system in a good as new condition with respect to maintainable failure modes. Therefore, it can supposed that $p_k(t)$ depends on the maintainable failure mode rather than on non-maintainable failure modes and that, with respect to time, $p_k(t)$ is a function that depends only on $t-(k-1)T$, $(k-1)T \le t < kT$. That is, preventive maintenance actions reset to zero the function $p_k(t)$. From the above reasoning, it is natural to state that for the function $p_k(t)$ the following equality should hold:

$$p_k(jT + t) = p_k(sT + t), \quad j, s = 0, 1, \ldots, N-1, \quad j \neq s, \quad 0 \le t < T. \qquad (3)$$

From now on it will be supposed that (3) holds and it will be written $p(t)$ instead of $p_k(t)$ to specify the system failure rate. Therefore, the system failure rate will be given in the following way

$$r_{k,T}(t) = h(t) + \lambda(t-(k-1)T) + p(t-(k-T)h(t), \quad (k-1)T \le t < kT \qquad (4)$$

and the improvement factors for the failure rate will take the following form:

$$\gamma_k(T) = \frac{\lambda(T) - \lambda(0) + h(kT)[p(T) - p(0)]}{h(kT) + \lambda(T) + p(T)h(kT)}. \qquad (5)$$

In Section 4, the dependence modelling is further explicated.

After some algebraic transformations it can be obtained that inequality (5) is equivalent to

$$[h((k+1)T - h(kT)]\{[1 + p(0)]\lambda(T) - [1 + p(T)]\lambda(0)\} > 0.$$

Hence, if the system failure rate has the form (4) and the following conditions hold:

1. $h(t)$ is increasing,
2. $[1+p(0)]\lambda(T) > [1+p(T)]\lambda(0)$ for all $T > 0$,

then for T fixed $\gamma_k(T)$ is decreasing as a function of k. That is, if both conditions presented above hold then the improvement due to preventive maintenance actions decreases in time. This improvement decrease would justify a system replacement after a number of preventive maintenance actions in a cycle. Note that the second condition holds in the two following special cases: (1) if $\lambda(0) = 0$ and $\lambda(t)$ is increasing, or (2) if $p(t) = p = \text{constant}$ and $\lambda(t)$ is increasing.

It is easy to see that if the following condition holds for all $T > 0$:

$$\lim_{k\to\infty} = \frac{\lambda(T)}{h(kT)} = 0$$

then as k tends to infinity the improvement factors of the maintenance action tend to a lower bound:

$$\lim_{k\to\infty} \gamma_k(T) = \frac{p(T) - p(0)}{1 + p(T)}.$$

Thus, if $h(t)$ is increasing then for cycles large enough the improvement factor of maintenance actions tends to a value which depends only on the function $p(t)$. This is intuitively correct since if $\lambda(T)$ is negligible in relation to $h(t)$ then system failure II type-proneness is driven mainly by the function $p(t)$ and the effect of preventive maintenance actions will be primarily to remove failure II type-proneness due to non-maintainable failure modes degradation.

2.4. Effect of preventive maintenance actions on the system failure rate

In this subsection, the effect of preventive maintenance actions on the system failure rate is studied, namely how a preventive maintenance action affects the increasing rate (first derivative) of the system failure rate.

Let us suppose that the last preventive maintenance action in the cycle was made at time kT (or, if $k = 0$, that the system was replaced at the beginning of the cycle). Hence, the system failure rate at time $t \in ((k+1)T,(k+2)T]$, which will denoted by $r_{nm}(t)$, is given by

$$\Gamma_{nm}(t) = h(t) + \lambda(t - kT) + p(t - kT)h(t). \qquad (6)$$

Suppose otherwise that at time $(k+1)T$ the system is preventively maintained. In this case, it is obtained that the system failure rate at time $t \in ((k+1)T, (k+2)T]$, which will be denoted by $r_{wm}(t)$ is given by

$$r_{wm}(t) = h(t) + \lambda(t - (k+1)T) + p(t - (k+1)T)h(t). \qquad (7)$$

Note from Eqs. (5)–(7) that the improvement of the system failure rate due to preventive maintenance actions depends on the time of the preventive maintenance action.

Note further that

$$r'_{nm}(t) - r'_{wm}(t) = \lambda'(t - kT) - \lambda'(t - (k+1)T) + h'(t)[p(t - kT) - p(t - (k+1)T)] + h(t)[p'(t - kT) - p'(t - (k+1)T)].$$

Therefore, since $t-kT>t-(k+1)T$, $(k+1)T \le t < (k+2)T$, $T>0$, it follows that if the system failure rate has the form (4) and the following conditions hold:

1. $h(t)$ is increasing,
2. $\lambda(t)$ is convex increasing,
3. $p(t)$ is convex non-decreasing,

then $r'_{nm}(t) > r'_{wm}(t)$, where $r_{nm}(t)$ and $r_{wm}(t)$ are given by Eqs. (6) and (7), respectively. That is, under these three conditions on the failure rates $h(t)$ and $\lambda(t)$ and on the function $p(t)$, which are fairly general, it is obtained that preventive maintenance actions diminish the system failure rate increase rate. This effect of the maintenance action on the system failure rate has been modelled previously using the system effective age [3].

Now the derivatives in subsequent periods between preventive maintenance actions are compared. The system failure rate at time $y \le T$ after the $(k-1)$th and after kth, $k=2,3,\ldots$, preventive maintenance action (or, if $k=1$, the system failure rate at time $t<T$ after the system replacement at time 0 of the cycle and after the first preventive maintenance action in a cycle), are given by

$$r_{k,T}((k-1)T+t) = h((k-1)T+t) + \lambda(t) + p(t)h((k-1)T+t)$$

and

$$r_{k+1,T}(kT+t) = h(kT+t) + \lambda(t) + p(t)h(kT+t),$$

respectively.

Hence, the increase in the maintainable failure rate between times $t<T$ after the $(k-1)$th and after the kth, $k=2,3,\ldots$ preventive maintenance action (or, if $k=1$, between times $t<T$ after the system replacement at time 0 of the cycle and after the first preventive maintenance action in a cycle), is given by

$$p(t)[h(kT+t) - h((k-1)T+t)], \quad t<T, \tag{8}$$

and the corresponding increase in the system failure rate is given by

$$[h(kT+t) - h((k-1)T+t)][1+p(t)], \quad t<T. \tag{9}$$

It is obtained that

$$\begin{aligned} &r'_{k+1,T}(kT+t) - r'_{k,T}((k-1)T+t) \\ &\quad = [1+p(t)][h'(kT+t) - h'((k-1)T+t)] \\ &\qquad + p'(t)[h(kT+t) - h((k-1)T+t)]. \end{aligned}$$

Therefore, if the system failure rate has the form (4) and the following conditions hold

1. $h(t)$ is convex increasing,
2. $p(t)$ is non-decreasing,

then $r'_{k+1,T}(kT+t) > r'_{k,T}((k-1)T+t)$

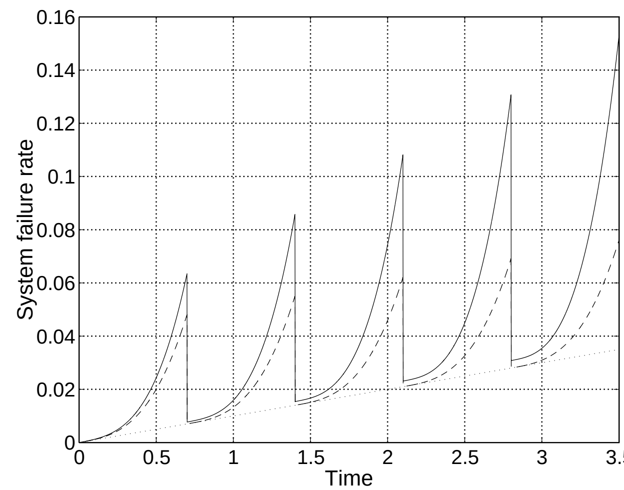

Fig. 1. Example of system failure rate as a function of time for $T=0.7$. The solid curve represents the system failure rate. The dashed curve represents the system failure rate if maintainable and non-maintainable failure modes are independent. The dotted line represents the non-maintainable failure rate.

That is, these two conditions are sufficient for being increasing the increase rate of the failure rate at the same time of subsequent periods between preventive maintenance actions. This increase of the failure rate derivative can be expected in practice when only a subcomponent of a series system is replaced. The system failure rate decreases in correspondence with the replaced component but system degradation corresponding to components non-replaced leads to an increase of the rate of increase of the system failure rate. This behavior of the failure rate derivative after preventive maintenance actions has been modelled previously using adjustment factors [16].

Fig. 1 represents an example of the shape of the system failure rate as a function of time. Decreases of system failure rate correspond to preventive maintenance actions. Note that as time passes the maintainable failure rate (i.e. the difference between the solid and dotted curves) after a preventive maintenance action increases in comparison with the same time after a previous preventive maintenance action. This increase is in correspondence with the modelling assumptions: the higher the failure rate of non-maintainable failure modes the higher its effect on the failure rate of maintainable failure modes.

3. Cost analysis of the maintenance policy

It will be supposed that the cost of a minimal repair due to a type I failure or a type II failure is the same, equal to c_m. It will be assumed that when both a type I and a type II failure occur simultaneously then the total cost of the system minimal repair is $2c_m$. There is not economic dependence in system repairs. The extension of the model presented in this paper to include different repair costs for type I, type II and multiple failures (that is, a type I and type II failure at the same time) is rather straightforward.

The maintenance policy cost per time unit for an infinite time span is given by the cost rate $C(T,N)$ [21].

$$C(T,N) = \frac{C_C(T,N)}{NT} = \frac{1}{NT}\left[c_r + (N-1)c_p + c_m \sum_{k=1}^{N} \int_{(k-1)T}^{kT} r_{k,T}(t)\mathrm{d}t\right],\ T>0, \tag{10}$$

where

$$\sum_{(k=1)}^{N} \int_{(k-1)T}^{kT} r_{k,T}(t)\mathrm{d}t = H(NT) + N\Lambda(T) + \sum_{k=1}^{N} \int_0^T p(t)h((k-1)T+t)dt. \tag{11}$$

3.1. Optimal value of T

The first derivative of the cost rate with respect to T is given by the following equation.

$$\frac{\partial}{\partial T} C(T,N) = \frac{1}{NT^2}\left[T \frac{\partial}{\partial T} C_C(T,N) - C_C(T,N)\right] = \frac{1}{T}\left[\frac{1}{N}\frac{\partial}{\partial T} C_C(T,N) - C(T,N)\right],\ T>0. \tag{12}$$

Hence the optimal value of T is a positive solution of

$$T \frac{\partial}{\partial T} C_C(T,N) = C_C(T,N)$$

or equivalently the optimal value of T is a positive solution of

$$\frac{1}{N}\frac{\partial}{\partial T} C_C(T,N) = C(T,N), \tag{13}$$

where

$$\frac{\partial}{\partial T} C_C(T,N) = Nh(NT) + N\lambda(T) + \sum_{k=1}^{N}\left[(k-1)\int_0^T p(t)h'((k-1)T+t)\mathrm{d}t + p(T)h(kT)\right]. \tag{14}$$

From Eq. (11) it is obtained that

$$\frac{\partial^2}{\partial T^2} \sum_{k=1}^{N} \int_{(k-1)T}^{kT} r_{k,T}(t)\mathrm{d}t = N^2h'(NT) + N\lambda'(T) + \sum_{k=1}^{N}\left[(k-1)^2\int_0^T p(t)h''((k-1)T+t)\lambda(t)\mathrm{d}t + (2k-1)p(T)h'(kT) + h(kT)p'(T)\right]. \tag{15}$$

If $h(t)$ is convex increasing, $\lambda(t)$ is increasing and $p(t)$ is non-decreasing then the right hand side of (15) is positive and hence

$$\frac{\partial}{\partial T}\left[T\frac{\partial C_C(T,N)}{\partial T}\right] > \frac{\partial C_C(T,N)}{\partial T},\ T>0. \tag{16}$$

Suppose there is a value of $T=T^* > 0$ for which condition (13) holds. Hence from inequality (16) it follows that for $T>T^*$

$$T\frac{\partial}{\partial T} C_C(T,N) > C_C(T,N)$$

and for $0<T<T^*$

$$T\frac{\partial}{\partial T} C_C(T,N) < C_C(T,N).$$

Therefore, T^* is unique. From (12) it follows that for $T>T^*$

$$\frac{\partial}{\partial T} C(T,N) > 0$$

and for $0<T<T^*$

$$\frac{\partial}{\partial T} C(T,N) < 0.$$

Hence T^* is minimizes globally the cost function $C(T,N)$. Furthermore,

$$\lim_{T\to 0} T\frac{\partial}{\partial T} C_C(T,N) = 0$$

and

$$C_C(T,N)|_{T=0} = c_r + (N-1)c_p.$$

Therefore, if the value of N is fixed and the following conditions hold:

1. $\lambda(t)$ is increasing,
2. $h(t)$ is convex increasing,
3. $p(t)$ is non-decreasing,

then for a given value of N there is at most one value $T^*>0$ for which

$$\frac{1}{N}\frac{\partial}{\partial T} C_C(T,N)|_{T=T^*} = C(T^*,N). \tag{17}$$

If such T^* exists then the cost rate (10) attains at T^* its global minimum. Besides if $h(t)$, $\lambda(t)$ and $p(t)$ are continuous and $c_r>0$ then $T^*>0$ always exists.

Note that the case $c_r=0$ would correspond to system replacements at no cost. If $c_r>0$ it follows that $C_C(T,N)|_{T=0}>0$ and hence from the continuity of $h(t)$, $\lambda(t)$ and $p(t)$ it follows that always exist $T^*>0$ for which (17) holds.

The necessary condition for the optimal T is given by

$$c_{\mathrm{m}}\left\{Nh(NT)+N\lambda(T)+\sum_{k=1}^{N}\left[(k-1)\int_0^T p(t)h'((k-1)T+t)\mathrm{d}t + p(T)h(kT)\right]\right\}=\frac{1}{T}\left[c_{\mathrm{r}}+(N-1)c_{\mathrm{p}} + c_{\mathrm{m}}\left(H(NT)+N\Lambda(T)+\sum_{k=1}^{N}\int_0^T p(t)h((k-1)T+t)\mathrm{d}t\right)\right]. \tag{18}$$

3.2. Optimal value of N

Let denote

$\prod(k,T)=\int_{(k-1)T}^{kT} r_{k,T}(t)\mathrm{d}t.$

The approach of Nakagawa [15] will be used. For the optimal value of N the following condition should hold:

$$C(T,N+1)\geq C(T,N). \tag{19}$$

If the optimal value of N is greater than 1, then the following condition should hold also:

$$C(T,N-1)>C(T,N). \tag{20}$$

Let define the function $L(N,T)$ in the following way:

$$L(N,T)=\begin{cases} N\prod(N+1,T)-\sum_{k=1}^{N}\prod(k,T) & \text{if } N=1,2,\ldots,\\ 0 & \text{if } N=0.\end{cases}$$

Condition (19) is equivalent to

$$L(N,T)\geq\frac{c_{\mathrm{r}}-c_{\mathrm{p}}}{c_{\mathrm{m}}}.$$

Condition (20) is equivalent to

$$\frac{c_{\mathrm{r}}-c_{\mathrm{p}}}{c_{\mathrm{m}}}>L(N-1,T).$$

Hence for the optimal N the following condition holds

$$L(N,T)\geq\frac{c_{\mathrm{r}}-c_{\mathrm{p}}}{c_{\mathrm{m}}}>L(N-1,T). \tag{21}$$

Consider that the value of T is fixed. It is obtained that

$$\prod(k+1,T)-\prod(k,T)=\Delta H(k+1,T)-\Delta H(k,T) + \int_0^T p(t)[h(kT+t)\mathrm{d}t-h((k-1)T+t)]\mathrm{d}t,$$

where

$$\Delta H(k,T)=\int_{(k-1)T}^{kT} h(t)\mathrm{d}t.$$

If $h(t)$ is increasing then for $T>0$ fixed it holds that $\Delta H(k,T)$ is increasing when k increases and $h(kT+t)-h((k-1)T+t)>0$, $t\geq 0$. Thus

$$L(N,T)-L(N-1,T)=N\left[\prod(N+1,T)-\prod(N,T)\right]>0.$$

Hence for T fixed $L(N,T)$ is increasing when N increases and there is, for T fixed, a unique value of N for which condition (21) holds. Therefore, if $h(t)$ is increasing then for $T>0$ fixed there is only one value of $N=N^{*}\geq 1$ for which condition (21) holds.

4. Choice of $p(t)$

It is natural to suppose that the probability that a type I failure provoke a type II failure depends on the existence of a coupling mechanism [11] and possibly also on the degradation with respect to maintainable failure modes. Therefore, it can be argued that the following form is a good model for $p(t)$:

$$p(t)=p_0+\delta_0\lambda(t-(k-1)T),\quad (k-1)T\leq t<kT,\quad \delta_0,p_0\geq 0 \tag{22}$$

or equivalently

$$p((k-1)T+t)=p_0+\delta_0\lambda(t),\quad 0\leq t<T,\quad \delta_0,p_0\geq 0,$$

where p_0 and δ_0 model the effect of the coupling mechanisms and $\lambda(t)$ embodies the effect of maintainable failure modes degradation on occurrences of multiple failures, i.e. type I and type II failures at the same time.

To justify the form (22) let us suppose that $p(t)$ is a probability and that failures modes correspond to subcomponents of a series system. Let us suppose further that the system has multiple failures at time t if and only if a type I failure at time t provoke instantaneously a type II failure. Hence the contribution of the degradation of maintainable failure modes to multiple failures can be supposed to be a function of the failure rate of the maintainable modes. This leads to the factor $\lambda(t-(k-1)T)$, $(k-1)T\leq t<kT$, in (22). The contribution of the coupling mechanism to the probability of multiple failures can be supposed, as a first approximation, to be constant as a function of time. This leads to the factor δ_0 when multiple failures depend on some coupling mechanism and also on the degradation with respect to maintainable failure modes. The term p_0 in (22) would consider the case when multiple failures depend only on some kind of coupling mechanism (and not on the degradation with respect to maintainable failure modes). Note that if $p(t)$ has the form (22) and it is supposed to be a probability then the maintenance policies (T,N) should be restricted to values of T for which $0\leq p(T)\leq 1$. For a reference on coupling factors and mechanisms see [11].

As it has been said before $p(t)$ can be interpreted as a measure of the vulnerability of maintainable failure modes with respect to type I failures. It can be expected that this vulnerability is a function of the failure rate $\lambda(t)$.

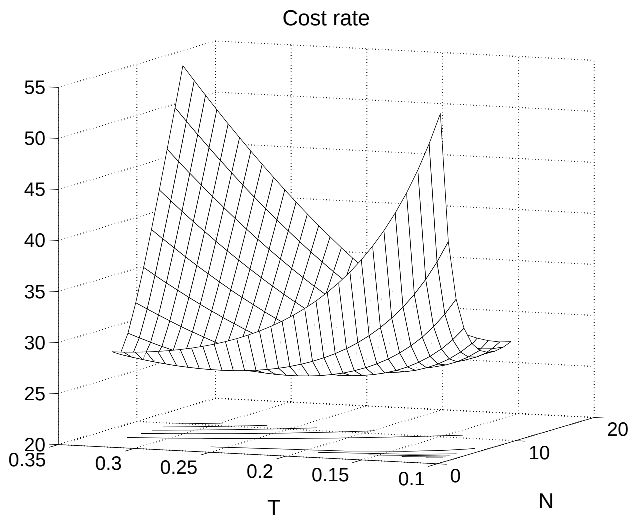

Fig. 2. Cost rate as a function of T and N.

Since equation (22) is a linear function of $\lambda(t)$ hence the form (22) could be used to model the effect of non-maintainable failure modes on maintainable failure modes, even if $p(t)$ is not considered to be a probability and failure modes do not correspond to subcomponents of a series system.

Note that if $p(t)$ has the form the form (22) then $p(t)$ has the same monotonicity and convexity properties as $\lambda(t)$ if $\delta_0 > 0$.

δ_0 will be expressed in the following way

$$\delta_0 = \delta \cdot \left[\lambda \left(\int_0^\infty e^{-\Lambda(t)} \mathrm{d}t \right) \right]^{-1},$$

to state the model as a function of a scalar δ rather than as a function of δ_0 which should be in time units.

If the maintenance policy described in this paper is used then the model presented would be useful when there is statistical information on single and simultaneous failures of the two-component series system, and when the failure rates $h(t)$ and $\lambda(t)$ are known or can be estimated. In practice simultaneous failures would correspond to failures of different components in a short period of time. Simultaneous failures would give information on coupling of failures types provided that coupling mechanisms can be determined. Note that if $\delta_0 = 0$ in (22) then a natural estimator of p_0 is the total number of simultaneous failures divided by the total number of failures of non-maintainable failure modes. For the estimation of δ_0, data is needed on times of multiple failures after preventive (imperfect) maintenance actions because δ_0 and $\lambda(t)$ appear multiplied in (22).

5. Weibull distribution

Suppose maintainable failure modes when $p(t) = 0$ and non-maintainable failure modes have a Weibull distribution such that $\Lambda(t) = \alpha t^\beta$ and $H(t) = at^b$. In this case δ_0 has the form

$$\delta_0 = \frac{\delta}{\beta \alpha^{(1/\beta)} \left[\Gamma \left(1 + \frac{1}{\beta} \right) \right]^{\beta - 1}}.$$

In order to simplify, the analysis it will be considered $b = 2$. That is, it will be considered that the failure rate of non-maintainable failure modes is a linear function of time. Then the system failure rate is given by

$$\begin{aligned} r_{k,T}(t) =& 2at[1 + p_0 + \alpha\beta\delta_0(t-(k-1)T)^{\beta-1}] \\ & + \alpha\beta(t-(k-1)T)^{\beta-1}, \quad (k-1)T \le t < kT. \end{aligned}$$

Hence, it is obtained

$$\begin{aligned} \sum_{k=1}^{N} \int_{(k-1)T}^{kT} r_{k,T}(t)\mathrm{d}t =& a(NT)^2 + N\alpha T^\beta \\ & + 2aNT^2 \left\{ p_0 \frac{N}{2} + \delta_0 \alpha T^{\beta-1} \left[\frac{(N-1)}{2} + \frac{\beta}{\beta+1} \right] \right\}. \end{aligned}$$

Table 1
Optimal policy and minimum cost rate for different values of c_r and $\delta = 2$

c_r	2	5	10	20	30	40	50
T^*	0.262	0.208	0.216	0.188	0.180	0.162	0.164
N^*	1	3	4	7	9	12	13
$N^* \cdot T^*$	0.262	0.624	0.864	1.316	1.620	1.944	2.132
$C(T^*, N^*)$	13.5	20.2	26.9	35.8	42.5	48.1	53.1

Table 2
Optimal policy and minimum cost rate for different values of c_r and $\delta = 1$

c_r	2	5	10	20	30	40	50
T^*	0.282	0.224	0.235	0.226	0.212	0.199	0.201
N^*	1	3	4	6	8	10	11
$N^* \cdot T^*$	0.282	0.672	0.940	1.356	1.696	1.990	2.211
$C(T^*, N^*)$	12.9	19.3	25.6	34.2	40.7	46.1	50.9

Therefore, if $\beta>1$ then, for given N, the optimal value of T is the unique solution of the following equation.

$$\begin{aligned}&T\left\{\frac{c_r}{c_m}+(N-1)\frac{c_p}{c_m}+a(NT)^2+N\alpha T^{\beta}\right.\\&\left.+2aNT^2\left\{p_0\frac{N}{2}+\delta_0\alpha T^{\beta-1}\left[\frac{(N-1)}{2}+\frac{\beta}{\beta+1}\right]\right\}\right\}\\&\quad=2aN^2T+N\alpha\beta T^{\beta-1}+2aNT\left\{p_0N+\delta_0\alpha(\beta+1)T^{\beta-1}\right.\\&\left.\times\left[\frac{(N-1)}{2}+\frac{\beta}{\beta+1}\right]\right\}.\end{aligned}$$

6. Numerical examples

Consider the following data $\alpha=3$, $\beta=2.2$, $a=2$, $b=2$, $p_0=0.1$, $c_m=4$ and $c_p=1$.

Fig. 2 represents the cost rate as a function of T and N for $\delta=2$ and $c_r=10$. Note from this figure that the cost rate is rather sensitive to the decision parameters.

Numerical results considering $\delta=2$ are presented in Table 1. Note from Table 1 that the number of preventive maintenance actions in a cycle, the cycle length and the minimal cost rate are increasing when c_r increases, as expected.

For $c_r=5$ it is obtained that the improvement factors of the two preventive maintenance actions in a cycle are 0.626 for the first preventive maintenance action and 0.530 for the second preventive maintenance action. For $c_r=50$ the improvement factors of the 12 preventive maintenance actions are, from the first one to the last one, 0.598, 0.490, 0.441, 0.412, 0.393, 0.380, 0.370, 0.363, 0.357, 0.352, 0.348 and 0.345. Note that in both cases improvement factors are decreasing.

Numerical results considering $\delta=1$ are presented in Table 2. Note that when $\delta=1$ the number of preventive maintenance actions in a cycle is equal or smaller than when $\delta=2$. The cycle length is greater when $\delta=1$. This is intuitively correct from the assumptions since the deterioration of maintainable failure modes due to non-maintainable failure modes is greater when δ increases. Therefore, preventive maintenance actions and system replacements are to be made more frequently as δ increases.

7. Conclusions

In this paper, a model has been presented to consider imperfect preventive maintenance policies of systems for which failures modes can be divided into maintainable and non-maintainable failure modes. It has been assumed that maintainable and non-maintainable failure modes are dependent. It has been studied how preventive maintenance actions affect the system failure rate and conditions have been given for which optimal policies exist. The model presented in this paper permits to calculate the improvement factor of imperfect preventive maintenance actions as a function of the time of preventive maintenance actions.

A specific dependence structure by means of the failure rates of maintainable and non-maintainable failure modes has been used. Future research is needed to analyze the problem addressed here by using other dependence structures. The model presented in this paper could be extended to sequential preventive maintenance policies.

References

[1] Barlow R, Proschan F. Statistical theory of reliability and life testing. Probability models. To begin with; 1981.
[2] Brown M, Proschan F. Imperfect repair. J Appl Probab 1983;20:851–9.
[3] Canfield RV. Cost optimization of periodic preventive maintenance. IEEE Trans Reliab 1986;35:78–81.
[4] Cooke R, Paulsen J. Concepts for measuring maintenance performance and methods for analysing competing failure models. Reliab Eng Syst Safe 1997;55:135–41.
[5] Doyen L, Gaudoin O. Classes of imperfect repair models based on reduction of failure intensity or virtual age. Reliab Eng Syst Safe 2004;84(1):45–56.
[6] Høyland A, Rausand M. System reliability theory. Models and statistical methods. London: Wiley; 1994.
[7] Jhang JP, Sheu SH. Opportunity-based age replacement policy with minimal repair. Reliab Eng Syst Safe 1999;64:339–44.
[8] Lie CH, Chun YH. An algorithm for preventive maintenance policy. IEEE Trans Reliab 1986;35:71–5.
[9] Lin D, Zuo MJ, Yam RCM. Sequential imperfect preventive maintenance models with two categories of failure modes. Naval Res Logist 2001;48:172–83.
[10] Malik MAK. Reliable preventive maintenance scheduling. AIIE Trans 1979;11:221–8.
[11] Mosleh A, Rasmuson DM, Marshall FM. Guidelines on modeling common-cause failures in probabilistic risk assessment. USA: Idaho National Engineering Laboratory; 1998.
[12] Lhorente B, Lugtigheid D, Knights PF, Santana A. A model for optimal armature maintenance in electrical haul truck wheel motors: a case study. Reliab Eng Syst Safe 2004;84:209–18.
[13] Murthy DNP, Nguyen DG. Study of two-component systems with failure interactions. Naval Res Logist Q 1985;32:239–47.
[14] Nakagawa T. A summary of imperfect preventive maintenance policies with minimal repair. RAIRO Oper Res 1980;14:249–55.
[15] Nakagawa T. Periodic and sequential preventive maintenance policies. J Appl Probab 1986;23:536–42.
[16] Nakagawa T. Sequential imperfect preventive maintenance policies. IEEE Trans Reliab 1988;37:295–8.
[17] Nelsen RB. An introduction to copulas. Berlin: Springer; 1998.
[18] Nguyen DG, Murthy DNP. Optimal preventive maintenance policies for repairable systems. Oper Res 1981;29:1181–94.
[19] Paulsen J, Cooke R, Nyman R. Comparative evaluation of maintenance performance using subsurvival functions. Reliab Eng Syst Safe 1997;58:157–63.
[20] Pham H, Wang H. Optimal imperfect maintenance models. In: Pham H, editor. Handbook of reliability engineering. London: Springer; 2003. p. 397–414.
[21] Ross SM. Stochastic processes. London: Wiley; 1983.
[22] Salinas-Torres VH, Pereira CAB, Tiwari RC. Bayesian nonparametric estimation in a series system or a competing risk model. Nonparametric Stat 2002;14(4):449–58.
[23] Scarf PA, Deara M. Block replacement policies for a two-component system with failure dependency. Naval Res Logist 2003;50:70–87.
[24] Zhao YX. On preventive maintenance policy of a critical reliability level for system subject to degradation. Reliab Eng Syst Safe 2003; 79(3):301–8.