universität freiburg

Machine Learning for Stochastics **Summer semester 2025**

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Tutorial 1

Exercise 1 (3 Points)**.**
Suppose a stationary Markov Decision Model with $\beta = 1$ is given with the following properties:

1. there exists a set $G \subset E$ such that $r(x,a) = 0$ and $Q(\{x\} \mid x,a) = 1$ for all $x \in G$ and $a \in D(x)$,
2. for each $x \in E$, there exists a finite $N(x) \leq N$ such that $\mathbb{P}^{\pi}_x(X_{N(x)} \in G) = 1$ for all policies $\pi \in F^N$.

Define $J(x) := J_{N(x)}(x)$.

(a) Show that $J(x) = g(x)$ for $x \in G$ and $J(x) = (\mathcal{T}J)(x)$ for $x \notin G$.

(b) Show that if $f \in F$ satisfies

$$(\mathcal{T}J)(x) = (\mathcal{T}_f J)(x), \quad x \notin G,$$

and $f(x) \in D(x)$ arbitrary for $x \in G$, then then the stationary policy $(f, \dots, f) \in F^N$ is optimal.

(c) Show that the red-and-black gambling model (Tutorial 2, Exercise 2) satisfies the assumptions of a terminating Markov decision model.

Exercise 2 (3 Points)**.**
Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. Let $\mathbb{X}$ and $\mathbb{Y}$ be Polish spaces equipped with their respective Borel σ-algebras $\mathcal{X}$ and $\mathcal{Y}$. Consider two random variables: a *state variable* X taking values in $\mathbb{X}$, and an *observable variable* Y taking values in $\mathbb{Y}$.
Assume there exist a σ-finite measure ν on $(\mathbb{Y},\mathcal{Y})$ and a measurable function $\lambda : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}_+$ such that the joint law of (X,Y) satisfies

$$P_{X,Y}(dx,dy) = \lambda(x,y)\, \nu(dy)\, P_X(dx).$$

Show that for any measurable set $A \in \mathcal{X}$ and for P_Y-almost every $y \in \mathbb{Y}$,

$$\mathbb{P}(X \in A \mid Y = y) = \frac{\int_A \lambda(x,y)\, P_X(dx)}{\int_{\mathbb{X}} \lambda(x,y)\, P_X(dx)}.$$

Hint: Use the disintegration theorem, which applies due to the assumption that $\mathbb{X}$ and $\mathbb{Y}$ are Polish spaces.

Exercise 3 (5 Points)**.**

(a) **Exponential–Gamma Model:** Let $X_1,\ldots,X_n$ be independent observations from an exponential distribution with unknown rate $\theta > 0$,

$$f(x \mid \theta) = \theta e^{-\theta x}, \quad x > 0.$$

Assume a Gamma prior $\theta \sim \mathrm{Gamma}(\alpha,\beta)$, with density

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \quad \theta > 0.$$

Show that the posterior distribution of θ given the data is again Gamma, with parameters α_n, β_n expressed in terms of α,β, and the data.

(b) **Gaussian–Gaussian Model:** Let $X_1,\ldots,X_n$ be i.i.d. observations from a normal distribution with unknown mean μ and known variance σ^2. Assume a Gaussian prior $\mu \sim \mathcal{N}(\mu_0,\tau^2)$.

Show that the posterior distribution of μ given the data is Gaussian, with mean μ_n and variance τ_n^2 depending on μ_0,τ^2,σ^2, and the sample mean $\bar{X}_n$.

Hint: Use Bayes' theorem and unnormalized densities to simplify the algebra. In the Gaussian case, complete the square in the exponent.