universität freiburg

Machine Learning for Stochastics

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Tutorial 1

Exercise 1 (3 Points)**.**
A fair coin is tossed repeatedly until the first head appears. If the first head occurs on the n-th toss, the player receives a reward of 2^{n-1} euros. Define the random variable associated with the lottery by $X : \Omega \to \{1,2,4,8,\dots\}$.

(a) Compute the probability $\mathbb{P}(X = 2^{n-1})$ for each $n \geq 1$.

(b) Show that the expected value of the lottery is infinite, i.e., $\mathbb{E}[X] = +\infty$.

(c) Suppose now that the player evaluates the lottery using a utility function. Compute the expected utility and the certainty equivalent for the following utility functions:

  (i) $U(x) = \sqrt{x}$,

  (ii) $U(x) = \log(x)$.

Hint: Given an increasing and concave utility function $U : E \to \mathbb{R}$, the *certainty equivalent* CE is defined as the solution of the equation

$$U(CE) = \mathbb{E}[U(X)].$$

Exercise 2 (3 Points)**.**
An investor is said to follow a *mean-variance* criterion if they prefer a lottery Y over a lottery X whenever $\mathbb{E}[X] \leq \mathbb{E}[Y]$ and $\mathrm{Var}(X) \geq \mathrm{Var}(Y)$. When X and Y are normally distributed, the mean-variance criterion is equivalent to the expected utility approach.
Show that if $X \sim \mathcal{N}(\mu,\sigma^2)$ and $Y \sim \mathcal{N}(\nu,\tau^2)$, then

$$\mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)]$$

for all increasing and concave utility functions U if and only if $\nu \geq \mu$ and $\tau \leq \sigma$.
Hint: The condition $\mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)]$ for all increasing and concave U means that Y *second-order stochastically dominates* X. Recall that Y second-order stochastically dominates X if and only if

$$\int_{-\infty}^{z} F_X(t)\,dt \geq \int_{-\infty}^{z} F_Y(t)\,dt \quad \text{for all } z \in \mathbb{R},$$

where F_X and F_Y denote the cumulative distribution functions of X and Y, respectively.

Exercise 3 (3 Points)**.**
Let $X = (X_n)_{n \geq 0}$ be a time-inhomogeneous Markov process on a measurable state space $(E,\mathscr{E})$. For all $n \leq m$, for all $x \in E$, and for all $A \in \mathscr{E}$, define

$$Q_{n,m}(x,A) := \mathbb{P}(X_m \in A \mid X_n = x).$$

Show that, for all $n \leq k \leq m$, the following equation holds:

$$Q_{n,m}(x,A) = \int_E Q_{k,m}(y,A)\, Q_{n,k}(x,dy).$$

Hint: Use the law of iterated expectations (also known as the tower property of conditional expectation).

Exercise 4 (3 Points)**.**
Let (E,A,D_n,Q_n,r_n,g_N) be a Markov Decision Process (MDP) over a finite time horizon $N < \infty$, satisfying the integrability assumption

$$\sup_\pi \mathbb{E}_x^\pi \left[\sum_{k=n}^{N-1} r_k(X_k,f(X_k))^+ + g_N(X_N)^+ \right] < \infty.$$

Let $\pi = (f_0,f_1,\ldots,f_{N-1})$ be an N-stage policy for the MDP. For each $n = 0,\ldots,N-1$, define the operator

$$(\mathcal{T}_n^f v)(x) := r_n(x,f(x)) + \int_E v(x')\, Q_n(dx' \mid x,f(x)),$$

for all measurable functions $v : E \to [-\infty,+\infty)$.
Show that, for all $n = 0,\ldots,N-1$, it holds:

$$V_n^\pi(x) = \left(\mathcal{T}_n^{f_n} \mathcal{T}_{n+1}^{f_{n+1}} \cdots \mathcal{T}_{N-1}^{f_{N-1}} g_N \right)(x),$$

and that

$$V_N^\pi(x) = g_N(x).$$