

## Tutorial 1

### Exercise 1 (3 Points).

A fair coin is tossed repeatedly until the first head appears. If the first head occurs on the  $n$ -th toss, the player receives a reward of  $2^{n-1}$  euros. Define the random variable associated with the lottery by  $X : \Omega \rightarrow \{1, 2, 4, 8, \dots\}$ .

- (a) Compute the probability  $\mathbb{P}(X = 2^{n-1})$  for each  $n \geq 1$ .
- (b) Show that the expected value of the lottery is infinite, i.e.,  $\mathbb{E}[X] = +\infty$ .
- (c) Suppose now that the player evaluates the lottery using a utility function. Compute the expected utility and the certainty equivalent for the following utility functions:
  - (i)  $U(x) = \sqrt{x}$ ,
  - (ii)  $U(x) = \log(x)$ .

*Hint:* Given an increasing and concave utility function  $U : E \rightarrow \mathbb{R}$ , the *certainty equivalent*  $CE$  is defined as the solution of the equation

$$U(CE) = \mathbb{E}[U(X)].$$

### Exercise 2 (3 Points).

An investor is said to follow a *mean-variance* criterion if they prefer a lottery  $Y$  over a lottery  $X$  whenever  $\mathbb{E}[X] \leq \mathbb{E}[Y]$  and  $\text{Var}(X) \geq \text{Var}(Y)$ . When  $X$  and  $Y$  are normally distributed, the mean-variance criterion is equivalent to the expected utility approach.

Show that if  $X \sim \mathcal{N}(\mu, \sigma^2)$  and  $Y \sim \mathcal{N}(\nu, \tau^2)$ , then

$$\mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)]$$

for all increasing and concave utility functions  $U$  if and only if  $\nu \geq \mu$  and  $\tau \leq \sigma$ .

*Hint:* The condition  $\mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)]$  for all increasing and concave  $U$  means that  $Y$  *second-order stochastically dominates*  $X$ . Recall that  $Y$  second-order stochastically dominates  $X$  if and only if

$$\int_{-\infty}^z F_X(t) dt \geq \int_{-\infty}^z F_Y(t) dt \quad \text{for all } z \in \mathbb{R},$$

where  $F_X$  and  $F_Y$  denote the cumulative distribution functions of  $X$  and  $Y$ , respectively.

**Exercise 3** (3 Points).

Let  $X = (X_n)_{n \geq 0}$  be a time-inhomogeneous Markov process on a measurable state space  $(E, \mathcal{E})$ . For all  $n \leq m$ , for all  $x \in E$ , and for all  $A \in \mathcal{E}$ , define

$$Q_{n,m}(x, A) := \mathbb{P}(X_m \in A \mid X_n = x).$$

Show that, for all  $n \leq k \leq m$ , the following equation holds:

$$Q_{n,m}(x, A) = \int_E Q_{k,m}(y, A) Q_{n,k}(x, dy).$$

*Hint:* Use the law of iterated expectations (also known as the tower property of conditional expectation).

**Exercise 4** (3 Points).

Let  $(E, A, D_n, Q_n, r_n, g_N)$  be a Markov Decision Process (MDP) over a finite time horizon  $N < \infty$ , satisfying the integrability assumption

$$\sup_{\pi} \mathbb{E}_x^{\pi} \left[ \sum_{k=n}^{N-1} r_k(X_k, f(X_k))^+ + g_N(X_N)^+ \right] < \infty.$$

Let  $\pi = (f_0, f_1, \dots, f_{N-1})$  be an  $N$ -stage policy for the MDP. For each  $n = 0, \dots, N-1$ , define the operator

$$(\mathcal{T}_n^f v)(x) := r_n(x, f(x)) + \int_E v(x') Q_n(dx' \mid x, f(x)),$$

for all measurable functions  $v : E \rightarrow [-\infty, +\infty)$ .

Show that, for all  $n = 0, \dots, N-1$ , it holds:

$$V_n^{\pi}(x) = \left( \mathcal{T}_n^{f_n} \mathcal{T}_{n+1}^{f_{n+1}} \dots \mathcal{T}_{N-1}^{f_{N-1}} g_N \right) (x),$$

and that

$$V_N^{\pi}(x) = g_N(x).$$