number theory

1. problem 1 Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, m_3, \ldots$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)\ldots\left(1 + \frac{1}{m_k}\right) \tag{1}$$

geometry

2. Problem 2. A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:
   * no line passes through any point of the configuration;
   * no region contains points of both colours.

   Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines

geometry

3. Problem 3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1, on CA and C_1, on AB analogously, using the excircles opposite B and C. respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$, lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

   The excircle of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.

geometry

4. problem4. Let ABC be an acute-angled triangle with orthocentre$H$, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the fect of the altitudes from B and C, respectively. Denote by w_1 the circumcircle of BWN, and let X be the point on wy such that WX is a diameter of w_1 Analogously, denote by w_2 the circumcircle of CWM. and let Y be the point on such that WY is a diameter of Prove that X, Y and Hare collinear.

number theory

5. Problem 5. Let $Q_{>0}$ be the set of positive rational mumbers. Let f : $Q_{>0} \to R$ be a function satisfying the following three conditions:

   (a) for all $x, y \epsilon Q{>}0$, we have $f(x) f(y) \geq f(xy)$

   (b) for all $x, y \epsilon Q{>}0$,we have$f(x+y) \geq f(x) + f(y)$

   (c) there exists a rational number $a{>}1$ such that $f(a) = a$.
   prove that $F(x) = x$ for all $x \epsilon Q{>}0$.

combinatorics

6. Problem 6. Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots n$ such that each label is used exactly once, two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called beautiful if, for any four labels $a{<}b{<}c{<}d$ with $a+d = b+c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c

   Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x+y \leq nandgcd(x, y) = 1$. Prove that

   $m = n + 1$

number theory

7. problem1 let $a_{0<}a_{1<}a_{2<}\dots$ be an infinite sequence of positive integers.prove that there exists a unique integer $n \geq 1$such that

$$a_{n<}\frac{a_0 + a1 + \ldots + a_n}{n}{<}a_{n+1}. \tag{2}$$

geometry

8. Problem 2. let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

geometry

9. Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$ Point His the foot of the perpendicular from A to BD. Points S and T lie on sides $ABandAD$, respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90°, \angle THC - \angle DTC = 90°$.
Prove that line BD is tangent to the circumcircle of triangle TSH.

geometry

10. Problem 4. Points $PandQ$lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines $BMandCN$ intersect on circumcircle of triangle ABC

number theory

11. Problem 5. For each positive integer n, the Bank of Cape Town ienes coins of denomination$\frac{1}{n}$ Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$ prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

geometry

12. Problem 6. A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cats the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n. in any set of a lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

    Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

geometry

13. Problem 1. We say that a finite set S of points in the plane is balanced if, for any two different points $AandB$ in S, there is a point Cin Ssuch that $AC = BC$. We say that S is centre-free if for any three different points A, B and C in S, there is no point P in S such that $PA = PB = PC$

    (a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

    (b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

geometry

14. Problem 2. Determine all triples (a, b, c) of positive integers such that each of the numbers
    $ab - c, bc - a, ca - b$
    is a power of 2
    (A power of 2 is an integer of the form 2^n,Where n is a non-negative integer).

geometry

15. Problem 3. Let ABC be an acute triangle with AB>AC Let I be its circumcircle, H its orthocentre, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q he the point on T such that $\angle HQA = 90$, and let K be the point on T such that $\angle HKQ = 90°$. Assume that the points$A, B, C, KandQ$ are all different, and lie on T in this order.

    Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

geometry

16. Problem 4. Triangle ABC has circumcircle Ω and circumcentre O. A circle T with centre. A intersects the segment BC at points $DandE$, such that $B, D, E$and Care all different and lie on line BC in this onter. Let $FandG$be the points of intersection of $Tand\Omega$. such that $A.FB.CandG$lie on Ωin this order. Let K he the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA Suppose that the lines $FKandGL$ are different and intersect at the point X. Prove that X lies on the line AO.

geometry

17. Problem 5. Let R be the set of real numbers. Determine all functions $f : R \to R$ satisfying the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x) \tag{3}$$
$$\tag{4}$$

for all real numbers x and y

geometry

18. problem6 the sequence $a_1, a_2, \ldots$ of an integers satisfies the following conditions;

    (a) $1 \leq a_j \leq 2015$ for all $j \geq 1$;
    (b) $k + a_k \neq l + a_l$ for all $1 \leq k < l$.

    prove that there exist two positive integers $bandN$ such that

    $\left|\sum_{j=m+1}^{n} (aj - b)\right| \leq 1007^2$
    for all integers $mandn$ satisfying $n > m \geq N$