On A New Heavy Tailed Pareto-Weibull Distribution and it's Regression Model

Submitted by - Buddepu Pavan
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Supervised by - Dr.Deepesh Bhati

Department of Statistics,
Central University of Rajasthan
Bandar Sindri, Kishangarh, Ajmer

Contents

Introduction

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- In actuarial statistics and finance, the classical Pareto distribution has been considered to be better than other models because it provides a good narration of the random behavior of large claims.

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- An Expectation-Maximization (EM) type algorithm is used to facilitate the ML estimation procedure for the proposed model.

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- To study the EM type algorithm with the regression sturucture to the parameters of the proposed mixed model.
- To assess the applicability pf proposed EM type algorithm for new heavy tailed mixed model and using a real world insurance data sets.

Proposed Problem

- To examine the versatility of the proposed EM type algorithm using a real world insurance data set.

Methodology

Methodology

Pareto-Weibull distribution

In order to proposed the model, for the sake of completeness, we define the following: Let random variable X follow Weibull(ϕ, τ) having density

$$g_X(x) = \frac{\tau}{\phi} e^{-\left(\frac{x}{\phi}\right)^\tau} \left(\frac{x}{\phi}\right)^{\tau-1}, \quad x > 0, \phi > 0, \tau > 0,$$

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The conditional distribution of $Y|X = x$ follows Pareto(x, α) with

density

$$f_{Y|X}(y|x) = \frac{\alpha x^{\alpha}}{y^{\alpha+1}}, \quad y > x, \alpha > 0. \tag{2}$$

Pareto-Weibull distribution

We say that the random variable Y follows a Pareto-Weibull (PW) distribution if it admits the stochastic representation:

$$Y|X \sim \mathit{Pareto}(x, \alpha), \quad X \sim \mathit{Weibull}(\phi, \tau),$$

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The unconditional distribution of Y will be denoted by $Y \sim PW(\alpha, \phi, \tau)$

and its density is given as

$$f_Y(y; \alpha, \phi, \tau) = \int_0^y \frac{\alpha x^{\alpha-1} \left(\tau e^{-\left(\frac{x}{\phi}\right)^{\tau}} \left(\frac{x}{\phi}\right)^{\tau} \right)}{y^{\alpha+1}} dx.$$

Methodology

Pareto-Weibull distribution

The probability density function of Pareto-Weibull distribution is given by

$$ {}_{PW}f_Y(y;\alpha,\phi,\tau) = \frac{\alpha\phi^{\alpha}}{y^{\alpha+1}}\left(\Gamma\left(\frac{\alpha+\tau}{\tau}\right) - \Gamma\left(\frac{\alpha+\tau}{\tau},\left(\frac{y}{\phi}\right)^{\tau}\right)\right) $$

Methodology

Distribution Proporties

Let $g(y) = \log(f(y))$ then the $g(y)$ can be written as

$$_{PW}g_y(y;\alpha,\phi,\tau) = \log\left(\frac{\alpha\phi^{\alpha}}{y^{\alpha+1}}\left(\Gamma\left(\frac{\alpha+\tau}{\tau}\right) - \Gamma\left(\frac{\alpha+\tau}{\tau}, \left(\frac{y}{\phi}\right)^{\tau}\right)\right)\right)$$

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After equating the first derivative of the above equation to zero, we have following necessary condition for the existing of the mode M as

$$-\phi^{(\tau+\alpha)}\left(\Gamma\left(\frac{\alpha+\tau}{\tau}\right) - \Gamma\left(\frac{\alpha+\tau}{\tau}, \left(\frac{y}{\phi}\right)^{\tau}\right)\right)(\alpha+1) - \tau y^{\alpha+\tau}$$

$$e^{-(\frac{y}{\phi})^{\tau}} = 0$$

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$e^{-(\frac{y}{\phi})^{\tau}} = 0$

It is easy to verify that, $f(0) = 0$ and $f(+\infty) = 0$, the Pareto-Weibull distribution is always uni-modal.

Pareto-Weibull Regression Model

In this study, we consider two mixed Pareto regression models with non constant variance, can be given as follows. Let us assume that Z_i are independent and identically distributed (i.i.d) random variables and $Y_i|Z_i$ be the claim from a i^{th} policyholder, $i = 1, 2, ..., n$, are i.i.d random variables follows a one-parameter Pareto distribution having probability density function (pdf) as

$$f_{Y|Z}(y_i|z_i) = \frac{\alpha_i z_i^{\alpha_i}}{y_i^{\alpha_i+1}}, \tag{3}$$

where $y_i \geq z_i$, $\alpha_i > 1$

Methodology

Pareto-Weibull Regression Model The expected value and variance of the $Y_i|z_i$ are given by

$$\mathbb{E}(Y_i|z_i) = \frac{\alpha_i z_i}{(\alpha_i - 1)}$$

Methodology

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$$\mathbb{E}(Y_i|z_i) = \frac{\alpha_i z_i}{(\alpha_i - 1)}$$

$$\mathbb{V}ar(Y_i|z_i) = \frac{\alpha_i z_i^2}{(\alpha_i - 2)(\alpha_i - 1)^2}$$

Pareto-Weibull Regression Model The unconditional distribution of

Y_i having mixed Pareto distribution with pdf

$$f(y_i) = \int_0^y f(y_i|z_i)g(z_i;\tau_i)\,dz_i$$

Methodology

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$$f(y_i) = \int_0^y f(y_i|z_i)g(z_i;\tau_i)\,dz_i \tag{4}$$

Pareto-Weibull regression model is generated by considering scale parameter of the Pareto distribution follows unit mean Weibull distribution having parameter ϕ_i with pdf given by

$$g(z_i;\phi_i) = \phi_i.\left[\Gamma\left(1+\frac{1}{\phi_i}\right)\right] e^{-\left(z_i\left[\Gamma\left(1+\frac{1}{\phi_i}\right)\right]\right)^{\phi_i}} \left(z_i\left[\Gamma\left(1+\frac{1}{\phi_i}\right)\right]\right)^{\phi_i-1} \tag{5}$$

$z_i > 0$ and $\mathbb{E}(Z_i) = 1$

EM Algorithm

EM Algorithm

- Using equation (4), the pdf of two-parameter Pareto-Weibull distribution can be given by

$$PWf_y(y_i;\phi_i,\alpha_i) = \frac{\alpha_i}{y_i^{\alpha_i+1}.\left[\Gamma(1+\frac{1}{\phi_i})\right]^{\alpha_i}}.\left[\Gamma\left(\frac{\alpha_i+\phi_i}{\phi_i}\right) - \Gamma\left(\frac{\alpha_i+\phi_i}{\phi_i}\left(y_i\left[\Gamma(1+\frac{1}{\phi_i})\right]\right)^{\phi_i}\right)\right]$$

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where $y_i > 0$, $\alpha_i > 0$ and $\phi_i > 0$. The parameter α_i and ϕ_i contains the covariate information as $\alpha_i = \exp(\beta_1^\top \mathbf{x}_{1,i})$ $\phi_i = \exp(\beta_2^\top \mathbf{x}_{2,i})$

EM Algorithm

Pareto-Weibull Distribution

- We make use of EM algorithm to estimate unknown parameters of the proposed model

$$l_c(\theta; y_i, x_i) = \sum_{i=1}^{n} [\log(\alpha) + \alpha \log(x_i) - (\alpha + 1) \log(y_i)] + \sum_{i=1}^{n} \left[\log(\tau) - \log(\phi) - \left(\frac{x_i}{\phi}\right)^{\tau} + (\tau - 1)(\log(x_i) - \phi)\right] \tag{7}$$

EM Algorithm

- E-step: The E-step is used to fill the missing data. It computes the expected value of $l_c(\theta; Y, X)$ given the observed data, Y, and the current parameter estimate, θ^r say. The conditional expectation of complete log-likelihood function given in (7), say Q function can be written as

$$Q(\theta; \theta^{(r)}) = \mathbb{E}_{X_i}(l_c(\theta; Y, X) | Y, \theta^{(r)})$$

$$Q(\theta; \theta^{(r)}) = \sum_{i=1}^{n} \left[\log(\alpha^{(r)}) + \alpha^{(r)}.\mathbb{E}(\log(x_i)) - (\alpha^{(r)} + 1).\log(y_i)\right] + \sum_{i=1}^{n} \left[\log(\tau^{(r)}) - \log(\phi^{(r)}) - \left(\frac{\mathbb{E}(x_i^{\tau^{(r)}})}{\phi^{(r)}\tau^{(r)}}\right) + (\tau^{(r)} - 1)(\mathbb{E}(\log(x_i)) - \phi^{(r)})\right] \tag{8}$$

EM Algorithm

- Expectation step consisting computational part of conditional expectation of some functions of unobserved random variable X that are needed for the maximization. Using the current estimates $\hat{\alpha}^{(r)}$, $\hat{\phi}^{(r)}$ and $\hat{\tau}^{(r)}$, we calculate the pseudo-values,

$$t_i = \mathbb{E}(\log(X_i)|y_i;\hat{\alpha}^{(r)},\hat{\phi}^{(r)},\hat{\tau}^{(r)}) = \frac{\int_0^y \log(x_i)\frac{\alpha^{(r)}x_i^{\alpha^{(r)}}}{y_i^{\alpha^{(r)}+1}}\cdot\frac{\tau^{(r)}}{\phi^{(r)}}e^{-\left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}}}\left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}-1}dx_i}{\int_0^y \frac{\alpha^{(r)}x_i^{\alpha^{(r)}}}{y_i^{\alpha^{(r)}+1}}\cdot\frac{\tau^{(r)}}{\phi^{(r)}}e^{-\left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}}}\left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}-1}dx_i}$$

EM Algorithm

$$s_i = \mathbb{E}(X_i^{\hat{\tau}^{(r)}} | y_i; \hat{\alpha}^{(r)}, \hat{\phi}^{(r)}, \hat{\tau}^{(r)}) = \frac{\int_0^y x_i^{\hat{\tau}^{(r)}} \frac{\alpha^{(r)} x_i^{\alpha^{(r)}}}{y_i^{\alpha^{(r)}+1}} \cdot \frac{\tau^{(r)}}{\phi^{(r)}} e^{-\left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}}} \left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}-1} dx_i}{\int_0^y \frac{\alpha^{(r)} x_i^{\alpha^{(r)}}}{y_i^{\alpha^{(r)}+1}} \cdot \frac{\tau^{(r)}}{\phi^{(r)}} e^{-\left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}}} \left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}-1} dx_i}$$

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$$w_i = \mathbb{E}(X_i^{\hat{\tau}^{(r)}}\log(X_i)|y_i;\hat{\alpha}^{(r)},\hat{\phi}^{(r)},\hat{\tau}^{(r)}) = \frac{\int_0^y x_i^{\hat{\tau}^{(r)}} \log(x_i) \frac{\alpha^{(r)}x_i^{\alpha^{(r)}}}{y_i^{\alpha^{(r)}+1}} \cdot \frac{\tau^{(r)}}{\phi^{(r)}} e^{-\left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}}} \left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}-1} dx_i}{\int_0^y \frac{\alpha^{(r)}x_i^{\alpha^{(r)}}}{y_i^{\alpha^{(r)}+1}} \cdot \frac{\tau^{(r)}}{\phi^{(r)}} e^{-\left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}}} \left(\frac{x_i}{\phi^{(r)}}\right)^{\tau^{(r)}-1} dx_i}$$

EM Algorithm

- M-step: The M-step consists of maximizing over parameter vector θ using the conditional expectations computed in E-step. This step generates complete data after the expectation step and updates the parameters of the model. That is, we set

$$\theta^{(r+1)} = \max Q(\theta; \theta^{(r)})$$

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$$\theta^{(r+1)} = \max Q(\theta; \theta^{(r)})$$

  Where $\theta^{(r)}$ be the estimates of the parameter vector θ at the r^{th} iteration.
- The updated values of the parameters are

$$\hat{\alpha}^{(r+1)} = \frac{n}{(\sum_{i=1}^{n} \log(y_i) - \sum_{i=1}^{n} t_i)}$$

EM Algorithm

$$\hat{\phi}^{(r+1)} = \left(\frac{\sum_{i=1}^{n} s_i}{n} \right)^{\frac{1}{\hat{\tau}^{(r)}}}$$

$$\frac{n}{\hat{\tau}^{(r+1)}} + \sum_{i=1}^{n} t_i - n \log(\hat{\phi}^{(r)}) - \frac{\sum_{i=1}^{n} w_i}{(\hat{\phi}^{(r)})^{\hat{\tau}^{(r+1)}}} + \frac{\log(\hat{\phi}^{(r)}) \sum_{i=1}^{n} s_i}{(\hat{\phi}^{(r)})^{\hat{\tau}^{(r+1)}}} = 0 \quad (9)$$

- Solving (9) for the $\hat{\tau}^{(r+1)}$, we can obtain the improved estimate of $\hat{\tau}^{(r+1)}$.

EM Algorithm

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- Solving (9) for the $\hat{\tau}^{(r+1)}$, we can obtain the improved estimate of $\hat{\tau}^{(r+1)}$.
- Finally, iterate between the E-step and the M-step until some convergence criterion is satisfied, for example the relative change in log-likelihood between two successive iterations is smaller than 10^{-12}.

EM algorithm

Pareto-Weibull Regression Model

- The complete data log-likelihood takes the form

$$l_c(\theta) = \sum_{i=1}^{n} [\log(\alpha_i) + \alpha_i . \log(z_i) - (\alpha_i + 1) . \log(y_i)]$$
$$+\sum_{i=1}^{n} \left[\log(\phi_i) + \log\left(\Gamma\left(1 + \frac{1}{\phi_i}\right)\right) + (\phi_i - 1) . \left(\log(z_i) + \log\left(\Gamma\left(1 + \frac{1}{\phi_i}\right)\right)\right) - z_i . \Gamma\left(1 + \frac{1}{\phi_i}\right)\right]$$

- The expectation of $\log(z_i)$ and z_i are needed for the process of M-step.

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- The expectation of $\log(z_i)$ and z_i are needed for the process of M-step.
- E-step: The required expectations for $i = 1, 2...n$ can be computed as

$$t_i = \mathbb{E}_{Z_i}\left[\log(Z_i)|y_i; \theta^{(r)}\right]$$

EM Algorithm

$$t_i = \frac{\int_0^y \log(z_i)\frac{\alpha_i^{(r)} z_i^{\alpha_i^{(r)}}}{y_i^{\alpha_i^{(r)}+1}}.\phi_i^{(r)}.\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right] e^{-\left(z_i\left[\Gamma(1+\frac{1}{\phi_i^{(r)}})\right]\right)^{\phi_i^{(r)}}} \left(z_i\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right]\right)^{\phi_i^{(r)}-1} dz_i}{\int_0^y \frac{\alpha_i^{(r)} z_i^{\alpha_i^{(r)}}}{y_i^{\alpha_i^{(r)}+1}}.\phi_i^{(r)}.\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right] e^{-\left(z_i\left[\Gamma(1+\frac{1}{\phi_i^{(r)}})\right]\right)^{\phi_i^{(r)}}} \left(z_i\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right]\right)^{\phi_i^{(r)}-1} dz_i}$$

EM Algorithm

$$t_i = \frac{\int_0^y \log(z_i)\,\frac{\alpha_i^{(r)} z_i^{\alpha_i^{(r)}}}{y_i^{\alpha_i^{(r)}+1}}.\phi_i^{(r)}.\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right] e^{-\left(z_i\left[\Gamma(1+\frac{1}{\phi_i^{(r)}})\right]\right)^{\phi_i^{(r)}}}\left(z_i\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right]\right)^{\phi_i^{(r)}-1} dz_i}{\int_0^y \frac{\alpha_i^{(r)} z_i^{\alpha_i^{(r)}}}{y_i^{\alpha_i^{(r)}+1}}.\phi_i^{(r)}.\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right] e^{-\left(z_i\left[\Gamma(1+\frac{1}{\phi_i^{(r)}})\right]\right)^{\phi_i^{(r)}}}\left(z_i\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right]\right)^{\phi_i^{(r)}-1} dz_i} \tag{11}$$

and

$$w_i = \mathbb{E}_{Z_i}\left[Z_i | y_i; \theta^{(r)}\right]$$

$$w_i = \frac{\int_0^y (z_i)\,\frac{\alpha_i^{(r)} z_i^{\alpha_i^{(r)}}}{y_i^{\alpha_i^{(r)}+1}}.\phi_i^{(r)}.\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right] e^{-\left(z_i\left[\Gamma(1+\frac{1}{\phi_i^{(r)}})\right]\right)^{\phi_i^{(r)}}}\left(z_i\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right]\right)^{\phi_i^{(r)}-1} dz_i}{\int_0^y \frac{\alpha_i^{(r)} z_i^{\alpha_i^{(r)}}}{y_i^{\alpha_i^{(r)}+1}}.\phi_i^{(r)}.\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right] e^{-\left(z_i\left[\Gamma(1+\frac{1}{\phi_i^{(r)}})\right]\right)^{\phi_i^{(r)}}}\left(z_i\left[\Gamma\left(1+\frac{1}{\phi_i^{(r)}}\right)\right]\right)^{\phi_i^{(r)}-1} dz_i} \tag{12}$$

where $\phi_i^{(r)} = \exp(\mathrm{x}_2^\top \beta_2^{(r)})$.

EM Algorithm

- M-step: Using the numerical approximate value of t_i, update the regression parameters β_1 using Newton-Raphson method.

$$h_2(\beta_2) = \left[1 + \left(\left(1 - \Gamma\left(1 + \frac{1}{\phi_i^{(r)}}\right)^{\phi_i^{(r)}} . w_i^{\phi_i^{(r)}}\right)\left(\phi_i^{(r)} . \log\left(\Gamma\left(1 + \frac{1}{\phi_i^{(r)}}\right)\right) - F\left(1 + \frac{1}{\phi_i^{(r)}}\right) + \phi_i^{(r)} . t_i\right)\right)\right] \mathbf{x}_{2,ij} \tag{13}$$

$$H_2(\beta_2) = [A_1 + A_2 + A_3 + A_4 + A_5]\, \mathbf{x}_{2,ij}\mathbf{x}_{2,ij}^\top = X_2^\top W_2 X_2 \tag{14}$$

where $i = 1, 2, 3, ...n$ and $j = 1, 2, 3....p_2$.

EM Algorithm

- The matrix W_2 can be written as $W_2 = diag\{A_1 + A_2 + A_3 + A_4 + A_5\}$.

EM Algorithm

- The matrix W_2 can be written as $W_2 = diag\{A_1 + A_2 + A_3 + A_4 + A_5\}$.

Where $A_1 = \phi_i^{(r)} \log(\Gamma(1 + \frac{1}{\phi_i^{(r)}})) - \Psi^{(0)}(1 + \frac{1}{\phi_i^{(r)}})$, $A_2 = \frac{\Psi^{(1)}\left(1+\frac{1}{\phi_i^{(r)}}\right)}{\phi_i^{(r)}}$,
$A_3 = \phi_i^{(r)} \log(z_i)$,

$A_4 =$
$\log(z_i)\left[1 + \phi_i^{(r)}.\log(z_i) + \phi_i^{(r)}.\log(\Gamma(1 + \frac{1}{\phi_i^{(r)}})) - \Psi^{(0)}(1 + \frac{1}{\phi_i^{(r)}})\right].z_i^{\phi_i^{(r)}}\left(\Gamma(1 + \frac{1}{\phi_i^{(r)}})\right).\phi_i^{(r)}$ and
$A_5 = z_i^{\phi_i^{(r)}}.\left(\Gamma(1 + \frac{1}{\phi_i^{(r)}})\right)\left[\left(\phi_i^{(r)}\right)^2.\log\left(\Gamma\left(1 + \frac{1}{\phi_i^{(r)}}\right)\right)\left(1 + \phi_i^{(r)}.\log(z_i) + \phi_i^{(r)}.\log\left(\Gamma\left(1 + \frac{1}{\phi_i^{(r)}}\right)\right)\right)\right.$

EM Alogrithm

- The improved estimates of $\beta_2^{(r)}$ can be obtained using Newton Raphson method.

EM Alogrithm

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- Finally, iterate between the E-step and the M-step until some convergence criterion is satisfied, for example the relative change in log-likelihood between two successive iterations is smaller than 10^{-12}.

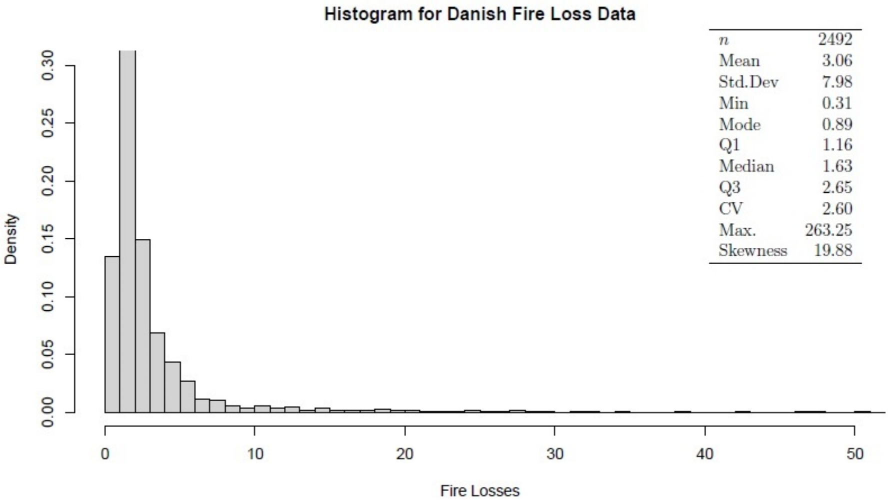
Real Data

Real Data

Pareto-Weibull Distribution

- we consider the well-known Danish fire insurance dataset that consists of 2492 fire insurance losses in millions of Danish kroner (DKr) from the years 1980 to 1990 (both inclusive), adjusted to reflect 1985 values.

Real Data



n	2492
Mean	3.06
Std.Dev	7.98
Min	0.31
Mode	0.89
Q1	1.16
Median	1.63
Q3	2.65
CV	2.60
Max.	263.25
Skewness	19.88

Figure: Histrogram for the claims

Real Data

Table: Value-at-Risk (VaR) for the Danish fire insurance data set at security level γ

	Model						
γ	PW	PL	Pareto	E-IG	CL-P	CW-P	Emperical
0.05	0.908	0.99	0.119	0.122	0.869	0.902	0.904
0.1	0.968	1.087	0.245	0.252	0.94	0.957	0.964
0.15	1.017	1.165	0.381	0.391	0.992	1.010	1.02
0.25	1.123	1.309	0.681	0.701	1.080	1.115	1.157
0.35	1.257	1.465	1.035	1.065	1.281	1.249	1.329
0.45	1.434	1.658	1.459	1.504	1.455	1.427	1.516
0.55	1.679	1.918	1.988	2.051	1.696	.673	1.735
0.65	2.047	2.301	2.68	2.764	2.053	2.043	2.049
0.75	2.668	2.937	3.66	3.773	2.653	2.668	2.645
0.85	3.989	4.252	5.276	5.421	3.915	4.002	3.884
0.95	9.475	9.425	9.344	9.446	9.039	9.571	8.406
0.99	33.655	30.254	17.103	16.561	30.793	34.334	24.613

Real Data

Table: LEV for the Danish fire insurance data set

u	PW	PL	Pareto	E-IG	CL-P	CW-P	Empirical
1	0.957	0.964	0.660	0.678	0.909	0.987	0.935
2	1.387	1.210	0.974	0.992	1.418	1.527	1.551
3	1.594	1.452	1.297	1.315	1.669	1.801	1.839
5	1.755	1.604	1.472	1.517	.942	2.107	2.155
8	1.925	1.782	1.668	1.713	2.158	2.355	2.387
10	1.982	1.835	1.668	1.741	2.250	2.462	2.483
15	2.135	1.976	1.668	1.810	2.401	2.642	2.653
21	2.195	2.035	1.668	1.811	2.513	2.778	2.762
40	2.353	2.165	1.668	1.811	2.697	3.007	2.919
70	2.475	2.259	1.668	1.811	2.829	3.176	2.997
110	2.552	2.305	1.668	1.811	2.920	3.297	3.045
170	2.620	2.350	1.668	1.811	2.997	3.400	3.093
270	2.694	2.408	1.668	1.811	3.067	3.497	3.113

Real Data

Table: Results for `Danish` fire insurance Data Set

Model	Parameter	NLL	AIC	BIC	K-S	Reference
PW	3	3839.43	7684.86	7689.05	0.049	Proposed Model
CW-P	3	3840.38	7686.76	7690.95	0.052	CalderÃn and Kwok (2016)
CL-P	3	3865.86	7737.72	7741.90	0.032	CalderÃn and Kwok (2016)
PL	3	3868.15	7742.30	7746.49	0.106	Colombi (1990)
Pareto	3	5051.91	10109.82	10114.01	0.290	–
E-IG	3	5088.45	10182.91	10187.09	0.283	–

Real Data

Pareto-Weibull Regression Model

- we consider the vehicle insurance losses data set based on one-year vehicle insurance policies taken out in 2004 and 2005, The original data set comprises of 67,856 policies and we consider 4,624 policies having at least one claim.

Real Data

Pareto-Weibull Regression Model

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Table: Summary of continuous explanatory variable and response variable of the vehicle insurance loss data set

Statistic	CLMSIZE	EXPSR
Minimum	0.2	0.002
Maximum	55.922	0.999
Q_1	0.354	0.411
Q_3	2.091	0.832
Median	0.761	0.637
Mean	2.014	0.611
Skewness	5.041	-0.346
Kurtosis	43.215	2.128

Real Data

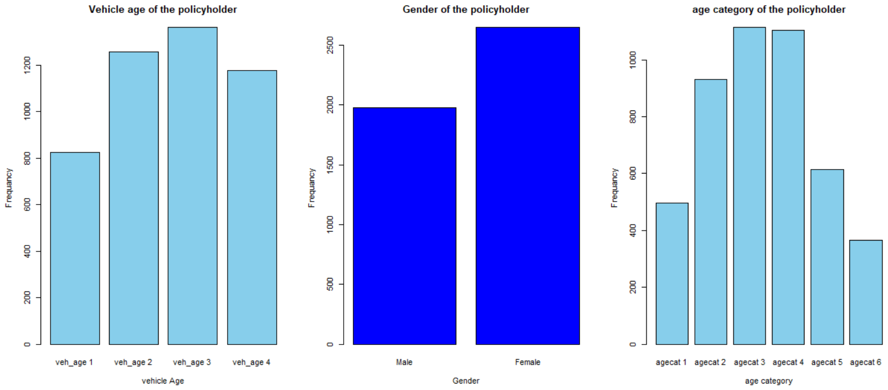

Figure: summary of categorical explanatory variable

Real Data

Table: Parameter estimates of the fitted models

Models		
Estimates	PW (α_i, ϕ_i)	PW (α, ϕ_i)
Coefficients β_1		
Intercept ($\beta_{1,0}$)	-0.0186	–
EXPSR ($\beta_{1,1}$)	0.7645	–
VEHAGE 2 ($\beta_{1,2}$)	-0.0561	–
VEHAGE 3 ($\beta_{1,3}$)	-0.1283	–
VEHAGE 4 ($\beta_{1,4}$)	-0.2055	–
GENDR 1 ($\beta_{1,5}$)	0.1411	–
AGECAT 2 ($\beta_{1,6}$)	0.2474	–
AGECAT 3 ($\beta_{1,7}$)	0.2839	–
AGECAT 4 ($\beta_{1,8}$)	0.2894	–
AGECAT 5 ($\beta_{1,9}$)	0.3996	–
AGECAT 6 ($\beta_{1,10}$)	0.3562	–
Coefficients β_2		
Intercept ($\beta_{2,0}$)	0.2454	0.1493
EXPSR ($\beta_{2,1}$)	-0.0404	0.0524
VEHAGE 2 ($\beta_{2,2}$)	0.0242	0.0241
VEHAGE 3 ($\beta_{2,3}$)	0.0704	0.0622
VEHAGE 4 ($\beta_{2,4}$)	0.1072	0.0918
GENDR 1 ($\beta_{2,5}$)	-0.0499	-0.0354
AGECAT 2 ($\beta_{2,6}$)	-0.0645	-0.0328
AGECAT 3 ($\beta_{2,7}$)	-0.0703	-0.0291
AGECAT 4 ($\beta_{2,8}$)	-0.0821	-0.0364
AGECAT 5 ($\beta_{2,9}$)	-0.0974	-0.0493
AGECAT 6 ($\beta_{2,10}$)	-0.0027	0.0282

Model Comparison

- The model fit of the proposed mixed Pareto regression model having regression structure to both the parameters (α_i, ϕ_i) for the vehicle insurance dataset is compared with the mixed Pareto regression model with regression structure to only one parameter (α, ϕ_i).

Model Comparison

- The model fit of the proposed mixed Pareto regression model having regression structure to both the parameters (α_i, ϕ_i) for the vehicle insurance dataset is compared with the mixed Pareto regression model with regression structure to only one parameter (α, ϕ_i).
- For the selection of best fitted model we have used three model selection criterion namely Deviance, AIC and SBC (also known as BIC). The formulae involved in the computation of above mentioned models selection criterion are

$$DEV = -2l(\hat{\theta})$$

where $l(\hat{\theta})$ is the maximum of the log-likelihood and $\hat{\theta}$ is the vector of the estimated model parameters.

Real Data

- The AIC can be computed as

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Real Data

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- The SBC is given by

$$SBC = -2l(\hat{\theta}) + \log(n) \times df$$

where n is sample size of the data set and df is the number of fitted parameters of the model.

Real Data

Table: Values of Negative Log-Likelihood Function, Global deviance, AIC and SBC for vehicle insurance dataset

Model	df	NLL	Deviance	AIC	SBC
PW (α_i, ϕ_i)	22	7375.29	14750.61	14794.61	14936.26
PW (α, ϕ_i)	12	7433.33	14866.66	14890.66	14967.93

Conclusions

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Conclusions

- We have proposed the new heavy tailed distribution known as Pareto-Weibull distribution generated by considering the scale parameter of the Pareto density follows Weibull distribution.
- We have illustrated the application of Pareto-Weibull distribution to two real world data sets.
- By comparing Pareto-Weibull distribution with other existing distribution we conclude that the Pareto-Weibull performs better and can be a considerable heavy tailed distribution

Conclusions

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Conclusions

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- One real life insurance data set namely vehicle insurance data set is provided to emphasize on the applicability of the proposed mixed Pareto regression model and to asses the utility of our proposed estimation methodology.
- The results obtained for vehicle insurance data set suggest that the Pareto-Weibull regression model (α_i, ϕ_i) performs better than the Pareto-Weibull regression model (α, ϕ_i).

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Thank you