Confidence intervals for quantities averaged over quantum circuits

Karl Mayer

August 2020

The following scenario is often encountered when running experiments on quantum computers. A number N_c different quantum circuits is selected, and each circuit is repeated N_t times. Each individual trial may be counted as either a success or failure, according to some criteria that may depend on the quantum circuit. We are interested in estimating the probability of success, averaged over the distribution with which the circuits are sampled from. Such an estimate may take the form of a confidence interval. In the following analysis, we assume that for a each circuit, the success probabilities on each trial are independent and identically distributed.

Let X_i be the true probability of the ith circuit succeeding. X_i is a random variable since it depends on the randomly chosen circuit. The true average success probability of the N_c random circuits is

$$Y = \frac{1}{N_c}\sum_{i=1}^{N_c} X_i$$

Hoeffding's inequality is the bound on the probability of Y deviating from its expectation by a given amount. This is given by

$$\mathbb{P}(|Y - \mathbb{E}(Y)| \geq \epsilon_1) \leq 2e^{-N_c\epsilon_1^2} \tag{1}$$

If we set the RHS equal to δ_1 and solve for ϵ_1, we find that $[Y - \epsilon_1, Y + \epsilon_1]$ is a $1 - \delta_1$ confidence interval for $\mathbb{E}(Y)$, with

$$\epsilon_1 = \sqrt{\frac{1}{2N_c}\log(2/\delta_1)} \tag{2}$$

In a real experiment, it is not possible to determine X_i exactly. Instead X_i must be estimated by repeating the circuit N_c times. For $j \in [N_c]$, let X_{ij} denote the random variable taking the value of 0 or 1 depending on if the jth trial of the the ith circuit fails or succeeds. The estimate of X_i is denoted by $\hat{X}_i$ and given by

$$\hat{X}_i = \frac{1}{N_t}\sum_{j=1}^{N_t} X_{ij} \tag{3}$$

We can then estimate the quantity Y by

$$\hat{Y} = \frac{1}{N_c} \sum_{i=1}^{N_c} \hat{X}_i = \frac{1}{N_c N_t} \sum_{ij} X_{ij} \tag{4}$$

Applying Hoeffding's inequality again, we get

$$\mathbb{P}(|\hat{Y} - Y| \geq \epsilon_2) = \delta_2 \tag{5}$$

with

$$\epsilon_2 = \sqrt{\frac{1}{2N_c N_t} \log(2/\delta_2)} \tag{6}$$

Combining this equation with Eq. (1) gives

$$\mathbb{P}(|\hat{Y} - \mathbb{E}(Y)| \geq \epsilon_1 + \epsilon_2) \leq \delta_1 + \delta_2 \tag{7}$$

Letting $\epsilon = \epsilon_1 + \epsilon_2$ and $\delta = \delta_1 + \delta_2$, we arrive at

$$\mathbb{P}(|\hat{Y} - \mathbb{E}(Y)| \geq \epsilon) \leq \delta \tag{8}$$

For a fixed confidence $1 - \delta$, the precision parameter ϵ can be minimized subject to the constraint $\delta_1 + \delta_2 = \delta$. This leads to a transcendental equation that can be solved numerically. A more conservative confidence interval is obtained by setting $\delta_1 = \delta_2 = \delta/2$, which results in

$$\epsilon = \sqrt{\frac{1}{2N_c} \log(2/\delta)} \left(1 + \frac{1}{\sqrt{N_t}}\right) \tag{9}$$