Turing model for transition to turbulence in wall-bounded shear flows

The equations simulated in the model are :

$$\frac{\partial M}{\partial t} + M\frac{\partial M}{\partial x} = f(M,W) + D_M\frac{\partial^2 M}{\partial x^2} \tag{1}$$

$$\frac{\partial W}{\partial t} + M\frac{\partial W}{\partial x} = g(M,W) + D_W\frac{\partial^2 W}{\partial x^2} + \sigma\eta W \tag{2}$$

where D_M, D_W are the diffusivity, η is white noise drawn from a Gaussian distribution of zero mean and unit standard deviation, σ is the amplitude of the noise. The functions $f(M,W)$ and $g(M,W)$ are given by :

$$f(M,W) = \alpha_M(M_0 - M) - \beta_M(M - \bar{M})W^4 + \sigma_M W^2 \tag{3}$$

$$g(M,W) = -\alpha_W W + \beta_W(M - \bar{M})W^3 - \gamma_W W^3 - \sigma_M MW \tag{4}$$

where :

$$M_0 = 1\,,$$

$$\begin{aligned} \alpha_M &= f_1\frac{\lambda_M}{R}\,, \quad \beta_M = f_2\frac{\sigma_U^2\sigma_V^2}{\lambda_U\lambda_V^2}R^3\,, \quad \bar{M} = f_3\frac{\lambda_V\sigma_W}{\sigma_U\sigma_V}\frac{1}{R} \\ \alpha_W &= f_4\frac{\lambda_W}{R}\,, \quad \beta_W = f_5\frac{\sigma_U\sigma_V\sigma_W}{\lambda_U\lambda_V}R^2\,, \quad \gamma_W = f_6\frac{\sigma_V^2}{\lambda_V}R \end{aligned} \tag{5}$$

$$\begin{aligned} \lambda_M &= 2.47\,, \quad \lambda_U = 5.2\,, \quad \lambda_V = 7.67\,, \quad \lambda_W = 7.13 \\ \sigma_M &= 0.31\,, \quad \sigma_U = 1.29\,, \quad \sigma_V = 0.22\,, \quad \sigma_W = 0.68 \end{aligned} \tag{6}$$

The control parameter for the model is R, and the model parameters $[f_1, f_2, f_3, f_4, f_5, f_6]$ are used for tuning the model. The default values set in the code are $[f_1, f_2, f_3, f_4, f_5, f_6] = [5, 5, 0.2, 0.2, 0.2, 5]$ and $D_M = 10$, $D_W = 0.2$. The user manual is structured based on the multiple tabs in the GUI :

1 Tab-1

This tab is used for configuring the model and is depicted in fig 1

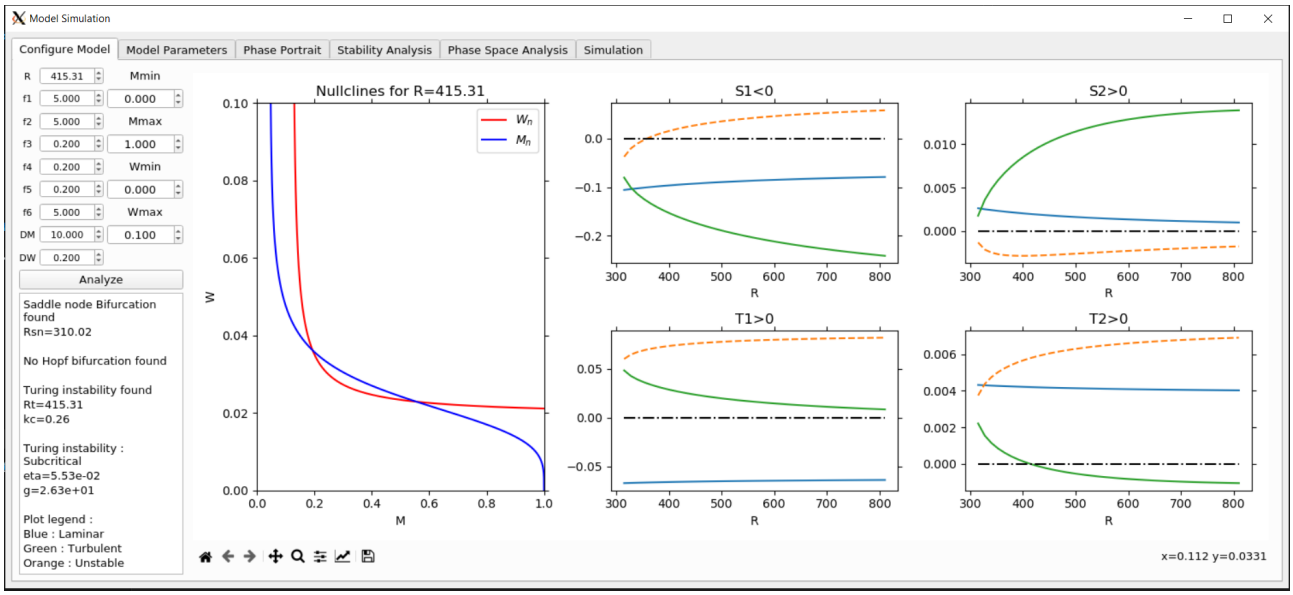

Figure 1: Tab-1 : Configuring the model

The conditions for a Turing instability are :

$$f_M + g_W < 0 \tag{7}$$

$$f_M g_W - g_M f_W > 0 \tag{8}$$

$$D_W f_M + D_M g_W > 0 \tag{9}$$

$$(D_W f_M + D_M g_W)^2 - 4 D_M D_W (f_M g_W - g_M f_W) > 0 \tag{10}$$

where the subscripts of f, g indicate the partial derivatives with respect to the variable indicated in the subscript. These constraints are shown as $S1, S2$ for Eq 7,8 and $T1, T2$ for Eq 9,10. Clicking the "Analyze" button will compute the constraints for different values of R and evaluate the different bifurcations of the system such as the saddle-node bifurcation, Hopf bifurcation if any and Turing instability. These are reported in the bottom left panel. The entire computation will take a couple of minutes.

The nullclines of the system for any value of R given the tuning parameter values are depicted in the (M, W) phase plane. The nullclines $f(M, W) = 0$ and $g(M, W) = 0$ are depicted as M_n, W_n in blue and red respectively. The entries $Mmin, Wmin, \cdots$ are used to set the extents of the phase plane. The bottom left of the panel with all the plots has options or zooming at any individual plot, saving a particular plot, adjusting the axes etc.

2 Tab-2

The second tab is used for setting the initial condition for the simulation. This tab is depicted in fig 2.

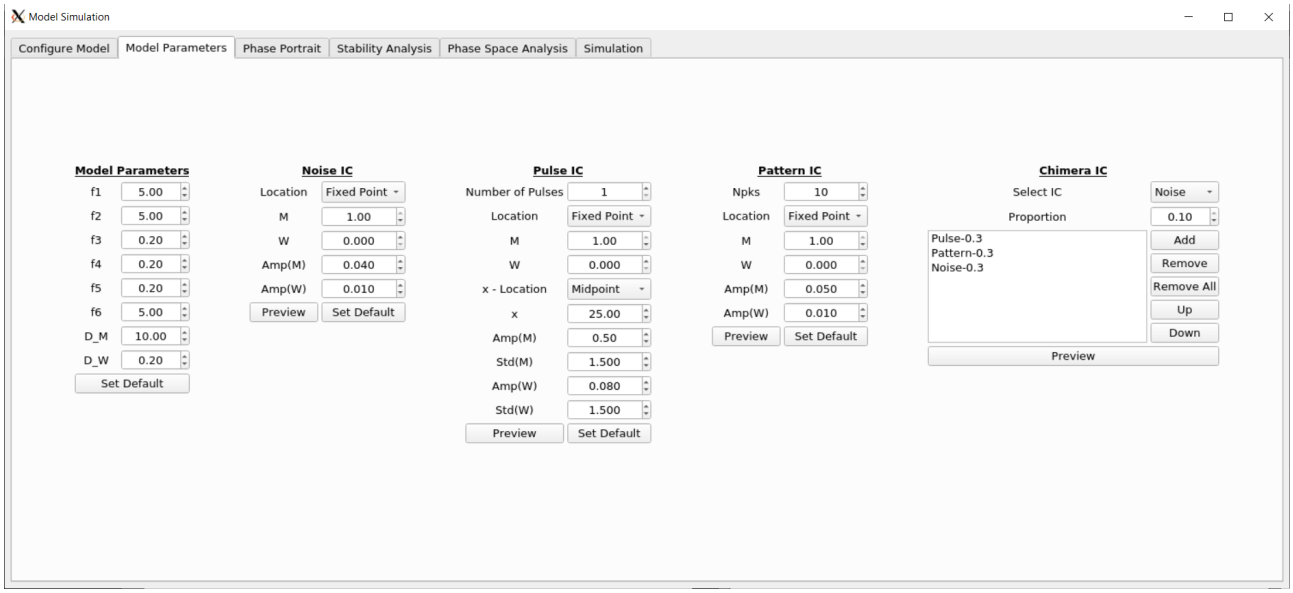

Figure 2: Tab-2 : Model parameters

The model parameters can be changed in the left most section of the tab. **The values indicated in this tab will be used in all the subsequent tabs**. The button "Set Default" resets the parameters to the default values. Three different types of initial conditions such as noise, pulse and pattern can be initialized. The length of the domain is set in the tab labelled "Simulation". The amplitudes, location, number of pulses, number of waves in the pattern (Npks) can all be changed to suit the simulation as required by the user. Each of the initial conditions can be previewed as $M(x), W(x)$ and in the (M, W) plane parametrized by x. Alternatively, a "Chimera" can be constructed by assembling different initial conditions applicable only to a portion of the domain set with the option "Proportion" in the "Chimera IC" section. Once the type of initial condition and its proportion in the domain are set, click the "Add" button to include it in the list. The list can be reorganized with the "Up" and "Down" buttons or particular entry can be removed with the "Remove" button.

3 Tab-3

This tab is utilized for viewing the phase portrait of the model as depicted in fig 3

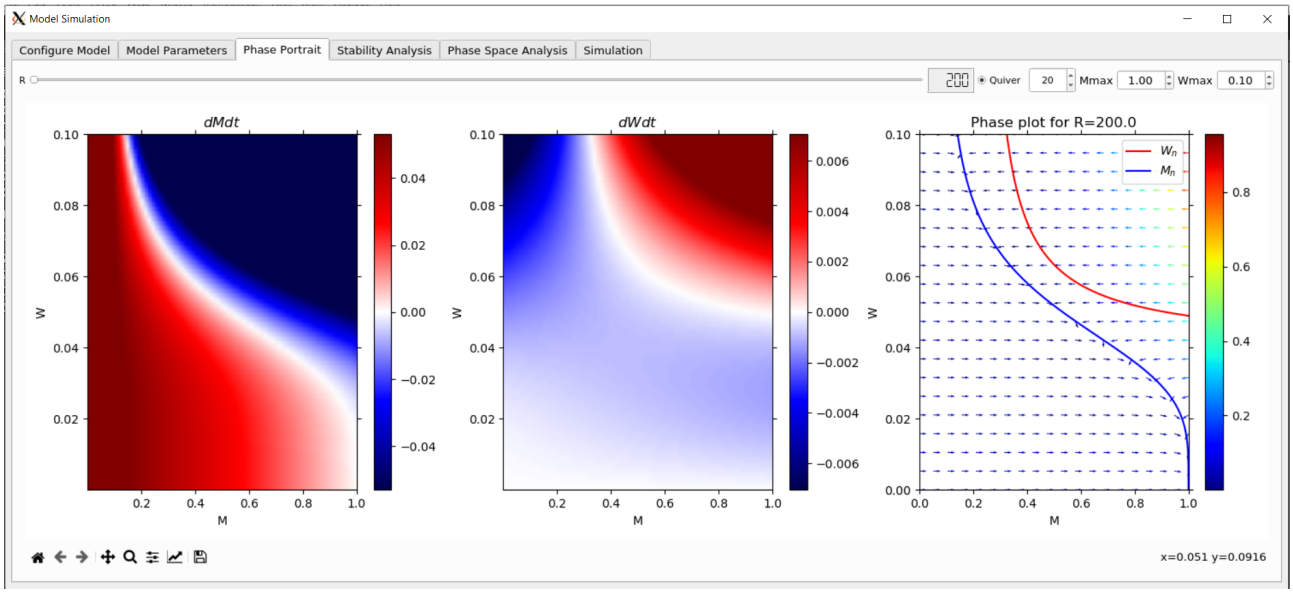

Figure 3: Tab-3 : Phase Portrait

The panels from left to right indicate the isocontours of dM/dt,dW/dt and the phase portrait with the normalized phase space velocity vectors and nullclines. The value of R can be changed with the slider at the top of the tab. The resulting value of R is displayed at the end of the slider. The extents of the phase portrait and the number of quivers to be displayed can be modified suitably.

4 Tab-4

The third tab is labelled "Stability analysis" as depicted in fig 4

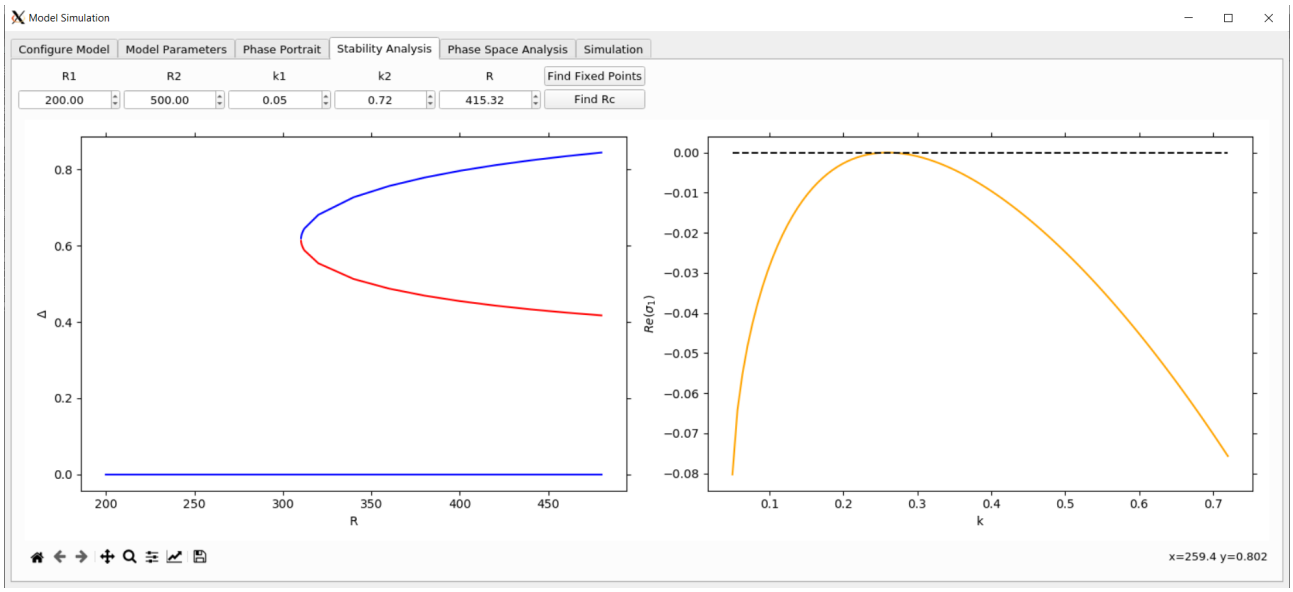

Figure 4: Tab-4 : Stability analysis

To find the fixed points of the system click the button "Find Fixed points". This will evaluate all the fixed points in the range $(R1, R2)$ and plot them with their distance Δ to the trivial fixed point $(M, W) = (1, 0)$ in the left panel.

Clicking the button "Find Rc" evaluates the critical value of R at which the eignevalue of the linearised problem touches the neutral axis. The resulting value of R is depicted in the entry labelled R. Simultaneously, the plot of $Re(\sigma_1)$ vs k is shown in the right panel, where σ is the eigenvalue of the linearised equations and k is the wavenumber. This computation is carried out for the range $(k1, k2)$ specified by the user.

5 Tab-5

The fourth tab titled "Phase Space Analysis" is depicted in fig 5 and is used to study the kinetics encoded in the functions f, g. This is done by considering the dynamics in time i.e the equations:

$$\frac{dM}{dt} = f(M, W) \tag{11}$$

$$\frac{dW}{dt} = g(M, W) + \sigma\eta W \tag{12}$$

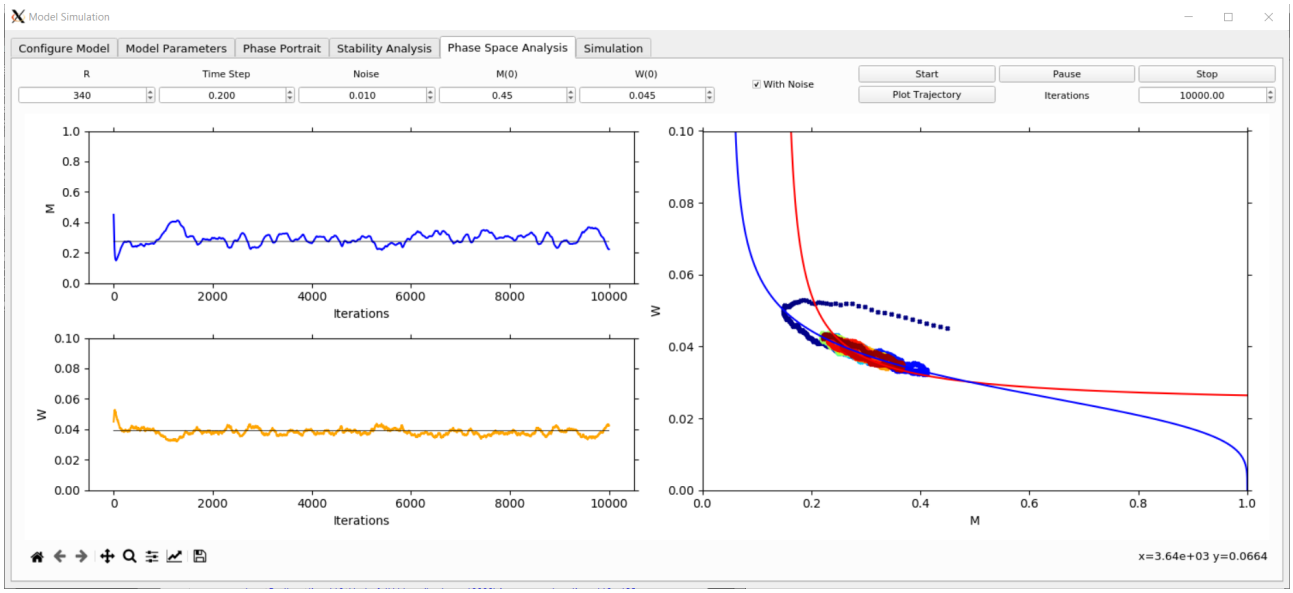

Figure 5: Tab-5 : Phase Space Analysis

By checking OR unchecking the box labelled "Noise", time stepping of Eq 11 and Eq 12 are done with or without noise respectively. The deterministic system is time stepped with the fourth order Runge-Kutta method, while the Euler-Maruyama time stepping is used for the stochastic equations. The time step to be used and initial condition are input near the top panel in the entries labelled "Time step", "M(0)" and "W(0)" respectively.

Simulation up to N iterations (input in the entry labelled "Iterations") is initiated with the button "Plot Trajectory". This plots the time evolution of Eq 11 and Eq 12 as $M(t), W(t)$ in the left panels (top and bottom respectively). The trajectory in the (M, W) phase plane is traced in the right panel. The *turbulent* (upper branch) fixed point of the system is also plotted in the left panels as a thin black line.

Instead of plotting the trajectory for N iterations, an animation tracing the time evolution at every time step can be started by clicking the "Start" button. The "Pause" and "Stop" buttons function as their name suggests.

6 Tab-6

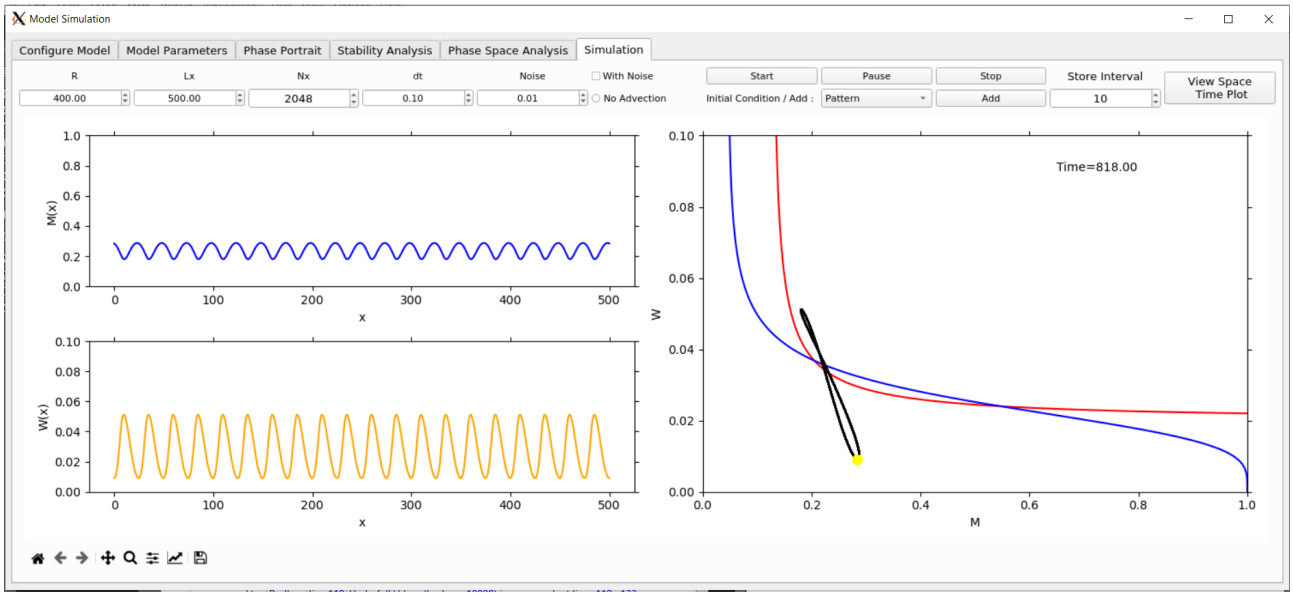

Figure 6: Tab-6 : Simulation

The last tab titled "Simulation" depicted in fig 6 is used for simulating Eq 1 and Eq 2 as an initial value problem. The spectral toolbox *Dedalus* is used for the simulation. Simulation parameters such as value of R, length of the domain L_x, domain discretization points N_x, time step dt and value of noise amplitude σ labelled "Noise" are input at the top of the tab. Two options are available :

- Simulate with or without noise
- Simulate with or without advection i.e either include or exclude the term $M\partial M/\partial x$ and $M\partial W/\partial x$ in the equations.

The initial condition is chosen from the drop down list labelled "Initial condition". This constructs the chosen initial condition according to the parameters set in the "Model Parameters" tab.

To start the simulation press the "Start" button . The "Pause" and "Stop" buttons perform actions as suggested by their names. Once started, no changes to the simulation parameters are allowed in between the simulation except for changing the time step.

The option "Self" for the initial condition sets the final state of a stopped simulation as the initial condition for the fresh simulation. It can be used only when the GUI has not been closed and a simulation has been stopped. This retains the state displayed and uses it for the next simulation. Except for L_x and N_x any other parameter or simulation option can be changed with the "Self" option.

The results of the simulation are stored in a folder labelled "case" within the same directory where the Python scripts are running. The results are saved at equal intervals of time as indicated in the entry labelled "Storage Interval". Space-time plots are plotted with the button "View Space time plots". **The results are overwritten unless they are copied to another directory or the "case" folder renamed**.