CS215 : Home Work Assignment -2

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Contents

Chapter 1

Question 1

1. (a) (i)

1) a)

(i) Let X be the random variable denoting the number of tests in round 2

X takes values ks $(k \in [0, n/s])$.

$$\underbrace{P(X = ks)}_{\text{We need } k \text{ pools to be positive.}} = {}^{n/s}C_k \times \Big(\underbrace{1 - (1-p)^s}_{\text{Probability that pool is positive}}\Big)^k \times \left[(1-p)^s\right]^{n/s - k}$$

$$E[X] = \sum_{k=0}^{n/s} ks \cdot P(X = ks)$$

$$= \sum_{k=0}^{n/s} ks \cdot {}^{n/s}C_k \, (1-p)^{n-ks} \cdot \left(1 - (1-p)^s\right)^k$$

$$= s \cdot \frac{n}{s} \cdot \left(1 - (1-p)^s\right)^1 \cdot \underbrace{\sum_{k=0}^{n/s} {}^{n/s - 1}C_{k-1} \left[(1-p)^s\right]^{(n/s - 1) - (k-1)} \cdot \left(1 - (1-p)^s\right)^{k-1}}_{1}$$

$$\boxed{E[X] = n\left(1 - (1-p)^s\right)}$$

$\therefore$ expected total no. of tests for this method $= n/s + n\left(1 - (1-p)^s\right)$

($\because$ n/s is fixed no. of tests in Round-1)

Note: 'p' is the probability of person with covid positive

Assuming p is calculated for large population.

(ii) $T(s) = n/s + nps$ as $p \to 0$

$$T'(s) = np - \frac{n}{s^2} = 0$$

$$s = 1/\sqrt{p} \quad (\because s > 0)$$

$$T''(s) = \frac{2n}{s^3} > 0 \quad (s > 0)$$

$\therefore s = 1/\sqrt{p}$ gives the least $T(s)$

$$T(1/\sqrt{p}) = 2n\sqrt{p}.$$

(iii)

1) a) (iii)

$$T(s) = \frac{n}{s} + n\left(1 - (1-p)^s\right) < n$$

$$\Rightarrow \frac{1}{s} + \cancel{1} - (1-p)^s < \cancel{1}$$

$$\Rightarrow (1-p)^s > 1/s$$

$$\Rightarrow (1-p) > s^{-1/s}$$

$$\Rightarrow p < 1 - s^{-1/s}$$

$f(s) = 1 - s^{-1/s}$ has maximum at $s = e$. , but s can take only integer values.

$$f(2) = 1 - 2^{-1/2} \approx 0.292$$

$$f(3) = 1 - 3^{-1/3} \approx 0.306$$

$\therefore$ p takes a max value $(0.306 - \delta)$ $\{\delta$ is negligible$\}$ for $s = 3$, if $T(s) < n$.

1) b) Here,

We are assuming np people are positive & $n(1-p)$ people are negative with Covid 19.

(i) Probability that a genuinely healthy person participates in pool that is negative

$$= \left[\text{Probability that all ~~no other~~ positive people are absent in this group}\right] * \left[\text{Probability that this healthy person is present in the group}\right]$$

$$\boxed{P_{req} = (1-\pi)^{np} \cdot \pi}$$

(ii) maximizing P_{req},

$$\frac{\partial P_{req}}{\partial \pi} = 0$$

$$\Rightarrow (1-\pi)^{np} + \pi \cdot np\,(1-\pi)^{np-1} \cdot (-1) = 0$$

$$\Rightarrow (1-\pi)^{np-1}\left(1-\pi-np\pi\right) = 0$$

$$\Rightarrow \pi = 1/np+1$$

$$\frac{\partial^2 P_{req}}{\partial \pi^2} = (1-\pi)^{np-1}\left(-(np+1)\right) + (np-1)(1-\pi)^{np-2}\left(1-(np+1)\pi\right)(-1) < 0$$

(when $\pi = 1/np+1$.)

$$\boxed{\text{optimal } \pi = \frac{1}{np+1}}$$

(iii) let, a healthy person participates in i pools out of T_1.

$$P(\text{All } i \text{ pools are positive}) = {}^{T_1}C_i\, \pi^i \left(1-(1-\pi)^{np}\right)^i \cdot (1-\pi)^{T_1-i}$$

$$P_{req} = \sum_{i=0}^{T_1} {}^{T_1}C_i \cdot \pi^i \left(1-(1-\pi)^{np}\right)^i \cdot (1-\pi)^{T_1-i}$$

$$P_{req} = \left[1-\pi(1-\pi)^{np}\right]^{T_1}, \quad \text{where } \pi = 1/np+1$$

$\therefore$ Probability that all pools that a healthy person participates in, tests positive

$$\approx \left[1-\frac{e^{\frac{-np}{np+1}}}{np+1}\right]^{T_1}$$

(iv) $E(X_{R_1}) \approx T_1$ where X_{R_i} is a random variable that represents the number of tests in Round-i

{I am neglecting the fact that there may be some pools out of T_1 that are null.}

$E(X_{R_2})$ = (expected no. of tests of ~~negative~~ positive subjects) + (expected no. of tests of negative subjects who participate in all positive pools).

$$= \underbrace{(\sim np)}_{\text{This is number of positive subjects.}} + \underbrace{\left[n(1-p)\left(1-\frac{e^{-np/np+1}}{np+1}\right)^{T_1}\right]}_{\text{number of healthy people} \times \text{Probability he participates in all positive pools.}}$$

$$E(X) = T_1 + np + n(1-p)\left(1-\frac{e^{-np/np+1}}{np+1}\right)^{T_1}$$

(v)

$$\frac{\partial E(X)}{\partial T_1} = 1 + n(1-p)\left(1-\frac{e^{-np/np+1}}{np+1}\right)^{T_1} \cdot \ln\left(1-\frac{e^{-np/np+1}}{np+1}\right) = 0$$

$$T_1 = \frac{\ln\left(\dfrac{-1}{n(1-p)\ln\left(1-\frac{e^{-np/np+1}}{np+1}\right)}\right)}{\ln\left(1-\frac{e^{-np/np+1}}{np+1}\right)}$$

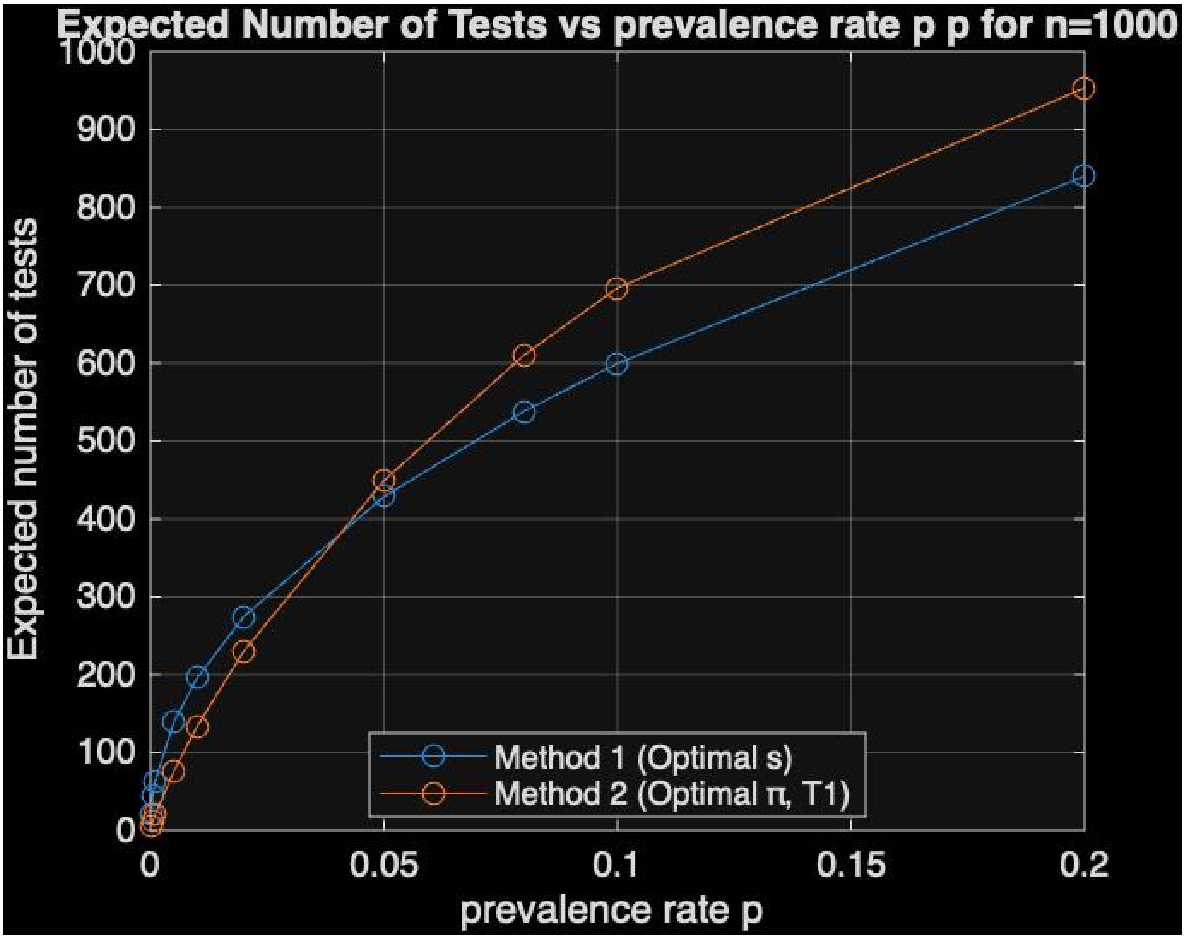
So,

$$T_1 = \frac{\ln\left(\dfrac{-1}{n(1-p)\ln(1-K)}\right)}{\ln(1-K)} \quad \text{where } K = \frac{e^{-np/np+1}}{np+1}$$

By resubstituting T_1 into $E(X)$,

$$E(X) = np - \frac{1 + \ln\left[-n(1-p)\ln(1-K)\right]}{\ln(1-K)} \quad \text{where } K = \frac{e^{-np/np+1}}{np+1}$$

1. (c)



Comments:

The two plots intersect at some value of p around 0.04.

- If disease spread ,i.e. p is small ($p < 0.05$), both methods needs almost equal number of tests.
- After some value of $p(> 0.1)$, method-1 works better.

As Covid-19 has smaller p values, both work similarly with method-2 needing slightly lesser number of tests.

Chapter 2

Question 2

X & Y are independent r.v with PDFs $f_X(\cdot)$ & $f_Y(\cdot)$ respectively.

$Z = X\cdot Y$

$$F_Z(z) = P(X\cdot Y \le z)$$

let, $I = [a, a+da]$.

$$P(X \in I) = f_X(a)\cdot da$$

Case-1: $a > 0$,

$$P(XY \le z) = P\left(Y \le \frac{z}{a} \mid X \in I\right) \qquad I = [a, a+da]$$

$$= P\left(Y \le \frac{z}{a}\right)\cdot P(X \in I) \qquad X, Y \text{ are independent.}$$

$$= [f_X(a)\,da]\cdot\left(F_Y(z/a)\right)$$

Case-2: $a < 0$,

$$P(X\cdot Y \le z) = P\left(Y \ge \frac{z}{a} \mid X \in I\right)$$

$$= \left[1 - P\left(Y \le \frac{z}{-|a|}\right)\right]\cdot P(X \in I)$$

$$= \left[1 - F_Y(z/-|a|)\right]\cdot f_X(a)\cdot da.$$

$$F_Z(z) = \int_{-\infty}^{0} \left(1 - F_Y(z/a)\right) f_X(a)\,da + \int_{0}^{\infty} \left(F_Y(z/a)\right)\cdot f_X(a)\,da$$

differentiate on both sides.

$$f_Z(z) = \int_{-\infty}^{0} f_X(a)\left(-f_Y(z/a)\cdot\frac{1}{a}\right)da + \int_{0}^{\infty} f_X(a)\left(f_Y(z/a)\cdot\frac{1}{a}\right)da$$

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} \frac{f_X(a)\cdot f_Y(z/a)}{|a|}\,da}$$

$\therefore$ PDF of $Z = X\cdot Y$ $\boxed{f_Z(z) = \int_{-\infty}^{\infty} \frac{f_X(x)\cdot f_Y(z/x)}{|x|}\,dx}$ or $\boxed{f_Z(z) = \int_{-\infty}^{\infty} \frac{f_X(z/y)\cdot f_Y(y)}{|y|}\,dy}$

Chapter 3

Question 3

3) Given, $\{x_i\}_{i=1}^{n}$ are independent sampling from a PDF.

- The estimated value by its definition = $\int_{-\infty}^{\infty} f_X(x)\cdot x\, dx = E[x]$

- As the x_i s are samples from the PDF, the estimate of $E[x]$ $(\hat{x})$ is

$$\boxed{\hat{x} = \frac{\sum x_i}{n}} //$$

The $\hat{x} = \frac{\sum f_X(x_i)\cdot x_i}{n}$ is estimating the $E[x\cdot f_X(x)]$.

Intution of Sampling justifies that $\hat{x} = \frac{\sum x_i}{n}$ is the correct estimate of $E(x)$.

Chapter 4

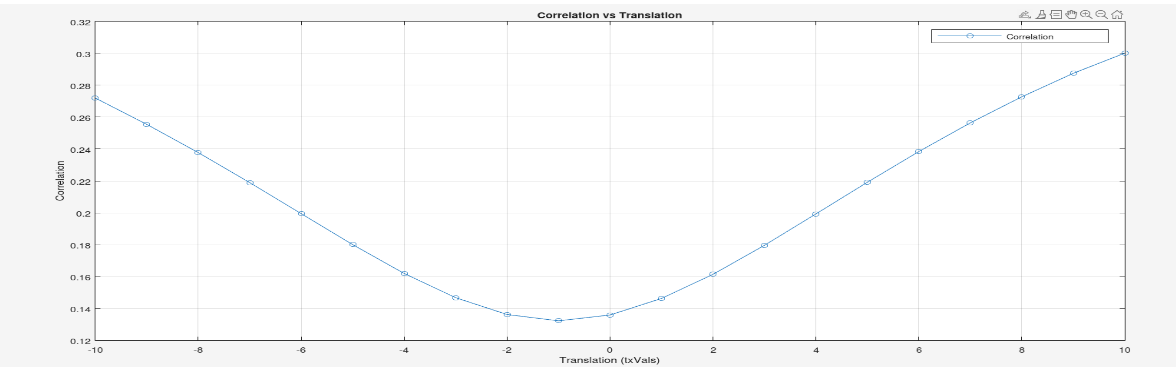
Question 4

Running Instructions:

- Save the first image as T1.jpg and second image as T2.jpg.
- Run the code in matlab to see 9 plots.

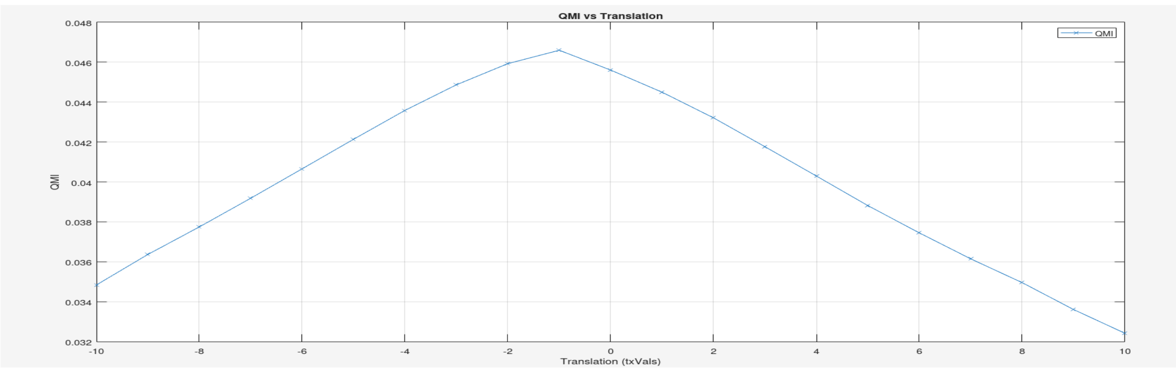
Plots show the relationship between different similarity measures (computed between Image 1 and the shifted version of Image 2) and the amount of translation applied to Image 2. (Note that we are changing image 2 after every three plots.)

- **Plot 1: Correlation vs Translation**



  - The correlation values are positive but remain close to zero, indicating weak dependence.
  - As the translation increases in either direction, the correlation rises, although it remains relatively low overall.

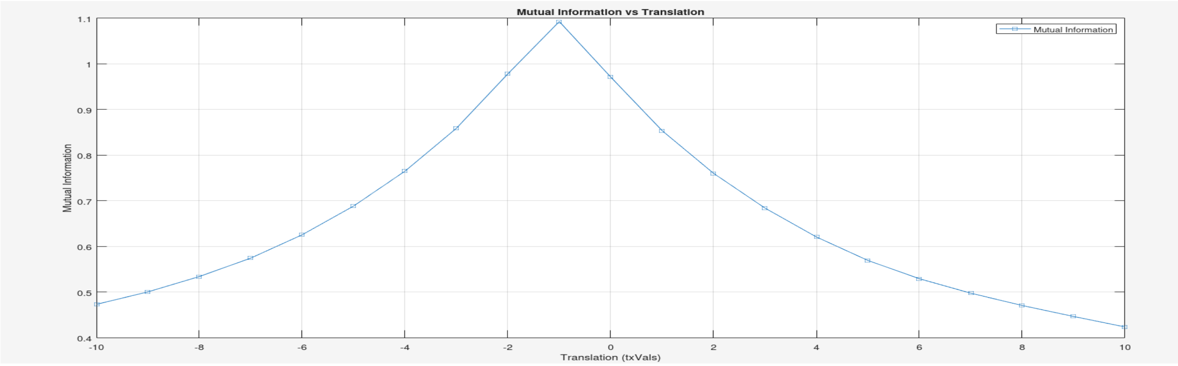
- **Plot 2: QMI vs Translation**



  - QMI reaches its maximum value at left shift by 1px, indicating best alignment at that shift.

– The value decreases symmetrically as translation increases, showing sensitivity to misalignment.

- **Plot 3: Mutual Information vs Translation**

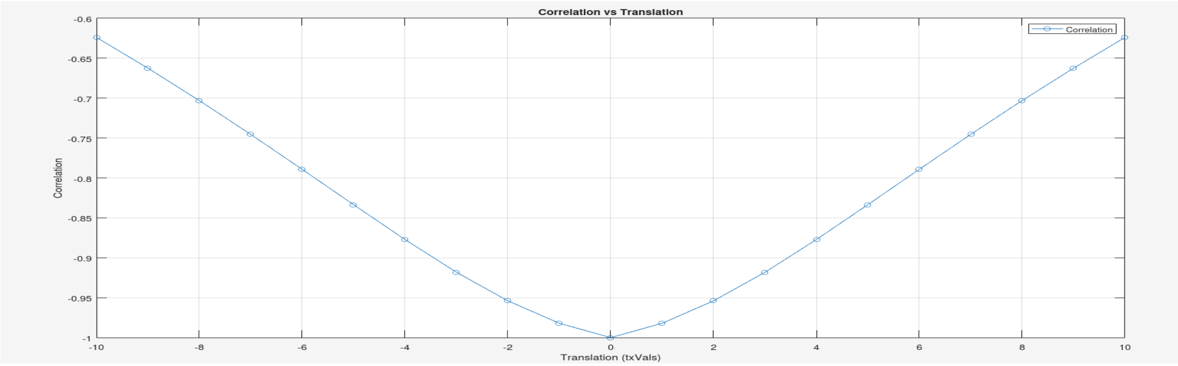


  – MI is maximized at left shift by 1px, confirming that alignment preserves the most shared information.

  – Like QMI, MI does not indicate the direction of translation, only the degree of alignment.

Now $image2 = 255 - image1$

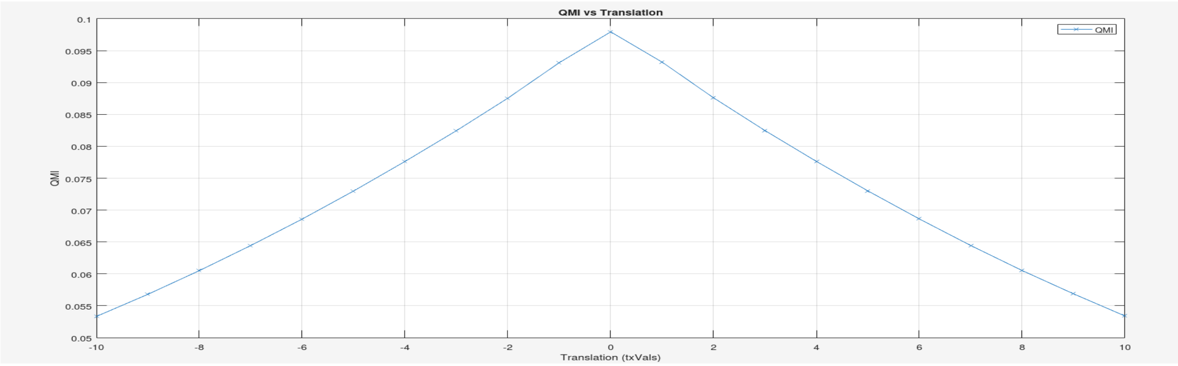
- **Plot 4: Correlation vs Translation**



  – Correlation captures only linear similarity, so inversion flips its sign and results in strong negative correlation.

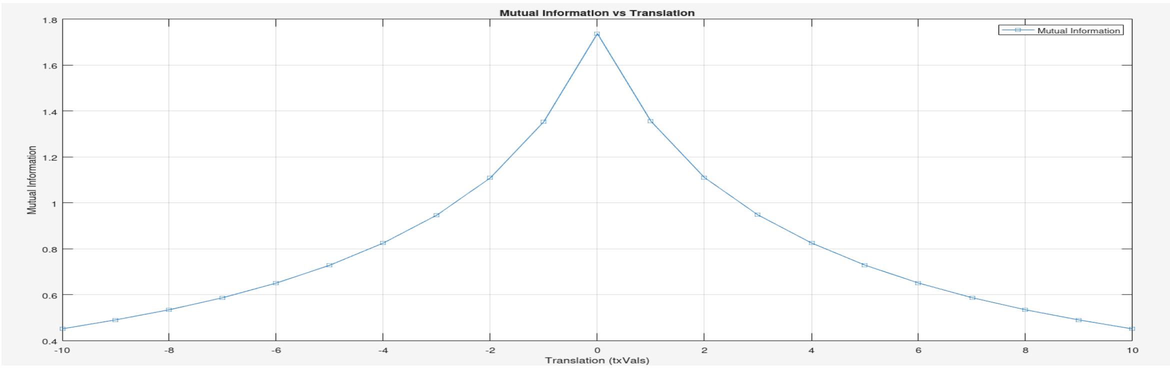
  – The strongest negative correlation occurs at zero shift, indicating perfect inverse alignment. As translation increases, the correlation moves closer to zero, meaning the dependency weakens with misalignment.

- **Plot 5: QMI vs Translation**



- QMI attains its maximum value at zero shift, showing that Image 1 and Image 2 share the strongest dependency when aligned(as image 2 is inverse of image 1, they share maximum similarity(ignoring the sign)).
- The decreases symmetrically with translation, due to misalignment. also it does not indicate direction of translation.

- **Plot 6: Mutual Information vs Translation**

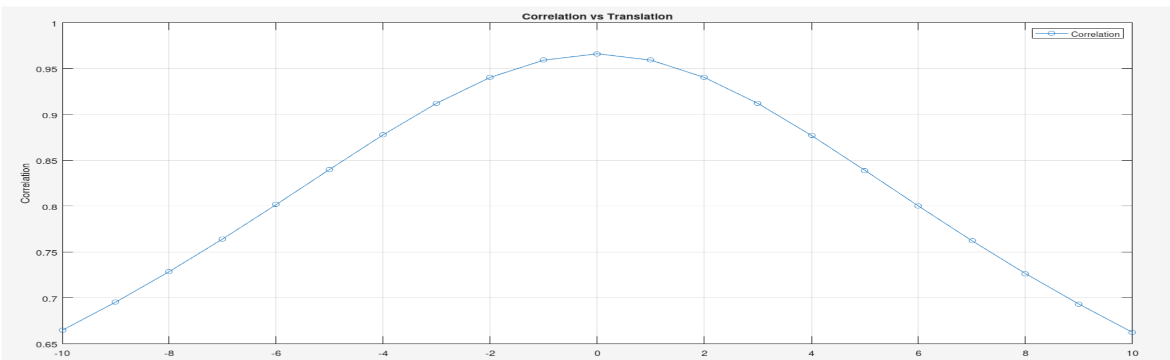


- Mutual Information is highest at zero translation, reflecting the maximum shared information between the original and inverted image.
- Like QMI, MI decreases steadily as translation increases, and it does not indicate the direction of translation.

Here, Image 2 is defined as:

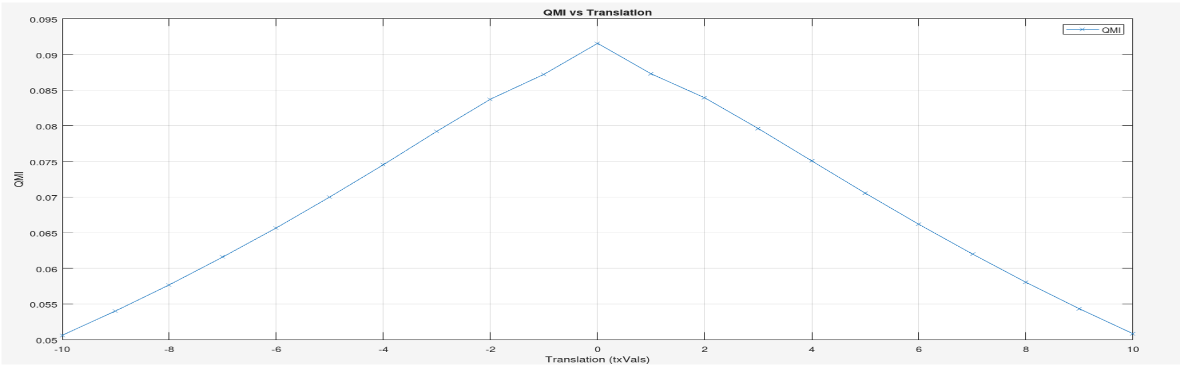
$$I_2 = \frac{255 \cdot (I_1^2)}{\max(I_1^2)} + 1$$

- **Plot 7: Correlation vs Translation**



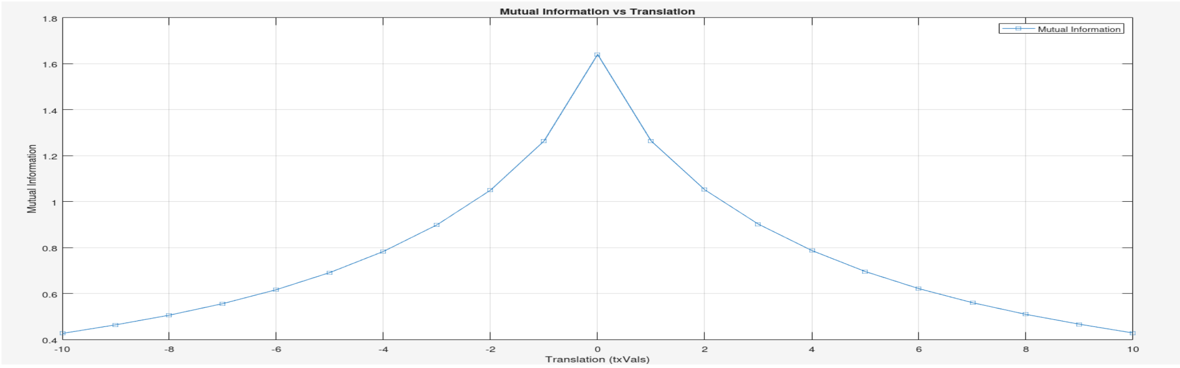
- The correlation values are very high (close to 1), since Image 2 is a monotonic quadratic transformation of Image 1.
- Maximum correlation occurs at zero shift, showing strongest alignment.
- As translation increases, correlation decreases but remains positive and relatively strong.

- **Plot 8: QMI vs Translation**



- QMI achieves its maximum at zero shift, reflecting high dependency between Image 1 and its quadratic transform.
- The measure decreases symmetrically with translation, confirming sensitivity to misalignment.

- **Plot 9: Mutual Information vs Translation**



- MI peaks at zero translation, confirming that maximum shared information is preserved when the images are aligned.
- MI decreases significantly as the translation grows, showing the misalignment decreasing the shared information.

Observations:

- **Correlation:** Correlation is a linear similarity measure. It achieves its maximum at zero translation when the images are identical. In the case of inversion ($I_2 = 255 - I_1$), correlation flips sign, giving strong negative correlation at zero shift. For monotonic transformations (eg: quadratic), correlation remains high and positive. As misalignment increases, the correlation value decreases.
- **Quadratic Mutual Information (QMI):** QMI consistently achieves its maximum at zero translation, regardless of how Image 2 is derived from image 1 (original, inverted, or transformed). It decreases symmetrically as translation increases, but does not indicate the *direction* of the shift. Unlike correlation, QMI still captures strong similarity under inversion and non-linear transformations.
- **Mutual Information (MI):** MI is also maximum at zero translation, confirming that alignment preserves the most shared information. Like QMI, it is symmetric with respect to translation and symmetric with shift direction. MI decreases significantly as translation grows, showing high sensitivity to misalignment. It captures both linear and non-linear dependencies, making it more general than correlation.

Chapter 5

Question 5

To prove: $P(X \geq x) \leq e^{-tx} \phi_X(t)$ for $t>0$

$P(X \geq x) = P(e^{tX} \geq e^{tx})$ for $t>0$;

because if $e^{tX} \geq e^{tx}$ as e^x is monotonically ~~exponent~~ increasing function,

$tX \geq tx$

$X \geq x$ as $t>0$ hence we can do $e^{tX} \geq e^{tx}$ instead of $X \geq x$

$P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}}$ [by markov's inequality, $P(Y \geq a) \leq \frac{E(Y)}{a}$]

$P(X \geq x) = P(e^{tX} \geq e^{tx}) \leq e^{-tx} \phi_X(t)$ [as $\phi_X(t) = E(e^{tX})$]

$\Rightarrow \boxed{P(X \geq x) \leq e^{-tx} \phi_X(t)}$, for $t>0$

To prove: $P(X \leq x) \leq e^{-tx} \phi_X(t)$ for $t<0$

$P(X \leq x) = P(e^{tX} \geq e^{tx})$ for $t<0$

as $X \leq x$

$tX \geq tx$ as $t<0$

$e^{tX} \geq e^{tx}$ as e^x is monotonically increasing function.

$P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}}$ [by markov's inequality, $P(Y \geq a) \leq \frac{E(Y)}{a}$]

$\Rightarrow P(X \leq x) = P(e^{tX} \geq e^{tx}) \leq e^{-tx} \phi_X(t)$

$\Rightarrow \boxed{P(X \leq x) \leq e^{-tx} \phi_X(t)}$, for $t<0$

Given $X = X_1 + X_2 + \dots + X_n$, where $X_1, X_2, \dots X_n$ are independent random variables.

$E(X_i) = P_i$, $\sum_{i=1}^{n} P_i = \mu$

$\Rightarrow P(X > (1+\delta)\mu) \leq P(X \geq (1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \cdot \phi_X(t)$ for any $t > 0$, $\delta > 0$

$\Rightarrow P(X > (1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \cdot \phi_X(t)$ — ①

$\phi_X(t) = E(e^{tX}) = E(e^{t(X_1 + X_2 + \dots + X_n)}) = E(e^{tX_1} \cdot e^{tX_2} \dots e^{tX_n})$

$= E(e^{tX_1}) \cdot E(e^{tX_2}) \dots E(e^{tX_n})$ [as $X_1, X_2, \dots X_n$ are independent random variables]

Given, $X_1, X_2, \dots X_n$ are Bernouli Random variables

$E(X_i) = P_i$

$E(e^{tX_i}) = e^{t(1)} P_i + e^{t(0)} (1 - P_i)$

$= P_i (e^t - 1) + 1$

$\leq e^{P_i(e^t - 1)}$ [Since, $1 + x \leq e^x$]

$\Rightarrow \phi_X(t) \leq e^{P_1(e^t - 1)} \cdot e^{P_2(e^t - 1)} \cdot \dots \cdot e^{P_n(e^t - 1)}$

$\leq e^{(P_1 + P_2 + \dots + P_n)(e^t - 1)}$

$\phi_X(t) \leq e^{\mu(e^t - 1)}$

Substituting in ①,

$P(X > (1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \cdot \phi_X(t) \leq e^{-t(1+\delta)\mu} \cdot e^{\mu(e^t - 1)}$

$\therefore \boxed{P(X > (1+\delta)\mu) \leq \dfrac{e^{\mu(e^t - 1)}}{e^{t(1+\delta)\mu}}}$, for any $t \geq 0$, $\delta > 0$

$$\therefore \; P(x > (1+\delta)\mu) \le \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}} \quad - (2)$$

To further tighten the bound, we need to find the minimum value of RHS, when $t>0$,

let, $f(t) = \dfrac{e^{\mu(e^t-1)}}{e^{(1+\delta)\mu t}} = e^{\mu(e^t-1)-(1+\delta)\mu t}$

when $f(t)$ is minimum, $f'(t) = 0$

$\Rightarrow e^{\mu(e^t-1)-(1+\delta)\mu t} \cdot [\mu e^t - (1+\delta)\mu] = 0$

$\Rightarrow \mu e^t = (1+\delta)\mu \quad \Rightarrow e^t = (1+\delta)$

let's consider $f''(t)$ at $e^t = (1+\delta)$

$f''(t) = \underbrace{\left(e^{\mu(e^t-1)-(1+\delta)\mu t}\right)}_{\text{always positive because } e^x \text{ is always positive}} \cdot \left[(\mu e^t - (1+\delta)\mu)^2 + (\mu e^t)\right]$

$\Rightarrow \left((\mu(1+\delta) - (1+\delta)\mu)^2 + \mu(1+\delta)\right) e^{\mu(e^t-1)-(1+\delta)\mu t}$

$\Rightarrow (\mu(1+\delta))\, e^{\mu(e^t-1)-(1+\delta)\mu t} > 0$ as $\mu > 0$, (as $P_i > 0$) $1+\delta > 0$

So $f''(t)$ when $e^t = (1+\delta) > 0$

$\therefore e^t = 1+\delta$ is the minimum,

Substituting in inequality (2),

$$\boxed{P(x > (1+\delta)\mu) \le \frac{e^{\mu(1+\delta-1)}}{(1+\delta)^{\mu(1+\delta)}}}$$

$$\therefore \boxed{P(x > (1+\delta)\mu) \le \frac{e^{\mu\delta}}{(1+\delta)^{\mu(1+\delta)}}}$$

Chapter 6

Question 6

Given n independent coin tosses,

heads probability $= p$

tails probability $= 1-p$

let e_i be the event in which the first head appears on ith trail. Random variable $T = i$, if e_i occurs.

$E(T) = \sum_{i=1}^{n} i\, P(e_i)$ by the definition of expectation of random variable.

$E(T) = \sum t\, P(T=t)$
$= \sum i\, P(e_i)$

Probability of $e_i \Rightarrow$

first head appears on ith trail, the before $(i-1)$ trails have tails

$\Rightarrow P(e_i) = \underbrace{(1-p)(1-p)\cdots(1-p)}_{(i-1)\text{ times}}(p)$ as $P(\text{tail}) = 1-p$

$= (1-p)^{i-1}p$

$E(T) = \sum_{i=1}^{n} i(1-p)^{i-1}p$

let $g = 1 + 2(1-p) + 3(1-p)^2 + \cdots + n(1-p)^{n-1}$

$(1-p)g = (1-p) + 2(1-p)^2 + \cdots + (n-1)(1-p)^{n-1} + n(1-p)^n$

$g - (1-p)g = 1 + (1-p) + (1-p)^2 + \cdots + (1-p)^{n-1} - n(1-p)^n$

$\cancel{g} - \cancel{g} + pg = \dfrac{1-(1-p)^n}{1-(1-p)} - n(1-p)^n$ [by using gp formula, $1 + x + x^2 + \cdots x^{n-1} = \dfrac{1-x^n}{1-x}$, if $x<1$]

$pg = \dfrac{1-(1-p)^n}{p} - n(1-p)^n$ — ①

$E(T) = p\left(\sum_{i=1}^{n} i(1-p)^{i-1}\right)$

$= p\left(1 + 2(1-p) + 3(1-p)^2 + \cdots (n)(1-p)^{n-1}\right)$

$= p\, g$ (as $g = 1 + 2(1-p) + 3(1-p)^2 + \cdots + n(1-p)^{n-1}$)

$\boxed{E(T) = \dfrac{1-(1-p)^n}{p} - n(1-p)^n}$ from ①