1. The shown 2-R manipulator in Figure 1 consists of mass-less rods of length L1 and L2 and masses m1 and m2.

(a) Find the Kinetic and Potential Energies of m1, m2 and the entire manipulato r.

a, Find the kinetic and potential energies of m_1 and m_2 and the entire manipulator
To find K.E and P.E we need to know their general formulas.

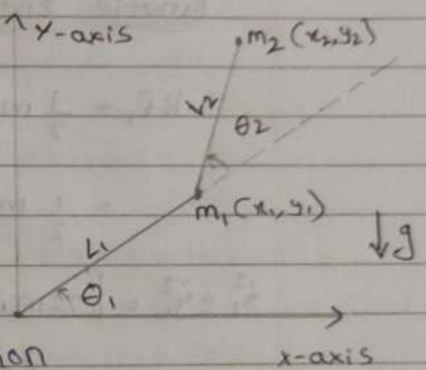
$$\boxed{K.E = \frac{1}{2} m v^2} \qquad \boxed{P.E = mgh.}$$

m = mass of object
v = velocity of object.

m = mass of object
g = gravity acting on object
h = height of object

Finding Kinematic Energy of m_1

Inorder to find kinetic Energy of m_1 we need the v value.
To determine v value of m_1 let assume velocity as v_1 for v_1



To find velocity we use position
By derivating position coordinates (x_1, y_1) we will have ~~velocity~~

$$v_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 \\ L_1 \cos\theta_1 \end{bmatrix} \dot{\theta}_1 \rightarrow \text{velocity of } m_1$$

we need to calculate Position of ~~m~~ m_1 given by x_1 & y_1.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 \\ L_1 \sin\theta_1 \end{bmatrix} \rightarrow \text{Position of } m_1$$

Similarly finding kinematic Energy of m_2.

Similarly we will find position of m_2 and it's velocity of m_2.

let it's position coordinates are x_2, y_2
and its velocity is $\dot{x}_2\ \dot{y}_2$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) \\ L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) \end{bmatrix} \rightarrow \text{position of } m_2$$

velocity of m_2

$$v_2 = \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

By these Information we will calculate kinetic energy of m_1

$$K.E_1 = \frac{1}{2} m_1 v_1^2$$

$$= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2)$$

$$\dot{x}_1^2 + \dot{y}_1^2 = [-L_1\sin\theta_1]^2(\dot{\theta}_1)^2 + [L_1\cos\theta_1]^2(\dot{\theta}_1)^2$$

$$\dot{x}_1^2 + \dot{y}_1^2 = (\dot{\theta}_1)^2[+L_1^2\sin^2\theta_1 + L_1^2\cos^2\theta_1]$$

$$\dot{x}_1^2 + \dot{y}_1^2 = (\dot{\theta}_1)^2[(L_1)^2[\sin^2\theta_1 + \cos^2(\theta_1)]]$$

$$\dot{x}_1^2 + \dot{y}_1^2 = (\dot{\theta}_1)^2[(L_1)^2(1)]$$

$$\boxed{\dot{x}_1^2 + \dot{y}_1^2 = (\dot{\theta}_1)^2(L_1^2)}$$

replace $\dot{x}_1^2 + \dot{y}_1^2$ in $K.E_1$

$$\boxed{K.E_1 = \frac{1}{2} m_1 L_1^2 (\dot{\theta}_1)^2} \quad \text{—— (1)}$$

These is Kinetic Energy of m_1

~~to~~ similarly for $K.E_2$

$\sin(C+D) = \sin C \cos D + \cos C \sin D)$

$$v_2 = \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix}$$

$$(v_2)^2 = \dot{x}_2^2 + \dot{y}_2^2$$

$$= \left\{\left[-L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) - L_2\sin(\theta_1+\theta_2)\right](\dot{\theta}_1)\right\}^2$$

$$= \left[-L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) - L_2\sin(\theta_1+\theta_2)\right]^2 (\dot{\theta}_1)^2$$

$$+ L_1^2\sin^2\theta_1 + L_2^2\left[\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1\right]^2 - L_2 [$$

$$\Rightarrow \left\{-L_2\sin(\theta_1+\theta_2)\left[-L_1\sin\theta_1 + 1\right]\right\}^2$$

$$= L_2^2\left[\sin^2(\theta_1+\theta_2)\right]^2\left[L_1^2\sin^2\theta_1 + 1 + 2L_1\sin\theta_1\right]$$

$$L_2^2\left[\sin(\theta_1)\cos\theta_2 + \sin\theta_2\cos\theta_1\right]^2\left[L_1^2\sin^2\theta_1 + 1 + 2L_1\sin\theta_1\right]$$

$$L_2^2\left[\sin^2\theta_1\cos^2\theta_2 + \sin^2\theta_2\cos^2\theta_1 + 2\sin\theta_1\sin\theta_2\cos\theta_1\cos\theta_2\right]$$

$$L_2^2\left[\sin^2\theta_1\cos^2\theta_2\right] + L_2^2\left[\sin^2\theta_2\cos^2\theta_1\right] + 2L_2^2\sin\theta_1\sin\theta_2\cos\theta_1\cos\theta_2\,]$$

I have done this d simplication in rough because it take a lot page & it look very messup here

$$v_2 = \boxed{\dot{x}^2 + \dot{y}^2 = (L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)(\dot{\theta}_1^2) + 2(L_2^2 + L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2 + L_2^2\dot{\theta}_2^2}$$

$$K.E_2 = \frac{1}{2}m_2 v_2^2$$

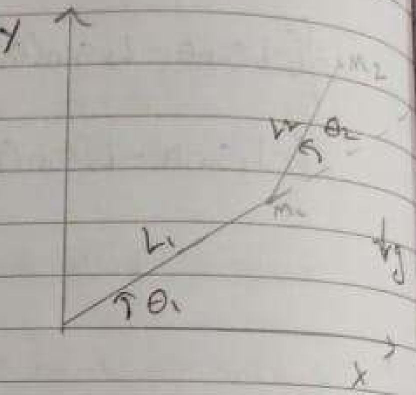
$$= \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$\boxed{K.E_2 \Rightarrow \frac{1}{2}m_2\left[(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\dot{\theta}_1^2 + 2(L_2^2 + L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2 + L_2^2\dot{\theta}_2^2\right]} \quad (2)$$

These expression is K.E of m_2

Finding Potential Energy of m_1

P.E of each mass depend only on its height and or on its y-coordinates.
let $P.E_1$ as Potential Energy of m_1.



$\boxed{P.E_1 = m_1 g h}$

Here h is height of m_1
m_1 has (x_1, y_1) coordinates
y_1 represents height &
x_1 represents ~~weight~~ width. of m_1

So, $P.E_1 = m_1 g(y_1)$

How to find y_1

$\boxed{y_1 = L_1 \sin\theta_1}$ (Length of m_1) × its angle wrt to x-axis.

$\boxed{P.E_1 = m_1 g(L_1 \sin\theta_1)}$ — (3)

This Potential energy of m_1

Similarly finding Potential energy of m_2.

For m_2 also we will have Cooridinate x_2, y_2
Similarly Here also y_2 is height of m_2
let $P.E_2$ is potential energy of m_2

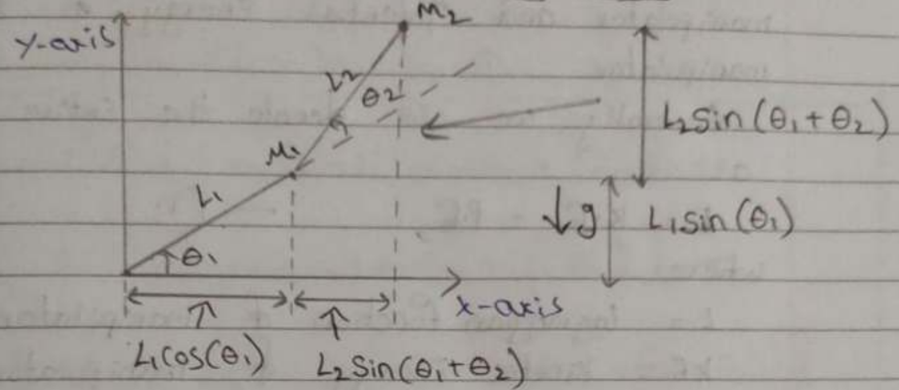
$P.E_2 = m_2 g(y_2)$.

y_2 is point to find height we need to sum y_1 & y_2

$y_2 = L_2 C$

$y_2 = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)$

$$\boxed{P.E_2 = m_2 g\left(L_1 \sin(\theta_1) + L_2 \sin(\theta_1+\theta_2)\right)} \quad -④$$



This is the image that How I got y_2 (height of m_2).

<u>now finding Energy of Entire manipulator.</u>

Using Equation ① & ② & ③ & ④ we will find Entire manipolator Energy.

Total Energy = $K.E_1 + K.E_2 + P.E_1 + P.E_2$.

$$P.E = \frac{1}{2} m_1 L_1^2 (\dot{\theta}_1)^2 + \frac{1}{2} m_2 \left[(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)(\dot{\theta}_1)^2 + 2(L_2^2 + L_1L_2\cos\theta_2)\right.$$

$$\left.\dot{\theta}_1\dot{\theta}_2 + L_2^2\dot{\theta}_2^2\right] + m_1 g(L_1 \sin\theta_1) + m_2 g\left(L_1 \sin(\theta_1) + L_2(\sin(\theta_1+\theta_2))\right)$$

$T.E = K.E + P.E$ (Sum of kinetic Energy of manipolator & P.E of manipulator)

$$T.E = \frac{1}{2}\left[\underbrace{m_1 L_1^2 (\dot{\theta}_1)^2 + m_2\left[(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)(\dot{\theta}_1)^2 + 2(L_2^2 + L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2\right]}_{K.E}\right]$$

$$+\left[\underbrace{m_1 g(L_1 \sin(\theta_1)) + m_2 g\left(L_1 \sin(\theta_1) + L_2(\sin(\theta_1+\theta_2))\right)}_{P.E}\right]$$

(b) Define the Lagrangian function for the manipulator

(C) Using the Lagrangian function derive the Euler lagrange Dynamics for the manipulator.

Using the Lagrangian function, we can derive the Euler - lagrange dynamics for the manipulator by applying the principle of least action.

This principle states that path of system minimizes the action, which is the difference between K.E and P.E. The Euler - lagrange Equation are obtained by setting the first variation of the zero.

These equations describe the relationship between generalized coordinates and forces of the manipulator

Lagrangian function:-

$$L(\theta,\dot{\theta}) = \underbrace{K(\theta,\dot{\theta})}_{\text{kinetic Energy}} - \underbrace{P(\theta)}_{\text{potential Energy}}$$

when we apply partial derivation to above Equation we get it as

$$\boxed{\tau_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i}} \quad \text{where } i=1,2$$

τ = Torque.

and also we apply derivation wrt to time.

θ_i = generalized coordinate.

$\dot{\theta}_i$ = time derivative of generalized coordinate

τ_i = generalized force.

derivate of lagrangian function twice

(1) onces with respect to generalized velocity

& once with respect to generalized coordinate.

2, Then we need to substract the second derivative from the first, & set it equal to generalized force. Which gives torque for coordinate.

Entire potential energy of manipulator given by

$$P.E = m_1 g L_1 \sin\theta_1 + m_2 g (L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2))$$

(from previous question (a)

replace K.E & P.E in Eq (1)

$$\boxed{\begin{aligned} L = \Big[& \frac{1}{2} m_1 L_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 (L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2)\,\theta^2 + \\ & 2(L_1^2 + L_1L_2\cos\theta_2)\,\dot\theta_1 \dot\theta_2 + L_2^2 \dot\theta_2^2 \}\Big] - \\ & \left[m_1 g L_1 \sin\theta_1 + m_2 g (L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2)) \right] \end{aligned}}$$

This is lagrangian function

$$L(\theta,\dot\theta) = \sum_{i=1}^{2} (k_i - P_i)$$

k_i = kinetic Energy of i^{th} link.

P_i = Potential Energy of i^{th} link.

$L(\theta,\dot\theta)$ = lagrangian function

(or)

$$L(\theta,\dot\theta) = K(\theta,\dot\theta) - P(\theta).$$

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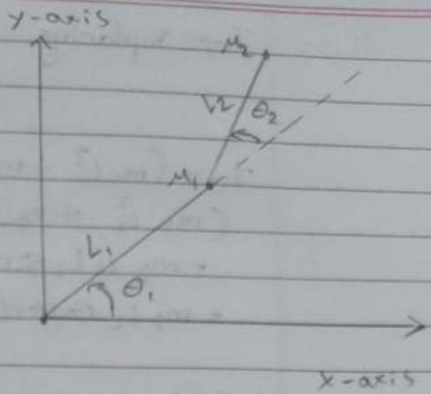
Finding joint 1 torque τ_1 at m_1



$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

Now, finding $\frac{\partial L}{\partial \dot{\theta}_1}$ & $\frac{\partial L}{\partial \dot{\theta}_1}$ wrt L.

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{\partial}{\partial \dot{\theta}_1}\Big[\frac{1}{2}(m_1L_1^2 + m_2L_1^2 + m_2L_2^2 + 2m_2L_1L_2\cos\theta_2)\dot{\theta}_1^2 + \frac{1}{2}(m_2L_2^2\dot{\theta}_2^2) + m_2L_2^2 + m_2L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2$$
$$-g(m_1L_1 + m_2L_1)\sin\theta_1 - g(m_2L_2 \underbrace{\cancel{\sin\theta_1\sin\theta_2}}_{\sin(\theta_1+\theta_2)})\Big]$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{\cancel{2}}{\cancel{2}}(m_1L_1^2 + m_2L_1^2 + m_2L_2^2 + 2m_2L_1L_2\cos\theta_2)\dot{\theta}_1 + (m_2L_2^2 + m_2L_1L_2\cos\theta_2)\dot{\theta}_2(1)$$

$$\boxed{\frac{\partial L}{\partial \dot{\theta}_1} = (m_1L_1^2 + m_2L_1^2 + m_2L_2^2 + 2m_2L_1L_2\cos\theta_2)\dot{\theta}_1 + (m_2L_2^2 + m_2L_1L_2\cos\theta_2)\dot{\theta}_2}$$

$$\boxed{\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = {} & (m_1L_1^2 + m_2L_1^2 + m_2L_2^2 + 2m_2L_1L_2\cos\theta_2)\ddot{\theta}_1 \\ & + (m_2L_2^2 + m_2L_1L_2\cos\theta_2)\ddot{\theta}_2 \\ & - 2m_2L_1L_2\sin(\theta_2)\dot{\theta}_1\dot{\theta}_2 - m_2L_1L_2\sin\theta_2\dot{\theta}_2^2\end{aligned}}$$

Now finding $\frac{\partial L}{\partial \theta_1}$

$$\frac{\partial L}{\partial \theta_1} = \frac{\partial}{\partial \theta_1}\Big[\frac{1}{2}(m_1L_1^2 + m_2L_1^2 + m_2L_2^2 + 2m_2L_1L_2\cos\theta_2)\dot{\theta}_1^2 + \frac{1}{2}(m_2L_2^2\dot{\theta}_2^2) + m_2L_2^2 + m_2L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2$$
$$-g(m_1L_1 + m_2L_1)\sin\theta_1 - g(m_2L_2 \underbrace{\sin\theta_1\sin\theta_2}_{\sin(\theta_1+\theta_2)})\Big]$$

$$\boxed{\frac{\partial L}{\partial \theta_1} = -g(m_1L_1 + m_2L_1)\cos\theta_1 - g(m_2L_2 \underbrace{\cos\theta_1\cos\theta_2}_{\cos(\theta_1+\theta_2)})}$$

$\tau_1 \rightarrow$ replacing $\frac{\partial L}{\partial \dot{\theta}_1}$ & $\frac{\partial L}{\partial \theta_1}$ in τ_1

$$\tau_1 = (m_1 L_1^2 + m_2 L_1^2 + m_2 L_2^2 + 2m_2 L_1 L_2 \cos\theta_2)\ddot{\theta}_1 +$$
$$(m_2 L_2^2 + m_2 L_1 L_2 \cos\theta_2)\ddot{\theta}_2 - 2m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2$$
$$- m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_2^2 + g(m_1 L_1 + m_2 L_1)\cos\theta_1$$
$$+ m_2 L_2 \cancel{\cos\theta_1 \cos\theta_2} \cdot \cos(\theta_1 + \theta_2)$$

similarlly we will find τ_2 for joint of m_2.

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{\partial}{\partial \dot{\theta}_2}\left[\frac{1}{2}(m_1 L_1^2 + m_2 L_1^2 + m_2 L_2^2 + 2m_2 L_1 L_2 \cos\theta_2)\dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 \right.$$
$$+ (m_2 L_2^2 + m_2 L_1 L_2 \cos\theta_2)\dot{\theta}_1 \dot{\theta}_2 - g(m_1 L_1 + m_2 L_1)\sin\theta_1$$
$$\left. - g m_2 L_2 \sin(\theta_1 + \theta_2)\right]$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{\cancel{2}}{\cancel{2}} m_2 L_2^2 \dot{\theta} + (m_2 L_1 L_2 \cos\theta_2)\dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{d}{dt}\left(m_2 L_2^2 + m_2 L_1 L_2 \cos(\theta_2)\dot{\theta}_1\right) - m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\Rightarrow m_2 L_2^2 + m_2 L_1 L_2 \cos(\theta_2) \cdot \ddot{\theta}_1 - m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\boxed{\frac{\partial L}{\partial \dot{\theta}_2} = m_2 L_2^2 \dot{\theta}_2 + (m_2 L_2^2 + m_2 L_1 L_2 C_2)\dot{\theta}_1}$$

$$\boxed{\frac{\partial L}{\partial \theta_2} = -m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_1^2 - m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 - g m_2 L_2 \cos(\theta_1+\theta_2)}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 L_2^2 \ddot{\theta}_2 + (m_2 L_2^2 + m_2 L_1 L_2 \cos\theta_2)\ddot{\theta}_1 - m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2}$$

$$\tau_2 = (m_2 L_2^2 + m_2 L_1 L_2 \cos\theta_2)\ddot{\theta}_1 + m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_1^2 + g m_2 L_2 \cos(\theta_1 + \theta_2)$$

$$\therefore \cancel{\tau} = (m_2(\theta), \ddot{\theta}) + c(\theta, \dot{\theta}) + g(\dot{\theta})$$

$$\therefore \tau = m(\theta)(\dot{\theta}) + c(\theta, \dot{\theta}) + g(\dot{\theta})$$

Both τ_1 & τ_2 represent or follow's τ
So By using lagrangian function we derive the Euler lagrange Dynamics for manipulator
as τ_1 & τ_2 { τ as

$$\tau = m(\theta)(\dot{\theta}) + c(\theta, \dot{\theta}) + g(\dot{\theta})$$

$$\tau_1 = (m_1 L_1^2 + m_2 L_1^2) + m_2 L_2^2 + 2 m_2 L_1 L_2 (\cos\theta_2)\ddot{\theta}_1 + (m_2 L_2^2 + m_2 L_1 L_2 (\cos\theta_2))\ddot{\theta}_2 - 2 m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_2^2 + g(m_1 L_1 + m_2 L_1)\cos\theta_1 + m_2 L_2 \cos(\theta_1 + \theta_2)$$

$$\tau_2 = (m_2 L_2^2 + m_2 L_1 L_2 (\cos\theta_2))\ddot{\theta}_2 + m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \sin\theta_2 \dot{\theta}_1^2 + g m_2 L_2 \cos(\theta_1 + \theta_2)$$

(d) Clearly show the Mass matrix(M) ,the combined

Coriolis and Centripetal terms matrix (C), the gravity term matrix(G) for the standard Euler-Lagrange model for the manipulator.

$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + g(\theta) = \tau$

where $\theta = [\theta_1^T \ \theta_2^T]^T$

d. Clearly show the mass matrix (m), the Combined coriolis & Centripetal terms matrix (c), the gravity term matrix (G) for the standard Euler - lagrange model for the manipulator.

$$m(\theta)(\dot{\theta}) + C(\theta, \dot{\theta}) + g(\theta) = \tau$$

$$\text{where } \theta = [\theta_1^T \ \theta_2^T]^T$$

The Euler- lagrange model of a 2R manipulator is derived from the lagrangian function which is the difference between of Kinetic & potential energies of system.

$$K.E = \frac{1}{2} \dot{\theta}^T m(\theta)\dot{\theta}$$

$$P.E = m_1 L_1 \sin\theta_1 - m_2 (L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2))$$

where $m(\theta)$ is the mass matrix which depends on the link length, masses & moments of inertia.

Lagrangian Function

$$L = K.E - P.E.$$

we got Euler -lagrange Equation of motion as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \tau$$

where τ is vector of generalized Force applied to System

Expanding the term in Equation we get.

$$m(\theta)\dot{\theta} + C(\theta, \dot{\theta}) + g(\dot{\theta}) = \tau$$

where $C(\theta, \dot{\theta})$ is matrix of coriolis and centripetal terms

$g(\theta)$ is vector of gravity term

The Explict Expression for these terms can be found by using formulas

$$m_{ij}(\theta) = \frac{\partial^2 T}{\partial \theta_i \partial \theta_j}$$

$$C_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \left(\frac{\partial m_{ij}}{\partial \theta_k} + \frac{\partial m_{ik}}{\partial \theta_j} - \frac{\partial m_{kj}}{\partial \theta_i} \right) \dot{\theta}_k.$$

$$g_i(\theta) = \frac{\partial v}{\partial \theta_i}$$

where

n = number of degree of freedom.

i, j, k are indicts raging from 1 to n.

Now, for 2R mainpulator (given diagram) we have it's n value as 2. then it's

~~matrix~~

mass matrixs is

$$\boxed{m(\theta) = \begin{bmatrix} m_1 L_1^2 + m_2 (L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2) + & m_2(L_1L_2\cos\theta_2 + L_2^2) \\ m_2(L_1L_2\cos\theta_2 + L_2^2) & m_2 L^2 \end{bmatrix}}$$

coriolis & centripetal matrix:-

$$\boxed{C(\theta, \dot{\theta}) = \begin{bmatrix} -m_2 L_1 L_2 \sin\theta_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ m_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix}}$$

velocity production term

gravity ~~velo~~ vector:-

$$g(\theta) = \begin{bmatrix} -(m_1+m_2) g L_1 \sin\theta_1 & -m_2 g L_2 \sin(\theta_1+\theta_2) \\ -m_2 g L_2 \sin(\theta_1+\theta_2) \end{bmatrix}$$

$m(\theta)(\ddot{\theta}) + c(\theta, \dot{\theta})$ can be derived by
$f_i = m_i a_i$ for each point mass.

θ is expressed by (x_1, y_1) & (x_2, y_2)

$$f_1 = \begin{bmatrix} f_{x_1} \\ f_{y_1} \\ f_{z_1} \end{bmatrix} = m_2 \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{z}_1 \end{bmatrix}$$

Here f_1 is force acting on m_1

$$= m_1 \begin{bmatrix} -L_1 \dot{\theta}_1^2 \cos\theta_1 & -L_1 \ddot{\theta}_1 \sin\theta_1 \\ -L_1 \dot{\theta}_1^2 \sin\theta_1 & L_1 \ddot{\theta}_1 \cos\theta_1 \\ \theta \end{bmatrix}$$

Here z is 0 because manipulator is 2DoF
Similarly for f_2

$$f_2 = m_2 \begin{bmatrix} -L_1 \dot{\theta}_1 \cos\theta_1 - L_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \cos(\theta_1+\theta_2) - L_1 \ddot{\theta}_1 \sin\theta_1 - L_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \sin(\theta_1+\theta_2) \\ -L_1 \dot{\theta}_1^2 \sin\theta_1 - L_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \sin(\theta_1+\theta_2) + L_1 \ddot{\theta}_1 \cos\theta_1 + L_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \cos(\theta_1+\theta_2) \\ 0 \end{bmatrix}$$

* Quadratic term containing $\dot{\theta}_i$, $\dot{\theta}_j$ $i \neq j$ are called Coriolis term.
* Quadratic term containing $\dot{\theta}_1^2$ called Centripetal terms.

To know combined Coriolis & Centripetal term matrix (C) gravity term matrix (G)

Let's consider an example.
consider arm at configuration $(\theta_1, \theta_2) = 0$.
$(0, \pi/2)$

i.e $\cos\theta_1 = \sin(\theta_1 + \theta_2) = 1$

$\sin\theta_1 = \cos(\theta_1 + \theta_2) = 0$.

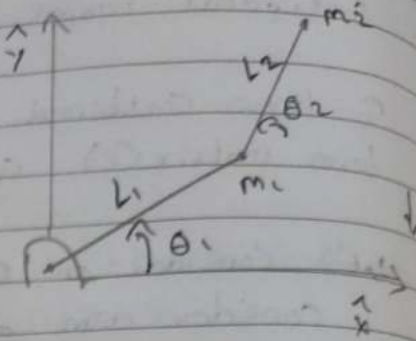
Assuming $\ddot{\theta} = 0$ the acceleration $(\ddot{x}_2, \ddot{y}_2)$ of m_2 equation can be written as.

$$\begin{bmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -L_1 \dot{\theta}_1^2 \\ -L_2 \dot{\theta}_1^2 - L_2 \dot{\theta}_2^2 \end{bmatrix}}_{\text{Centripetal terms}} + \underbrace{\begin{bmatrix} 0 \\ -2L_2 \dot{\theta}_1 \dot{\theta}_2 \end{bmatrix}}_{\text{Coriolis term.}}$$

Properties:-
$m(\theta)$ is mass matrix.
* $m(\theta)$ is positive definite.
  ($x^T m(\theta)\, x > 0$ for all $x \neq 0$)
* $m(\theta)$ is symmetric ($m = M^T$)
* $m(\theta)$ depends on θ

(e) Assuming 1.1 as the compact formulation for the Euler-Lagrange model you have derived, derive the control input τ desired .

c) $m(\theta)\ddot{\theta} + C(\theta,\dot{\theta}) + g(\dot{\theta}) = \tau$ as the compact formulation for Euler - lagrange model you have derived, derive the control input τ desired.



$m(\theta)$ = mass matrix

$\ddot{\theta}$ = second derivative of joint angle wrt to time.

$C(\theta,\dot{\theta})$: Coriolis & Centrifugal effect due to rotation

$g(\theta)$ = gravitational force.

For given 2R manipulator Equation will be

$$m(\theta)\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + C(\theta,\dot{\theta}) + g(\theta) = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix}$$

So, we need to solve τ_1 & τ_2 to control input τ desired.

<u>mass matrix</u> $m(\theta)$

$$m(\theta) = \begin{bmatrix} m_1 L_1^2 + m_2(L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2 & m_2(L_1^2 + L_1L_2\cos(\theta_2)) \\ m_2(L_2^2 + L_1L_2\cos\theta_2) & m_2 L_2^2 \end{bmatrix}$$

For simplicity Here I am considering $C(\theta,\dot{\theta})$ as 0

Coriolis & Centrifugal effect $C(\theta,\dot{\theta})$:

This say's derivatives of θ & $\dot{\theta}$ for now no velocity dependent effects

$$K(\theta) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}$$

Date
Page

$C(\theta, \dot{\theta}) = 0$ (because we don't know θ & $\dot{\theta}$ value of 2R manipulator

Gravitational force $g(\theta)$:-

$$g(\theta) = \begin{bmatrix} m_1 L_1 + m_2(L_1+L_2) g \sin(\theta_1) - m_2 L_2 g \sin(\theta_1+\theta_2) \\ m_2 L_2 g \sin(\theta_1+\theta_2) \end{bmatrix}$$

Then τ_1 & τ_2 will be.

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \underbrace{\begin{bmatrix} m_1 L_1^2 + m_2(L_1^2+L_2^2+2L_1L_2\cos(\theta_2)) & m_2(L_2^2+L_1L_2\cos(\theta_2)) \\ m_2(L_2^2 + L_1L_2\cos(\theta_2)) & m_2 L_2^2 \end{bmatrix}}_{M(\theta)} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{(\ddot{\theta})}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{C(\theta,\dot{\theta})} + \underbrace{\begin{bmatrix} (m_1 L_1 + m_2(L_1+L_2))\, g \sin(\theta_1) - m_2 L_2 \cdot g \cdot \sin(\theta_1+\theta_2) \\ m_2 \cdot L_2 \cdot g \cdot \sin(\theta_1+\theta_2) \end{bmatrix}}_{g(\dot{\theta})}$$

(or)

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1 L_1^2 + m_2(L_1^2+L_2^2+2L_1L_2\cos\theta_2) & m_2(L_2^2+L_1L_2\cos\theta_2) \\ m_2(L_2^2 + L_1L_2\cos\theta_2) & m_2 L_2^2 \end{bmatrix} +$$

$$\begin{bmatrix} -m_2 L_1 L_2 \sin\theta_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ m_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} +$$

$$\begin{bmatrix} (m_1 L_1 + m_2(L_1+L_2))g \sin(\theta_1) - m_2 L_2 g \sin(\theta_1+\theta_2) \\ m_2 L_2 g \sin(\theta_1+\theta_2) \end{bmatrix}$$

τ_1, τ_2 is τ desired Here control input are $M(\theta)$, $\ddot{\theta}$, $C(\theta, \dot{\theta})$ $g(\dot{\theta})$ Based on this we will get τ desired

Here, 1 and 2 is τ desired.

In order to derive control input, I have assumed that
$C(\theta, \dot{\theta})$ as 0 for easy purpose If we have an extracted value,
we can substitute all the values and derive the desired
And also here, control inputs are
$M(\theta)\ \theta$
$C(\theta, \dot{\theta})$
$g(\theta)$
Because based on this control input, we will derive the τ
desired

I TRIED MY LEVEL BEST TO GIVE THIS ANSWER.
THANK YOU
NOTE : IN MY HAND WRITTEN IMAGE I HAVE
WRITTEN I1 I2 BUT THEIR ARE L1 AND L2 SOMEWHERE PLEASE
NEGLECT IT

LINK OF THIS DOC

https://docs.google.com/document/d/1IyIDJAB3xzfyocg9A5W6_WD0GtvPdKDQ6u4G2I8zdnQ/edit?usp=sharing

THE ABOVE LINK IS MY ROUGH ASSIGNMENT LINK AND IT IS MY LINK TO WHERE I HAVE ATTEMPTED THIS ASSIGNMENT AT FIRST