Cahn-Hilliard Solver for One Dimension

What solver can do:

1. Compute numerical solution of Cahn-Hilliard equation for any initial condition and display as a animation/movie. The solution is constructed/approximated using spectral methods .
2. Compute numerical solution using ODE obtained in [1] for fast transitions and display as animation.
3. Explore metastable states (use switching scheme). It used both PDE and ODE with continuous adaptation of time, and display as animation.

What it can not do:

1. Numerical blow up if the initial condition is not monotone stationary solution (for example $h = [0.2,\, 0.8]$ $h = [0.3,\, 0.7]$.
2. Switching scheme cannot take more than 4 transition layers (But easily extendable to $N-$ transition layers).

Cahn-Hilliard equation is a fourth order highly non-linear partial differential equation originally proposed to describe phase separation in a 2 component system, with $u = u(x,t)$ representing concentration of one of the two components with conservation of mass in the domain $\Omega = [0,1]$ assumed sufficiently smooth and bounded. Often the evolution of these systems is so slow that the pattern formation or evolution dynamics becomes exponentially small in time and these states are called "Meta-Stable states". It is only after a exponentially long period of time that the system converges into a stable system.In this solver, we give the switching scheme which can explore metastable states. We consider the one dimensional Cahn-Hilliard Equation with constant mobility

$$u_t = (-\epsilon^2 u_{xx} + W'(u))_{xx} \text{ for } \quad 0 \le x \le 1,\ t \ge 0$$

$$u_x = u_{xxx} = 0 \text{ at } x = 0, 1$$

where ϵ is an interaction distance and $W(u)$ is a double well bi-stable potential function with equal minima having stable equilibrium at -1 and +1. All the $W(u)$ is taken to be $W(u) = \frac{1}{4}(u^2 - 1)^2$ in the numerics. It is very important to note that Cahn-Hilliard Equation conserve mass.

$$\int_0^1 u(x,t).dx = m \text{ where } -1 < m < 1 \quad \text{[Mass conservation equation].}$$

The Solver contains solution of Cahn-Hilliard Equation using three different methods

1. **System of ODE** proposed in [1] as:

$$
\begin{aligned}
\dot{h}_1 &= \frac{1}{4l_2}\left(\alpha^3 - \alpha^1\right) + O(\epsilon\alpha) \\
\dot{h}_2 &= \frac{1}{4l_2}\left(\alpha^3 - \alpha^1\right) + \frac{1}{4l_3}\left(\alpha^4 - \alpha^2\right) + O(\epsilon\alpha) \\
\dot{h}_3 &= \frac{1}{4l_3}\left(\alpha^4 - \alpha^2\right) + \frac{1}{4l_4}\left(\alpha^5 - \alpha^3\right) + O(\epsilon\alpha) \\
&\ \ \vdots \\
\dot{h}_N &= \frac{1}{4l_N}\left(\alpha^{N+1} - \alpha^{N-1}\right) + \frac{1}{4l_{N+1}}\left(\alpha^{N+2} - \alpha^{N}\right) + O(\epsilon\alpha) \\
\dot{h}_{N+1} &= \frac{1}{4l_{N+1}}\left(\alpha^{N+2} - \alpha^{N}\right) + O(\epsilon\alpha),
\end{aligned}
$$

where h_i is the position of layers (Cahn-Hilliard Equation forms layered structure for most inputs). And α is as defined as;

$$\alpha^j_{\pm}(r) = \frac{1}{2}k^2_{\pm}A^2_{\pm}exp\left(-\frac{A_{\pm}l_i}{\epsilon}\right)\left[1 + O\left(r^{-1}exp\left(-\frac{A_{\pm}}{2\epsilon}\right)\right)\right],$$

where $A_{\pm} = W''(\pm 1) > 0$ (Refer [1] for more details), and

$$k_{\pm} = 2exp\left[\int_0^1 \frac{A_{\pm}}{2\sqrt{W(\pm s)}} - \frac{1}{1-s}ds\right].$$

This is a dynamical system (ODE) which describes the movement of layers position when they are well separated.

2. **Spectral Method** proposed by Eyre [2] in one dimension (not a trivial extension from two dimension to one dimension) .

3. Combination of ODE and PDE to form **continuously adaptive switching scheme** to explore metastable states.

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References:

[1] P. W. Bates and J. P. Xun. Metastable patterns for the Cahn-Hilliard equation: Part 2. layer dynamics and slow invariant manifold. Journal of Differential Equations, 117(1):165 – 216, 1995.

[2] David J. Eyre. Unconditionally gradient stable time marching the Cahn-Hilliard equation. MRS Proceedings, 529:39, 1998.