Continuity

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Contents

Consider a closed bounded interval of the real line, then:

$$\text{Continuously differentiable} \subseteq \text{Lipschitz continuous} \subseteq \text{absolutely continuous}$$
$$\subseteq \text{continuous and bounded variation} \subseteq \text{differentiable almost everywhere}$$

1 Continuity

Theorem 1 (First Fundamental Theorem of Calculus) *Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all x in $[a, b]$, as*

$$F(x) = \int_a^x f(t)dt$$

Then F is uniformly continuous on $[a, b]$ and differential on (a, b), and

$$\forall x \in (a, b) : F'(x) = f(x)$$

Corollary. *If f is real-valued continuous function on $[a, b]$ and F is antiderivative of f in $[a, b]$, then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

Theorem 2 (Second Fundamental Theorem of Calculus) *Let f be a real-valued function on a closed interval $[a, b]$ and F be a continuous function on $[a, b]$ which is an antiderivative of f in (a, b):*

$$F'(x) = f(x)$$

If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

NB! *Note that we assume that F is continuous, but f may not be.*

2 Uniform Continuity

3 Bounded Variation

4 Lipschitz Continuity

5 Absolute Continuity

Definition 1 (Absolute Continuity) *Let I be an interval in the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is absolutely continuous on I if for every positive number ε, there is a positive number δ such that whenever a* *finite sequence of pair-wise disjoint sub-intervals* (x_k, y_k) *on* I *with* $x_k < y_k \in I$ *satisfies*

$$\sum_k (y_k - x_k) < \delta$$

then

$$\sum_k |f(y_k) - f(x_k)| < \varepsilon$$

Collection of absolutely continuous function on I *is denoted* $AC(I)$

Equivalent definitions

1. f is absolutely continuous
2. f has a derivative f' almost everywhere, the derivative is Lebesque integrable, and

$$\forall x \in [a,b] : f(x) = f(a) + \int_a^b f'(t)dt$$

3. There exists a Lebesque integrable function g on $[a,b]$, such that

$$\forall x \in [a,b] : f(x) = f(a) + \int_a^b g(t)dt$$

If these conditions are satisfied then $g = f'$ almost everywhere.

Definition 2 (Fundamental Theorem of Lebesque integral Calculus) f *is absolutely continuous if and only if there exists a Lebesque integrable function* g *on* $[a,b]$*, such that*

$$\forall x \in [a,b] : f(x) = f(a) + \int_a^b g(t)dt$$