№1

$$p_\theta(x) = \begin{cases} ae^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$L(\vec{x};\theta) = \prod_{i=1}^{n} ae^{-ax_i} \mathbb{1}_{\{x_i \geq 0\}} = a^n e^{-a\sum_{i}^{n} x_i} \mathbb{1}_{\{x_{(1)} \geq 0\}}$$

$$LL = n\log a - a\sum_{1}^{n} x_i$$

$$\frac{\partial LL}{\partial a} = \frac{n}{a} - \sum x_i = 0$$

$$\hat{a} = \frac{n}{\sum x_i} = \frac{1}{(\overline{x})}$$

a) $\theta = \frac{1}{a}$

$$\hat{\theta} = \frac{1}{\hat{a}} = \overline{X}$$

$$E\overline{X} = \left(\frac{1}{a}\right)^1 \cdot \frac{\Gamma(n+1)}{n\Gamma(n)} = \frac{1}{a} \frac{n\cancel{\Gamma(n)}}{n\cancel{\Gamma(n)}} = \frac{1}{a} \quad - \text{ НРМД} = \tilde{\theta} = \hat{\theta}$$

b) $\theta = a^2$

$$E\frac{1}{(\overline{X})^2} = n^2\left(\frac{1}{a}\right)^{-2} \cdot \frac{\Gamma(n-2)}{\Gamma(n)} = n^2 a^2 \frac{\cancel{\Gamma(n-2)}}{(n-2)(n-1)\cancel{\Gamma(n-2)}} =$$

$$= \frac{n^2}{(n-1)(n-2)} a^2 \quad - \hat{\theta} \text{ смещённая оценка}$$

$$\tilde{\theta} = \frac{(n-1)(n-2)}{n^2}\hat{\theta} \Rightarrow E\tilde{\theta} = a^2 \Rightarrow \text{несмещённая оценка от ПВС} \Rightarrow \text{НРМД}$$

№2

а) $\theta = \lambda$

$$L(\vec{x}, \theta) = \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} = \underbrace{\lambda^{\sum x_i} e^{-n\lambda}}_{g_\theta(T(\vec{x}))} \underbrace{\prod \frac{1}{x_i!}}_{h(\vec{x})}$$

МДС: $\sum_{i=1}^{n} x_i$ – полная, т.к. исходное семейство эксп.

$$LL = \sum x_i \cdot \log\lambda - n\lambda - \lg\left(\prod_{i=1}^{n} x_i!\right)$$

$$\frac{\partial LL}{\partial \lambda} = \frac{\sum x_i}{\lambda} - n = 0 \implies \lambda = \bar{x}$$

$\hat{\theta} = \bar{x}$ – ОМП

$E\bar{x} = \lambda \implies \hat{\theta}$ – НРМД

№3

$$L(\vec{x}, a, \delta^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(x_i-a)^2}{2\delta^2}} = \frac{1}{(2\pi)^{n/2}\delta^n} \cdot$$

$$\cdot \exp\left(-\frac{1}{2\delta^2} \sum_{i=1}^{n} (x_i^2 - 2ax_i + a^2)\right) =$$

$$= \frac{1}{(2\pi)^{n/2}\delta^n} \exp\left(-\frac{1}{2\delta^2}\left(\sum_{i=1}^{n} x_i^2 - 2a\sum_{i=1}^{n} x_i + na^2\right)\right)$$

МДС: $\left(\sum_{i=1}^{n} x_i ; \sum_{i=1}^{n} x_i^2\right) \leftrightarrow (\bar{x}, s^2)$ – полная

$$LL(\vec{x}; a; \delta^2) = -\frac{n}{2}\lg(2\pi) - n\lg\delta - \frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i - a)^2$$

$$\frac{\partial LL}{\partial a} = -2\sum_{i=1}^{n}(x_i - a) = 0$$

$$\hat{a} = \frac{\sum x_i}{n} = \bar{x}$$

а) $\theta = a \Rightarrow \hat{\theta} = \bar{x}$

$E\bar{x} = a$ – несмещённая оценка $(\hat{\theta})$

$\Rightarrow \hat{\theta}$ – НРМД

б) $\theta = a^2$

$\hat{\theta} = \bar{x}^2$ – ОМП

$$E(\bar{x}^2) = \frac{1}{n^2}E\left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\right) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E(x_i x_j) =$$

$$E(x_i x_j) = \begin{cases} i = j & - Ex_1^2 = (Ex_1)^2 + Dx_1 = a^2 + \delta^2 \\ i \neq j & - Ex_1 x_2 = Ex_1 Ex_2 = a^2 \end{cases}$$

$= \frac{1}{n^2}\left(n(a^2 + \delta^2) + (n^2 - n)a^2\right) = a^2 + \frac{\delta^2}{n} \Rightarrow \hat{\theta}$ – смещённая оценка

$$Es^2 = \frac{n-1}{n}\delta^2$$

$\tilde{\theta} = \hat{\theta} - \frac{s^2}{n-1} \Rightarrow E\tilde{\theta} = a^2 \Rightarrow \tilde{\theta}$ – НРМД