д/з от 03.10.20

№1

а) $p_\theta(x) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$L = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum x_i}$$

$$LL = n \ln(\theta) - \theta \sum x_i$$

$$\frac{\partial}{\partial \theta} LL = \frac{n}{\theta} - \sum x_i$$

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$$

б) $p_\theta(x) = \begin{cases} \frac{1}{1-\theta}, & x \in [\theta, 1] \\ 0, & x \notin [\theta, 1] \end{cases}$

$$p_\theta(x) = \frac{1}{1-\theta} \cdot \mathbb{1}_{\{x \geq \theta\}} \cdot \mathbb{1}_{\{x \leq 1\}}$$

$$L(x, \theta) = \prod_{i=1}^{n} \frac{1}{1-\theta} \mathbb{1}_{\{x \geq \theta\}} \cdot \mathbb{1}_{\{x \leq 1\}} =$$

$$= \frac{1}{(1-\theta)^n} \mathbb{1}_{\{x_{(1)} \geq \theta\}} \cdot \mathbb{1}_{\{x_{(n)} \leq 1\}}.$$

$$\begin{cases} \frac{1}{(1-\theta)^n} \to max \\ \theta \leq x_{(1)} \\ x_{(n)} \leq 1 \end{cases} \Rightarrow \hat{\theta} = x_{(1)}$$

g) $p_\theta(x) = \begin{cases} \frac{1}{3\theta} & x \in [-\theta, 2\theta] \\ 0 & x \notin [-\theta, 2\theta] \end{cases}$

$$p_\theta(x) = \frac{1}{3\theta} \cdot \mathbb{1}_{\{x \geq -\theta\}} \cdot \mathbb{1}_{\{x \leq 2\theta\}}$$

$$L(\vec{x}, \theta) = \prod_{i=1}^{n} \frac{1}{3\theta} \mathbb{1}_{\{x_i \geq -\theta\}} \cdot \mathbb{1}_{\{x_i \leq 2\theta\}} =$$

$$= \frac{1}{(3\theta)^n} \cdot \mathbb{1}_{\{x_{(1)} \geq -\theta\}} \cdot \mathbb{1}_{\{x_{(n)} \leq 2\theta\}}$$

$$\begin{cases} \frac{1}{(3\theta)^n} \to \max \\ x_{(1)} \geq -\theta \\ x_{(n)} \leq 2\theta \end{cases} \Rightarrow \hat{\theta} = \max\left(|x_{(1)}|, \left|\frac{x_{(n)}}{2}\right|\right)$$

и) $p_\theta(x) = \frac{1}{2b} e^{-|x|/b}$, $x \in \mathbb{R}$, $\theta = b$

$$L(\vec{x}, b) = \prod_{i=1}^{n} \frac{1}{2b} e^{-|x|/b} = \frac{1}{(2b)^n} e^{-\frac{1}{b}\sum_{i=1}^{n}|x_i|}$$

$$LL(\vec{x}, b) = -\frac{1}{n} \ln(2b) - \frac{1}{b}\sum x_i$$

$$\frac{\partial}{\partial b} LL = -\frac{1}{2bn} + \frac{\sum_1^n |x_i|}{b^2}$$

$$\hat{b} = 2n \cdot \sum_{i=1}^{n} |x|$$

з) $p_\theta(x) = \frac{1}{2} e^{-|x-a|}$, $x \in \mathbb{R}$, $\theta = a$

$$L = \prod_{i=1}^{n} \frac{1}{2} e^{-|x-a|} = \frac{1}{2^n} e^{-\sum_1^n |x_i - a|}$$

$$LL = -n\ln 2 - \sum_{i=1}^{n} |x_i - a|$$

$$|x_{(i)} - a|' = \begin{cases} -1, & x_{(i)} < a \\ 1, & x_{(i)} > a \end{cases}$$

$$|x_{(i)} - a|' = \operatorname{sign}(x_{(i)} - a)$$

$$\frac{\partial}{\partial a} LL = -\sum_{i=1}^{n} \operatorname{sign}(x_{(i)} - a)$$

$$\begin{cases} \hat{a} = x_{(\frac{n+1}{2})}, & n - \text{нечёт.} \\ \hat{a} \in [x_{(\frac{n}{2})}, x_{(\frac{n}{2}+1)}] & - n - \text{чётн.} \end{cases}$$

$$\Rightarrow \hat{a} = Z_{n, 1/2}$$

№2

б) $q_\theta(x) = C_m^x p^x (1-p)^{m-x}, \; x = 0, 1, \dots, m, \quad \theta = p^2$

$$L = \prod_1^m C_m^{x_i} p^{x_i} (1-p)^{m-x_i} = \prod_1^n \frac{m!}{x_i!(m-x_i)!} p^{x_i} (1-p)^{m-x_i} =$$

$$= \left(\prod_1^m C_m^{x_i}\right) p^{\Sigma x_i} (1-p)^{m^2 - \Sigma x_i}$$

$$LL = \ln\left(\prod_1^m C_m^{x_i}\right) + \ln(p) \Sigma x_i + \ln(1-p) \cdot (m^2 - \Sigma x_i)$$

$$\frac{\partial}{\partial p} LL = \frac{\Sigma x_i}{p} - \frac{m^2 - \Sigma x_i}{1-p}$$

$$\Sigma x_i - \hat{p} \Sigma x_i = \hat{p}(m^2 - \Sigma x_i)$$

$$\hat{p} m^2 = \Sigma x_i$$

$$\hat{p} = \frac{\Sigma x_i}{m^2}$$

г) $q_\theta(x) = p(1-p)^x, \; x = 0, \dots, \dots \quad \theta = p$

$$L = \prod_1^n p(1-p)^{x_i} = p^n (1-p)^{\Sigma x_i}$$

$$\frac{\partial}{\partial p} = n p^{n-1} (1-p)^{\Sigma x_i} - (\Sigma x_i) p^n (1-p)^{(\Sigma x_i) - 1}$$

$$n p^{n-1} (1-\hat{p})^{\Sigma x_i} = (\Sigma x_i) \hat{p}^n (1-\hat{p})^{\Sigma x_i - 1}$$

$$n(1-\hat{p}) = (\Sigma x_i) \hat{p}$$

$$n - n\hat{p} = \hat{p} \Sigma x_i$$

$$\hat{p} = \frac{n}{n + \Sigma x_i} = 1 + \frac{1}{x}$$