

## Curs 3

$$\frac{dx}{dt} = f(t, x) \quad f(\cdot, \cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

PROP (ec. integrală asociată unei ec. diferențiale)

$$f(\cdot, \cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ cont, } \frac{dx}{dt} = f(t, x)$$

Atunci  $\varphi(\cdot) : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$  graph( $\cdot$ )  $\subset D$  este soluție  $\Leftrightarrow$

$$1. \varphi(\cdot) \text{ cont}$$

$$2. \varphi(t) = \varphi(t_0) + \int_{t_0}^t f(s, \varphi(s)) ds, \forall t, t_0 \in I$$

T. Peano (Existența locală a sol)

$$\text{Fie } f(\cdot, \cdot) : D = \bar{D} \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ continuă, } \frac{dx}{dt} = f(t, x)$$

Atunci  $f(\cdot, \cdot)$  admite EL pe  $D$  ( $\exists (t_0, x_0) \in D \ni \varphi(\cdot) : I_0 \in \Theta(t_0) \rightarrow \mathbb{R}^n$  sol cu  $\varphi(t_0) = x_0$ )

Nrm: Fie  $(t_0, x_0) \in D$

Planul dem: 1.  $I_0 = [t_0 - a, t_0 + a]$ ,  $a = ?$

2.  $\varphi_m(\cdot) : I_0 \rightarrow \mathbb{R}^n$  sir

egal mărginit  
egal uniform continuu

3. ~~T. Arzela-Ascoli~~

$\exists \varphi_m \xrightarrow{u} \varphi(\cdot)$ ,  $\varphi(\cdot) : I_0 \rightarrow \mathbb{R}^n$  continuă

4.  $\varphi(\cdot)$  soluția

1.  $(t_0, x_0) \in D = \bar{D} \Rightarrow \exists \delta, \gamma > 0$  a.i  $\overline{B}_g(t_0) \times \overline{B}_g(x_0) \subset D$

$$k := \max \{ \gamma, \|f(t, x)\| \}$$

$$(t, x) \in \overline{B}_g(t_0) \times \overline{B}_g(x_0)$$

$$k = 0 \Rightarrow f(t, x) = 0 \Rightarrow \varphi(t) \equiv x_0$$

$$k > 0 \quad a := \min \left\{ \delta, \frac{\gamma}{K} \right\}, \quad I_0 = [t_0 - a, t_0 + a]$$

$m \geq 1$ ,  $\varphi_m(\cdot) : I_0 \rightarrow \mathbb{R}^n$

$$\varphi_m(t) = \begin{cases} x_0, & t \in [t_0 - \frac{a}{m}, t_0 + \frac{a}{m}] \\ t - \frac{a}{m} & \end{cases}$$

$$x_0 + \int_{t_0}^t f(s, \varphi_m(s)) ds, \quad t \in [t_0 + \frac{a}{m}, t_0 + a]$$

$$x_0 + \int_{t_0}^{t + \frac{a}{m}} f(s, \varphi_m(s)) ds, \quad t \in [t_0 - a, t_0 - \frac{a}{m}]$$

Akkumulativ ca:

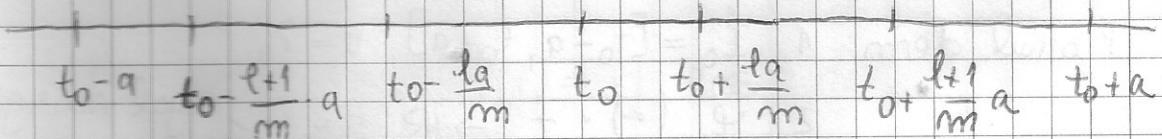
a)  $\varphi_m(\cdot)$  binne märginit

b)  $\varphi_m(t) \in \overline{B}_\rho(x_0)$ ,  $\forall m \geq 1, \forall t \in I_0$ .

c)  $\varphi_m(\cdot)$  este legal Lipschitz

$$\|\varphi_m(t') - \varphi_m(t'')\| \leq k |t' - t''| \quad \forall t', t'' \in I_0$$

a)



b) Inductie relativ la intervalul  $[t_0 - \frac{l-1}{m}a, t_0 + \frac{l}{m}a]$

$l=1$ :  $\varphi_m(t) \equiv x_0$  ok

$l \mapsto l+1$ : Pp ca  $t \in [t_0 + \frac{l}{m}a, t_0 + \frac{l+1}{m}a] \Rightarrow$

$$\frac{t-a}{m} \in [t_0 + \frac{l-1}{m}a, t_0 + \frac{l}{m}a]$$

$$\Rightarrow \varphi_m(t) = x_0 + \int_{t_0}^t f(s, \varphi_m(s)) ds \text{ ok.}$$

$$l=1 \quad \varphi_m(t) = x_0 \in \overline{B}_\rho(x_0)$$

$l \mapsto l+1$  Pp. ca  $t \in [t_0 + \frac{l}{m}a, t_0 + \frac{l+1}{m}a]$

$$\|t - \frac{a}{m}\|$$

$$\leq \left| \int_{t_0}^{t-\frac{a}{m}} f(s, \varphi_m(s)) ds \right| \leq k \left| t - \frac{a}{m} - t_0 \right| \leq k \cdot \frac{a}{m} \cdot a \leq k \cdot \frac{a}{m} \cdot \frac{a}{k} = \frac{a^2}{m}$$

$\in \overline{B_\rho(x_0)}$

$= \gamma$

$$\varphi_m(t) \in \overline{B_\rho(x_0)} \quad \forall m \neq t \quad \|\varphi_m(t)\| \leq \|\varphi_m(t) - x_0\| + \|x_0\| \leq \gamma + \|x_0\|, \quad \forall m \neq t$$

c)  $t', t'' \in \overset{\circ}{I}_0$ . De ex, pp că  $t', t'' \in [t_0 + \frac{a}{m}, t_0 + a]$

$$\|\varphi_m(t') - \varphi_m(t'')\| = \left\| x_0 + \int_{t_0}^{t-\frac{a}{m}} f(s, \varphi_m(s)) ds \right\| -$$

$$- \left\| \int_{t_0}^{t''-\frac{a}{m}} f(s, \varphi_m(s)) ds \right\| =$$

$$= \left\| \int_{t''-\frac{a}{m}}^{t-\frac{a}{m}} f(s, \varphi_m(s)) ds \right\| \leq \left| \int_{t''-\frac{a}{m}}^{t-\frac{a}{m}} \|\varphi(s, \varphi_m(s))\| ds \right| \leq k \cdot |t' - t''|$$

ok.

Analog celelalte cazuri

3. T. Arzela-Ascoli  $\Rightarrow \exists \varphi_{m_p}(\cdot) \xrightarrow[p \rightarrow \infty]{u} \varphi(\cdot)$ ,  $\varphi(\cdot): \overset{\circ}{I}_0 \rightarrow \overline{B_\rho(x_0)}$  continuă

$$4. \varphi_{m_p}(t) = \begin{cases} x_0, & t \in [t_0 - \frac{a}{m_p}, t_0 + \frac{a}{m_p}] \\ x_0 + \int_{t_0}^t f(s, \varphi_{m_p}(s)) ds + \int_t^{t+\frac{a}{m_p}} f(s, \varphi_{m_p}(s)) ds, & t \in [t_0 + \frac{a}{m_p}, t_0 + a] \\ x_0 + \int_{t_0}^t f(s, \varphi_{m_p}(s)) ds + \int_t^{t+\frac{a}{m_p}} f(s, \varphi_{m_p}(s)) ds, & t \in [t_0 - a, t_0 - \frac{a}{m_p}] \end{cases}$$

$$\begin{aligned}
 & \left. \begin{aligned}
 & f_{mp}(.) \xrightarrow{u} \varphi(.) \\
 & \varphi(.,.) \text{ cont}
 \end{aligned} \right\} \Rightarrow \int_{t_0}^t g(s, f_{mp}(s)) ds \Rightarrow \int_{t_0}^t g(s, \varphi(s)) ds \\
 & t - \frac{a}{mp} \\
 & \left| \int_{t_0}^t g(s, f_{mp}(s)) ds \right| \leq \left| \int_{t_0}^{t - \frac{a}{mp}} \|g(s, f_{mp}(s))\| ds \right| \leq k \cdot \frac{a}{m} \\
 & \xrightarrow{m \rightarrow \infty} 0 \\
 & mp \rightarrow \infty \quad \varphi(t) = x_0 + \int_{t_0}^t g(s, \varphi(s)) ds, \forall t \in I \\
 & \varphi(t) = \varphi(t_0) + \int_{t_0}^t g(s, \varphi(s)) ds, \forall t \in I \\
 & \varphi(.) \text{ continua} \quad \Rightarrow
 \end{aligned}$$

Obs

$$D = D^\circ, f(\cdot, \cdot) \text{ cont}$$

$$1) \quad x' = \alpha(t), \quad \alpha(t) = \begin{cases} 1, & t \in Q \\ 0, & t \in RI \setminus Q \end{cases}$$

There are E.2 in million punct

$$2) \quad x' = \text{sgn}(x), \quad \text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

are EU1 for firecare point.

## Functii local Lipschitz

Derm: 1)  $g(\cdot) : G \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  s.m Lipschitz (global),  
 G dacă  $\exists L > 0$  a.s.  $\|g(x_1) - g(x_2)\| \leq L \|x_1 - x_2\|$ ,

2)  $g(\cdot): G \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  s.m local Lipschitz în  $x_0 \in G$  dacă  
 $\exists \mu > 0, \exists L > 0$  a.t  $\|g(x_1) - g(x_2)\| \leq L \|x_1 - x_2\|$   
 $\forall x_1, x_2 \in B_\mu(x_0) \cap G$

Obs: 1.  $g(\cdot)$  Lipschitz  $\Rightarrow g(\cdot)$  local Lipschitz  $\Rightarrow g$  continuă

2.  $g(x) = x^2$  este local Lipschitz, nu este global Lipschitz

$g(x) = x^{\frac{1}{3}}$  este continuă, dar nu este local Lipschitz

Prop  $g(\cdot): G = \overset{0}{G} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ . Atunci  $g$  este local Lipschitziană  
 $\Leftrightarrow g(\cdot)|_{G_0}$  este (global) Lipschitz  $\forall G_0 \subset G$  compact

Dem:  $\Leftarrow$  "Evident"

" $\Rightarrow$ "

Pp.că  $\exists G_0 \subset G$  compact a.t  $g(\cdot)|_{G_0}$  nu este Lipschitz

$\forall m \in \mathbb{N} \exists x_m^1, x_m^2 \in G_0$  a.t  $\|g(x_m^1) - g(x_m^2)\| >$   
 $m \|x_m^1 - x_m^2\|$  (1)

$$\Rightarrow \|x_m^1 - x_m^2\| < \frac{1}{m} \|g(x_m^1) - g(x_m^2)\| \leq$$

$$\leq \frac{1}{m} (\|g(x_m^1)\| + \|g(x_m^2)\|)$$

$g(\cdot)$  local Lipschitz  $\Rightarrow g(\cdot)$  continuă,  $G_0$  compactă,

$$K := \max_{x \in G_0} \|g(x)\|$$

$x_m^1, x_m^2 \in G_0$  compactă  $\Rightarrow \exists x_{m_\ell}^1 \rightarrow x^1 \in G_0$

$\exists x_{m_\ell}^2 \rightarrow x^2 \in G_0$

$$(*) \|x_m^1 - x_m^2\| \leq \frac{2K}{m}, m_\ell \rightarrow \infty \Rightarrow \|x^1 - x^2\| \leq 0$$

$g(\cdot)$  local Lipschitz im  $x_0 \in G \Rightarrow \exists \eta > 0 \exists L > 0$  a.

$$\|g(x_1) - g(x_2)\| \leq L \|x_1 - x_2\|, \forall x_1, x_2 \in B_\eta(x_0)$$

~~$x_{m_\ell}^1, x_{m_\ell}^2 \rightarrow x_0 \Rightarrow \exists l_0 \in \mathbb{N}, l \geq l_0,$~~

$$x_{m_\ell}^1, x_{m_\ell}^2 \in B_{\eta}(x_0)$$

$$\|g(x_{m_\ell}^1) - g(x_{m_\ell}^2)\| \leq L \|x_{m_\ell}^1 - x_{m_\ell}^2\| \quad (2)$$

$$(1), (2) \Rightarrow \lim_{m_\ell} \|x_{m_\ell}^1 - x_{m_\ell}^2\| \leq L \|x_{m_\ell}^1 - x_{m_\ell}^2\|, m_\ell \rightarrow$$

PROP  $G = \overset{\circ}{G} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ , derivabilă cu  $Dg(\cdot)$  mărginită

Atunci  $g(\cdot)$  local Lipschitziană pe  $G$

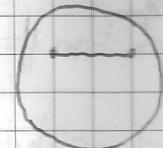
Ieșire: T. medie:  $\|g(x) - g(y)\| = \sup_{z \in [x,y]} \|Dg(z)\| \cdot \|x - y\|$

$x, y \in G, [x, y] \subset G$

$$[x, y] = \{x + s(y - x); s \in [0, 1]\}$$

Fie  $x_0 \in G = \overset{\circ}{G} \Rightarrow \exists \eta > 0$  a.t.  $\overline{B}_\eta(x_0) \subset$

$$L := \sup_{z \in \overline{B}_\eta(x_0)} \|Dg(z)\| \text{ g.e.d.}$$



(Convexitate)

Consecință:  $g(\cdot): G = \overset{\circ}{G} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ , de clasă  $C^1$  atunci  $g(\cdot)$  local Lipschitz.

Functie local Lipschitz în mapare cu variabila a două

$f(\cdot, \cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  să fie local Lipschitz în mapare cu

$$\|f(t, x_1) - f(t, x_2)\| \leq L \cdot \|x_1 - x_2\|, \forall (t, x_1), (t, x_2) \in D_0$$

PROP Fie  $f(\cdot, \cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  continuă.

Atunci  $f(\cdot, \cdot)$  local Lipschitz (II) (pe  $D$ )  $\Leftrightarrow \forall \overset{\circ}{D} \subset D$  compactă  $\exists L > 0$  a.ș.  $\|f(t, x_1) - f(t, x_2)\| \leq L \cdot \|x_1 - x_2\|, \forall (t, x_1), (t, x_2) \in D_0$ .

Bem: Ex!

PROP Fie  $f(\cdot, \cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \in C^1(\underline{II})$

$\left[ \exists D_2 f(t, x) \ni (t, x) \rightarrow D_2 f(t, x) \text{ continuă} \right] \Rightarrow$   
 deriv. în raport cu variabilea a două  $\left( \frac{\partial}{\partial x} f(t, x) \right)$   
 $f(\cdot, \cdot)$  local Lipschitz (II)

### Lema Bellmann - Gronwall

$M > 0, u(\cdot), v(\cdot) : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_+$  continue,  $t_0 \in I$ . Dacă:

$$u(t) \leq M + \left| \int_{t_0}^t u(s)v(s)ds \right|, \forall t \in I \text{ atunci}$$

$$u(t) \leq M \cdot e^{\left| \int_{t_0}^t v(s)ds \right|}, \forall t \in I$$

## Seminar 3

### Ecuatii Bernoulli

$$\frac{dx}{dt} = a(t)x + b(t) \cdot x^\alpha, \quad a(\cdot), b(\cdot) : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \text{ cont.,}$$

$\alpha \in \mathbb{R} \setminus \{0, 1\}$

### Algoritm

1. Cons. ec. lich. asociata:  $\frac{dx}{dt} = a(t)x$

Scriem sol. generala:  $x(t) = c \cdot e^{A(t)}$

2. Variatia constantelor

Caut sol de forma  $x(t) = c(t) \cdot e^{A(t)}$

$$x(t) \text{ sol} \Rightarrow \frac{dc}{dt} = e^{(\alpha-1)A(t)} \cdot b(t) \cdot c^\alpha$$

Algoritm

$$\Rightarrow c(t) = \dots$$

$$x(t) = \dots$$

Ex: Să se determine sol. generală:

$$1) x' = x^2 \cdot e^t - 2x$$

$$2) t x' + x = t^5 x^3 \cdot e^t$$

$$3) t x' = 2t^2 \sqrt{x} + 4x$$

$$4) x' + \frac{x}{t} = \frac{1}{t^2 \cdot x^2}$$

$$1) x' = -2x + x^2 \cdot e^t.$$

$$\frac{dx}{dt} = -2x + x^2 \cdot e^t$$

$$\frac{dx}{dt} = -2x$$

$$\overline{x(t)} = c \cdot e^{-2 \cdot \frac{t^2}{2}} \Rightarrow \overline{x(t)} = c \cdot e^{-t^2}$$

$$x(t) = c \cdot e^{-2t}$$

Caut sol. de forma  $x(t) = c(t) \cdot e^{-2t}$

Pentru conditia ca  $c(t) \cdot e^{-2t}$  solutie

Revins aici cu  
③, ④ pt. a  
determina sol.  
finală

$$(c(t) \cdot e^{-2t})' = (c(t) \cdot e^{-2t})^2 \cdot e^t - 2(c(t) \cdot e^{-2t})$$

$$\Leftrightarrow c'(t) e^{-2t} + \cancel{c(t) \cdot (-2) \cdot e^{-2t}} =$$

$$= c^2(t) e^{-3t} - \cancel{2c(t) e^{-2t}}$$

$$\Leftrightarrow c'(t) e^{-2t} = c^2(t) e^{-3t}$$

$$\Leftrightarrow c'(t) = c^2(t) e^{-t}$$

$$\frac{dc}{dt} = c^2 e^{-t} \text{ ecuatie cu variabile separate}$$

$$c^2 = 0 \Rightarrow c=0 \Rightarrow c(t)=0. \oplus$$

$\varphi_1 = 0$  sol stationara

~~$$\frac{dc}{c^2} = e^{-t} dt$$~~

$$\int \frac{dc}{c^2} = \int e^{-t} dt \Leftrightarrow -\frac{1}{c} = -e^{-t} + k, k \in \mathbb{R}$$

$$\Rightarrow c(t) = \frac{1}{e^{-t} + k}, k \in \mathbb{R}. \oplus$$

$$x_0(t) = 0$$

$$x_R(t) = e^{-2t} \cdot \frac{1}{e^{-t} + k}, k \in \mathbb{R}.$$

$$2) t\dot{x}^1 + x = t^5 \cdot x^3 \cdot e^t$$

$$\dot{x}^1 = \frac{1}{t}(t^5 \cdot x^3 \cdot e^t - x)$$

$$\dot{x}^1 = t^4 \cdot x^3 \cdot e^t - \frac{x}{t}, t \neq 0, t > 0$$

$$\frac{dx}{dt} = \underbrace{t^4 \cdot x^3 \cdot e^t}_{b(t) \cdot x^3} - \underbrace{\frac{x}{t}}_{x \cdot a(t)}$$

$$\bar{x}(t) = -\frac{x}{t}$$

$$\bar{x}(t) = c_1 e^{\int -\frac{1}{t} dt} \Rightarrow \bar{x}(t) = c_1 e^{-\ln t}$$

$$\Rightarrow \bar{x}(t) = \frac{c_1}{t}$$

Căutăm sol:  $x(t) = c(t) \cdot \frac{1}{t}$

$$(c(t) \cdot \frac{1}{t})' = t^4 \cdot (c(t) \cdot \frac{1}{t})^3 \cdot e^t - (c(t) \cdot \frac{1}{t}) \cdot \frac{1}{t}$$

$$\Leftrightarrow c'(t) \cdot \frac{1}{t} - c(t) \cdot \frac{1}{t^2} = t^4 \cdot c^3(t) \cdot \frac{1}{t^3} \cdot e^t - c(t) \cdot \frac{1}{t^2}$$

$$\Leftrightarrow c'(t) \cdot \frac{1}{t} = t^4 \cdot c^3(t) \cdot \frac{1}{t^3} \cdot e^t$$

$$\Leftrightarrow c'(t) \cdot \frac{1}{t} = t \cdot c^3(t) \cdot e^t$$

$$c'(t) = t^2 \cdot c^3(t) \cdot e^t$$

$c'(t) = c^3(t) \cdot t^2 \cdot e^t \rightarrow$  Ecuație cu variabile separate.

$$\frac{dc}{dt} = c^3 \cdot t^2 \cdot e^t$$

$c^3 = 0 \Rightarrow c = 0, q_0 = 0$  sol. stacionară

$c(t) \equiv 0$ .

$$dc = t^2 e^t dt$$

$$\int \frac{dc}{c^3} = \int t^2 \cdot e^t dt$$

$$-\frac{1}{2} \cdot \frac{1}{c^2} = t^2 e^t - \int 2t e^t dt$$

$$\Leftrightarrow -\frac{1}{2c^2} = t^2 \cdot e^t - 2t \cdot e^t + \int 2e^t dt$$

$$\Leftrightarrow -\frac{1}{2c^2} = t^2 \cdot e^t - 2t \cdot e^t + 2e^t + k, k \in \mathbb{R}$$

$$\Rightarrow c(t) = \pm \frac{1}{\sqrt{-2(t^2 e^t - 2t e^t + 2e^t) + k}}, k \in \mathbb{R}$$

$$x_0(t) = 0$$

$$x_{\neq}(t) = \pm \frac{1}{t \sqrt{...}}$$

### Ecuatii Riccati

$$\frac{dx}{dt} = a(t)x^2 + b(t)x + c(t), \quad a(t), b(t), c(t), t \in \mathbb{R} \rightarrow \mathbb{R}$$

cont.

$\exists \varphi_0(\cdot)$  solutie.

Algoritm: SV  $y = x - \varphi_0(t)$  [A  $x(\cdot)$  sol SV def

$y(\cdot)$  după regulă  $y(t) = x(t) - \varphi_0(t)$

$\Rightarrow$  Ec. Bernoulli în  $y \Rightarrow$  Algoritm  $\Rightarrow y(t) = \dots$   
 $\Rightarrow x(t) = y(t) + t = \dots$

Sol: 1)  $x' = x^2 - 2x e^t + e^{2t} + e^t, \varphi_0(t) = e^{t^2}$  sol

2)  $x' = x^2 + 6x - 4t + 11, \varphi_0(t) = 2t - 3$  sol

3)  $x' + x^2 - \frac{2}{t}x + \frac{2}{t^2} = 0, \varphi_0(t) = \frac{2}{t}$  sol

$$1) \frac{dx}{dt} = x^2 - 2e^t x + e^{2t} + e^t$$

$$\varphi_0(t) = e^t \text{ so}$$

$$sv \circ y = x - e^t$$

$$\cancel{\Rightarrow y(t) = (x - e^t)^2 - 2e^t(x - e^t) + e^{2t} + e^t}$$

$$\cancel{\Rightarrow y(t) = x^2 - 2xe^t + e^{2t} - 2e^tx + 2e^t + e^{2t} + e^t}$$

$$x(t) = y(t) + e^t$$

$$(y(t) + e^t)' = (y(t) + e^t)^2 - 2 \cdot (y(t) + e^t) \cdot e^t + e^{2t} + e^t$$

$$= y^2(t) + e^{2t} + 2y(t)e^t - 2y(t)e^t - 2e^{2t}$$

$$+ e^{2t} + e^t$$

$$\cdot (y(t) + e^t)' = y^2(t) + e^t$$

$$\dot{y}(t) + e^t = y^2(t) + e^t$$

$$\dot{y}(t) = y^2(t)$$

$$\frac{dy}{dt} = y^2 \rightarrow \text{ec. voor sep}$$

$$y^2 = 0 \Rightarrow y = 0 \Rightarrow y_0(t) = 0$$

$$\frac{dy}{y^2} = \cancel{\frac{dt}{dt}} \Rightarrow \int \frac{dy}{y^2} = \int \cancel{\frac{dt}{dt}} \Rightarrow -\frac{1}{y} = t + k, k \in \mathbb{R}$$

$$\Rightarrow y_k(t) = -\frac{1}{t+k}, k \in \mathbb{R}$$

$$\Rightarrow x(t) = y(t) + e^t$$

$$x_0(t) = e^t$$

$$2) \quad x' = x^2 + 6x - 4t^2 + 11, \varphi_0(t) = 2t - 3 \text{ sol}$$

$$\text{sv: } y(t) = x - 2t + 3$$

$$x(t) = y(t) + 2t - 3.$$

$$(y(t) + 2t - 3)' = (y(t) + 2t - 3)^2 + 6(y(t) + 2t - 3) - 4t^2 + 11$$

$$\Leftrightarrow y'(t) + 2 = (y(t) + 2t - 3)(y(t) + 2t - 3) -$$

$$+ 6 \cdot y(t) + 12t - 18 - 4t^2 + 11$$

$$\cancel{(y(t) + 2t - 3)^2} = \cancel{y^2(t) + 2t \cdot y(t) - 3 \cdot y(t) + 2t \cdot y(t) + y(t)^2 - 6t^2} - 6yt(t) - 6t + 9 - 6y(t) + 12t - 18 - 4t^2 + 11$$

$$\cancel{(y(t) + 2t - 3)^2} = \cancel{y^2(t) + 2t \cdot y(t) - 3 \cdot y(t) + 2t \cdot y(t) + y(t)^2 - 6t^2} - 6yt(t) - 6t + 9 - 6y(t) + 12t - 18 - 4t^2 + 11$$

$$\Leftrightarrow y'(t) + 2 = y^2(t) + 2t \cdot y(t) - 3 \cdot y(t) + 2t \cdot y(t) + y(t)^2 - 6t^2 - 6t - 3y(t) - 6t + 9 + 6y(t) + 12t - 18 - 4t^2 + 11$$

$$\Leftrightarrow y'(t) = y^2(t) + 4t \cdot y(t)$$

$$\cancel{\frac{dy}{dt}} = ht \cdot y + y^2 \rightarrow \text{Ec. Bernoulli}'$$

$$\frac{dy}{dt} = ht \cdot y, \text{ scriem sol. generală: } \bar{y}(t) = c \cdot e^{\int ht dt}$$

$$\bar{y}(t) = c \cdot e^{zt^2} \quad (\Rightarrow \bar{y}(t) = c \cdot e^{\frac{2t^2}{4t}})$$

Caut sol. de forma  $y(t) = c(t) \cdot e^{4t}$

Peste tot...

Nu  $\cdot e^{4t} : ($

$$(c(t) \cdot e^{4t})' = 4t \cdot (c(t) \cdot e^{4t}) + (c(t) \cdot e^{4t})^2$$

$$c'(t) \cdot e^{4t} + 4 \cdot c(t) \cdot e^{4t} = 4t \cdot c(t) \cdot e^{4t} + c^2(t) \cdot e^{8t}$$

$$c'(t) \cdot e^{4t} = c^2(t) \cdot e^{8t}$$

$$c'(t) = c^2(t) \cdot e^{4t}$$

$$\underline{\frac{dc}{c^2}} = e^{4t} \rightarrow \text{Ec. var. separabile}$$

$$\frac{dc}{dt} = -\frac{1}{c} e^{4t} \Rightarrow -\frac{1}{c} = \frac{1}{4} e^{4t} + k, k \in \mathbb{R}$$

$$c(t) = \frac{1}{e^{\frac{4t}{4} + k}}$$

$$\frac{dc}{dt} = -\frac{1}{c^2}$$

$$\frac{dc}{c^2} = e^{2t^2} dt \Rightarrow \int \frac{dc}{c^2} = \int e^{2t^2} dt$$

$$\Rightarrow -\frac{1}{c} = \int_0^t e^{2s^2} ds + k, k \in \mathbb{R}$$

$$c(t) = -\frac{1}{\int_0^t e^{2s^2} ds + k}, k \in \mathbb{R}$$

$$y_0(t) = 0$$

$$y_R(t) = \dots$$

$$y(t) = y_0(t) + x_R(t) = 2t - 3 + x_R(t)$$

$$x_0(t) = 2t - 3$$

$$x_R(t) = -\frac{e^{2t^2}}{\int_0^t e^{2s^2} ds + k}, k \in \mathbb{R}$$

Ecuatii omogene

$$\frac{dx}{dt} = f\left(\frac{x}{t}\right), f(\cdot) : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$$

Algoritm

SV:  $y = \frac{x}{t} \quad (\forall x(t))$  sol s.v def functie  $y(\cdot)$  după regula

$y(t) = \frac{x(t)}{t} \Rightarrow \frac{dy}{dt} = \frac{f(y)-y}{t}$  și vari. son  $\rightarrow$  rezolv. Algoritm

$$\text{Ex: 1) } 2t^2x' = t^2 + x^2$$

$$2) x' = \frac{x + \sqrt{t+x}}{t}$$

$$3) t \cdot x' = x + \sqrt{x^2 - t^2}$$

$$4) x' = \frac{2tx}{t^2 - x^2}$$

$$1) 2t^2 \cdot x' = t^2 + x^2, t^2 \neq 0, t > 0 \quad | :2t^2$$

$$x' = \frac{1}{2} + \frac{1}{2} \left( \frac{x}{t} \right)^2$$

$$\text{sv } y = \frac{x}{t} \Rightarrow y(t) = \frac{x(t)}{t}$$

$$x(t) = t \cdot y(t) \Leftrightarrow (t \cdot y(t))' = \frac{1}{2} + \frac{1}{2} \cdot y^2(t)$$

$$\Leftrightarrow y(t) + t \cdot y'(t) = \frac{1}{2} + \frac{1}{2} \cdot y^2(t)$$

$$\Leftrightarrow y'(t) = \frac{1}{t} \left( \frac{1}{2} + \frac{1}{2} y^2(t) - y(t) \right) \Rightarrow \frac{dy}{dt} = \frac{(y-1)^2}{2t}$$

$$(y-1)^2 = 0 \Rightarrow y = 1 \Rightarrow y_0(t) \equiv 1$$

$$\Rightarrow \frac{dy}{\frac{1}{2} + \frac{1}{2} y^2(t)} = dt \Leftrightarrow \int \frac{dy}{(y-1)^2} = \frac{1}{2} \int dt \Rightarrow \frac{1}{1-y} = \frac{1}{4} t^2 + k \quad k \in \mathbb{R}$$

$$y(k) = -\frac{1}{\frac{1}{4} t^2 + k} - 1, k \in \mathbb{R}$$

$$x_0(t) = t$$

$$x_k(t) = t \cdot \left( -\frac{1}{\frac{1}{4} t^2 + k} - 1 \right), k \in \mathbb{R}$$

$$2) x' = \frac{x + \sqrt{t}x}{t}$$

$$\Leftrightarrow x' = \frac{x}{t} + \sqrt{\frac{x}{t^2}} \Leftrightarrow x' = \frac{x}{t} + \sqrt{\frac{x}{t}}, t \neq 0, t > 0$$

$$SV: \frac{x}{t} = y \Rightarrow y(t) = \frac{x(t)}{t} \Rightarrow x(t) = y(t) \cdot t.$$

$$(y(t) \cdot t)' = y(t) + \sqrt{y(t)}$$

$$y'(t) \cdot t + y(t) = y(t) + \sqrt{y(t)}$$

$$y'(t) = \frac{\sqrt{y(t)}}{t} \Rightarrow y' = \frac{\sqrt{y}}{t}$$

$$\Rightarrow \frac{dy}{dt} = \frac{\sqrt{y}}{t} \Leftrightarrow dy = dt \sqrt{\frac{y}{t}}$$

$$\frac{dy}{\sqrt{y}} = \frac{dt}{t}$$

$$\int \frac{dy}{\sqrt{y}} = \int \frac{dt}{t} \Rightarrow$$

$$\Rightarrow 2\sqrt{y} = \ln t + c, c \in \mathbb{R}.$$

$$y = \left( \frac{\ln t + c}{2} \right)^2, c \in \mathbb{R}.$$

$$x(t) = \left( \frac{\ln t + c}{2} \right)^2 \cdot t, c \in \mathbb{R}.$$

## Cvrs 4

### Lema Bellman - Gromov

Fie  $M > 0$ ,  $t_0 \in I \subseteq \mathbb{R}$ ,  $u(\cdot), v(\cdot) : I \rightarrow \mathbb{R}$ , continue

Dacă:  $u(t) = M + \left| \int_{t_0}^t u(s) v(s) ds \right|$ ,  $\forall t \in I$  atunci  $u(t) \leq M \cdot e^{\left| \int_{t_0}^t v(s) ds \right|}$ ,  $\forall t \in I$

$$M \cdot e^{\left| \int_{t_0}^t v(s) ds \right|}, \quad \forall t \in I$$

$$\text{Dem: } u(t) \leq M + \left| \int_{t_0}^t u(s) v(s) ds \right| \rightarrow e^{\left| \int_{t_0}^t v(s) ds \right|} \cdot e^{\left( \int_{t_0}^t v(s) ds \right)} = \psi(t)$$

Anătărим că  $\psi(t) \leq M$ ,  $\forall t \Rightarrow$  g.e.d.

$$I_+ = \{t \in I, t \geq t_0\}, I_- = \{t \in I, t < t_0\}$$

Anătărим că  $\psi(\cdot)$  este l.p. pe  $I_+$  și  $\bar{P}$  pe  $I_-$ ,  $\psi(t_0) = \psi(t)$

$$\rightarrow \psi(t) = M, \forall t$$

Anătărим de exemplu pt.  $t \in I_+$

$$\psi(t) = \left( M + \int_{t_0}^t u(s) v(s) ds \right) \cdot e^{- \int_{t_0}^t v(s) ds}$$

$$\psi(t) = u(t) = u(t) \cdot v(t) \cdot e^{- \int_{t_0}^t v(s) ds} + \left( M + \int_{t_0}^t u(s) v(s) ds \right)$$

$$= e^{- \int_{t_0}^t v(s) ds} \cdot (-v(t)) = \frac{u(t)}{e^{\int_{t_0}^t v(s) ds}} \geq 0$$

$$\leq 0, \forall t \rightarrow \psi(\cdot) \text{ desc. pe } I_+$$

### Teorema Cauchy - Lipschitz

Fie  $f(\cdot, \cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  cont., local Lipschitz (L)

$\frac{dx}{dt} = f(t, x)$ . Atunci  $f(\cdot, \cdot)$  admite EUL pe  $D$  ( $\forall (t_0, x_0) \in D$ )

$\exists D, \exists I_0 = [t_0 - a, t_0 + a] \in \mathcal{D}(t_0) \exists \varphi(\cdot) : I_0 \rightarrow \mathbb{R}^n$

Soluție cu  $\varphi(t_0) = x_0$ .

Dem: Fie  $(t_0, x_0) \in D = \overset{\circ}{D}$ ,  $f(\cdot, \cdot)$  cont  $\Rightarrow \exists I_0 = [t_0 - a, t_0 + a] \in \mathcal{D}(t_0)$

$\exists \varphi(\cdot) : I_0 \rightarrow \mathbb{R}^n$  sol cu  $\varphi(t_0) = x_0$ .

Pentru  $\forall t \in I_0 \exists \delta > 0$  s.t.  $\forall x \in \mathbb{R}^n$  cu  $|x - x_0| < \delta$  avem  $|f(t, x) - f(t, x_0)| < \epsilon$

$\varphi(\cdot)$  soluție  $\Rightarrow$  ec. integrală asociată

$$\varphi(t) = \varphi(t_0) + \int\limits_{t_0}^t f(s, \varphi(s)) ds, \forall t \in I_0.$$

$$\varphi_1(\cdot) \text{ soluție } \Rightarrow -|| - \varphi_1(t) = \varphi_1(t_0) + \int\limits_{t_0}^t f(s, \varphi_1(s)) ds, \forall t \in I_0.$$

$$\varphi(t) - \varphi_1(t) = \int\limits_{t_0}^t f(s, \varphi(s)) ds - \int\limits_{t_0}^t f(s, \varphi_1(s)) ds,$$

$$= \int\limits_{t_0}^t [f(s, \varphi(s)) - f(s, \varphi_1(s))] ds$$

$$\|\varphi(t) - \varphi_1(t)\| = \left\| \int\limits_{t_0}^t [f(s, \varphi(s)) - f(s, \varphi_1(s))] ds \right\| \leq$$

$\leq u(t)$

$$\leq \left\| \int\limits_{t_0}^t \|f(s, \varphi(s)) - f(s, \varphi_1(s))\| ds \right\|$$

- Fie  $D_0 = \{(t, \varphi(t)), (t, \varphi_1(t)); t \in I_0 = [t_0-a, t_0+a]\} \subset D$  compactă ( $\varphi(\cdot), \varphi_1(\cdot)$  cont  $[t_0-a, t_0+a]$  compact) imaginea unui int. compact printr-o fct. cont este tot compact

$f(\cdot, \cdot)$  local Lipschitz (II),  $D_0 \subset D$  compact  $\Rightarrow f(\cdot) / D_0$  este

Lipschitz  $\Rightarrow \exists L > 0$  a.t.  $\|f(t, x_1) - f(t, x_2)\| \leq L \|x_1 - x_2\|$ ,  $\forall (t, x_1), (t, x_2) \in D_0$ .

$$u(t) \leq \left| \int\limits_{t_0}^t L \|\varphi(s) - \varphi_1(s)\| ds \right| \stackrel{\text{Lema B.G}}{=} \left| \int\limits_{t_0}^t L u(s) ds \right| \quad \forall t$$

$$\begin{cases} u(t) \leq 0, e^{-t} = 0, \forall t \\ u(t) \geq 0, \forall t \\ (\text{e normă}) \end{cases} \Rightarrow u(t) \equiv 0 \Rightarrow \text{q.e.d}$$

Obs:  $x^t = x^{2/3} + a, a \in \mathbb{R}$

Ecuatii diferențiale de ordin superior

Existența și unicitatea soluțiilor

Def a)  $f(\cdot, \cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  def. ec. dif. de

$$(1) \quad x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

b)  $\varphi(\cdot) : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  s.m. soluție a ec. dacă este de  $n$  derivabilă și  $\varphi^{(m)}(t) = f(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t))$ ,  $\forall t \in I$

Mai general,  $F(\cdot, \cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ ,

F

Metodă generală de studiu (sistemul canonnic asociat)

$$(1) \quad x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

$$(2) \quad \begin{cases} \frac{dx_1}{dt} = x_2 \\ \vdots \\ \frac{dx_{n-1}}{dt} = x_n \end{cases} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

$$\begin{pmatrix} \frac{dx_2}{dt} = x_3 \\ \vdots \\ \frac{dx_n}{dt} = x_1 \end{pmatrix} \quad f(t, (x_1, \dots, x_m)) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_1 \end{pmatrix}$$

$$\frac{dx_m}{dt} = x_1$$

$$\frac{dx_1}{dt} = f(t, x_1, \dots, x_m)$$

$$(2) \quad \frac{dX}{dt} = \tilde{f}(t, X), \quad \tilde{f}(\cdot, \cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Prop. (de echivalență)

$\varphi(\cdot) : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  e soluție a ec. (1)  $\Leftrightarrow$

$$\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot)) : I \rightarrow \mathbb{R}^n \text{ e sol a ec.}$$

Denum "  $\Rightarrow$ "

$$\varphi'(t) \equiv \varphi'(t)$$

$$\varphi''(t) \equiv \varphi''(t)$$

$$(\varphi^{(n-2)}(t))' \equiv \varphi^{(n-1)}(t) \equiv \varphi^{(n-1)}(t)$$

$$(\varphi^{(n-1)}(t))' \equiv f(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t)) = \varphi^{(n)}(t)$$

$\Rightarrow \varphi(\cdot)$  sol. a ec (1)

" $\Leftarrow$ " Fie  $\tilde{\varphi}(\varphi_1(\cdot), \varphi_2(\cdot), \dots, \varphi_m(\cdot))$  sol. a ec. (2)

$$\varphi_1(t) \equiv \varphi_2(t) \Rightarrow \varphi_2(t) = \varphi_1'(t)$$

$$\varphi_2'(t) \equiv \varphi_3(t) \Rightarrow \varphi_3(t) \equiv \varphi_2'(t) = \varphi_1''(t)$$

$$\varphi_m'(t) \equiv \varphi_m(t) \Rightarrow \varphi_m(t) \equiv \varphi_{m-1}'(t) = (\varphi_1^{(n-2)}(t))' = \varphi_1^{(n-1)}(t)$$

$$\varphi_m(t) \equiv f(t, \varphi_1(t), \dots, \varphi_{m-1}(t))$$

$$\varphi_1^{(n)}(t) = (\varphi_1^{(n-1)}(t))' \equiv f(t, \varphi_1(t), \varphi_1'(t), \dots, \varphi_1^{(n-1)}(t)),$$

$\varphi_1(\cdot)$  sol. a ec. (1)

### Problema Cauchy

Problema Cauchy pt. ec (2). Se dau

$$f(t, \cdot) \rightarrow \frac{dx}{dt} = \tilde{f}(t, x)$$

$(t_0, \tilde{x}_0) \in D$  - condiție inițială

Se caută  $\tilde{\varphi}(\cdot) : \tilde{t} \in \mathbb{R} \rightarrow \mathbb{R}^m$  sol. a ec. (2) cu  $\tilde{\varphi}(t_0) = \tilde{x}_0$ .

$$\tilde{x}_0 \in \mathbb{R}^m, \tilde{x}_0 = (\tilde{x}_0, \tilde{x}_0', \dots, \tilde{x}_0^{n-1})$$

Prop. de echivalență  $\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$  cu  $\varphi(\cdot)$  sol. a ec. (1) :  $x^{(n)} = f(t, \varphi, \varphi', \dots, \varphi^{(n-1)})$ .

$$\Rightarrow \varphi(\cdot) \text{ sol. a ec (1) cu } \varphi(t_0) = \tilde{x}_0, \varphi'(t_0) = \tilde{x}_0', \dots, \varphi^{(n-1)}(t_0) = \tilde{x}_0^{n-1}$$

### Problema Cauchy pt. ec (1)

Se dau  $\rightarrow f(t, \cdot, \cdot) \rightarrow \exists \overset{(n)}{x} = f(t, x, x', \dots x^{(n-1)})$

$\downarrow (t_0, x_0, x'_0, \dots x_0^{(n-1)}) \in D$  condiție initială

Se căută  $\varphi_0(\cdot) : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$  sol. a ec. cu  $\varphi(t_0) = x_0$ ,

$$\varphi'(t_0) = x'_0, \dots \varphi^{(n-1)}(t_0) = x_0^{(n-1)}$$

În acest caz spurnem că  $f(\cdot)$  este sol. a pb. Cauchy

$$(f, t_0, x_0, x'_0, \dots x_0^{(n-1)})$$

### I.T. Peano (pt. ec. de ordinul superior)

Fie  $f(t, \cdot, \cdot) : D = \bar{D} \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  cont. deg.  $\overset{(n)}{x} = f(t, x, x', \dots x^{(n-1)})$

Atunci  $f(t, \cdot, \cdot)$  admite ~~E.L~~ E.L pe  $D$  ( $\forall (t_0, (x_0, x'_0, \dots x_0^{(n-1)})$ )

$\exists \varphi(\cdot) : I_0 \in \mathcal{V}(t_0) \rightarrow \mathbb{R}^n$  soluție a ec.  $\varphi(t_0) = x_0, \varphi'(t_0) = x'_0$   
 $\varphi^{(n-1)}(t_0) = x_0^{(n-1)}$

Denum: Fie  $(t_0, (x_0, x'_0, \dots x_0^{(n-1)})) \in D, \overset{n}{x}_0 = (x_0, x'_0, \dots x_0^{(n-1)})$

$(t_0, \overset{n}{x}_0) \in D$

(2)  $\frac{d\overset{n}{x}}{dt} = \overset{n}{f}(t, \overset{n}{x}), \overset{n}{x} = (x_1, \dots x_m), \overset{n}{f}(t, (x_1, \dots x_m))$

$$= (x_2, x_3, \dots x_m, f(t, x_1, \dots x_m))$$

$(t_0, \overset{n}{x}_0) \in D = \bar{D}, \overset{n}{f}(\cdot, \cdot)$  cont. ( $f(\cdot, \cdot)$  cont.)  $\xrightarrow{\text{T.Peano}}$  pt. (2)

$t_0 \in \mathcal{V}(t_0) \rightarrow \mathbb{R}^n$  sol. a ec. (2) cu  $\varphi(t_0) = \overset{n}{x}_0$

### Prop. de echivalență

$\overset{n}{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$  în  $\varphi(\cdot)$  sol. a ec (1)

$$\overset{n}{\varphi}(t_0) = \overset{n}{x}_0 \Leftrightarrow \varphi(t_0) = x_0, \varphi'(t_0) = x'_0, \varphi^{(n-1)}(t_0) = x_0^{(n-1)}$$

g.e.d

## T. Cauchy-Lipschitz (pt. ec. de ordin superior)

Fie  $f(t, \cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  cont., local Lipschitz (I),

$$x^{(m)} = f(t, x, x', \dots, x^{(m-1)}) \quad (1)$$

Atunci  $\forall (t_0, (x_0, x'_0, \dots, x_0^{(n-1)})) \in D \exists I: I_0 \in \mathcal{V}(t_0) \rightarrow \mathbb{R}$

sol. cu  $\varphi(t_0) = x_0, \varphi'(t_0) = x'_0, \dots, \varphi^{(n-1)}(t_0) = x_0^{(n-1)}$

Dem: Fie  $(t_0, (x_0, x'_0, \dots, x_0^{(n-1)})) \in D$ ,

$$\tilde{x}_0 := (x_0, x'_0, \dots, x_0^{(n-1)})$$

$$(2) \frac{d\tilde{x}}{dt} = \tilde{f}(t, \tilde{x}), \quad \tilde{x} = (x_1, x_2, \dots, x_m), \quad \tilde{f}(t, (x_1, \dots, x_m)), = \\ (x_2, x_3, \dots, x_m, f(t, x_1, \dots, x_m))$$

$f(\cdot, \cdot)$  cont., local Lipschitz (I)  $\Rightarrow \tilde{f}(\cdot, \cdot)$  cont. local Lip (I)  
 $D = \overset{\circ}{D}$

$\Rightarrow$  T. Cauchy-Lipschitz pt. ec. (2)

$\Rightarrow \exists (t_0, \tilde{x}_0) \in D \exists I: I_0 \in \mathcal{V}(t_0) \rightarrow \mathbb{R}^n$  sol. a ec. (2),

$$\tilde{\varphi}(t_0) = \tilde{x}_0$$

Prop. de echivalență  $\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$  cu  $\varphi(\cdot)$  sol. a ec. (1)

$$\tilde{\varphi}'(t_0) = \tilde{x}_0' \Leftrightarrow \varphi(t_0) = x_0, \varphi'(t_0) = x'_0, \dots, \varphi^{(n-1)}(t_0) = x_0^{(n-1)} \rightarrow \text{q.e.d.}$$

## Continuitatea sol. locale în raport cu datele initiale și parametriu

$\frac{d\tilde{x}}{dt} = \tilde{f}(t, \tilde{x}), \quad \tilde{f}(\cdot, \cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  admite EUL pe  $D$

~~$\forall (t_0, \tilde{x}_0) \in D \exists I: \varphi_{t_0, \tilde{x}_0}(\cdot) : I_{t_0, \tilde{x}_0} \subseteq \mathcal{V}(t_0) \rightarrow \mathbb{R}^n$~~

tau csi

Def: S.m. curentul local asociat câmpului vectorial

$f(\cdot, \cdot)$  în  $(t_0, x_0) \in D$  funcția  $\alpha(\cdot, \cdot, \cdot): I_1 \times I_0 \times G_0 \in$

$\mathcal{V}(t_0, t_0, x_0) \rightarrow \mathbb{R}^n$  cu prop. că  $\forall (\bar{t}, \bar{x}) \in I_0 \times G_0$

$\alpha(\cdot, \bar{t}, \bar{x})$  sol a pb Cauchy ( $f, \bar{t}, \bar{x}$ )

Def: s.m. câmp vectorial parametrizat  $f(\cdot, \cdot, \cdot, \cdot): D \subseteq \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^k \rightarrow \mathbb{R}^n$  a.t.  $\forall \lambda \in \text{pr}_3 D$   $f(\cdot, \cdot, \lambda)$  c.v.

a treia componentă

$$\text{def } \frac{dx}{dt} = f(t, x, \lambda)$$

→ familie parametrizată de ec. diferențiale

Def: s.m. curent local parametrizat asociat c.v.p  $f(\cdot, \cdot, \cdot)$

$(t_0, x_0, \lambda_0) \in D$  funcția  $\alpha(\cdot, \cdot, \cdot, \cdot, \cdot): I_1 \times I_0 \times G_0 \times \Lambda_0 \rightarrow$

$\in \mathcal{V}(t_0, t_0, x_0, \lambda_0)$  a.t.  $\forall (\bar{t}, \bar{x}, \bar{\lambda}) \in I_0 \times G_0 \times \Lambda_0$

$\alpha(\cdot, \bar{t}, \bar{x}, \bar{\lambda}): I_0 \rightarrow \mathbb{R}^n$  e sol. Cauchy  $f(\cdot, \cdot, \bar{\lambda}), \bar{t}, \bar{x}$

T. (E.U și continuitatea curentului local parametrizat)

Fie  $f(\cdot, \cdot, \cdot, \cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^k \rightarrow \mathbb{R}^n$  cont., local lipschitz

(I)  $\frac{dx}{dt} = f(t, x, \lambda)$ . Atunci  $\forall (t_0, x_0, \lambda_0) \in D \exists !$

$\alpha(\cdot, \cdot, \cdot, \cdot, \cdot): I_1 \times I_0 \times G_0 \times \Lambda_0 \in \mathcal{V}(t_0, t_0, x_0, \lambda_0) \rightarrow \mathbb{R}^n$   
continuă. Curent local parametrizat

Demn (Schită): Fie  $(t_0, x_0, \lambda_0) \in D$ .

1. E:  $= I_1 \times I_0 \times G_0 \times \Lambda_0 = ?$

( $t_0 \sim \bar{t} \sim \tau \sim \cdots$ )

$\exists \delta, \gamma, \eta > 0 \quad \overline{B}_\delta(t_0) \times \overline{B}_\gamma(x_0) \times \overline{B}_\eta(\lambda_0) \subset D$

$\therefore D_0$  compactă

$$K := \max_{(t, x, \lambda) \in D_0} \|f(t, x, \lambda)\|$$

$$(t, x, \lambda) \in D_0$$

$$K = 0 \Rightarrow d(t, \delta, \gamma, \lambda) = \gamma$$

$$K > 0 \quad a := \min \left\{ \delta, \frac{\gamma}{4K} \right\} \quad R := \frac{\gamma}{2}$$

$$E = \overline{B}_a(t_0) \times \overline{B}_\gamma(t_0) \times \overline{B}_R(x_0) \times \overline{B}_\eta(\lambda_0)$$

$f(\cdot, \cdot, \cdot)$  local Lipschitz ( $\underline{T}$ )  
 $D_0$  - compactă

$$\|f(t, x_0, \lambda) - f(t, x_2, \lambda)\| \leq L \|x_1 - x_2\|, \forall (t, x_1, \lambda), (t, x_2, \lambda) \in D_0$$

2.  $d_m(\cdot, \cdot, \cdot, \cdot) : E \rightarrow \mathbb{R}^n$  sirul aproximărilor successive  
 ale lui Picard

$$d_0(t, \delta, \gamma, \lambda) = \gamma$$

$$d_m(t, \delta, \gamma, \lambda) = \gamma + \int_0^t f(s, d_m(s, \delta, \gamma, \lambda), \lambda) ds,$$

$$m \geq 1$$

a)  $d_m(\cdot, \cdot, \cdot, \cdot)$  continuă și m (ind. după m)

b)  $d_m(t, \delta, \gamma, \lambda) \in \overline{B}_\gamma(x_0), \forall m, \forall (t, \delta, \gamma, \lambda) \in E$

c)  $d_m(\cdot, \cdot, \cdot, \cdot)$  și uniform Cauchy  $\Rightarrow d_m(\cdot, \cdot, \cdot, \cdot) \xrightarrow[m \rightarrow \infty]{\text{u}} d(\cdot, \cdot, \cdot, \cdot)$

$d(\cdot, \cdot, \cdot, \cdot) : E \rightarrow \mathbb{R}^n$  cont

3.  $d(\cdot, \cdot, \cdot, \cdot)$  current local param.

4. unicitatea ( $\Leftrightarrow$  dem T. Cauchy-Lipschitz)

## Seminar 4

Ec. de ordin superior care admit reduserea ordinului

1)  $F(t, x^{(k)}, x^{(k+1)}, \dots, x^{(n)}) = 0, k \geq 1 \rightarrow$  Nu apare  $x$ , doar derivatele sale.

S.V.  $y = x^{(k)} \rightarrow F(t, y, y', \dots, y^{(n-k)}) = 0$

2)  $F(t, \frac{x'}{x}, \frac{x''}{x}, \dots, \frac{x^{(n)}}{x}) = 0 \rightarrow$  Ec. omogenă

S.V.  $y = \frac{x'}{x} \Rightarrow G(t, y, y', \dots, y^{(n-1)}) = 0$

3)  $F(x, x', \dots, x^{(n)}) = 0$  (Autonomă)

S.V.  $x' = y(x) \Rightarrow G(x, y, y', \dots, y^{(n-1)}) = 0$

4) Ec. Euler

$F(x, t \cdot x', t^2 x'', \dots, t^n x^{(n)}) = 0$

S.V.  $|t| = e^{\frac{s}{n}} \Rightarrow G(y, y', \dots, y^{(n)}) = 0$ .

Ecuații liniare de ordinul al doilea cu coezi constanți

$$x'' + a \cdot x' + b = 0, a, b \in \mathbb{R}$$

$$\text{Ec. caracteristică: } \lambda^2 + a \cdot \lambda + b = 0 < \begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}$$

Dacă  $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 \neq \lambda_2 \Rightarrow$  sol generală este

$$x(t) = c_1 \cdot e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t}, c_1, c_2 \in \mathbb{R}$$

$$\text{Dacă } \lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \Rightarrow x(t) = (c_1 \cdot e^{\lambda t} + c_2 \cdot t \cdot e^{\lambda t})$$

$$\text{Dacă } \lambda_1 = \lambda_2 = \alpha + i \cdot \beta$$

$$\lambda_2 = \alpha - i \cdot \beta, \beta > 0$$

sol. generală

$$\Rightarrow x(t) = c_1 \cdot e^{\alpha t} \cos \beta t + c_2 \cdot e^{\alpha t} \sin \beta t$$

Ex Să se determine sol. generală

$$(1) x''' + x'' = 0$$

$$SV: y = \mathfrak{X}'' \Rightarrow y' = \mathfrak{X}''$$

$$\mathfrak{X}'' = c \cdot e^{-t}$$

$$\mathfrak{X}' + y = 0 \Rightarrow \mathfrak{X}' = -y \Rightarrow y(t) = c \cdot e^{-t}, c \in \mathbb{R}$$

$$\mathfrak{X}'(t) = \int c \cdot e^{-t} dt = -c \cdot e^{-t} + c_1, c_1 \in \mathbb{R}$$

$$\mathfrak{X}(t) = \int (-c \cdot e^{-t} + c_1) dt = c \cdot e^{-t} + c_1 \cdot t + c_2, \\ c, c_1, c_2 \in \mathbb{R}$$

b)  $t \cdot \mathfrak{X}'' + \mathfrak{X}' + t = 0$

$\mathfrak{X}(t) \equiv 0$  verifică ec.

c)  $t \cdot \mathfrak{X} \cdot \mathfrak{X}'' + t \cdot (\mathfrak{X}')^2 - 2c \cdot \mathfrak{X}' = 0$ . |  $\mathfrak{X} \neq 0$ , căutăm sol. care sunt diferențe

$$\frac{t \cdot \mathfrak{X} \cdot \mathfrak{X}''}{\mathfrak{X}^2} + \frac{t \cdot (\mathfrak{X}')^2}{\mathfrak{X}^2} - \frac{\mathfrak{X}' \cdot \mathfrak{X}'}{\mathfrak{X}} = 0$$

$$\frac{t \mathfrak{X}''}{\mathfrak{X}} + t \left( \frac{\mathfrak{X}'}{\mathfrak{X}} \right)^2 - \frac{\mathfrak{X}'}{\mathfrak{X}} = 0 \text{ ec. omogenă}$$

$$y = \frac{\mathfrak{X}'}{\mathfrak{X}} \Rightarrow y(t) = \frac{\mathfrak{X}'(t)}{\mathfrak{X}(t)}$$

$$\mathfrak{X}'(t) = y(t) \cdot \mathfrak{X}(t)$$

$$\mathfrak{X}''(t) = y'(t) \cdot \mathfrak{X}(t) + y(t) \cdot \mathfrak{X}'(t)$$

$$\Leftrightarrow \frac{\mathfrak{X}''(t)}{\mathfrak{X}(t)} = y'(t) + \frac{(\mathfrak{X}'(t))^2}{\mathfrak{X}^2(t)} \Leftrightarrow \frac{\mathfrak{X}''(t)}{\mathfrak{X}(t)} = y'(t) + y^2(t)$$

$$t(y' + y^2) + t \cdot y^2 - y = 0$$

$$y' = \frac{-2ty^2 + y}{t}, t \neq 0$$

$$\frac{dy}{dt} = \frac{y}{t} - 2y^2$$

$$Ec. \limică \text{arc. } \bar{y} = \frac{y}{t} \Rightarrow \bar{y}(t) = c \cdot e^{\int \frac{1}{t} dt} = c \cdot t$$

"Variatia constantelor"

$$(c(t) \cdot t)' = \frac{c(t) \cdot t'}{t} - 2c^2(t) \cdot t^2$$

$$c(t) \cdot t + c(t) = c(t) - 2c^2(t) \cdot t^2$$

$$c'(t) = -2c^2(t) \rightarrow \text{ec. Var. Separabile.}$$

$$\frac{dc}{dt} = -2c^2 \cdot t \Rightarrow c^2 = 0 \Rightarrow c = 0 \Rightarrow c(t) \equiv 0$$

~~soluție~~

Sol. staționară

$$-\frac{dc}{c^2} = 2t dt \Rightarrow \int -\frac{dc}{c^2} = \int 2t dt \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{c} = t^2 + k, k \in \mathbb{R} \Rightarrow c(t) = \frac{1}{t^2 + k}, k \in \mathbb{R}.$$

~~soluție~~

$$y_0(t) = 0$$

$$y_k(t) = \frac{t}{t^2 + k}, k \in \mathbb{R}$$

$$y = \frac{x}{x} \Rightarrow \frac{dx}{x} = 0 \Rightarrow x' = 0 \Rightarrow x_0(t) = c_1, c_1 \in \mathbb{R}$$

$$\frac{x'}{x} = \frac{t}{t^2 + k} \Rightarrow x' = \frac{t}{t^2 + k} \cdot x. \rightarrow \text{ec. liniară în sol}$$

$$x(t) = c_2 \cdot e^{\int \frac{t}{t^2 + k} dt} = c_2 \cdot e^{\frac{1}{2} \int \frac{2t}{t^2 + k} dt} =$$

$$= c_2 e^{\frac{1}{2} \ln |t^2 + k|} = c_2 \sqrt{t^2 + k}, c_1, c_2 \in \mathbb{R}.$$

$$d) t^2 x x'' - (x - t \cdot x')^2 = 0$$

$$e) x \cdot x'' + 1 = (x')^2$$

$$x' = y(x)$$

Se caută o fct. y a.i.  $x'(t) = y(x(t))$ ,  $\forall t \in$  domeniul lui s

$$x''(t) = y'(x(t)) \cdot x'(t) = y'(x(t)) \cdot y(x(t))$$

$$x \cdot y'(x) \cdot y(x) + 1 = y^2(x)$$

$$y'(x) = (y^2(x) - 1) \cdot \frac{1}{x \cdot y(x)}$$

$$\frac{dy}{dx} = \frac{y^2 - 1}{xy} \Rightarrow \frac{y^2 - 1}{y} = 0 \Leftrightarrow y^2 = 1 \Rightarrow y = \pm 1$$

$$y_1(x) \equiv 1, y_2(x) \equiv -1$$

$$\frac{y}{y^2 - 1} dy = \frac{dx}{x} \Leftrightarrow \frac{1}{2} \int \frac{2y}{y^2 - 1} dy = \int \frac{dx}{x} \Leftrightarrow$$

$$\frac{1}{2} \ln|y^2 - 1| = \ln|x| + C, C \in \mathbb{R}$$

$$\Leftrightarrow \ln k, k \in \mathbb{R}^*$$

$$\Leftrightarrow \sqrt{|y^2 - 1|} = k \cdot |x|, k \in \mathbb{R}^*$$

$$\Leftrightarrow |y^2 - 1| = k^2 \cdot x^2 \Rightarrow y^2 - 1 = c_1 x^2, c_1 \in \mathbb{R}^*$$

$$\Rightarrow y(x) = \pm \sqrt{c_1 x^2 + 1}, c_1 \in \mathbb{R}^*$$

$$x' = 1 \Rightarrow x_1(t) \equiv t + c_2, c_2 \in \mathbb{R}$$

$$x' = -1 \Rightarrow x_2(t) \equiv -t + c_3, c_3 \in \mathbb{R}$$

$$x' = \pm \sqrt{c_1 x^2 + 1} \rightarrow \text{Ec. var. Separabile}$$

g)  $x \cdot x'' + (x')^2 = 0$

g)  $t^2 \cdot x'' - 4t x' + 6x = 0$

Ec. Euler

S.V:  $|t| = e^s \Rightarrow t = e^s, t > 0$

$$t = -e^s, t < 0$$

P.p.  $t > 0 : t = e^s$

$$t^2 \cdot x''(t) - 4t x'(t) + 6x(t) = 0$$

$$e^{2s} x''(e^s) - 4e^s x'(e^s) + 6x(e^s) = 0$$

$$y(s) = \alpha(e^s) \Leftrightarrow x(t) = y(\ln t)$$

$x(t) \neq x(e^s) \rightarrow \text{asymptote}$

$$e^{2s} x''(e^s) - 4e^2 x'(e^s) + 6x(e^s) = 0.$$

$$y'(s) = x'(e^s) \cdot e^s \Rightarrow x'(e^s) = \frac{y'(s)}{e^s} = y'(s) \cdot e^{-s}$$

$$x''(e^s) = y''(s) \cdot e^{-s} - y'(s) \cdot e^{-s}$$

$$x''(e^s) = y''(s) \cdot e^{-2s} - y'(s) \cdot e^{-2s}$$

$$e^{2s}(y''(s) \cdot e^{-2s} - y'(s) \cdot e^{-2s}) - 4 \cdot e^s \cdot y'(s) \cdot e^{-s} + 6 \cdot y(s) =$$

$$y''(s) - 5y'(s) + 6y(s) = 0.$$

$$\lambda^2 - 5\lambda + 6 = 0.$$

$$\Delta = 25 - 24 = 1.$$

$$\lambda_1 = \frac{5-1}{2} = 2, \quad \lambda_2 = \frac{5+1}{2} = 3.$$

$$y(t) = C_1 \cdot e^{2t} + C_2 \cdot e^{3t}, \quad C_1, C_2 \in \mathbb{R}.$$

$$x(t) = y(\ln t) \Rightarrow x(t) = C_1 \cdot e^{2\ln t} + C_2 \cdot e^{3\ln t}$$

$$\Rightarrow x(t) = C_1 \cdot t^2 + C_2 \cdot t^3, \quad C_1, C_2 \in \mathbb{R}$$

h)  $t^2 x'' + t \cdot x' + x = 0.$   
Ec. Euler

$$|t| = e^s, \quad \text{pp. } t > 0$$

$$\ln t = -e^s$$

$$t^2 \cdot x''(t) + t \cdot x'(t) + x(t) = 0.$$

$$e^{2s} \cdot x''(t) - e^s x'(t) + x(t) = 0.$$

$$\Rightarrow \mathfrak{X}'(-e^s) = \frac{-y'(s)}{e^s} \Rightarrow \mathfrak{X}'(-e^s) = -e^{-s} \cdot y'(s)$$

$$\mathfrak{X}''(-e^s) = (-e^{-s} \cdot y'(s))'$$

$$\Leftrightarrow \mathfrak{X}''(-e^s) = (-e^{-s} \cdot y''(s)) + (e^{-s} \cdot y'(s)) \cdot e^{-s}$$

$$\Leftrightarrow \mathfrak{X}'''(s) = -y''(s) \cdot e^{-2s} + y'(s) \cdot e^{-2s}$$

$$e^{2s} \cdot (-y''(s) \cdot e^{-2s} + y'(s) \cdot e^{-2s}) - e^s \cdot (-e^{-s} \cdot y'(s)) + y(s) = 0$$

$$\Leftrightarrow -y''(s) + y'(s) + y(s) = 0$$

$$y^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

~~Wurzeln ausrechnen~~

$$\Rightarrow y(s) = (C_1 \cdot e^{0 \cdot s} \cos 1 \cdot s + C_2 \cdot e^{0 \cdot s} \sin 1 \cdot s)$$

$$y(s) = C_1 s + C_2 \sin s, C_1, C_2 \in \mathbb{R}$$

$$\mathfrak{X}(t) = C_1 \cos(\theta \ln t) + C_2 \sin(\theta \ln t), C_1, C_2 \in \mathbb{R}$$

$$i) t^2 \mathfrak{X}'' + 5t \cdot \mathfrak{X}' - 5\mathfrak{X} = 0$$

$$j) t^2 \mathfrak{X}'' - t \cdot \mathfrak{X}' - 2\mathfrak{X} = 0$$