Ecuații diferențiale

curs 8: 21.11.2017.

Ecuații liniare pe $\mathbb{R}^n$ cu coeficienți constanți

Def: Fie matricea $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\frac{dx}{dt} = Ax$. În coordonate avem:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j \quad , i = \overline{1,n}$$ sistem de ecuații liniare cu coeficienți constanți.

Caz particular $A(t) \equiv A \in L(\mathbb{R}^n, \mathbb{R}^n)$.

Teoremă (E.U.G): $\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, $\exists! \varphi(\cdot): \mathbb{R} \to \mathbb{R}^n$ soluție cu $\varphi(t_0) = x_0$.

$S_A := \{ \varphi(\cdot): \mathbb{R} \to \mathbb{R}^n; \varphi(\cdot) \text{ soluție cu } x' = Ax \}$

$S_A \subset C^1(\mathbb{R}, \mathbb{R}^n)$ subspațiu vectorial cu $\dim(S_A) = n$.

Obs: $\varphi(\cdot)$ soluție $\Rightarrow \varphi'(t) \equiv A\varphi(t) \Rightarrow \varphi''(t) \equiv A\varphi'(t) \equiv A^2 \varphi(t) = \dots \Rightarrow$

$\Rightarrow \varphi^{(n)}(t) \equiv A^n \varphi(t)$, $\forall n \in \mathbb{N}$.

Dacă $\varphi(\cdot)$ analitică $\Rightarrow \varphi(t) = \sum_{k\geq 0} \frac{t^k}{k!} \varphi^{(k)}(0) \equiv \sum_{k \geq 0} \frac{t^k}{k!} A^k \varphi(0) = \sum_{k\geq 0} \frac{(tA)^k}{k!} \varphi(0)$

Exponențiala unei aplicații liniare

Prop: $\forall A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\exists \sum_{k\geq 0} \frac{A^k}{k!} = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{A^k}{k!} =: \exp(A) \ (= e^A)$

Dem: (schiță)

Inelele $L(\mathbb{R}^n, \mathbb{R}^n) \simeq M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$

$\sum_{k\geq 0} \| u_k \| < \infty \Rightarrow \sum_{k\geq 0} u_k$ convergentă

$u_k = \frac{A^k}{k!}$

$\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$; $\|AB\| \leq \|A\| \cdot \|B\| \Rightarrow \|A^k\| \leq \|A\|^k, \forall k$

$\sum_{k\geq 0} \left\| \frac{A^k}{k!} \right\| \leq \sum_{k \geq 0} \frac{\|A\|^k}{k!} = e^{\|A\|}$

Prop: ⓐ $\exp(0) = I_n$

ⓑ $AB = BA \Rightarrow \exp(A+B) = \exp(A)\cdot\exp(B)$

ⓒ $\forall A \in L(\mathbb{R}^n, \mathbb{R}^n), \exists (\exp(A))^{-1} = \exp(-A)$

Prop (Legătura cu ecuațiile exponențiale):

$A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $t \to \exp(tA)$ este derivabilă $(\exp(tA))' = A\exp(tA)$

$(\equiv (\exp(tA))A)$

Dem:

$$u(t) = \sum_{k\geq 0} u_k(t)$$

$\sum_{k\geq 0} u_k'(t)$ este uniform convergentă pe orice compactă $\Rightarrow \exists\ u'(t) = \sum$

$$\exp(tA) = \sum_{k\geq 0} \frac{(tA)^k}{k!}, \quad u_k(t) = \frac{t^k A^k}{k!}$$

$$\sum_{k\geq 1} \frac{k t^{k-1} A^k}{k!} = \sum_{k\geq 1} \frac{t^{k-1} A^k}{(k-1)!} \equiv A \sum_{k\geq 1} \frac{t^{k-1} A^{k-1}}{(k-1)!} = A\exp(tA)$$

Corolar: $\varphi(\cdot) \in S_A \Leftrightarrow \exists\ x_0 \in \mathbb{R}^n$ astfel încât $\varphi(t) = (\exp(tA))x_0$

Dem: „$\Leftarrow$" Evident din propoziția anterioară

„$\Rightarrow$" Fie $x_0 := \varphi(0)$

Fie $\psi(t) = (\exp(tA))x_0$

$\left.\begin{array}{l}\varphi(\cdot) \text{ soluție}, \ \varphi(0) = x_0 \\ \psi(\cdot) \text{ soluție}, \ \psi(0) = x_0\end{array}\right\} \overset{U.E.}{\Longrightarrow} \varphi(t) \equiv \psi(t)$

Valori proprii. Vectori proprii. Soluții la ecuațiilor liniare pe $\mathbb{R}^n$ cu coeficienți constanți

Def: Fie $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma(A) = \{\lambda \in \mathbb{C};\ \det(A - \lambda I_n) = 0\}$, $\lambda \in \sigma(A)$ se numește valoare proprie a lui A.

$VP_A(\lambda) = \{u \in \mathbb{C}^n \setminus \{0\};\ (A - \lambda I_n)u = 0\}$, $u \in VP_A(\lambda)$ se numește vector

Prop 1: $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

$$\lambda \in \sigma(A) \cap \mathbb{R} \Rightarrow VP_A(\lambda) \cap \mathbb{R}^n \neq \phi$$

Dem:

$$u = v + iw \in VP_A(\lambda) \Rightarrow (A - \lambda I_n)\cdot u = 0 \ , (A - \lambda I_n)(v + iw) = 0$$

$$(A - \lambda I_n)v + i(A - \lambda I_n)w = 0 \Rightarrow \left.\begin{array}{l}(A - \lambda I_n)v = 0 \\ (A - \lambda I_n)w = 0\end{array}\right\} \Rightarrow v \text{ și } w \text{ nu pot fi simultan egale cu } 0 \ (\Rightarrow u = 0, \text{ⅹ})$$

De exemplu: $v \neq 0 \Rightarrow v \in VP_A(\lambda) \cap \mathbb{R}^n$

Prop 2: $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

$$\left.\begin{array}{l}\lambda = \alpha + i\beta \in \sigma(A) \\ u = v + iw \in VP_A(\lambda)\end{array}\right\} \Rightarrow \begin{array}{l}\bar{\lambda} = \alpha - i\beta \in \sigma(A) \\ \bar{u} = v - iw \in VP_A(\bar{\lambda})\end{array}$$

Dem:

$$\lambda \in \sigma(A) \Rightarrow \det(A - \lambda I_n) = 0 \Rightarrow 0 = \overline{\det(A - \lambda I_n)} = \det\overline{(A - \lambda I_n)} = $$
$$= \det(A - \bar{\lambda} I_n)$$

$$u \in VP_A(\lambda) \quad (A - \lambda I_n)\cdot u = 0 \Rightarrow 0 = \overline{(A - \lambda I_n)u} = \overline{(A - \lambda I_n)}\cdot \bar{u} = (A - \bar{\lambda} I_n)\cdot \bar{u}$$

Prop 3: (legătura cu ecuațiile diferențiale)

Fie $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ ce definește ecuația $\frac{dx}{dt} = Ax$.

① Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$ și $u_\lambda \in VP_A(\lambda) \cap \mathbb{R}^n \Rightarrow \varphi_\lambda(t) = e^{\lambda t}\cdot u_\lambda$, $\varphi_\lambda(t) \in S_A$

② Dacă $\lambda \in \sigma(A)$, $u_\lambda \in VP_A(\lambda) \Rightarrow \begin{array}{l}\varphi_\lambda(t) = Re(e^{\lambda t}\cdot u_\lambda) \\ \varphi_{\bar{\lambda}}(t) = Im(e^{\lambda t}\cdot u_\lambda)\end{array}$, $\varphi_\lambda(\cdot) \cdot \varphi_{\bar{\lambda}}(\cdot) \in S_A$

Dem:

① $\varphi_\lambda(t) = e^{\lambda t}\cdot u_\lambda$

$$(\varphi_\lambda(t))' = \lambda e^{\lambda t}\cdot u_\lambda = e^{\lambda t}\underline{\underline{\lambda u_\lambda}} = e^{\lambda t} A u_\lambda = A e^{\lambda t} u_\lambda = A\varphi_\lambda(t)$$

$$u_\lambda \in VP_A(\lambda) \ , \ (A - \lambda I_n)u_\lambda = 0 \Rightarrow \underline{\underline{Au_\lambda = \lambda u_\lambda}}$$

② Fie $\tilde{\varphi}_\lambda(t) := e^{\lambda t}\cdot u_\lambda$ (la fel ca la 1.) $\tilde{\varphi}_\lambda'(t) = A\tilde{\varphi}_\lambda(t)$

$= e^{\alpha t}\left(v\cos\beta t - w\, i \sin\beta t + i\left(v \sin\beta t + w\cos\beta t\right)\right) =$

$= e^{\alpha t}\left(v\cos\beta t - w\sin\beta t\right) + i e^{\alpha t}\left(v\sin\beta t + w\cos\beta t\right)$

Fie $\varphi_\lambda(t) := \mathrm{Re}\left(\tilde{\varphi}_\lambda(t)\right) = e^{\alpha t}\left(v\cos\beta t - w\sin\beta t\right)$

$\varphi_{\bar\lambda}(t) := \mathrm{Im}\left(\tilde{\varphi}_\lambda(t)\right) = e^{\alpha t}\left(v\sin\beta t - w\sin\beta t\right)$

$$\left.\begin{array}{l}\tilde{\varphi}_\lambda(t) = \varphi_\lambda(t) + i\,\varphi_{\bar\lambda}(t)\\ \tilde{\varphi}'_\lambda(t) = A\tilde{\varphi}_\lambda(t)\end{array}\right\} \Rightarrow \varphi'_\lambda(t) + i\,\varphi'_{\bar\lambda}(t) = A\varphi_\lambda(t) + iA\varphi_{\bar\lambda}(t) \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l}\varphi'_\lambda(t) = A\varphi_\lambda(t)\\ \varphi'_{\bar\lambda}(t) = A\varphi_{\bar\lambda}(t)\end{array}\right\} \Rightarrow \varphi_\lambda(\cdot), \varphi_{\bar\lambda}(\cdot) \in S_A$$

Prop 4: (vectori proprii liniari dependenți): $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

① $\left.\begin{array}{l}\lambda_1, \dots, \lambda_k \in \sigma(A) \text{ și } u_j \in VP_A(\lambda_j),\ j \in \overline{1,k}\\ \lambda_j \neq \lambda_p,\ \forall j \neq p\end{array}\right\} \Rightarrow \{u_1, \dots, u_k\} \subset \mathbb{C}^k$ liniar independenți

② $\left.\begin{array}{l}\lambda_1, \dots, \lambda_m \in \sigma(A) \cap \mathbb{R} \text{ și } u_j \in VP_A(\lambda_j) \cap \mathbb{R}^n,\ j = \overline{1,m}\\ \lambda_j = \alpha_j + i\beta_j \in \sigma(A)\ \ j = \overline{m+1,k},\ u_j = v_j + i w_j \in VP_A(\lambda_j),\ j = \overline{m+1,k}\end{array}\right\} \Rightarrow$

$\Rightarrow \{u_1, \dots, u_m, u_{m+1}, \dots, u_k\} \subset \mathbb{R}^n$ liniar independenți

Dem: ① Inducție după k.

$k = 1$, $u_1 \in VP_A(\lambda_1) \Rightarrow u_1 \neq 0$ ok.

$k \mapsto k+1$, $A \,|\, c_1 u_1 + \dots + c_k u_k + c_{k+1} u_{k+1} = 0 \Rightarrow c_j = 0, \forall j = \overline{1, k+1}$

$$c_1 A u_1 + \dots + c_k \underbrace{A u_k}_{\lambda_k u_k} + c_{k+1} \underbrace{A u_{k+1}}_{\lambda_{k+1} u_{k+1}} = 0$$

$$c_1 \lambda_1 u_1 + \dots + c_k \lambda_k u_k + c_{k+1} \lambda_{k+1} u_{k+1} = 0$$

$$c_1 \lambda_{k+1} u_1 + \dots + c_k \lambda_{k+1} u_k + c_{k+1} \lambda_{k+1} u_{k+1} = 0$$

$c_1(\lambda_1 - \lambda_{k+1}) u_1 + \dots + c_k(\lambda_k - \lambda_{k+1}) u_k = 0 \Rightarrow$ ipoteza de inducție $\Rightarrow$

$\Rightarrow c_j(\lambda_j - \lambda_{k+1}) = 0,\ j = \overline{1,k} \Rightarrow c_j = 0,\ j = \overline{1,k} \Rightarrow c_{k+1} = 0$

②. $$\sum_{j=1}^{m} c_j u_j + \sum_{j=m+1}^{k} (c_j v_j + k_j w_j) = 0 \overset{?}{\Rightarrow} c_j = 0,\ k_j = 0$$

$j = \overline{m+1, k}$ $\begin{cases} u_j = v_j + i w_j \\ \bar{u}_j = v_j - i w_j \end{cases} \Rightarrow \begin{matrix} v_j = \frac{1}{2}(u_j + \bar{u}_j) \\ w_j = \frac{1}{2i}(u_j - \bar{u}_j) \end{matrix} \Rightarrow$

$$\Rightarrow \sum_{j=1}^{m} c_j u_j + \sum_{j=m+1}^{k} \left[\frac{c_j}{2}(u_j + \bar{u}_j) + \frac{k_j}{2i}(u_j - \bar{u}_j) \right] = 0$$

$$\sum_{j=1}^{m} c_j u_j + \sum_{j=m+1}^{k} \left[\left(\frac{c_j}{2} + \frac{k_j}{2i} \right) u_j + \left(\frac{c_j}{2} - \frac{k_j}{2} \right) \bar{u}_j \right] = 0 \overset{1)}{\Rightarrow}$$

$$\Rightarrow c_j = 0, j = \overline{1, m} \quad \begin{cases} \frac{c_j}{2} + \frac{k_j}{2i} = c \\ \frac{c_j}{2} - \frac{k_j}{2i} \end{cases}, j = \overline{m+1, k} \Rightarrow \begin{matrix} c_j = 0 \\ k_j = 0 \end{matrix}, \ j = \overline{m+1, k}$$

Teoremă (Structura soluțiilor în cazul valorilor proprii simple):

$A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\frac{dx}{dt} = Ax$. Presupunem $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ distincte. Definim funcția, pentru $\lambda \in \sigma(A)$, $\varphi_\lambda(t) = \begin{cases} e^{\lambda t} \cdot u_\lambda, \text{ dacă } \lambda \in \sigma(A) \cap \mathbb{R}, u_\lambda \in VP_A(\lambda) \cap \mathbb{R}^n \\ \left.\begin{matrix} Re(e^{\lambda t} \cdot u_t) \\ Im(e^{\lambda t} u_t) \end{matrix}\right\}, \text{ dacă } \lambda = \alpha + i\beta \in \sigma(A), \beta > 0, u_\lambda \in VP_A(\lambda) \end{cases}$

Atunci $\{\varphi_\lambda(\cdot)\}_{\lambda \in \sigma(A)} \subset S_A$ sistem fundamental de soluții.

Dem:

$\varphi_\lambda(\cdot) \in S_A$, $\forall \lambda \in \sigma(A)$ (Prop. 3)

n = soluții $\Rightarrow$ este suficient să verificăm $\{\varphi_\lambda(\cdot)\}_{\lambda \in \sigma(A)} \subset S_A$ liniar indepen.

$\Updownarrow$ Prop (Soluții liniar independente)

$= \{\varphi_\lambda(0)\}_{\lambda \in \sigma(A)} \subset \mathbb{R}^n$ liniar independente

$= \{u_\lambda ; \lambda \in \sigma(A) \cap \mathbb{R}, u_\lambda \in VP_A(\lambda) \cap \mathbb{R}^n\} \cup \{Re(u_\lambda), Im(u_\lambda) ; \lambda = \alpha + i\beta \in \sigma(A), u_\lambda \in VP_A(\lambda)\} \subset \mathbb{R}^n$ liniar independent

Algoritm: (cazul valorilor proprii simple) $\frac{dx}{dt} = Ax$

① Rezolvă ecuația caracteristică: $\det(A - \lambda I_n) = 0 \Rightarrow \sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ distincte

② Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$ caută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$ astfel încât $(A - \lambda I_n) u_\lambda = 0$. Scrie $\varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$

③ Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$ caută $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ astfel încât $(A - \lambda I_n) u_\lambda = 0$. Scrie soluțiile: $\varphi_\lambda(t) = Re(e^{\lambda t} \cdot u_\lambda)$

$$\varphi_{\bar\lambda}(t) = Im(e^{\lambda t} \cdot u_\lambda)$$

④ Renumerotează $\{\varphi_\lambda(\cdot)\}_{\lambda \in \sigma(A)} = \{\varphi_1(\cdot), \dots, \varphi_\lambda(\cdot)\} \subset S_A$ sistem fundamental de soluții. Scrie soluția generală $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in R$, $i = \overline{1,n}$

Ecuații diferențiale

seminar 8 : 21. 11. 2017.

Algoritm (cazul valorilor proprii simple): $\frac{dx}{dt} = Ax$, $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

1. Rezolvă ecuația caract. $\det(A - \lambda I_n) = 0 \Rightarrow \sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ distincte
2. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$ caută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$ astfel încât $(A - \lambda I_n) \cdot u_\lambda = 0$
   Se scrie soluția $\varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$
3. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$ caută $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ astfel încât $(A - \lambda I_n) u_\lambda = 0$.
   Se scriu soluțiile: $\varphi_\lambda(t) = Re(e^{\lambda t} \cdot u_\lambda)$
   $\varphi_{\bar{\lambda}}(t) = Im(e^{\lambda t} \cdot u_\lambda)$
4. Renumerotează $\{\varphi_\lambda(\cdot)\}_{\lambda \in \sigma(A)} = \{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$ sistem fundamental de soluții. Se scrie soluția generală $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in \mathbb{R}$, $i = \overline{1,n}$

Exercițiu: Să se determine soluția generală:

1) $$\begin{cases} x' = x + z - y \\ y' = x + y - z \\ z' = 2x - y \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\det\left[\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = 0$$

$$\det\begin{pmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{pmatrix} = -\lambda(1-\lambda)^2 - 1 + 2 - 2(1-\lambda) - (1-\lambda) - \lambda =$$

$$= -\lambda(1 - 2\lambda + \lambda^2) \cancel{+1} - 2 + 2\lambda \cancel{-1} \cancel{+\lambda} \cancel{-\lambda}$$

$$= -\lambda + 2\lambda^2 - \lambda^3 - 2 + 2\lambda = -\lambda^3 + 2\lambda^2 + \lambda - 2$$

$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$

$$-\lambda^3+\lambda^2+\lambda^2-\lambda+2\lambda-2=-\lambda^2(\lambda-1)+\lambda(\lambda-1)+2(\lambda-1)=$$

$$=(\lambda-1)(\lambda^2+\lambda-2)=0$$

$$\left.\begin{array}{l}\lambda_1=1\\ \Delta=1+4\cdot 2=9\\ \lambda_2=2\\ \lambda_3=-1\end{array}\right\}\text{rădăcini reale, simple și distincte}$$

• $\lambda=1$, se caută $u=?$ astfel încât $(A-I_3)u=0$

$$u=\begin{pmatrix}a\\b\\c\end{pmatrix}\quad \begin{pmatrix}0&-1&1\\1&0&-1\\2&-1&-1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-b+c\\a-c\\2a-b-c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\iff\begin{cases}-b+c=0 & \Rightarrow b=c\\ a-c=0 & \Rightarrow c=a\\ 2a-b-c=0\end{cases}\Rightarrow u=\begin{pmatrix}a\\a\\a\end{pmatrix}$$

$a\in\mathbb{R}\setminus\{0\}$

$a=1\Rightarrow u=\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow$ soluția $\varphi_1(t)=e^t\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$

• $\lambda=2$, se caută $u=?$ astfel încât $(A-2I_3)\cdot u=0$

$$u=\begin{pmatrix}a\\b\\c\end{pmatrix}\quad \begin{pmatrix}-1&-1&1\\1&-1&-1\\2&-1&-2\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}-a-b+c=0\\ a-b-c=0\\ 2a-b-2c=0\end{cases}\Rightarrow b=0\quad\Rightarrow a=c\quad\Rightarrow u=\begin{pmatrix}a\\0\\a\end{pmatrix}$$

Pt. $a=1$ avem soluția $\varphi_2(t)=e^{2t}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}$

• $\lambda=-1$, se caută $u=?$ astfel încât $(A+I_3)\cdot u=0$. $\quad u=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$$\begin{pmatrix}2&-1&1\\1&2&-1\\2&-1&1\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\iff\begin{cases}2a-b+c=0\\ a+2b-c=0\\ 2a-b+c=0\end{cases}\Rightarrow\begin{cases}3a+b=0\Rightarrow b=-3a\\ c=a+2b=-5a\end{cases}$$

$\Rightarrow u=\begin{pmatrix}a\\-3a\\-5a\end{pmatrix}$. Pt. $a=1$ avem $\varphi_3(t)=e^{-t}\cdot\begin{pmatrix}-1\\3\\5\end{pmatrix}$

Soluția generală este: $\varphi(t) = c_1 \varphi_1(t) + c_2 \varphi_2(t) + c_3 \varphi_3(t)$

$$\varphi(t) = c_1 e^{t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2 \cdot e^{2t}\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3 e^{-t}\begin{pmatrix}-1\\3\\5\end{pmatrix}$$

$$\varphi(t) = \begin{pmatrix} c_1 e^{t} + c_2 e^{2t} - c_3 e^{-t} \\ c_1 e^{t} + 3c_3 e^{-t} \\ c_1 e^{t} + c_2 e^{2t} + 5c_3 e^{-t} \end{pmatrix}$$

$$\Rightarrow \quad x(t) = c_1 e^{t} + c_2 e^{2t} - c_3 e^{-t}$$

$$y(t) = c_1 e^{t} + 3c_3 e^{-t}$$

$$z(t) = c_1 e^{t} + c_2 e^{2t} + 5c_3 e^{-t}$$

Exerciții:

2. $\begin{cases} x' = 2x + y \\ y' = x + 3y - z \\ z' = 2y + 3z - x \end{cases}$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

$$A - \lambda I_3 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 & 0 \\ -1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)^2 - (3-\lambda) + 2(2-\lambda) + 3 - \lambda$$

$$= (2-\lambda)(9 - 6\lambda + \lambda^2) - 3 + \lambda + 4 - 2\lambda + 3 - \lambda$$

$$= (2-\lambda)(\lambda^2 - 6\lambda + 10)$$

$\lambda_1 = 2$

$\lambda_{2,3} = 3 \pm i$

- $\lambda = 2$, caută $u = ?$ astfel încât $(A - 2I_3) \cdot u = 0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} b = 0 \\ a + b - c = 0 \Rightarrow a = c \end{cases} \Rightarrow u = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix}$$

$a=1 \Rightarrow u=\begin{pmatrix}1\\0\\1\end{pmatrix} \Rightarrow \varphi_1(t)=e^{2t}\begin{pmatrix}1\\0\\1\end{pmatrix}$

- $\lambda = 3+i$, căutam $u=?$ astfel încât $\left(A-(3+i)I_3\right)\cdot u=0$.

$u=\begin{pmatrix}a\\b\\c\end{pmatrix}$ $\quad\begin{pmatrix}-1-i & 1 & 0\\ 1 & -i & -1\\ -1 & 2 & -i\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{cases}-a-ai+b=0 & \Rightarrow b=-a(1+i)\\ a-ib-c=0 & \Rightarrow c=a-ib=a(2-i)\\ -a+2b-ic=0\end{cases} \Rightarrow u=\begin{pmatrix}a\\a(1+i)\\a(2-i)\end{pmatrix}$$

Pentru $a=1$ avem: $u=\begin{pmatrix}1\\1+i\\2-i\end{pmatrix}$

$$\tilde{\varphi}(t)=e^{(3+i)t}\cdot\begin{pmatrix}1\\1+i\\2-i\end{pmatrix}=e^{3t}\cdot e^{it}\cdot\begin{pmatrix}1\\1+i\\2-i\end{pmatrix}=e^{3t}\cdot(\cos t+i\sin t)\begin{pmatrix}1\\1+i\\2-i\end{pmatrix}$$

$$=e^{3t}\cdot\begin{pmatrix}\cos t+i\sin t\\ \cos t-\sin t+i(\sin t+\cos t)\\ 2\cos t+\sin t+i(2\sin t+\cos t)\end{pmatrix}=$$

$$=e^{3t}\begin{pmatrix}\cos t\\ \cos t-\sin t\\ 2\cos t+\sin t\end{pmatrix}+i\cdot e^{3t}\begin{pmatrix}\sin t\\ \sin t+\cos t\\ 2\sin t+\cos t\end{pmatrix}$$

$$\varphi_2(t)=e^{3t}\begin{pmatrix}\cos t\\ \cos t-\sin t\\ 2\cos t+\sin t\end{pmatrix}\qquad \varphi_3(t)=e^{3t}\begin{pmatrix}\sin t\\ \sin t+\cos t\\ 2\sin t+\cos t\end{pmatrix}$$

Exercițiu: $\begin{cases}x'=x-2y\\ y'=3y+x\end{cases}$ $\quad A=\begin{pmatrix}1 & -2\\ 1 & 3\end{pmatrix}$

$$\det(A-\lambda I_2)=\begin{vmatrix}1-\lambda & -2\\ 1 & 3-\lambda\end{vmatrix}=(1-\lambda)(3-\lambda)+2=3-\lambda-3\lambda+\lambda^2+2=$$
$$=\lambda^2-4\lambda+5$$

$\lambda^2-4\lambda+5=0$

$\Delta = 16-20=-4 \Rightarrow \lambda_{1,2}=2\pm i$

- $\lambda = 2+i$, se caută $u=?$ astfel încât $(A-(2+i)I)\cdot u=0$.

$u=\begin{pmatrix} a \\ b\end{pmatrix}$ $\begin{pmatrix} -1-i & -2 \\ 1 & 1-i\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix}=\begin{pmatrix} 0 \\ 0\end{pmatrix} \Leftrightarrow \begin{cases} -a-ai-2b=0 \\ a+b-bi=0 \Rightarrow a=b(1+i)\end{cases}$

$\Rightarrow u=\begin{pmatrix} b(1+i) \\ b\end{pmatrix}$

Pentru $b=1 \Rightarrow u=\begin{pmatrix} 1+i \\ 1\end{pmatrix}$. Fie $\tilde{\varphi}(t)=e^{(2+i)t}\begin{pmatrix} 1+i \\ 1\end{pmatrix}=e^{2t}\cdot e^{it}\begin{pmatrix} 1+i \\ 1\end{pmatrix}$

$=e^{2t}(\cos t+i\sin t)\begin{pmatrix} 1+i \\ 1\end{pmatrix}=e^{2t}\begin{pmatrix} \cos t+\sin t \quad +i(\sin t-\cos t) \\ \cos t+i\sin t\end{pmatrix}=$

$=e^{2t}\begin{pmatrix} \cos t+\sin t \\ \cos t\end{pmatrix}+i\cdot e^{2t}\begin{pmatrix} \sin t-\cos t \\ \sin t\end{pmatrix}$

$\varphi_1(t)=e^{2t}\begin{pmatrix} \cos t+\sin t \\ \cos t\end{pmatrix}$ $\varphi_2(t)=e^{2t}\begin{pmatrix} \sin t-\cos t \\ \sin t\end{pmatrix}$

Varianta a II-a: $\begin{cases} x'=x-2y \\ y'=3y+x \Rightarrow x=y'-3y\end{cases}$

$(y'-3y)'=y'-3y-2y$

$y''-3y'=y'-5y$

$y''-3y'-y'+5y=0$

$y''-4y'+5y=0$ → ecuație liniară cu coeficienți constanți

Ecuația caracteristică este: $z^2+4z+5=0$

$z_{1,2}=2\pm i$

$y(t)=c_1 e^{2t}\cos t+c_2 e^{2t}\sin t$, $c_1,c_2\in\mathbb{R}$

$x(t)=2c_1e^{2t}\cos t+c_1e^{2t}(-\sin t)+2c_2e^{2t}\sin t+c_2e^{2t}\cos t-$

$-3c_1e^{2t}\cos t-3c_2e^{2t}\sin t=(-c_1+c_2)e^{2t}\cos t+(-c_1-c_2)e^{2t}\sin t$

Exercițiu:
$$\begin{cases} x' = 2x + y \\ y' = x + 3y - z \\ z' = -x + 2y + 3z \end{cases}$$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

$$\det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix}$$

Exercițiu:
$$\begin{cases} x' = x - y - z \\ y' = x + y \\ z' = 3x + z \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \Rightarrow \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)^3 + 3(1-\lambda) + (1-\lambda) = (1-\lambda)\left[(1-\lambda)^2 + 3 + 1\right] = (1-\lambda)(1 - 2\lambda + \lambda^2 + 4) =$$
$$= (1-\lambda)(\lambda^2 - 2\lambda + 5)$$

$1 - \lambda = 0 \Rightarrow \lambda_1 = 1$

$\lambda^2 - 2\lambda + 5 = 0$

$\Delta = 4 - 20 = -16 \Rightarrow \lambda_2 = 1 + 2i$

$\sqrt{\Delta} = 4i \qquad \lambda_3 = 1 - 2i$

- $\lambda_1 = 1$, căutăm $u = ?$ astfel încât $(A - I_3) \cdot u = 0$. $\quad u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} b = 0 \\ a - c = 0 \\ -a + 2b = 0 \end{cases} \Rightarrow \begin{matrix} c = -b \\ a = 0 \end{matrix}$$

$\Rightarrow u = \begin{pmatrix} 0 \\ b \\ -b \end{pmatrix}$. Pentru $b = 1$ avem $u = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

$\ldots (t) = e^{t}\begin{pmatrix} 0 \\ 1 \\ \ldots \end{pmatrix}$