SEMINAR 8+9

Examples of vector spaces.

1) Show that the Abelian group $(\mathbb{R}_+^*, \cdot)$ is an $\mathbb{R}$-vector space with the external operation $*$ defined by

$$\alpha * x = x^\alpha,\ \alpha \in \mathbb{R},\ x \in \mathbb{R}_+^*\,.$$

2) Let V be a K-vector space an let M be a set. Show that V^M is a K-vector space with the pointwise operations on V^M, i.e.

$$(f+g)(x) = f(x) + g(x),\ (\alpha f)(x) = \alpha f(x),\ \forall f, g \in V^M,\ \forall \alpha \in K.$$

3) Can one organize a finite set M as a vector space over an infinite field K?

4) Let $p \in \mathbb{N}$ be a prime. Can one organize the Abelian group $(\mathbb{Z}, +)$ as a vector space over the field $(\mathbb{Z}_p, +, \cdot)$?

5) Which of the following subsets is a subspace in the space mentioned nearby:
a) $A = \{(x,y) \in \mathbb{R}^2 \mid ax + by = 0\}$, $(a, b \in \mathbb{R}$ are given) in ${}_{\mathbb{R}}\mathbb{R}^2$;
b) $D = [-1, 1]$ in ${}_{\mathbb{R}}\mathbb{R}$;
b') $D' = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ in ${}_{\mathbb{R}}\mathbb{R}^2$;
b") $D'' = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$ in ${}_{\mathbb{R}}\mathbb{R}^n$;
c) $P_n(\mathbb{R}) = \{f \in \mathbb{R}[X] \mid \mathrm{grad} f \leq n\}$ in ${}_{\mathbb{R}}\mathbb{R}[X]$ ($n \in \mathbb{N}$ is given);
d) $B = \{f \in \mathbb{R}[X] \mid \mathrm{grad} f = n\}$ in ${}_{\mathbb{R}}\mathbb{R}[X]$ ($n \in \mathbb{N}$ is given)?

6) Let V be a K-vector space, $A \leq_K V$ and $C_V A = V \setminus A$.
i) Is $C_V A$ a subspace in ${}_K V$?
ii) What about $C_V A \cup \{0\}$?

7) Let V be a K-vector space, $S \leq_K V$ and $x, y \in V$. We denote $\langle S, x\rangle = \langle S \cup \{x\}\rangle$. Show that if $x \in V \setminus S$ and $x \in \langle S, y\rangle$ then $y \in \langle S, x\rangle$.

8) Let V be a K-vector space and $\alpha, \beta, \gamma \in K$, $x, y, z \in V$ such that $\alpha\gamma \neq 0$ and $\alpha x + \beta y + \gamma z = 0$. Show that $\langle x, y\rangle = \langle y, z\rangle$.

9) Let V, V' be K-vector spaces, $f : V \to V'$ a linear map, $A \leq_K V$ and $A' \leq_K V'$. Show that:
a) $f(A) = \{f(a) \in V' \mid a \in A\} \leq_K V'$;
b) $\overset{-1}{f}(A') = \{x \in V \mid f(x) \in A'\} \leq_K V$.

10) In the $\mathbb{R}$-vector space $\mathbb{R}^\mathbb{R} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ we consider

$$\mathbb{R}_o^\mathbb{R} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is odd}\},\ \mathbb{R}_e^\mathbb{R} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is even}\}.$$

Show that $\mathbb{R}_o^\mathbb{R}$ şi $\mathbb{R}_e^\mathbb{R}$ are subspaces of $\mathbb{R}^\mathbb{R}$ and $\mathbb{R}^\mathbb{R} = \mathbb{R}_o^\mathbb{R} \oplus \mathbb{R}_e^\mathbb{R}$.

11) Show that the property of being a direct summand is transitive.