SEMINAR 1

Vector Algebra

1.1 Elementary Operations with Vectors

Problem 1.1. Given a tetrahedron $ABCD$, find the sums of the vectors:

1) $\overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DC}$;

2) $\overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{DC}$;

3) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{CD}$.

Problem 1.2. Given a pyramid with vertex at S and base a parallelogram $ABCD$ whose diagonals intersect at the point O, prove the vector equality:

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}.$$

Problem 1.3. Let $ABCD$ be a tetrahedron, prove that $\overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{AC}$. Is this statement true for any four points in space?

Problem 1.4. Point O is the centre of a regular hexagon $ABCDEF$. Express the decompositions of the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}$ in terms of the vectors $\mathbf{p} = \overrightarrow{OE}$ and $\mathbf{q} = \overrightarrow{OF}$.

Problem 1.5. Prove that if M, N, P, Q are the midpoints of the sides of a quadrilateral $ABCD$, then $\overrightarrow{MN} + \overrightarrow{PQ} = 0$.

Problem 1.6. Points E and F are the midpoints of the diagonals of a quadrilateral $ABCD$. Prove that

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD}) = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{CB}).$$

Problem 1.7. Let E and F be the midpoints of the sides AB and CD of a quadrilateral $ABCD$. Prove that

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{BC} + \overrightarrow{AD})$$

and use this property to prove the midline theorem in a trapezium.

Problem 1.8. Given a regular hexagon $C_1C_2C_3C_4C_5C_6$. Prove that

$$\overrightarrow{C_1C_2} + \overrightarrow{C_1C_3} + \overrightarrow{C_1C_4} + \overrightarrow{C_1C_5} + \overrightarrow{C_1C_6} = 3\overrightarrow{C_1C_4}.$$

Problem 1.9. In triangle ABC, the bisector AD of angle A is drawn. Determine the decomposition of the vector $\overrightarrow{AD}$ in terms of the vectors $\mathbf{c} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AC}$.

Problem 1.10. The chords AB and CD of a circle with centre O intersect orthogonally at the point P. Prove the relation

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = 2\overrightarrow{PO}.$$

Problem 1.11. Given a trapezium $ABCD$ in which the base AB is k times ($k > 1$) larger than the smaller base CD. Let M and N be the midpoints of the bases. Find the decompositions of the vectors $\overrightarrow{AC}, \overrightarrow{MN}$ and $\overrightarrow{BC}$ in terms of the vectors $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.

Problem 1.12. Let A', B', C' be the midpoints of the sides of an arbitrary triangle ABC and let O be an arbitrary point in the plane of the triangle. Prove the relation

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

Problem 1.13. Points M, N, P are, respectively, the midpoints of the sides AB, BC, CA of triangle ABC. Determine the vectors $\overrightarrow{BP}, \overrightarrow{AN}, \overrightarrow{CM}$ in terms of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Problem 1.14. In Figure **??** the parallelepiped $ABCDEFGH$ is depicted. Let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{AD}$ and $\mathbf{w} = \overrightarrow{AE}$. Express the vectors $\overrightarrow{AG}$, $\overrightarrow{EC}$, $\overrightarrow{HB}$ and $\overrightarrow{DF}$ in terms of the vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

Problem 1.15. Prove that the medians of a triangle are concurrent and that the sum of the vectors, which have their origins at the point of intersection of the medians and their endpoints at the vertices of the triangle, is the zero vector.

Problem 1.16. The coordinates of the vertices A, B, C of the parallelogram $ABCD$ are known with respect to an arbitrary coordinate system. Determine the coordinates of the fourth vertex (D) in each of the following cases:

1) $A(2, 3), B(1, 4), C(0, -2)$;

2) $A(-2, -1), B(3, 0), C(1, -2)$.

Problem 1.17. The coordinates of the vertices A and B and the coordinates of the centroid G of triangle ABC are known. Determine the coordinates of the vertex C of the triangle in each of the following cases:

1) $A(4, 1), B(3, -2), G(0, 2)$;

2) $A(3, 5), B(-1, -3), C(1, 1)$.

Problem 1.18. Given a trapezium $ABCD$, in which $\overrightarrow{DC} = k\overrightarrow{AB}$. The points M and N are the midpoints of the bases AB and DC, and P is the intersection point of the diagonals AC and BD of the trapezium.

1) Taking the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as a basis, determine the components of the vectors $\overrightarrow{CB}, \overrightarrow{MN}, \overrightarrow{AP}, \overrightarrow{PB}$.

2) Taking the vectors $\overrightarrow{PA}$ and $\overrightarrow{PB}$ as a basis, determine the components of the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$.

Problem 1.19. In the plane, three vectors are given by their components with respect to an arbitrary basis: $\mathbf{a}(4, -2)$, $\mathbf{b}(3, 5)$, $\mathbf{c}(-2, -12)$. Express the vector $\mathbf{c}$ as a linear combination of the vectors $\mathbf{a}$ and $\mathbf{b}$.

Problem 1.20. Given the non-collinear vectors $\mathbf{a}$ and $\mathbf{b}$. Prove that the system of vectors $\mathbf{m} = 3\mathbf{a} - \mathbf{b}$, $\mathbf{n} = 2\mathbf{a} + \mathbf{b}$, $\mathbf{p} = \mathbf{a} + 3\mathbf{b}$ is linearly dependent, and that the vectors $\mathbf{n}$ and $\mathbf{p}$ are non-collinear. Express the vector $\mathbf{m}$ in terms of the vectors $\mathbf{n}$ and $\mathbf{p}$.

Problem 1.21. Point M is the centroid of triangle ABC. Express:

1) the vector $\overrightarrow{MA}$ in terms of the vectors $\overrightarrow{BC}$ and $\overrightarrow{CA}$;

2) the vector $\overrightarrow{AB}$ in terms of the vectors $\overrightarrow{MB}$ and $\overrightarrow{MC}$;

3) the vector $\overrightarrow{OA}$ in terms of the vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, and $\overrightarrow{OM}$, where O is an arbitrary point in space.

1.2 The Scalar Product of Vectors

Problem 1.22. Determine the lengths of the diagonals of a parallelogram constructed on the vectors $\mathbf{a} = 2\mathbf{m} + \mathbf{n}$ and $\mathbf{b} = \mathbf{m} - 2\mathbf{n}$, where $\mathbf{m}$ and $\mathbf{n}$ are vectors of length 1 and $\angle(\mathbf{m}, \mathbf{n}) = 60°$.

Problem 1.23. Find the angle between the vectors $\mathbf{a} = 2\mathbf{m} + 4\mathbf{n}$ and $\mathbf{b} = \mathbf{m} - \mathbf{n}$, where $\mathbf{m}$ and $\mathbf{n}$ are unit vectors, and $\angle(\mathbf{m}, \mathbf{n}) = 120°$.

Problem 1.24. The length of the hypotenuse AB of a right-angled triangle ABC is equal to c. Calculate the sum

$$S = \overrightarrow{AB} \cdot \overrightarrow{AC} + \overrightarrow{BC} \cdot \overrightarrow{BA} + \overrightarrow{CA} \cdot \overrightarrow{CB}.$$

Problem 1.25. Determine the angle formed by the diagonals of the parallelogram constructed on the vectors
$\mathbf{a}(2, 1, 0)$ and $\mathbf{b}(0, -2, 1)$.

Problem 1.26. Determine the real number λ such that the cosine of the angle formed by the vectors

$$\mathbf{p} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$$

and

$$\mathbf{q} = 3\mathbf{i} + \mathbf{j}$$

is equal to $\frac{5}{12}$.

Problem 1.27. A vector $\mathbf{p}$ is perpendicular to the vectors $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 18\mathbf{i} - 22\mathbf{j} - 5\mathbf{k}$ and forms an obtuse angle with the Oy axis. Determine the components of $\mathbf{p}$, given that $\|\mathbf{p}\| = 14$.

Problem 1.28. A vector $\mathbf{p}$ is perpendicular to the vectors $\mathbf{a}(4, -2, -3)$ and $\mathbf{b}(0, 1, 3)$ and forms an acute angle with the Ox axis. Determine the components of $\mathbf{p}$ provided that $\|\mathbf{p}\| = 26$.

Problem 1.29. Given three vectors $\mathbf{a}(4, 1, 5)$, $\mathbf{b}(0, 5, 2)$ and $\mathbf{c}(-6, 2, 3)$, determine a vector $\mathbf{x}$ such that $\mathbf{x} \cdot \mathbf{a} = 18$, $\mathbf{x} \cdot \mathbf{b} = 1$, $\mathbf{x} \cdot \mathbf{c} = 1$.

Problem 1.30. In an equilateral triangle ABC, with side length equal to one, $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$, $\overrightarrow{AB} = \mathbf{c}$. Calculate

$$\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}.$$

Problem 1.31. In space, a quadrilateral $ABCD$ is given, such that $\overrightarrow{AB} = (1, 2, -2)$, $\overrightarrow{BC} = (-2, -1, -2)$ and $\overrightarrow{CD} = (-1, -2, 2)$. Prove that the quadrilateral is a square.

Problem 1.32. The lengths of the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are equal. Determine the angle φ between them, provided that the vectors $\mathbf{p} = \mathbf{a} + 3\mathbf{b}$ and $\mathbf{q} = 5\mathbf{a} + 3\mathbf{b}$ are perpendicular.

Problem 1.33. Determine the angle, in radians, between the vectors $\mathbf{u}$ and $\mathbf{v}$ in the following cases:

(a) $\mathbf{u} = (1, 0, 1), \mathbf{v} = (-2, 10, 2)$;

(b) $\mathbf{u} = (3, 3, 0), \mathbf{v} = (2, 1, -2)$;

(c) $\mathbf{u} = (-1, 1, 1), \mathbf{v} = (1, 1, 1)$;

(d) $\mathbf{u} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right), \mathbf{v} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}\right)$;

(e) $\mathbf{u} = (300, 300, 0), \mathbf{v} = (-2000, -1000, 2000)$.

Problem 1.34. Determine the vector $\mathbf{u}$ such that $\|\mathbf{u}\| = \sqrt{2}$, the measure in degrees of the angle between $\mathbf{u}$ and $(1, -1, 0)$ is $45°$, and $\mathbf{u}$ is perpendicular to the vector $(1, 1, 0)$.

Problem 1.35. Calculate $\overrightarrow{AB} \cdot \overrightarrow{DA}$, given that $ABCD$ is a regular tetrahedron with edge length equal to 1.

Problem 1.36. Calculate $\|2\mathbf{u} + 4\mathbf{v}\|^2$, given that $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 2$, and the measure in radians of the angle between $\mathbf{u}$ and $\mathbf{v}$ is equal to $\frac{2\pi}{3}$.

Problem 1.37. Let A, B, C be three points in $\mathbb{R}^3$ and let $\mathbf{c} = \overrightarrow{BA}$ and $\mathbf{a} = \overrightarrow{BC}$. Prove that the vector $\mathbf{u} = \frac{\mathbf{c}}{\|\mathbf{c}\|} + \frac{\mathbf{a}}{\|\mathbf{a}\|}$ is parallel to the bisector of the angle $\widehat{ABC}$. Interpret the result by relating it to a well-known property of the rhombus.

Problem 1.38. Determine the vector $\mathbf{u}$ such that $\|\mathbf{u}\| = 3\sqrt{3}$, and $\mathbf{u}$ is perpendicular to the vectors $\mathbf{v} = (2, 3, -1)$ and $\mathbf{w} = (2, -4, 6)$. Among the vectors $\mathbf{u}$ that satisfy these conditions, which one forms an acute angle with the vector $(1, 0, 0)$?

Problem 1.39. Determine the vector $\mathbf{u}$, perpendicular to the vectors $\mathbf{v} = (4, -1, 5)$ and $\mathbf{w} = (1, -2, 3)$, and which satisfies $\mathbf{u} \cdot (1, 1, 1) = -1$.

1.3 The Vector Product and the Mixed Product

Problem 1.40. Determine $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.

Problem 1.41. Given the vectors $\mathbf{a}(3, -1, -2)$ and $\mathbf{b}(1, 2, -1)$. Calculate:

$$\mathbf{a} \times \mathbf{b},\ (2\mathbf{a} + \mathbf{b}) \times \mathbf{b},\ (2\mathbf{a} + \mathbf{b}) \times (2\mathbf{a} - \mathbf{b}).$$

Problem 1.42. Determine the distances between the parallel sides of the parallelogram constructed on the vectors $\overrightarrow{AB}(6, 0, 2)$ and $\overrightarrow{AC}(1.5, 2, 1)$.

Problem 1.43. Determine the vector $\mathbf{p}$, knowing that it is perpendicular to the vectors $\mathbf{a}(2, 3, -1)$ and $\mathbf{b}(1, -1, 3)$ and that it satisfies the equation

$$\mathbf{p} \cdot (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 51.$$

Problem 1.44. Given the points $A(1, 2, 0)$, $B(3, 0, -3)$ and $C(5, 2, 6)$. Calculate the area of triangle ABC.

Problem 1.45. Given the points $A(1, -1, 2)$, $B(5, -6, 2)$ and $C(1, 3, -1)$. Determine the altitude of triangle ABC, dropped from vertex B onto side AC.

Problem 1.46. Given the vectors $\mathbf{a}(2, -3, 1)$, $\mathbf{b}(-3, 1, 2)$ and $\mathbf{c}(1, 2, 3)$. Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Problem 1.47. Let $ABCD$ be a convex quadrilateral. Prove that if the diagonal AC bisects the diagonal BD, then the triangles ACB and ACD have equal areas.

Problem 1.48. Let P and Q be the midpoints of the non-parallel sides BC and AD of a trapezium $ABCD$. Prove that the triangles APD and CQB have equal area.

Problem 1.49. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of the vertices of a triangle ABC with respect to a point O. Determine the area of triangle ABC in terms of these vectors.

Problem 1.50. Determine whether the triplet of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is right-handed or left-handed, given that

$$\mathbf{a} = \mathbf{i} + \mathbf{j},\ \mathbf{b} = \mathbf{i} - \mathbf{j},\ \mathbf{c} = \mathbf{k}.$$

Problem 1.51. Prove that the points $A(1, 2, -1)$, $B(0, 1, 5)$, $C(-1, 2, 1)$ and $D(2, 1, 3)$ are coplanar.

Problem 1.52. Determine the volume of the tetrahedron with vertices at the points $A(2, -1, 1)$, $B(5, 5, 4)$, $C(3, 2, -1)$ and $D(4, 1, 3)$.

Problem 1.53. A tetrahedron of volume 5 has three of its vertices at the points $A(2, 1, -1)$, $B(3, 0, 1)$ and $C(2, -1, 3)$. The fourth vertex, D, is located on the Oy axis. Determine the coordinates of the point D.

Problem 1.54. Given three vectors $\mathbf{a}(8, 4, 1)$, $\mathbf{b}(2, 2, 1)$ and $\mathbf{c}(1, 1, 1)$. Determine the vector $\mathbf{d}$, of length 1, which forms equal angles with the vectors $\mathbf{a}$ and $\mathbf{b}$, is perpendicular to the vector $\mathbf{c}$ and is oriented in such a way that the triplets of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$ have the same orientation (i.e. they are both right-handed or both left-handed).

Problem 1.55. Given two vectors $\mathbf{a}(11, 10, 2)$ and $\mathbf{b}(4, 0, 3)$. Find a unit vector $\mathbf{c}$, orthogonal to the vectors $\mathbf{a}$ and $\mathbf{b}$, such that the triplet of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is right-handed.

Problem 1.56. Let ABC be a triangle and let E and F be the midpoints of the sides AB and AC, respectively. Through C draw a line parallel to AB which intersects BE at P. Prove that

$$\text{Area}\,\triangle FEP = \text{Area}\,\triangle FCE = \frac{1}{2}\triangle ABC.$$

Problem 1.57. Let $ABCD$ be a convex quadrilateral in the plane. Prove that

$$\text{Area}\,ABCD = \frac{1}{2}\left\|\overrightarrow{AC} \times \overrightarrow{BD}\right\|.$$

Problem 1.58. Let $ABCD$ be a convex quadrilateral in the plane such that

$$\overrightarrow{AB} = \mathbf{b},\ \overrightarrow{AD} = \mathbf{d},\ \overrightarrow{AC} = m\mathbf{b} + p\mathbf{d},$$

where m and p are two real numbers. Prove that the area of the quadrilateral is given by the formula

$$\text{Area}\,ABCD = \frac{1}{2}|m + p| \cdot \|\mathbf{b} \times \mathbf{d}\|.$$

Problem 1.59. Let $ABCD$ be a convex quadrilateral in the plane such that $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{CD} = \mathbf{c}$. Then the area of the quadrilateral is given by the formula

$$\text{Area}\,ABCD = \frac{1}{2}\|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} - \mathbf{c} \times \mathbf{a}\|.$$

Problem 1.60. Determine the areas of the triangles with vertices at the points:

(a) $(0,0,0)$, $(1,2,3)$ and $(2,-1,4)$;

(b) $(1,0,0)$, $(0,1,0)$ and $(1,1,1)$;

(c) $(-1,2,3)$, $(2,-1,-1)$ and $(1,1,-1)$;

(d) $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$.

Problem 1.61. Determine the volumes of the tetrahedra with vertices at the points:

(a) $(0,0,0)$, $(1,1,-1)$, $(1,-1,1)$ and $(-1,1,1)$;

(b) $(-1,0,1)$, $(2,-1,0)$, $(3,2,5)$ and $(1,2,1)$.

Problem 1.62. Prove that the volume of the tetrahedron with vertices at the points (x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_2) and (x_4,y_4,z_4) is equal to the absolute value of

$$\frac{1}{6}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}.$$

Problem 1.63. Prove that the volume of the tetrahedron whose vertices have the position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ is given by the formula

$$\text{Vol} = \frac{1}{6}\left|(\mathbf{b},\mathbf{c},\mathbf{d}) + (\mathbf{c},\mathbf{a},\mathbf{d}) + (\mathbf{a},\mathbf{b},\mathbf{d}) - (\mathbf{a}\cdot\mathbf{b},\mathbf{c})\right|.$$

Deduce from this a criterion for the coplanarity of the points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$.

SEMINAR 2

The Line in the Plane

Problem 2.1. Write the parametric equations of a line that:

(i) passes through $M_0(1, 2)$ and is parallel to the vector $\mathbf{a}(3, -1)$;

(ii) passes through the origin and is parallel to the vector $\mathbf{b}(3, 4)$;

(iii) passes through $A(1, 7)$ and is parallel to the Oy axis;

(iv) passes through the points $M_1(2, 4)$ and $M_2(2, -5)$.

Problem 2.2. A line is given by the parametric equations $x = 1 - 4t, y = 2 + t$. Determine the direction vector of the line.

Problem 2.3. Write the equation of a line that:

(i) has a slope of $k = -5$ and passes through the point $A(1, -2)$;

(ii) has a slope of $k = 8$ and intersects the Oy axis at a segment of length 2;

(iii) passes through the point $A(-2, 3)$ and forms an angle of $60°$ with the Ox axis;

(iv) passes through the point $B(1, 7)$ and is perpendicular to the vector $\mathbf{n}(4, 3)$.

Problem 2.4. The triangle ABC is given with vertices $A(1, 1)$, $B(-2, 3)$, and $C(4, 7)$. Write the equations of the sides of this triangle as well as the equation of the median passing through vertex A.

Problem 2.5. Write the equation of the line passing through the point $A(-2, 5)$ and cutting equal segments on the coordinate axes.

Problem 2.6. The midpoints of the sides of a triangle are $M_1(1, 2)$, $M_2(3, 4)$, $M_3(5, -1)$. Find the equations of the sides.

Problem 2.7. A triangle has vertices $A(1, 5)$, $B(-4, 3)$, and $C(2, 9)$. Determine the equation of the altitude from vertex A to side BC.

Problem 2.8. Determine the reflection of the point $A(10, 10)$ with respect to the line $3x + 4y - 20 = 0$.

Problem 2.9. Find the equation of the line passing through the point $A(3, 1)$ and forming an angle of $45°$ with the line $2x + 3y - 1 = 0$.

Problem 2.10. Find the vertices and angles of the triangle whose sides are given by the equations $x + 3y = 0$, $x = 3$, and $x - 2y + 3 = 0$.

Problem 2.11. A triangle has vertices $A(1, -2)$, $B(5, 4)$, and $C(-2, 0)$. Find the equation of the internal bisector and the external bisector corresponding to angle A.

Problem 2.12. Show that the figure bounded by the lines $x-3y+1 = 0$, $x-3y+12 = 0$, $3x+y-1 = 0$, and $3x + y + 10 = 0$ is a square. Calculate its area.

Problem 2.13. Find the equations of the altitudes of the triangle defined by the lines:

$$x + 2y - 1 = 0, \quad 5x + 4y - 17 = 0, \quad x - 4y + 11 = 0,$$

without determining the coordinates of the vertices.

Problem 2.14. Find the distances from the points $O(0, 0)$, $A(1, 2)$, and $B(-5, 7)$ to the line:

$$\Delta: \ 6x + 8y - 15 = 0.$$

Problem 2.15. Find the equation of the line passing through the point $A(8, 9)$, for which the segment on the line between the lines $x - 2y + 5 = 0$ and $x - 2y = 0$ has a length of 5.

Problem 2.16. Find the distance between the parallel lines:

1) $x - 2y + 3 = 0$ and $2x - 4y + 7 = 0$;

2) $3x - 4y + 1 = 0$ and $x = 1 + 4t, y = 3t$;

3) $x = 2 - t, y = -3 + 2t$ and $x = 2s, y = 5 - 4s$.

Problem 2.17. Determine the equation of the bisector of the angle formed by the lines $\Delta_1 : x+2y-11 = 0$ and $\Delta_2 : 3x - 6y - 5 = 0$ that contains the point $A(1, -3)$.

Problem 2.18. Establish the equations of the sides of a triangle, given one of its vertices, $B(2, -1)$, as well as the equation of an altitude: $3x - 4y + 27 = 0$ and the equation of a bisector: $x + 2y - 5 = 0$, originating from different vertices.

SEMINAR 3

The Line and the Plane in Space

Problem 3.1. Write the parametric equations of the plane that passes through:

1) the point $M_0(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(1,2,3)$, $\mathbf{a}_2(0,3,1)$;

2) the point $A(1,2,1)$ and is parallel to the vectors $\mathbf{i}, \mathbf{j}$;

3) the point $A(1,7,1)$ and is parallel to the plane xOz;

4) the points $M_1(5,3,2)$, $M_2(1,0,1)$ and is parallel to the vector $\mathbf{a}(1,3,-3)$;

5) the point $A(1,5,7)$ and the axis Ox;

6) the origin and the points $M_1(1,0,1)$, $M_2(-2,-3,1)$.

Problem 3.2. Write the general equation of the plane starting from its parametric equations:

(a)

$$\begin{cases} x & = 2+3u-4v, \\ y & = 4-v, \\ z & = 2+3u; \end{cases}$$

(b)

$$\begin{cases} x = u+v, \\ y = u-v, \\ z = 5+6u-4v. \end{cases}$$

Problem 3.3. Determine the parametric equations of the plane starting from its general equation:

(a) $3x-6y+z=0$;

(b) $2x-y-z-3=0$.

Problem 3.4. Determine the equation of the plane that passes through the point $A(3,5,-7)$ and cuts equal length segments on the coordinate axes.

Problem 3.5. Given the vertices of a tetrahedron: $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$. Write the equation of the plane that passes through the edge AB and through the midpoint of edge CD.

Problem 3.6. Determine whether the following planes intersect, are parallel or coincide:

(a) $x - y + 3z + 1 = 0$ and $2x - y + 5z - 2 = 0$;

(b) $2x + 4y + 2z + 4 = 0$ and $4x + 2y + 4z + 8 = 0$;

(c)
$$\begin{cases} x = u + 2v, \\ y = 1 + v, \\ z = u - v \end{cases}$$
and
$$\begin{cases} x = 2 + 3u' + v', \\ y = 1 + u' + v', \\ z = 2 - 2v'. \end{cases}$$

Problem 3.7. Prove that the parallelepiped having three non-parallel faces lying in the planes $2x + y - 2z + 6 = 0$, $2x - 2y + z + 8 = 0$ and $x + 2y + 2z + 10 = 0$ is rectangular.

Problem 3.8. Determine the orthogonal projection of the point $A(1, 3, 5)$ onto the line of intersection of the planes $2x + y + z - 1 = 0$ and $3x + y + 2z - 3 = 0$.

Problem 3.9. Determine the equation of a plane, knowing that the point $A(1, -1, 3)$ is the orthogonal projection of the origin onto this plane.

Problem 3.10. Determine the distance between the parallel planes $x - 2y - 2z + 7 = 0$ and $2x - 4y - 4z + 17 = 0$.

Problem 3.11. Determine the equation of a plane that is parallel to the plane $2x - 2y - z - 6 = 0$ and located at a distance of 7 units from it. Is the solution unique?

Problem 3.12. Determine the parametric equations of the line that passes through:

1. the point $M_0(2, 0, 3)$ and is parallel to the vector $\mathbf{a}(3, -2, -2)$;
2. the point $A(1, 2, 3)$ and is parallel to the axis Ox;
3. the points $M_1(1, 2, 3)$ and $M_2(4, 4, 4)$.

Problem 3.13. Given the vertices $A(1, 2, -7), B(2, 2, -7), C(3, 4, -5)$ of a triangle. Write the equations of the internal bisector of angle A.

Problem 3.14. Determine the parametric equations of the line defined by the planes $x + y + 2z - 3 = 0$ and $x - y + z - 1 = 0$.

Problem 3.15. Prove that the line $x = 1 - 2t, y = 3t, z = -2 + t$ is parallel to the line $x = 7 + 4s, y = 5 - 6s, z = 4 - 2s$ and determine the distance between them.

Problem 3.16. Prove that the line $x = -3t, y = 2 + 3t, z = 1$ intersects the line $x = 1 + 5s, y = 1 + 13s, z = 1 + 10s$ and determine the coordinates of the point of intersection.

Problem 3.17. For which value of the parameter m does the line $x = -1 + 3t, y = 2 + mt, z = -3 - 2t$ have no common points with the plane $x + 3y + 3z - 2 = 0$?

Problem 3.18. For which values of the real parameters a and d is the line
$$\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$$
contained in the plane $ax + y - 2z + d = 0$?

Problem 3.19. For which values of the real parameters a and c is the line

$$\begin{cases} 3x - 2y + z + 3 = 0 \\ 4x - 3y + 4z + 1 = 0 \end{cases}$$

perpendicular to the plane $ax + 8y + cz + 2 = 0$?

Problem 3.20. Determine the equation of the plane passing through the origin and the line $x = 1 + 3t, y = -2 + 4t, z = 5 - 2t$.

Problem 3.21. Determine the parametric equations of the line that passes through the point $A(3, -2, -4)$, is parallel to the plane $3x - 2y - 3z - 7 = 0$, and intersects the line $x = 2 + 3t, y = -4 - 2t, z = 1 + 2t$.

Problem 3.22. Determine the orthogonal projection of the point $A(2, 11, -5)$ onto the plane $x + 4y - 2z + 7 = 0$.

Problem 3.23. Determine the symmetric point of $P(6, -5, 5)$ with respect to the plane $2x - 3y + z - 4 = 0$.

Problem 3.24. Determine the symmetric point of $Q(4, -5, 4)$ with respect to the plane passing through the lines

$$\begin{cases} x + y + z - 3 = 0, \\ x - y + z - 1 = 0 \end{cases}$$

and

$$\begin{cases} x + z = 0, \\ y = 0 \end{cases}.$$

Problem 3.25. Verify that the line $x = 8 + 5t, y = 1 + 2t, z = 6 + 4t$ intersects the line $x = 11 + 3s, y = 2 + s, z = 4 - 2s$ and determine the equation of the plane defined by them.

Problem 3.26. Determine the equation of the plane that passes through the point $A(1, 2, -2)$ and is perpendicular to the line

$$\frac{x+3}{4} = \frac{y-6}{-6} = \frac{z-3}{2}.$$

Problem 3.27. Determine the symmetric point of $P(-3, 1, -2)$ with respect to the line

$$\begin{cases} 4x - 3y - 13 = 0, \\ y - 2z + 5 = 0 \end{cases}.$$

Problem 3.28. Determine the equations of the line that passes through the intersection point between the plane $x + y + z - 1 = 0$ and the line $x = t, y = 1, z = -1$, belongs to the given plane, and is perpendicular to the given line.

Problem 3.29. Determine the equations of the common perpendicular of the lines

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{2}$$

and $x = -1 + 3t, y = 2 + 2t, z = 1$.

Problem 3.30. Determine the relative position of the planes:

a) $\begin{cases} 2x - 5y + 3z - 7 = 0, \\ x + 4y - 2z - 7 = 0, \\ x - 22y + 12z - 9 = 0. \end{cases}$

b) $\begin{cases} 2x - 4y + 4z - 7 = 0, \\ x + 3y + 2z - 5 = 0, \\ -3x + 6y - 6z - 5 = 0. \end{cases}$

c) $\begin{cases} -x - y + z - 3 = 0, \\ 2x + y - 3z + 12 = 0, \\ x + 3y + z - 9 = 0. \end{cases}$

Problem 3.31. Determine whether the lines (d_1) and (d_2) are skew and, if so, write the equations of the common perpendicular and compute its length.

a) $(d_1)\ \begin{cases} x - y + z + 1 = 0, \\ 2x - y - z + 2 = 0 \end{cases}$ and $(d_2)\ \begin{cases} 3x + y + z = 0, \\ x + y - 2z - 1 = 0. \end{cases}$

b) $(d_1)\ \begin{cases} 3x - 2y - 1 = 0, \\ y + 3z - 7 = 0 \end{cases}$ and $(d_2)\ \begin{cases} 2x - 3y + 6 = 0, \\ x + z - 4 = 0. \end{cases}$

Problem 3.32. Determine the equations of the planes passing through the points $P(0, 2, 0)$ and $Q(-1, 0, 0)$ and which form an angle of $60°$ with the Oz axis.

Problem 3.33. Determine the distance between the parallel lines

$$\frac{x+3}{3} = \frac{y+2}{2} = \frac{z-8}{-2} \quad \text{and} \quad \begin{cases} x = -1 + 3t, \\ y = -1 + 2t, \\ z = -2 - 2t. \end{cases}$$

SEMINAR 4

Conics, quadrics and generated surfaces

4.1 Conics

Problem 4.1. Determine the equation of an ellipse whose foci lie on the Oy axis and are symmetric with respect to the origin in each of the following cases:

1) the semi-axes are equal to 5 and 3, respectively;

2) the distance between the foci is $2c = 6$, and the major axis is equal to 10;

3) the major axis is equal to 26, and the eccentricity is $\varepsilon = \frac{12}{13}$.

Problem 4.2. Write the equations of the tangents to the ellipse

$$\frac{x^2}{10} + \frac{y^2}{5} = 1$$

which are parallel to the line

$$3x + 2y + 7 = 0.$$

Problem 4.3. Write the equations of the tangents to the ellipse

$$x^2 + 4y^2 = 20$$

which are perpendicular to the line

$$(d) : 2x - 2y - 13 = 0.$$

Problem 4.4. Write the equations of the tangents to the ellipse

$$\frac{x^2}{30} + \frac{y^2}{24} = 1$$

which are parallel to the line

$$4x - 2y + 23 = 0$$

and determine the distance between them.

Problem 4.5. From the point $A\left(\frac{10}{3}, \frac{5}{3}\right)$ tangents are drawn to the ellipse

$$\frac{x^2}{10} + \frac{y^2}{5} = 1.$$

Write their equations.

Problem 4.6. From the point $C\,(10, -8)$ tangents are drawn to the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Determine the equation of the chord connecting the points of contact.

Problem 4.7. An ellipse passes through the point $A(4, -1)$ and is tangent to the line $x + 4y - 10 = 0$. Determine the equation of the ellipse, knowing that its axes coincide with the coordinate axes.

Problem 4.8. Determine the equation of an ellipse whose axes coincide with the coordinate axes and which is tangent to the lines $3x - 2y - 20 = 0$ and $x + 6y - 20 = 0$.

Problem 4.9. Determine the equations of the tangents to the hyperbola

$$\frac{x^2}{20} - \frac{y^2}{5} = 1$$

which are perpendicular to the line

$$4x + 3y - 7 = 0.$$

Problem 4.10. Determine the equations of the tangents to the hyperbola

$$\frac{x^2}{16} - \frac{y^2}{64} = 1$$

which are parallel to the line

$$10x - 3y + 9 = 0.$$

Problem 4.11. A hyperbola passes through the point $M(\sqrt{6}, 3)$ and is tangent to the line $9x+2y-15 = 0$. Determine the equation of the hyperbola, knowing that its axes coincide with the coordinate axes.

Problem 4.12. Determine the equation of a parabola with vertex at the origin if the axis of the parabola is the Ox axis and the parabola passes through the point $A(9, 6)$.

Problem 4.13. Find the locus of the points from which perpendicular tangents can be drawn to the parabola $y^2 = 2px$.

Problem 4.14. Determine the canonical equation of a parabola, knowing that it is tangent to the line $3x - 2y + 4 = 0$ and determine the point of tangency.

Problem 4.15. Determine the canonical equation of a parabola, knowing that the tangent parallel to the line $5x - 4y - 2 = 0$ passes through the point $A(4, 7)$.

Problem 4.16. From the point $A(5, 9)$ tangents are drawn to the parabola $y^2 = 5x$. Determine the equation of the chord joining the points of tangency.

Problem 4.17. Determine the equation of the tangent to the parabola $y^2 = 20x$ that makes an angle of 45° with the positive direction of the Ox axis.

Problem 4.18. Write the equation of the tangent to the parabola $y^2 = 4ax$ which intersects the coordinate axes at points equidistant from the origin.

4.2 Quadrics

Problem 4.19. Determine the points of intersection between the ellipsoid

$$\frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1 = 0$$

and the line

$$x = 4 + 2t,\ y = -6 - 3t,\ z = -2 - 2t.$$

Problem 4.20. Write the equation of the tangent plane to the one-sheet hyperboloid

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

at the point $M(2, 3, 1)$. Show that this tangent plane intersects the surface along two real lines and compute the angle between the two lines.

Problem 4.21. Write the equations of the tangent planes at the points of intersection between the line $x = y = z$ and:

a) the elliptic paraboloid $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 9z$;

b) the hyperbolic paraboloid $\dfrac{x^2}{2} - \dfrac{y^2}{4} = 9z$.

Problem 4.22. Write the equations of the tangent planes to:

a) the elliptic paraboloid $\dfrac{x^2}{5} + \dfrac{y^2}{3} = z$;

b) the hyperbolic paraboloid $x^2 - \dfrac{y^2}{4} = z$,

that are parallel to the plane

$$x - 3y + 2z - 1 = 0.$$

Problem 4.23. Determine the straight-line generators of the hyperbolic paraboloid $4x^2 - 9y^2 = 36z$ that pass through the point $P(3\sqrt{2}, 2, 1)$.

Problem 4.24. Write the equations of the straight-line generators of the hyperbolic paraboloid

$$\frac{x^2}{16} - \frac{y^2}{4} = z$$

that are parallel to the plane

$$3x + 2y - 4z = 0.$$

Problem 4.25. Find the straight-line generators of the surface

$$\frac{x^2}{36} + \frac{y^2}{9} - \frac{z^2}{4} = 1$$

that are parallel to the plane

$$x + y + z = 0.$$

Problem 4.26. Find a point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad a > b > c > 0.$$

such that the tangent plane at that point cuts equal length segments on the coordinate axes.

Problem 4.27. Find the points on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad a > b > c > 0.$$

where the normals intersect the Oz axis.

Problem 4.28. What conditions must the semi-axes of an ellipsoid satisfy so that its normals pass through its centre?

Problem 4.29. Find the locus of the points on the quadric

$$y^2 - z^2 = 2x$$

through which mutually perpendicular straight-line generators pass.

Problem 4.30. Find the equation of the projection onto the xOy plane of the curve of intersection between the ellipsoid

$$\frac{x^2}{1} + \frac{y^2}{9} + \frac{z^2}{4} - 1 = 0$$

and the plane

$$x + y + z - 1 = 0.$$

Problem 4.31. Find the locus of the points M on the surface $x^2 - y^2 = z$ for which the normal at M to the surface forms a constant angle with the Oz axis. Show that the projection of this locus onto the xOy plane is a circle and find its equation.

Problem 4.32. Determine the equation of the one-sheet hyperboloid having the coordinate axes as its symmetry axes, which is tangent to the plane

$$6x - 3y + 2z - 6 = 0$$

and for which the line

$$\begin{cases} 4x - z - 5 = 0, \\ 6x + 5z + 9 = 0 \end{cases}$$

is a straight-line generator.

Problem 4.33. Write the equation of the normal at the point $P(-2, 2, -1)$ to the quadric

$$\frac{x^2}{8} - \frac{y^2}{2} + \frac{z^2}{2} + 1 = 0.$$

Determine the coordinates of the point where the normal intersects the surface for the second time.

Problem 4.34. Determine the planes that contain the line

$$\frac{x+1}{2} = \frac{y}{-1} = \frac{z}{0}$$

and are tangent to the two-sheet hyperboloid

$$x^2 + 2y^2 - z^2 + 1 = 0.$$

Problem 4.35. Find the shortest distance between the elliptic paraboloid

$$\frac{x^2}{12} + \frac{y^2}{4} = z$$

and the plane

$$x - y - 2z = 0.$$

Problem 4.36. Show that the line

$$\frac{x-2}{0} = \frac{y-3}{-1} = \frac{z-6}{2}$$

is tangent to the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} - 1 = 0$$

and determine the coordinates of the point of tangency.

4.3 Generated surfaces

Problem 4.37. Determine the equation of the conical surface with vertex at the point $(0, 0, h)$ and whose generators rest on the lemniscate

$$(x^2 + y^2)^2 = a^2(x^2 - y^2), \quad z = 0.$$

Problem 4.38. Determine the equation of the conical surface with vertex at the point $(0, 0, h)$ and whose generators rest on the lemniscate

$$(x^2 + y^2)^2 = a^2(x^2 - y^2), \quad z = 0.$$

Problem 4.39. Find the equation of the conical surface with vertex at the point $(0, 0, -h)$ whose generators are tangent to the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z.$$

Problem 4.40. Find the equation of the cone with vertex at $V(1, 1, 1)$ and having as directrix the ellipse defined by the equations

$$y^2 + z^2 = 1, \quad x + y + z = 1.$$

Problem 4.41. Find the equation of the conical surface with vertex at point $A(0, -a, 0)$ and having as directrix the curve $x^2 = 2py, \ z = h$.

Problem 4.42. Three parallel lines are given:

$$x = y = z, \ x + 1 = y = z - 1, \ x - 1 = y + 1 = z - 2.$$

Write the equation of the circular cylinder that contains these lines.

Problem 4.43. Write the equation of the cylinder circumscribed around the sphere $x^2 + y^2 + z^2 = 1$, knowing that its generators make equal angles with the three coordinate axes.

Problem 4.44. Find the equation of the cylindrical surface having generators parallel to the line

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

and directrix the parabola $y^2 = 4x$, $z = 0$.

Problem 4.45. Find the equation of the conoidal surface generated by a line that remains parallel to the plane $x + z = 0$, rests on the Ox axis and on the circle $x^2 + y^2 = 1,\ z = 0$.

Problem 4.46. Find the equation of the surface of revolution obtained by rotating the line $x - y = a,\ z = 0$ about the line $x = y = z$.

Problem 4.47. Write the equation of the cone with vertex at the origin and whose directrix is the curve defined by the equations

$$x = 1,\ y^2 + z^2 - 2z = 0.$$

Problem 4.48. Write the equation of the cone with vertex at the origin which has three generators coinciding with the coordinate axes.

Problem 4.49. Write the equation of the cone of revolution around the axis $y = 1,\ x = 2 + pz - p^2$, knowing that this cone has the generator $y = 1,\ z = p$. Determine p, knowing that this cone passes through the origin.

Problem 4.50. Find the equation of the surface generated by the rotation of the curve $y = \sin x,\ z = 0$ around the Ox axis.

Problem 4.51. Find the equation of the conoidal surface generated by a line that remains parallel to the xOy plane, rests on the Oz axis, and is tangent to the sphere

$$x^2 + y^2 + z^2 - 2Rx = 0.$$

Problem 4.52. Find the equation of the surface generated by the curve $z = e^{-x^2},\ y = 0$ by rotation around the Oz axis.

Problem 4.53. Find the equation of the cone with vertex $V(0, -a, 0)$ and having as directrix the circle

$$x^2 + y^2 + z^2 = a^2,\ y + z = a.$$

Problem 4.54. Find the equation of the surface of revolution obtained by rotating the curve $x^2 + y^2 = z^3,\ y = 0$ around the Oz axis.

Problem 4.55. Find the locus of the points that are at a distance equal to 1 from the line $x = y = z$.

Problem 4.56. Find the equation of the conoid generated by a line that remains parallel to the plane $z = 0$ and rests on the line $x = 0,\ y = a$ and on the parabola $z^2 - 2px = 0,\ y = 0$.

SEMINAR 5

The differential geometry of curves (1)

Problem 5.1. Let us consider the circle of equation

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$

and C – the diametrically opposed point of the origin on this circle. An arbitrary halfline OE, starting from the origin, intersects the circle at D and the tangent to the circle at the point C – at the point E.

Through D and E we construct parallel lines to the axes Ox and Oy, respectively. The two lines intersect at a point M. Find the locus of M, when the halfline OE varies (*Maria Agnesi's witch*).

Problem 5.2. A circle of radius a rolls, without sliding, on a straight line. Find the equations of the locus of a point invariably attached to the circle (*the cycloid*).

Problem 5.3. A circle of radius r rolls, without sliding, on a circle of radius R, *outside* it. Find the equations of the locus of a point invariably attached to the mobile circle (*the epicycloid*).

Problem 5.4. Show that the support of the curve

$$\mathbf{r} = (\sin 2t, 1 - \cos 2t, 2\cos t), \quad t \in [0, 2\pi)$$

lies on a sphere and find the center and the radius of this sphere.

Problem 5.5. Check whether the following pairs of parameterized curves are equivalent or not:

a) $\mathbf{r} : [0, 1) \to \mathbb{R}^3,\ \mathbf{r}(t) = (\cos t, \sin t, 2t + 3)$ şi $\boldsymbol{\rho} : [1, \infty) \to \mathbb{R}^3$,

$$\boldsymbol{\rho}(u) = \left(\cos\frac{u-1}{u+1}, \sin\frac{u-1}{u+1}, \frac{5u+1}{u+1}\right);$$

b) $\mathbf{r} : (2, \infty) \to \mathbb{R}^3,\ \mathbf{r}(t) = \left(e^t + 1, 3e^t - 2, \ln t + 1\right)$ şi $\boldsymbol{\rho} : (e^2, \infty) \to \mathbb{R}^3$,

$$\boldsymbol{\rho}(u) = (u + 1, 3u - 2, \ln\ln u + 1);$$

c) $\mathbf{r} : [0, 1] \to \mathbb{R}^3,\ \mathbf{r}(t) = (f_1(t), f_2(t), t)$ şi $\boldsymbol{\rho} : [0, 4] \to \mathbb{R}^3$,

$$\boldsymbol{\rho}(u) = (g_1(u), g_2(u), 2u + 1).$$

Problem 5.6. Find the length of the arc of the cycloid $\mathbf{r}(t) = (a(t - \sin t), a(1 - \cos t))$, between $t = 0$ and $t = 2\pi$.

Problem 5.7. Find a natural parameterization of the conical helix $\mathbf{r}(t) = \left(e^t \cos t, e^t \sin t, e^t\right)$.

Problem 5.8. Write down the equations of the tangent and the equation of the normal plane for the following curves, at the indicated points.

a) $\mathbf{r}(t) = (1/\cos t, \operatorname{tg} t, at)$, for $t = \pi/4$;

b) $\mathbf{r}(t) = \left(e^t, e^{-t}, t^2\right)$, for $t = 1$;

c) $\mathbf{r}(t) = \left(e^t \cos t, e^t \sin t, e^t\right)$, for $t = 0$.

Problem 5.9. Find the equations of the tangent to the curve $\mathbf{r}(t) = \left(a(t - \sin t), a(1 - \cos t), 4a \sin \dfrac{t}{2}\right)$ for $t = \pi/2$. What is the angle between this tangent and the z-axis?

Problem 5.10. Show that all the tangents to the curve

$$\begin{cases} x^2 + y^2 - z^2 - 1 = 0, \\ x^2 - y^2 - z^2 - 1 = 0, \end{cases}$$

are parallel to the plane xOy (provided, of course, these tangents exist).

Problem 5.11. At which point the tangent to the parabola $y = x^2 - 6x + 5$ is perpendicular on the line of equation $x - 2y + 8 = 0$?

Problem 5.12. Find the equation of the tangent and that of the normal for the following plane curves:

a) $\mathbf{r}(t) = (t^3 - 2t, t^2 + 1)$, for $t = 1$;

b) $y = \operatorname{tg} x$, at the points of abscissae $0, \pi/4$;

c) $x^3 + y^3 - 3axy = 0$, at the point $A(3a/2, 3a/2)$;

d) $(x^2 + y^2)x - ay^2 = 0$, at the point $A(a/2, a/2)$.

Problem 5.13. Is there any point on the curve $y = x^3$ at which the tangent makes an angle of $3\pi/4$ with the x-axis?

Problem 5.14. Show that all the normal planes to the curve $\mathbf{r} : [0, 2\pi) \to \mathbb{R}^3$,

$$\mathbf{r}(t) = \left(\sin t, \cos t, 2 \sin \frac{t}{2}\right),$$

pass through a fixed point and fin the coordinates of this point.

Problem 5.15. Find the points of the parameterized curve

$$\begin{cases} x = t + \ln t, \\ y = t^2, & t > 0, \\ z = 1 - t, \end{cases}$$

at which the tangents are parallel to the plane of equation

$$12x - y + 14z + 5 = 0.$$

Problem 5.16. Find the tangents to the parameterized curve

$$\begin{cases} x = \dfrac{t^4}{4}, \\ y = \dfrac{t^3}{3}, \\ z = \dfrac{t^2}{2} \end{cases}$$

which are parallel to the plane

$$x + 3y + 2z - 7 = 0.$$

Problem 5.17. Consider the Viviani curve, with the parameterization

$$\begin{cases} x = \dfrac{a}{2}(1 + \cos t), \\ y = \dfrac{a}{2}\sin t, \\ z = a \sin \dfrac{t}{2}. \end{cases}$$

Find the minimal angle between the tangents to the curve and the z-axis.

Problem 5.18. Find the length of the closed curve

$$\mathbf{r}(t) = \left(a\cos^3 t, a\sin^3 t, a\cos 2t\right), \quad t \in [0, 2\pi].$$

Problem 5.19. Find a natural parameteriozation of the curve Determinaţi parametrizarea

$$\mathbf{r}(t) = \left(e^t, e^{-t}, t\sqrt{2}\right),$$

assuming that $s(0) = 0$.

Problem 5.20. Find the length of the arc of the curve $x^3 = 3a^2 y$, $2xz = a^2$, lying between the planes $3y = a$ and $y = 9a$. Write down the equations of the tangents to the curve at the extremities of ths arc.

Problem 5.21. Prove that the tangents to the curve $x^2 = 3y$, $2xy = 9z$ make a constant angle with a fixed direction. Find a versor of the fixed direction and the constant angle.

Problem 5.22. Compute the length of the arc of the curve from the previous problem for $x \in [0, 3]$.

Problem 5.23. Check whether there is a point on the curve

$$\begin{cases} z = x^2 + y^2, \\ y = x \end{cases}$$

at which the tangent is perpendicular on the plane $2x - y - 3 = 0$.

Problem 5.24. Find the equations of the tangent and the equation of the normal plane at th e curve

$$\begin{cases} x^2 + y^2 + z^2 = 25, \\ x + z = 5 \end{cases}$$

at the point $\left(2, 2\sqrt{3}, 3\right)$.

Problem 5.25. Write down the equations of the tangent and the equation of the normal plane for the following curves, at the indicated points.

a) $\mathbf{r}(t) = (1/\cos t, \operatorname{tg} t, at)$, for $t = \pi/4$;

b) $\mathbf{r}(t) = \left(e^t, e^{-t}, t^2\right)$, for $t = 1$;

c) $\mathbf{r}(t) = \left(e^t \cos t, e^t \sin t, e^t\right)$, for $t = 0$.

Problem 5.26. Find the equations of the tangent to the curve $\mathbf{r}(t) = \left(a(t - \sin t), a(1 - \cos t), 4a \sin \dfrac{t}{2}\right)$ for $t = \pi/2$. What is the angle between this tangent and the z-axis?

Problem 5.27. Show that all the tangents to the curve

$$\begin{cases} x^2 + y^2 - z^2 - 1 = 0, \\ x^2 - y^2 - z^2 - 1 = 0, \end{cases}$$

are parallel to the plane xOy (provided, of course, these tangents exist).

Problem 5.28. At which point the tangent to the parabola $y = x^2 - 6x + 5$ is perpendicular on the line of equation $x - 2y + 8 = 0$?

Problem 5.29. Find the equation of the tangent and that of the normal for the following plane curves:

a) $\mathbf{r}(t) = (t^3 - 2t, t^2 + 1)$, for $t = 1$;

b) $y = \operatorname{tg} x$, at the points of abscissae $0, \pi/4$;

c) $x^3 + y^3 - 3axy = 0$, at the point $A(3a/2, 3a/2)$;

d) $(x^2 + y^2)x - ay^2 = 0$, at the point $A(a/2, a/2)$.

Problem 5.30. Is there any point on the curve $y = x^3$ at which the tangent makes an angle of $3\pi/4$ with the x-axis?

Problem 5.31. Show that all the normal planes to the curve $\mathbf{r} : [0, 2\pi) \to \mathbb{R}^3$,

$$\mathbf{r}(t) = \left(\sin t, \cos t, 2 \sin \frac{t}{2}\right),$$

pass through a fixed point and fin the coordinates of this point.

Problem 5.32. Find the points of the parameterized curve

$$\begin{cases} x = t + \ln t, \\ y = t^2, \qquad\qquad t > 0, \\ z = 1 - t, \end{cases}$$

at which the tangents are parallel to the plane of equation

$$12x - y + 14z + 5 = 0.$$

Problem 5.33. Find the tangents to the parameterized curve

$$\begin{cases} x = \dfrac{t^4}{4}, \\ y = \dfrac{t^3}{3}, \\ z = \dfrac{t^2}{2} \end{cases}$$

which are parallel to the plane

$$x + 3y + 2z - 7 = 0.$$

Problem 5.34. Consider the Viviani curve, with the parameterization

$$\begin{cases} x = \dfrac{a}{2}(1 + \cos t), \\ y = \dfrac{a}{2} \sin t, \\ z = a \sin \dfrac{t}{2}. \end{cases}$$

Find the minimal angle between the tangents to the curve and the z-axis.

Problem 5.35. Find the length of the closed curve

$$\mathbf{r}(t) = \left(a\cos^3 t, a\sin^3 t, a\cos 2t\right), \quad t \in [0, 2\pi].$$

Problem 5.36. Find a natural parameteriozation of the curve Determinaţi parametrizarea

$$\mathbf{r}(t) = \left(e^t, e^{-t}, t\sqrt{2}\right),$$

assuming that $s(0) = 0$.

Problem 5.37. Find the length of the arc of the curve $x^3 = 3a^2y$, $2xz = a^2$, lying between the planes $3y = a$ and $y = 9a$. Write down the equations of the tangents to the curve at the extremities of ths arc.

Problem 5.38. Prove that the tangents to the curve $x^2 = 3y$, $2xy = 9z$ make a constant angle with a fixed direction. Find a versor of the fixed direction and the constant angle.

Problem 5.39. Compute the length of the arc of the curve from the previous problem for $x \in [0, 3]$.

Problem 5.40. Check whether there is a point on the curve

$$\begin{cases} z = x^2 + y^2, \\ y = x \end{cases}$$

at which the tangent is perpendicular on the plane $2x - y - 3 = 0$.

Problem 5.41. Find the equations of the tangent and the equation of the normal plane at th e curve

$$\begin{cases} x^2 + y^2 + z^2 = 25, \\ x + z = 5 \end{cases}$$

at the point $\left(2, 2\sqrt{3}, 3\right)$.

Problem 5.42. Write down the equation of the oscuating plane to the following curves:

a) $y^2 = x, x^2 = z$ în punctul $(1, 1, 1)$;

b) $y = \varphi(x), z = a\varphi(x) + b$;

c) $x^2 = 2az, y^2 = 2bz$;

d) $x^2 + z^2 = a^2, y^2 + z^2 = b^2$;

e) $\mathbf{r}(t) = \left(e^t, e^{-t}, t\sqrt{2}\right)$.

Problem 5.43. Show that if all the osculating planes of a curve pass through a fixed point, then the curve is plane.

Problem 5.44. Find the versors of the Frenet frame for the curve $\mathbf{r}(t) = \left(t, t^2, t^3\right)$ at the origin.

Problem 5.45. Find the versors of the Frenet frame for the curve

$$\mathbf{r}(t) = (a(t - \sin t), a(1 - \cos t), 4a \cos t),$$

where a a strictly positive real constant, at an arbitrary point.

Problem 5.46. Write down the equations of the faces of the Frenet frame for the curve

$$\mathbf{r}(t) = \left(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}, \ln|\sin t|\right)$$

at the point $t = \pi/2$.

Problem 5.47. Write down the equations of the edges of the Frenet frame at an arbitrary point of the curve

$$\mathbf{r}(t) = \left(\frac{1}{2}\sin^2 t, \frac{1}{2}\left(t + \sin t \cos t\right), \sin t\right).$$

Problem 5.48. Write down the equations of the principal normal and those of the binormal to the curve

$$\mathbf{r}(t) = \left(\frac{t^2}{2}, \frac{2t^3}{3}, \frac{t^4}{2}\right)$$

at the point $M\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}\right)$.

Problem 5.49. Find the points of the curve

$$\mathbf{r}(t) = \left(\frac{1}{t}, \ln t, t\right), \quad t > 0,$$

at which the principal normal is parallel to the plane

$$5x + 2y - 5z - 4 = 0.$$

Problem 5.50. Consider the curve

$$\mathbf{r}(t) = \left(\frac{1}{2}t^2, \frac{2\sqrt{2}}{3}t^{3/2}, t\right).$$

1) Find the versors of the tangent and principal normal, as well as the curvature at the point $t = 2$;

2) Write down the equations of the tangent and that of the rectificant plane, at the same point.

Problem 5.51. Find the curvature of the curve

$$\mathbf{r}(t) = \left(t - \sin t, 1 - \cos t, 4 \sin \frac{t}{2}\right).$$

Problem 5.52. Consider the curve

$$\mathbf{r}(t) = (\cos t, \sin t, -\ln|\cos t|)\,.$$

a) Find the curvature at an arbitrary point of the curve;

b) Find the versors of the principal normal and the binormal;

c) Write down the equations of the binormal and that of the rectificant plane at the point $t = 0$.

Problem 5.53. Considering the skew curve

$$\mathbf{r}(t) = \left(t^2 - 1, t^2, -\ln t\right),$$

write down the equations of the edges and faces of the Frenet frame at $t = 1$. Find the curvature and torsion at the same point.

Problem 5.54. Find the points of the skew curve

$$\mathbf{r}(t) = \left(\frac{1}{t}, t, 2t^2 - 1\right)$$

at which the binormals are perpendicular aon the line D of equations

$$\begin{cases} x + y = 0 \\ 4x - z = 0 \end{cases}.$$

Problem 5.55. Find the points of the curve

$$\begin{cases} x^2 - y = 0, \\ x^3 - z = 0, \end{cases}$$

at which the principal normals are perpendicular on the line of equations

$$\begin{cases} 8y - 3z = 0, \\ 11x - 6z = 0. \end{cases}$$

Find the curvature and torsion of the curve at these points.

Problem 5.56. Find the torsion of the curve

$$\mathbf{r}(t) = (\cosh t \cos t, \cosh t \sin t, t).$$

Problem 5.57. Find the points of the curve

$$\mathbf{r}(t) = (\cos^3 t, \sin^3 t, \cos 2t)$$

where the curvature takes a minimal value.

Problem 5.58. Find the smooth function f for which the curve

$$\mathbf{r}(t) = (f(t), a \cos t, a \sin t)\,,$$

is planar, where a is a strictly positive constant.

Problem 5.59. If d is the distance from the origin to the osculating plane at an arbitrary point of the curve

$$\mathbf{r}(t) = \left(e^t \sin\left(t\sqrt{3}\right), e^t \cos\left(t\sqrt{3}\right), e^{-2t}\right),$$

while χ is the torsion of the curve at that point, prove that

$$\frac{d^2}{\chi} = \text{const} = \frac{3\sqrt{3}}{2}.$$

Problem 5.60. Prove that for the conical helix

$$\mathbf{r}(t) = (t\cos(a\ln t), t\sin(a\ln t), bt),$$

where a and b are strictly positive constants, while $t > 0$, the ratio between the curvature and the torsion at an arbitrary point of the curve is constant.

SEMINAR 6

Seminary

Problem 6.1. Find the envelope of the family of circles

$$(x-\lambda)^2 + y^2 = 4\lambda k,$$

where k is a strictly positive constant.

Problem 6.2. Let us consider the family of straight lines cutting the x and y coordinate axes at the point A and B, respectively, such that the segment AB has a constant length, equal to a. Find the envelope of this family of lines.

Problem 6.3. Find the envelopes of the following families of curves

$$y^2 - (x-\lambda)^3 = 0$$

and

$$3(y-\lambda)^2 - 2(x-\lambda)^3 = 0.$$

Problem 6.4. Find the envelope of the family of curves of equation

$$\frac{x^2}{\lambda} + \frac{y^2}{\mu} = 1,$$

where

$$\lambda + \mu = 1.$$

Problem 6.5. Find the signed curvature of the curve

$$y = a \arccos \frac{a-x}{a} + \sqrt{2ax - x^2},$$

at $x = a/2$.

Problem 6.6. Show that the curvature radius of the catenary

$$y = a \cosh \frac{x}{a}$$

at an arbitrary point M of the curve is equal to the length of the segment of normal lying between M and the x-axis.

Problem 6.7. Show that the curvature radius of the cycloid

$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t) \end{cases}$$

at an arbitrary point M of the curve is equal to twice the length of the segment of the normal to the curve at that point lying between M and the x-axis.

Problem 6.8. Find the parametric equations, as well as the implicit equation of the evolute of the curve

$$\mathbf{r}(t) = \left(3\sin t - 2\sin^3 t, 3\cos t - 2\sin^3 t\right).$$

Problem 6.9. Find the signed curvature at an arbitrary point of the astroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Problem 6.10. Prove that all the normals of the curve

$$\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t) \end{cases}$$

($a > 0$), lie at the same distance with respect to the origin.

Problem 6.11. Show that the curve

$$y^2(x + y - 1) = x^3$$

is the envelope of the family of straight lines

$$(\lambda^3 - 3\lambda)x + (\lambda^3 + 2)y = \lambda^3.$$

Problem 6.12. Find the envelope of the family of curves

(a) $x = u^2 + v^2, \quad y = uv(u + v)$;

(b) $x = u + v^2, \quad u^2 + v$

when

(i) the parameter of the family is u;

(ii) the parameter of the family is v.

Problem 6.13. Consider the curve

$$\begin{cases} x = t - \operatorname{sh} t \cosh t, \\ y = 2\cosh t. \end{cases}$$

(a) Find the signed curvature of the curve at an arbitrary point.

(b) Find the evolute of the curve.

Problem 6.14. Find the evolute of the curve

$$\begin{cases} x = a\left(\cos t + \ln \operatorname{tg}\dfrac{t}{2}\right), \\ y = a\sin t, \end{cases}$$

with $a > 0$ (the tractrix).

Problem 6.15. Find the equation of the osculating circle and the evolute of the curve

$$\begin{cases} x = \dfrac{t^2}{t^2+1}, \\ y = \dfrac{t^3}{t^2+1}. \end{cases}$$

Problem 6.16. Find the parametric equations of the curves given by the intrinsic equations:

(a) $k_{\pm}(s) = \dfrac{1}{\sqrt{2as}}$, cu $a > 0$;

(b) $k_{\pm}(s) = \dfrac{1}{\sqrt{16a^2 - s^2}}$, cu $a > 0$.

Problem 6.17. Let $M(u = 1, v = \pi/2)$ be a point of the surface

$$\begin{cases} x = u + \cos v, \\ y = u - \sin v, \\ z = \lambda u. \end{cases}$$

a) Write the equations of the tangents and normal planes to the curves $u = 1$ and $v = \pi/2$ at the point M.

b) Find the angle between the curves $u = 1$ and $v = \pi/2$.

c) Show that the tangent to the curve $u = \sin v$ at M coincides to the tangent to the curve $u = 1$ at the same point.

Problem 6.18. Find the equation of the tangent plane at the surface

$$\mathbf{r}(u, v) = (2u - v, u^2 + v^2, u^3 - v^3)$$

at the point $M(3, 5, 7)$.

Problem 6.19. Write the equation of the tangent plane and those of the normal at the surface

$$\mathbf{r}(u, v) = (u + v, u - v, uv)$$

at the point $M(u = 2, v = 1)$.

Problem 6.20. Write the equation of the tangent plane and those of the right helicoid

$$\mathbf{r}(u, v) = (u \cos v, u \sin v, av),$$

where a is a strictly positive real number, $u > 0$ and $v \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, at the points where the normal is parallel to the vector $\mathbf{p}(1, 2, 3)$.

Problem 6.21. Prove that the curve

$$\boldsymbol{\rho}(t) = \left(\frac{2}{1+t}, \frac{2}{1-t}, t\right), \quad t \neq \pm 1,$$

lies on the surface

$$z = \frac{1}{x^2} - \frac{1}{y^2}, \quad x, y \neq 0,$$

and that at each point of the curve its osculating plane coincides wit hthe tangent plane to the surface at that point.

Problem 6.22. Find the equation of the tangent plane and those of the normal at the surface

$$\mathbf{r}(u,v) = (u, u^2 - 2uv, u^3 - 3u^2v).$$

at the point $M(1,3,4)$.

Problem 6.23. Write the equations of the tangent planes and those of the normals at the following surfaces at the indicated points:

(a) $z = x^3 + y^3$ at the point $M(1,2,9)$;

(b) $x^2 + y^2 + z^2 = 169$ at the point $M(3,4,12)$;

(c) $x^2 - 2y^2 - 3z^2 - 4 = 0$ at the point $M(3,1,-1)$.

Problem 6.24. Find the equations of the normal at the surface

$$\mathbf{r}(u,v) = (u+v, u-v, uv+3)$$

that passes throug h the origin.

Problem 6.25. Write the equation of the tangent plane at the pseudosphere

$$\mathbf{r}(u,v) = \left(a \sin u \cos v, a \sin u \sin v, a\left(\ln \operatorname{tg} \frac{u}{2} + \cos u\right)\right),$$

at an arbitrary point.

Problem 6.26. Find the equation of the tangent plane at the surface $xyz = 1$ which is parallel to the plane $x + y + z - 3 = 0$.

Problem 6.27. Write the equation of the tangent plane at the torus

$$\mathbf{r}(u,v) = ((7 + 5\cos u)\cos v, (7 + 5\cos u)\sin v, 5\sin u)$$

at the point $M(u_0, v_0)$ for which $\cos u_0 = 3/5$ and $\cos v_0 = 4/5$, where $0 < u, v < \pi/2$.

Problem 6.28. Find the first fundamental form for the following surfaces:

(a) $\mathbf{r}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, where f and g are two smooth functions (rotation surface around the z-axis);

(b) $\mathbf{r}(u,v) = (R\cos u \cos v, R\cos u \sin v, R\sin u)$ (the sphere);

(c) $\mathbf{r}(u,v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ (the torus);

(d) $\mathbf{r}(u,v) = \left(a\sin u\cos v, a\sin u\sin v, a\left(\ln \operatorname{tg}\frac{u}{2} + \cos u\right)\right)$ (the pseudosphere);

(e) $\mathbf{r}(u,v) = \left(a\cosh\frac{u}{a}\cos v, a\cosh\frac{u}{a}\sin v, u\right)$ (the catenoid);

(f) $\mathbf{r}(u,v) = (u\cos v, u\sin v, av)$ (the right helicoid).

Problem 6.29. Find the equations of the curves making a constant angle α with the meridians of the sphere (the $v = const$ in the parameterization from the previous problem (the loxodroms of the sphere).

Problem 6.30. On the hyperbolic paraboloid

$$\mathbf{r}(u,v) = (u, v, uv)$$

we consider the curves $\boldsymbol{\rho}_1(t) = (\cos t, \sin t)$ and $\boldsymbol{\rho}_2(s) = (s, ks)$, where $k \neq 0$.

(a) Find the intersection points between the two curves and find the angle θ between them.

(b) For what values of k are the curves orthogonal?

Problem 6.31. Prove that the curve $u = \ln t, v = 2 \operatorname{arctg} t, t > 0$ from the sphere

$$\mathbf{r}(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$$

intersects the meridians $u = const$ under a constant angle, equal to $\pi/4$.

Problem 6.32. Compute the length of the curve $v = au$ from the surface

$$\mathbf{r}(u, v) = \big(u^2 + v^2, u^2 - v^2, uv\big)$$

between the intersection points with the curves $u = 1$ and $v = 1$.

Problem 6.33. Compute the area of the portion of the sphere with the parameterization

$$\mathbf{r} = (R \cos v \cos u, R \sin v \cos u, R \sin u),$$

corresponding to $u \in \left[0, \dfrac{\pi}{2}\right]$, $v \in \left[0, \dfrac{\pi}{2}\right]$.

Problem 6.34. Compute the area of the portion of the torus with the parameterization

$$\mathbf{r}(u, v) = ((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u),$$

with $a, b > 0$, corresponding to $u \in [u_1, u_2]$, $v \in [v_1, v_2]$.

Problem 6.35. On a surface with the coefficients of the first fundamental form given by $E = 1$, $F = 0$, $G = u^2 + v^2$, find the perimeter and the interior angles of the curvilinear triangle determined by the curves $u = \pm\dfrac{1}{2}v^2$, $v = 1$.

Problem 6.36. Find the angle between the curves $u + v = 0$ and $u - v = 0$ on the helicoid

$$\mathbf{r}(u, v) = (u \cos v, u \sin v, av)$$

Problem 6.37. Find the curves that divide in equal parts the angles between the coordinate curves of the following surfaces:

(a) the sphere $\mathbf{r}(u, v) = (R \cos u \cos v, R \cos u \sin v, R \sin u)$;

(b) the right helicoid $\mathbf{r}(u, v) = (u \cos v, u \sin v, av)$;

(c) the developable helicoid $\mathbf{r}(u, v) = (u \cos v, u \sin v, u + v)$.

SEMINAR 7

Seminary

Problem 7.1. Find the second fundamental form for the following surfaces:

(a) $\mathbf{r}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, where f and g are two smooth functions (surface of revolution around the z-axis);

(b) $\mathbf{r}(u,v) = (R\cos u\cos v, R\cos u\sin v, R\sin u)$ (the sphere);

(c) $\mathbf{r}(u,v) = ((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u)$ (the torus);

(d) $\mathbf{r}(u,v) = \left(a\sin u\cos v, a\sin u\sin v, a\left(\ln\operatorname{tg}\frac{u}{2} + \cos u\right)\right)$ (the pseudosphere);

(e) $\mathbf{r}(u,v) = \left(a\cosh\dfrac{u}{a}\cos v, a\cosh\dfrac{u}{a}\sin v, u\right)$ (the catenoid);

(f) $\mathbf{r}(u,v) = (u\cos v, u\sin v, av)$ (the right helicoid).

Problem 7.2. Find the principal curvatures at the vertices of the hyperboloid with two sheets

$$\mathbf{r}(u,v) = \left(\frac{a}{2}\left(v-\frac{1}{v}\right)\cos u, \frac{b}{2}\left(v-\frac{1}{v}\right)\sin u, \frac{c}{2}\left(v+\frac{1}{v}\right)\right).$$

Hint: Choose local parameterizations of the surface. Beware that you should use different parameterizations around the two vertices. □

Problem 7.3. Find the elliptic, parabolic and hyperbolic points of the torus.

Problem 7.4. Find the total curvature of the paraboloid

$$z = \frac{1}{2}\left(\frac{x^2}{p} + \frac{y^2}{q}\right).$$

Problem 7.5. Find the total curvature of the surface

$$\mathbf{r}(u,v) = (u\cos v, u\sin v, av).$$

Along which curves on the surface is the total curvature constant?

Problem 7.6. Find the normal curvature of the coordinate lines for the following surfaces:

a) the helicoid: $x = u\cos v,\ y = u\sin v,\ z = av$, cu $u > 0, v \in [0, 2\pi)$, $a \neq 0$;

b) the sphere: $x = r\cos u\cos v,\ y = r\cos u\sin v,\ z = r\sin u$, $u \in [0, 2\pi), \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$;

c) the catenoid: $x = \sqrt{a^2+u^2}\cos u,\ y = \sqrt{a^2+u^2}\sin v,\ z = a\ln\left(u+\sqrt{a^2+u^2}\right)$, $u > 0, v \in [0, 2\pi)$;

d) the surface of revolution: $x = e^{-\frac{u^2}{2}}\cos v,\ y = e^{-\frac{u^2}{2}}\sin v,\ z = u$, $u \in \mathbb{R}, v \in [0, 2\pi)$.

Problem 7.7. Show that the umbilical points of the surface

$$\mathbf{r}(u,v) = \left(\frac{1}{2}u^2 + v, u + \frac{1}{2}v^2, uv\right)$$

lie on the curves $u = v$ and $u + v + 1 = 0$.

Problem 7.8. Find the total curvature of the pseudosphere

$$\mathbf{r}(u,v) = \left(a\sin u\cos v, a\sin u\sin v, a\left(\ln\operatorname{tg}\frac{u}{2} + \cos u\right)\right),$$

$u \in [0, \pi), v \in [0, 2\pi)$, with $a > 0$, at an arbitrary point.

Problem 7.9. Consider the surface

$$\mathbf{r}(u,v) = \left(u\cos v, u\sin v, \sqrt{2p\left(u - \frac{p}{2}\right)}\right),$$

where $u > 0, v \in [0, 2\pi)$ and $p > 0$. Show rhat the ratio of the principal curvature of the surface is constant.

Problem 7.10. Find the lines of curvature of the right helicoid

$$\mathbf{r}(u,v) = (u\cos v, u\sin v, a\cdot v).$$

Problem 7.11. Find the lines of curvature of the paraboloid

$$\mathbf{r}(u,v) = ((u+v)\sqrt{p}, (u-v)\sqrt{q}, 2uv),$$

where p and q are strictly positive constants.

Problem 7.12. Find the lines of curvature of the surface of revolution

$$\mathbf{r}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where f and g are arbitrary smooth functions.

Problem 7.13. Find the envelope of the family of planes

$$(x - a\cos\lambda)^2 + (y - a\sin\lambda)^2 + z^2 = b^2,$$

where a and b are stricly positive real numbers.

Problem 7.14. Find the envelope of the family of planes

$$3\lambda^2 x - 3\lambda y + z - \lambda^3 = 0$$

and find the equations of its regression edge.

Problem 7.15. Find the equation of the develoopable surface having the curve

$$\mathbf{r}(t) = (6t, 3t^2, 2t^3)$$

as regression edge.

Problem 7.16. Find the equation of the developable surface having the helix

$$\mathbf{r}(t) = (a\cos t, a\sin t, bt)$$

as regression edge.

Problem 7.17. Establish whether the ruled surface

$$\mathbf{r}(u, v) = \left(e^u + v\cos u, e^{-u} + v\sin u, \sqrt{2}u\right)$$

is developable or not.

Problem 7.18. Consider the surface

$$\mathbf{r}(u, v) = (\cos v - (u + v)\sin v, \sin v + (u + v)\cos v, u + 2v)\,.$$

(i) Show that the surface is ruled and point out a directing curve and the corresponding line generators. Is the surface developable?

(ii) Find the principal curvature of the surface, as well as its parabolic and planar points.

Problem 7.19. Consider the family of planes

$$\lambda(x + y) - \lambda^2 z - 1 = 0.$$

(i) Find the envelope of the family. Does the envelope have a regression edge?

(ii) Find the nature of the points of the envelope, as well as its asymptotic lines.

(iii) Find the principal curvatures of the envelope at an arbitrary point.

Problem 7.20. Find the most general expression of the function $b(\lambda)$ frio wghich the envelope of the family of planes

$$(1 + \lambda)x + \lambda^2 y + \lambda z - b(\lambda) = 0$$

is a conical surface.

Hint. The regression edge of the envelpe should degenerate to a point. □

Problem 7.21. Find the regression edge of the envelope of the family of planes

$$x\sin\lambda - y\cos\lambda + z - k\lambda = 0,$$

where k is a non-vanishing real constant.

Problem 7.22. Consider the surface

$$\mathbf{r}(u, v) = \left(u + (v - 1)\sin u, 1 + (v - 1)\cos u, a\left(\frac{u^2}{2} + v\right)\right),$$

where a is a non-vanishing constant.

(i) Show that the surface is ruled, indicating the directing curve and the line generators.

(ii) Find all the space curves whose tangents are parallel to the generators of the surface.

(iii) Among the curves from the previous point, identify thos of constant curvature.

(iv) Find the asymptotic lines of the surface.

Problem 7.23. Find the regression edge of the developable surface which is the envelope of the family of tangent planes to the hyperbolic paraboloid $z = xy$, along the intersection line between this paraboloid and the cylinder $x^2 = y$.

Problem 7.24. Prove that on a surface the coordinate lines are, also, lines of curvature iff $F = D' = 0$, i.e. iff both fundamental forms are diagonal in the chosen parameterization.

Problem 7.25. Prove that the coordinate lines on the surface

$$\mathbf{r}(u,v) = \left(3u - u^3 + 3uv^2, v^3 - 3u^2v - 3v, 3(u^2 - v^2)\right)$$

are lines of curvature.

Problem 7.26. Find the lines of curvature of the surface

$$\mathbf{r}(u,v) = (u^2 + v^2, u^2 - v^2, v).$$

Problem 7.27. Find the asymptotic lines of the surface

$$\mathbf{r}(u,v) = \left(u^2 + v, u^3 + uv, u^4 + \frac{2}{3}u^2v\right).$$

Problem 7.28. Find the asymptotic lines of the catenoid

$$\mathbf{r}(u,v) = (\cosh u \cos v, \cosh u \sin v, u).$$

Problem 7.29. Find the asymptotic lines of the surface

$$z = xy^3 - yx^3$$

passing through the point $M(1,2,6)$

Problem 7.30. Find the asymptotic lines of the surface

$$\mathbf{r}(u,v) = \left(3(u+v), 3(u^2+v^2), 2(u^3+v^3)\right). \tag{7.0.1}$$

Problem 7.31. For the surface

$$\mathbf{r}(u,v) = \left(u\left(3v^2 - u^2 - \frac{1}{3}\right), v\left(3u^2 - v^2 - \frac{1}{3}\right), 2uv\right),$$

find the lines of curvature, the asymptotic lines, the total curvature and the mean curvature at an arbitrary point.