1) prove: $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$

$$f(x,y) = \ln(2\sqrt{x^2+y^2} - x) - \ln y$$

$$x \frac{\partial f}{\partial x}(x,y) + y \frac{\partial f}{\partial y}(x,y) = 0 \quad \forall (x,y) \in \mathbb{R} \times (0,0)$$

solution:

$$\left.\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \frac{2 \cdot \frac{2x}{2\sqrt{x^2+y^2}} - 1}{2\sqrt{x^2+y^2} - x} = \frac{2x - \sqrt{x^2+y^2}}{\sqrt{x^2-y^2}(2\sqrt{x^2-y^2} - x)} \\ \frac{\partial f}{\partial y}(x,y) &= \frac{2 \cdot \frac{2y}{2\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2} - x} - \frac{1}{y} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \quad x \frac{\partial f}{\partial x}(x,y) + y \frac{\partial f}{\partial y} = \frac{2x^2 - x\sqrt{x^2+y^2} + 2y^2}{\sqrt{x^2+y^2}(2\sqrt{x^2+y^2} - x)} - 1 = 1 - 1 = 0$$

2) $f : (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$

$f(x,y,z) = (xy + z^2)\cos\left(\frac{yz}{x^2}\right)$ satisfies

$$x \frac{\partial f}{\partial x}(x,y,z) + y \frac{\partial f}{\partial y}(x,y,z) + z \frac{\partial f}{\partial z}(x,y,z) = 2 f(x,y,z) \quad \forall (x,y,z) \in (0,\infty) \times \mathbb{R}^2$$

$$x \frac{\partial f}{\partial x}(x,y,z) = y \cdot \cos\frac{yz}{x^2} - (xy+z^2)\sin\left(\frac{yz}{x^2}\right) \cdot \frac{(-2)yz}{x^3} = y\cos\left(\frac{yz}{x^2}\right) + \sin\left(\frac{yz}{x^2}\right)\frac{2yz}{x^3}(xy+z^2)$$

$$y \frac{\partial f}{\partial y}(x,y,z) = x \cos\left(\frac{yz}{x^2}\right) - (xy+z^2)\sin\left(\frac{yz}{x^2}\right) \cdot \frac{z}{x^2}$$

$$z \frac{\partial f}{\partial z}(x,y,z) = 2z \cdot \cos\frac{yz}{x^2} - (xy+z^2)\sin\left(\frac{yz}{x^2}\right)\frac{y}{x^2}$$

4) Let $r > 0$, $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < r^2\}$ and $f : A \to \mathbb{R}$ $\quad f(x,y) = 2\ln\left(\frac{r\sqrt{8}}{r^2 - x - y^2}\right)$, prove that

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = e^{f(x,y)} \quad \forall (x,y) \in A$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial f}{\partial x}(x,y)=\frac{-2\cdot(-2x)}{r^2-x^2-y^2}=\frac{4x}{r^2-x^2-y^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{4y}{r^2-x^2-y^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(x,y)=\frac{4(r^2-x^2-y^2)-4x(-2x)}{(r^2-x^2-y^2)^2}=\frac{4(r^2-x^2-y^2)+8x^2}{(r^2-x^2-y^2)^2}=\frac{4(r^2+x^2-y^2)}{(r^2-x^2-y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{4(r^2-x^2+y^2)}{(r^2-x^2-y^2)^2}$$

$$\Rightarrow$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{8r^2}{(r^2-x^2-y^2)^2}=e^{\left(2\ln\frac{r\sqrt{8}}{(r^2-x^2-y^2)}\right)}=e^{f(x,y)}$$

5) Prove $f:(0,\infty)\times\mathbb{R}\to\mathbb{R}$, $f(x,y)=(x^2+y^2)\arctan\frac{y}{x}$ satisfies

$$x^2\frac{\partial^2 f}{\partial x^2}(x,y)+2xy\frac{\partial^2 f}{\partial x\partial y}(x,y)+y^2\frac{\partial^2 f}{\partial y^2}(x,y)=2\cdot f(x,y)\quad\forall (x,y)\in(0,\infty)\times\mathbb{R}$$

$$\boxed{(\arctan x)'=\frac{1}{x^2+1}}$$

$$\frac{\partial f}{\partial x}(x,y)=2x\arctan\frac{y}{x}+(x^2+y^2)\cdot\left(\frac{-\frac{y}{x^2}}{\frac{y^2}{x^2}+1}\right)=$$

$$=2x\arctan\frac{y}{x}-(x^2+y^2)\cdot\frac{y}{x^2+y^2}=2x\arctan\frac{y}{x}-y$$

$$\frac{\partial f}{\partial y}(x,y)=2y\arctan\frac{y}{x}+(x^2+y^2)\cdot\frac{\frac{1}{x}}{\frac{y^2}{x^2}+1}=2y\arctan\frac{y}{x}+(x^2+y^2)\frac{\frac{1}{x}}{\frac{x^2+y^2}{x^2}}=2y\arctan\frac{y}{x}+x$$

$$\frac{\partial^2 f}{\partial x^2}(x,y)=2\arctan\frac{y}{x}+2x\cdot\left(\frac{-y}{x^2+y^2}\right)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=2\arctan\frac{y}{x}+2y\left(\frac{x}{x^2+y^2}\right)$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=2y\cdot\frac{-y}{x^2+y^2}+1$$

$$x^2\frac{\partial^2 f}{\partial x^2}(x,y)+2xy\frac{\partial^2 f}{\partial x\partial y}(x,y)+y^2\frac{\partial^2 f}{\partial y^2}(x,y)=\ldots$$

6) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x\sqrt{x^2+y^2}$, determine $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\nabla f(3,4)$ and $df(3,4)$

sol: $\frac{\partial f}{\partial x}(x,y) = \sqrt{x^2+y^2} + x \cdot \frac{2x}{2\sqrt{x^2+y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$

$\frac{\partial f}{\partial y}(x,y) = x \cdot \frac{2y}{2\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$

$$\boxed{\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)}$$

$\nabla f(3,4) = \left(\frac{34}{5}, \frac{12}{5}\right)$

$df(3,4) \in L(\mathbb{R}^2, \mathbb{R})$

$df(3,4)(h_1,h_2) = h_1 \frac{\partial f}{\partial x}(3,4) + h_2 \frac{\partial f}{\partial y}(3,4) = \frac{34}{5}h_1 + \frac{12}{5}h_2$