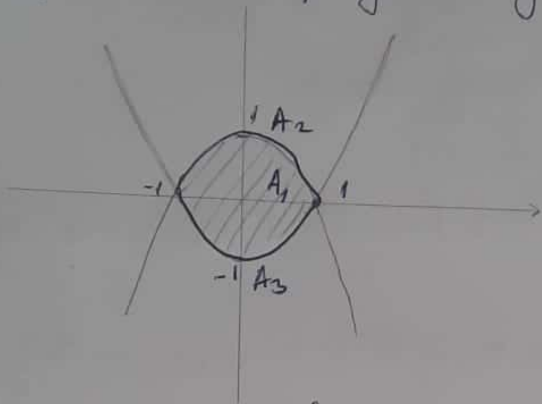
$\boxed{1}$ $A = \{ (x,y) \in \mathbb{R}^2 \mid x^2 - 1 \le y \le 1 - x^2 \}$

$f : A \to \mathbb{R} \;;\; f(x,y) = x^2 - y^2$



Fie $m_1 = \min\limits_{(x,y)\in A_1} f(x,y)$ $\qquad A_1 = \operatorname{int}(A) \Rightarrow$ mulțime deschisă

$M_1 = \max\limits_{(x,y)\in A_1} f(x,y)$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x = 0 \Rightarrow x = 0 \\ \frac{\partial f}{\partial y} &= -2y \Rightarrow y = 0 \end{aligned}\right\} \Rightarrow (0,0) \text{ este singurul pct critic pt } f$$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2 \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = 0 \\ \frac{\partial^2 f}{\partial y^2} &= -2 \end{aligned}\right| \Rightarrow H = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \text{ (hessiana)}$$

$$\left.\begin{aligned} \Delta_1 &= 2 \\ \Delta_2 &= \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow$ nu putem folosi Criteriul lui Sylvester

Fie șirul $(x_n)_{n \ge 1}$, $x_n = \left(\frac{1}{n}, 0\right) \Rightarrow f(x_n) = \frac{1}{n^2} > 0 = f((0,0)) \Rightarrow$

$\Rightarrow (0,0)$ nu poate fi pct de maxim

Fie șirul $(y_n)_{n \ge 1}$, $y_n = \left(0, \frac{1}{n}\right) \Rightarrow f(y_n) = -\frac{1}{n^2} < 0 = f((0,0)) \Rightarrow$

$\Rightarrow (0,0)$ nu poate fi pct de minim

$\Rightarrow$ $(0,0)$ punct șa $\Rightarrow$ funcția f își atinge extremele pe bd A

Fie $m_2 = \min\limits_{(x,y)\in A_2} f(x,y) = \min\limits_{\substack{y=1-x^2\\ x\in[-1,1]}} f(x,y)$

$M_2 = \max\limits_{(x,y)\in A_2} f(x,y) = \max\limits_{\substack{y=1-x^2\\ x\in[-1,1]}} f(x,y)$

$y = 1-x^2 \Rightarrow f(x,y) = x^2-(1-x^2)^2 = x^2 - x^4 + 2x^2 - 1 = -x^4+3x^2-1 = g(x)$

$g'(x) = -4x^3+6x = 0 \Rightarrow x(6-4x^2)=0 \Rightarrow x\in\{0, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}\}$

$\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}} \notin [-1,1]$ $\Rightarrow$ $x\in\{0\}$

x	-1		0		1
$g(x)$	$g(-1)$	$\searrow$	-1	$\nearrow$	$g(1)$
$g'(x)$	$-$	$- - - - -$	0	$+ + + + +$	

$\Rightarrow m_2 = -1$

$M_2 = \max\{g(-1), g(1)\} = \max\{1,1\} = 1$

Fie $m_3 = \min\limits_{(x,y)\in A_3} f(x,y) = \min\limits_{\substack{y=x^2-1\\ x\in[-1,1]}} f(x,y)$

$M_3 = \max\limits_{(x,y)\in A_3} f(x,y) = \max\limits_{\substack{y=1-x^2\\ x\in[-1,1]}} f(x,y)$

$y = x^2-1 \Rightarrow f(x,y) = x^2-(x^2-1)^2 = -x^4+3x^2-1 = g(x)$

Analog ca la A_2 se arata că $m_3=-1$, $M_3=1$

$\Rightarrow$ $\boxed{\begin{array}{l} m = \min\limits_{(x,y)\in A} f(A) = \min\{m_2, m_3\} = -1 \\ M = \max f(A) = \max\{M_2, M_3\} = 1 \end{array}}$

② ~~$\forall$~~ $f : \mathbb{R}^2 \to \mathbb{R}$

~~(1) • $\forall y \in \mathbb{R}$ $f_y(x) : \mathbb{R} \to \mathbb{R}$ $f_y(x) = f(x,y)$ continua pe $\mathbb{R}$~~

~~(2) • $\exists \alpha > 0$ a.î. $|f(x,y) - f(x,y')| \le \alpha |y-y'|$ $\forall x, y, y' \in \mathbb{R}$~~

~~(2) $\Rightarrow$ $f_x : \mathbb{R} \to \mathbb{R}$, $f_x(y) = f(x,y)$ este funcție Lipschitz $\Rightarrow$~~

~~$\Rightarrow$ f_x este continuă~~

~~Fie $\varepsilon > 0$, f~~

• Fie $(x_0, y_0) \in \mathbb{R}^2$, Fie $\varepsilon > 0$

Fie f_y continuă $\Rightarrow$ $\exists \delta > 0$ a.î. $\forall x \in \mathbb{R}$ cu $|x - x_0| < \delta$

$|f_{y_0}(x) - f_{y_0}(x_0)| < \frac{\varepsilon}{2}$

$\Leftrightarrow$ $\exists \delta > 0$ a.î. $\forall x \in \mathbb{R}$ cu $|x - x_0| < \delta$

$|f(x_0, y_0) - f(x, y_0)| < \frac{\varepsilon}{2}$

Fie $y \in \mathbb{R}$ a.î. $|y - y_0| < \frac{\varepsilon}{2\alpha}$

• Atunci $\forall x \in \mathbb{R}$ $|f(x,y_0) - f(x,y)| \le \alpha |y - y_0| < \frac{\alpha \varepsilon}{2\alpha} = \frac{\varepsilon}{2}$

Fie $x \in (x_0 - \delta, x_0 + \delta)$ și $y \in (y_0 - \frac{\varepsilon}{2\alpha}, y_0 + \frac{\varepsilon}{2\alpha})$

Atunci $|f(x,y) - f(x_0,y_0)| = |f(x,y) - f(x,y_0) + f(x,y_0) - f(x_0,y_0)|$

$\le |f(x,y) - f(x,y_0)| + |f(x,y_0) - f(x_0,y_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists r = \min\{\delta, \frac{\varepsilon}{2\alpha}\} > 0$ a.î. $\forall (x,y) \in \mathbb{R}^2$

cu $\|(x,y) - (x_0,y_0)\| < r$, $|f(x,y) - f(x_0,y_0)| < \varepsilon$ $\Rightarrow$

$\Rightarrow$ f continua în (x_0, y_0) $\}$
(x_0, y_0) punct arbitrar $\}$ $\Rightarrow$ f continuă pe $\mathbb{R}^2$

③ $I=\iiint_A \frac{1}{z^2}\,dx\,dy\,dz$; $A=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le 1,\ z\ge \frac{1}{2}\}$

Trecem la coordonate sferice

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi\end{cases} \Rightarrow \frac{D(x,y,z)}{D(r,\theta,\varphi)} = r^2\sin\varphi$$

~~$A=\{(r,\theta,\varphi)\in$~~

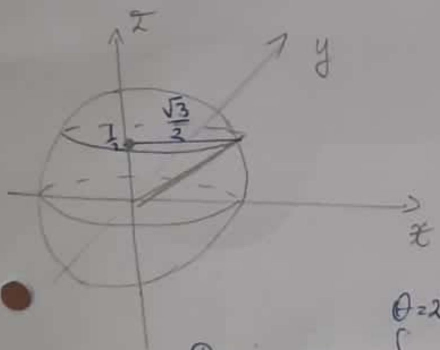
$x^2+y^2+z^2\le 1 \Rightarrow r^2\le 1 \Rightarrow r\le 1$

$z\ge \frac{1}{2} \Rightarrow r\cos\varphi \ge \frac{1}{2} \Rightarrow r\ge \frac{1}{2\cos\varphi}$

$\Rightarrow r\in\left[\frac{1}{2\sin\varphi}, 1\right]$

$\theta\in[0,2\pi]$

$\varphi\in[0, \operatorname{arctg}\sqrt{3}] \Rightarrow \varphi\in\left[0,\frac{\pi}{3}\right]$



Deci $I = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{3}}\int_{r=\frac{1}{2\cos\varphi}}^{r=1} \frac{1}{r^2\cos^2\varphi}\cdot r^2\sin\varphi\, d\theta\, d\varphi\, dr$

$= \int_0^{2\pi} d\theta \cdot \int_{\varphi=0}^{\varphi=\frac{\pi}{3}}\frac{\sin\varphi}{\cos^2\varphi}\cdot\left(\int_{\frac{1}{2\cos\varphi}}^{1} dr\right)d\varphi$

$= 2\pi\cdot\int_0^{\frac{\pi}{3}}\frac{\sin\varphi}{\cos^2\varphi}\left(1-\frac{1}{2\cos\varphi}\right)d\varphi$

$= 2\pi\left(\int_0^{\frac{\pi}{3}}\frac{\sin\varphi}{\cos^2\varphi} - \frac{1}{2}\int_0^{\frac{\pi}{3}} \operatorname{tg}\varphi\cdot\frac{1}{\cos^2\varphi}\,d\varphi\right)$

$$I = 2\pi \left(-\int_0^{\pi/3} \frac{\cos'(\varphi)}{\cos^2\varphi}\, d\varphi - \frac{1}{2}\int_0^{\pi/3} tg\varphi \cdot tg'(\varphi)\, d\varphi \right)$$

$$= 2\pi \left(\frac{1}{\cos\varphi}\Big|_0^{\pi/3} - \frac{1}{4} tg^2\varphi \Big|_0^{\pi/3} \right)$$

$$= 2\pi \left(\frac{1}{\frac{1}{2}} - 1 - \frac{1}{4}(3-0) \right) = 2\pi\left(2 - 1 - \frac{3}{4}\right) \Rightarrow$$

$$\Rightarrow \boxed{I = \frac{\pi}{2}}$$

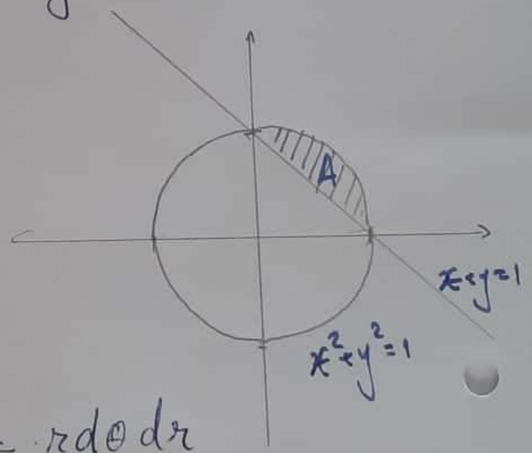
④ $I = \iint_A \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy$, $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1 \le x+y\}$

Trecem la coordonate polare $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \frac{D(x,y)}{D(r,\theta)} = r$

$x^2+y^2 \le 1 \Rightarrow r^2 \le 1 \Rightarrow r \le 1$

$x+y \ge 1 \Rightarrow r(\sin\theta + \cos\theta) \ge 1$

$\Rightarrow r \ge \frac{1}{\sin\theta + \cos\theta}$



$\theta \in [0, \frac{\pi}{2}]$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\sin\theta+\cos\theta}}^{1} \frac{1}{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}} \cdot r\, d\theta\, dr$$

$$I = \int_0^{\frac{\pi}{2}} \left(\int_{\frac{1}{\sin\theta+\cos\theta}}^{1} \frac{1}{r} \cdot r\, dr \right) d\theta = \int_0^{\pi/2} r \Big|_{\frac{1}{\sin\theta+\cos\theta}}^{1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{\sin\theta+\cos\theta}\right) d\theta = \int_0^{\pi/2} 1\, d\theta - \int_0^{\pi/2} \frac{1}{\sin\theta+\cos\theta}\, d\theta$$

$$= \pi/2 - \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta + \cos\theta}\, d\theta$$

$t = tg\frac{\theta}{2} \Rightarrow \theta = 2arctg\, t \Rightarrow d\theta = \frac{2}{1+t^2}dt$

$\sin\theta = \frac{2tg\frac{\theta}{2}}{1+tg^2\frac{\theta}{2}} = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-tg^2\frac{\theta}{2}}{1+tg^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2}$

$I = \frac{\pi}{2} - \int_0^1 \frac{1}{\frac{2t+1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}dt = \frac{\pi}{2} - 2\int_0^1 \frac{1}{2-(t-1)^2}dt$

$= \frac{\pi}{2} - 2 \cdot \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right| \Bigg|_{t=0}^{t=1}$

$= \frac{\pi}{2} - \frac{1}{\sqrt{2}}\ln(1) + \frac{1}{\sqrt{2}}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1} \Rightarrow \boxed{I = \frac{\pi}{2} + \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}}$

5. $f(x,y,z) = \underbrace{2x\ln z}_{:=X}\, dx + \underbrace{\frac{e^y}{z}}_{:=Y}dy + \underbrace{\frac{ax^2z+be^y}{z^2}}_{:=Z} dz$

$\int f$ nu depinde de drum $\Rightarrow \begin{cases} \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} \\ \frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x} \\ \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y} \end{cases}$

$\frac{\partial X}{\partial y} = 0 ;$ $\quad \frac{\partial X}{\partial z} = \frac{2x}{z}$

$\parallel$

$\frac{\partial Y}{\partial z} = 0 ,$ $\quad \frac{\partial Z}{\partial x} = \frac{2ax}{z}$ $\Bigg\} \Rightarrow 2 = 2a = a = 1$

$\left.\begin{array}{l} \frac{\partial Y}{\partial z} = -\frac{e^y}{z^2} \\ \frac{\partial Z}{\partial y} = \frac{be^y}{z^2} \end{array}\right\} \Rightarrow b = -1$

Deci $f(x,y,z) = 2x\ln z\, dx + \frac{e^y}{z} dy + \frac{x^2 z - e^y}{z^2} dz$

Căutăm $F : \mathbb{R}^2 \times (0,\infty) \to \mathbb{R}$ a.î.

$$\begin{cases} \frac{\partial F}{\partial x} = X \\ \frac{\partial F}{\partial y} = Y \\ \frac{\partial F}{\partial z} = Z \end{cases} \Rightarrow \frac{\partial F}{\partial x} = 2x\ln z \Rightarrow F = \int 2x \ln z\, dx = x^2 \ln z + \varphi(y,z)$$

φ derivabilă

$$\frac{\partial F}{\partial y} = Y \Rightarrow \frac{\partial}{\partial y}\left(x^2 \ln z + \varphi(y,z)\right) = \frac{e^y}{z} \Rightarrow$$

$$\Rightarrow \frac{\partial \varphi}{\partial y}(y,z) = \frac{e^y}{z} \Rightarrow \varphi(y,z) = \int \frac{e^y}{z} dy$$

$$\Rightarrow \varphi(y,z) = \frac{e^y}{z} + \psi(z), \quad \psi \text{ derivabilă}$$

$$\Rightarrow F(x,y,z) = x^2 \ln z + \frac{e^y}{z} + \psi(z) \Rightarrow \frac{\partial F}{\partial z} = \frac{x^2}{z} - \frac{e^y}{z^2} + \psi'(z)$$

dar $\frac{\partial F}{\partial z} = Z = \frac{x^2}{z} - \frac{e^y}{z^2}$

$$\Rightarrow \psi'(z) = 0 \Rightarrow \psi(z) = 0$$

$\Rightarrow$ O primitivă a lui f este $F(x,y,z) = \cancel{\frac{x^3}{z} + e^y}$ $x^2 \ln z + \frac{e^y}{z}$

$$\Rightarrow \int_{(1,0,1)}^{(-1,1,e)} f = F(-1,1,e) - F(1,0,1)$$

$$= \left(1 \cdot \ln e + \frac{e}{e}\right) - \left(1 \cdot \ln 1 + \frac{1}{1}\right)$$

$$= (1+1) - 1 = 1$$

$$\boxed{\int_{(1,0,1)}^{(-1,1,e)} f = 1}$$