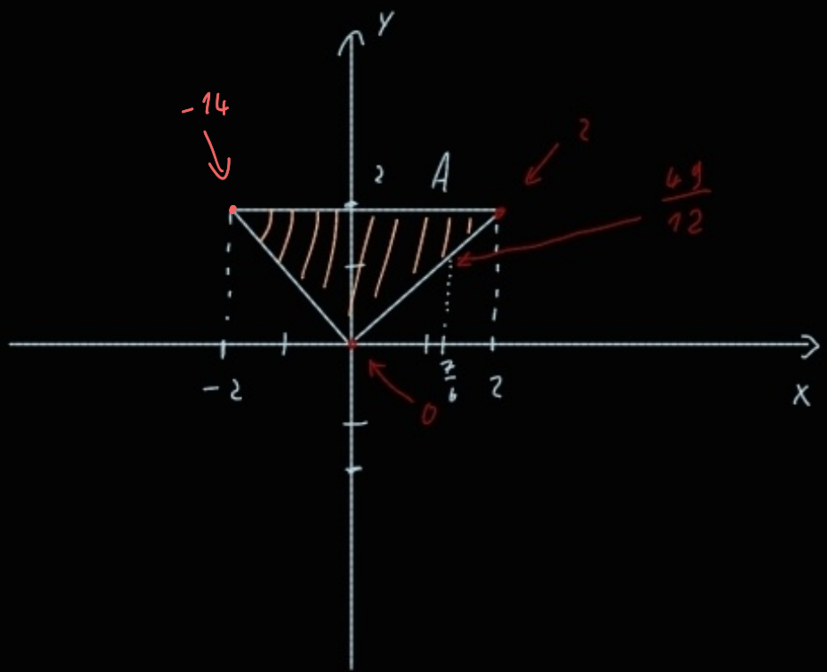
4) consider $A=\{(x,y)\in\mathbb{R}^2 \mid |x|\le |y| \le 2\}$ and $f: A\to\mathbb{R}$

$f(x,y)=4x+3y-2x^2-y^2$ determine $\min f(A)$ $\max f(A)$



A is compact and has non-empty interior

$m=\min f(A)$, $M=\max f(A)$

$m_1=\min f(bd(A))$ $M_1=\max f(bd(A))$ $C=\{(x,y)\in \operatorname{int} A \mid \nabla(x,y)=(0,0)\}$

$m_2=\min f(C)$ $M_2=\max f(C)$

$bd\,A=\{(x,x) \mid x\in[0,2\} \cup \{(-x,x) \mid x\in[0,2]\} \cup \{(x,2) \mid x\in[-2,2]\}$

$f(x,x)=-3x^2+7x$

x	0	$\frac{7}{6}$	2
$-3x^2+7x$	0 $\nearrow$	$\frac{49}{12}$	$\searrow$ 2

$f\left(\frac{7}{6}\right)=\frac{49}{12}$

$\min=0$

$\max=\frac{49}{12}$

$f(-x,x)=-3x^2-x$

x	0	2
$-3x^2-x$	0 $\searrow$	-14

$\min=-14$

$\max=0$

$f(x,2)=-2x^2+4x+2$

x	-2	1	2
$-2x^2+4x+2$	-14 $\nearrow$	4	$\searrow$ 2

$\min=-14$

$\max=4$

$$\nabla f(x,y)=(0,0) \iff \begin{cases} \frac{\partial f}{\partial x}(x,y)=4-4x=0 & \Rightarrow x=1 \\ \frac{\partial f}{\partial y}(x,y)=3-2y=0 & \Rightarrow y=\frac{3}{2} \end{cases}$$

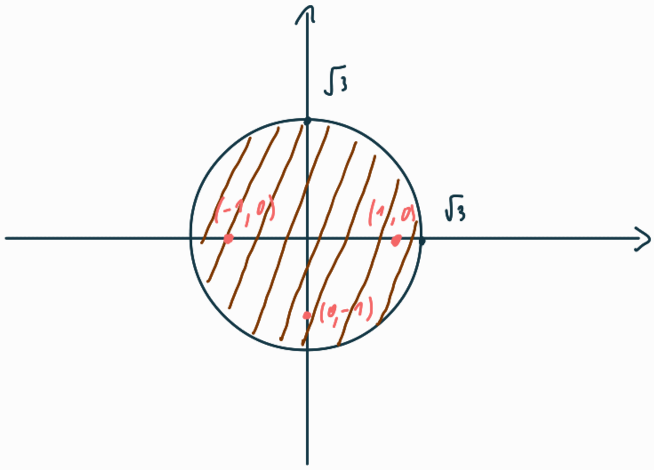
$$C=\left\{\left(1,\frac{3}{2}\right)\right\} \Rightarrow m_2=M_2=f\left(1,\frac{3}{2}\right)=\frac{17}{4}$$

$m=\min\{m_1,m_2\}=-14$ $\quad M=\max\{M_1,M_2\}=\frac{17}{4}$

1) $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 3\}$ and $f:A\to\mathbb{R}$, $f(x,y)=y^2+2y-2x^2y$

determine $\min f(A)$, $\max f(A)$

solution



$\left.\begin{array}{l}A \text{ is compact} \\ f \text{ is con.}\end{array}\right\} \overset{W.}{\Rightarrow} f$ is bounded and reaches its bounds on A

let $m_1=\min f(\mathrm{bd}A)$, $M_1=\max f(\mathrm{bd}A)$ $\quad C=\{(x,y)\in \mathrm{int}A \mid \nabla f(x,y)=(0,0)\}$

$m_2=\min f(C)$, $M_2=\max f(C)$, $m=\min f(A)$, $M=\max f(A)$

$\mathrm{bd}A=\{(x,y)\in\mathbb{R} \mid x^2+y^2=3\}=\{(x,y)\in\mathbb{R}^2 \mid x^2=3-y^2\}$

$$f(\pm\sqrt{3-y^2},y)=y^2+2y-2y(3-y^2)=\underbrace{2y^3+y^2-4y}_{g(y)}$$

$y\in[-\sqrt{3},\sqrt{3}]$

$g'(y)=6y^2+2y-4=0 \iff 3y^2+y-2=0 \quad y_1=-1,\ y_2=\frac{2}{3}$

y	$-\sqrt{3}$		-1		$\frac{2}{3}$		$\sqrt{3}$
$g'(y)$	$+$	$+$	0	$-$	0	$+$	$+$
$g(y)$	$3-2\sqrt{3}$	$\nearrow$	3	$\searrow$	$\frac{-44}{27}$	$\nearrow$	$3+2\sqrt{3}$

$$g(-\sqrt{3}) = -6\sqrt{3}+3+4\sqrt{3} = 3-2\sqrt{3} = -0.464\ldots$$

$$g(-1) = 3$$

$$g\left(\frac{2}{3}\right) = \frac{-44}{27} \simeq -1.629\ldots$$

$$g(\sqrt{3}) = 6\sqrt{3}+3-4\sqrt{3} = 3+2\sqrt{3}$$

$$m_1 = \frac{-44}{27} \qquad M_1 = 3+2\sqrt{3}$$

$$\nabla f(x,y) = (0,0) \iff \begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Rightarrow \begin{cases} -4xy = 0 & \Rightarrow x=0 \text{ or } y=0 \\ 2y+2-2x^2 = 0 & \Rightarrow \end{cases}$$

$$x=0 \Rightarrow 2y+2=0 \Rightarrow y=-1$$

$$y=0 \Rightarrow 2-2x^2=0 \Rightarrow x=\pm 1$$

$$C = \{(-1,0),(1,0),(0,-1)\}$$

$$\left.\begin{aligned} f(0,-1) &= -1 \\ f(1,0) &= 0 \\ f(-1,0) &= 0 \end{aligned}\right\} \Rightarrow m_2 = -1 \qquad M_2 = 0$$

$m = \frac{-44}{27}$ reached at $\left(\pm\sqrt{\frac{23}{3}}, \frac{2}{3}\right)$

$M = 3+2\sqrt{3}$ reached at $(0,\sqrt{3})$

2) consider $f(x,y,z) = x+y+z$ and $C = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1 \text{ and } 2x+y+2z=1\}$

$\min f(C)$, $\max f(C)$

solution: $\left.\begin{array}{l} C \text{ is compact} \\ f \text{ is cont} \end{array}\right\} \overset{W.}{\Rightarrow}$ f is bounded and reaches its bounds on C $\Rightarrow$

$\Rightarrow \exists$ 2 point $(a,b,c)\in C$ and $(a',b',c')\in C$ s.t

$\left.\begin{array}{l} f(a,b,c) = \min f(C) \\ f(a',b',c') = \max f(C) \end{array}\right\}$ by the Lagrange multipliers rule $\Rightarrow$

(L.m.r.)

$\Rightarrow \exists \lambda_0, \mu_0, \lambda_0', \mu_0' \in \mathbb{R}$ s.t. $a, b, c, \lambda_0, \mu_0, a', b', c', \lambda_0', \mu_0'$ are crit. points of the Lagrange function

let $F_1(x,y,z) = x^2+y^2+z^2-1$

$F_2(x,y,z) = 2x+y+2z-1$

$L(x,y,z,\lambda,\mu_0) = f(x,y,z) + \lambda_0 F_1(x,y,z) + \mu_0 F_2(x,y,z) = x+y+z+\lambda(x^2+y^2+z^2-1)^2 +$

$+\mu(2x+y+2z-1)$

the crit. points of L are solution to the system

$$\begin{cases} \frac{\partial L}{\partial x}(x,y,z,\lambda,\mu) = 1+2\lambda x+2\mu = 0 & (1) \\ \frac{\partial L}{\partial y}(\dots) = 1+2\lambda y+\mu = 0 & (2) \\ \frac{\partial L}{\partial z}(\dots) = 1+2\lambda z+2\mu = 0 & (3) \\ \frac{\partial L}{\partial \lambda}(\dots) = x^2+y^2+z^2-1 = 0 & (4) \\ \frac{\partial L}{\partial \mu}(\dots) = 2x+y+2z-1 = 0 & (5) \end{cases}$$

$(1)-(3) \to 2\lambda(x-z) = 0$

$\Rightarrow \lambda = 0 \Rightarrow \left.\begin{matrix}\mu = \frac{1}{2} \\ \mu = -1\end{matrix}\right\}$ contra.

$\Rightarrow x = z \begin{cases} 2x^2+y^2-1=0 \Rightarrow 2x^2+1-8x+16x^2 = 0 \Rightarrow 18x^2-8x = 0 \Rightarrow \\ 4x+y-1=0 \Rightarrow y = 1-4x \end{cases}$

$\Rightarrow 2x(9x-4) = 0 \Rightarrow x_1 = 0 \quad y_1 = 1 \quad z_1 = 0 \quad \lambda_1 = -\frac{1}{4} \quad \mu_1 = -\frac{1}{2}$

$x_2 = \frac{4}{9} \quad y_2 = \frac{-7}{9} \quad z_2 = \frac{4}{9} \quad \lambda_2 = \dots \quad \mu_2 = \dots$

$\Rightarrow$ the critical points of L are $(0, 1, 0, -\frac{1}{4}, -\frac{1}{2})$ and $(\frac{4}{9}, \frac{-7}{9}, \frac{4}{9}, \dots, \dots)$

$f(0,1,0) = 1 = \max f(C)$

$f(\frac{4}{9}, -\frac{7}{9}, \frac{4}{9}) = \frac{1}{9} = \min f(C)$

3) consider $f(x,y,z) = x^2+y^2+z^2-2x+2\sqrt{2}\cdot y+2z$ and $B=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le 1\}$

determine $\min f(B)$, $\max f(B)$

solution:

$\left.\begin{array}{l} B \text{ is compact} \\ f \text{ is con.} \end{array}\right\} \overset{W}{\Rightarrow} f$ is bounded and reaches its bounds on B

$\Rightarrow \exists\, (a,b,c)\in B$ and $(a',b',c')\in B$ s.t. $f(a,b,c)=\min f(B)$
$f(a',b',c')=\max f(B)$

if $(a,b,c)\in \operatorname{int} B$ or $(a',b',c')\in\operatorname{int} B \overset{\text{Fermat}}{\Rightarrow} \nabla f(a,b,c)=(0,0,0)$ or $\nabla f(a',b',c')=(0,0,0)$

$\nabla f(x,y,z)=(0,0,0) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 2x-2=0 \\ \frac{\partial f}{\partial y}(\dots) = 2y-2\sqrt{2}=0 \\ \frac{\partial f}{\partial z}(\dots) = 2z+2=0 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=-\sqrt{2} \\ z=-1 \end{cases} \Rightarrow \text{the only crit. point of } f \text{ is}$$

$(1,-\sqrt{2},-1)\notin \operatorname{int} B \Rightarrow \quad (a,b,c),(a',b',c')\in \operatorname{bd} B=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$

$(a,b,c),(a',b',c')$ are constrained extrema for $f \overset{\text{L.m.1}}{\Rightarrow}$

$\Rightarrow \exists\, \lambda_0, \lambda_0'\in\mathbb{R}$ s.t (a,b,c,λ_0) and (a,b,c,λ_0') are crit. points of the

Langrange function

let $F(x,y,z)=x^2+y^2+z^2-1$ and $L(x,y,z,\lambda)=f(x,y,z)+\lambda F(x,y,z)=$

$=x^2+y^2+z^2-2x+2\sqrt{2}y+2z+\lambda(x^2+y^2+z^2-1)$

crit. points for L are solutions for $\nabla L(x,y,z,\lambda)=0$

$$\Rightarrow \begin{cases} \frac{\partial L}{\partial x}(x,y,z,\lambda) = 2x-2+2\lambda x=0 & \Rightarrow 2x(\lambda+1)=2 \Rightarrow x=\frac{1}{\lambda+1} \\ \frac{\partial L}{\partial y}(\dots) = 2y+2\sqrt{2}+2\lambda y=0 & \Rightarrow 2y(\lambda+1)=-2\sqrt{2} \Rightarrow y=\frac{-\sqrt{2}}{\lambda+1} \\ \frac{\partial L}{\partial z}(\dots) = 2z+2+2\lambda z=0 & \Rightarrow 2z(\lambda+1)=-2 \Rightarrow z=\frac{-1}{\lambda+1} \\ \frac{\partial L}{\partial z}(\dots) = x^2+y^2+z^2-1=0 & \Rightarrow \frac{4}{(\lambda+1)^2}=1 \Rightarrow \lambda+1=\pm 2 \end{cases}$$

- $\lambda + 1 = 2 \Rightarrow x = \frac{1}{2}$, $y = \frac{-\sqrt{2}}{2}$, $z = \frac{-1}{2}$, $\lambda = 1$
- $\lambda + 1 = -2 \Rightarrow x = \frac{-1}{2}$, $y = \frac{\sqrt{2}}{2}$, $z = \frac{1}{2}$, $\lambda = -3$

$\Rightarrow$ the critical points of L are $\left(\frac{1}{2}, \frac{-\sqrt{2}}{2}, \frac{-1}{2}, 1\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, -3\right)$

$f\left(\frac{1}{2}, \frac{-\sqrt{2}}{2}, -\frac{1}{2}\right) = -3$

$f\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right) = 5$