Geometry Formulas Cheatsheet

Based on provided sources

June 13, 2025

Contents

1 Vectors

1.1 Basic Properties and Operations

- **Vector Segment Sum**: $\vec{AB} + \vec{BC} = \vec{AC}$ [?, 1.4.2]p. 90]
- **Projection of Sum of Vectors**: $\mathrm{pr}_{\vec{u}}(\mathbf{a} + \mathbf{b}) = \mathrm{pr}_{\vec{u}}\mathbf{a} + \mathrm{pr}_{\vec{u}}\mathbf{b}$ [?, 1.4.3]p. 94]
- **Projection of Scalar Product**: $\mathrm{pr}_{\vec{u}}(\lambda\mathbf{a}) = \lambda\mathrm{pr}_{\vec{u}}\mathbf{a}$ [?, 1.4.4]p. 96]
- **Projection of Linear Combination**: $\mathrm{pr}_{\vec{u}}(\sum_{i=1}^{k} \lambda_i \mathbf{a}_i) = \sum_{i=1}^{k} \lambda_i \mathrm{pr}_{\vec{u}}\mathbf{a}_i$ []itemize

1.2 Linear Dependence and Independence

- **Linear Independence Condition**: Vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are linearly independent if $\sum_{i=1}^{k} \lambda_i \mathbf{a}_i = \mathbf{0}$ implies $\lambda_1 = \cdots = \lambda_k = 0$ [?, p. 97]
- **Vector as Linear Combination (Plane)**: Given non-collinear vectors $\mathbf{e}_1, \mathbf{e}_2$ in a plane, any vector $\mathbf{a}$ in the plane can be uniquely decomposed as $\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2$ [?, 1.5.4]p. 99].
- **Vector as Linear Combination (Space)**: Given linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in space, any vector $\mathbf{a}$ in space can be uniquely decomposed as $\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ [?, 1.5.10]p. 100].
- **Coordinates of a Vector defined by two points**: If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then $\vec{AB} = (x_2 - x_1, y_2 - y_1)$ [?, p. 110]
- **Collinearity of two vectors**: Vectors $\mathbf{a}(x_1, y_1)$ and $\mathbf{b}(x_2, y_2)$ are collinear iff $x_2 = \lambda x_1, y_2 = \lambda y_1$ for some $\lambda \in \mathbb{R}$ (or $x_2/x_1 = y_2/y_1$ if denominators are non-zero) [?, 1.7.3]p. 111].

1.3 Scalar Product

- **Definition**: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ [?, 1.11.1]p. 128].
- **In Components (Orthonormal Basis)**: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ [?, 1.11.12]p. 129].
- **Magnitude of a Vector**: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ [?, 1.11.13]p. 129].
- **Distance Between Two Points**: $d(M, M') = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}$ [?, p. 129]
- **Cosine of Angle Between Vectors**: $\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$ [?, p. 130]
- **Generalized Pythagorean Theorem**: For a triangle with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and angle A opposite to $\mathbf{a}$, $|\mathbf{a}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2|\mathbf{b}||\mathbf{c}| \cos A$ [?, p. 131]

1.4 Vector Product (Cross Product)

- **Magnitude (Area of Parallelogram)**: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ (Area of parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$) [?, p. 135] Area of triangle is half of this.
- **In Components (Orthonormal Basis)**: $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$ [?, 1.12.8]p. 138].
- **Determinant Form**: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ [?, 1.12.9]p. 138].
- **Double Vector Product (Lagrange's Identity)**: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ [?, 1.12.13]p. 140].
- **Sine Rule for Triangles**: $\frac{|\mathbf{a}|}{\sin A} = \frac{|\mathbf{b}|}{\sin B} = \frac{|\mathbf{c}|}{\sin C}$ [?, 1.12.15]p. 141].
- **Jacobi Identity**: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$ [?, 1.12.18]p. 143].

1.5 Mixed Product (Scalar Triple Product)

- **Definition**: $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ [**?**, 1.13.1]p. 144].
- **Geometric Interpretation**: Volume of parallelepiped formed by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (signed) [**?**, p. 144]
- **Volume of Tetrahedron**: $\mathrm{Vol}_{OABC} = \pm\frac{1}{6}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ [**?**, 1.13.2]p. 146].
- **Coplanarity Condition**: Three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar iff $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$ [**?**, 1.13.2]p. 147].
- **In Components (Orthonormal Basis)**: $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ [**?**, 1.13.7]p. 149].

1.6 Point Combinations

- **Sum of Point and Vector**: $Q = P + \mathbf{v}$ means $\vec{PQ} = \mathbf{v}$ [**?**, 1.5.17]p. 103].
- **Point P for sum of OA vectors**: If $\sum_{i=1}^{n} \alpha_i = 1$, then $\vec{OP} = \sum_{i=1}^{n} \alpha_i \vec{OA_i}$ is independent of origin O [**?**, p. 104]

2 Lines in the Plane

2.1 Equations of a Line

- **General Equation**: $Ax + By + C = 0$ [**?**, 2.2.1]p. 161].
- **Slope-Intercept Form**: $y = kx + b$ [**?**, 2.2.2]p. 161].
- **Intercept Form**: $\frac{x}{a} + \frac{y}{b} - 1 = 0$ [**?**, p. 162]
- **Line Through Two Points** $M_0(x_0, y_0), M_1(x_1, y_1)$: $\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0}$ [**?**, 2.3.5]p. 163].
- **Bundle of Lines (Pencil of Lines)**: $\alpha(A_1x + B_1y + C_1) + \beta(A_2x + B_2y + C_2) = 0$ (for lines through intersection of L_1, L_2) [**?**, 2.5.3]p. 165].

2.2 Distances and Angles

- **Distance from a Point** $M_0(x_0, y_0)$ **to a Line** $Ax + By + C = 0$: $d = \frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$ [**?**, 2.6.6]p. 170].
- **Distance Between Two Parallel Lines (Normal Form** $x\cos\alpha + y\sin\alpha - p = 0$**)**: $d = |p_1 - p_2|$ (same side of origin) or $d = p_1 + p_2$ (origin between lines) [**?**, 2.7.7, 2.7.8]p. 171].
- **Angle Between Two Lines (General Forms)**: $\cos\theta = \frac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}}$ [**?**, 2.8.3]p. 172].
- **Perpendicular Lines Condition**: $A_1A_2 + B_1B_2 = 0$ [**?**, 2.8.4]p. 172].
- **Bisectors of Angles Between Two Lines**: $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ [**?**, 2.9.5]p. 174].

3 Planes in Space

3.1 Equations of a Plane

- **General Equation**: $Ax + By + Cz + D = 0$ [**?**, p. 180]
- **Plane Through Origin**: $Ax + By + Cz = 0$ [**?**, p. 180]
- **Planes Parallel to Coordinate Axes**:
  * Parallel to Oz: $Ax + By + D = 0$
  * Parallel to Oy: $Ax + Cz + D = 0$
  * Parallel to Ox: $By + Cz + D = 0$ []itemize
  * **Vector Equation (Mixed Product Form)**: $(\mathbf{r} - \mathbf{r}_0, \mathbf{v}, \mathbf{w}) = 0$ (plane through $\mathbf{r}_0$ parallel to $\mathbf{v}, \mathbf{w}$) [**?**, 3.1.7]p. 181].

* **Plane Through Three Points** $M_1(x_1,y_1,z_1), M_2(x_2,y_2,z_2), M_3(x_3,y_3,z_3)$: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$ [**?**, 3.1.10]p. 182].
* **Plane Equation by Intercepts**: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$ [**?**, p. 183]
* **Hesse Normal Form (Normal Form)**: $x\cos\alpha+y\cos\beta+z\cos\gamma-p=0$ [**?**, 3.1.14]p. 184].
* **Normalizing Factor** λ: $\lambda=\pm\frac{1}{\sqrt{A^2+B^2+C^2}}$ (sign opposite to D if $D\neq 0$) [**?**, 3.1.15]p. 185].
* **Plane Bundle (Pencil of Planes)**: $\alpha(A_1x+B_1y+C_1z+D_1)+\beta(A_2x+B_2y+C_2z+D_2)=0$ (for planes through intersection line of P_1,P_2) [**?**, 3.3.8]p. 195].
* **Pencil of Planes (through a point** $S(x_0,y_0,z_0)$**)**: $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$ [**?**, 3.3.9]p. 196].

3.2 Distances and Angles

* **Coplanarity Condition for Four Points** M_1,M_2,M_3,M_4: $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0$ [**?**, 3.1.12]p. 183].
* **Distance from a Point** $M_0(x_0,y_0,z_0)$ **to a Plane** $Ax+By+Cz+D=0$: $d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$ [**?**, 3.1.19]p. 187].
* **Angle Between Two Planes**: $\cos\theta=\pm\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$ [**?**, 3.1.22]p. 188].
* **Perpendicular Planes Condition**: $A_1A_2+B_1B_2+C_1C_2=0$ [**?**, 3.1.23]p. 189].
* **Parallel Planes Condition**: $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$ [**?**, 3.1.24]p. 189].

3.3 Lines in Space

* **Vector Equation**: $\mathbf{r}(t)=\mathbf{r}_0+t\mathbf{a}$ [**?**, 3.2.1]p. 189].
* **Parametric Equations**: $x=x_0+lt, y=y_0+mt, z=z_0+nt$ [**?**, 3.2.2]p. 190].
* **Canonical Equations**: $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ [**?**, 3.2.4]p. 190].
* **Line as Intersection of Two Planes**: $\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$ [**?**, p. 191]
* **Angle Between Line and Plane**: $\sin\phi=\frac{|Al+Bm+Cn|}{\sqrt{A^2+B^2+C^2}\sqrt{l^2+m^2+n^2}}$ [**?**, p. 204]

4 Conics (Canonical Forms)

4.1 Ellipse

* **Definition**: Sum of distances from any point M to two foci F_1,F_2 is constant $2a$, i.e., $F_1M+F_2M=2a$. Distance between foci is $2c$, with $c<a$ [**?**, 4.1]p. 211].
* **Foci Coordinates**: $F_1(-c,0)$, $F_2(c,0)$ [**?**, p. 211]
* **Distances to Foci**: $F_1M=\sqrt{(x+c)^2+y^2}$, $F_2M=\sqrt{(x-c)^2+y^2}$ [**?**, 4.1.1]p. 212].
* **Canonical Equation**: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $b^2=a^2-c^2$ [**?**, 4.1.6]p. 213].
* **Focal Rays (in terms of eccentricity** $e=c/a$**)**: $r_1=a+ex$, $r_2=a-ex$ [**?**, p. 216]
* **Tangent Equation at** $M_0(x_0,y_0)$ **(Duplication Method)**: $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$ [**?**, 4.1.14]p. 218].

4.2 Hyperbola

* **Definition**: Absolute value of difference of distances from any point M to two foci F_1,F_2 is constant $2a$, i.e., $|F_1M-F_2M|=2a$. Distance between foci is $2c$, with $0<a<c$ [**?**, 4.2]p. 219].

* **Foci Coordinates**: $F_1(-c, 0)$, $F_2(c, 0)$ [?, p. 220]
* **Canonical Equation**: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = c^2 - a^2$ [?, 4.2.6]p. 222].
* **Focal Rays (in terms of eccentricity $e = c/a$)**: For $x \geq a$: $F_1M = ex + a$, $F_2M = ex - a$. For $x \leq -a$: $F_1M = -ex - a$, $F_2M = -ex + a$ [?, 4.2.9, 4.2.10]p. 222].

4.3 Parabola

* **Definition**: Locus of points equidistant from a fixed line (directrix Δ) and a fixed point (focus F) [?, 4.3]p. 225].
* **Focus and Directrix Coordinates**: Focus $F(p/2, 0)$, Directrix $x = -p/2$ [?, p. 225]
* **Canonical Equation**: $y^2 = 2px$ [?, 4.3.3]p. 226].
* **Tangent Equation at $M_0(x_0, y_0)$ (Duplication Method)**: $yy_0 = p(x + x_0)$ [?, p. 229]

5 Quadrics (Canonical Forms)

5.1 Ellipsoid

* **Equation**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ [?, 5.2.2]p. 232].
* **Tangent Plane at $M_0(x_0, y_0, z_0)$**: $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$ [?, 5.2.7]p. 244].

5.2 Second-Degree Cone

* **Equation**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ [?, p. 245]
* **Tangent Plane at $M_0(x_0, y_0, z_0)$**: $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = 0$ [?, 5.3.3]p. 248].

5.3 One-Sheeted Hyperboloid

* **Equation**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ [?, 5.4.1]p. 250].
* **Straight-Line Generators (First Family λ-generators)**: $\begin{cases} \lambda(\frac{x}{a} + \frac{z}{c}) = \mu(1 + \frac{y}{b}) \\ \mu(\frac{x}{a} - \frac{z}{c}) = \lambda(1 - \frac{y}{b}) \end{cases}$ [?, 5.4.4]p. 256]. (Simplified forms with one parameter are also given in the source).
* **Straight-Line Generators (Second Family η-generators)**: $\begin{cases} \alpha(\frac{x}{a} + \frac{z}{c}) = \beta(1 - \frac{y}{b}) \\ \beta(\frac{x}{a} - \frac{z}{c}) = \alpha(1 + \frac{y}{b}) \end{cases}$ [?, 5.4.7]p. 258].

5.4 Two-Sheeted Hyperboloid

* **Equation**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ [?, 5.5.1]p. 262].
* **Tangent Plane at $M_0(x_0, y_0, z_0)$**: $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = -1$ [?, p. 266]

5.5 Elliptic Paraboloid

* **Equation**: $\frac{x^2}{p} + \frac{y^2}{q} = 2z$ [?, 5.6.1]p. 267].
* **Tangent Plane at $M_0(x_0, y_0, z_0)$**: $\frac{xx_0}{p} + \frac{yy_0}{q} = z + z_0$ [?, 5.6.5]p. 269].

5.6 Hyperbolic Paraboloid

* **Equation**: $\frac{x^2}{p} - \frac{y^2}{q} = 2z$ [?, 5.7.1]p. 270].
* **Rectilinear Generators (First Family)**: $\begin{cases} \lambda(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}) = 2\mu z \\ \mu(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}) = \lambda \end{cases}$ [?, 5.7.3]p. 274].
* **Rectilinear Generators (Second Family)**: $\begin{cases} \lambda(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}) = \mu \\ \mu(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}) = 2\lambda z \end{cases}$ [?, 5.7.4]p. 274].
* **Tangent Plane at $M_0(x_0, y_0, z_0)$**: $\frac{xx_0}{p} - \frac{yy_0}{q} = z + z_0$ [?, 5.7.5]p. 275].

5.7 Cylinders

- **Elliptic Cylinder**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ [?, 5.8.1]p. 275].
- **Tangent Plane to Elliptic Cylinder at** $M_0(x_0, y_0, z_0)$: $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$ [?, 5.8.5]p. 281].
- **Hyperbolic Cylinder**: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ [?, 5.9.1]p. 281].
- **Tangent Plane to Hyperbolic Cylinder at** $M_0(x_0, y_0, z_0)$: $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$ [?, 5.9.4]p. 284].
- **Parabolic Cylinder**: $y^2 = 2px$ [?, 5.10.1]p. 285].
- **Tangent Plane to Parabolic Cylinder at** $M_0(x_0, y_0, z_0)$: $yy_0 = p(x + x_0)$ [?, 5.10.6]p. 288].

6 Generated Surfaces

6.1 Cylindrical Surfaces

- **Generator Equations**: For directrix line (P1=0, P2=0), generators are $\begin{cases} P_1(x, y, z) = \lambda \\ P_2(x, y, z) = \mu \end{cases}$ [?, 6.1.1]p. 298].

6.2 Conical Surfaces

- **Generator Equations**: For vertex at intersection of P1=0, P2=0, P3=0, generators are $\begin{cases} P_1 = \lambda P_3 \\ P_2 = \mu P_3 \end{cases}$ [?, 6.2.2]p. 307].

6.3 Conoidal Surfaces

- **Generator Equations**: For directrix line (P1=0, P2=0) and director plane P=0, generators are $\begin{cases} P_1 = \lambda P_2 \\ P = \mu \end{cases}$ [?, 6.3.3]p. 313].

6.4 Surfaces of Revolution

- **Generating Circle Equations**: For rotation axis D and curve C, the generating circle is $\begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda^2 \\ lx + my + nz = \mu \end{cases}$ [?, 6.4.3]p. 319].
- **Equation of Surface of Revolution (General Form)**: $F\left(\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}, lx + my\right.$ 0 [?, 6.4.7]p. 321].

7 Curves in Space (Differential Geometry)

7.1 Representations of Curves

- **Parametric Representation of Plane Curve**: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ [?, 7.4.1]p. 333].
- **Explicit Representation of Space Curve**: $\begin{cases} y = f(x) \\ z = g(x) \end{cases}$ (global parametrization $x = t, y = f(t), z = g(t)$) [?, 7.4.9]p. 335].

7.2 Tangent and Normal Plane

* **Vectorial Equation of Tangent Line**: $\mathbf{R}(\theta) = \mathbf{r}(t_0) + \theta \mathbf{r}'(t_0)$ [?, 7.5.1]p. 337].
* **Parametric Equations of Tangent Line (Space Curve)**: $\begin{cases} X(\theta) = x(t_0) + \theta x'(t_0) \\ Y(\theta) = y(t_0) + \theta y'(t_0) \\ Z(\theta) = z(t_0) + \theta z'(t_0) \end{cases}$ [?, 7.5.6]p. 340].
* **Parametric Equations of Tangent Line (Plane Curve)**: $\begin{cases} X(\theta) = x(t_0) + \theta x'(t_0) \\ Y(\theta) = y(t_0) + \theta y'(t_0) \end{cases}$ [?, 7.5.7]p. 340].
* **Vectorial Equation of Normal Plane**: $(\mathbf{R} - \mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = 0$ [?, 7.5.4]p. 340].
* **Equation of Normal Plane (Space Curve - expanded)**: $(X - x)x' + (Y - y)y' + (Z - z)z' = 0$ [?, 7.5.9]p. 341].
* **Equation of Normal Line (Plane Curve - expanded)**: $(X - x)x' + (Y - y)y' = 0$ [?, 7.5.10]p. 341].
* **Tangent Line (Explicit Space Curve $y = f(x), z = g(x)$)**: $\frac{X-x}{1} = \frac{Y-f(x)}{f'(x)} = \frac{Z-g(x)}{g'(x)}$ [?, 7.5.11]p. 342].
* **Normal Plane (Explicit Space Curve $y = f(x), z = g(x)$)**: $(X-x)+(Y-f(x))f'(x)+(Z - g(x))g'(x) = 0$ [?, 7.5.12]p. 342].
* **Tangent Line (Explicit Plane Curve $y = f(x)$)**: $Y - f(x) = f'(x)(X - x)$ [?, 7.5.14]p. 342].
* **Normal Line (Explicit Plane Curve $y = f(x)$)**: $Y - f(x) = -\frac{1}{f'(x)}(X - x)$ [?, 7.5.16]p. 343].
* **Derivatives for Implicit Curve ($F(x, y, z) = 0, G(x, y, z) = 0$)**: $f' = \frac{D(F,G)/D(z,x)}{D(F,G)/D(y,z)}$, $g' = \frac{D(F,G)/D(x,y)}{D(F,G)/D(y,z)}$ [?, 7.5.19]p. 344].
* **Tangent Line (Implicit Space Curve $F = 0, G = 0$)**: $\frac{X-x_0}{D(F,G)/D(y,z)} = \frac{Y-y_0}{D(F,G)/D(z,x)} = \frac{Z-z_0}{D(F,G)/D(x,y)}$ [?, 7.5.21]p. 345].

7.3 Osculating Plane

* **Equation**: $(\mathbf{R} - \mathbf{r}(t_0), \mathbf{r}'(t_0), \mathbf{r}''(t_0)) = 0$ [?, 7.6.1]p. 346].
* **Determinant Form**: $\begin{vmatrix} X - x_0 & Y - y_0 & Z - z_0 \\ x_0' & y_0' & z_0' \\ x_0'' & y_0'' & z_0'' \end{vmatrix} = 0$ [?, 7.6.2]p. 346].

7.4 Frenet Frame and Formulas

* **Frenet Frame Versors (for natural parameter s)**: $\vec{\tau}(s) = \mathbf{r}'(s)$ $\vec{\nu}(s) = \frac{\mathbf{r}''(s)}{|\mathbf{r}''(s)|}$ $\vec{\beta}(s) = \vec{\tau}(s) \times \vec{\nu}(s)$ [?, 7.8.2]p. 348].
* **Frenet Frame Versors (for general parameter t)**: $\vec{\tau}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ $\vec{\beta}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}$ $\vec{\nu}(t) = \vec{\beta}(t) \times \vec{\tau}(t)$ [?, 7.8.3, 7.8.4]p. 349].
* **Frenet Formulas (for natural parameter s)**: $\vec{\tau}'(s) = k(s)\vec{\nu}(s)$ $\vec{\nu}'(s) = -k(s)\vec{\tau}(s) + \chi(s)\vec{\beta}(s)$ $\vec{\beta}'(s) = -\chi(s)\vec{\nu}(s)$ [?, 7.10.3]p. 352]. (Note: χ is torsion, k is curvature).
* **Curvature (k)**: $k = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$ [?, p. 354]
* **Torsion (χ)**: $\chi = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{|\mathbf{r}' \times \mathbf{r}''|^2}$ [?, 7.10.5]p. 354].
* **Circle Condition**: If torsion $\chi = 0$ and curvature $k = k_0$ (constant strictly positive), curve support lies on a circle of radius $1/k_0$ [?, p. 355-356]
* **Curve on Sphere Condition**: If naturally parameterized curve $\mathbf{r}(s)$ has support on sphere of radius a centered at origin, then $k \geq 1/a$ [?, p. 356]
* **General Helix Condition (Lancret's Theorem)**: A space curve with curvature $k > 0$ is a general helix iff $\chi/k = \text{const}$ [?, p. 357]

* **Bertrand Curve Relation**: For Bertrand mates $\mathbf{r}$ and $\mathbf{r}^*$, $\mathbf{r}^*(s) = \mathbf{r}(s) + a(s)\vec{\nu}(s)$ [**?**, 7.10.7]p. 359].
* **Involute of a Curve (Parameterized)**: $\vec{\gamma}(t) = \mathbf{r}(t) + (c - s(t))\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, where $s(t)$ is arc length [**?**, p. 370]
* **Logarithmic Spiral (Polar Equation)**: $r = C \cdot e^{\theta}$ [**?**, 9.2.3]p. 377].
* **Catenary (Natural Equation)**: $R = a + s^2/a$ [**?**, 9.2.8]p. 379].
* **Tractrix (Natural Equation)**: $R^2 + a^2 = a^2 e^{-2s/a}$ [**?**, 9.2.12]p. 380].

8 Surfaces (Differential Geometry)

8.1 Representations and Basic Concepts

* **Parametrized Surface Regularity Condition**: $\mathbf{r}'_u \times \mathbf{r}'_v \neq \mathbf{0}$ [**?**, 1]p. 32].
* **Parametric Equations of Surface**: $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ [**?**, p. 341]
* **Tangent Vector to a Curve on Surface**: $\vec{\rho}'(t_0) = u'(t_0)\mathbf{r}'_u(u_0, v_0) + v'(t_0)\mathbf{r}'_v(u_0, v_0)$ [**?**, p. 391]
* **Tangent Plane to Implicit Surface** $F(x, y, z) = 0$: $(X - x_0)F'_x + (Y - y_0)F'_y + (Z - z_0)F'_z = 0$ [**?**, p. 394] (Normal vector is grad$F$).
* **Normal to Implicit Surface** $F(x, y, z) = 0$: $\frac{X - x_0}{F'_x} = \frac{Y - y_0}{F'_y} = \frac{Z - z_0}{F'_z}$ [**?**, p. 394]
* **Normal to Sphere** S^2_R: Tangent space $T_a S^2_R$ is orthogonal to radius vector $\mathbf{a}$ [**?**, p. 394] grad$F = 2\{x, y, z\} = 2\mathbf{a}$.
* **Parametrization Compatible with Orientation**: $\mathbf{n}(\mathbf{a}) = \frac{\mathbf{r}'_u \times \mathbf{r}'_v}{|\mathbf{r}'_u \times \mathbf{r}'_v|}$ [**?**, p. 395]

8.2 Fundamental Forms and Curvatures

* **First Fundamental Form** ϕ_1: $\phi_1(\mathbf{p}, \mathbf{q}) = \mathbf{p} \cdot \mathbf{q}$ [**?**, 10.10.1]p. 406].
* **Coefficients of First Fundamental Form**: $E(u, v) = \mathbf{r}'_u \cdot \mathbf{r}'_u$ $F(u, v) = \mathbf{r}'_u \cdot \mathbf{r}'_v$ $G(u, v) = \mathbf{r}'_v \cdot \mathbf{r}'_v$ [**?**, p. 407]
* **Matrix of First Fundamental Form**: $G = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ [**?**, p. 407]
* **Length of Curve Segment on Surface**: $L = \int_{t_1}^{t_2} \sqrt{E(t)u'^2 + 2F(t)u'v' + G(t)v'^2}dt$ [**?**, p. 410]
* **Cosine of Angle Between Curves on Surface**: $\cos\theta = \frac{Eu'_1u'_2 + F(u'_1v'_2 + u'_2v'_1) + Gv'_1v'_2}{\sqrt{Eu'^2_1 + 2Fu'_1v'_1 + Gv'^2_1}\sqrt{Eu'^2_2 + 2Fu'_2v'_2 + Gv'^2_2}}$ [**?**, p. 412]
* **Area of Parametrized Surface**: $A = \iint_D \sqrt{EG - F^2}dudv = \iint_D |\mathbf{r}'_u \times \mathbf{r}'_v|dudv$ [**?**, 10.10.2]p. 413, 415].
* **Shape Operator (Weingarten Map)** A: $A(\mathbf{h}) = -\mathrm{d}\mathbf{n}(\mathbf{h})$ (differential of spherical map) [**?**, 10.9]p. 403].
* **Matrix of Shape Operator (Natural Basis)**: $A = G^{-1}H = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ [**?**, 10.11.3, 10.11.4]p. 416].
* **Second Fundamental Form** ϕ_2: $\phi_2(\xi, \eta) = -\phi_1(A(\xi), \eta)$ [**?**, 10.12.1]p. 418].
* **Coefficients of Second Fundamental Form**: $D = \mathbf{n} \cdot \mathbf{r}''_{u^2}$ $D' = \mathbf{n} \cdot \mathbf{r}''_{uv}$ $D'' = \mathbf{n} \cdot \mathbf{r}''_{v^2}$ [**?**, 10.12.2]p. 420]. (Sometimes denoted L, M, N or e, f, g).
* **Normal Curvature** k_n: $k_n = \frac{\phi_2(\mathbf{p},\mathbf{p})}{\phi_1(\mathbf{p},\mathbf{p})}$ [**?**, p. 422]
* **Euler's Formula for Normal Curvature**: $k_n(\mathbf{e}) = k_1\cos^2\theta + k_2\sin^2\theta$, where k_1, k_2 are principal curvatures and θ is angle with principal direction $\mathbf{e}_1$ [**?**, 10.16.1]p. 426].
* **Total (Gaussian) Curvature** K_t and **Mean Curvature** K_m: $K_t = k_1k_2 = \det A$ $K_m = \frac{1}{2}(k_1 + k_2) = -\frac{1}{2}\mathrm{Tr}A$ [**?**, 10.16.2, 10.16.3]p. 427, 428].

* **Principal Curvatures from** K_t, K_m: $k_1 = K_m + \sqrt{K_m^2 - K_t}$ $k_2 = K_m - \sqrt{K_m^2 - K_t}$ [**?**, 10.16.4, 10.16.5]p. 431, 432].
* **Classification of Points**:
  - Elliptic: $K_t > 0$
  - Parabolic: $K_t = 0$
  - Hyperbolic: $K_t < 0$ [**?**, p. 432]
* **Gauss's Theorema Egregium**: $K_t = \frac{DD'' - (D')^2}{EG - F^2}$ [**?**, 10.18.1]p. 454]. (States that K_t depends only on E, F, G and their derivatives).
* **Weingarten's Formulae**: Express derivatives of normal vector $\mathbf{n}'_u, \mathbf{n}'_v$ in terms of $\mathbf{r}'_u, \mathbf{r}'_v$ and fundamental form coefficients [**?**, 10.17.11]p. 436].
* **Gauss's and Codazzi-Mainardi's Equations**: Compatibility conditions for first and second fundamental forms [**?**, 10.17.14, 10.17.15]p. 438, 439].

8.3 Geodesics

* **Geodesic Curvature** k_g: $k_g = k \sin\theta$, where θ is angle between osculating plane and normal to surface [**?**, p. 461]
* **Geodesic Curvature (in terms of fundamental forms)**: Given a local parameterization $u = u(t), v = v(t)$, $k_g = \frac{1}{\sqrt{EG - F^2}} \left[\dot{u}(\ddot{v}E + \dot{u}\dot{v}F - \dot{v}^2 G_u - \dot{u}^2 E_v) - \dot{v}(\ddot{u}G + \dot{u}\dot{v}F - \dot{u}^2 E_v - \dot{v}^2 F_u)\right]$ (simplified form 10.19.10) [**?**, 10.19.9, 10.19.10]p. 466].
* **Differential Equations of Geodesics**: $\begin{cases} u'' + u'^2\Gamma^1_{11} + 2u'v'\Gamma^1_{12} + v'^2\Gamma^1_{22} = 0 \\ v'' + u'^2\Gamma^2_{11} + 2u'v'\Gamma^2_{12} + v'^2\Gamma^2_{22} = 0 \end{cases}$ [**?**, 10.19.12]p. 468].
* **Geodesics of the Sphere**: Arcs of great circles [**?**, p. 470]
* **Liouville Surfaces Metric**: $ds^2 = (U(u) + V(v))(du^2 + dv^2)$ [**?**, 10.19.13]p. 473].
* **Geodesic Equation for Liouville Surfaces**: $\frac{du}{\sqrt{U(u) - a}} = \pm\frac{dv}{\sqrt{V(v) + a}} + b$ (integrable form) [**?**, 10.19.17]p. 475].

8.4 Special Classes of Surfaces

* **Ruled Surface Parametrization**: $\mathbf{r}(u, v) = \vec{\gamma}(u) + v\mathbf{b}(u)$ [**?**, 11.1.1]p. 479].
* **First Fundamental Form for Ruled Surface**: $E = |\vec{\gamma}' + v\mathbf{b}'|^2$, $F = (\vec{\gamma}' + v\mathbf{b}') \cdot \mathbf{b}$, $G = |\mathbf{b}|^2 = 1$ [**?**, 11.1.3]p. 480].
* **Total Curvature of Ruled Surface**: $K_t = -\frac{(\vec{\gamma}', \mathbf{b}, \mathbf{b}')^2}{|\vec{\gamma}' + v\mathbf{b}'|^2 |\mathbf{b}|^2}$ [**?**, 11.1.7]p. 483].
* **Envelope of Family of Surfaces** $F(x, y, z, \lambda) = 0$: Add condition $F'_\lambda(x, y, z, \lambda) = 0$ [**?**, 11.1.12]p. 484].
* **Developable Surface Condition (Ruled Surface)**: $(\vec{\gamma}', \mathbf{b}, \mathbf{b}') = 0$ [**?**, 11.1.14]p. 485].
* **Developable Ruled Surfaces are Cylindrical, Conical, or Tangent Developables** [**?**, p. 493]
* **Envelope of Normal Planes (Space Curve)**: $\begin{cases} (\mathbf{R} - \mathbf{r}(s)) \cdot \vec{\tau}(s) = 0 \\ (\mathbf{R} - \mathbf{r}(s)) \cdot \vec{\nu}(s) = 1/k(s) \\ (\mathbf{R} - \mathbf{r}(s)) \cdot \vec{\beta}(s) = \frac{k'(s)}{k^2(s)\chi(s)} \end{cases}$ [**?**, 11.1.29 - 11.1.32]p. 494-495].
* **Envelope of Rectifying Planes (Space Curve)**: $\begin{cases} (\mathbf{R} - \mathbf{r}(s)) \cdot \vec{\nu}(s) = 0 \\ (\mathbf{R} - \mathbf{r}(s)) \cdot (\chi(s)\vec{\beta}(s) - k(s)\vec{\tau}(s)) = 0 \end{cases}$ [**?**, 11.1.36, 11.1.37]p. 496].
* **Minimal Surface Condition (for isothermic parametrization** $E = G = \lambda^2, F = 0$**)**: $D + D'' = 0$ [**?**, p. 501]
* **Catalan's Theorem**: The only ruled minimal surface (not a plane) is the right helicoid [**?**, p. 509]

* **Pseudosphere Parametric Equations**: $\begin{cases} x = a \sin u \cos v \\ y = a \sin u \sin v \\ z = a \left(-\ln \tan \frac{u}{2} + \cos u \right) \end{cases}$ [**?**, 11.3.1]p. 516].