SEMINAR 11

1) Give a necessary and sufficient condition for the vectors $v_1 = (a_1, b_1)$, $v_2 = (a_2, b_2)$ to form a basis for the $\mathbb{R}$-vector space $\mathbb{R}^2$. What does this condition mean from geometrical point of view? Using the condition established, find infinitely many bases for ${}_{\mathbb{R}}\mathbb{R}^2$. Is there any basis of $\mathbb{R}^2$ for which the coordinates of a vector $v = (x, y)$ are exactly x and y? Show that $v_1 = (1, 0)$ and $v_2 = (1, 1)$ form a basis of $\mathbb{R}^2$ and find the coordinates of $v = (x, y)$ in this basis.
Homework: Formulate and solve a similar problem for the $\mathbb{R}$-vector space $\mathbb{R}^3$.

2) Show that the vectors $(1, 2, -1)$, $(3, 2, 4)$, $(-1, 2, -6)$ from ${}_{\mathbb{R}}\mathbb{R}^3$ are linearly dependendent and find a dependency relation between them.

3) Determine the values of $a \in \mathbb{R}$ for which the vectors $v_1 = (a, 1, 1)$, $v_2 = (1, a, 1)$, $v_3 = (1, 1, a)$ form a basis of ${}_{\mathbb{R}}\mathbb{R}^3$.

Homework: Which of the following systems of vectors from $\mathbb{R}^3$:
a) $((1, 0, -1), (2, 5, 1), (0, -4, 3))$;
b) $((2, -4, 1), (0, 3, -1), (6, 0, 1))$;
c) $((1, 2, -1), (1, 0, 3), (2, 1, 1))$;
d) $((-1, 3, 1), (2, -4, -3), (-3, 8, 2))$;
e) $((1, -3, -2), (-3, 1, 3), (-2, -10, -2))$
are bases for the $\mathbb{R}$-vector space $\mathbb{R}^3$?

4) Show that in the $\mathbb{R}$-vector space $M_2(\mathbb{R})$ the matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

form a basis and determine the coordinates of $A = \begin{pmatrix} -2 & 3 \\ 4 & -2 \end{pmatrix}$ in this basis.

5) In the $\mathbb{Q}$-vector space $\mathbb{Q}^3$ we consider the vectors

$$a = (-2, 1, 3),\ b = (3, -2, -1),\ c = (1, -1, 2),\ d = (-5, 3, 4),\ e = (-9, 5, 10).$$

Does the following equality $\langle a, b\rangle = \langle c, d, e\rangle$ hold?

6) In the $\mathbb{R}$-vector space $\mathbb{R}^4$ one considers the subspaces:
a) $S = \langle u_1, u_2\rangle$, with $u_1 = (1, 1, 0, 0)$, $u_2 = (1, 0, 1, 1)$,
$T = \langle v_1, v_2\rangle$, with $v_1 = (0, 0, 1, 1)$, $v_2 = (0, 1, 1, 0)$;
b) $S = \langle u_1, u_2, u_3\rangle$, with $u_1 = (1, 2, -1, -2)$, $u_2 = (3, 1, 1, 1)$, $u_3 = (-1, 0, 1, -1)$,
$T = \langle v_1, v_2\rangle$, with $v_1 = (-1, 2, -7, -3)$, $v_2 = (2, 5, -6, -5)$;
c) $S = \langle u_1, u_2\rangle$, with $u_1 = (1, 2, 1, 0)$, $u_2 = (-1, 1, 1, 1)$,
$T = \langle v_1, v_2\rangle$, with $v_1 = (2, -1, 0, 1)$, $v_2 = (1, -1, 3, 7)$.
d) **(optional)** $S = \langle u_1, u_2, u_3\rangle$, cu $u_1 = (1, 2, 1, -2)$, $u_2 = (2, 3, 1, 0)$, $u_3 = (1, 2, 2, -3)$,
$T = \langle v_1, v_2, v_3\rangle$, cu $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 0, 1, -1)$, $v_3 = (1, 3, 0, -3)$.
Find a basis and the dimension for each of the subspaces S, T, $S + T$ and $S \cap T$.