1) limits:

a) $\lim\limits_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) = 0$

$$0 \le \left|xy\sin\left(\frac{1}{x^2+y^2}\right) - 0\right| = |x|\cdot|y|\cdot\overbrace{\left|\sin\frac{1}{x^2+y^2}\right|}^{\le 1} \le |x|\cdot|y|$$

$$0$$

b) $\lim\limits_{(x,y)\to(0,2)} \frac{\sin(xy)}{x} = \lim\limits_{(x,y)\to(0,2)} \frac{\sin(xy)}{xy}\cdot y = 1\cdot 2 = 2$ $\qquad \lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

c) $\lim\limits_{(x,y)\to(5,0)} \frac{e^{xy}-1}{e^y-1} = \left(\frac{0}{0}\right) = \lim\limits_{(x,y)\to(5,0)} \frac{e^{xy}-1}{xy}\cdot\frac{y}{e^y-1}\cdot x = 1\cdot 1\cdot 5 = 5$ $\qquad \lim\limits_{t\to 0}\frac{e^t-1}{t} = 1$

d) $\lim\limits_{(x,y)\to(\frac{\pi}{4},\frac{\pi}{4})} \frac{x-y}{\sin x-\sin y} = \lim\limits_{(x,y)\to(\frac{\pi}{4},\frac{\pi}{4})} \frac{x-y}{2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)} = \lim\limits_{(x,y)\to(\frac{\pi}{4},\frac{\pi}{4})} \frac{\frac{x-y}{2}}{\sin\left(\frac{x-y}{2}\right)}\cdot\frac{1}{\cos\left(\frac{x+y}{2}\right)} = 1\cdot\frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$

e) $\lim\limits_{(x,y)\to(a,a)} \frac{\frac{x+y}{2}-\frac{2xy}{x+y}}{\frac{x+y}{2}-\sqrt{xy}} = \lim\limits_{(x,y)\to(a,a)} \frac{\frac{(x+y)^2-4xy}{2(x+y)}}{\frac{x+y-2\sqrt{xy}}{2}} = \lim\limits_{(x,y)\to(a,a)} \frac{(x+y)^2-4xy}{(x+y)\cdot(x+y-2\sqrt{xy})} =$

$a>0$

$= \lim\limits_{(x,y)\to(a,a)} \frac{(x-y)^2}{(x+y)(\sqrt{x}-\sqrt{y})^2} = \lim\limits_{(x,y)\to(a,a)} \frac{(\sqrt{x}-\sqrt{y})^2(\sqrt{x}+\sqrt{y})^2}{(x+y)(\sqrt{x}-\sqrt{y})^2} = \frac{4a}{2a} = 2$

f) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} =$

let $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$

$$\left.\begin{array}{ll} \lim\limits_{k\to\infty}\left(\frac{1}{k},\frac{1}{k}\right) = (0,0) & \lim\limits_{k\to\infty} f\left(\frac{1}{k},\frac{1}{k}\right) = \lim\limits_{k\to\infty} 0 = 0 \\ \lim\limits_{k\to\infty}\left(\frac{1}{k},0\right) = (0,0) & \lim\limits_{k\to\infty} f\left(\frac{1}{k},0\right) = \lim\limits_{k\to\infty} 1 = 1 \end{array}\right\} \Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$$

g) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$ $\qquad |a-b| \le |a|+|b|$

$$0 \le \left|\frac{x^3-y^3}{x^2+y^2} - 0\right| = \frac{|x^3-y^3|}{x^2+y^2} \le \frac{|x^3|+|y^3|}{x^2+y^2} = \underbrace{\frac{x^2}{x^2+y^2}}_{\le 1}|x| + \underbrace{\frac{y^2}{x^2+y^2}}_{\le 1}|y| \le |x|+|y|$$

$$0$$

h) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy}$ let $f(x,y)=\frac{x^3+y^3}{xy}$

$\lim\limits_{l\to\infty}\left(\frac{1}{l},\frac{1}{l}\right)=(0,0)$

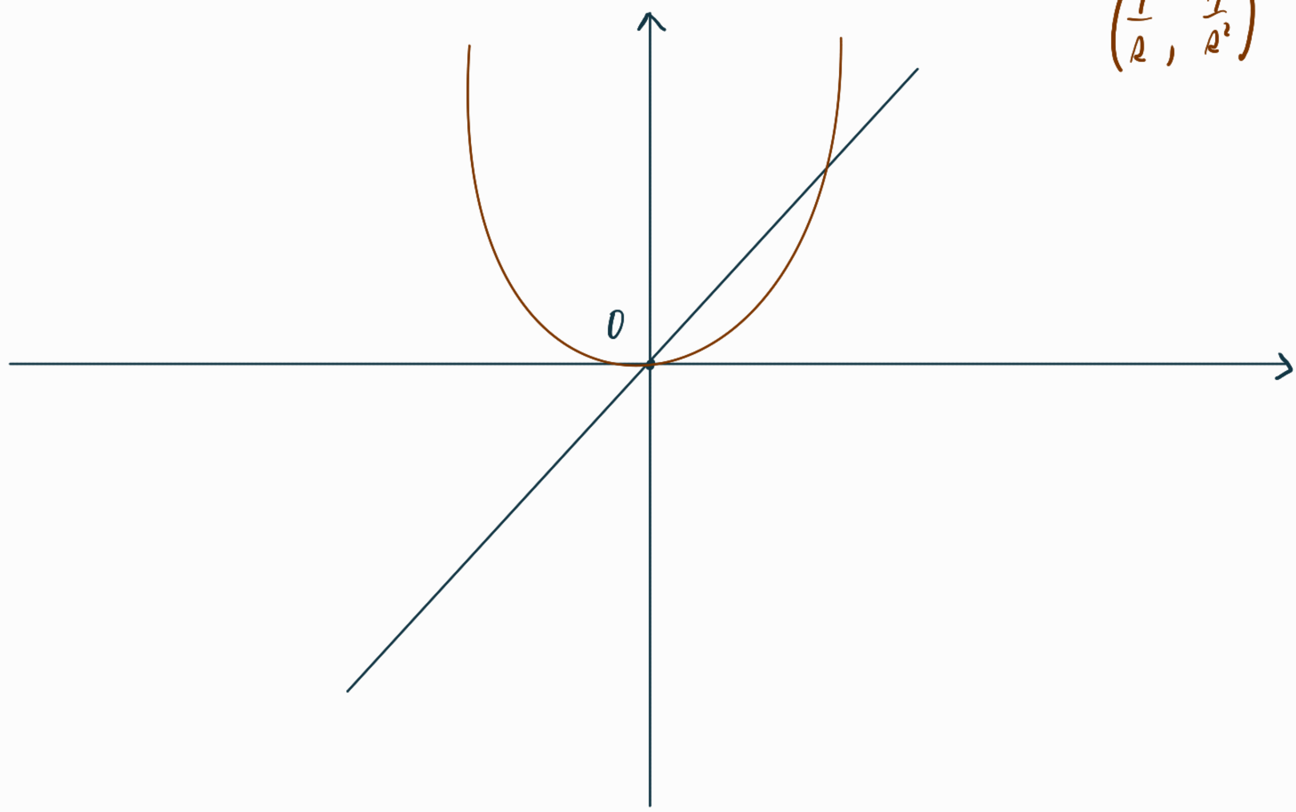
$\lim\limits_{l\to\infty}\left(\frac{1}{l^2},\frac{1}{l}\right)=(0,0)$

$\left.\begin{aligned}&\lim\limits_{l\to\infty} f\left(\frac{1}{l},\frac{1}{l}\right)=\lim\limits_{l\to\infty}\frac{2}{l}=0\\ &\lim\limits_{l\to\infty} f\left(\frac{1}{l^2},\frac{1}{l}\right)=\lim\limits_{l\to\infty}\frac{\frac{1}{l^6}+\frac{1}{l^3}}{\frac{1}{l^3}}=\lim\limits_{l\to\infty}\frac{1}{l^3}+1=0+1=1\end{aligned}\right\}\Rightarrow$

$\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

$\left(\frac{1}{l},\frac{1}{l}\right)$

$\left(\frac{1}{l},\frac{1}{l^2}\right)$



2) Let $B\subseteq\mathbb{R}^3$ be a closed set, let $f_1,\dots f_p: B\to\mathbb{R}$ and $g_1,\dots g_q: B\to\mathbb{R}$ be con. functions. Prove that that set

$A=\{x \mid x\in B, f_i(x)=0\ \forall i\in\{1,p\}, g_j(x)\le 0\ \forall j\in\{1,q\}$ is closed$\}$

? A closed $\Leftrightarrow$ the limit of every convergent sequence of points of A belongs also to A

Let (x_l) be an arbitrary convergent sequence of points in A

Let $\lim\limits_{l\to\infty} x_l = x$ $x_l\in A \Rightarrow x_l\in B,\ \forall l\ge 1$, B closed $\Rightarrow x\in B$

$?\, x\in A \Leftrightarrow \begin{cases}\bullet\ x\in B\\ \bullet\ \forall i\in\{1,p\}\ f_i(x)=0\\ \bullet\ \forall j\in\{1,q\}\ g_j(x)\le 0\ (*)\end{cases}$

Let $i \in \{1, p\}$ $\left.\begin{matrix} \\ \end{matrix}\right\} \Rightarrow f_i(x_k) = 0 \quad \forall k \geq 1 \quad \Rightarrow \lim_{k \to \infty} f_i(x_k) = 0$

$x_k \in A$

but $\lim_{k \to \infty} f_i(x_k) = f_i(x)$ $\Rightarrow f_i(x) = 0$

↳ f_i is con. at x

(*) the same

3) Prove that the following set are compact:

a) $A = \{(x, y) \mid (x, y) \in \mathbb{R}^2, \ x \geq 0, \ y \geq 0, \ x + y \leq 1\}$

b) $A = \{(x, y) \mid (x, y) \in \mathbb{R}^2, \ |x| \leq y \leq 2\}$

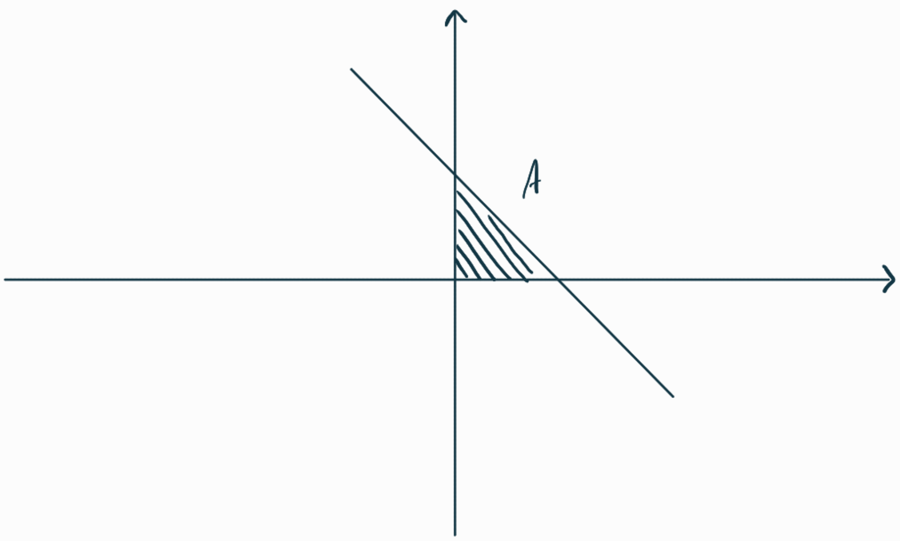
c) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0, \ y \geq 0, \ z \geq 0, \ x + y + z = 1\}$

solution:

A is compact $\Leftrightarrow$ A bounded and closed

a) the fact that A is closed follows by exercise (2) taking $B = \mathbb{R}^2$ closed

$$\left.\begin{aligned} g_1(x, y) &= -x \\ g_2(x, y) &= -y \\ g_3(x, y) &= x + y - 1 \end{aligned}\right\} \text{con. on } B$$

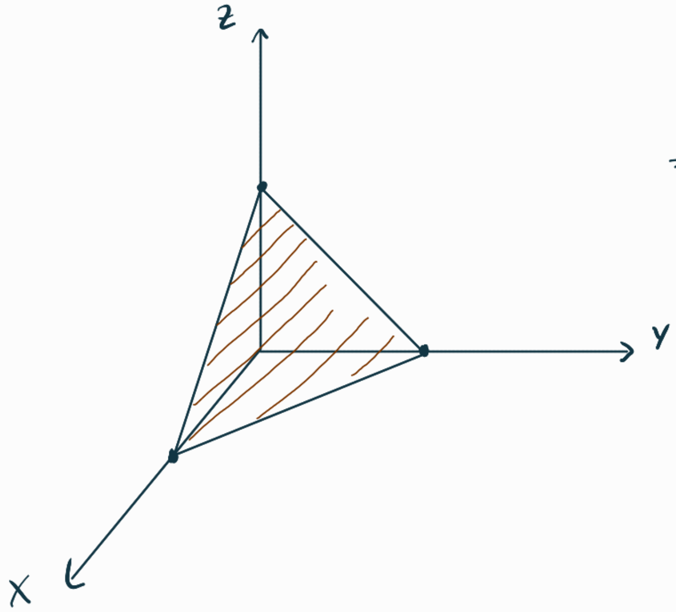


$\Rightarrow$ bounded

c) The fact that A is closed follows by exercise (2) taking $B = \mathbb{R}^3$ closed

$$\left.\begin{aligned} f_1(x, y, z) &= x + y + z - 1 \\ g_1(x, y, z) &= -x \\ g_2(x, y, z) &= -y \end{aligned}\right\} \text{con. on } B$$

$g_3(x, y, z) = -z$



$\Rightarrow A$ bounded