1) let $a=(-1,2,-2)\in\mathbb{R}^3$ and $b=(2,3,2)\in\mathbb{R}^3$, determine:

$a+b=(1,5,0)$

$a-b=(-3,-1,-4)$

$-3a+b=(5,-3,8)$

if $a=(a_1,\dots a_n)$ and $b=(b_1,\dots b_n)$ then $\langle a,b\rangle:=a_1b_1+\dots+a_nb_n$

$\langle a,b\rangle=0 \Rightarrow a\perp b$

def: $\|a\|=\sqrt{\langle a,a\rangle}=\sqrt{a_1^2+\dots+a_n^2}$

$\|a\|=\sqrt{9}=3$

$\|b\|=\sqrt{17}$

def: $d(a,b)=\|a-b\|$

$d(a,b)=6$

2) let $a=(2,3)\in\mathbb{R}^2$ and $b=(2,-1)$

$$\cos(\widehat{a,b})=\frac{\langle a,b\rangle}{\|a\|\cdot\|b\|}=\frac{1}{\sqrt{13}\cdot\sqrt{5}}=\frac{1}{\sqrt{65}} \Rightarrow m(\widehat{a,b})=\arccos\frac{1}{\sqrt{65}}$$

3) prove the Cauchy-Schwartz inequality: $\forall x,y\in\mathbb{R}^n$ one has

$|\langle x,y\rangle|\le\sqrt{\langle x,x\rangle}\cdot\sqrt{\langle y,y\rangle}$ (1)

solution:

let $x=(x_1,\dots x_n)$, $y=(y_1,\dots y_n)$

$(1) \Leftrightarrow |x_1y_1+\dots+x_ny_n|\le\sqrt{(x_1^2+\dots+x_n^2)}\cdot\sqrt{(y_1^2+\dots+y_n^2} \Leftrightarrow$

$\Leftrightarrow (x_1y_1+\dots+x_ny_n)^2\le(x_1^2+\dots+x_n^2)\cdot(y_1^2+\dots+y_n^2)$ (2)

consider $f:\mathbb{R}\to\mathbb{R}$ $f(t)=(x_1t-y_1)^2+\dots+(x_nt-y_n)^2$

$f(t)=t^2(x_1^2+\dots+x_n^2)-2t(x_1y_1+\dots+x_ny_n)+(y_1^2+\dots+y_n^2)$

if $x_1^2+\dots+x_n=0 \Rightarrow x_1=\dots=x_n=0 \Rightarrow$ (2) holds with „="

if $x_1^2+\dots+x_n^2>0 \Rightarrow f$ is 2nd degree polynomial $\Big\} \Rightarrow$

$\Rightarrow f(t)\geq 0 \ \forall t\in\mathbb{R}$

$\Rightarrow \Delta\leq 0$, but $\Delta=4(x_1y_1+\dots+x_ny_n)^2-4(x_1^2+\dots x_n^2)(y_1^2+\dots y_n^2) \Rightarrow (2)$ holds

4) the Euclidian norm on $\mathbb{R}^n$ is defined by $\|x\|=\sqrt{\langle x,x\rangle}$, $\forall x\in\mathbb{R}^n$

prove that the Euclidian norm satisfies the triangle inequality

$\|x+y\|\leq\|x\|+\|y\|$ (1)

solution:

let $x,y\in\mathbb{R}^n$, $x=(x_1,\dots x_n)$, $y=(y_1,\dots y_n)$

$(1) \Leftrightarrow \sqrt{\langle x+y,x+y\rangle}\leq\sqrt{\langle x,x\rangle}+\sqrt{\langle y,y\rangle} \Leftrightarrow$

$\langle x+y\rangle \leq \langle x,x\rangle+2\sqrt{\langle x,x\rangle\langle y,y\rangle}+\langle y,y\rangle \Leftrightarrow$

$\cancel{\langle x,x\rangle}+\langle x,y\rangle+\langle x,y\rangle+\cancel{\langle y,y\rangle}\leq\cancel{\langle x,x\rangle}+2\sqrt{\langle x,x\rangle\langle y,y\rangle}+\cancel{\langle y,y\rangle}$

$\Rightarrow 2\langle x,y\rangle\leq 2\sqrt{\langle x,x\rangle\langle y,y\rangle}$ holds because:

$\langle x,y\rangle\leq|\langle x,y\rangle|\leq\sqrt{\langle x,x\rangle\langle y,y\rangle}$

Properties of the inner product on $\mathbb{R}^n$:

1) $\langle x+y,z\rangle=\langle x,z\rangle+\langle y,z\rangle \ \forall x,y,z\in\mathbb{R}^n$

2) $\langle \lambda x,y\rangle=\lambda\langle x,y\rangle \ \forall x,y\in\mathbb{R}^n, \forall \lambda\in\mathbb{R}$

3) $\langle x,y\rangle=\langle y,x\rangle \ \forall x,y\in\mathbb{R}^n$

4) $\langle x,x\rangle>0 \ \forall x\in\mathbb{R}^n$, $x\neq 0_n$, $0_n=(\underbrace{0,\dots 0}_{n \text{ times}})$

5) using the properties of the inner product prove that $\| \ \|$ statisfies the parallelogram identity: $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2 \ \forall x,y\in\mathbb{R}^n$

solution:

$x,y\in\mathbb{R}$

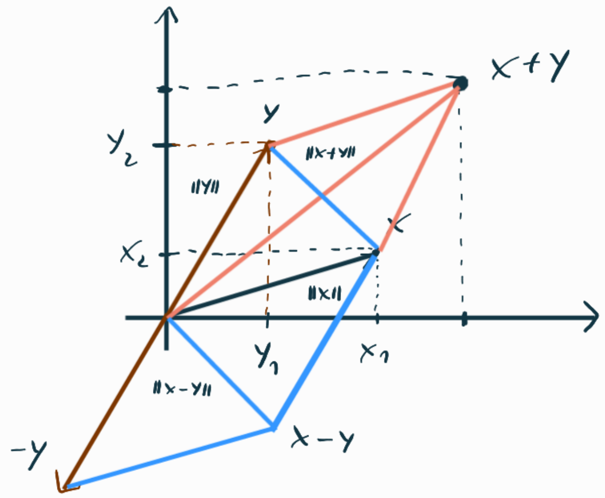
$\|x+y\|^2=\langle x+y,x+y\rangle\overset{1)}{=}\langle x,x+y\rangle+\langle y,x+y\rangle=\langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle$

$=\langle x,x\rangle+\langle x,y\rangle+\langle x,y\rangle+\langle y,y\rangle=\langle x,x\rangle+2\langle x,y\rangle+\langle y,y\rangle$

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x, x-y\rangle + \langle -y, x-y\rangle = \langle x,x\rangle + \langle x,-y\rangle + \langle -y, x\rangle + \langle -y,-y\rangle$$
$$= \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle - \langle y,-y\rangle = \langle x,x\rangle - \langle x,y\rangle - \langle x,y\rangle + \langle y,y\rangle =$$
$$= \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle$$

$$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = \langle x+x\rangle + 2\langle x,y\rangle + \langle y,y\rangle + \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle =$$
$$= 2\langle x,x\rangle + 2\langle y,y\rangle = 2\|x\|^2 + 2\|y\|^2 \quad \text{true}$$

geometric interpretation:




$x = (x_1, x_2)$

$y = (y_1, y_2)$

in a parallelogram (para.) the sum of the squares of the two diagonals = the sum of the squares of the four sides

6) in $\mathbb{R}^2$ we have

- the Euclidean norm $\|x\| = \sqrt{x_1^2 + x_2^2}$
- the Minkowski norm $\|x\|_1 = |x_1| + |x_2|$
- the Tchebyshev norm $\|x\|_\infty = \max(|x_1|, |x_2|)$

$\overline{B}(a, r) = \{x \in \mathbb{R}^n \mid \|x-a\| \le r\}$ the closed ball about a with radius r

Draw the closed balls in $\mathbb{R}^2$ about the origin, with $r=1$, w.r.t the norms mentioned above

Solution: $\overline{B}(0_2, 1) = \{x \in \mathbb{R}^2 \mid \|x - 0_2\| \le 1\} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \le 1\}$

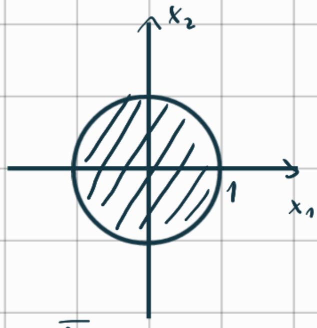
Euclidean ball

$\overline{B}_1(0_2, 1) = \{x \in \mathbb{R}^2 \mid \|x - 0_2\| < 1\} = \{(x_1, x_2) \mid |x_1| + |x_2| \le 1\}$

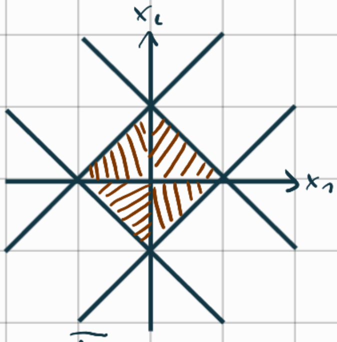
Minkowski ball

$\overline{B}_\infty(0_2, 1) = \{x \in \mathbb{R}^2 \mid \|x - 0_2\| < 1\} = \{(x_1, x_2) \mid \max(|x_1|, |x_2|) \le 1\}$

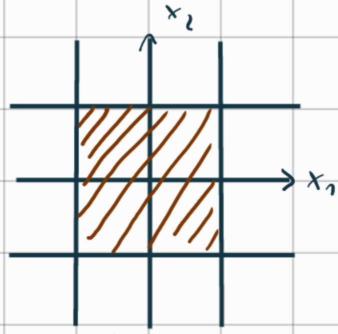
Tchebyshev ball




$\overline{B}(0_2, 1)$




$\overline{B}_1(0_2, 1)$




$\overline{B}_\infty(0_2, 1)$

$C_{\underline{I}}: x_1 + x_2 \leq 1$

$C_{\underline{II}}: -x_1 + x_2 \leq 1$

$\max(|x_1|, |x_2|) \Rightarrow$

$$\Rightarrow \begin{cases} |x_1| \leq 1 \\ \text{and} \\ |x_2| \leq 2 \end{cases} \Rightarrow \begin{cases} -1 \leq x_1 \leq 1 \\ -1 \leq x_2 \leq 1 \end{cases}$$