1) a) $f(x,y,z) = 2x^2 - xy + 2xz - y + y^3 + z^2$

solution: the crit. points of f are solutions to the system:

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 4x - y + 2z = 0 \Rightarrow -4z - y + 2z = 0 \Rightarrow x = -2z \\ \frac{\partial f}{\partial y}(x,y,z) = -x - 1 + 3y^2 = 0 \Rightarrow z - 1 + 3 \cdot 4z^2 = 0 \Rightarrow 12z^2 + z - 1 = 0 \Rightarrow z_1 = -\frac{1}{3} \\ \qquad z_2 = \frac{1}{4} \\ \frac{\partial f}{\partial z}(x,y,z) = 2x + 2z = 0 \Rightarrow x = -z \end{cases}$$

$\Rightarrow$ the crit. points are $\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right), \left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)$

$$H(f)(x,y,z) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 6y & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$H(f)\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 4$, $\Delta_2 = 15$ $\Delta_3 = 14$ $\Rightarrow \left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ is local min.

$$H(f)\left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 4 \\ \Delta_2 = -13 \\ \Delta_3 = -14 \end{array}\right\} \Rightarrow \left(-\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \text{ saddlepoint}$$

c) $f(x,y,z) = x^2 + y^2 + z^2 - 2xyz$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 2x - 2yz = 0 \\ \frac{\partial f}{\partial y}(x,y,z) = 2y - 2xz = 0 \\ \frac{\partial f}{\partial z}(x,y,z) = 2z - 2xy = 0 \end{cases} \Leftrightarrow \begin{cases} x = yz \\ y = xz \\ z = xy \end{cases} \Leftrightarrow y = yz^2 \Rightarrow$$

$\Leftrightarrow \left\{ y(1-z^2)=0 \right.$ $\Rightarrow$ $y=0 \Rightarrow x=0,\ z=0 \Rightarrow (0,0,0)$ crit. point

$z=1 \Rightarrow x=y \Rightarrow y^2=1 \Rightarrow y=\pm 1$

$\Rightarrow (1,1,1), (-1,-1,1)$ crit. points

$z=-1 \Rightarrow x=-y \Rightarrow y^2=1 \Rightarrow y=\pm 1$

$\Rightarrow (1,1,-1), (-1,-1,-1)$ crit. points

$$H(f)(x,y,z)=\begin{pmatrix} 2 & -2z & -2y \\ -2z & 2 & -2x \\ -2y & -2x & 2 \end{pmatrix}$$

$$H(f)(0,0,0)=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ $\Delta_1=2 \quad \Delta_2=4 \quad \Delta_3=8 \Rightarrow (0,0,0)$ local min.

$$H(f)(1,1,1)=\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$ $\Delta_1=2,\ \Delta_2=0 \Rightarrow$ Sylvester's test does not apply

$$d^2f(1,1,1)(h_1,h_2,h_3)=2h_1^2+2h_2^2+2h_3^2-4h_1h_2-4h_1h_3-4h_2h_3=E$$

if $h_1=1,\ h_2=h_3=0$ then $E=2>0$

if $h_1=h_2=h_3=1$ then $E=-6<0$

$\Big\} \Rightarrow d^2f(1,1,1)$ is indef. q. form $\Rightarrow$

$\Rightarrow (1,1,1)$ is a saddlepoint

d) $f(x,y)=x^4+y^4-4(x-y)^2$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x}(x,y)=4x^3-8\cdot 1\cdot(x-y)=0 \\ \dfrac{\partial f}{\partial y}(x,y)=4y^3+8\cdot 1\cdot(x-y)=0 \end{cases} \downarrow +$$

$\Rightarrow 4x^3+4y^3=0 \Rightarrow x^3=-y^3=(-y)^3 \Rightarrow$

$\Rightarrow x=-y$

$\Rightarrow 4y^3 - 16y = 0 \Rightarrow 4y(y^2-4) = 0 \Rightarrow y = 0 \Rightarrow x = 0 \quad (0,0)$ crit. point

$y = 2 \Rightarrow x = -2 \quad (-2,2)$ — // —

$y = -2 \Rightarrow x = 2 \quad (2,-2)$ — // —

$$H(f)(x,y) = \begin{pmatrix} 12x^2-8 & 8 \\ 8 & 12y^2+8 \end{pmatrix}$$

$$H(f)(0,0) = \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix} \quad \Delta_1 = -8 \quad \Delta_2 = 0 \Rightarrow \text{Sylvester's test does not apply}$$

$d^2f(0,0)(h_1,h_2) = -8h_1^2 - 8h_2^2 + 16h_1h_2 = -8(h_1^2+h_2^2-2h_1h_2)$ n. sdef. q. form that is not n. def. $\Rightarrow$

$\Rightarrow$ we can't tell the nature of $(0,0)$ based on $d^2f(0,0)$

$$\left.\begin{array}{l} f(x,x) = 2x^4 > 0 = f(0,0) \quad \forall x \in \mathbb{R}\setminus\{0\} \Rightarrow (0,0) \text{ not local max.} \\ f(x,0) = x^4 - 4x^2 = x^2(x^2-4) < 0 = f(0,0) \quad \forall x \in (-2,2)\setminus\{0\} \Rightarrow \\ \qquad \Rightarrow (0,0) \text{ not a local min} \end{array}\right\} \Rightarrow$$

$\Rightarrow (0,0)$ saddle point

$$H(f)(-2,2) = H(f)(2,-2) = \begin{pmatrix} 40 & 8 \\ 8 & 40 \end{pmatrix} \quad \Delta_1 = 40 \quad \Delta_2 > 0 \Rightarrow$$

$\Rightarrow (2,-2)$ and $(-2,2)$ are local min. points

2) Calculate the following integrals over closed cells (rectangles or paralelopipeds)

a) $I = \int_1^6 \int_2^3 \frac{1}{(x+y)^2}\, dx\, dy \longleftarrow$ double integral over the rectangle $[1,6]\times[2,3]$

$$I = \int_{x=1}^{x=6} \left(\int_{y=2}^{y=3} \frac{1}{(x+y)^2}\, dy \right) dx$$

$$I = \int_{y=2}^{y=3} \left(\int_{x=1}^{x=6} \frac{1}{(x+y)^2}\, dx \right) dy$$

iterated integrals

$$I = \int_{x=1}^{x=6} \left(\int_{y=2}^{y=3} \frac{1}{(x+y)^2}\, dy \right) dx = \int_{x=1}^{x=6} \left(\frac{-1}{x+y} \Big|_{y=2}^{y=3} \right) dx =$$

$$= \int_1^6 \left(\frac{-1}{x+3} + \frac{1}{x+2} \right) dx = -\log(x+3)\Big|_1^6 + \log(x+2)\Big|_1^6 = -\ln 9 + \ln 4 + \ln 8 - \ln 3$$

$$= \ln \frac{32}{27}$$

b) $$I = \int_0^1 \int_0^1 \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}}\, dx\, dy = \int_0^1 \left(\int_0^1 \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}}\, dx \right) dy =$$

$$= \int_0^1 \left(\frac{1}{2} \cdot \left(-\frac{2}{1}\right) \cdot \frac{1}{(1+x^2+y^2)^{\frac{1}{2}}} \Big|_0^1 \right) dy =$$

OR

$$1+x^2+y^2 = t$$
$$2x\, dx = dt$$

$$\Rightarrow I = \int_{y=0}^{y=1} \left(\int_{t=1+y^2}^{t=2+y^2} t^{-\frac{3}{2}} \cdot \frac{1}{2}\, dt \right) dy =$$

$$= \int_0^1 \left(\frac{1}{2} \cdot \frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} \Big|_{t=1+y^2}^{t=2+y^2} \right) dy$$

$$= \int_0^1 \left(-\frac{1}{\sqrt{2+y^2}} + \frac{1}{\sqrt{1+y^2}} \right) dy = -\ln\left(y+\sqrt{y^2+2}\right)\Big|_0^1 - \ln\left(y+\sqrt{1+y^2}\right)\Big|_0^1 =$$

$$= -\ln\left(\frac{2+\sqrt{2}}{1+\sqrt{3}}\right)$$

c) $$I = \int_1^2\int_1^2\int_1^2 \frac{1}{(x+y+z)^3}\, dx\, dy\, dz = \int_{x=1}^{x=2} \int_{y=1}^{y=2} \left(\frac{(x+y+z)^{-2}}{-2} \Big|_{z=1}^{z=2} dy \right) dx =$$

$$= \frac{1}{2} \int_{x=1}^{x=2} \int_{y=1}^{y=2} \left(\frac{-1}{(x+y+2)^2} + \frac{1}{(x+y+1)^2}\, dy \right) dx = \frac{1}{2} \int_{x=1}^{x=2} \left(\frac{1}{x+y+2}\Big|_1^2 - \frac{1}{x+y+1}\Big|_1^2 \right) dx =$$

$$= \frac{1}{2} \int \left(\frac{1}{x+4} - \frac{1}{x+3} - \frac{1}{x+3} + \frac{1}{x+2} \right) dx = \frac{1}{2} \left(\ln(x+4) + \ln(x+2) - 2\ln(x+3) \right)\Big|_1^2 =$$

$$= \frac{1}{2} \ln \frac{128}{125}$$