1) $f = f(u,v) : \mathbb{R}^2 \to \mathbb{R}$, $F : \mathbb{R}^3 \to \mathbb{R}$, $F(x,y,z) = f(x^2 - y + 2yz, \ z^3 e^{xy})$

express the part. derivative of F in terms of first order part. derivatives of f

solution: $F = f \circ g$ where $g(x,y,z) = (\underbrace{x^2 - y + 2yz^2}_{u(x,y,z)}, \ \underbrace{z^3 e^{xy}}_{v(x,y,z)})$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial F}{\partial z} = \dots$$

$$\frac{\partial F}{\partial x}(x,y,z) = \frac{\partial f}{\partial u}(g(x,y,z)) \cdot \underbrace{\frac{\partial u}{\partial x}(x,y,z)}_{2x} + \frac{\partial f}{\partial v}(g(x,y,z)) \cdot \underbrace{\frac{\partial v}{\partial y}(x,y,z)}_{yz^3 e^{xy}} =$$

$$= 2x \frac{\partial f}{\partial u}(x^2 - y + 2yz^2, \ z^3 e^{xy}) + yz^3 e^{xy} \frac{\partial f}{\partial v}(x^2 - y + 2yz^3, \ z^3 e^{xy})$$

$$\frac{\partial F}{\partial y}(x,y,z) = (-1 + 2z^2) \frac{\partial f}{\partial u}(x^2 - y + 2yz^2, \ z^3 e^{xy}) + xz^3 e^{xy} \frac{\partial f}{\partial v}(x^2 - y + 2yz^2, \ z^3 e^{xy})$$

$$\frac{\partial F}{\partial z}(x,y,z) = \dots$$

2) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be diff on $\mathbb{R}^3$ and let $F : \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = f(\cos x + \sin y, \ \sin x + \cos y, \ e^{x-y})$

a) prove: if f is con. diff. on $\mathbb{R}^3$ then F is con. diff on $\mathbb{R}^3$

b) determine $dF(\frac{\pi}{2}, \frac{\pi}{2})$ if $J(f)(1,1,1) = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix}$

solution: $F = f \circ g$ where $g(x,y) = (\underbrace{\cos x + \sin y}_{u(x,y)}, \underbrace{\sin x + \cos y}_{v(x,y)}, \underbrace{e^{x-y}}_{w(x,y)})$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial f}{\partial u}(g(x,y)) \cdot \underbrace{\frac{\partial u}{\partial x}(x,y)}_{-\sin x} + \frac{\partial f}{\partial v}(g(x,y)) \cdot \underbrace{\frac{\partial v}{\partial x}(x,y)}_{\cos x} + \frac{\partial f}{\partial w}(g(x,y)) \cdot \underbrace{\frac{\partial w}{\partial x}(x,y)}_{e^{x-y}}$$

$$\Rightarrow \frac{\partial F}{\partial x}(x,y) = (-\sin x) \cdot \frac{\partial F}{\partial u}(\cos x + \sin y, \sin x + \cos y, e^{x-y}) + \cos x \frac{\partial F}{\partial v}(\dots) + e^{x-y} \frac{\partial F}{\partial w}(\dots) \Bigg\} \Rightarrow$$

f is con. diff. on $\mathbb{R}^3 \Rightarrow \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}$ are con. diff. on $\mathbb{R}^3$

$\Rightarrow \frac{\partial F}{\partial x}$ is con. on $\mathbb{R}^3$ $\Bigg\} \Rightarrow F$ is con. diff. on $\mathbb{R}^2$

analoguously $\frac{\partial F}{\partial y}$ is con. diff on $\mathbb{R}^2$

b) $dF(\frac{\pi}{2}, \frac{\pi}{2}) \in L(\mathbb{R}^2, \mathbb{R}^2)$ $\quad [dF(\frac{\pi}{2}, \frac{\pi}{2})] = J(F(\frac{\pi}{2}, \frac{\pi}{2})) =$

$= J(f)(g(\frac{\pi}{2}, \frac{\pi}{2})) \cdot J(g)(\frac{\pi}{2}, \frac{\pi}{2}) = J(f)(1, 1, 1) \cdot J(g)(\frac{\pi}{2}, \frac{\pi}{2}) = *$

$$J(g)(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix} = \begin{pmatrix} -\sin x & \cos y \\ \cos x & -\sin y \\ e^{x-y} & -e^{x-y} \end{pmatrix}$$

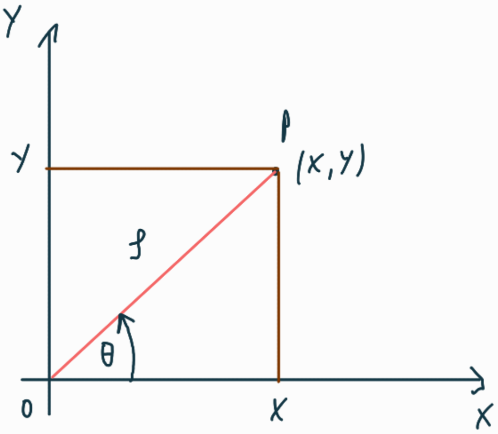
$$* = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix} \Rightarrow dF(\tfrac{\pi}{2}, \tfrac{\pi}{2})(h_1, h_2) = \begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \begin{pmatrix} 3h_1 - 7h_2 \\ h_1 - 2h_2 \end{pmatrix}$$

$dF(\frac{\pi}{2}, \frac{\pi}{2})(h_1, h_2) = (3h_1 - 7h_2, h_1 - 2h_2)$

3) with the aid of polar coordinates determine all diff. functions

$f : (0,\infty) \times (0,\infty) \to \mathbb{R}$, satisfying $x \frac{\partial f}{\partial x}(x,y) + y \frac{\partial f}{\partial y}(x,y) = \sqrt{x^2+y^2} \quad \forall (x,y) \in (0,\infty) \times (0,\infty)$ (1)



(x, y) = Cartesians coordinates of P

(ρ, θ) = polar coordinates of P

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$$

solution: assume that f satisfies (1)

consider the function defined by $F(\rho, \theta) = f(\rho\cos\theta, \rho\sin\theta)$

$F : (0, \infty) \times (0, \frac{\pi}{2}) \to \mathbb{R}$

$F = f \circ g$ where $g(\rho, \theta) = (\underbrace{\rho\cos\theta}_{x(\rho,\theta)}, \underbrace{\rho\sin\theta}_{y(\rho,\theta)})$

$$\frac{\partial F}{\partial \rho} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \rho} \qquad \frac{\partial F}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$\frac{\partial F}{\partial \rho}(\rho, \theta) = \frac{\partial f}{\partial x}(g(\rho,\theta)) \cdot \underset{\cos\theta}{\underset{\downarrow}{\frac{\partial x}{\partial \rho}(\rho,\theta)}} + \frac{\partial f}{\partial y}(g(\rho,\theta)) \cdot \underset{\sin\theta}{\underset{\downarrow}{\frac{\partial y}{\partial \rho}(\rho,\theta)}}$$

$$\frac{\partial F}{\partial \rho}(\rho, \theta) = \cos\theta \frac{\partial f}{\partial x}(\rho\cos\theta, \rho\sin\theta) + \sin\theta \frac{\partial f}{\partial y}(\rho\cos\theta, \rho\sin\theta) \;/\cdot \rho \Rightarrow$$

$$\Rightarrow \rho \frac{\partial F}{\partial \rho}(\rho, \theta) = \rho\cos\theta \cdot \frac{\partial f}{\partial x}(\rho\cos\theta, \rho\sin\theta) + \rho\sin\theta \cdot \frac{\partial f}{\partial y}(\rho\cos\theta, \rho\sin\theta) \overset{(1)}{=} \rho$$

$$\Rightarrow \frac{\partial F}{\partial \rho}(\rho, \theta) = 1 \quad \Rightarrow \quad F(\rho, \theta) = \int 1\, d\rho = \rho + h(\theta)$$

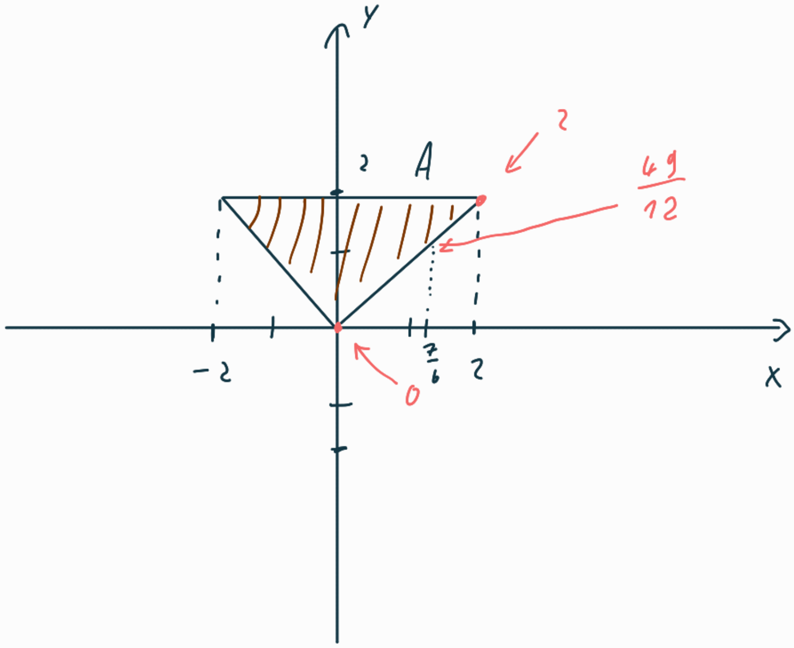
$F = f \circ g \,/ \circ g^{-1} \Rightarrow f = F \circ g^{-1}$

$g: \begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$ $\quad g^{-1}: \begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan \frac{y}{x} \end{cases}$

$f(x,y) = \sqrt{x^2+y^2} + h(\arctan(\frac{y}{x}))$, where $h: (0; \frac{\pi}{2}) \to \mathbb{R}$ is an arbitrary diff. function

4) consider $A = \{(x,y) \in \mathbb{R}^2 \mid |x| \le y \le 2\}$ and $f: A \to \mathbb{R}$

$f(x,y) = 4x + 3y - 2x^2 - y^2$ determine $\min f(A)$ $\max f(A)$



A is compact and has non-empty interior

$m = \min f(A)$, $M = \max f(A)$

$m_1 = \min f(bd(A))$ $\quad M_1 = \max f(bd(A))$ $\qquad C = \{(x,y) \in \operatorname{int} A \mid \nabla(x,y) = (0,0)\}$

$m_2 = \min f(C)$ $\quad M_2 = \max f(C)$

$bd\,A = \{(x,x) \mid x \in [0,2\} \cup \{(-x,x) \mid x \in [0,2]\} \cup \{(x,2) \mid x \in [-2,2]\}$

$f(x,x) = -3x^2 + 7x$

x	0	$\frac{7}{6}$	2
$-3x^2+7x$	0 $\nearrow$	$\frac{49}{12}$	$\searrow$ 2

$f(\frac{7}{6}) = \frac{49}{12}$

$\min = 0$

$\max = \frac{49}{12}$