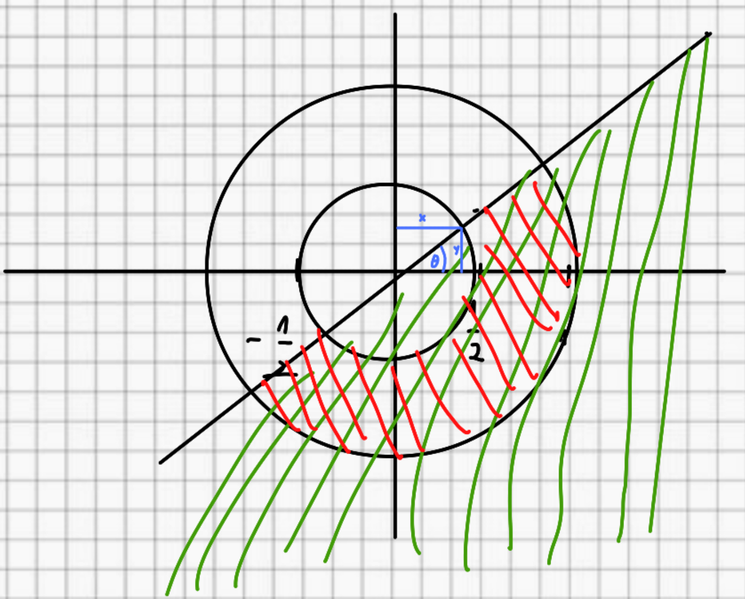
1. Calculate $I = \iint_A x \cdot \sqrt{1-x^2-y^2}\, dx\, dy$ where $A = \{(x,y) \in \mathbb{R}^2 \mid$

$1 \le 4(x^2+y^2) \le 4$

$\sqrt{3} \cdot x \ge 3y$



$y \le \frac{\sqrt{3}}{3} x$

30^0

$x = \rho \cos(\theta) \quad \rho \in [\frac{1}{2}, 1]$

$y = \rho \cos(\theta) \quad \theta \in [0, \frac{\pi}{6}] \cup [\frac{7\pi}{6}, 2\pi]$

$\theta \in [\frac{-5\pi}{6}, \frac{\pi}{6}]$

$$I = \int_{\frac{1}{2}}^{1} \int_{-\frac{5\pi}{6}}^{\frac{\pi}{6}} \rho \cos\theta \cdot \sqrt{1-\rho^2} \cdot \rho \, d\rho \, d\theta =$$

$$= \int_{\frac{1}{2}}^{1} \rho^2 \sqrt{1-\rho^2}\, d\rho \cdot \int_{-\frac{5\pi}{6}}^{\frac{\pi}{6}} \cos\theta \, d\theta = \frac{1}{4} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 \sin^2 t \cdot \cos^2 t \, dt =$$

$\sin\theta \Big/_{-\frac{5\pi}{6}}^{\frac{\pi}{6}}$

$\rho = \sin t$

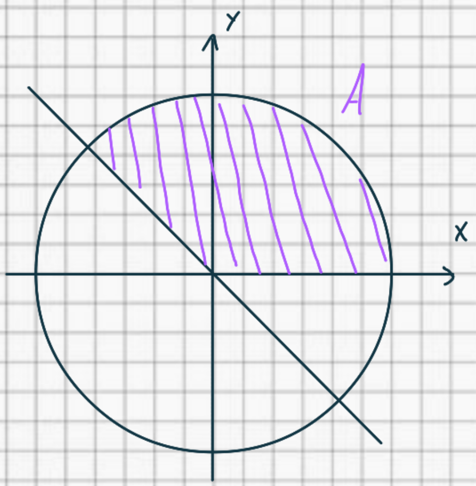
$$= \frac{1}{4} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2(2t)\, dt = \frac{1}{4} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1 - \cos(4t)}{2}\, dt =$$

$$\boxed{\begin{aligned} \sin^2(\alpha) &= \frac{1-\cos(2\alpha)}{2} \\ \cos^2(\alpha) &= \frac{1+\cos(2\alpha)}{2} \end{aligned}}$$

$$\frac{1}{8} \cdot \left(t - \frac{\sin(4t)}{4} \right) \Big/_{\frac{\pi}{2}}^{\frac{\pi}{6}} =$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}}{64}$$

2) $I = \iint_A \frac{y}{x^2+y^2+3}\,dx\,dy \qquad A = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1,\ x+y\ge 0,\ y\ge 0\}$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \rho\in[0,1] \quad \theta\in\left[0,\tfrac{3\pi}{4}\right]$$

$$I = \int_0^1\int_0^{\frac{3\pi}{4}} \frac{\rho\sin\theta}{\rho^2+3}\,\rho\,d\rho\,d\theta = \int_0^1 \frac{\rho^2}{\rho^2+3}\,d\rho \int_0^{\frac{3\pi}{4}} \sin\theta\,d\theta =$$

$$= \int_0^1 1-\frac{3}{\rho^2+3}\,d\rho\ \underbrace{(-\cos\theta)\Big|_0^{\frac{3\pi}{4}}}_{\frac{\sqrt2}{2}+1=\frac{\sqrt2+2}{2}} = \frac{\sqrt2+2}{2}\left(\rho - 3\cdot\frac{1}{\sqrt3}\cdot\operatorname{arctg}\frac{\rho}{\sqrt3}\right)\Big|_0^1 =$$

$$= \frac{\sqrt2+2}{2}\cdot\left(1-\sqrt3\cdot\frac{\pi}{6}\right)$$

3) $I = \iiint_A \frac{1}{x^2+y^2+z^2+1}\,dx\,dy\,dz \quad , \quad A = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le 1\}$

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases} \qquad \begin{aligned} &\rho\in[0,1] \\ &\varphi\in[0,\pi] \\ &\theta\in[0,2\pi] \end{aligned}$$

$$I = \int_0^1\int_0^{\pi}\int_0^{2\pi} \frac{1}{\rho^2+1}\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^1 \frac{\rho^2}{\rho^2+1}\,d\rho \cdot \underbrace{\int_0^{\pi}\sin\varphi\,d\varphi}_{-\cos\varphi\big|_0^{\pi}=1+1=2} \cdot \underbrace{\int_0^{2\pi} 1\,d\theta}_{2\pi} =$$

$$= 4\pi\int_0^1\left(1-\frac{1}{\rho^2+1}\right)d\rho = 4\pi\left(\rho-\operatorname{arctg}\rho\right)\Big|_0^1 = 4\pi\left(1-\frac{\pi}{4}\right)$$

4) $I = \iiint_A \frac{z}{(x^2+y^2+1)^2}\,dx\,dy\,dz \qquad A = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le 1,\ z\ge 0\}$

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\theta \end{cases} \qquad \begin{aligned} &\rho\in[0,1] \\ &\varphi\in[0,\pi] \\ &\theta\in[0,2\pi] \end{aligned}$$

$$I = \int_0^1\int_0^\pi\int_0^{2\pi} \frac{\rho\cos\theta}{(\rho^2\sin^2\varphi+1)^2}\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = \int_0^1\int_0^\pi \frac{\rho^3\sin\varphi\cos\varphi}{(\rho^2\sin^2\varphi+1)}\,d\rho\, d\varphi \cdot \underbrace{\int_0^{2\pi} d\theta}_{2\pi} =$$

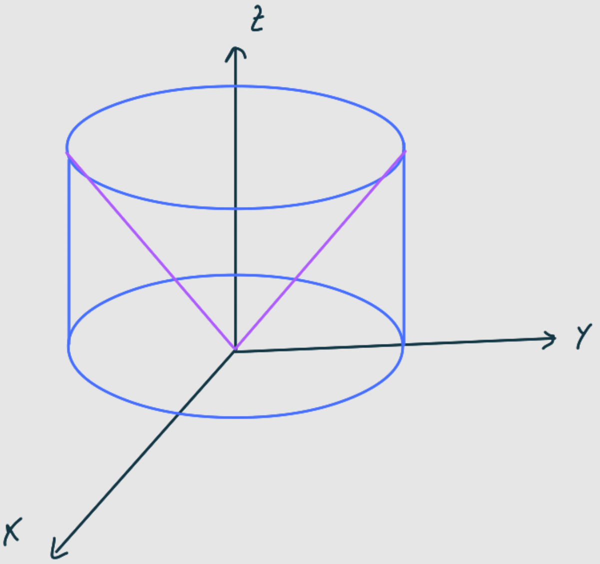
$$= 2\pi\int_0^1\left(\int_0^\pi \frac{\rho^3\sin\varphi\cos\varphi}{(\rho^2\sin^2\varphi+1)^2}\,d\varphi\right)d\rho = 2\pi\int_0^1 \frac{1}{2}\left(\int_1^{\rho^2+1}\frac{\rho\,dt}{t^2}\,dt\right)d\rho =$$

$$t = \rho^2\sin^2\varphi + 1$$

$$dt = 2\rho^2\sin\varphi\cos\varphi\, d\varphi$$

$$= 2\pi\int_0^1\left(\frac{-\rho}{t}\Big|_1^{\rho^2+1}\right)d\rho = 2\pi\int_0^1\left(\frac{-\rho}{\rho^2+1}+\rho\right)d\rho = \pi\left(-\frac{1}{2}\ln(\rho^2+1)+\rho^2\right)\Big|_0^1 = \frac{\pi}{2}(1-\ln 2)$$

5) $I = \iiint_A 2z(x^2+y^2)\,dx\,dy\,dz$, $A = \{(x,y,z)\in\mathbb{R}^3 \mid z\geq 0,\ x^2+y^2=1,\ z=\sqrt{x^2+y^2}\}$

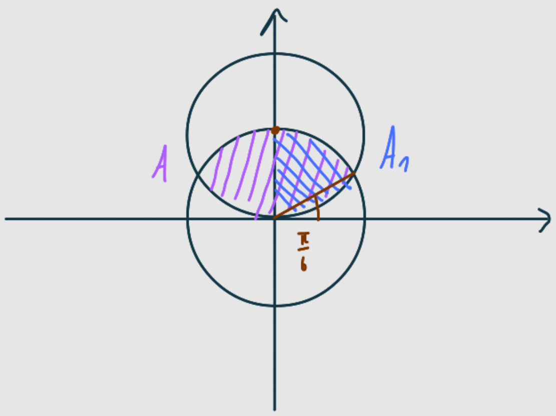


$$\begin{cases} x = \rho\cos\theta & \rho\in[0,1] \\ y = \rho\sin\theta & \theta\in[0,2\pi] \\ z = z & z\in[0,\rho] \end{cases}$$

$$I = \int_0^1\int_0^{2\pi}\left(\int_0^\rho 2z\,\rho^2\cdot\rho\,dz\right)d\rho\,d\theta = \int_0^1\int_0^{2\pi}\rho^3\left(\int_0^\rho 2z\,dz\right)d\rho\,d\theta =$$

$$= \int_0^1\int_0^{2\pi}\rho^5\,d\rho\,d\theta = \int_0^1\rho^5\,d\rho\int_0^{2\pi}d\theta = \frac{\pi}{3}$$

6. calculate the Jordan measure (area) of $A = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 1,\ x^2+y^2\leq 2y\}$



$$m(A) = \iint_A dx\,dy = 2\iint_{A_1} dx\,dy$$

$x = \rho\cos\theta$ if $\theta\in\left[\frac{\pi}{6},\frac{\pi}{2}\right]$ $\rho\in[0,1]$

$y = \rho\sin\theta$ if $\theta\in\left[0,\frac{\pi}{6}\right]$ $\rho\in[0,2\sin\theta]$

$$m(A) = 2\int_0^1 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \rho \, d\rho \, d\theta + 2\int_0^{\pi} \left(\int_0^{2\sin\theta} \rho \, d\rho \right) d\theta = etc$$