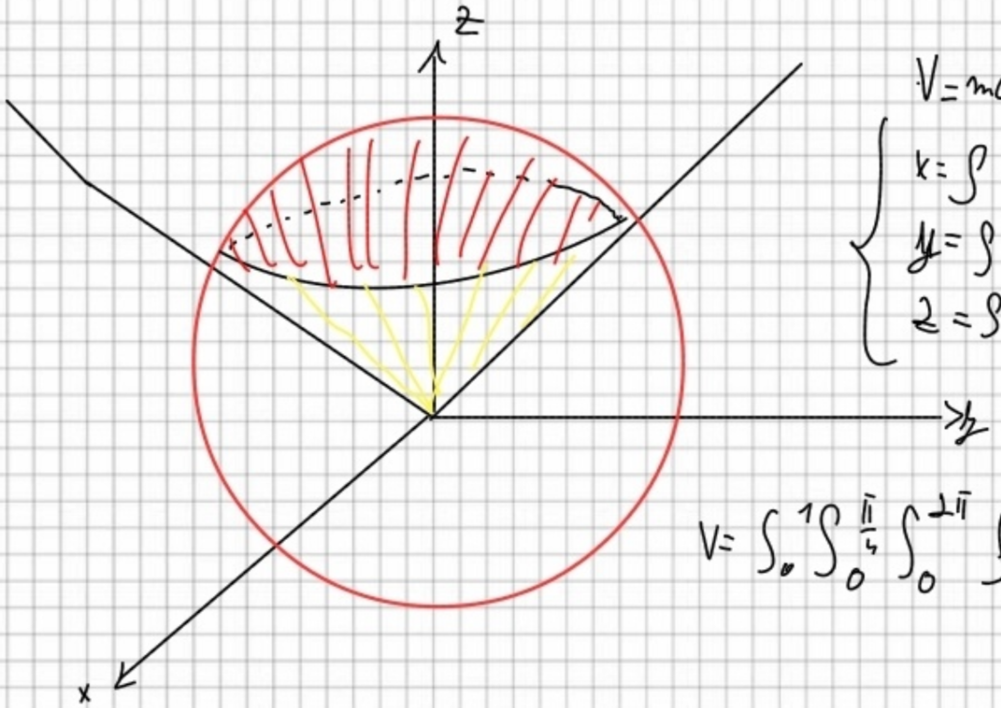
1) Calculate the volume (Jordan-measure) of the solid bounded by the cone $z=\sqrt{x^2+y^2}$ and by the sph $x^2+y^2+z^2=1$



$V=m(A)=\iiint_A dx\,dy\,dz$

$$\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\cos\theta \\ z=\rho\cos\varphi \end{cases}\quad \rho\in[0,1],\ \varphi\in[0,\tfrac{\pi}{4}],\ \theta\in[0,2\pi]$$

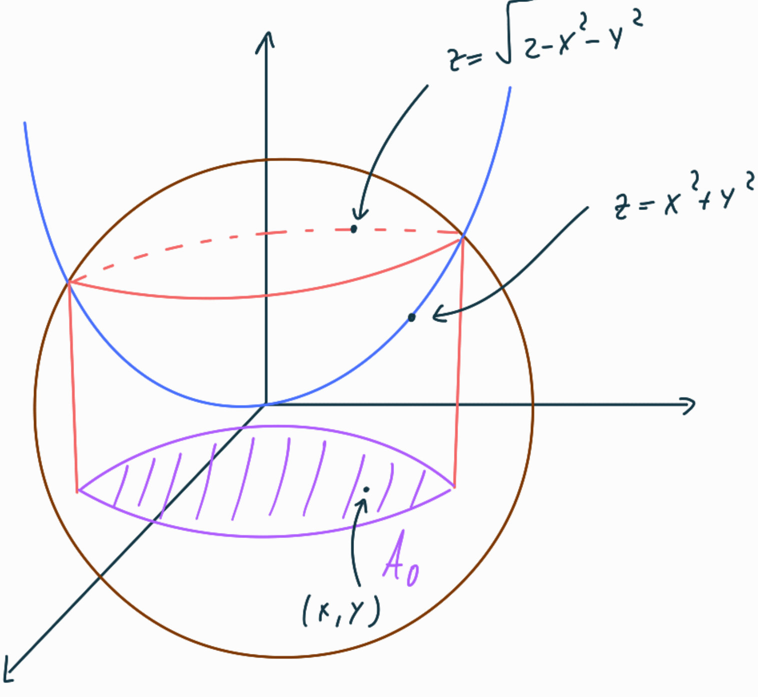
$$V=\int_0^1\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\rho^2\sin\varphi\,d\rho\,d\varphi\,dt$$

$$\int_0^1\rho^2 d\rho\cdot\int_0^{\frac{\pi}{4}}\sin\varphi\,d\varphi\cdot\int_0^{2\pi}1\,d\theta=\frac{1}{3}\cdot 2\pi\cdot -\cos\varphi\Big/_0^{\frac{\pi}{4}}=$$

$$=\frac{2\pi}{3}\cdot\left(1-\frac{\sqrt{2}}{2}\right)$$

2) volume and centre of mass of $A=\{(x,y,z)\in\mathbb{R}^3 \mid z=x^2+y^2 \text{ and } x^2+y^2+z^2=2\}$

$V=m(A)=\iiint_A dx\,dy\,dz$



$$V = \iint_{A_0}\left(\int_{z=x^2+y^2}^{z=\sqrt{2-x^2-y^2}} dz\right)dx\,dy = \iint_{A_0}\left(\sqrt{2-x^2-y^2}-x^2-y^2\right)dx\,dy =$$

$$\begin{cases} z = x^2+y^2 \\ x^2+y^2+z^2 = 2 \end{cases} \Rightarrow z^2+z=2 \Rightarrow z^2+z-2=0 \qquad \begin{matrix} z_1 = 1 \\ z_2 = -2 \end{matrix}$$

$$\begin{cases} x^2+y^2 = 1 \\ z = 1 \end{cases}$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \begin{matrix} \rho \in [0,1] \\ \theta \in [0,2\pi] \end{matrix}$$

$$V = \int_0^1\int_0^{2\pi}\left(\sqrt{2-\rho^2}-\rho^2\right)\rho\,d\rho\,d\theta = \int_0^1\left(\sqrt{2-\rho^2}-\rho^2\right)\rho\,d\rho\cdot\int_0^{2\pi}d\theta =$$

$$= 2\pi\left(\int_0^1 \rho\sqrt{2-\rho^2}\,d\rho - \int_0^1\rho^3\,d\rho\right) = 2\pi\left(\int_0^{\sqrt{2}} t^2\,dt - \frac{1}{4}\right) = 2\pi\left(\frac{2\sqrt{2}}{3}-\frac{1}{3}-\frac{1}{4}\right) =$$

$$= 2\pi\left(\frac{2\sqrt{2}}{3}-\frac{7}{12}\right)$$

$$\int_0^1 \rho\sqrt{2-\rho^2}\,d\rho$$

$$t = \sqrt{2-\rho^2}$$

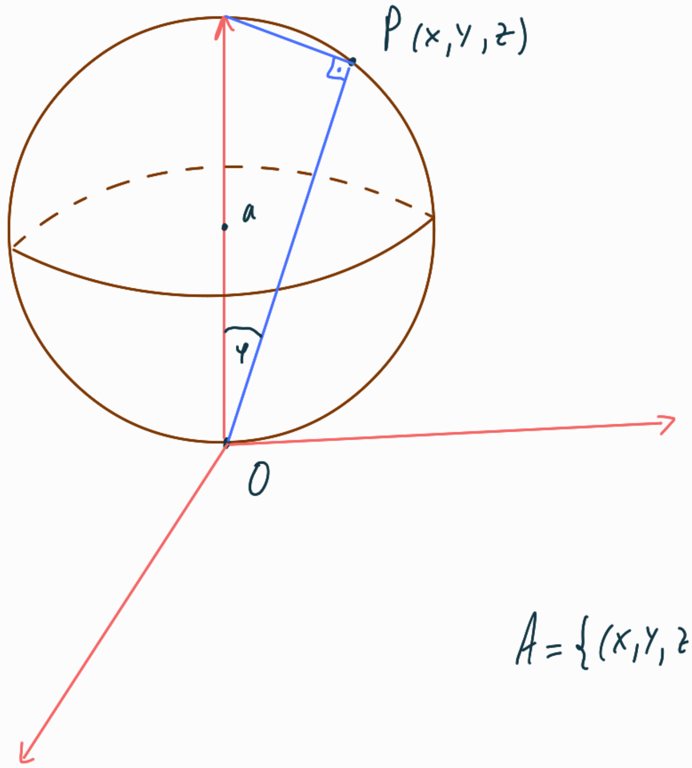
$$t^2 = 2-\rho^2$$

$$2t\,dt = -2\rho\,d\rho$$

the z axis is a symmetry axis for A $\Rightarrow G \in Oz \Rightarrow x_G = y_G = 0$

$$z_G = \frac{\iiint_A z\,dx\,dy\,dz}{V} = \text{etc.}$$

3) calculate the mass of a ball who's density is proportional to the distance from that point to some fixed point on the boundary of the ball (ball of radius $\underline{a}$)



$$\delta(P) = k \cdot |OP|$$

$$\delta(x,y,z) = k \cdot \sqrt{x^2+y^2+z^2}$$

$$m = \iiint_A \delta(x,y,z)\,dx\,dy\,dz$$

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+(z-a)^2 \le a^2\}$$

$$m = k \iiint_A \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz =$$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases} \qquad \begin{matrix} \cancel{\rho \in [0, 2a]} \\ \cancel{\varphi \in [0, \frac{\pi}{2}]} \\ \cancel{\theta \in [0, 2\pi]} \end{matrix} \qquad \begin{matrix} \rho \in [0, 2a\cos\varphi] \\ \varphi \in [0, \frac{\pi}{2}] \\ \theta \in [0, 2\pi] \end{matrix}$$

$$x^2+y^2+z^2 \le 2az$$

$$\rho \le 2a\cos\varphi \Rightarrow \theta \in [0, 2\pi]$$

$$\Rightarrow \cos\varphi \ge 0 \Rightarrow \varphi \in [0, \tfrac{\pi}{2}] \Rightarrow \rho \in [0, 2a\cos\varphi]$$

$$m = k \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left(\int_0^{2a\cos\varphi} \rho \cdot \rho^2 \sin\varphi \, d\rho \right) d\theta\, d\varphi =$$

$$= k \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin\varphi \left(\frac{\rho^4}{4} \Big|_0^{2a\cos\varphi} \right) d\varphi\, d\theta =$$

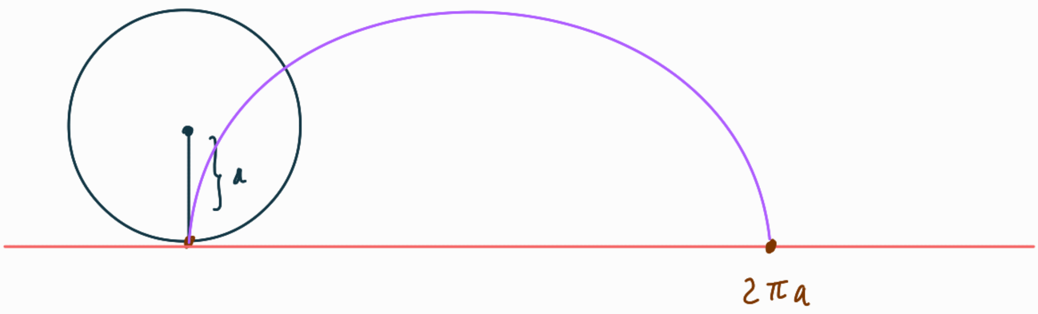
$$= k \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin\varphi \, 4a^4 \cos^4\varphi \, d\varphi\, d\theta = 4ka^4 \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \sin\varphi \cdot \cos^4\varphi \, d\theta =$$

$$= 8k\pi a^2 \int_0^1 t^4\,dt = \frac{8k\pi}{5} a^4$$

$t = \cos\theta$

4) cycloid is the plane curve whose parametrization is $x(t) = a(t - \sin t)$
$y(t) = a(1 - \cos t)$

$t \in [0, 2\pi]$, $a > 0$; calculate the arclength of the cycloids

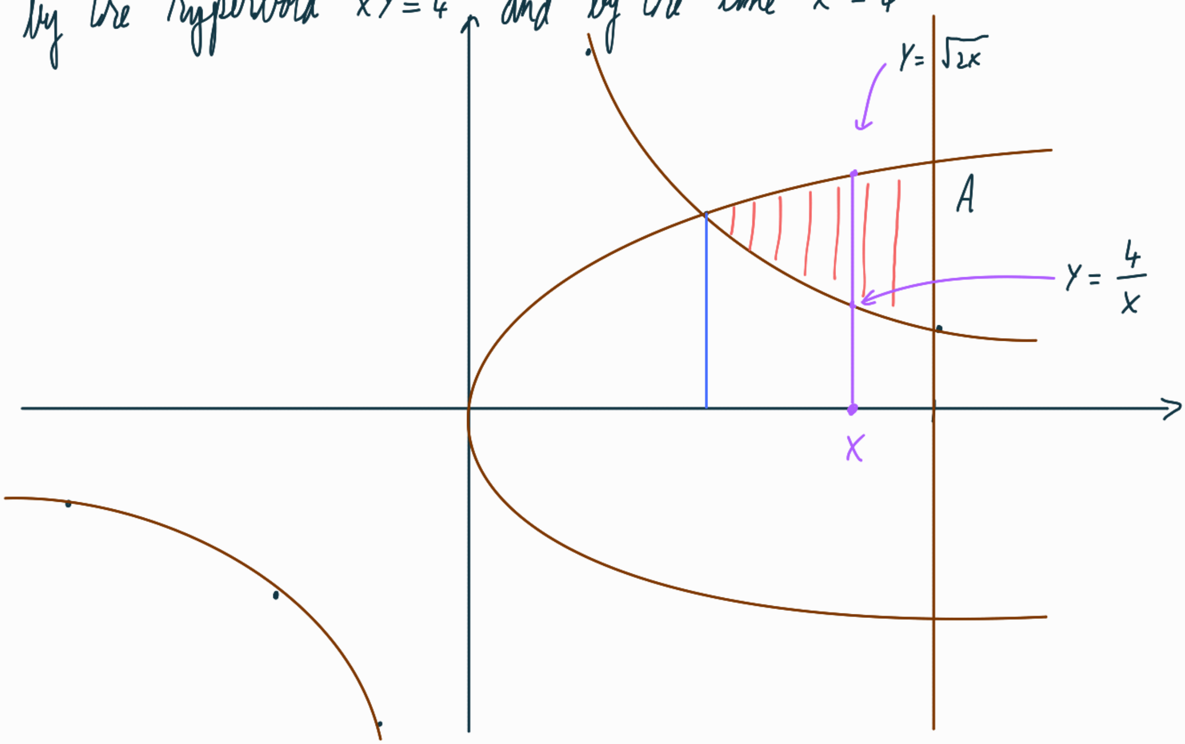


$$l(\gamma) = \int_0^{2a\pi} \| \gamma'(t) \| \, dt = \int_0^{2a\pi} 2a \sin\frac{t}{2} \, dt = 2a \left(-2 \cos\frac{t}{2} \Big|_0^{2a\pi} \right) = 2a \cdot (2+2) = 8a$$

$\gamma'(t) = (a(1 - \cos t), a(\sin t))$

$$\|\gamma'(t)\|^2 = a^2(1-\cos t)^2 + a^2 \sin^2 t = a^2(1 - 2\cos t + 1) = 2a^2(1 - \cos t) =$$
$$= 2a^2 \left(2 \sin^2 \frac{t}{2}\right) = 4a^2 \sin^2\frac{t}{2} \Rightarrow \|\gamma'(t)\| = 2a \sin\frac{t}{2}$$

5) calculate $I = \iint_A \frac{x}{y} \, dx\, dy$ if A is the set bounded by the parabola $y^2 = 2x$, by the hyperbola $xy = 4$, and by the line $x = 4$



$$\begin{cases} y^2 = 2x \\ xy = 4 \Rightarrow x = \frac{4}{y} \end{cases} \qquad y^2 = \frac{8}{y} \Rightarrow y^3 = 2^3 \Rightarrow y = 2$$

$$I = \int_{x=2}^{x=4} \left(\int_{y=\frac{4}{x}}^{y=\sqrt{2x}} \frac{y}{x}\, dy \right) dx = \int_{x=2}^{x=4} x \ln y \Big|_{y=\frac{4}{x}}^{y=\sqrt{2x}} dx =$$

$$= \int_2^4 x \left(\ln \sqrt{2x} - \ln \frac{4}{x} \right) dx = \int_2^4 x \ln \frac{x\sqrt{2x}}{4}\, dx =$$

$$= \int_2^4 x \left(\frac{1}{2} \ln 2 - 2 \ln 2 + \frac{3}{2} \ln x \right) dx = etc$$