SEMINAR 1 ANALIZĂ

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The Euclidian space $\mathbb{R}^n$

$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}, i = \overline{1,n}\}$

$+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ addition of vectors

$\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ scalar multiplication

$\forall x, y \in \mathbb{R}^n, \alpha \in \mathbb{R}$

$x + y = (x_1 + y_1, \dots, x_n + y_n)$

$\alpha \cdot x = (\alpha \cdot x_1, \alpha x_2, \dots, \alpha \cdot x_n)$

$(\mathbb{R}^n, +, \cdot)$ is an $\mathbb{R}$-vector space

Operations

Def 1 The SCALAR PRODUCT ON $\mathbb{R}^n$

$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$\forall x, y \in \mathbb{R}^n$,

$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$

Def 2 $\|\cdot\| : \mathbb{R}^n \longrightarrow [0, \infty)$

$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$

The EUCLIDIAN NORM ON $\mathbb{R}^n$

Def 3 The EUCLIDIAN DISTANCE

$d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0, \infty)$

$\forall x, y \in \mathbb{R}^n \quad d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

Particular cases:

$n = 2, \ x = (x_1, x_2) \in \mathbb{R}^2$

$\|x\| = \|(x_1, x_2)\| = \sqrt{\langle (x_1, x_2), (x_1, x_2) \rangle} =$

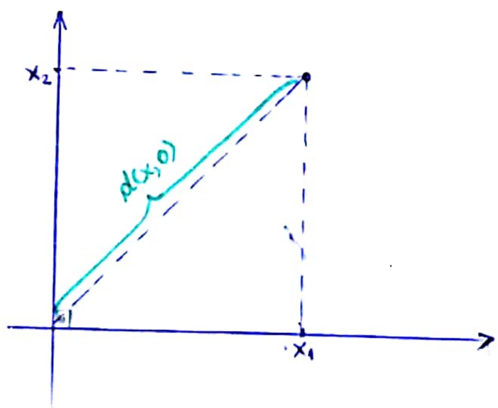
$= \sqrt{\langle x_1, x_1 \rangle + 2 \langle x_1, x_2 \rangle + \langle x_2, x_2 \rangle} \quad |(\,)^2$

$\|x\|^2 = \|x_1\|^2 + 2 \langle x_1, x_2 \rangle + \|x_2\|^2$ Graphic

$d(x, 0) = \|x - 0\| = \|x\|$

$d = \sqrt{x_1^2 + x_2^2}$ because $x_1 \perp x_2$

$\langle x_1, x_2 \rangle = 0$



The EUCLIDIAN distance in $\mathbb{R}^2$ is the distance from the point to the origin of the space.

Exercices

1 a). Let $a=(3,-2,-4)$, $b=(8,6,3)$ $\in \mathbb{R}^3$. Determine $a+b$, $a-b$, $-3a+b$, $<a,b>$, $\|a\|$, $\|b\|$, $\|a-b\|$

$a+b=(11,4,-1)$

$a-b=(-5,-8,-7)$

$-3a+b=(-1,12,15)$

$<a,b> = 24-12-12=0$

$\|a\| = \sqrt{9+4+16} = \sqrt{29}$

$\|b\| = \sqrt{64+36+9} = \sqrt{109}$

$\|a-b\| = \sqrt{25+64+49} = \sqrt{138}$

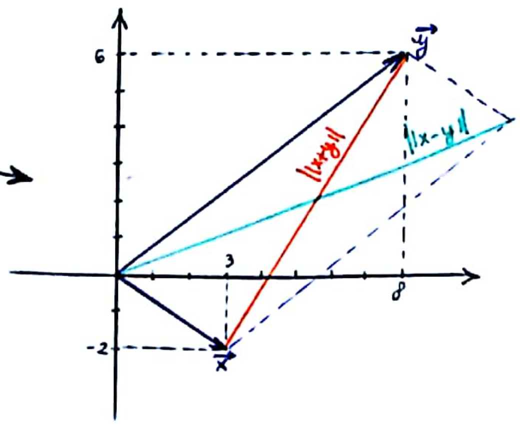
b). $x=(3,-2)$

$y=(8,6)$

$\|x-y\| = \sqrt{25+64} = \sqrt{89}$

$\|x+y\| = \sqrt{121+16} = \sqrt{137}$

Graphic



AXIOMS

Let X be a real vector space.

$<\cdot,\cdot> : X \times X \to \mathbb{R}$ is said to be a SCALAR PRODUCT if:

(SP_1) $<x+y, z> = <x,z> + <y,z>$ $\forall x,y,z \in X$ SP = scalar product

(SP_2) $<\alpha x, y> = \alpha <x,y>$ $\forall \alpha \in \mathbb{R}, \forall x \in X$

(SP_3) $<x,y> = <y,x>$ $\forall x,y \in X$

(SP_4) $<x,x> > 0$ $\forall x \in X \setminus \{0_x\}$

$(X, <\cdot,\cdot>)$ is called a PREHILBERT SPACE

! Remark: $(SP_3) \Rightarrow <x, 0_x> = <0_x, x>$

$\|\cdot\| : X \to [0,\infty)$

$(X, \|\cdot\|)$ is a NORMED SPACE

(N_1) $\|x\|=0 \iff x=0_x$ N = the NORM of x

(N_2) $\|\alpha x\| = |\alpha| \, \|x\|$ $\forall x \in X$

(N_3) $\|x+y\| \leq \|x\| + \|y\|$ $\forall x,y \in X$

$d : X \times X \to [0,\infty)$

(d_1) $d(x,y)=0 \iff x=y$ d = a metric on X

(d_2) $d(x,y) = d(y,x)$ $\forall x,y \in X$

(d_3) $d(x,z) \leq d(x,y) + d(y,z)$ $\forall x,y,z \in X$

The scalar product, the norm and the distance are particular examples of the above notions.

Exercices

2) Prove that the scalar product on $\mathbb{R}^m$ is ... ?

$\langle \cdot,\cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \quad \langle x, y\rangle = x_1y_1 + \dots + x_m y_m$

(SP_1) $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle \quad \forall x,y,z \in \mathbb{R}^m$

Let $x = (x_1,\dots,x_m)$
$y = (y_1,\dots,y_m) \quad \in \mathbb{R}^m$
$z = (z_1,\dots,z_m)$

$\langle x+y, z\rangle = \langle (x_1,\dots,x_m) + (y_1,\dots,y_m), (z_1,\dots,z_m)\rangle =$

$= \langle (x_1+y_1,\dots,x_m+y_m), (z_1,\dots,z_m)\rangle =$

by the def. of the scalar product
$= (x_1+y_1)z_1 + \dots + (x_m+y_m)z_m$

distributivity of $\cdot$ w.r.t. $+$ on $\mathbb{R}$
$= x_1z_1 + y_1z_1 + \dots + x_mz_m + y_mz_m$

commutativity on $\mathbb{R}$
$= x_1z_1 + x_2z_2 + \dots + x_mz_m + y_1z_1 + y_2z_2 + \dots + y_mz_m =$

$= \langle x,z\rangle + \langle y,z\rangle$

(SP_2) $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle \quad \alpha \in \mathbb{R},\ x,y \in \mathbb{R}^m$

Let $x = (x_1,\dots,x_m)$
$y = (y_1,\dots,y_m)$
$\alpha \in \mathbb{R}$

$\langle \alpha x, y\rangle = \langle \alpha(x_1,\dots,x_m), (y_1,\dots,y_m)\rangle$

scalar multiplication
$= \langle (\alpha x_1,\dots,\alpha x_m), (y_1,\dots,y_m)\rangle$

scalar product
$= (\alpha x_1)y_1 + (\alpha x_2)y_2 + \dots + (\alpha x_m)y_m$

associativity
$= \alpha(x_1y_1) + \alpha(x_2y_2) + \dots + \alpha(x_my_m)$

distributivity
$= \alpha(x_1y_1 + x_2y_2 + \dots + x_my_m) =$

$= \alpha\langle x,y\rangle$

(SP_3) $\langle x,y\rangle = \langle y,x\rangle \quad \forall x,y \in \mathbb{R}^m$

Let $x = (x_1,\dots,x_m)$
$y = (y_1,\dots,y_m) \quad \in \mathbb{R}^m$

$\langle x,y\rangle = \langle (x_1,\dots,x_m), (y_1,\dots,y_m)\rangle$

$= x_1y_1 + \dots + x_my_m \overset{\text{comm.}}{=} y_1x_1 + \dots + y_mx_m =$

$= \langle y,x\rangle$

(SP_4) $\langle x,x\rangle > 0 \quad \forall x \in \mathbb{R}^m \setminus \{0_m\}$ → zero m

Let $x = (x_1,\dots,x_m) \in \mathbb{R}^m$

$\langle x,x\rangle = \underbrace{x_1^2}_{\geq 0} + \dots + \underbrace{x_m^2}_{\geq 0} \geq x_k^2$

because $x \neq 0_m$, $\exists k = \overline{0,m}$ s.t. $x_k \neq 0 \Rightarrow x_k^2 > 0$

$\Rightarrow \langle x,x\rangle > 0$

! Remark:

$(SP_3) \Rightarrow \langle x, 0_m\rangle = \langle 0_m, x\rangle = 0$

$\xRightarrow{(SP_4)} \langle x,x\rangle \geq 0 \quad \forall x \in \mathbb{R}^m$

! Remark:

$\langle x,y\rangle = 0 \not\Rightarrow x=0$ or $y=0$

$\langle (1,0),(0,1)\rangle = 1\cdot 0 + 0\cdot 1 = 0$

$\langle (1,-1),(1,1)\rangle = 1\cdot(-1) + 1\cdot 1 = -1+1 = 0$

The scalar product does not behave like the multiplication of real numbers.
When $\langle x,y\rangle = 0$ it means that $x \perp y$ and the vectors are called ORTOGONAL.

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Exercise 3

a). $\forall x, y \in \mathbb{R}^m$, $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$

b). $\forall x, y \in \mathbb{R}^m$, $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

The PARALLELOGRAM EQUALITY (egalitatea paralelogramului)

c). $\forall x, y \in \mathbb{R}^m$, $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$

CBS inequality (Chauchy - Bunakowski - Swartz inequality)

a). Let $x = (x_1, \dots, x_m)$
$y = (y_1, \dots, y_m)$

$\langle x, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle$

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$$\overset{SP_2}{=} \langle x, x+y\rangle + \langle y, x+y\rangle \underset{SP_3}{\overset{SP_1}{=}} \overbrace{\langle x, x\rangle}^{\|x\|^2} + \langle x, y\rangle + \langle x, y\rangle + \overbrace{\langle y, y\rangle}^{\|y\|^2} =$$

$$\overset{SP_3}{=} \|x\|^2 + \|y\|^2 + 2\cdot\langle x, y\rangle$$

Alternative proof:

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle (x_1, \dots, x_m) + (y_1, \dots, y_m), (x_1, \dots, x_m) + (y_1, \dots, y_m)\rangle$

$= \langle (x_1+y_1, \dots, x_m+y_m), (x_1+y_1, \dots x_m+y_m)\rangle$

$\overset{def\ sp}{=} (x_1+y_1)^2 + (x_2+y_2)^2 + \dots + (x_m+y_m)^2$

$= x_1^2 + 2x_1y_1 + y_1^2 + \dots + x_m^2 + 2x_my_m + y_m^2$

$= x_1^2 + \dots + x_m^2 + y_1^2 + \dots + y_m^2 + 2\cdot(x_1y_1 + \dots + x_my_m)$

$= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$

b). $\|x-y\|^2 = \|x+(-y)\|^2 = \|x\|^2 + \|-y\|^2 + 2\langle x, -y\rangle$

$= \|x\|^2 + \|(-1)y\|^2 - 2\langle x, y\rangle$

$= \|x\|^2 + |-1|\cdot\|y\|^2 - 2\langle x, y\rangle$

$= \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$

$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

c). Case 1 $\quad x = 0_m$ or $y = 0_m \Leftrightarrow \|x\| = 0$ or $\|y\| = 0 \Rightarrow \|x\|\cdot\|y\| = 0 = \langle x, y\rangle$

Case 2 $\quad x \ne 0_m$ and $y = 0_m$

We introduce the vector $z = x - \dfrac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y$

z is ortogonal vector on y

$$\langle z, y\rangle = \left\langle x - \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y, y\right\rangle \underset{SP_2}{\overset{SP_1}{=}} \langle x, y\rangle - \left\langle \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y, y\right\rangle \overset{SP_2}{=} \langle x, y\rangle - \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot\langle y, y\rangle =$$

$$= \langle x, y\rangle - \langle x, y\rangle = 0 \Rightarrow z \perp y$$

$x = z + \dfrac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y$

$$\langle x, x\rangle = \left\langle z + \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y, z + \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y\right\rangle =$$

? $\leftarrow$

$$= \left\langle z + z + \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y\right\rangle + \left\langle \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y, z + \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y\right\rangle$$

$$= \langle z, z\rangle + \left\langle z, \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y\right\rangle + \left\langle \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y, \frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y\right\rangle$$

$$= \|z\|^2 + \left\|\frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot y\right\|^2 + 2\cdot\frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot\langle z, y\rangle =$$

$$= \|z\|^2 + \frac{\langle x, y\rangle^2}{\langle y, y\rangle^2}\|y\|^2 + 2\,\frac{\langle x, y\rangle}{\langle y, y\rangle}\cdot\overset{=0}{\langle y, z\rangle} =$$

$\langle y, y\rangle^2 = \|y\|^4 \Rightarrow \|y\|^2$

$\langle x, x\rangle = \|z\|^2 + \dfrac{\langle x, y\rangle^2}{\|y\|^2}$

$\Downarrow$

$\|x\|^2 \ge \dfrac{\langle x, y\rangle^2}{\|y\|^2} \Rightarrow \|x\|^2\cdot\|y\|^2 \ge \langle x, y\rangle^2 \Rightarrow \boxed{\|x\|\cdot\|y\| \ge |\langle x, y\rangle|}$

- interior
- exterior
- margine
- pct. de acumulare

4) $\|\cdot\|_\infty : \mathbb{R}^m \longrightarrow \mathbb{R}$

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_m|\}$$
$$= \max\{|x_i| : i=\overline{1,m}\} \quad \forall x=(x_1,\dots,x_m) \in \mathbb{R}^m$$

Prove that this norm $\|\cdot\|_\infty$ is a CEBÎŞEV NORM (uniform norm)

(N_1) $\|x\|_\infty = 0 \Leftrightarrow x = 0_m$

Let $x=(x_1,\dots,x_m) \in \mathbb{R}^m$

$$\|x\|_\infty = 0 \Leftrightarrow \max\{|x_1|, |x_2|, \dots, |x_m|\} = 0$$
$$\Leftrightarrow |x_1| = |x_2| = \dots = |x_m| = 0$$
$$\Leftrightarrow x_1 = x_2 = \dots = x_m = 0$$
$$\Leftrightarrow x = 0_m$$

(N_2) $\|\alpha \cdot x\|_\infty = |\alpha| \cdot \|x\|_\infty$

Let $\alpha \in \mathbb{R}$

$x=(x_1,\dots,x_m) \in \mathbb{R}^m$

$$\|\alpha x\|_\infty = \|\alpha(x_1,\dots,x_m)\|_\infty = \|(\alpha x_1,\dots,\alpha x_m)\|_\infty$$
$$= \max\{|\alpha x_1|, |\alpha x_2|, \dots, |\alpha x_m|\}$$
$$= \max\{|\alpha||x_1|, \dots, |\alpha|\cdot|x_m|\}$$
$$= |\alpha| \cdot \max\{|x_1|, \dots, |x_m|\}$$
$$= |\alpha| \cdot \|x\|_\infty$$

$|\alpha x| = |\alpha|\,|x|$

modulul produsului = produsul modulelor

(N_3) $\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$

! We will prove next time that this norm is not --- ? from a scalar product
Therefore, we cannot apply SP-write.

$$\|x+y\|_\infty = \max\{|x_1+y_1|, \dots, |x_m+y_m|\}$$

COUNTABLE SET $\Rightarrow \exists\, k \in \overline{1,m}$ s.t. $\|x+y\|_\infty = |x_k+y_k| \le |x_k| + |y_k| \le$

$$\le \max\{|x_1|,\dots,|x_m|\} + \max\{|y_1|,\dots,|y_m|\} = \|x\|_\infty + \|y\|_\infty$$

$$\Rightarrow \|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$$

Exercise 1

Let $p,q>1$ be s.t. $\frac{1}{p}+\frac{1}{q}=1$

Prove:

a). <u>YOUNG</u> $\forall a,b\in[0,\infty)\quad a\cdot b\le \frac{a^p}{p}+\frac{b^q}{q}$

b). <u>HÖLDER</u> $\forall a_1,\dots,a_m$, $\forall b_1,\dots,b_m \in[0,\infty)$

$$\sum_{k=1}^{m} a_k b_k \le \left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}}\cdot\left(\sum_{k=1}^{m} b_k^q\right)^{\frac{1}{q}}$$

(a) Fix $b\in[0,\infty)$

We define $f:[0,\infty)\to\mathbb{R}$, $f(x)=x\cdot b-\frac{x^p}{p}-\frac{b^q}{q}$; $p,q>1$ (p,q are fixed from the beginning)

f is differentiable $\Rightarrow\exists\, f':(0,\infty)\to\mathbb{R}$

$f'(x)=b-x^{p-1}$, $\forall x\in(0,\infty)$

$f'(x)=0$

$b-x^{p-1}=0$

$x^{p-1}=b\Rightarrow x=\sqrt[p-1]{b}\Rightarrow x=b^{\frac{1}{p-1}}$

x	0		$b^{\frac{1}{p-1}}$		∞
$f'(x)$	$+\ +$	$+\ +\ +$	0	$-\ -\ -\ -$	$-$
$f(x)$	$\nearrow$		$f(b^{\frac{1}{p-1}}) = 0$	$\searrow$	

$$f(b^{\frac{1}{p-1}})=b^{\frac{1}{p-1}}\cdot b-\frac{b^{\frac{p}{p-1}}}{p}-\frac{b^q}{q}=b^{\frac{p}{p-1}}-b^{\frac{p}{p-1}}\cdot\frac{1}{p}-\frac{b^q}{q}=$$

$$=b^{\frac{p}{p-1}}\underbrace{\left(1-\frac{1}{p}\right)}_{\frac{1}{q}}-\frac{b^q}{q}=\frac{1}{q}\left(b^{\frac{p}{p-1}}-b^q\right)$$

$\frac{1}{p}+\frac{1}{q}=1$

$\frac{1}{q}=1-\frac{1}{p}\Leftrightarrow\frac{1}{q}=\frac{p-1}{p}\Rightarrow q=\frac{p}{p-1}$

In conclusion, $x=b^{\frac{1}{p-1}}$ is a global maximum of f.

Therefore, $f(x)\le f(b^{\frac{1}{p-1}})\Leftrightarrow f(x)\le 0$

$\Leftrightarrow x\cdot b-\frac{x^p}{p}-\frac{b^q}{q}\le 0\quad\forall x\in[0,\infty)$

We notice that no restriction is imposed on b. This means that the inequality holds for $\forall x\in[0,\infty)$.

b) $A = \left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}}$

$B = \left(\sum_{k=1}^{m} b_k^2\right)^{\frac{1}{2}}$

Case 1: $A=0$ or $B=0$

$\Leftrightarrow a_1 = a_2 = \dots = a_m = 0$ or $b_1 = b_2 = \dots = b_m = 0$ $\Rightarrow \sum_{k=1}^{m} a_k b_k = 0 \leq 0 = A \cdot B$ ✓

Case 2: $A \neq 0$ and $B \neq 0$

$$\sum_{k=1}^{m} a_k \cdot b_k \leq A \cdot B \Leftrightarrow \frac{\sum_{k=1}^{m} a_k \cdot b_k}{A \cdot B} \leq 1 \Leftrightarrow \sum_{k=1}^{m} \frac{a_k}{A} \cdot \frac{b_k}{B} \leq 1 \quad (*)$$

Fix $k \in \overline{1,m}$. We apply Young's inequality for $a = \frac{a_k}{A}$ and $b = \frac{b_k}{B}$

$$\frac{a_k}{A} \cdot \frac{b_k}{B} \leq \frac{\left(\frac{a_k}{A}\right)^p}{p} + \frac{\left(\frac{b_k}{B}\right)^2}{2} = \frac{a_k^p}{p \cdot A^p} + \frac{b_k^2}{2 \cdot B^2} \quad \Big| \Sigma$$

$$\sum_{k=1}^{m} \frac{a_k}{A} \cdot \frac{b_k}{B} \leq \sum_{k=1}^{m} \left(\frac{a_k^p}{p \cdot A^p} + \frac{b_k^2}{2 \cdot B^2}\right) = \left(\frac{1}{p \cdot A^p} \cdot \sum_{k=1}^{m} a_k^p\right) + \left(\frac{1}{2 \cdot B^2} \cdot \sum_{k=1}^{m} b_k^2\right) =$$

$$= \frac{1}{p \cdot A^p} \cdot A^p + \frac{1}{2 \cdot B^2} \cdot B^2 \overset{ip}{=} 1 \Leftrightarrow \text{HÖLDER } (*)$$

c) **MINKOWSKI's inequality:** $\forall p \geq 1$, $\forall a_1, \dots, a_m \in [0, \infty)$, $\forall b_1, \dots, b_m$

$$\left(\sum_{k=1}^{m} (a_k + b_k)^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{m} b_k^p\right)^{\frac{1}{p}}$$

Case 1: $p = 1$

$$\sum_{k=1}^{m} (a_k + b_k) = \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} b_k$$

Case 2: $p > 1$

$$A = \sum_{k=1}^{m} (a_k + b_k)^p = \sum_{k=1}^{m} (a_k + b_k) \cdot (a_k + b_k)^{p-1} =$$

$$= \sum_{k=1}^{m} a_k \cdot (a_k + b_k)^{p-1} + \sum_{k=1}^{m} b_k (a_k + b_k)^{p-1}$$

We apply twice HÖLDER inequality

First, $\sum_{k=1}^{m} a_k (a_k + b_k)^{p-1} \leq \left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}} \cdot \left\{\sum_{k=1}^{m} \left[(a_k + b_k)^{p-1}\right]^2\right\}^{\frac{1}{2}}$

$$\left(\sum_{k=1}^{m} b_k^p\right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^{m} (a_k + b_k)^{2(p-1)}\right)^{\frac{1}{2}}$$

In conclusion, $A \leq \left(\sum_{k=1}^{m} (a_k + b_k)^{2(p-1)}\right)^{\frac{1}{2}} \left[\left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{m} b_k^p\right)^{\frac{1}{p}}\right] \quad \frac{1}{p} + \frac{1}{2} = 1 \Leftrightarrow$

$\Leftrightarrow 1 - \frac{1}{p} = \frac{1}{2} \Leftrightarrow \frac{p-1}{p} = \frac{1}{2} \Leftrightarrow 2(p-1) = p \Rightarrow A \leq \left(\sum_{k=1}^{m} (a_k + b_k)^p\right)^{\frac{1}{2}} \left[\left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{m} b_k^p\right)^{\frac{1}{p}}\right]$

$A \leq A^{\frac{1}{2}} \cdot \left[\left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{m} b_k^p\right)^{\frac{1}{p}}\right]$ • if $A \neq 0 \Leftrightarrow \underset{A^{\frac{1}{p}}}{\underset{\parallel}{A^{1-\frac{1}{2}}}} \leq \left(\sum_{k=1}^{m} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{m} b_k^p\right)^{\frac{1}{p}}$

• if $A = 0 \Leftrightarrow a_1 = \dots = a_m = b_1 = \dots = b_m = 0 \Leftrightarrow 0 \leq 0$

Ex 2 Let $p \geq 1$.

a). Prove that $\|\cdot\|_p : \mathbb{R}^n \to [0, \infty)$
$\|x\| = \|(x_1, \dots, x_n)\|_p = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}} = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$ is a norm on $\mathbb{R}^n$ (called the p-norm)

b). The p-norm is not obtained from a scalar product, unless $p = 2$. $(n \geq 2)$.

a) N_1: $\|x\|_p = 0 \overset{?}{\Leftrightarrow} x = 0_n$

Let $|x| = (x_1, \dots, x_n) \in \mathbb{R}^n$ s.t. $\|x\|_p = 0 \Leftrightarrow (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}} = 0 \Leftrightarrow$
$\Leftrightarrow x_k = 0, \forall k = \overline{1,n} \Leftrightarrow x = 0_n$

N_2: $\|\alpha x\|_p = |\alpha| \|x\|_p \quad , \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$

Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

$$\begin{aligned}
\|\alpha x\|_p &= \|\alpha \cdot (x_1, \dots, x_n)\|_p = \\
&= \|(\alpha x_1, \dots, \alpha x_n)\|_p = \\
&= (|\alpha x_1|^p + \dots + |\alpha x_n|^p)^{\frac{1}{p}} = \\
&= (|\alpha|^p \cdot |x_1|^p + \dots + |\alpha|^p \cdot |x_n|^p)^{\frac{1}{p}} = \\
&= (|\alpha|^p \cdot (|x_1|^p + \dots + |x_n|^p))^{\frac{1}{p}} = \\
&= |\alpha| \cdot (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}} = \\
&= |\alpha| \cdot \|x\|_p
\end{aligned}$$

N_3: $\|x+y\|_p \leq \|x\|_p + \|y\|_p \quad \forall x, y \in \mathbb{R}^n$

Let $\left.\begin{array}{l} x = (x_1, \dots, x_n) \\ y = (y_1, \dots, y_n) \end{array}\right\} \in \mathbb{R}^n$

$$\begin{aligned}
\|x+y\|_p &= \|(x_1, \dots, x_n) + (y_1, \dots, y_n)\|_p = \\
&= \|(x_1+y_1, \dots, x_n+y_n)\|_p = \\
&= (|x_1+y_1|^p + |x_2+y_2|^p + \dots + |x_n+y_n|^p)^{\frac{1}{p}} = \\
&= \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}}
\end{aligned}$$

$\forall k = \overline{1,n} \quad |x_k + y_k| \leq |x_k| + |y_k| \Rightarrow |x_k + y_k|^p \leq (|x_k| + |y_k|)^p$

$$\Rightarrow \|x+y\|_p \leq \left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}} = \|x\|_p + \|y\|_p$$

We apply MINKOWSKI inequality for $a_k = |x_k|$ and $b_k = |y_k|$

From $N_1, N_2, N_3 \Rightarrow \|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

! Remark:

Let X be a real vector space. A norm $\|\cdot\|: X \to [0,\infty)$ is said to be obtained from a scalar product if $\exists$ scalar product $\langle\cdot,\cdot\rangle: X \to \mathbb{R}$ s.t. $\|x\| = \sqrt{\langle x, x\rangle}$

In practice, in order to verify if a norm is obtain from a scalar product, we verify if the parallelogram equality is satisfied.

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x,y \in \mathbb{R}^n$$

If it is not, than the norm is not connected to the scalar product.

(b) We consider the first two vectors from the canonical base.

$$e_1 = (1,0,0,\dots,0), \quad e_2 = (0,1,0,\dots,0) \in \mathbb{R}^n$$

We write the parallelogram equality for e_1 and e_2.

$$e_1+e_2 = (1,1,0,\dots,0), \quad e_1 - e_2 = (1,-1,0,\dots,0) \in \mathbb{R}^n$$

Let $p \geq 1$ be arbitrarily chosen.

$$\|e_1+e_2\|_p = \left(|1|^p + |1|^p + |0| + \dots + |0|\right)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

$$\|e_1-e_2\|_p = \left(|1|^p + |-1|^p + |0 + \dots + 0|\right)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

$$\|e_1\|_p = \left(|1|^p + |0| + \dots + |0|\right)^{\frac{1}{p}} = 1^{\frac{1}{p}} = 1$$

$$\|e_2\|_p = 1^{\frac{1}{p}} = 1$$

In order to parallelogram equality to hold, we must have:

$$(2^{\frac{1}{p}})^2 + (2^{\frac{1}{p}})^2 = 2\cdot 1^2 + 2\cdot 1^2 \iff$$

$$\iff (2^{\frac{2}{p}}) + (2^{\frac{2}{p}}) = 4 \iff 2^{\frac{2}{p}} = 2 \iff p = 2$$

If $p \neq 2$, the parallelogram equality FAILS. So, if $p \neq 2$, the p-norm is NOT obtained from a scalar product.

Taking into account the proof from the last seminar for

$$\langle\cdot,\cdot\rangle: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad \langle x,y\rangle = x_1y_1 + \dots + x_ny_n = \sum_{k=1}^{n} x_k y_k$$

it is a natural scalar product on $\mathbb{R}^n$ and the Euclidean norm $\|x\|_2 = \sqrt{\langle x,x\rangle} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ which coincides with p-norm ($p=2$) is derived from the scalar product (**derivă din produsul scalar**). Thus, the p-norm derives from a scalar product $\iff p = 2$.

Ex 3 Let us recall $\|\cdot\|_\infty: \mathbb{R}^n \to [0,\infty)$ from the last seminar

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\} = \max\{|x_k| \mid k \in \overline{1,n}\} \quad \forall x \in \mathbb{R}^n$$

Prove that the uniform norm is NOT obtained from a scalar product.

$$\|e_1\|_\infty = 1 \qquad \|e_2\|_\infty = 1$$

$$\|e_1+e_2\|_\infty = \|(1,1,0,\dots,0)\|_\infty = \max\{|1|,|1|,|0|,|0|,\dots,|0|\} = 1$$

$$\|e_1-e_2\|_\infty = \|(1,-1,0,\dots,0)\|_\infty = \max\{|1|,|-1|,|0|,\dots,|0|\} = 1$$

We apply the parallelogram equality and we obtain:

$$1^2 + 1^2 = 2\cdot 1^2 + 2\cdot 1^2 \iff 2 = 4 \ \text{⚡}$$

Therefore, the uniform norm is NOT obtain from a scalar product.

Ex 4 Prove that $\forall x \in \mathbb{R}^m$, $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$

Let $x = (x_1, \dots, x_m) \in \mathbb{R}^m$.

$\lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty} \left(|x_1|^p + |x_2|^p + \dots + |x_m|^p\right)^{\frac{1}{p}}$

$\|x\|_\infty = \max\{|x_1|, \dots, |x_m|\} \Rightarrow \exists\, j = \overline{1,m}$ s.t. $\|x\|_\infty = |x_j|$

$\Leftrightarrow \forall k = \overline{1,m}, \quad |x_k| \le |x_j| \quad \Big|^p$ ridicăm la puterea p inegalitatea

$\Leftrightarrow \forall k = \overline{1,m}, \quad |x_k|^p \le |x_j|^p$

$\Rightarrow |x_1|^p + \dots + |x_m|^p \le \underbrace{|x_j|^p + \dots + |x_j|^p}_{m \text{ times}} = m \cdot |x_j|^p \Rightarrow$

$\Rightarrow \|x\|_p \le (m \cdot |x_j|^p)^{\frac{1}{p}} = m^{\frac{1}{p}} \cdot |x_j| = m^{\frac{1}{p}} \cdot \|x\|_\infty$

Thus, $\|x\|_p \le m^{\frac{1}{p}} \cdot \|x\|_\infty \Rightarrow \lim_{p\to\infty} \|x\| \le \lim_{p\to\infty} \left(m^{\frac{1}{p}} \cdot \|x\|_\infty\right) = \|x\|_\infty \cdot \lim_{p\to\infty} \underbrace{m^{\frac{1}{p}}}_{=1} = \|x\|_\infty$ (1)

$\Leftarrow \quad \|x\|_\infty = |x_j| = \left(|x_j|^p\right)^{\frac{1}{p}} \le \underbrace{\left(|x_1|^p + \dots + |x_m|^p\right)^{\frac{1}{p}}}_{\|x\|_p}$

$\|x\|_\infty \le \|x\|_p$

$\lim_{p\to\infty} \|x\|_\infty \le \lim_{p\to\infty} \|x\|_p$ (2)

(1), (2) $\xrightarrow[\text{theorem}]{\text{Sandwich}} \lim_{p\to\infty} \|x\|_p = \|x\|_\infty$

SEMINAR 3 ANALIZĂ 12.03.2018.

TOPOLOGY on $\mathbb{R}^m$

Let $a \in \mathbb{R}^m$, $r > 0$

$B(a, r) = \{x \in \mathbb{R}^m : \|x - a\| < r\}$

↳ the ball of radius r, about a $\quad V \in \mathcal{V}(x)$ if $\exists\, r > 0$ s.t. $B(x, r) \subseteq V$

Let $A \subseteq \mathbb{R}^m$.

- $\operatorname{int} A = \{x \in \mathbb{R}^m : A \in \mathcal{V}(x)\} = \{x \in \mathbb{R}^m : \exists\, r_x > 0 \text{ s.t. } B(x, r_x) \subseteq A\}$
  $= \{x \in \mathbb{R}^m : \exists\, V_x \in \mathcal{V}(x),\ V_x \subseteq A\}$
- $\operatorname{ext} A = \operatorname{int}(\mathbb{R}^m \setminus A)$
- $\operatorname{cl} A = \{x \in \mathbb{R}^m : \forall\, V \in \mathcal{V}(x),\ V \cap A \ne \emptyset\}$
  ↳ the set of cluster or adherent (closure) points.
  $= \{x \in \mathbb{R}^m : \forall\, r > 0,\ B(x, r) \cap A \ne \emptyset\}$

pt. contraexemple lucrăm cu bile
pt simple exemple lucrăm cu vecinătăți

- $A' = \{x \in \mathbb{R}^m : \forall\, V \in \mathcal{V}(x),\ V \cap A \setminus \{x\} \ne \emptyset\}$
  $\forall\, r > 0,\ B(x, r) \cap A \setminus \{x\} \ne \emptyset\}$
- $\operatorname{bd} A = \{x \in \mathbb{R}^m : \forall\, V \in \mathcal{V}(x),\ V \cap A \ne \emptyset \text{ and } V \cap (\mathbb{R}^m \setminus A) \ne \emptyset\}$
  $\forall\, r > 0,\ B(x, r) \cap A \ne \emptyset \text{ and } B(x, r) \cap (\mathbb{R}^m \setminus A) \ne \emptyset\}$

PROPERTIES

$\operatorname{int} A \subseteq A \subseteq \operatorname{cl} A$

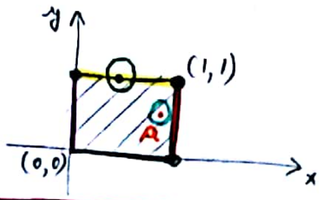
Ex 1 Determine int, ext, bd, cl, A' for:

a). $A = [0,1] \times [0,1] \subseteq \mathbb{R}^2$

b). $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$

c). $A = \left\{ \left(1 + \frac{1}{n}\right)^n \right.$ → homework

a)



• distanța de la punct până când întâlnește fiecare din laturile pătratului

$\boxed{\textbf{! Remark: } A_1, A_2, \dots, A_n \subseteq \mathbb{R} \quad \text{int}(A_1 \times A_2 \times \dots \times A_n) = \text{int}\, A_1 \times \text{int}\, A_2 \times \dots \times \text{int}\, A_n}$

$\text{int}\, A = (\text{int}[0,1]) \times (\text{int}[0,1]) = (0,1) \times (0,1)$

Let $a = (x,y) \in (0,1) \times (0,1)$. $\exists\, r_a = \min\{1-x, x, 1-y, y\} > 0$, $x, y \neq 1$, $x, y \neq 0$

$\Rightarrow B(a, r_a) \subseteq [0,1] \times [0,1] \Rightarrow a \in \text{int}\, A$

On the border of the square, each ball has at least its half out of the square (its half outside)

⇒ It is not contained in the interior.

$\text{ext}\, A = \text{int}(\mathbb{R}^2 \setminus A)$

$\text{bd}\, A = \{x \in \mathbb{R}^2 . \forall r > 0, B(x,r) \cap A \neq \emptyset \text{ and } B(x,r) \cap (\mathbb{R}^2 \setminus A) \neq \emptyset\} =$

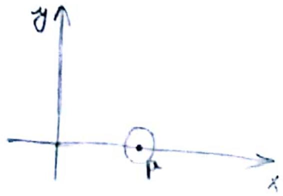
exteriorul lui $A \neq$ complementara lui A (deschis) dar uneori sunt egale

$= (\{0\} \times [0,1]) \cup (\{1\} \times [0,1]) \cup ([0,1] \times \{0\}) \cup ([0,1] \times \{1\})$

$\text{cl}\, A = A = \text{int}\, A \cup \text{bd}\, A$

$A' = \text{cl}\, A - \text{Izo}\, A = \text{cl}\, A = A$.

b)



interiorul unui punct $= \emptyset$

$A = Ox$

$\text{int}\, A = \text{int}(A) = \text{int}(\mathbb{R} \times \{0\}) = \text{int}(\mathbb{R}) \times \text{int}(\{0\}) = \mathbb{R} \times \emptyset = \emptyset$

$\text{ext}\, A = \text{int}(\mathbb{R}^2 \setminus A) = \mathbb{R}^2 \setminus A$ (un spațiu despărțit de o dreaptă ⇒ două subspații mărg. de dreaptă ⇒ Case 1: bilă deasupra, Case 2: bilă sub)

$= (\mathbb{R} \times (-\infty, 0)) \cup (\mathbb{R} \times (0, \infty))$

$\text{bd}\, A = A$

Let $a\,(x,0) \in A$ (a point) random $r > 0$ be fixed

$B(a,r) \cap A = (x-r, x+r) \times \{0\}$

$B(a,r) \cap (\mathbb{R}^2 \setminus A) = B(a,r) \setminus \{(x-r, x+r) \times \{0\}\} \neq \emptyset$

a and r were both taken randomly → $\text{bd}\, A = A$.

$\text{cl}\, A = \text{int}\, A \cup \text{bd}\, A = A$

Exercise 2

a). $cl\, A = \mathbb{R}^m \setminus int(\mathbb{R}^m \setminus A)$

b). $cl\, A = A \cup A'$

c). $cl\, A = int\, A \cup bd\, A$

a) $\boxed{\subseteq}$ We prove $cl\, A \subseteq \mathbb{R}^m \setminus int(\mathbb{R}^m \setminus A)$

Let $x \in cl A \Leftrightarrow \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset$ ①

Assume by contradiction that $x \notin \mathbb{R}^m \setminus (int\, \mathbb{R}^m \setminus A) \Leftrightarrow$

$\Leftrightarrow x \in int(\mathbb{R}^m \setminus A) \Leftrightarrow \mathbb{R}^m \setminus A \in \mathcal{V}(x)$ ②

①, ② $\Rightarrow \mathbb{R}^m \setminus A \cap A \neq \emptyset$ ↯ $\Rightarrow x \in \mathbb{R}^m \setminus (int(\mathbb{R}^m \setminus A))$

$\boxed{\supseteq}$ We prove that $\mathbb{R}^m \setminus int(\mathbb{R}^m \setminus A) \subseteq cl\, A$

Let $x \in \mathbb{R}^m \setminus int(\mathbb{R}^m \setminus A) \Leftrightarrow x \notin int(\mathbb{R}^m \setminus A)$ ③

Assume by contradiction that $x \notin cl A \Leftrightarrow$

$\Leftrightarrow \exists V \in \mathcal{V}(x)$ s.t. $V \cap A = \emptyset \Rightarrow \exists V \in \mathcal{V}(x)$ s.t. $V \subseteq \mathbb{R}^m \setminus A$

$\Leftrightarrow x \in int\, \mathbb{R}^m \setminus A$ ④

③, ④ $\Rightarrow$ ↯ $\Rightarrow \supseteq$

From $\subseteq$ and $\supseteq$, there $\exists$ equality $=$.

b) $\boxed{\subseteq}$ We prove : $cl\, A \subseteq A \cup A'$

Let $x \in cl A$.

Case 1: $x \in A \subseteq A \cup A'$

Case 2: $x \in cl\, A \setminus A \Leftrightarrow \left.\begin{matrix}\forall V \in \mathcal{V}(x) & V \cap A \neq \emptyset \\ & x \notin A\end{matrix}\right\} \Rightarrow$

$\forall V \in \mathcal{V}(x),\ V \cap (A \setminus \{x\}) \neq \emptyset \Leftrightarrow x \in A'$

Case 1, Case 2 $\Rightarrow$

$\boxed{\supseteq}$ We prove that $A \cup A' \subseteq cl\, A$

Let $x \in A \cup A'$

Case 1: $x \in A \subseteq cl A$

Case 2: $x \in A' \setminus A \Leftrightarrow \left.\begin{matrix}\forall V \in \mathcal{V}(x), & V \cap (A \setminus \{x\}) \neq \emptyset \\ & x \in A\end{matrix}\right\} \Rightarrow$??

The reunion of $A \cup A'$ is NOT necessarily disjoint sets. A and A' are disjoint iff. $A = Iz\, A$.

$\Leftrightarrow \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset \Rightarrow x \in cl A$ (because $A \setminus \{x\} = A$)

c) $\boxed{\subseteq}$ We prove that $cl\, A \subseteq (int\, A) \cup (bd\, A)$

Let $x \in cl A \Leftrightarrow \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset$

Assume by contradiction that $x \notin (int\, A) \cup (bd\, A) \Leftrightarrow$

$\Leftrightarrow x \in \mathbb{R}^m \setminus ((int\, A) \cup (bd\, A)) \Leftrightarrow x \in (\mathbb{R}^m \setminus int\, A) \cap (\mathbb{R}^m \setminus bd\, A)$

$\Leftrightarrow \begin{cases} x \in \mathbb{R}^m \setminus int\, A \\ x \in \mathbb{R}^m \setminus bd\, A \end{cases} \Leftrightarrow \begin{cases} x \notin int\, A \\ x \notin bd\, A \end{cases} \Leftrightarrow \begin{cases} A \notin \mathcal{V}(x) \text{ ①} \\ \exists V_0 \in \mathcal{V}(x) \text{ s.t. } V_0 \cap A = \emptyset \\ \quad \text{or } V_0 \cap \mathbb{R}^m \setminus A = \emptyset \end{cases}$

We know that $V_0 \cap A \neq \emptyset \Rightarrow V_0 \cap (\mathbb{R}^m \setminus A) = \emptyset \Rightarrow V_0 \subseteq A \Rightarrow A \in \mathcal{V}(x)$ ②

①, ② $\Rightarrow$ ↯

$\boxed{\supseteq}$ We prove that $(int A) \cup (bd\, A) \subseteq cl A$

Let $x \in (int\, A) \cup (bd A)$

Case 1: $x \in (int\, A) \subseteq A \subseteq cl\, A$ ✓

Case 2: $x \in (bd\, A) \Leftrightarrow \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset$ and $V \cap (\mathbb{R}^m \setminus A) \neq \emptyset$

$\Rightarrow x \in cl A$ ✓ $\Rightarrow \boxed{\supseteq}$

Proposition: The closer operator enjoys the following monotony property:

If $A \subseteq B \Rightarrow cl\, A \subseteq cl\, B$

proof: Let $x \in cl\, A$

- Case 1: $x \in A \subseteq B \subseteq cl\, B$ ✓
- Case 2: $x \notin A$

$$\left.\begin{array}{l} x \in cl\, A \Leftrightarrow \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset \\ A \subseteq B \end{array}\right\} \Rightarrow V \cap B \neq \emptyset \Leftrightarrow x \in cl\, B \checkmark$$

Case 1, Case 2 $\Rightarrow cl\, A \subseteq cl\, B$

Ex 3

$\forall A, B \subseteq \mathbb{R}^m$, $cl\,(A \cup B) = cl\, A \cup cl\, B$.

$\boxed{\subseteq}$ We prove that $cl\,(A \cup B) \subseteq cl\, A \cup cl\, B$

Let $x \in cl\,(A \cup B) \Leftrightarrow \forall V \in \mathcal{V}(x),\ V \cap (A \cup B) \neq \emptyset \Leftrightarrow$ ①

$\Leftrightarrow \forall V \in \mathcal{V}(x), (V \cap A) \cup (V \cap B) \neq \emptyset$

Assume by contradiction that $x \notin cl\, A \cup cl\, B \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x \notin cl\, A \\ \text{and} \\ x \notin cl\, B \end{cases} \Leftrightarrow \begin{cases} \exists V_A \in \mathcal{V}(x) \text{ s.t. } V_A \cap A = \emptyset \\ \exists V_B \in \mathcal{V}(x) \text{ s.t. } V_B \cap B = \emptyset \end{cases}$$

Let us denote $W = V_A \cap V_B \in \mathcal{V}(x)$

$$\Rightarrow \begin{cases} W \cap A = \emptyset \\ W \cap B = \emptyset \end{cases}$$

But $W \cap (A \cup B) = (W \cap A) \cup (W \cap B) = \emptyset \cup \emptyset = \emptyset$ ②

①, ② $\Rightarrow$ ↯

$\boxed{\supseteq}$ We prove that $cl\, A \cup cl\, B \subseteq cl\,(A \cup B)$

$$\left.\begin{array}{l} A \subseteq A \cup B \Rightarrow cl\, A \subseteq cl(A \cup B) \\ B \subseteq A \cup B \Rightarrow cl\, B \subseteq cl(A \cup B) \end{array}\right\} \Rightarrow cl\, A \cup cl\, B \subseteq cl\,(A \cup B)$$

! **Remark**: We should be careful concerning the reunion of infite sets, because the closure of the reunion might not be the reunion of the closure (see calculus 1 $\Rightarrow$ open and closed sets).

$\rightarrow$ try to give a contraexample of this!

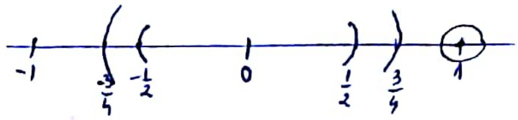
SEMINAR 4 ANALIZĂ

• Topology on $\mathbb{R}^n$ •

We prove last time that $cl A \cup cl B = cl(A\cup B)$.
The question is if this property can be extended to an infinite reunion of sets.

If $I \subseteq \mathbb{N}$? $cl\left(\bigcup_{k\in I} A_k\right) \overset{?}{=} \bigcup_{k\in I} cl(A_k)$

$\forall k \in \mathbb{N}$ $A_k = \left(-1+\frac{1}{2k}, 1-\frac{1}{2k}\right)$



$\bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k\in\mathbb{N}} \left(1-\frac{1}{k}, 1+\frac{1}{k}\right)$

$$\boxed{cl\left(\bigcup_{k\in\mathbb{N}} A_k\right) = [-1, 1]}$$

$1 \in cl\left(\bigcup_{k\in\mathbb{N}} A_k\right) \Leftrightarrow \forall V \in \mathcal{V}(1),\ V \cap cl\left(\bigcup_{k\in\mathbb{N}} A_k\right) \neq \emptyset$

$cl\left(-1+\frac{1}{2k}, 1-\frac{1}{2k}\right) = \left[-1+\frac{1}{2k}, 1-\frac{1}{2k}\right]$ $\Downarrow$ $\exists r>0$ s.t. $B(1,r) \cap cl\left(\bigcup_{k\in\mathbb{N}} A_k\right) \neq \emptyset$

$\lim_{k\to\infty} 1-\frac{1}{2k} = 1 \Rightarrow \exists k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0$
$x \in B(1,r)$
$\in A_k$

$\bigcup_{k\in\mathbb{N}} \left[1-\frac{1}{2k}, 1+\frac{1}{2k}\right] = (-1, 1)$

Assume that $1 \in \bigcup_{k\in\mathbb{N}} \left[-1+\frac{1}{2k}, 1-\frac{1}{2k}\right]$

$\Rightarrow \exists k_0 \in \mathbb{N} \Rightarrow 1 \in \left[-1+\frac{1}{2k_0}, 1-\frac{1}{2k_0}\right]$

$\Rightarrow 1 \leq 1-\frac{1}{2k_0} \Leftrightarrow \frac{1}{2k_0} \leq 0$ ↯

Exercise 1.

$\boxed{*}$ Let $A, B \subseteq \mathbb{R}^n$. Prove that:

a). If $A\cup B = \mathbb{R}^n$ then $cl A \cup int B = \mathbb{R}^n$

b). If $A \cap B = \emptyset$ then $cl A \cap int B = \emptyset$

a. $cl A \cup int B \subseteq \mathbb{R}^n$

$\boxed{\supseteq}$ $\mathbb{R}^n \overset{?}{\subseteq} cl A \cup int B$

$x \in \mathbb{R}^n$

I. $x \in int B \Rightarrow x \in cl B \cup int B$

II. $x \in \mathbb{R}^n \setminus int B = \mathbb{R}^n \setminus int(\mathbb{R}^n \setminus A)$

b. Assume $\exists x \in cl A \cap int B \Rightarrow$
$\Rightarrow x \in cl A \Leftrightarrow \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset$
$x \in int B \Leftrightarrow B \in \mathcal{V}(x)$
$\Rightarrow B \cap A \neq \emptyset$ ↯

$\Rightarrow int B \cap cl A = \emptyset$

Exercise 2 Let $A, B \subseteq \mathbb{R}^m$. Prove:

a). $\text{int}(A \setminus B) \subseteq \text{int} A \setminus \text{int} B$

b). $\text{cl} A \setminus \text{cl} B \subseteq \text{cl}(A \setminus B)$

c). give ex. for the strict inclusion in both cases.

a. Let $x \in \text{int}(A \setminus B)$.

$\exists V \in \mathcal{V}(x), V \subseteq A \setminus B \subseteq A \Rightarrow x \in \text{int} A$

Assume that $x \in \text{int} B \Rightarrow \exists W \in \mathcal{V}(x), W \subseteq B$.

$T = V \cap W \in \mathcal{V}(x)$ s.t. $T \overset{1}{\subseteq} A \setminus B$, $T \overset{2}{\subseteq} B$ $\Big\} \Rightarrow$ ↯ *mulțimea vidă nu e vecinătate pentru nimeni

1 and 2 $\Rightarrow T = \emptyset$

b. $\text{cl} A \setminus \text{cl} B \subseteq \text{cl}(A \setminus B)$

Let $x \in \text{cl} A \setminus \text{cl} B$

I. $x \in \text{cl} A \Leftrightarrow \forall V \in \mathcal{V}(x), V \cap A \neq \emptyset$

II. $x \notin \text{cl} \Leftrightarrow \exists W \in \mathcal{V}(x), W \cap B = \emptyset$ $\Big\} \Rightarrow W \cap A \neq \emptyset$ and $W \cap B = \emptyset$

Assume that $x \notin \text{cl}(A \setminus B) \Rightarrow$

$\Rightarrow \exists T \in \mathcal{V}(x)$ s.t. $T \cap (A \setminus B) = \emptyset$

$S = T \cap W \in \mathcal{V}(x)$

$S \cap (A \setminus B) = \emptyset \Rightarrow S \subseteq \mathbb{R}^m \setminus (A \setminus B) \Rightarrow S \subseteq B \cup (\mathbb{R}^m \setminus A)$

$S \cap A \neq \emptyset$ (2)

$S \cap B = \emptyset$

$S \cap B = \emptyset$ $\Big\} \Rightarrow S \subseteq \mathbb{R}^m \setminus A \Rightarrow S \cap A = \emptyset$ (1)

(1) and (2)

c. Let $A = [1,2]$, $B = [1,2)$

$\text{cl} A = [1,2] = \text{cl} B$

$\text{cl} A \setminus \text{cl} B = \emptyset$ $\quad \Rightarrow \text{cl} A \setminus \text{cl} B \neq \text{cl}(A \setminus B)$

$A \setminus B = \{2\} = \text{cl}(A \setminus B)$

We prove that $\text{int}(A \setminus B) \subset \text{int} A \setminus \text{int} B$

$A = \mathbb{R}^m$, $B = \{0\}$

$\text{int}(\mathbb{R} \setminus \{0\}) = (-\infty, 0) \cup (0, \infty)$

$\text{int} \mathbb{R} = \mathbb{R}$, $\text{int}\{0\} = \emptyset$

$\text{int} \mathbb{R} \setminus \text{int}\{0\} = \mathbb{R} \neq \text{int} \mathbb{R} \setminus \{0\}$

- Compact sets -

Exercise 1 Let $(x_k) \subseteq \mathbb{R}^m$ be a convergent sequence with $x = \lim_{k \to \infty} x_k$.

Prove that $A = \{x\} \cup \{x_k \mid k \in \mathbb{N}\}$ is a compact set.

We are going to prove the compactness by using the def.

Let $I \subseteq \mathbb{N}$

G_i be open, $\forall i \in I$ be s.t. $A \subseteq \bigcup_{i \in I} G_i$

Let $x \in A \Rightarrow \exists i_0 \in I$ s.t. $x \in G_{i_0}$, G_{i_0} - open $\Rightarrow G_{i_0} \in \mathcal{V}(x)$

$x = \lim_{k \to \infty} x_k \Leftrightarrow$

$\Leftrightarrow$ by applying the ch. th. of neighbourhoods for $G_{i_0} \Leftrightarrow \exists k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0, x_k \in G_{i_0}$

$\forall k \in \{1, 2, \dots, k_0 - 1\}$, $\exists i_k \in I$ s.t. $i_k \in G_{i_k} \Rightarrow A \subseteq G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_{k_0-1}} \cup G_{i_0}$

which is FINITE COVERING OF A

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According to the def., A is a compact set.

Exercise 2 Let $A, B \subseteq \mathbb{R}^n$, $A+B = \{x \in \mathbb{R}^n \mid \exists a \in A, \exists b \in B$ s.t. $x = a+b\}$ called MINKOWSKI SUM of two sets.

a). If A is closed and B is compact $\Rightarrow A+B$ is closed.

b). Give an example for which both A and B are closed, but $A+B$ is not closed.

! Remark: $A - B \neq A \setminus B$

ex.: $A = \{(1,2,3)\}$, $B = \{(1,0,0)\}$, $A, B \subseteq \mathbb{R}^3$

We have: $A - B = \{(0,2,3)\}$, $A \setminus B = A$.

a). A is closed $\Leftrightarrow$ for $\forall (x_k) \in A$ convergent with $x = \lim_{k\to\infty} x_k$, it holds that $x \in A$

B is compact if $\Leftrightarrow (b_k) \subseteq B$, $\exists (b_{k_j})_{j \in \mathbb{N}}$ a subseq. s.t. $b = \lim_{j\to\infty} b_{k_j} \in B$

We prove that $A+B$ is closed $\Leftrightarrow \forall (x_k) \subseteq A+B$ with $\lim_{k\to\infty} x_k = x \Rightarrow x \in A+B$

Let $(x_k) \subseteq A+B$ with $\lim_{k\to\infty}$

- cont. poate -

b). According to a theorem, a set is compact if $\Leftrightarrow$ closed AND bounded. In order for a closed set not to be compact, it has to be unbounded.

Let A, B be closed.

$A = \mathbb{Z}$

$B = \{2+\frac{1}{2}, 3+\frac{1}{3}, \dots, m+\frac{1}{m}, \dots\}$

$\frac{1}{2} = 2+\frac{1}{2} - 2 = \underbrace{2+\frac{1}{2}}_{\in B} + \underbrace{(-2)}_{\in A} \Rightarrow \in A+B$

$\frac{1}{m} = m + \frac{1}{m} - m \in A+B$

$\vdots$

$\Rightarrow (\frac{1}{m})_{m\geq 2} \in A+B$

Assume by contradiction that $A+B$ closed $\Rightarrow 0 = \lim_{m\to\infty} \frac{1}{m} \in A+B \Leftrightarrow$

$\Leftrightarrow \exists a \in \mathbb{Z}, \exists m \in \mathbb{N}, m \geq 2$ s.t. $0 = a + m + \frac{1}{m} \Leftrightarrow -a - m = \frac{1}{m} \Rightarrow \begin{cases} -a-m \in \mathbb{Z} \\ \frac{1}{m} \notin \mathbb{Z} \end{cases} \Rightarrow \lightning$

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Exercise 3 Let $A, B \subseteq \mathbb{R}^m$, $d(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\}$

a). If $A = \{a\}$, B - closed $\Rightarrow \exists\, b \in B$ s.t. $d(A,B) = d(a,b)$

! The infimum becomes the minimum

B closed $\Leftrightarrow \forall (b_k) \subseteq B$, $b = \lim\limits_{k\to\infty} b_k \Rightarrow b \in B$

because $A = \{a\} \Rightarrow \exists\, (b_k) \subseteq B$ s.t. $d(A,B) = \lim\limits_{k\to\infty} d(a, b_k) \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon > 0,\ \exists\, k_0 \in \mathbb{N}$ s.t. for $\forall\, k \geq k_0$, $|d(a,b_k) - d(A,B)| \leq \varepsilon$

for $\varepsilon = 1$, $\exists\, k_1 \in \mathbb{N}$ s.t. $k \geq k_1$, $|d(a,b_k) - d(A,B)| \leq 1$

$-1 \leq d(a,b_k) - d(A,B) \leq 1 \Rightarrow d(a,b_k) \leq \underbrace{1 + d(A,B)}_{>0} \Rightarrow$

$\Rightarrow \forall\, k \geq k_1$, $d(a - b_k) \leq r$

$\forall\, k \geq k_1$, $b_k \in \overline{B}(a, r)$

— continue next time —

Seminar 5

$d(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\}$

b) $\left.\begin{array}{l} A \text{ compact} \\ B \text{ closed} \end{array}\right\} \Rightarrow \begin{array}{l} \exists\, a \in A \\ \exists\, b \in B \end{array}$ s.t. $d(A,B) = d(a,b)$

c). Give an example of A and B closed, where the infimum is not attained

b) $\exists\, (a_k) \subset A$, $(b_k) \subset B$ s.t. $d(A,B) = \lim\limits_{k\to\infty} d(a_k, b_k)$

$\left.\begin{array}{l} A \text{ compact} \\ (a_k) \subset A \end{array}\right\} \Rightarrow \exists\, (a_{k_j})_{j\in\mathbb{N}}$ convergent subseq. of (a_k) in $A \Leftrightarrow$

$\Leftrightarrow \exists\, a \in A$ s.t. $\lim\limits_{j\to\infty} (a_{k_j}) = a$

A compact $\Leftrightarrow$ bounded and closed $\Rightarrow \exists\, r > 0$ s.t. $A \subset \overline{B}(a, r)$

$\lim\limits_{k\to\infty} (a_k, b_k) = d(A,B) \Leftrightarrow \forall \varepsilon > 0,\ \exists\, k_\varepsilon \in \mathbb{N}$ s.t. $\forall\, k \geq k_\varepsilon$, $|d(a_k, b_k) - d(A,B)| < \varepsilon$

For $\varepsilon = 1$, $\exists\, k_1 \in \mathbb{N}$ s.t. $k \geq k_1$, $|d(a_k, b_k) - d(A,B)| \leq 1$

$-1 \leq d(a_k, b_k) - d(A,B) \leq 1 \Rightarrow$

$\Rightarrow d(a_k, b_k) \leq 1 + d(A,B)$

$d(b_k, a) \leq d(b_k, a_k) + d(a_k, a) \leq 1 + d(A,B) + \underbrace{d(a_k, a)}_{\leq r} \leq \underbrace{1 + d(A,B) + r}_{t}$

$b_k \in \overline{B}(a, t)$

Hence $(b_k) \subset \underbrace{B \cap \overline{B}(a,t)}_{\text{compact set}}$

$\left.\begin{array}{l} B \text{ closed} \\ \overline{B}(a,t) \text{ closed} \end{array}\right\} \Rightarrow B \cap \overline{B}$ closed & bounded

bounded

$(b_{k_j})_{j\in\mathbb{N}} \subset B \cap \overline{B}(a,t)$

because we are in a compact set $\Rightarrow \exists\, (b_{k_{j_l}})_{l\in\mathbb{N}}$ a convergent subseq of (b_{k_j}) in $B \cap \overline{B}(a,t)$

Hence, $\lim\limits_{l\to\infty} (b_{k_{j_l}}) = b \in B \cap \overline{B}(a,t)$, $\exists$ (it exists)

$d(A,B) = \lim\limits_{k\to\infty} d(a_k, b_k) = \lim\limits_{l\to\infty} d(a_{k_l}, b_{k_l}) = d(a,b)$

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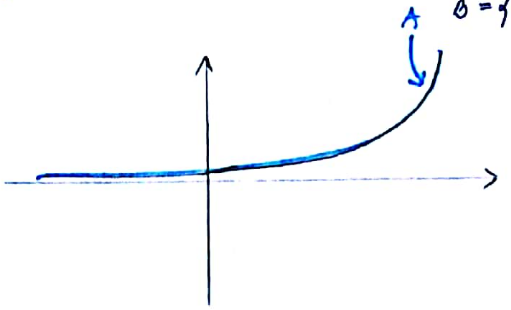
c) We give an example in $\mathbb{R}^2$, where $A = \text{graph } e^x$
$B = \{(x,0) : x \in \mathbb{R}\} = Ox$ axis.



$d(A,B) = \inf d(e^x, 0) = \inf e^x = 0$

but $e^x \neq 0 \ \forall x \in \mathbb{R}$

Because the $\inf e^x = 0$, in order to be attained, the sets A and B must have a common point (which is impossible)

Ex 1 According to Weierstrass th. if $A \subset \mathbb{R}^n$ is compact $\Rightarrow$ each cont. function $f: A \to \mathbb{R}^n$ is bounded. Prove the reverse statement:

If $A \subset \mathbb{R}^n$ is a set satisfying that each continuous function $f: A \to \mathbb{R}$ continuous is bounded, then A is compact $\mathbb{R}^n$ (bounded and closed).

$\boxed{\text{Case I}}$ Assume by contradiction that A is NOT bounded.

$B(0,r) = \{x \in \mathbb{R}^n \mid \|x - 0\| < r\} = \{x \in \mathbb{R}^n \mid \|x\| < r\}$

We define the function $f: A \to \mathbb{R}$, $f(x) = \|x\|$.

Because f is a continuous function $\Rightarrow f$ is bounded $\Rightarrow f(A)$ is a bounded subset of $\mathbb{R}$

$\Leftrightarrow f(A) = \{f(x), x \in A\} = \{\|x\| \mid x \in A\}$ is bounded $\Rightarrow$

$\Rightarrow \exists t \in \mathbb{R}$ s.t. $\|x\| \leq t$ for $\forall x \in A \Leftrightarrow$

$\Leftrightarrow x \in B(0,t), \forall x \in A. \Leftrightarrow A \subseteq B(0,t)$ which is a CONTRADICTION to our assumption $\Rightarrow$

$\Rightarrow A$ is bounded

$\boxed{\text{Case II.}}$ Assume by contradiction that A is not closed $\Leftrightarrow$

$\Leftrightarrow \exists a_k \subseteq A$ convergent with $\lim_{k\to\infty} a_k = a \notin A \Rightarrow$

$\Rightarrow \forall x \in A, \ \|x - a\| \neq 0 \Rightarrow$

$\Rightarrow f: A \to \mathbb{R}, f(x) = \dfrac{1}{\|x-a\|}$ is well defined and moreover, is continuous. $\Rightarrow$

$\Rightarrow f(A)$ is bounded

$f(A) = \left\{\dfrac{1}{\|x-a\|}, x \in A\right\}$ is bounded $\Rightarrow \exists t > 0$ s.t. $f(A) \subseteq B(0,t) \Leftrightarrow$

$\Leftrightarrow \dfrac{1}{\|x-a\|} < t \ \forall x \in A$ (1)

$\lim_{k\to\infty} \|a_k - a\| = 0_+ \Rightarrow$

$\Rightarrow \lim_{k\to\infty} \dfrac{1}{\|a_k - a\|} = \infty$

$\Rightarrow \forall \varepsilon > 0, \exists k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0, \dfrac{1}{\|a_k - a\|} > \varepsilon$ (2)

$$\left.\begin{array}{l} A \ni a_{k_0} \xrightarrow{(1)} \dfrac{1}{\|a_{k_0} - 0\|} < t \\ \xrightarrow[\varepsilon := t]{(2)} \dfrac{1}{\|a_{k_0} - a\|} > t \end{array}\right\} \Rightarrow \text{CONTRADICTION} \Rightarrow A \text{ closed}$$

$\Rightarrow A$ closed

Case I and Case II $\Rightarrow A$ is compact (bounded and closed)

Exercise 2

a). $\lim_{(x,y)\to(0,2)} \frac{\sin(x\cdot y)}{x} = \lim_{(x,y)\to(0,2)} \frac{\sin(x\cdot y)}{x\cdot y}\cdot y =$ $\quad \boxed{\lim_{U(x)\to 0} \frac{\sin(U(x))}{U(x)} = 1}$

$= \lim_{(x,y)\to(0,2)} \frac{\sin(xy)}{xy} \cdot \lim_{y\to 2} y = 1\cdot 2 = 2$

b). $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} =$

Consider $y = mx$, $m \neq 0$

$\lim_{x\to 0} f(x)\cdot mx = \lim_{x\to 0} \frac{x^2-(mx)^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{x^2(1-m^2)}{x^2(1+m^2)} = \frac{1-m^2}{1+m^2}$

It depends on $m \Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$

We prove that this limit does not exist, using the def. of the limits of functions (with seq.)
$\exists\, l \in \bar{\mathbb{R}}$ s.t. $\forall (a_k, b_k) \subseteq \mathbb{R}^2$ with $\lim_{k\to\infty} f(a_k, b_k) = (0,0) \Rightarrow \lim_{k\to\infty} f(a_k, b_k) = l$
S

In order to prove that the limit does not exist, we emphasize two seqs. in $\mathbb{R}^2$
$((a_k),(b_k))_{k\geq 1}$, $((u_k, v_k))_{k\geq 1} \subseteq \mathbb{R}^2$ with $\lim_{k\to\infty}(a_k, b_k) = \lim_{k\to\infty}(u_k, v_k) = (0,0)$
but $\lim_{k\to\infty}(a_k, b_k) \neq \lim_{k\to\infty}$

In the particular case

$a_k = \frac{1}{k}$ $\quad m = 1$

$b_k = (1)\cdot\frac{1}{k} = \frac{1}{k}$

$(a_k, b_k) = \left(\frac{1}{k}, \frac{1}{k}\right) \to (0,0)$

the continuous seq. 0

$\lim_{k\to\infty} f(a_k, b_k) = \lim_{k\to\infty} f\left(\frac{1}{k}, \frac{1}{k}\right) =$

$= \lim_{k\to\infty} \frac{\frac{1}{k^2} - \frac{1}{k^2}}{\frac{1}{k^2} + \frac{1}{k^2}} = \lim_{k\to\infty} 0 = 0$ $\quad \neq$ (not equal)

$\left.\begin{array}{l} u_k = \frac{1}{k} \\ v_k = 2\cdot\frac{1}{k} \end{array}\right\} \to (0,0)$ $\quad \lim_{k\to\infty} f(u_k, v_k) = \lim_{k\to\infty} \frac{\frac{1}{k^2} - \frac{4}{k^2}}{\frac{1}{k^2} + \frac{4}{k^2}} = -\frac{3}{5}$

$\Rightarrow \lim_{k\to\infty}(a_k, b_k) \neq \lim_{k\to\infty}(u_k, v_k)$

$\Leftrightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$

c). $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

$0 \leq |f(x,y) - 0| = \left|\frac{x^3+y^3}{x^2+y^2}\right|$

$0 \leq \left|\frac{x^3+y^3}{x^2+y^2}\right| \leq \frac{|x^3|+|y^3|}{|x^2+y^2|} = \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1}\cdot|x| + \underbrace{\frac{y^2}{x^2+y^2}}_{\leq 1}\cdot|y|$

$\leq |x| + |y|$

$$0 \le \frac{|x^3+y^3|}{|x^2+y^2|} \le |x|+|y|$$

$$\searrow \quad \downarrow \quad \swarrow$$
$$0$$

d). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x\cdot y}$

$y = m\cdot x$

$\Rightarrow \displaystyle\lim_{x\to 0} f(x, m\cdot x) = \lim_{x\to 0} \frac{x^{3}(1+m^3)}{x^2 m} = 0$

$y = m\cdot x^2$

$\Rightarrow \displaystyle\lim_{x\to 0} f(x, mx^2) = \lim_{x\to 0}\left(\frac{x^2}{mx^2} + \frac{m^2x^4}{x}\right) = \frac{1}{m} + 0 = \frac{1}{m}$

depends on $m \Rightarrow \nexists \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy}$

Alternative

Let $a_k = \frac{1}{k}$, $b_k = \frac{1}{k}$

$(a_k, b_k) \to (0,0)$

$\displaystyle\lim_{k\to 0} f(a_k, b_k) = \lim_{k\to 0} \frac{\left(\frac{1}{k}\right)^3 + \left(\frac{1}{k}\right)^3}{\left(\frac{1}{k}\right)^2} = \lim_{k\to\infty} \frac{2}{k} = 0$

Let $u_k = \frac{1}{k}$

$v_k = \frac{1}{k^2} \Rightarrow \displaystyle\lim_{k\to\infty} f(u_k, v_k) = \lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)^3 + \left(\frac{1}{k}\right)^6}{\left(\frac{1}{k}\right)^3} = \lim_{k\to\infty} \frac{\frac{k^3+1}{6}}{\left(\frac{1}{k}\right)^3} =$

$= \displaystyle\lim_{k\to\infty} \frac{\frac{k^3+1}{k^6}}{\left(\frac{1}{k}\right)^3} = \lim_{k\to\infty} \frac{k^3+1}{k^3} = 1 \neq 0$

! $\displaystyle\lim_{k\to\infty} (a_k, b_k) \neq \lim_{k\to\infty} (u_k, v_k)$

Norm of a linear map (norma unei aplicații liniare)

$f \in L(\mathbb{R}^n, \mathbb{R}^m)$, $\|f\| = \max\limits_{\substack{\|x\|=1 \\ (\mathbb{R}^n)}} \{ \|f(x)\|_{(\mathbb{R}^m)} \}$

Exercise 1

Let $a_1, \dots, a_m \in \mathbb{R}$ $\quad a = (a_1, \dots, a_m)$

$\varphi : \mathbb{R}^m \to \mathbb{R}$ $\quad \varphi(x_1, \dots, x_m) = \langle a, x\rangle$

Prove that $\|\varphi\| = \|a\| = \sqrt{a_1^2 + \dots + a_m^2}$.

$\|\varphi\| = \max\{|\varphi(x)| \mid \|x\| = 1\} = \max\{|\langle a, x\rangle| \mid x_1^2 + \dots + x_m^2 = 1\} = \max\{|\langle a, x\rangle| \mid x_1^2 + \dots + x_m^2 = 1\} =$

$= \max\{|a_1x_1 + \dots + a_mx_m| \mid x_1^2 + \dots + x_m^2 = 1\}$

Recall the Cauchy - Buniakowski - Swartz (CBS) theorem:

$|\langle a, x\rangle| \le \|a\| \cdot \|x\|$

We prove that $\max\{|a_1x_1 + \dots + a_mx_m|\} = \sqrt{a_1^2 + \dots + a_m^2}$ $\quad x_1^2 + \dots + x_m^2 = 1$

Case I. We prove that $\forall x \in \mathbb{R}^m$ with $x_1^2 + \dots + x_m^2 = 1 \Rightarrow |a_1x_1 + \dots + a_mx_m| \le \sqrt{a_1^2 + \dots + a_m^2}$

Case II. $\exists x \in \mathbb{R}^2$ with $\bar{x}_1^2 + \dots + \bar{x}_m^2 = 1$ and $|a_1\bar{x}_1 + \dots + a_m\bar{x}_m| = \sqrt{a_1^2 + \dots + a_m^2}$

L'Hopital

$\boxed{\text{Case I.}}$ Let $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ with $\|x\| = 1$

$\overset{\text{CBS}}{\Longrightarrow} |\langle a, x\rangle| \le \|a\| \cdot \|x\| = \|a\|$ ✓

Case 1. $\|a\| = 0 \Rightarrow |\langle a, x\rangle| \le 0 \Rightarrow \langle a, x\rangle = 0$

The inequality holds for all x.

Case 2. $\|a\| \ne 0 \Rightarrow \sqrt{a_1^2 + \dots + a_m^2}$

~~The equals~~ $\bar{x}_1 = \dfrac{a_1}{\sqrt{a_1^2+\dots+a_m^2}}, \dots, \bar{x}_m = \dfrac{a_m}{\sqrt{a_1^2+\dots+a_m^2}}$

$$\langle a, x\rangle = \left| a_1 \cdot \frac{a_1}{\sqrt{a_1^2+\dots+a_m^2}} + \dots + a_m \cdot \frac{a_m}{\sqrt{a_1^2+\dots+a_m^2}} \right| =$$

$$= \left| \frac{a_1^2+\dots+a_m^2}{\sqrt{a_1^2+\dots+a_m^2}} \right| = \sqrt{a_1^2+\dots+a_m^2}$$

! Homework:

Let $f \in L(\mathbb{R}^2, \mathbb{R}^2)$ $\quad [f] = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$

Determine $\|f\|$.

$$\|f\| = \max_{x_1^2+x_2^2=1} \|f(x)\| = \max_{x_1^2+x_2^2=1} \left\| \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \max_{x_1^2+x_2^2=1} \|(2x_1 - x_2, 3x_1 + x_2)\|$$

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$\varphi \in L(\mathbb{R}^m, \mathbb{R}^m)$ $\qquad \varphi(x) = [\varphi]\cdot\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in (\ )_m$

$[\varphi] \in \mathbb{R}^{m\times m}$

$\hookrightarrow$ which has on the i column, $\varphi(e_i)$

$\|\varphi\| = \max\limits_{\|x\|_m = 1} \|\varphi(x)\|_m$ different because $\varphi: \mathbb{R}^m \to \mathbb{R}^m$

Ex 1 Let $a_1, a_2, \dots, a_m \in \mathbb{R}$ and $\varphi \in L(\mathbb{R}^m, \mathbb{R}^m)$ having the matrix $[\varphi] = diag(a_1, \dots, a_m) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a_m \end{pmatrix}$

Determine $\|\varphi\|$.

$$\|\varphi\| = \max_{\|x\|=1} \|\varphi(x)\|_m = \max_{\|x\|=1} \left\|[\varphi]\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}\right\| = \max_{\sqrt{x_1^2+\dots+x_m^2}=1} \left\|\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ 0 & 0 & \cdots & a_m \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}\right\| =$$

$$= \max_{\sqrt{x_1^2+\dots+x_m^2}=1} \left\|\begin{pmatrix} x_1a_1 \\ x_2a_2 \\ \vdots \\ x_ma_m \end{pmatrix}\right\| = \max_{\sqrt{x_1^2+\dots+x_m^2}=1} \|(x_1a_1, \dots, x_m a_m)\| = \max_{\|x\|=1} \sqrt{(x_1a_1)^2 + (x_2a_2)^2 + \dots + (x_ma_m)^2} =$$

$$= \max_{\|x\|=1} \langle a^2, x^2\rangle$$

Let $T = \max\{|a_1|, \dots, |a_m|\}$

We prove that $\|\varphi\| = T$

Step 1: $\forall x \in \mathbb{R}^m$ with $\|x\| = 1 \Rightarrow \|\varphi(x)\| \le T$

$$\|\varphi(x)\| = \sqrt{x_1^2\underbrace{a_1^2}_{\le T^2} + \dots + x_m^2\underbrace{a_m^2}_{\le T^2}} \le \sqrt{x_1^2T^2 + \dots + x_m^2T^2} = \sqrt{T^2(x_1^2+\dots+x_m^2)} =$$

$$= T\sqrt{x_1^2+\dots+x_m^2} = T\underbrace{\|x\|}_{=1} = T$$

because the max is attained $\Rightarrow$

$\Rightarrow \exists\, i \in \overline{1,m}$ s.t. $T = |a_i|$

By taking $\bar{x} = e_i$ we obtained that $\|\bar{x}\| = 1$, $\|\varphi(\bar{x})\| = \|\varphi(e_i)\| =$

$$= \|\varphi(0, \dots, 1, 0, \dots, 0)\| = \sqrt{0\cdot a_1^2 + 0\cdot a_2^2 + \dots + 1\cdot a_i^2 + 0 + \dots} = \sqrt{a_i^2} = |a_i| = T$$

Ex 2 Let $\varphi \in L(\mathbb{R}^2, \mathbb{R}^2)$ having the matrix $[\varphi] = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$. Determine $\|\varphi\|$.

$$\|\varphi\| = \max_{\|x\|=1} \|\varphi(x)\| = \max_{\sqrt{x_1^2+x_2^2}=1} \left\|[\varphi]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\| = \max_{\sqrt{x_1^2+x_2^2}=1} \left\|\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\| = \max_{\sqrt{x_1^2+x_2^2}=1} \left\|\begin{pmatrix} 2x_1 - x_2 \\ 3x_1 + x_2 \end{pmatrix}\right\| =$$

$$= \max_{\sqrt{x_1^2+x_2^2}=1} \sqrt{(2x_1-x_2)^2 + (3x_1+x_2)^2} = \max_{\sqrt{x_1^2+x_2^2}=1} \sqrt{4x_1^2 - 4x_1x_2 + x_2^2 + 9x_1^2 + 6x_1x_2 + x_2^2} =$$

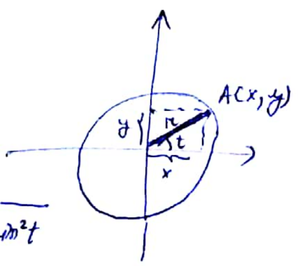
$$= \max_{\sqrt{x_1^2+x_2^2}=1} \sqrt{13x_1^2 + 2x_1x_2 + 2x_2^2}$$

When we shift from the centilinear coordinates xOy to the polar coordinates, we have

$r = d(O, A) \ge 0$

$t = \measuredangle(\vec{OA}, Ox) \in [0, 2\pi)$

$\begin{cases} \cos t = \frac{x}{r} \\ \sin t = \frac{y}{r} \end{cases} \Rightarrow \begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$. If $x^2+y^2 = 1 \Rightarrow \boxed{r = 1}$, $t \in [0, 2\pi)$



$$\|\varphi\| = \max_{t\in[0,2\pi)} \sqrt{13\cos^2 t + 2\cos t\sin t + 2\sin^2 t}$$

$$= \max_{t\in[0,2\pi)} \sqrt{11\cos^2 t + 2\cos t\sin t + 2}$$

$\sin^2 2t = 4\sin^2 t\cos^2 t = 4(1-\cos^2 t)\cos^2 t = \cos^2 t + 1 = \cos t\cdot\cos t + \sin\frac{\pi}{2}\cdot\sin\frac{\pi}{2} =$

$\cos 2t = \cos^2 - \sin^2 t = \cos^2 t - 1 + \cos^2 t = 2\cos^2 t - 1$

We consider the function $11\cos^2 t + \sin 2t + 2 = 11\cdot\frac{\cos 2t + 1}{2} + \sin 2t + 2 = \frac{11}{2}\cos 2t + \sin 2t + \frac{15}{2}$

We give a general proof for an equation $a\sin x + b\cos x = c \,\Big|\cdot \frac{1}{\sqrt{a^2+b^2}}$ (if $a\neq 0$ or $b\neq 0$)

$$\Leftrightarrow \frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}}$$

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = 1 \to \exists\, s \in [0,2\pi) \text{ s.t. } \cos s = \pm\frac{a}{\sqrt{a^2+b^2}} \Leftrightarrow \cos s\sin x + \sin s\cos x = \frac{c}{\sqrt{a^2+b^2}} \Leftrightarrow$$

$$\Leftrightarrow \sin(s+x) = \frac{c}{\sqrt{a^2+b^2}}$$

In our eq. $a = 1$ and $b = \frac{11}{2}$

$$\sqrt{a^2+b^2} = \sqrt{1+\frac{121}{4}} = \sqrt{\frac{125}{4}} = \frac{\sqrt{125}}{2} = \frac{5\sqrt5}{2}$$

$$g(t) = \frac{15}{2} + \frac{5\sqrt5}{2}\cdot\left(\frac{\frac{11}{2}}{\frac{5\sqrt5}{2}}\cdot\cos 2t + \frac{1}{\frac{5\sqrt5}{2}}\sin 2t\right) =$$

$$= \frac{15}{2} + \frac{5\sqrt5}{2}\sin(s+2t)$$

$$\|f\| = \max_{t\in[0,2\pi]}\sqrt{|g(t)|} = \sqrt{\frac{15}{2} + \frac{5\sqrt5}{2}\underbrace{\max_{t\in[0,2\pi)}[\sin(s+2t)]}_{1}} = \sqrt{\frac{15+5\sqrt5}{2}}$$

Ex 3 $A \subseteq \mathbb{R}^n$

$a \in \operatorname{int} A$

$f: A \to \mathbb{R}^m$

$$\lim_{x\to a}\frac{1}{\|x-a\|}\left[f(x) - f(a) - T(x-a)\right] = 0_m$$

If $\exists\, T \in L(\mathbb{R}^n, \mathbb{R}^m)$ it is ! $\overset{\text{not.}}{=} \boxed{d f(a)}$

0 point $\longrightarrow$ $d f(a): \mathbb{R}^n \to \mathbb{R}^m$
function

differentiable

If $n = m = 1$ and f is derivable at a

$$d f(a)(x) = x\cdot f'(a) \quad \boxed{\forall x \in \mathbb{R}}$$

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$

$f = (f_1, \dots, f_m)$

$f_j: \mathbb{R}^n \to \mathbb{R},\ \forall\, j = \overline{1,m}$

Let $v \in \mathbb{R}^n$

$$\lim_{t\to 0}\frac{f(a+t\cdot v) - f(a)}{t} = f'(a, v)$$

$v = e_i$

$$f'(a, e_i) = \frac{\partial f}{\partial x_i}$$

diferențială
derivată după o direcție
derivată parțială
matricea lui Jacoby
gradientul
diferența dintre
legătura dintre
legătura dintre

Ex 4 Let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = z^2 e^{xy} + x\sin(yz^2)$

Det. all the second partial derivatives.

Ex 5 Let $u: \mathbb{R}^2 \to \mathbb{R}$, $u(x,y) = 2\cos^2\left(y - \frac{x}{2}\right)$.

Prove that
$$2\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial x\,\partial y}(x,y) = 0$$

Ex 6 Let $z \in \mathbb{R}$ and $f: \mathbb{R}\times(0,\infty) \to \mathbb{R}$, $f(x,y) = y^z\cdot e^{-\frac{x^2}{4y}}$. Prove that
$$x^2\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial x}\left(x^2\cdot\frac{\partial f}{\partial x}(x,y)\right) \quad \forall (x,y) \in \mathbb{R}\times(0,\infty)$$

Ex 4 Let $(x,y,z) \in \mathbb{R}^3$

$$\frac{\partial f}{\partial x}(x,y,z) = f'_x(x,y,z) = \left(z^2 e^{xy} + x \sin(y \cdot z^2)\right)'_x$$
$$= z^2 (e^{xy})'_x + \sin(yz^2) =$$
$$= z^2 \cdot e^{xy} \cdot y + \sin(yz^2)$$

$$\frac{\partial f}{\partial y}(x,y,z) = f'_y(x,y,z) = \left(z^2 e^{xy} + x \sin(yz^2)\right)'_y$$
$$= z^2 \cdot e^{xy} \cdot x + x \cos(yz^2) \cdot z^2$$

$$\frac{\partial f}{\partial z}(x,y,z) = f'_z(x,y,z) = \left(z^2 e^{xy} + x \sin(yz^2)\right)'_z$$
$$= e^{xy} 2z + x \cos(yz^2) 2yz$$

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y,z) = \left(z^2 e^{xy} y + \sin(yz^2)\right)'_x = f''_{xx}$$
$$= z^2 y e^{xy} y + 0 = z^2 y^2 e^{xy}$$

$$f''_{yx} = \frac{\partial^2 f}{\partial y \partial x}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y,z)\right) = \left(yz^2 e^{xy} + \sin(yz^2)\right)'_y = z^2 e^{xy} + y z^2 e^{xy} x + \cos(yz^2) z^2$$

$$f''_{zx}(x,y,z) = \frac{\partial^2 f}{\partial z \partial x}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}(x,y,z)\right) = \left(yz^2 e^{xy} + \sin(yz^2)\right)'_z = e^{xy} y 2z + \cos(yz^2) \cdot 2yz.$$

$$f'_{xy}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y,z)\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right.$$

We notice that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

de facut acasă

Ex 5 - poză am făcut la tablă

$\boxed{\text{Ex } 6}$ Let $(x,y) \in \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = \left(y^z \cdot e^{-\frac{x^2}{4y}}\right)'_x = y^z \cdot e^{\frac{-x^2}{4y}} \cdot \left(-\frac{x^2}{4y}\right)'_x = y^z \cdot e^{-\frac{x^2}{4y}} \cdot \frac{(-2x)}{4y} = \frac{-x \cdot y^{z-1}}{2} \cdot e^{-\frac{x^2}{4y}}$$

$$\frac{\partial f}{\partial y}(x,y) = \left(y^z \cdot e^{-\frac{x^2}{4y}}\right)'_y = z \cdot y^{z-1} \cdot e^{-\frac{x^2}{4y}} + y^z \cdot e^{-\frac{x^2}{4y}} \cdot \left(\frac{-x^2}{4y}\right)'_y = z y^{z-1} \cdot e^{-\frac{x^2}{4y}} + y^z \cdot e^{\frac{-x^2}{4y}} \cdot \frac{1}{y^2} \cdot \frac{x^2}{4} =$$

$$= y^{z-2} \cdot e^{\frac{-x^2}{4y}} \left(zy + \frac{x^2}{4}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{-x^3 \cdot y^{z-1}}{2} \cdot e^{-\frac{x^2}{4y}}\right) = \frac{y^{z-1}}{2}\left[\left(x^3 e^{-\frac{x^2}{4y}}\right)'_x\right] =$$

$=$ — — — poză tablă

SEMINAR 7

$\lim_{t \to 0} \frac{1}{t} \cdot$ —— ?

Ex 1 Let $r>0$, $A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<r^2\}$

$f: A \to \mathbb{R}$, $f(x,y) = 2 \ln \frac{r\sqrt{8}}{r^2-x^2-y^2} = 2\ln r\sqrt{8} - \ln(r^2-x^2-y^2)$

Prove that $\forall (x,y)\in A$ $\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = e^{f(x,y)}$

Let $(x,y)\in A$ $\frac{\partial f}{\partial x}(x,y) = -2\frac{1}{r^2-x^2-y^2}(-2x) = \frac{4x}{r^2-x^2-y^2}$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 4\,\frac{1}{r^2-x^2-y^2}$$

$$\frac{\partial f}{\partial x^2}(x,y) \quad ?$$

se făcut deart

Ex 2 Let $m\geq 2$, $m\in\mathbb{N}$, $z<0$, $k>0$

Consider $f:\mathbb{R}^m \to \mathbb{R}$, $f(x) = k\cdot \|x\|^z$

Determine LAPLACE's OPERATOR ASSOCIATED to f.

$$\Delta f = \sum_{j=1}^{m} \frac{\partial^2 f}{\partial x_j^2} = \frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_m^2}$$

Let $x=(x_1,\dots,x_m)\in\mathbb{R}^m$

$\|x\| = \sqrt{x_1^2+\dots+x_m^2}$

$f(x_1,\dots,x_m) = k(x_1^2+\dots+x_m^2)^{\frac{z}{2}}$

Let $j\in\{1,\dots,m\}$

$$\underset{j\in\{1,\dots,m\}}{\frac{\partial f}{\partial x_j}}(x_1,\dots,x_m) = \left(k\cdot(x_1^2+\dots+x_m^2)^{\frac{z}{2}}\right)'_{x_j} =$$

$$= k\cdot\frac{z}{2}(x_1^2+\dots+x_m^2)^{\frac{z}{2}-1}\cdot 2(x_1^2+\dots+x_m^2)'_{x_j} =$$

$$= k\cdot\frac{z}{2}(x_1^2+\dots+x_m^2)^{\frac{z}{2}-1}\cdot 2x_j$$

$$\frac{\partial^2 f(x_1,\dots,x_m)}{\partial x_j^2} = \left[k\cdot z\cdot x_j\cdot(x_1^2+\dots+x_m^2)^{\frac{z}{2}-1}\right]'_{x_j}$$

$$\frac{\partial^2 f}{\partial x_j^2}(x_1,\dots,x_m) = k\cdot z\cdot\left[(x_1^2+\dots+x_m^2)^{\frac{z}{2}-1} + x_j\cdot\left(\frac{z}{2}-1\right)\cdot(x_1^2+\dots+x_m^2)^{\frac{z}{2}-2}\cdot 2x_j\right]$$

$$= k\cdot z\left[\|x\|^{z-2} + (z-2)\cdot x_j^2\cdot\|x\|^{z-4}\right]$$

$$= k\cdot z\cdot\|x\|^{z-4}\left[\|x\|^2 + (z-2)\cdot x_j^2\right]$$

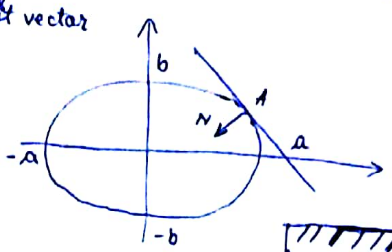
$$\Delta f(x_1,\dots,x_m) = \sum_{j=1}^{m} \frac{\partial^2 f}{\partial x_j^2}(x_1,\dots,x_m) = kz\,\|x\|^{z-4}\left[\|x\|^2+(z-2)x_1^2\right]+\dots+kz\,\|x\|^{z-4}\left[\|x\|+(z-2)x_m^2\right]=$$

$$= kz\cdot\|x\|^{z-4}\left[m\cdot\|x\|^2+(z-2)\|x\|^2\right] = kz\,\|x\|^{z-2}\left[m+(z-2)\right]$$

3) Let $a, b>0$ $\quad f:\mathbb{R}^2\to\mathbb{R}$ $\quad f(x,y) = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$

Determine the derivative of f at $A=\left(\frac{a}{\sqrt2},\frac{b}{\sqrt2}\right)$ on the direction of the versor of the inner normal at this point to the ellipse of the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

unit vector



$\vec{m} = \frac{N}{\|N\|} =$

= see below

Recall that given $a\in\mathbb{R}^n$ $\quad f:\mathbb{R}^n\to\mathbb{R}$
$v\in\mathbb{R}^n$

$$f'(a;v) = \lim_{t\to0}\frac{1}{t}\left(f(a+tv)-f(a)\right)$$

↳ the derivative of f at the point a, in the direction v

If f is differentiable at a, then it has directional derivatives and

$$f'(a;v) = df(a)(v) = v_1\frac{\partial f}{\partial x_1}(a)+\dots+v_m\frac{\partial f}{\partial x_m}(a) = \sum_{j=1}^{m} v_j\frac{\partial f}{\partial x_j}(a)$$

$$\lim_{h\to0}\frac{1}{\|h\|}\left(f(a+h)-f(a)-\underbrace{T(h)}_{df(a)}\right) = 0_m \qquad = \langle\nabla f(a), v\rangle$$

The tangent of the ellipse to the point A has the eq.:

$$\frac{x x_A}{a^2}+\frac{y y_A}{b^2}=1$$

$$\frac{x\frac{a}{\sqrt2}}{a^2}+\frac{y\frac{b}{\sqrt2}}{b^2}=1$$

$$tg:\ x\cdot\frac{1}{a\sqrt2}+y\cdot\frac{1}{b\sqrt2}=1$$

$$N=\left(-\frac{1}{a\sqrt2}, -\frac{b}{b\sqrt2}\right)$$

$$\|N\| = -$$

$$\vec{m} = \frac{N}{\|N\|} = \left(\frac{\sqrt{a^2+b^2}}{ab\sqrt2}\right)^{-1}\cdot\left(-\frac{1}{a\sqrt2}, -\frac{1}{b\sqrt2}\right) = -\left(\frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right)$$

$$f'(A,m) = m_1\cdot\frac{\partial f}{\partial x}(A)+m_2\cdot\frac{\partial f}{\partial y}(A)$$

$$\frac{\partial f}{\partial x}(x,y) = \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)'_x = -\frac{2x}{a^2} \cdot \frac{\partial f}{\partial y} = -\frac{2y}{b^2}$$

$$f'(A,m) = -\frac{b}{\sqrt{a^2+b^2}}\cdot\frac{-2\frac{a}{\sqrt2}}{a^2} - \frac{a}{\sqrt{a^2+b^2}}\cdot\frac{-2\cdot\frac{b}{\sqrt2}}{b^2} =$$

$$= \frac{\sqrt2}{\sqrt{a^2+b^2}}\left(\frac{b}{a}+\frac{a}{b}\right) = \frac{\sqrt2}{\sqrt{a^2+b^2}}\cdot\frac{b^2+a^2}{ab} = \frac{\sqrt{2(a^2+b^2)}}{ab}$$

4. Example of a function which is discontinuous at $(0,0)$ in $\mathbb{R}^2$ but has antiderivatives at $(0,0)$ on each direction. $\underset{(0,0)\text{ in }\mathbb{R}^2}{}$ directional

Let $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}\frac{x^2y}{x^6+y^2}, & (x,y)\neq 0_2\\ 0, & (x,y)=0_2\end{cases}$

a). Prove that f is discontinuous at 0_2

b). Prove that f posses directional derivatives at 0_2 on each random direction

(a) In order to prove that f is discont. to 0_2 we are going to use the def. More ~~speif~~ precisely, we specify a seq. $(a_k, b_k)_{k\in\mathbb{N}}\subseteq\mathbb{R}^2$, for which:

$$\lim_{k\to\infty}(a_k,b_k)=0_2$$

but for which

$$\lim_{k\to\infty} f(a_k,b_k)\neq f(0_2)$$

$$f\left(\tfrac{1}{k},\tfrac{1}{k}\right)=\frac{\frac{1}{k^3}}{\frac{1}{k^6}+\frac{1}{k^2}}=\frac{1}{k^3}\cdot\frac{k^6}{1+k^4}=\frac{k^3}{1+k^4}$$

We choose $(a_k,b_k)=\left(\frac{1}{k},\frac{1}{k^3}\right)$, $\forall k\in\mathbb{N}$

Then $\lim_{k\to\infty}(a_k,b_k)=0_2$

$\lim_{k\to\infty} f(a_k,b_k)=\lim_{k\to\infty}\left(\frac{1}{k^3}\cdot\frac{k^6}{2}\right)=\lim_{k\to\infty}\frac{k^3}{2}=\infty\neq 0=f(0_2)$ $\Rightarrow$

$\Rightarrow f$ is discontinuous at 0_2

(b) Let $v=(v_1,v_2)\in\mathbb{R}^2$ an arbitrarily chosen direction

$$f'(0_2;v)=\lim_{t\to 0}\frac{1}{t}\left[f(0_2+t\cdot v)-f(0_2)\right]=\lim_{t\to 0}\frac{1}{t}\cdot f(tv)=$$

$$=\lim_{t\to 0}\frac{1}{t}\cdot f(tv_1,tv_2)=$$

$$=\lim_{t\to 0}\frac{1}{t}\cdot\frac{t^2v_1^2\cdot t\,v_2}{t^6v_1^6+t^2v_2^2}=\lim_{t\to 0}\frac{v_1^2v_2}{t^4v_1^6+v_2^2}=\frac{v_1^2v_2}{v_2^2}=\frac{v_1^2}{v_2},\quad v_2\neq 0$$

if $v_2=0$ $\quad f'(0_2;(v_1,0))=\lim_{t\to 0}\frac{1}{t}\cdot f(tv_1,0)=\lim_{t\to 0}\frac{1}{t}\cdot\frac{t^2v_1\cdot 0}{t^6v_1^6}$

In conclusion, $f'(0_2;(v_1,v_2))=\begin{cases}\frac{v_1^2}{v_2}, & v_2\neq 0\\ 0, & v_2=0\end{cases}$

Let $A\subseteq\mathbb{R}^n$, $a\in\text{int}A$, $f:A\to\mathbb{R}^m$

If $\exists\, T\in L(\mathbb{R}^n,\mathbb{R}^m)$ s.t. $\lim_{x\to a}\frac{1}{\|x-a\|}\left[f(x)-f(a)-T(x-a)\right]=0_m$ we say that f is Fréchet differentiable at a,

$\lim_{h\to 0}\frac{1}{\|h\|}\left[f(a+h)-f(a)-T(h)\right]=0_m$

$\to df(a)\in L(\mathbb{R}^n,\mathbb{R}^m)$, $df(a)(x)\in\mathbb{R}^m$

Prove that if $f:\mathbb{R}^n\to\mathbb{R}^m$ is a linear map, then f is differentiable on $\mathbb{R}^n$ (at each point) and $df(a)=f$, $\forall a\in\mathbb{R}^n$ $\quad f\in L(\mathbb{R}^n,\mathbb{R}^m)$

$f\in L(\mathbb{R}^n,\mathbb{R}^m)\Leftrightarrow\forall\alpha,\beta\in\mathbb{R}$, $\forall x,y\in\mathbb{R}^n$, $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$

Let $A\in\mathbb{R}^m$

$$\lim_{x\to a}\frac{1}{\|x-a\|}\left[f(x)-f(a)-f(x-a)\right]=$$

$$=\lim_{x\to a}\frac{1}{\|x-a\|}\left[f(x)-f(a)-f(x\right.$$

$=?$

If $\exists\, T\in L(\mathbb{R}^n,\mathbb{R}^m)$, $\exists\,\omega:A\to\mathbb{R}^m$ with $\lim_{x\to a}\omega(x)=0_m$

$f(x)=f(a)+df(x-a)+\|x-a\|\,\omega(x)$, $\forall x\in A$

$\Rightarrow f$ is diff. at a.

Let $f: \mathbb{R}^n \to \mathbb{R}^m$, $f(tx) = t f(x)$, $\forall t > 0$, $\forall x \in \mathbb{R}^n$

Prove that if f is diff. at 0_n, then f is , ?? a linear function

* we are going to use the characterization with ω *

f is diff. at 0_n if $\exists\, \omega: \mathbb{R}^n \to \mathbb{R}^m$ s.t. $\lim\limits_{x \to 0_n} \omega(x) = 0_m$

and $f(x) = f(0_n) + df(0_n)(x) + \|x\| \cdot \omega(x)$ $\forall x \in \mathbb{R}^n$

$$\left.\begin{array}{l} \text{if } x = 0_n \\ \text{and} \\ t = 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} f(2 \cdot 0_n) = 2 f(0_n) \\ f(2\, 0_n) = f(0_n) \end{array}\right\} \Rightarrow 2f(0_n) = f(0_n) \Rightarrow f(0_n) = 0_m$$

$$\Rightarrow \forall x \in \mathbb{R}^n,\ f(x) = 0_m + \underbrace{2\, 0_n(x)}_{f(x)} + \|x\|\, \omega(x)$$

SEMINAR 8

23.04.2018.

f is differentiable at $a \in A$ $\quad (f: A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n)$

if $\exists\, df(a) \in L(\mathbb{R}^n, \mathbb{R}^m)$

$$\lim_{x \to a} \frac{1}{\|x-a\|} \left[f(x) - f(a) - df(a)(x-a) \right] = 0_m$$

$$\Updownarrow$$

$$\lim_{h \to 0_n} \frac{1}{\|h\|} \left[f(a+h) - f(a) - df(a)(h) \right] = 0_m$$

(T) $\exists\, \omega: A \to \mathbb{R}^m$ with $\lim\limits_{x \to a} \omega(x) = 0_m$ and

$$f(x) = f(a) + df(a)(x-a) + \|x-a\|\, \omega(x) \quad \forall x \in A$$

Ex 1 Let $A \subseteq \mathbb{R}^n$

$a = (a_1, \dots, a_n) \in \text{int}\, A$

$f: A \to \mathbb{R}^m$, with all partial derivatives w.r.t. a

Prove that f is differentiable at a

$$\Leftrightarrow \lim_{x \to a} \frac{1}{\|x-a\|} \left[f(x) - f(a) - \sum_{j=1}^{n} (x_j - a_j) \frac{\partial f}{\partial x_j}(a) \right] = 0_m \quad (*)$$

$\Rightarrow$ We know that f is differentiable at $a \Rightarrow$

$\Rightarrow \exists\, df(a) \in L(\mathbb{R}^n, \mathbb{R}^m)$ s.t. $\lim\limits_{x \to a} \frac{1}{\|x-a\|} [f(x) - f(a) - df(a)(x-a)] = 0_m \quad (*)$

Moreover, for $h = (h_1, \dots, h_n) \in \mathbb{R}^n$

$$df(a)(h) = h_1 \cdot \frac{\partial f}{\partial x_1}(a) + \dots + h_n \cdot \frac{\partial f}{\partial x_n}(a) = \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a)$$

$(*) = (*)$ $\quad h = x - a$ ✓

$\mathbb{R}$ $y = ax + b$

$\mathbb{R}^2$ $y = ax + by$ funcții liniare

$\mathbb{R}^3$ $ax + by + cz$

$\Leftarrow$ We define $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(h) = \sum\limits_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a) \in L(\mathbb{R}^n, \mathbb{R}^m) + (*) \Rightarrow (*) + T = df(a)$

because $\frac{\partial f}{\partial x_j}(a)$ is constant $\Rightarrow$ combinație liniară $\Rightarrow$

$\Rightarrow$ funcție liniară

The previous problem supplies an algorithm for studying the differentiability of a function at some point by using ?

Algorithm for differentiability of f at a, $f: A \to \mathbb{R}$.

I. We study the partial derivatives of f at a.

If at least one partial derivative does not exist $\Rightarrow$ STOP (Conclusion: f is NOT diff. at a)

Otherwise, compute $\frac{\partial f}{\partial x_j}(a)$, $\forall j = \overline{1,m}$ $\Rightarrow$ Go to II

II. Study

$$\ell = \lim_{x \to a} \frac{1}{\|x-a\|}\left[f(x) - f(a) - \sum_{j=1}^{m}(x_j - a_j)\frac{\partial f}{\partial x_j}(a)\right]$$

If $\ell = 0_m \Rightarrow f$ is diff. at a and $df(a)(h) = \sum_{j=1}^{m} h_j \frac{\partial f}{\partial x_j}(a)$

Otherwise

$(\ell \neq 0_m$ or $\nexists) \Rightarrow f$ is not diff. at a.

$$\ell = \lim_{h \to 0_m} \frac{1}{\|h\|}\left[f(a+h) - f(a) - df(a)(h)\right]$$

$$= \lim_{\substack{h_1 \to 0 \\ \vdots \\ h_m \to 0}} \frac{1}{\sqrt{h_1^2 + \dots + h_m^2}}\Big[f(a_1 + h_1, \dots, a_m + h_m) - f(a_1, \dots, a_m) - \left[h_1 \frac{\partial f}{\partial x_1}(a) + \dots + h_m \frac{\partial f}{\partial x_m}(a)\right]\Big]$$

! Remark: If $f: A \to \mathbb{R}^m$, $f = (f_1, \dots, f_m)$

We apply the algorithm for all $f_1, \dots, f_m$.

If one of them fails to be diff. at a $\Rightarrow f$ is not diff. at a

$df(a) = (df_1(a), \dots, df_m(a))$

Ex 2

$f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \sqrt[3]{x^3 - y^3}$

Let $(x,y) \in \mathbb{R}^2$, $\frac{\partial f}{\partial x}(x,y) = \frac{1}{3\sqrt[3]{(x^3-y^3)^2}} \cdot 3x^2 = \frac{x^2}{\sqrt[3]{(x^3-y^3)^2}}$ if $x^3 \neq y^3$

$\frac{\partial f}{\partial y}(x,y) = \frac{-y^2}{3\sqrt[3]{(x^3-y^3)^2}}$ if $x^3 \neq y^3$

$x^3 - y^3 = 0 \Leftrightarrow (x-y)(x^2+xy+y^2) = 0 \Leftrightarrow \begin{cases} x - y = 0 \Leftrightarrow x = y \\ x^2 + xy + y^2 = 0 \end{cases}$

If $x \neq y$ at least one of them $\neq 0 \Rightarrow$ we divide by one of them $| : x^2$

$\Leftrightarrow 1 + \frac{y}{x} + \frac{y^2}{x^2} = 0 \overset{\frac{y}{x} = t}{\Leftrightarrow} t^2 + t + 1 = 0 \}$ $\Delta = -3 < 0$ $\Rightarrow t_{1,2} \notin \mathbb{R}$

$\Rightarrow x^3 - y^3 = 0 \Leftrightarrow x = y$ (in $\mathbb{R}$)

We consider $A = \{(a,a) \in \mathbb{R}^2 \mid a \in \mathbb{R}\}$

- $\exists$ partial derivative
- both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous functions, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}: \mathbb{R}^2 \setminus A \to \mathbb{R}$

• $\mathbb{R}^2 \setminus A$ is open

From $\vdots \Rightarrow f$ is diff. on $\mathbb{R}^2\setminus A$

$\forall (x,y) \in \mathbb{R}^2 \setminus A$

$df(x,y) \in L(\mathbb{R}^2, \mathbb{R})$

$df(x,y)(h_1,h_2) = h_1 \cdot \frac{\partial f}{\partial x}(x,y) + h_2 \cdot \frac{\partial f}{\partial y}(x,y)$

The set A has not yet been analized.

• Let $(a,a) \in A$? $\frac{\partial f}{\partial x}(a,a) \exists$?

$\Rightarrow$ we study the limit

$$\lim_{x\to a} \frac{1}{x-a} \cdot [f(x,a) - f(a,a)] =$$

$$= \lim_{x\to a} \frac{1}{x-a} \sqrt[3]{x^3-a^3} =$$

$$= \lim_{x\to a} \frac{(x-a)^{\frac{1}{3}} (x^2+xa+a^2)^{\frac{1}{3}}}{x-a} = \lim_{x\to a} \frac{(x^2+xa+a^2)^{\frac{1}{3}}}{(x-a)^{\frac{2}{3}}} =$$

$$= \frac{\sqrt[3]{3a^2}}{0+} = \infty \notin \mathbb{R} \Rightarrow$$

$\Rightarrow f$ has not the partial derivative at (a,a) w.r.t. $x \Rightarrow$

$\Rightarrow f$ isn't diff. at (a,a) with $a \neq 0$.

The last to be checked is ? ?

We study the partial diff. at $(0,0)$? $\exists \frac{\partial f}{\partial x}(0,0)$?

$$\lim_{x\to 0} \frac{1}{x}[f(x,0) - f(0,0)] = \lim_{x\to 0} \frac{1}{x}\sqrt[3]{x^3} = 1 \ \exists \Rightarrow$$

$\Rightarrow f$ is partially diff. at $(0,0)$ w.r.t. x and $\frac{\partial f}{\partial x}(0,0) = 1$

$\frac{\partial f}{\partial y}(0,0)$: $\lim_{y\to 0} \frac{1}{y}(-y) = -1$

$$\lim_{h\to(0,0)} \frac{1}{\|h\|}\left[f(0_2+h) - f(0_2) - \left[h_1 \frac{\partial f}{\partial x}(0_2) + h_2 \frac{\partial f}{\partial y}(0_2)\right]\right]$$

$$= \lim_{\substack{h_1\to 0\\ h_2\to 0}} \frac{1}{\sqrt{h_1^2+h_2^2}} \left[\sqrt[3]{h_1^3-h_2^3} - 0 - [h_1 \cdot 1 + h_2(-1)]\right] =$$

$$= \lim_{\substack{h_1\to 0\\ h_2\to 0}} \frac{1}{\sqrt{h_1^2+h_2^2}} \left[\sqrt[3]{h_1^3+b} - h_1 + h_2\right]$$

$h_1 = \frac{1}{k} \xrightarrow[k\to\infty]{} 0$

We consider the seq. $(a_k, b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^2$ $a_k = \frac{1}{k}$ $b_k = -\frac{1}{k}$

$\downarrow$

0_2

$$\lim_{k\to\infty} \frac{\sqrt[3]{\frac{1}{k^3} - (-\frac{1}{k^3})} - (\frac{1}{k} \cdot 1 - \frac{1}{k})}{\sqrt{\frac{1}{k^2} + \frac{1}{k^2}}} = \lim_{k\to\infty} \frac{\sqrt[3]{2} - 2}{\sqrt{2}} \neq 0$$

E. Ex 3 $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\begin{cases} x^{\frac{4}{3}}\cdot\sin\frac{y}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Let $A=\{(0,a)\mid a\in\mathbb{R}\}$

$(x,y)\in\mathbb{R}^2\setminus A$

$\frac{\partial f}{\partial x}(x,y)=\frac{4}{3}x^{\frac{1}{3}}\cdot\sin\frac{y}{x}+x^{\frac{4}{3}}\cdot\cos\frac{y}{x}\cdot\left(-\frac{y}{x^2}\right)$

$\frac{\partial f}{\partial y}(x,y)=x^{\frac{4}{3}}\cdot\cos\frac{y}{x}\cdot\frac{1}{x}=x^{\frac{1}{3}}\cos\frac{y}{x}$

- f is partially diff. at
- $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ are cont. functions
- $\mathbb{R}^2\setminus A$ is OPEN

$\Rightarrow f$ is diff. on $\mathbb{R}^2\setminus A$

Let $(0,a)\in A$? $\frac{\partial f}{\partial x}(0,a)$?

$\lim\limits_{x\to 0}\frac{f(x,a)-f(0,a)}{x-0}=\lim\limits_{x\to 0}\frac{x^{\frac{4}{3}}\cdot\sin\frac{a}{x}-0}{x-0}=\lim\limits_{x\to 0}x^{\frac{1}{3}}\sin\frac{a}{x}=0\in\mathbb{R}\Rightarrow$

$\Rightarrow\frac{\partial f}{\partial x}(0,a)=0$

? $\frac{\partial f}{\partial y}(0,a)$?

$\lim\limits_{y\to 0}\frac{f(0,y)-f(0,a)}{y-0}=\lim\limits_{y\to 0}\frac{0}{y}=\lim\limits_{y\to 0}0\in\mathbb{R}\Rightarrow\frac{\partial f}{\partial y}(0,a)=0$

$l=\lim\limits_{\substack{h_1\to 0\\ h_2\to 0}}\frac{1}{\|h\|}\left[f((0,a)+(h_1,h_2))-f(0,a)-\left[h_1\cdot\frac{\partial f}{\partial x}(0,a)+h_2\cdot\frac{\partial f}{\partial y}(0,a)\right]\right]$

$l=\lim\limits_{\substack{h_1\to 0\\ h_2\to 0}}\frac{1}{\sqrt{h_1^2+h_2^2}}f(h_1,(a+h_2))=$

$=\frac{1}{\sqrt{h_1^2+h_2^2}}\cdot h_1^{\frac{4}{3}}\cdot\sin\frac{a+h_2}{h_1}$

We define the function $w:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}$, $W(h_1,h_2)=\dfrac{h_1^{\frac{4}{3}}\cdot\sin\frac{a+h_2}{h_1}}{\sqrt{h_1^2+h_2^2}}$

$0\leq|w(h_1,h_2)|\leq\frac{h_1^{\frac{4}{3}}}{\sqrt{h_1^2+h_2^2}}=\underbrace{\frac{h_1}{\sqrt{h_1^2+h_2^2}}}_{\leq 1}\cdot h_1^{\frac{1}{3}}\leq|h_1^{\frac{1}{3}}|$

$\to 0 \leftarrow$

$\Rightarrow\lim\limits_{\substack{h_1\to 0\\ h_2\to 0}}w(h_1,h_2)=0\Rightarrow$

$\Rightarrow f$ is differentiable at $(0,a)$

$df(0,a):\mathbb{R}^2\to\mathbb{R}$ ∄

$df(0,a)(h_1,h_2)=h_1\frac{\partial f}{\partial x}(0,a)+h_2\frac{\partial f}{\partial y}(0,a)=0$

$\Rightarrow$ is a linear function

In conclusion, f is diff. on $\mathbb{R}^2$

Theory:

$A \subseteq \mathbb{R}^m$
$a \in \text{Int}\, A$
$g : A \to B \subseteq \mathbb{R}^m$
$g = (g_1, \dots, g_m)$
$f : B \to \mathbb{R}^p$

If g is diff. at a and $g(a) \in \text{Int}\, B$
AND
f is diff. at $g(a)$
then $(f \circ g)$ is diff at a AND
$$d(f\circ g)(a) = df(g(a)) \circ dg(a)$$
$$J(f\circ g)(a) = J(f)(g(a)) \cdot J(g)(a)$$

$f = f(u_1, \dots, u_m)$
$\forall j = \overline{1, m}$
$$\frac{\partial (f\circ g)}{\partial x_j}(a) = \frac{\partial f}{\partial u_1}(g(a)) \cdot \frac{\partial g_1}{\partial x_j}(a) + \dots + \frac{\partial f}{\partial u_m}(g(a)) \cdot \frac{\partial g_m}{\partial x_j}(a)$$
$$= \sum_{i=1}^{m} \frac{\partial f}{\partial u_i} \cdot \frac{\partial g_i}{\partial x_j}$$

Ex 4 Let $f = f(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$ differentiable on $\mathbb{R}^3$
$F : \mathbb{R}^2 \to \mathbb{R}$
$F(x,y) = f(-3x+2y,\ x^2+y^2,\ 2x^3 - y^3)$

Determine the partial derivative of F in terms of the partial derivatives of f.

Proof: we introduce the function
$g : \mathbb{R}^3 \to \mathbb{R}^3$
$$g(x,y,z) = (\underbrace{-3x+2y}_{u(x,y)},\ \underbrace{x^2+y^2}_{v(x,y)},\ \underbrace{2x^3-y^3}_{w(x,y)})$$
$F = f \circ g$ $\quad u : \mathbb{R}^2 \to \mathbb{R}$ $\quad u(x,y) = -3x+2y$
$v : \mathbb{R}^2 \to \mathbb{R}$ $\quad v(x,y) = x^2+y^2$
$w : \mathbb{R}^2 \to \mathbb{R}$ $\quad w(x,y) = 2x^3 - y^3$

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial (f\circ g)}{\partial x}(x,y) = \frac{\partial f}{\partial u}(g(x,y)) \cdot \frac{\partial u}{\partial x}(x,y) +$$
$$+ \frac{\partial f}{\partial v}(g(x,y)) \cdot \frac{\partial v}{\partial x}(x,y) + \frac{\partial f}{\partial w}(g(x,y)) \cdot \frac{\partial w}{\partial x}(x,y)$$
$$= \frac{\partial f}{\partial u}(g(x,y)) \cdot (-3) + \cancel{\frac{\partial f}{\partial v}} + \frac{\partial f}{\partial v}(g(x,y)) \cdot 2x + \frac{\partial f}{\partial w}(g(x,y)) \cdot 6x^2$$

$$\frac{\partial F}{\partial y}(x,y) = -\text{"}- \cdot (2) + \text{ — '' — } (-2y) + \dots\dots \cdot (3y^2)$$

Ex5 Let $f = f(u, v): \mathbb{R}^2 \to \mathbb{R}$ diff. on $\mathbb{R}^2$

$F: \mathbb{R}^2 \to \mathbb{R} \quad F(x,y) = \sin(y + f(y^2; x))$

Let $x, y \in \mathbb{R}^2$ $\frac{\partial F}{\partial x}(x,y) = \sin(y + f(y^2,x))'_x =$

$= \cos(y + f(y^2,x)) \cdot (y + f(y^2,x))'_x =$

$= \cos(y + f(y^2,x)) \cdot f'_x(y^2,x)$ $\to$ $u: \mathbb{R}^2 \to \mathbb{R}$, $u(x,y) = y^3$, $v: \mathbb{R}^2 \to \mathbb{R}$, $v(x,y) = x$

We define the function $g: \mathbb{R}^2 \to \mathbb{R}^2$, $g(x,y) = (\overset{u}{y^2}, \overset{v}{x}) \Rightarrow$

$\Rightarrow$ we need the partial derivative of $(f \circ g)$ at (x,y)

$$\frac{\partial (f \circ g)}{\partial x}(x,y) = \frac{\partial f}{\partial u}(g(x,y)) \cdot \frac{\partial u}{\partial x}(x,y) + \frac{\partial f}{\partial v}(g(x,y)) \cdot \frac{\partial v}{\partial x}(x,y) =$$

$$= \frac{\partial f}{\partial u}(g(x,y)) \cdot 0 + \frac{\partial f}{\partial v}(g(x,y)) \cdot 1 =$$

$$= \frac{\partial f}{\partial v}(y^2, x)$$

In conclusion, $\frac{\partial F}{\partial x}(x,y) = \cos(y + f(y^2,x)) \cdot \frac{\partial f}{\partial v}(y^2,x)$

$$\frac{\partial F}{\partial y}(x,y) = \cos(y + f(y^2,x))\left(\frac{\partial f}{\partial y}(y^2,x) + 1\right)$$

$$\frac{\partial f}{\partial y}(y^2,x) = \frac{\partial f}{\partial u}(g(x,y)) \cdot \frac{\partial u}{\partial y}(x,y) + \frac{\partial f}{\partial v}(g(x,y)) \cdot \frac{\partial v}{\partial y}(x,y)$$

$$\frac{\partial f}{\partial u}(g(x,y)) \cdot 2y + \frac{\partial f}{\partial v}(g(x,y)) \cdot 0 = \frac{\partial f}{\partial u}(y^2,x) \cdot 2y$$

$$\to = \cos(y + f(y^2,x))\left(\frac{\partial f}{\partial u}(y^2,x) \cdot 2y\right)$$

SEMINAR 9

27. 04. 2018

Ex 1

Find the local extrema points of the function $f:\mathbb{R}^2\to\mathbb{R}$

Let $(x,y)\in\mathbb{R}^2$ $\quad f(x,y)=3x-3y-2x^3-xy^2+2x^2y+y^3$

$$\frac{\partial f}{\partial x}(x,y)=3-6x^2-y^2+4xy$$

$$\frac{\partial f}{\partial y}(x,y)=-3-2xy+2x^2+3y^2$$

We determine the critical points of the function f which are solution to:

$$0=\frac{\partial f}{\partial x}(x,y)\Leftrightarrow\frac{\partial f}{\partial y}(x,y)=0$$

$$\begin{cases}3-6x^2-y^2+4xy=0\\ -3-3xy+2x^2+3y^2=0\end{cases}\Leftrightarrow\begin{cases}3-6x^2-y^2+4xy=0\\ -4x^2+2xy+2y^2=0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}3-6x^2-y^2+4xy=0\\ 2x^2-xy-y^2=0\end{cases}$$

$\boxed{\text{Case I}}$ $x=y=0\Rightarrow 3=0$ ↯ (from the first equation)

$\Downarrow$

at least one of them is $\neq 0$.

$\boxed{\text{Case II}}$ $x\neq 0\Rightarrow 2-\frac{y}{x}-\frac{y^2}{x^2}=0$, $\frac{y}{x}=t$ $\Big\}\Rightarrow 2-t-t^2=0$

$\Delta=1-4\cdot 2\cdot(-1)=9$

$$\begin{cases}t_1=\frac{1+3}{-2}=-2\\ t_2=\frac{1-3}{-2}=1\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}\frac{y}{x}=-2\Rightarrow y=-2x\\ \text{or}\\ \frac{y}{x}=1\Rightarrow y=x\end{cases}$$

(a) $y=2x\Rightarrow 3-6x^2-4x^2+4x\cdot(-2x)=0\Leftrightarrow$

$\Leftrightarrow 3-10x^2-8x^2=0\Leftrightarrow$

$\Leftrightarrow 3-18x^2=0\ \Big|\cdot\frac{1}{3}\Leftrightarrow$

$\Leftrightarrow 1-6x^2=0\Leftrightarrow$

$\Leftrightarrow 6x^2=1\Leftrightarrow x^2=\frac{1}{6}\Leftrightarrow x=\pm\frac{1}{\sqrt6}\Rightarrow y=\mp\frac{2}{\sqrt6}$

$\Rightarrow\left(\frac{1}{\sqrt6},-\frac{2}{\sqrt6}\right)$ and $\left(-\frac{1}{\sqrt6},\frac{2}{\sqrt6}\right)$ are critical points of f

(b) $y=x\Rightarrow 3-6x^2-x^2+4x^2=0\Leftrightarrow$

$\Leftrightarrow 3-3x^2=0\ \Big|\cdot\frac{1}{3}\Rightarrow x=\pm1\Rightarrow y=\pm1$

$\Rightarrow(1,1)$ and $(-1,-1)$ are critical points of f

$\boxed{\text{Case III}}$ $y\neq0\Rightarrow 2\left(\frac{x}{y}\right)^2-\frac{x}{y}-1=0$; $u=\frac{x}{y}\Rightarrow$

$\Rightarrow 2u^2-u-1=0$; $\Delta=9\Rightarrow u_1=1$, $u_2=\frac{1-3}{4}=-\frac{1}{2}$

(a) $\frac{x}{y}=1\Rightarrow x=y\Rightarrow\boxed{\text{Case II}}$ (b)

(b) $\frac{x}{y}=-\frac{1}{2}\Rightarrow y=-2x\Rightarrow\boxed{\text{Case II}}$ (a)

In conclusion, the function f has 4 critical points and the set of the critical points is $\mathcal{C} = \left\{ \left(\frac{1}{\sqrt6}, -\frac{2}{\sqrt6}\right), \left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}\right), (1,1), (-1,1) \right\}$;

Let $(x,y) \in \mathbb{R}^2$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -12x + 4y \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -2y + 4x$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = -2x + 6y \qquad \frac{\partial^2 f}{\partial y \partial x}(x,y) = -2x + 4x$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \left(f'_y\right)'_x$$

$$H_f(x,y) = \quad -12$$

? The diagonal of the Hessian matrix $H_f(x,y)$ contains the 2^{nd} order partial derivatives with respect to the same variable

$$\frac{\partial^2 f}{\partial x^2} \quad \frac{\partial^2 f}{\partial y^2} \quad \frac{\partial^2 f}{\partial z^2} \quad \dots$$

Step III. We analize each critical point, using the matrix

$$H_f\left(\frac{1}{\sqrt6}, -\frac{2}{\sqrt6}\right) = -2\begin{pmatrix} \frac{6}{\sqrt6} + \frac{4}{\sqrt6} & -\frac{4}{\sqrt6} \\ -\frac{4}{\sqrt6} & \frac{1}{\sqrt6} + \frac{6}{\sqrt6} \end{pmatrix} =$$

$$= -2\begin{pmatrix} \frac{10}{\sqrt6} & -\frac{4}{\sqrt6} \\ -\frac{4}{\sqrt6} & \frac{7}{\sqrt6} \end{pmatrix}$$

$$\Delta_1 = -\frac{20}{\sqrt6} < 0$$

$$\Delta_2 = -2\cdot\left(\frac{70}{6} - \frac{16}{6}\right) = -2\cdot\frac{54}{6} = -\frac{54}{3} = -18 < 0$$

?

$$\partial^2 f\left(-\frac{1}{\sqrt2}, \frac{2}{\sqrt6}\right)(h_1, h_2) = \frac{-20}{\sqrt6}h_1^2 + \frac{8}{\sqrt6}h_1h_2 + \frac{8}{\sqrt6}h_2h_1 - \frac{14}{\sqrt6}h_2^2 =$$

$$= \frac{-20}{\sqrt6}h_1^2 + \frac{16}{\sqrt2}h_1h_2 - \frac{14}{\sqrt2}h_2^2$$

$$-\frac{2}{\sqrt6}\left(10h_1^2 - 8h_1h_2 + 7h_2^2\right)$$

$$h_1 = 0 \Rightarrow -\frac{2}{\sqrt6}\cdot 7h_2^2 < 0$$

– fă rost –

$g: \mathbb{R}^2 \to \mathbb{R}^3$

$g(x,y) = (\cos x + \sin y, \sin x + \cos y, e^{x-y})$

$g = (g_1, g_2, g_3)$

$g_1 : \mathbb{R}^2 \to \mathbb{R} \quad g_1(x,y) = \cos x + \sin y$

$g_2 : \mathbb{R}^2 \to \mathbb{R} \quad g_2(x,y) = \sin x + \cos y$

$g_3 : \mathbb{R}^2 \to \mathbb{R} \quad g_3(x,y) = e^{x-y}$

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial (f \circ g)}{\partial x}(x,y) = \frac{\partial f}{\partial g_1}(g(x,y)) \cdot \frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial g_2}(g(x,y)) \cdot \frac{\partial g_2}{\partial x}(x,y)$$

$$+ \frac{\partial f}{\partial g_3}(g(x,y)) \frac{\partial g_3}{\partial x}(x,y) =$$

$$= \frac{\partial f}{\partial g_1}(g(x,y)) \cdot (-\sin x) + \frac{\partial f}{\partial g_2}(g(x,y)) \cdot \cos x + \frac{\partial f}{\partial g_3}(g(x,y)) \cdot e^{x-y}$$

$$\frac{\partial F}{\partial y}(x,y) = \frac{\partial (f \circ g)}{\partial y}(x,y) = \frac{\partial f}{\partial g_1}(g(x,y)) \cdot \frac{\partial g_1}{\partial y}(x,y) + \frac{\partial f}{\partial g_2}(g(x,y)) \cdot \frac{\partial g_2}{\partial y}(x,y) +$$

$$+ \frac{\partial f}{\partial g_3}(g(x,y)) \cdot \frac{\partial g_3}{\partial y}(x,y) =$$

$$= \frac{\partial f}{\partial g_1}(\cos x + \sin x, \sin x + \cos y, e^{x-y}) \cdot \cos y + \frac{\partial f}{\partial g_2}(g(x,y)) \cdot (-\sin y) +$$

$$+ \frac{\partial f}{\partial g_2}(g(x,y)) \cdot e^{x-y} \cdot (-1)$$

The hypotheses that f is C_1 on $\mathbb{R}^3$ means that all first ordered partial derivatives are continuous on $\mathbb{R}^3$. More expicitly $\frac{\partial f}{\partial g_1}, \frac{\partial f}{\partial g_2}, \frac{\partial f}{\partial g_3}$ are continuous (they are functions)

Taking this into account $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are obtained by applying elementary operations on continuous functions $\Rightarrow$ they are continuous on $\mathbb{R}^2$ ~~and~~

6. From the hyp. that f is differentiable on $\mathbb{R}^3$ and we noticed that g is differentiable $\Rightarrow f\circ g$ is diff. as well and

$$d(f\circ g)(a) = d(f(g(a)) \cdot dg(a)$$

$$J(f\circ g)(a) = J(f)(g(a)) \cdot J(g)(a)$$

Recall that

$A \subseteq \mathbb{R}^m$
$a \in \operatorname{int} A$
$f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$
$j \in \{1, \dots, m\}$

$$\Rightarrow \frac{\partial f}{\partial x_j}(a) = \left(\frac{\partial f_1}{\partial x_j}(a), \dots, \frac{\partial f_m}{\partial x_j}(a)\right)$$

$$J(f)(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_m}(a) \\ \frac{\partial f_2}{\partial x_2}(a) & \dots & \frac{\partial f_2}{\partial x_m}(a) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(a) & \dots & \frac{\partial f_m}{\partial x_m}(a) \end{pmatrix} = \begin{pmatrix} \partial f_1 \to \\ \partial f_2 \to \\ \dots \\ \partial f_m \to \end{pmatrix}$$

$$d f(a)(h) = J(f)(a)(h)$$

We know that the differential of a function may be expressed ~~by~~ with

So In order to det $d(F)$, we study it's Jacoby matrix.

$$J(F)\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) = J(f\circ g)\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) =$$

Acording to the formula to Jacoby matrix for the com. function

$$= J(f)\left(g\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)\right) \cdot J(g)\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

$\to$ the Jacoby matrix of f at $g\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\to$ the Jacoby matrix of f at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$g\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) = (1, 1, 1)$$

$$= J(f)(1,1,1) \cdot J(g)\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 5 \end{pmatrix} \cdot J(g)\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

We determine the J matrix at $\left(g\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$ and ~~$g\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$~~

~~$J(g)\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$~~

$$J(g)\left(\frac{\pi}{2},\frac{\pi}{2}\right)=\begin{pmatrix} \frac{\partial g_1}{\partial x}\left(\frac{\pi}{2},\frac{\pi}{2}\right) & \frac{\partial g_1}{\partial y}\left(\frac{\pi}{2},\frac{\pi}{2}\right) \\ \frac{\partial g_2}{\partial x}\left(\frac{\pi}{2},\frac{\pi}{2}\right) & \frac{\partial g_2}{\partial y}\left(\frac{\pi}{2},\frac{\pi}{2}\right) \\ \frac{\partial g_3}{\partial x}\left(\frac{\pi}{2},\frac{\pi}{2}\right) & \frac{\partial g_3}{\partial y}\left(\frac{\pi}{2},\frac{\pi}{2}\right) \end{pmatrix}$$

$$J(F)\left(\frac{\pi}{2},\frac{\pi}{2}\right)=\begin{pmatrix} 1 & 0 & 1 \\ -3 & 2 & 5 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 8 & -7 \end{pmatrix}$$

$$dF\left(\frac{\pi}{2},\frac{\pi}{2}\right)(h_1,h_2)=\begin{pmatrix} 0 & -1 \\ 8 & -7 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}=\begin{pmatrix} -h_2 \\ 8h_1-7h_2 \end{pmatrix}=(-h_2, 8h_1-7h_2)$$

Ex 2 Let $A=\{(x,y)\in\mathbb{R}^2 \mid |x|\le y\}$

$f: A\to\mathbb{R}$ $\quad f(x,y)=5+4x+3y-2x^2-y^2$

Determine $\min f$

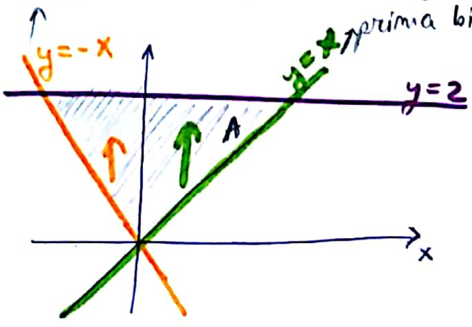
$\max f$

$$|x|=\begin{cases} x & : x\ge 0 \\ -x & : x<0 \end{cases}$$

$$A=\{(x,y)\in\mathbb{R}^2 : \begin{array}{ll} x\le y & \text{if } x\ge 0 \\ -x\le y & \text{if } x<0 \end{array}$$

a doua bisectoare

prima bisectoare



$\left.\begin{array}{l} A \text{ compact (bounded and closed)} \\ f \text{ is continuous} \end{array}\right\} \Rightarrow$

$\xrightarrow{\text{Weierstrass}}$ $\exists\, m=\min_{x\in A} f(x)$

$\exists\, M=\max_{x\in A} f(x)$

$m=\min\{m_1, m_2\}$

$M=\max\{M_1, M_2\}$

$m_1 \to$ min on int A

$m_2 \to$ min on bd A

$\text{bd}\,A = OC\cup OB\cup BC$ $\quad OC=\{(x,y)\in\mathbb{R}^2 : x=y,\ x\in[0,2]\}=\{(x,x): x\in[0,2]\}$

$OB=\{(x,y)\in\mathbb{R}^2 : y=-x,\ y\in[0,2]\}=\{(-y,y): y\in[0,2]\}$

$BC=\{(x,y): y=2,\ x\in[-2,2]\}=\{(x,2): x\in[-2,2]\}$

SEMINAR 10 week 10

07.05.2018

Ex 1

$$\int_0^{\pi-0} \frac{\cos x}{2+\cos x}\,dx = \int_0^{\infty} \frac{\frac{1-t^2}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt =$$

$$= \int_0^{\infty} \frac{\frac{1-t^2}{1+t^2}}{\frac{2(1+t^2)+1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt =$$

$$= \int_0^{\infty} \frac{2(1-t^2)}{(3+t^2)(1+t^2)}\,dt$$

We split the integral in two simple fractions.

$$\frac{2-2t^2}{(3+t^2)(1+t^2)} = \frac{\overset{1+t^2}{At+B}}{3+t^2} + \frac{\overset{3+t^2}{Ct+D}}{1+t^2} = \frac{(At+B)(1+t^2)+(Ct+D)(3+t^2)}{(3+t^2)(1+t^2)}$$

$$At+At^3+B+Bt^2+3Ct+Ct^3+3D+Dt^2 = 2-2t^2$$

$$t^3(A+C)+t^2(B+D)+t(A+3C)+B+3D = 2-2t^2$$

$$\begin{cases} A+C=0 \Rightarrow A=-C \Rightarrow A=0 \\ B+D=-2 \Rightarrow B=-2-D \\ A+3C=0 \Rightarrow -C+3C=0 \Leftrightarrow 2C=0 \Rightarrow C=0 \\ B+3D=2 \Rightarrow -2-D+3D=2 \end{cases}$$

$$-2+2D=2$$

$$2D=4 \Rightarrow D=2 \Rightarrow B=-2-2=-4$$

$$\Rightarrow \left.\begin{array}{l} A=0 \\ B=-4 \\ C=0 \\ D=2 \end{array}\right\} \Rightarrow \frac{-4}{3+t^2}+\frac{2}{1+t^2}$$

$$\int_0^{\infty} \frac{-4}{3+t^2}dt + \int_0^{\infty}\frac{2}{1+t^2}dt = -4\int_0^{\infty}\frac{1}{3+t^2}dt + 2\int \frac{1}{1+t^2}dt =$$

$$= -4\frac{1}{\sqrt3}\operatorname{arctg}\frac{t}{\sqrt3}\Big|_0^{\infty} + 2\operatorname{arctg} t\Big|_0^{\infty} =$$

$$= -4\cdot\frac{1}{\sqrt3}\left(\frac{\pi}{2}-0\right)+2\frac{\pi}{2} =$$

$$= \frac{-2\pi}{\sqrt3}+\pi$$

Integrable functions

RECALL

Recall that for

$$\operatorname{tg}\frac{x}{2}=t \Leftrightarrow \frac{x}{2}=\operatorname{arctg} t \Leftrightarrow x=2\operatorname{arctg} t \Leftrightarrow dx=\frac{2}{1+t^2}dt$$

$$\cos x = \frac{1-t^2}{1+t^2} \qquad \sin x = \frac{2t}{1+t^2}$$

Short cuts:

- $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ $\quad \cos x = t$
- $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ $\quad \sin x = t$
- $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

$$\begin{cases} tg\, x = u \\ \cos^2 x = \frac{1}{1+u^2} \\ \sin^2 x = \frac{u^2}{1+u^2} \end{cases} \qquad dx = \frac{1}{1+u^2}\,du$$

- $\int R(x, \sqrt{x^2+a^2})\,dx$ $\quad x = a\, tg\, t$ or $a\, ctg\, t$
- $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x+\sqrt{x^2+a^2}) + C$

Sometimes, for $\int_a^b f(x)\,dx$ we make $x = b+a-t$

Short CUTS

$\boxed{\text{Ex } 2}$ $\int_0^{\frac{\pi}{2}-0} \frac{2\cos x - 1}{1+\cos^2 x}\,dx = \int_0^{\infty} \frac{2\left(\frac{1}{1+u^2}\right)-1}{1+\left(\frac{1}{1+u^2}\right)} \cdot \left(\frac{1}{1+u^2}\right)du$

$=$

$$= \int_0^{\infty} \frac{2-1-u^2}{2+u^2} \cdot \frac{1}{1+u^2}\,du = \int_0^{\infty} \frac{1-u^2}{2+u^2} \cdot \frac{1}{1+u^2}\,du =$$

$$= \int_0^{\infty} \frac{1-u^2}{(2+u^2)(1+u^2)}\,du$$

P.S.: pt. polinoame cu $\Delta < 0$ facem astfel de împărțire a fracției. ex.: I. $2+u^2 = 0$, $\Delta < 0$ ✓ II. $1+u^2 = 0$, $\Delta < 0$ ✓

$$\frac{1-u^2}{(2+u^2)(1+u^2)} = \frac{\overset{1+u^2}{Au+B}}{2+u^2} + \frac{\overset{2+u^2}{Cu+D}}{1+u^2} \Rightarrow$$

$\Rightarrow Au + Au^3 + B + Bu^2 + 2Cu + Cu^3 + 2D + Du^2 = 1-u^2$

$u^3(A+C) + u^2(B+D) + u(A+2C) + B + 2D = 1-u^2$

$$\begin{cases} A+C = 0 \Rightarrow A = C = 0 \\ B+D = -1 \Rightarrow B = -1-D \\ A+2C = 0 \\ B+2D = 1 \Rightarrow -1-D+2D = 1 \end{cases}$$

$-1+D = 1$

$D = 2$

$B = -1-2 = -3$

$B = -3$

$$\int_0^{\infty} \frac{-3}{2+u^2}\,du + \int_0^{\infty} \frac{2}{1+u^2}\,du = -3\int_0^{\infty} \frac{1}{2+u^2}\,du + 2\int_0^{\infty} \frac{1}{1+u^2}\,du =$$

$$= -3\left(\frac{1}{\sqrt{2}} \operatorname{arctg} \frac{u}{\sqrt{2}} \Big|_0^{\infty}\right) + 2 \operatorname{arctg} u \Big|_0^{\infty} = \left(-\frac{3}{\sqrt{2}} + 2\right)\frac{\pi}{2}$$

Ex 3 $\int_0^1 \frac{1}{x^2+3}\cdot\frac{1}{\sqrt{x^2+3}}\,dx =$

We make the substitution that $x=\sqrt{3}\cdot tg\, t$

$dx = \frac{\sqrt{3}}{1+t^2}\,dt$

$x=0 \Rightarrow tg\, t = 0 \Rightarrow t=0$

$x=1 \Rightarrow \sqrt{3}\, tg\, t = 1 \Rightarrow tg\, t = \frac{1}{\sqrt{3}} \Rightarrow t=\frac{\pi}{6}$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{3\cdot tg^2 t+3}\cdot\frac{1}{\sqrt{3\, tg^2 t+3}}\cdot\frac{\sqrt{3}}{1+t^2}\,dt =$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{3\,\frac{\sin^2 t}{\cos^2 t}+3}\cdot\frac{1}{\sqrt{3\cdot\frac{\sin^2 t}{\cos^2 t}+3}}\cdot\frac{\sqrt{3}}{1+t^2}\,dt =$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos^2 t}{3}\cdot\frac{\sqrt{\cos^2 t}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\cos^2 t}\,dt$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos t}{3}\,dt = \frac{1}{3}\sin t\,\Big|_0^{\frac{\pi}{6}} = \frac{1}{3}\left(\frac{1}{2}-0\right) = \frac{1}{6}$$

Ex 4 $\int_0^{\pi} \frac{x\cdot\sin x}{1+\cos^2 x}\,dx =$

$x = \pi + 0 - t = \pi - t$

$\sin(\pi - t) = \sin\pi\cos t - \sin t\cos\pi =$

$= \sin t$

$\cos(\pi - t) = \cos\pi\cdot\cos t + \sin\pi\cdot\sin t =$

$= -\cos t$

$dx = -dt$

$x=0 \Rightarrow t=\pi$

$x=\pi \Rightarrow t=0$

$$= \int_{\pi}^{0} \frac{(\pi - t)\cdot\sin t}{1+\cos^2 t}\,dt = \int_0^{\pi}\frac{\pi\sin t}{1+\cos^2 t}\,dt - \underbrace{\int_0^{\pi}\frac{t\sin t}{1+\cos^2 t}\,dt}_{I}$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi}\frac{\sin t}{1+\cos^2 t}\,dt = -\frac{\pi}{2}\int_0^{\pi}\frac{(\cos t)'}{1-\cos^2 t}\,dt =$$

$$= -\frac{\pi}{2}\, arctg(\cos t)\,\Big|_0^{\pi} = -\frac{\pi}{2}\Big[arctg(-1) - arctg(\underset{1}{\underbrace{\cos 0}})\Big] =$$

$$= -\frac{\pi}{2}(-2)\cdot arctg\, 1 = \pi\, arctg\, 1 = \frac{\pi^2}{4}$$

REMARK: we notice that $R(+\sin t, \cos t) = R(-\sin t, \cos t) = -R(\sin t, \cos t)$
We could make the substitution

MULTIPLE INTEGRALS

1) Integration over nondegenerate compact intervals in $\mathbb{R}^n$

TIPS:

nedegenerat = nu se reduce la un singur punct

$\mathbb{R}^2$: înmulțește ce avem pe Ox și Oy și obținem un interval (dreptunghi)

$\mathbb{R}^3$: corpuri (paralelipipede)

$\mathbb{R}^4$: translația unui corp în timp

n.c.i. in $\mathbb{R}^n$ = Produse de intervale închise (putem face pt. suprafețe variate, nu doar pt. corpuri cu muchii drepte)

Ex 5 $I = \iint_A (2x+3y)\,dx\,dy$, where $A = [0,2] \times [0,3]$

According to a theorem in the lecture, the order of the integration can be chosen arbitrarily. Sometimes it may happen that a certain order is more convenient than the other.

$$I = \int_0^2 \int_0^3 (2x+3y)\,dx\,dy$$

For this particular case, there is no obvious reason for starting with one or another.

If we choose to start with x, we have:

$$I = \int_0^3 \Bigg(\underbrace{\int_0^2 (2x+3y)\,dx}_{I} \Bigg) dy$$

$$I = \int_0^2 (2x+3y)\,dx = x^2\Big|_0^2 + 3y\int_0^2 dx = x^2\Big|_0^2 + 3y\,x\Big|_0^2 =$$

$$= 4 + 3y \cdot x\Big|_0^2 =$$

$$= 4 + 6y$$

$$I = \int_0^3 (4+6y)\,dy = \left(4y + 6\frac{y^2}{2}\right)\Big|_0^3 = (4y+3y^2)\Big|_0^3 = 4\cdot 3 + 3\cdot 9 = 12 + 27 = 39.$$

Theory:

If $f: \mathbb{R}^n \to \mathbb{R}$ is such that $\forall (x_1, \dots, x_n) \in \mathbb{R}^n$

$f(x_1, \dots, x_n) = g_1(x_1) \cdot g_2(x_2) \cdot \dots \cdot g_n(x_n)$

where $g_i: \mathbb{R} \to \mathbb{R}$ are real functions than f is said to be with separable variables.

$$f(x,y,z) = g(x) \cdot h(y) \cdot u(z)$$

In this case

$$\underbrace{\int \dots \int}_{n\times} f(x_1, \dots, x_n)\,dx_1 \dots dx_n = \int g_1(x_1)\,dx_1 \cdot \dots \cdot \int g_n(x_n)\,dx_n$$

$$\iiint f(x,y,z)\,dx\,dy\,dz = \int g(x)\,dx \cdot \int h(y)\,dy \cdot \int u(z)\,dz$$

(în majoritatea cazurilor avem integrală în fct. de 3 variabile)

Ex 6 $\iiint_A \frac{xy^2}{1+z^2}\,dx\,dy\,dz \qquad A=[0,a]\times[0,b]\times[0,c]$

$a, b, c > 0$

!REMARK:
We notice that this time the function has separable variables and this is why:

$$\int_0^a x\,dx \cdot \int_0^b y^2\,dy \cdot \int_0^c \frac{1}{1+z^2}\,dz =$$

$$= \frac{x^2}{2}\Big|_0^a \cdot \frac{y^3}{3}\Big|_0^b \operatorname{arctg} z\Big|_0^c = \frac{a^2}{2}\cdot\frac{b^3}{3}\cdot\operatorname{arctg} c$$

Ex 7 $\iiint_A \frac{2z}{(y+x)^2}\,dx\,dy\,dz$, where $A=[1,2]\times[2,3]\times[0,2]$

$$I=\int_0^2 2z\,dz \int_1^2\int_2^3 \frac{1}{(x+y)^2}\,dx\,dy = z^2\Big|_0^2 \cdot \int_1^2\left(\int_2^3 \frac{1}{(x+y)^2}\,dy\right)dx$$

RECALL THAT: $\int u^n(x)\,u'(x)\,dx = \frac{u^{(n+1)}(x)}{n+1} + C$

$$\int_2^3 (x+y)^{-2}\,dy = \int_2^3 (x+y)^{-2}(x+y)'_y\,dy = \frac{(x+y)^{-1}}{-1}\Big|_2^3 =$$

$$= -\left(\frac{1}{x+3} - \frac{1}{x+2}\right)$$

$$I = 4\int_1^2\left(\frac{1}{x+2} - \frac{1}{x+1}\right)dx = 4\cdot\left(\ln(x+2)\right)\Big|_1^2 - \ln(x+3)\Big|_1^2 =$$

$$= 4\left(\ln(4)-\ln(3)-\ln(5)+\ln(4)\right) = 4\left(\ln\frac{16}{15}\right)$$

This was a case with just one separable variable

Ex 8 $I=\int_0^1\int_0^1 \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}}\,dx\,dy$

Just in this case is easy to start integrating ~~with dy~~ with respect to y.

We notice that $(1+x^2+y^2)'_y = 2y$

Then $\int_0^1 \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}}\,dy = \int_0^1 \frac{(1+x^2+y^2)'_y}{2}\cdot(1+x^2+y)^{-\frac{3}{2}}\,dy =$

$$= \frac{\frac{1}{2}(1+x^2+y^2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\Bigg|_0^1 = -(1+x^2+y^2)^{-\frac{1}{2}}\Big|_0^1 =$$

$$= -(1+x^2+1)^{-\frac{1}{2}} + (1+x^2)^{-\cancel{t}-\frac{3}{2}+1} = -(2+x^2)^{-\frac{1}{2}} + (1+x^2)^{-\frac{1}{2}} =$$

$$= -\frac{1}{\sqrt{2+x^2}} + \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{2+x^2}}$$

$$I = \int_0^1 \frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{2+x^2}}\, dx = \ln\left(x+\sqrt{1+x^2}\right)\Big|_0^1 - \ln\left(x+\sqrt{2+x^2}\right)\Big|_0^1 =$$

$$= \ln(1+\sqrt{2}) - \ln(1+\sqrt{3}) + \ln 2 = \ln\left(\frac{1+\sqrt{2}}{1+\sqrt{3}}\right) + \ln\sqrt{2}$$

$$= \ln\left(\frac{\sqrt{2}+2}{1+\sqrt{3}}\right)$$

$\frac{\pi^2}{4}$ (from $\boxed{\text{Ex. 4}}$)

$\boxed{\text{Ex 9}}$ $$I = \int_0^{\pi}\int_0^{\frac{\pi}{2}} \frac{x \sin x}{(2+\cos^2 x)(2+\cos^2 y)}\, dx\, dy = \underbrace{\int_0^{\pi} \frac{x \sin x}{(2+\cos^2 x)}\, dx}_{=\frac{\pi^2}{4}} \int_0^{\frac{\pi}{2}} \frac{1}{2+\cos^2 y}\, dy =$$

$$= \frac{\pi^2}{4} \int_0^{\frac{\pi}{2}} \frac{1}{2+\cos^2 y}\, dy$$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{2+\cos^2 y}\, dy$$

$$R(\sin y, \cos y) = \frac{1}{y+\cos^2 y}$$

$$R(-\sin y, -\cos y) = \frac{1}{1+(-\cos y)^2} = R(\sin y, \cos y) \Rightarrow$$

$\Rightarrow$ substituția $\operatorname{tg} y = u \Rightarrow \cos^2 y = \frac{1}{1+u^2}$

$$dy = \frac{1}{1+u^2}\, du$$

când $y = 0 \Rightarrow u = 0$ $\Rightarrow$

$y = \frac{\pi}{2} \Rightarrow u = \infty$

$$\Rightarrow I = \int_0^{\infty} \frac{1}{2+\frac{1}{2+u^2}} \cdot \frac{1}{2+u^2}\, du = \int_0^{\infty} \frac{1}{2+u^2}\, du \Rightarrow$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{u}{\sqrt{2}}\Big|_0^{\infty} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2\sqrt{2}}$$

$\boxed{\text{Ex 10}}$ $$\int_{\frac{1}{a}}^{a}\int_0^1 \frac{1}{x^2+y^2}\, dx\, dy, \quad a>0$$

$$= \int_{\frac{1}{a}}^{a} \left(\frac{1}{x} \operatorname{arctg} \frac{y}{x}\right)\Big|_0^1 dx = \int_{\frac{1}{a}}^{a} \left(\frac{1}{x}\left(\operatorname{arctg} \frac{1}{x}\right) dx\right) = \int_{t}^{\frac{1}{t}} t \cdot \operatorname{arctg} t \left(-\frac{1}{t^2}\right) dt$$

$$\frac{1}{x} = t \Rightarrow x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\, dt$$

$$I = +\int_{\frac{1}{t}}^{t} \frac{\operatorname{arctg} t}{t}\, dt = \int_{\frac{1}{t}}^{t} \frac{\operatorname{arctg} t}{t}\, dt$$

$$\boxed{\operatorname{arctg} t + \operatorname{arctg} \frac{1}{t} = \frac{\pi}{2}}$$ **Property** ^^

$\Downarrow$

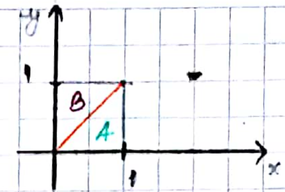
$$\operatorname{arctg} t = \frac{\pi}{2} - \operatorname{arctg} \frac{1}{t}$$

$$A = \int_{\frac{1}{t}}^{t} \frac{\pi}{2} t - A \Rightarrow A = \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{t^2}{2} \Big|_{\frac{1}{t}}^{t}$$

$$A = \frac{\pi}{8} \left(t^2 - \frac{1}{t^2}\right) = \frac{\pi}{8} \left(\left(\frac{1}{x}\right)^2 - \frac{1}{\left(\frac{1}{x}\right)^2}\right)$$

Subject: ANALIZĂ **Seminar 11** week 11 Date: 14 / 05 / 2018

1 $\int_0^1\int_0^1 \max(x,y)\,dx\,dy$



$$\max(x,y)=\begin{cases} x, & y\le x \\ y, & x<y \end{cases}$$

$$\begin{cases} x: & (x,y)\in A \\ y: & (x,y)\in B \end{cases}$$

$$\int_0^1\int_0^1 \max(x,y)\,dx\,dy=\int_A\int x\,d\,d+\iint_B y\,dx\,dy$$

Both Δ are normal domains w. respect to both Ox and Oy. (We have a normal domain when a second segment contained in a set parallel to one of the axis is unintrerupted.

$A=\{(x,y)\in[0,1]^2 \mid 0\le x\le 1,\ 0\le y\le x\}=\{(x,y)\in[0,1]^2 \mid 0\le y\le 1,\ y\le x\le 1\}$

$B=\{(x,y)\in[0,1]^2 \mid 0\le x\le 1,\ x\le y\le 1\}=\{(x,y)\in[0,1]^2 \mid 0\le x\le 1,\ x\le y\le 1\}$

↳ because in this case y depends to x, we start integrated with

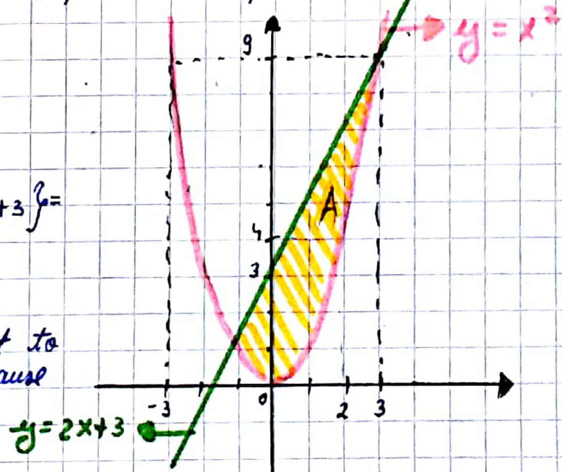
! Not all the sets are normal w.r.t. both axis.

$$=\int_0^1\int_0^x x\,dx\,dy+\int_0^1\int_x^1 y\,dx\,dy=\int_0^1\left(\int_0^x x\,dy\right)dx+\int_0^1\left(\int_x^1 y\,dy\right)dx=$$

$$=\int_0^1 x\cdot y\Big|_0^x dx+\int_0^1 \frac{y^2}{2}\Big|_x^1 dx=\int_0^1 x^2\,dx+\int_0^1\left(\frac{1}{2}-\frac{x^2}{2}\right)dx=\frac{x^3}{3}\Big|_0^1+\frac{1}{2}x\Big|_0^1-\frac{x^3}{\ }$$

(POZĂ TELEFON (am făcut la tablă))

2 $\iint_A \frac{1}{y+1}\,dx\,dy$, A = set bounded by the ~~parabola~~ parable $y=x^2$ and the line $y=2x+3$.



We find once again a normal domain w.r.t. Ox and Oy.

$A=\{(x,y)\in\mathbb{R}^2 \mid -\frac{3}{2}\le x\le 3,\ x^2\le y\le 2x+3\}=$

$=\{(x,y)\in\mathbb{R}^2 \mid 0\le y\le 9,\ \dots\}$

it is more dificult to consider the domain because of the x^2.

$$\iint_A \frac{1}{y+1}\,dx\,dy=\int_{-\frac{3}{2}}^{3}\int_{x^2}^{2x+3}\frac{1}{y+1}\,dx\,dy=\int_{-\frac{3}{2}}^{3}\left(\int_{x^2}^{2x+3}\frac{1}{y+1}\,dy\right)dx=$$

$$=\int_{-\frac{3}{2}}^{3}\left(\ln(y+1)\Big|_{x^2}^{2x+3}\right)dx=\int_{-\frac{3}{2}}^{3}\left[\ln(2x+4)-\ln(x^2+1)\right]dx$$

$$\int_{-\frac{3}{2}}^{3}\ln(2x+4)\,dx=\int_{-\frac{3}{2}}^{3}(x)'\cdot\ln(2x+4)\,dx=x\ln(2x+4)\Big|_{-\frac{3}{2}}^{3}-\int_{-\frac{3}{2}}^{3}x\cdot\frac{1}{2x+4}\cdot 2\,dx=$$

$$=3\ln 10-0-\int_{-\frac{3}{2}}^{3}\frac{x}{x+2}\,dx=3\ln 10-\int_{-\frac{3}{2}}^{3}dx+2\int_{-\frac{3}{2}}^{3}\frac{1}{x+2}\,dx=3\ln 10-3+\frac{3}{2}+2\ln(x+2)\Big|_{-\frac{3}{2}}^{3}=$$

$$\frac{x}{x+2}=\frac{x+2}{x+2}+\frac{-2}{x+2}$$

$$=\ln 1000-\frac{3}{2}+2\ln 5-2\ln\frac{1}{2}=$$

$$=-\frac{3}{2}+\ln 1000+\ln 25+4\ln 2=-\frac{3}{2}+\ln\dots$$

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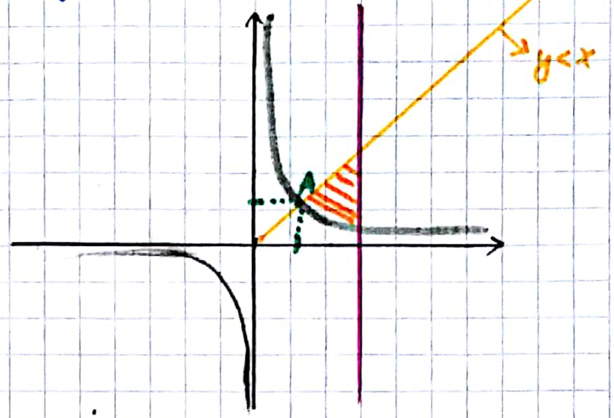
3 $\iint_A \frac{x}{y^2+2}\,dx\,dy$, A is the set bounded by the lines $x=\sqrt{3}$ and $y=x$ and the hyperbola $x\cdot y = 1$.



We are forced start integrated w.r.t. y because y depends to x.

$A=\{(x,y)\in\mathbb{R}^2 \mid 1\le x\le\sqrt{3},\ \frac{1}{x}\le y\le x\}$

$=\{(x,y)\in\mathbb{R}^2 \mid \frac{1}{\sqrt{3}}\le y\le\sqrt{3},\ y\le x\le\frac{1}{y}\}$

$x=\sqrt{3}\Rightarrow y=\frac{1}{3}$

$\operatorname{arctg} x + \operatorname{arcctg}\frac{1}{x}=\frac{\pi}{2}$

$\int_1^{\sqrt{3}}\int_{\frac{1}{x}}^{x}\frac{x}{1+y^2}\,dx\,dy=\int_1^{3}(x\cdot\operatorname{arctg} y)\Big|_{\frac{1}{x}}^{x}=\int_1^{\sqrt{3}} x\,(\operatorname{arctg} x-\operatorname{arctg}\frac{1}{x})\,dx=$

$=\int_1^{\sqrt{3}} x(2\operatorname{arctg} x-\frac{\pi}{2})\,dx=\int_1^{\sqrt{3}}(x^2)'\operatorname{arctg} x\,dx-\frac{\pi}{4}x^2\Big|_1^{\sqrt{3}}=$

$=x^2\operatorname{arctg} x\Big|_1^{\sqrt{3}}-\int_1^{\sqrt{3}}x^2\cdot\frac{1}{1+x^2}\,dx-\frac{\pi}{2}=3\operatorname{arctg}\sqrt{3}-\operatorname{arctg} 1-\left(\int_1^{\sqrt{3}}\frac{x^2+1}{x^2+1}\,dx-\int_1^{\sqrt{3}}\frac{1}{x^2+1}\,dx\right)-\frac{\pi}{2}$

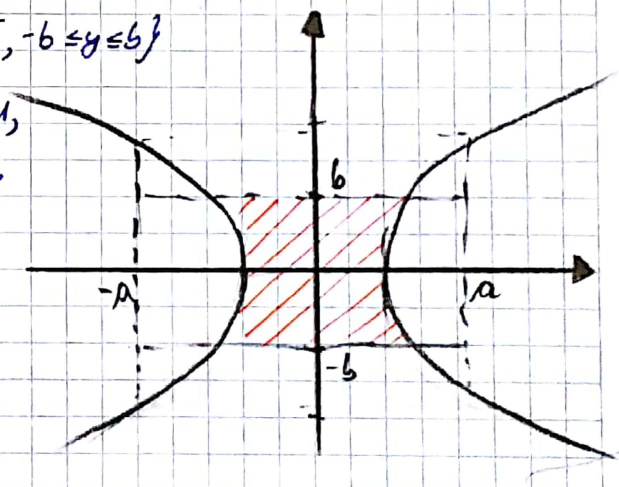
$=\frac{\pi}{2}-\frac{\pi}{4}-\sqrt{3}+1+\operatorname{arctg}\sqrt{3}-\operatorname{arctg}\sqrt{3}-\operatorname{arctg} 1-\frac{\pi}{2}=$

$=\frac{\pi}{4}-\sqrt{3}+1+\frac{\pi}{6}-\frac{\pi}{4}-\frac{\pi}{2}=\frac{\pi}{6}-\sqrt{3}+1-\frac{\pi}{2}=1-\sqrt{3}-\frac{\pi}{3}$

5 $A=\{(x,y)\in\mathbb{R}^2: -\frac{a}{b}\sqrt{b^2+y^2}\le x\le\frac{a}{b}\sqrt{b^2+y^2},\ -b\le y\le b\}$

$\iint_A\frac{x^2}{y^2+b^2}\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2: \frac{x^2}{a^2}-\frac{y^2}{b^2}\le 1,\ -b\le y\le b\}$

$a\,\&\,b>0$



$I=\int_{-b}^{b}\left(\int_{-\frac{a}{b}\sqrt{b^2+y^2}}^{\frac{a}{b}\sqrt{b^2+y^2}}\frac{x^2}{y^2+b^2}\,dx\right)dy=$

$=\int_{-b}^{b}\frac{1}{y^2+b^2}\cdot\frac{x^3}{3}\Big|_{-\frac{a}{b}\sqrt{b^2+y^2}}^{\frac{a}{b}\sqrt{b^2+y^2}}\,dy=$

$=\int_{-b}^{b}\frac{1}{y^2+b^2}\cdot\frac{2}{3}\cdot\frac{a^3}{b^3}(b^2+y^2)\sqrt{b^2+y^2}\,dy=\frac{2}{3}\cdot\frac{a^3}{b^3}\int_{-b}^{b}\sqrt{b^2+y^2}\,dy$

$I_1=\int_{-b}^{b}\sqrt{b^2+y^2}\,dy=\int_{-b}^{b}\frac{b^2+y^2}{\sqrt{b^2+y^2}}\,dy=b^2\int_{-b}^{b}\frac{1}{\sqrt{b^2+y^2}}\,dy+\int_{-b}^{b}\frac{y^2}{\sqrt{b^2+y^2}}\,dy=$

$=b^2\left(\ln(y+\sqrt{y^2+b^2})\right)\Big|_{-b}^{b}+\int_{-b}^{b}(\sqrt{b^2+y^2})'\,y\,dy=$

$=b^2\left[\ln(b+b\sqrt{2})-\ln(-b+b\sqrt{2})\right]+y\sqrt{b^2+y^2}\Big|_{-b}^{b}-\underbrace{\int_{-b}^{b}\sqrt{b^2+y^2}\,dy}_{I_1}$

$I_1=\frac{1}{2}b^2\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)+\frac{1}{2}(2b^2\sqrt{2})=\frac{1}{2}b^2\ln(\sqrt{2}+1)^2+b^2\sqrt{2}=b^2(\ln\sqrt{2}+1+\sqrt{2})$

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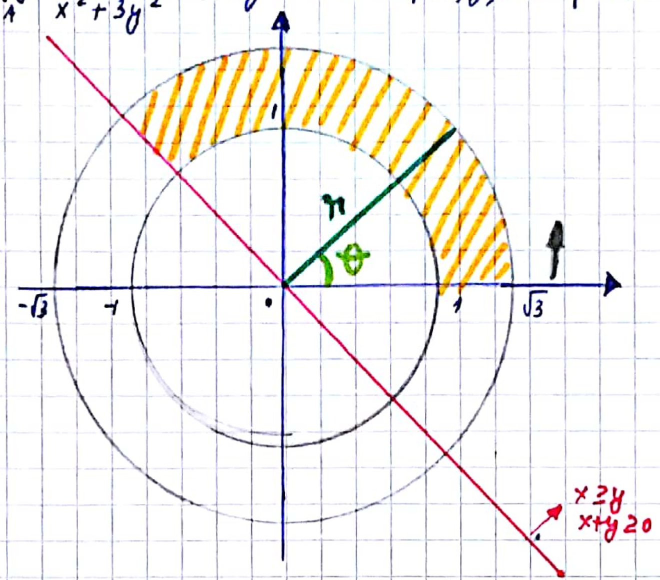
$$I_2 = \frac{1}{2} b^2 \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \frac{1}{2}\left(2b^2\sqrt{2}\right) = \frac{1}{2} b^2 \ln(\sqrt{2}-1)^2 + b^2\sqrt{2} =$$

$$= b^2\left[\ln(\sqrt{2}+1)+\sqrt{2}\right]$$

$$I = \frac{2}{3}\cdot\frac{a^3}{b^3}\cdot b^2\left[\ln(\sqrt{2}+1)+\sqrt{2}\right] = \frac{2}{3}\cdot\frac{a^3}{b}\left[\ln(\sqrt{2}+1)+\sqrt{2}\right].$$

5 $\iint_A \frac{x^2}{x^2+3y^2}\,dx\,dy \qquad A = \{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2 \le 3,\ x+y\ge 0,\ y\ge 0\}$



$x+y=0$

$x=0 \Rightarrow y=0$

$y=1 \Rightarrow x=-1$

$\left.\begin{matrix} y=0 \\ x\in\mathbb{R}\end{matrix}\right\} \rightarrow Ox$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases},\ \theta\in[0,2\pi],\ r\ge 0$

After changing the variables we replace them in all the position of the given set and take care NOT to exced $r(\sin\theta - \cos\theta)\ge 0$

$1 \le r^2\cdot 1 \le 3 \underset{r>0}{\iff} 1\le r\le\sqrt{3}$

$\begin{cases} r(\sin\theta+\cos\theta)\ge 0 \\ r\sin\theta \ge 0\end{cases} \iff \begin{cases}\cos\theta \ge -\sin\theta \\ \sin\theta\ge 0\end{cases} \iff 1+tg\frac{\theta}{2} - tg^2\frac{\theta}{2}\ge 0$

$\downarrow$

$\theta\in[0,\pi]$

$tg\frac{\theta}{2} = u \Rightarrow$

$\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} = \cos x \qquad \sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}$

$\Rightarrow 1+2u-u^2\ge 0;\ \Delta = 4+4 = 8$

$u = \frac{-2\pm\sqrt{8}}{-2} = 1\pm\sqrt{2} \Rightarrow$

$\Rightarrow u\in(1-\sqrt{2}, 1+\sqrt{2})$

$tg\frac{\theta}{2}\in(1-\sqrt{2}, 1+\sqrt{2})$

<u>Alternative</u>: $\theta\in[0,\pi]$

I. $\cos\theta > 0 \Rightarrow \theta\in\left[0,\frac{\pi}{2}\right]$

$1\ge -tg\,\theta \Rightarrow tg\,\theta \ge -1 \Rightarrow \theta \ge -\frac{\pi}{4}$

II. $\cos\theta < 0 \Rightarrow \theta\in\left(\frac{\pi}{2},\pi\right)$

$\cancel{1\ge tg\,\theta}\quad 1\le -tg\,\theta$

$\iff tg\,\theta \le -1 \Rightarrow \theta \le -\frac{\pi}{4} \Rightarrow \theta\le\frac{3\pi}{4}$

$$\int_1^{\sqrt{3}}\int_0^{\frac{3\pi}{4}} \frac{r^2\cos^2\theta}{r^2\cos^2\theta + 3r^2\sin^2\theta}\cdot r\cdot dr\,d\theta =$$

$$= \int_1^{\sqrt{3}} r\,dr\cdot\underbrace{\int_0^{\frac{3\pi}{4}}\frac{\cos^2\theta}{1+2\sin^2\theta}\,d\theta}_{I} = \frac{r^2}{2}\Big|_1^{\sqrt{3}}\cdot I$$

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$tg\,\theta = t \Rightarrow \cos^2\theta = \frac{1}{t^2+1}$ and $\sin^2\theta =$

Seminar 12 week 12 21.05.2018.

Polar coordinates

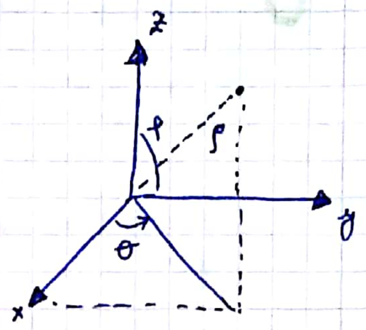
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \rho \geq 0,\ \theta \in [0, 2\pi] \qquad dx\,dy = \rho\,d\rho\,d\theta$$

Cylindric coordinates

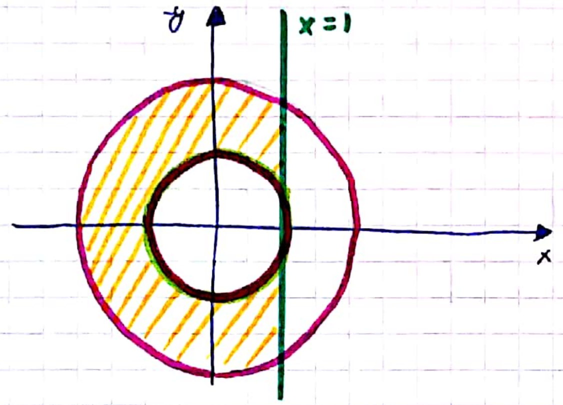
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \qquad \rho \geq 0,\ \theta \in [0, 2\pi] \qquad dx\,dy = \rho\,d\rho\,d\theta$$

Spherical coordinates

$$x = \rho\cos\theta\sin\varphi \qquad \rho \geq 0$$
$$y = \rho\sin\theta\sin\varphi \qquad \theta \in [0, 2\pi]$$
$$z = \rho\cos\varphi \qquad \varphi \in [0, 2\pi]$$



1) $\iint_A \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy \qquad A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2+y^2 \leq 4,\ x \leq 1\}$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \rho \geq 0,\ \theta \in [0, 2\pi] \qquad dx\,dy = \rho\,d\rho\,d\theta$$

$$\begin{cases} 1 \leq \rho^2\cos^2\theta + \rho^2\sin^2\theta \leq 4 \\ \rho\cos\theta \leq 1 \end{cases} \Leftrightarrow$$

We analyze the structure of the set A.

$$\Leftrightarrow \begin{cases} 1 \leq \rho^2 \leq 4 \\ \rho\cos\theta \leq 1 \end{cases} \Leftrightarrow \begin{cases} 1 \leq \rho \leq 2 \\ \rho\cos\theta \leq 1 \end{cases} \Leftrightarrow \begin{cases} 1 \leq \rho \leq 2 \\ \rho\cos\theta \leq 1 \end{cases} \Leftrightarrow \begin{cases} 1 \leq \rho \leq 2 \\ \cos\theta \leq \frac{1}{\rho} \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 \leq \rho \leq 2 \\ \rho \leq \frac{1}{\cos\theta} \end{cases} \Rightarrow 1 \leq \rho \leq \min\left\{2, \frac{1}{\cos\theta}\right\}$$

$$2 = \frac{1}{\cos\theta} \Leftrightarrow \cos\theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$$

$$\min\left\{2, \frac{1}{\cos\theta}\right\} = \begin{cases} \frac{1}{\cos\theta}, & \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ 2, & \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right] \cup \left\{\frac{5\pi}{3}\right\} \end{cases}$$

$$\int_0^{2\pi}\int_1^{\min\{2,\frac{1}{\cos\theta}\}} \frac{1}{\rho}\cdot\rho\,d\rho\,d\theta = \int_0^{2\pi}\int_1^{\min\{2,\frac{1}{\cos\theta}\}} d\rho\,d\theta = \int_0^{\frac{\pi}{3}}\left(\int_1^{\frac{1}{\cos\theta}} d\rho\right)d\theta + \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\left(\int_1^2 d\rho\right)d\theta +$$

$$+ \int_{\frac{5\pi}{3}}^{2\pi}\left(\int_1^{\frac{1}{\cos\theta}} d\rho\right)d\theta = \int_0^{\frac{\pi}{3}}\left(\frac{1}{\cos\theta} - 1\right)d\theta + \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} d\theta + \int_{\frac{5\pi}{3}}^{2\pi}\left(\frac{1}{\cos\theta} - 1\right)d\theta =$$

$$= -\int_0^{\frac{\pi}{3}} d\theta + \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} d\theta - \int_{\frac{5\pi}{3}}^{2\pi} d\theta + \int_0^{\frac{\pi}{3}} \frac{1}{\cos\theta} + \int_{\frac{5\pi}{3}}^{2\pi} \frac{1}{\cos\theta}\,d\theta =$$

$= -\frac{\pi}{3} + \frac{4\pi}{3} - \frac{\pi}{3} + \ldots$ see next page.

$R(-\cos\theta \sin\theta) = -R(\cos\theta \sin\theta) \Rightarrow$ $\begin{cases} \sin\theta = t \\ \cos\theta \, d\theta = dt \\ \theta = 0 \Rightarrow t = 0 \\ \theta = \frac{\pi}{3} \Rightarrow t = \frac{\sqrt{3}}{2} \end{cases}$

$$\int_0^{\frac{\pi}{3}} \frac{1}{\cos\theta} d\theta = \int_0^{\frac{\pi}{3}} \frac{\cos\theta}{\cos^2\theta} d\theta =$$

$$= \int_0^{\frac{\pi}{3}} \frac{\cos\theta}{1-\sin^2\theta} d\theta = \int_0^{\frac{\sqrt{3}}{2}} \frac{dt}{1-t^2} = \frac{1}{2} \ln \left| \frac{1-t}{1+t} \right| \Big|_0^{\frac{\sqrt{3}}{2}}$$

② $\iint_A \sqrt{x^2+y^2}\, dx dy \qquad A = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 2ax, \ a \geq 0 \}$

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \theta \in [0, 2\pi], \ \rho \geq 0, \ dxdy = \rho \, d\rho \, d\theta$

$\begin{cases} \rho^2\cos^2\theta + \rho^2\sin^2\theta < 2a\rho\cos\theta \\ a > 0 \end{cases} \Leftrightarrow \begin{cases} \rho^2 < 2a\rho\cos\theta \\ a > 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \rho < 2a\cos\theta \\ a > 0 \\ \rho \geq 0 \end{cases} \Rightarrow \cos\theta \geq 0 \Rightarrow \theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$

$$I = \int_0^{2a\cos\theta} \Big/ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho \cdot \rho \, d\rho \Big/ d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2a\cos\theta} \rho^2 \, d\rho \right) d\theta =$$

Because ρ depends on θ, we start integrated w.r.t. θ.

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\rho^3}{3} \Big|_0^{2a\cos\theta} \right) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8a^3\cos^3\theta}{3} d\theta =$$

$$= \frac{8a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \, d\theta$$

$R(-\cos\theta, \sin\theta) = -R(\cos\theta, \sin\theta) \Rightarrow \sin\theta = t$

$\cos\theta \, d\theta = dt$

$\theta = -\frac{\pi}{2} \quad t = -1$

$\theta = \frac{\pi}{2} \quad t = 1$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \cos\theta \, d\theta =$$

$$= \int_{-1}^{1} (1-t^2) dt = \left(t - \frac{t^3}{3} \right) \Big|_{-1}^{1} = 2 - \frac{2}{3} = \frac{4}{3}$$

③ $\iint_A (x^2+y^2)\, dx dy \qquad A = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 2ax, \ x^2+y^2 \leq 2ay, \ a > 0 \}$

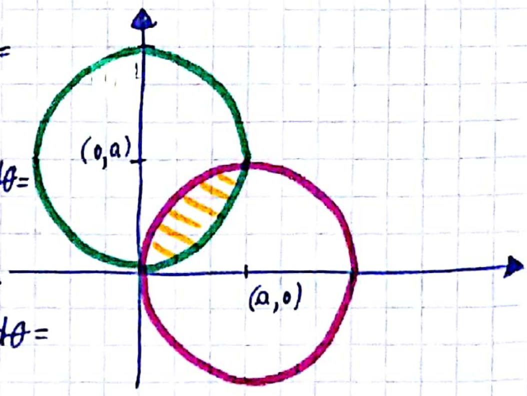
$\begin{cases} \rho^2\cos^2\theta + \rho^2\sin^2\theta \leq 2a\rho\cos\theta \\ \rho^2\cos^2\theta + \rho^2\sin^2\theta \leq 2a\rho\sin\theta \end{cases} \Leftrightarrow$ $\begin{cases} x = \rho\cos\theta & \theta \in [0, 2\pi] \\ y = \rho\sin\theta & \rho \geq 0 \\ dxdy = \rho \, d\rho \, d\theta \end{cases}$

$\Leftrightarrow \begin{cases} \rho^2 \leq 2a\rho\cos\theta \\ \rho^2 \leq 2a\rho\sin\theta \end{cases} \Big| \rho \Leftrightarrow \begin{cases} \rho \leq 2a\cos\theta \\ \rho \leq 2a\sin\theta \\ \rho \geq 0 \end{cases} \Leftrightarrow \begin{cases} 0 \leq \rho \leq 2a \min\{\cos\theta, \sin\theta\} \\ \cos\theta \geq 0 \\ \sin\theta \geq 0 \end{cases}$

$\Rightarrow \theta \in [0, \frac{\pi}{2}]$

$(x-a)^2+y^2 \le a^2$ discul de centru $(a,0)$ și rază a

$$\int_0^{\frac{\pi}{4}}\int_0^{2a\sin\theta} \rho^2\cdot\rho \, d\theta\, d\rho + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2a\cos\theta} \rho^2\cdot\rho\, d\theta\, d\rho =$$



$$= \int_0^{\frac{\pi}{4}}\left(\int_0^{2a\sin\theta} \rho^3\, d\rho\right) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^{2a\cos\theta}\rho^3\, d\rho\right) d\theta =$$

$$= \int_0^{\frac{\pi}{4}}\left(\frac{\rho^4}{4}\Big|_0^{2a\sin\theta}\right) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\rho^4}{4}\Big|_0^{2a\cos\theta} d\theta =$$

$$= \int_0^{\frac{\pi}{4}} 4a^4 \sin^4\theta \, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4a^4\cos^4\theta\, d\theta$$

$\sin^2 x + \cos^2 x = 1 \quad \Big| \cdot \frac{1}{\cos^2 x}$

$tg^2 x + 1 = \frac{1}{\cos^2 x}$ $\qquad \cos^2 x = \frac{1}{1+tg^2 x}$

$\sin^2 x = 1-\cos^2 x = \frac{tg^2 x}{1+tg^2 x}$

$tg\, x = t$

$\cos^2 x = \frac{1}{1+t^2}$

$\sin^2 x = \frac{t^2}{1+t^2}$

$dx = \frac{1}{1+t^2}dt \qquad tg\, x = t$

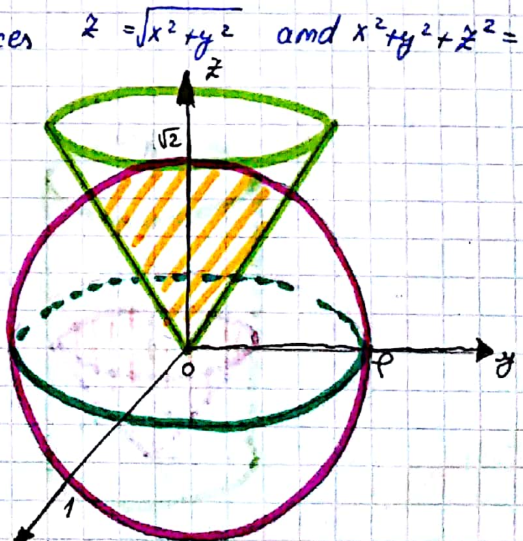
$x = arctg\, t$

$dx = \frac{1}{1+t^2}dt$

$$= \int_0^1 4a^4 \frac{t^4}{(1+t^2)^2}\cdot\frac{1}{1+t^2}dt + \int_1^\infty 4a^4\cdot\frac{1}{(1+t^2)^2}\cdot\frac{1}{(1+t^2)}dt = \dots$$

4) A = the set bounded by the surfaces $z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=1$

$$\iiint_A \frac{2z}{1+x^2+y^2}\,dx\,dy\,dz$$



$A = \{ z = \sqrt{x^2+y^2}$ ~~sa~~ $z \ge \sqrt{x^2+y^2}$

and $x^2+y^2+z^2=1\}$

$(0,0,1) \in$ cilindrului

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

$r \ge 0$

$\theta \in (0, 2\pi]$ $\quad dx\,dy\,dz = r^2\sin\varphi\, dr\,d\theta\,d\varphi$

$\varphi \in [0,\pi]$

$A = \{(x,y,z)\in \mathbb{R}^3 \mid x^2+y^2+z^2\le 1$ and $z^2 \ge x^2+y^2\}$

$x^2+y^2+z^2 \le 1 \iff r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\varphi \le 1$
$\iff r^2\sin^2\varphi + r^2\cos^2\varphi \le 1$
$\iff r^2 \le 1$
$\iff r \in [0,1]$

$r\sqrt{\sin^2 y} \le r\cos\varphi \longrightarrow \begin{cases} \cos\varphi \ge 0 \\ |\sin\varphi| \le \cos\varphi \\ \varphi \in [0,\pi] \end{cases} \Rightarrow \begin{array}{l} \varphi \in [0,\frac{\pi}{2}] \\ \sin\varphi \le \cos\varphi \end{array}$

$\iff \begin{cases} \varphi \in [0,\frac{\pi}{2}] \\ tg\,\varphi \le 1 \end{cases} \Rightarrow \varphi \in [0,\frac{\pi}{4}]$

$r \in [0,1]$
$\theta \in [0,2\pi]$
$\varphi \in [0;\frac{\pi}{4}]$

$$\int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{2r\cos\varphi \cdot r^2\sin\varphi}{1+r^2\sin^2\varphi}\,dr\,d\theta\,d\varphi = \int_0^1\int_0^{\frac{\pi}{4}} \frac{2r^3\cos\varphi\sin\varphi}{1+r^2\sin^2\varphi}\,dr\,d\varphi \cdot \underbrace{\int_0^{2\pi} d\theta}_{2\pi} =$$

$$= 2\pi\int_0^1 2r^3\,\frac{1}{2r^2}\,\ln\left(\frac{2+r^2}{2}\right)dr = 2\pi\int_0^1 r(\ln(2+r^2)-\ln 2)\,dr$$

$$\int_0^{\frac{\pi}{4}} \frac{\cos\varphi\sin\varphi}{1+r^2\sin^2\varphi} = \int_1^{\frac{2+r^2}{2}} \frac{\frac{1}{2r^2}}{t}\,dt = \frac{1}{2r^2}\left(\ln t\Big|_1^{\frac{2+r^2}{2}}\right)$$

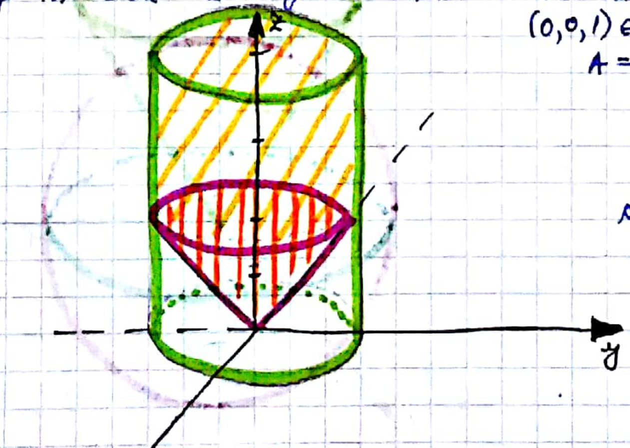
$$= \frac{1}{2r^2}\left(\ln\frac{2+r^2}{2} - 0\right)$$

$1+r^2\sin^2\varphi = t$
$0+r^2\cdot 2\sin\varphi\cos\varphi\,d\varphi = dt$
$\varphi = 0 \Rightarrow t = 1$
$\varphi = \frac{\pi}{4} \Rightarrow t = 1 + r^2\cdot\frac{1}{2} = \frac{2+r^2}{2}$

Putem separa o integrală dublă / triplă
- capetele sunt fixe (să fie mulțimi compacte)
- funcția poate fi descompusă și dat factor comun

5. $\iiint_A x^2y^2z^2\,dx\,dy\,dz$

A is the set bounded by the planes $z=0$ and $z=4$ inside of the cone $z^2=y^2=x^2$ and inside the cilinder $x^2+y^2=1$



$(0,0,1) \in$ conului

$A = \{(x,y,z)\in\mathbb{R}^3 \mid 0\le z\le 4,\ x^2+y^2\le 1,\ z^2 \ge x^2+y^2\}$

We shift to cilindric coordinates:
$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$

$\begin{cases} dx\,dy\,dz = \rho\,d\rho\,d\theta\,dz \\ \rho \ge 0 \\ \theta \in [0,2\pi] \end{cases}$

$\begin{cases} 0 \le z \le 4 \\ \rho^2\cos^2\theta + \rho^2\sin^2\theta \le 1 \\ z^2 \ge \rho^2\cos^2 + \rho^2\sin^2\theta \end{cases} \iff$

x

Seminar week 14 - ANALIZA

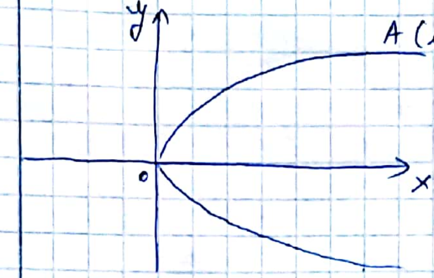
1. Line integrals of the first kind

$\gamma : [a,b] \to \mathbb{R}^m$

$f : I(\gamma) \to \mathbb{R}$ $\quad \int_\gamma f\, ds = \int_a^b (f \circ \gamma)(dt) \cdot \|\gamma'(t)\|\, dt$

$\|\gamma'(t)\| = \sqrt{(\ )^2 + \dots + (\ )^2}$

1). $\int_\gamma y\, ds$ where γ is the segment of the parabolla $y^2 = 2px$ starting from the origin to the point $A(a, \sqrt{2pa})$, $a>0$



$x(t) = t$

$y(t) = \pm\sqrt{2pt} = (2pt)^{\frac{1}{2}}$

$t \in [0, a]$

Curve γ is defined:

$\gamma : [0, a] \to \mathbb{R}^2$

$\gamma(t) = (x(t), y(t))$

$f:$

$f(\gamma(t)) = y$

$f(x(t), y(t)) = y(t)$

$x'(t) = 1$

$y'(t) = \frac{1}{2}(2pt)^{-\frac{1}{2}} 2p = p \cdot (2pt)^{-\frac{1}{2}}$

$\|\gamma'(t)\| = \sqrt{1 + p^2 \cdot (2pt)^{-1}} = \sqrt{1 + \frac{p^2}{2pt}}$

$\int_\gamma y\, ds = \int_0^a (2pt)^{\frac{1}{2}} \cdot \sqrt{1 + \frac{p^2}{2pt}}\, dt = \int_0^a \sqrt{2pt + p^2}\, dt =$

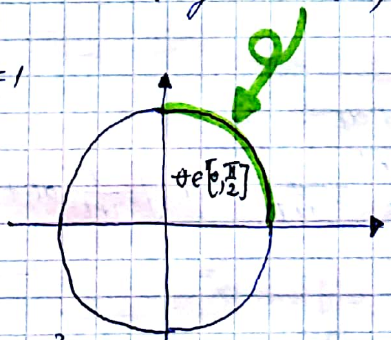
$= \int_0^a (2pt + p^2)^{\frac{1}{2}} \cdot (2pt + p^2)' \cdot \frac{1}{2p}\, dt = \frac{1}{8p} \cdot \frac{(2pt + p^2)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^a = \frac{1}{3p} \cdot (\dots)$

2) $\int_\gamma xy\, ds$ where γ is the first quarter of the ellipse having param. eq.

$\begin{cases} x = a\cos(\theta) \\ y = b\sin(\theta) \end{cases}$, $\theta \in (0, 2\pi)$

$a, b > 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\gamma : [0, \frac{\pi}{2}] \to \mathbb{R}^2$

$\gamma(\theta) = (x(\theta), y(\theta))$

$\gamma'(\theta) = (-a\sin(\theta), b\cos(\theta)) \Rightarrow \|\gamma'(\theta)\| = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

$\int_\gamma xy\, ds = \int_0^{\frac{\pi}{2}} a\cos\theta\, b\sin\theta \cdot \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$

$\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = q$

$a^2\sin^2\theta + b^2\cos^2\theta = q^2$

$2a^2\sin\theta\cos\theta + 2b^2\cos\theta\sin\theta\, dt = 2q\, dq$

$\theta = 0 \Rightarrow q = b$

$\theta = \frac{\pi}{2} \Rightarrow q = a$

$$\int_b^a a\cdot b\cdot \rho\cdot \frac{\rho}{a^2-b^2}\, d\rho = \frac{ab}{a^2-b^2}\int_b^a \rho^2\, d\rho = \frac{ab}{a^2-b^2}\cdot \frac{\rho^3}{3}\Big|_a^b$$

$$= \frac{ab}{a^2-b^2}\cdot\frac{b^3}{3} - \frac{ab}{a^2-b^2}\cdot\frac{a^3}{3} = \frac{ab^4-a^4b}{3(a^2-b^2)} = \frac{ab(b^3-a^3)}{3(a^2-b^2)}$$

3) $\int_\gamma xyz\, ds$ where $x=t$, $y=\frac{1}{3}\sqrt{8t^3}$, $z=\frac{1}{2}t^2$, $t\in[0,1]$

$\gamma:[0,1]\to\mathbb{R}^3$, $\gamma(t)=(x(t),y(t),z(t))$

$\gamma'(t)=(1,\ 3\sqrt{2}\, t^{\frac{1}{2}},\ t)$

$\|\gamma'(t)\| = \sqrt{1+2t+t^2} = \sqrt{(t+1)^2} = t+1$

$$\int_\gamma xyz\, ds = \int_0^1 t\left(\frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\right)\frac{t^2}{2}\cdot(t+1)\,dt =$$

$$= \frac{\sqrt{2}}{3}\int_0^1 t^{\frac{5}{2}}\, t^2\cdot(t+1)\,dt = \frac{\sqrt{2}}{3}\int_0^1 t^{\frac{5}{2}}\cdot t^3 + t^{1+\frac{3}{2}+2}\,dt =$$

$$= \frac{\sqrt{2}}{3}\int_0^1 t^{\frac{11}{2}} + t^{\frac{9}{2}}\,dt = \frac{\sqrt{2}}{3}\left(\int_0^1 t^{\frac{11}{2}}dt + \int_0^1 t^{\frac{9}{2}}dt\right) =$$

$$= \frac{\sqrt{2}}{3}\left(\frac{t^{\frac{11}{2}+1}}{\frac{11}{2}+1}\Big|_0^1 + \frac{t^{\frac{9}{2}+1}}{\frac{9}{2}+1}\Big|_0^1\right)$$

4) $\int_\gamma y\cdot e^x\, ds$, where $y = 2\,\text{arctg}\, t - t + 3$, $x=\ln(1+t^2)$, $t\in[0,1]$

$\gamma:[0,1]\to\mathbb{R}^2$, $\gamma(t)=(x(t),y(t))$

$\gamma'(t) = \left(\frac{1}{1+t^2}\cdot 2t,\ \frac{2}{1+t^2}-1\right)$

$\|\gamma'(t)\| = \sqrt{\frac{4t^2}{(1+t^2)}+\frac{1-t^2}{1+t^2}} = 1$

$$\int_0^1(2\,\text{arctg}\, t - t+3)\cdot e^{-\ln(1+t^2)}\cdot 1\,dt =$$

$$= \int_0^1(2\,\text{arctg}\, t - t + 3)\cdot\left(\frac{1}{1+t^2}\right) = 2\int_0^1\frac{\text{arctg}\, t}{1+t^2}\,dt - \int_0^1\frac{t}{1+t^2}\,dt + \int_0^1\frac{3}{1+t^2}\,dt$$

$$= \frac{\pi^2}{16} - \frac{\ln 2}{2} - \frac{3\pi}{4}$$

5) Cardioida este curba având rel. implicite:

$x^2+y^2 = \frac{a}{2}\left(x+\sqrt{x^2+y^2}\right)$, $a>0$

Să se det. lungimea cardioidei

$l=\int_\gamma ds$ we shift to polar coordinates
$x=\rho\cos\theta$
$y=\rho\sin\theta$
$\theta=(0,2\pi)$ $\rho>0$

$$\rho^2\cos^2\theta+\rho^2\sin^2\theta=\frac{a}{2}\left(\rho\cos\theta+\sqrt{\rho^2\cos^2\theta+\rho^2\sin^2\theta}\right)$$

$$\rho^2=\frac{a}{2}(\rho\cdot\cos\theta+\rho)$$

$$\rho=\frac{a}{2}(\cos\theta+1)=a\cdot\cos^2\frac{\theta}{2}$$

$$\cos\theta=\cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right)=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}=\cos^2\frac{\theta}{2}-1+\cos^2\frac{\theta}{2}=2\cos^2\frac{\theta}{2}-1$$

$$\cos^2\frac{\theta}{2}=\frac{\cos\theta+1}{2}$$

$$\begin{cases}x=a\cdot\cos^2\frac{\theta}{2}\cdot\cos\theta\\ y=a\cdot\cos^2\frac{\theta}{2}\cdot\sin\theta\end{cases},\ \theta\in(0,2\pi)$$

$$\gamma:(0,2\pi)\to\mathbb{R}^2,\ \gamma(\theta)=(x(\theta),y(\theta))$$

$$x'(\theta)=\left(a\left(2\cos\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\right)\cdot\frac{1}{2}+\cos\theta+\cos^2\frac{\theta}{2}(-\sin\theta)\right)\right)$$

$$=-a\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2}\cos\theta+\cos\frac{\theta}{2}\sin\theta\right)$$

$$=-a\cos\frac{\theta}{2}\sin\frac{3\theta}{2}$$

$$y'(\theta)=a\left(2\cos\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\right)\cdot\frac{1}{2}\cdot\sin\theta+\cos^2\frac{\theta}{2}\cdot\cos\theta\right)$$

$$=-a\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}\cdot\cos\theta+\left(-\sin\frac{\theta}{2}\sin\theta\right)\right)$$

$$=-a\cos\frac{\theta}{2}\cdot\cos\frac{3\theta}{2}$$

$$\|\gamma'(\theta)\|=\sqrt{a^2\cos^2\frac{\theta}{2}\sin^2\frac{3\theta}{2}+a^2\cos^2\frac{\theta}{2}\cos^2\frac{3\theta}{2}}$$

$$=a\cdot\left|\cos\frac{\theta}{2}\right|$$

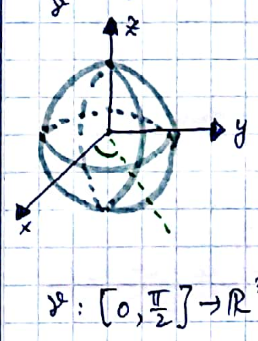
* $a\cos\frac{\theta}{2}$, $\theta\in(0,2\pi)$, $\frac{\theta}{2}\in(0,\pi)$

* $-a\cdot\cos\frac{\theta}{2}$, $\theta\in(\pi,2\pi)$, $\frac{\theta}{2}\in\left(\frac{\pi}{2},\pi\right)$

$$\int_\gamma ds=\int_0^\pi a\cdot\cos\frac{\theta}{2}\,d\theta+\int_\pi^{2\pi}a\cdot\cos\frac{\theta}{2}\,d\theta$$

$$=a\cdot\sin\frac{\theta}{2}\cdot2\Big|_0^\pi-a\cdot\sin\frac{\theta}{2}\,2\Big|_\pi^{2\pi}=4a$$

6) $\int_\gamma\frac{z^2}{x^2+y^2+1}\,ds$ where γ este curba simplă având drept imagine arcul din primul octanul cercului $x^2+y^2+z^2=1$, $x=y$



$$\begin{cases}x=\rho\sin\varphi\cos\theta\\ y=\rho\sin\varphi\sin\theta\\ z=\rho\cos\varphi\end{cases},\quad \rho>0,\ \theta\in[0,2\pi],\ \varphi\in[0,\pi]$$

$\rho^2=1\Rightarrow\rho=1$

$x=y\Leftrightarrow\rho\sin\theta\,\rho\cos\theta=\rho\sin\theta\,\rho\sin\theta$

$\cos\theta=\sin\theta\Rightarrow\theta=\frac{\pi}{4}$

$\varphi\in\left[0,\frac{\pi}{2}\right]$

$$\gamma:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}^3,\ \gamma(\varphi)=(x(\varphi),y(\varphi),z(\varphi))$$

$$x(\varphi)=1\cdot\sin\varphi\,\frac{\sqrt2}{2}=\frac{\sqrt2}{2}\sin\varphi\Rightarrow x'(\varphi)=\frac{\sqrt2}{2}\cdot\cos\varphi=y'(\varphi)$$

$$y(\varphi)=\frac{\sqrt2}{2}\cdot\sin\varphi\qquad z'(\varphi)=-\sin\varphi$$

$$z(\varphi)=\cos\varphi$$

$$\|\gamma'(\varphi)\| = \sqrt{\frac{1}{2}\cos^2\varphi + \frac{1}{2}\cos^2\varphi + \sin^2\varphi} = 1$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{\sin^2\varphi + 1}\, d\varphi$$

$$\sin^2\varphi + \cos^2\varphi = 1 \quad \Big| \cdot \frac{1}{\cos^2\varphi}$$

$$tg^2\varphi + 1 = \frac{1}{\cos^2\varphi}$$

$$\cos^2\varphi = \frac{1}{1+tg^2\varphi} \qquad \sin^2\varphi = 1 - \frac{1}{1+tg^2\varphi} = \frac{tg^2\varphi}{1+tg^2\varphi}$$

$$tg\,\varphi = u \Rightarrow \varphi = arctg\ u$$

$$d\varphi = \frac{1}{1+u^2}\, du$$

$$\varphi = 0 \Rightarrow u = 0$$
$$\varphi = \frac{\pi}{2} \Rightarrow u = \infty$$

$$I = \int_0^\infty \frac{1}{1+u^2} \cdot \left(\frac{1+u^2}{u^2} + 1\right) \cdot \frac{1}{1+u^2}\, du = \int_0^\infty \frac{1}{2u^2+1} \cdot \frac{1}{u^2+1}$$

- despărțire în fracții simple

$$= \int_0^\infty \left(-\frac{1}{1+t^2} + \frac{2}{1+2t^2}\right) = -arctg\ t\,\Big|_0^\infty + \frac{1}{\sqrt{2}}\, arctg\ \sqrt{2}\, t\,\Big|_0^\infty$$