1) $A \subseteq \mathbb{R}^n$, prove that:

a) $cl\, A = \mathbb{R}^n \setminus int(\mathbb{R}^n \setminus A)$

b) $cl\, A = (int\, A) \cup (bd\, A)$

c) $cl\, A = A \cup A'$

solution:

a) we prove that $cl\, A \overset{?}{\subseteq} \mathbb{R}^n \setminus int(\mathbb{R}^n \setminus A)$ (1)

let $x \in cl\, A$

assume by contradiction: $x \notin \mathbb{R}^n \setminus int(\mathbb{R}^n \setminus A) \Rightarrow x \in int(\mathbb{R}^n \setminus A) \Rightarrow$

$\Rightarrow \mathbb{R}_n \setminus A \subseteq \mathcal{V}(x) \Rightarrow \forall V \subseteq \mathcal{V}(x)$: it holds $V \cap A \neq \emptyset \Rightarrow$

$\Rightarrow (\mathbb{R}^n \setminus A) \cap A \neq \emptyset$ contradiction

The obtained contradiction shows that $x \in \mathbb{R}^n \setminus int(\mathbb{R}^n \setminus A)$, hence (1) holds

we claim that $\mathbb{R}^n \setminus int(\mathbb{R}^n \setminus A) \overset{?}{\subseteq} cl\, A$ (2)

let $y \in \mathbb{R}^n \setminus int(\mathbb{R}^n \setminus A) \Rightarrow y \notin int(\mathbb{R}^n \setminus A)$

assume by contradiction: $y \notin cl\, A \Rightarrow \exists V \in \mathcal{V}(y)$ s.t. $V \cap A = \emptyset \Rightarrow$

$\Rightarrow V \subseteq \mathbb{R}^n \setminus A \Rightarrow \mathbb{R}^n \setminus A \in \mathcal{V}(y) \Rightarrow y \in int(\mathbb{R}^n \setminus A) \Rightarrow$ contradiction

The obtained contradiction shows that $y \in int(\mathbb{R}^n \setminus A)$, hence (2) holds

By (1) and (2) it follows that a) holds

b) $cl\, A = (int\, A) \cup (bd\, A)$

From the definitions $\Rightarrow$ $int\, A \subseteq cl\, A$
$bd\, A \subseteq cl\, A \cap cl(\mathbb{R}^n \setminus A) \Rightarrow bd\, A \subseteq cl\, A$ $\Big\} \Rightarrow$

$\Rightarrow (int\, A) \cup (bd\, A) \subseteq cl\, A$ (3)

we claim that $cl\, A \overset{?}{\subseteq} (int\, A) \cup (bd\, A)$ (4)

let $p \in cl\, A$

assume by contra. $p \notin (int\, A) \cup (bd\, A) \Rightarrow p \notin int\, A$ and $p \notin bd\, A$

Since $p \notin bd\, A \Rightarrow \exists V \in \mathcal{V}(p)$ s.t. $V \cap A = \emptyset$ or $V \cap \mathbb{R}^n \setminus A = \emptyset$, $V \cap A = \emptyset$ can't

hold because $p \in A \Rightarrow V \cap \mathbb{R}^n \setminus A = \emptyset \Rightarrow V \subseteq A \Rightarrow A \in \mathcal{V}(p) \Rightarrow$
$\Rightarrow p \in \text{int} A$ contra.

The obtained contradiction shows that $p \in (\text{int} A) \cup (\text{bd} A) \Rightarrow (4)$ holds

By (3) and (4) b) holds

c) $\text{cl} A = A \cup A'$

From the definition $\left.\begin{array}{l} A \subseteq \text{cl} A \\ A' \subseteq \text{cl} A \end{array}\right\} \Rightarrow A \cup A' \subseteq \text{cl} A$ (5)

We claim that $\text{cl} A \overset{?}{\subseteq} A \cup A'$ (6)

let $x \in \text{cl} A$

Assume by contra $x \notin A \cup A' \Rightarrow x \notin A$ and $x \notin A' \Rightarrow$

$\exists V \in \mathcal{V}(x)$ s.t. $V \cap A \setminus \{x\} = \emptyset \Rightarrow V \cap A = \{x\}$ or $V \cap A = \emptyset$

$\left.\begin{array}{l} V \cap A = \{x\} \text{ can't hold because } x \notin A \\ V \cap A = \emptyset \text{ can't hold because } x \in \text{cl} A \end{array}\right\}$ contra.

The obtained contradiction shows that $x \in A \cup A' \Rightarrow (6)$ holds

By (5) and (6) c) holds

2) given 2 set $A, B \subseteq \mathbb{R}^n$ prove that:

a) If $A \cup B = \mathbb{R}^n$ then $(\text{cl} A) \cup (\text{int} B) = \mathbb{R}^n$

b) If $A \cap B = \emptyset$ then $(\text{cl} A) \cap (\text{int} B) = \emptyset$

solution:

a) Obviously $(\text{cl} A) \cup (\text{int} B) \subseteq \mathbb{R}^n$ (1)

We claim that $\mathbb{R}^n \overset{?}{\subseteq} (\text{cl} A) \cup (\text{bd} A)$ (2)

let $x \in \mathbb{R}^n$

Assume by contra. $x \notin \text{cl} A$ and $x \notin \text{bd} A \Rightarrow$

$\Rightarrow \exists V \subseteq \mathcal{V}(x)$ s.t. $V \cap A = \emptyset \Rightarrow \left.\begin{array}{l} V \subseteq \mathbb{R}^n \setminus A \\ \text{but } A \cup B = \mathbb{R}^n \Rightarrow \mathbb{R}^n \setminus A \subseteq B \end{array}\right\} \Rightarrow V \subseteq B \Rightarrow$

$\Rightarrow B \subseteq \mathcal{V}(x) \Rightarrow x \in \operatorname{int} B$ contra.

The obtained contr shows that $x \in \operatorname{cl} A \cup \operatorname{int} B \Rightarrow (2)$ holds

By (1) and (2) $\Rightarrow$ a) holds

b) $A \cap B = \emptyset$ then $\operatorname{cl} A \cap \operatorname{int} B = \emptyset$

Assume by contra. $\exists x \in \operatorname{cl} A \cap \operatorname{int} B \Rightarrow x \in \operatorname{cl} A$ and $x \in \operatorname{int} B \Rightarrow$

$\Rightarrow \forall V \subseteq \mathcal{V}(x)$ s.t. $V \cap A \neq \emptyset$ and $B \in \mathcal{V}(x) \Rightarrow B \cap A \neq \emptyset$ contra.

3) Prove that $\forall A, B \subseteq \mathbb{R}^n$ one has $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$

solution:

From the def. $\Rightarrow$ if $A_1 \subseteq A_2 \Rightarrow \operatorname{cl} A_1 \subseteq \operatorname{cl} A_2 \Rightarrow \left.\begin{array}{l} \operatorname{cl} A \subseteq \operatorname{cl}(A \cup B) \\ \operatorname{cl} B \subseteq \operatorname{cl}(A \cup B) \end{array}\right\} \Rightarrow$

$\Rightarrow \operatorname{cl} A \cup \operatorname{cl} B \subseteq \operatorname{cl}(A \cup B)$ (1)