COURSES 10+11

Linear maps

Definition 1. Let V and V' be vector spaces over K. The map $f : V \to V'$ is called a **(vector space) homomorphism** or a **linear map** (or a **linear transformation**) if

$$f(x + y) = f(x) + f(y)\,,\ \forall x, y \in V\,,$$

$$f(kx) = kf(x),\ \ \forall k \in K,\ \forall x \in V\,.$$

The **(vector space) isomorphism**, **endomorphism** and **automorphism** are defined as usual.

We will mainly use the name *linear map* or *K-linear map*.

Remarks 2. (1) When defining a linear map, we consider vector spaces over the same field K.
(2) If $f : V \to V'$ is a K-linear map, then the first condition from its definition tells us that f is a group homomorphism between $(V, +)$ and $(V', +)$. Thus we have

$$f(0) = 0' \text{ and } f(-x) = -f(x),\ \forall x \in V.$$

We denote by $V \simeq V'$ the fact that two vector spaces V and V' are isomorphic and

$$Hom_K(V, V') = \{f : V \to V' \mid f \text{ is a } K\text{-linear map}\}\,,$$

$$End_K(V) = \{f : V \to V \mid f \text{ is a } K\text{-linear map}\}\,,$$

$$Aut_K(V) = \{f : V \to V \mid f \text{ is a } K\text{-isomorphism}\}\,.$$

Theorem 3. Let V, V' be K-vector spaces. Then $f : V \to V'$ is a linear map if and only if

$$f(k_1v_1 + k_2v_2) = k_1f(v_1) + k_2f(v_2)\,,\ \forall k_1, k_2 \in K,\ \forall v_1, v_2 \in V.$$

Proof. □

One can easily prove by way of induction the following:

Corollary 4. If $f : V \to V'$ is a linear map, then

$$f(k_1v_1 + \cdots + k_nv_n) = k_1f(v_1) + \cdots + k_nf(v_n),\ \forall v_1, \ldots, v_n \in V,\ \forall k_1, \ldots, k_n \in K.$$

Examples 5. (a) Let V and V' be K-vector spaces and let $f : V \to V'$ be defined by $f(x) = 0'$, for any $x \in V$. Then f is a K-linear map, called the **trivial linear map**.
(b) Let V be a vector space over K. Then the identity map $1_V : V \to V$ is an automorphism of V.
(c) Let V be a vector space and $S \leq_K V$. Define $i : S \to V$ by $i(x) = x$, for any $x \in S$. Then i is a K-linear map, called the **inclusion linear map**.

(d) Let us consider $\varphi \in \mathbb{R}$. The map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x,y) = (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi),$$

i.e. the plane rotation with the rotation angle φ, is a linear map.
(e) If $a, b \in \mathbb{R}$, $a < b$, $I = [a, b]$, and $C(I, \mathbb{R}) = \{f : I \to \mathbb{R} \mid f$ continuous on $I\}$, then

$$F : C(I, \mathbb{R}) \to \mathbb{R},\ F(f) = \int_a^b f(x)dx$$

is a linear map.

As in the case of group homomorphisms, we have the following:

Theorem 6. Let V, V', V'' be K-vector spaces.
(i) If $f : V \to V'$ and $g : V' \to V''$ are K-linear maps (isomorphisms) then $g \circ f : V \to V''$ is a K-linear map (isomorphism).
(ii) If $f : V \to V'$ is an isomorphism of K-vector spaces then $f^{-1} : V' \to V$ is again an isomorphism of K-vector spaces.

Proof. □

Definition 7. Let $f : V \to V'$ be a K-linear map. Then the set

$$\mathrm{Ker} f = \{x \in V \mid f(x) = 0'\}$$

is called the **kernel** of the K-linear map f and the set

$$\mathrm{Im} f = \{f(x) \mid x \in V\}$$

is called the **image** of the K-linear map f.

Theorem 8. Let $f : V \to V'$ be a K-linear map. Then we have
1) $\mathrm{Ker} f \leq_K V$ and $\mathrm{Im} f \leq_K V'$.
2) f is injective if and only if $\mathrm{Ker} f = \{0\}$.

Proof. □

Theorem 9. Let $f : V \to V'$ be a K-linear map and let $X \subseteq V$. Then

$$f(\langle X \rangle) = \langle f(X) \rangle\,.$$

Proof. □

Theorem 10. Let V and V' be vector spaces over K. For any $f, g \in Hom_K(V, V')$ and for any $k \in K$, we consider $f + g, k \cdot f \in Hom_K(V, V')$,

$$(f + g)(x) = f(x) + g(x),\ \forall x \in V\,,$$

$$(kf)(x) = kf(x),\ \forall x \in V\,.$$

The above equalities define an addition and a sclar multiplication on $Hom_K(V,V')$ and $Hom_K(V,V')$ is a vector space over K.

Proof. □

Corollary 11. If V is a K-vector space, then $End_K(V)$ is a vector space over K.

Remarks 12. a) Let V be a K-vector space. From Theorem 6 one deduces that $End_K(V)$ is a subgroupoid of $(V^V,\circ)$ and from Example 5 (b) it follows that $(End_K(V),\circ)$ is a monoid. Moreover, the endomorphism composition $\circ$ is distributive with respect to endomorphism addition $+$, thus $End_K(V)$ also has a unitary ring structure, $(End_K(V),+,\circ)$.

b) The set $Aut_K(V)$ is the group of the units of $(End_K(V),\circ)$.

Bases. Dimension

Let $(K,+,\cdot)$ be a field and let V be a vector space over K.

Definition 13. We say that the vectors $v_1,\dots,v_n \in V$ are (or the set of vectors $\{v_1,\dots,v_n\}$ is):

(1) **linearly independent** in V if for any $k_1,\dots,k_n \in K$,

$$k_1v_1 + \cdots + k_nv_n = 0 \Rightarrow k_1 = \cdots = k_n = 0\,.$$

(2) **linearly dependent** in V if they are not linearly independent, that is,

$$\exists k_1,\dots,k_n \in K \text{ not all zero, such that } k_1v_1 + \cdots + k_nv_n = 0\,.$$

More generally, an infinite set of vectors of V is said to be:

(1) **linearly independent** if any finite subset is linearly independent.

(2) **linearly dependent** if there exists a finite subset which is linearly dependent.

Remarks 14. (1) A set consisting of a single vector v is linearly dependent if and only if $v = 0$.

(2) As an immediate consequence of the definition, we notice that if V is a vector space over K and $X,Y \subseteq V$ such that $X \subseteq Y$, then:

(i) If Y is linearly independent, then X is linearly independent.

(ii) If X is linearly dependent, then Y is linearly dependent. Thus, every set of vectors containing the zero vector is linearly dependent.

Theorem 15. Let V be a vector space over K. Then the vectors $v_1,\dots,v_n \in V$ are linearly dependent iff one of the vectors is a linear combination of the others, that is,

$$\exists j \in \{1,\dots,n\},\ \exists \alpha_i \in K:\ v_j = \sum_{\substack{i=1\\ i\neq j}}^{n} \alpha_i v_i.$$

Proof. See the seminar. □

Examples 16. (a) $\emptyset$ is linearly independent in any vector space.
(b) Let V_2 be the real vector space of all vectors (in the classical sense) in the plane with a fixed origin O. Recall that the addition is the usual addition of two vectors by the parallelogram rule and the external operation is the usual scalar multiplication of vectors by real scalars. Then:

(i) one vector v is linearly dependent in $V_2 \Leftrightarrow v = 0$;

(ii) two vectors are linearly dependent in $V_2 \Leftrightarrow$ they are collinear;

(iii) three vectors are always linearly dependent in V_2.

(c) Let V_3 be the real vector space of all vectors (in the classical sense) in the space with a fixed origin O. Then:

(i) one vector v is linearly dependent in $V_3 \Leftrightarrow v = 0$;

(ii) two vectors are linearly dependent in $V_3 \Leftrightarrow$ they are collinear;

(iii) three vectors are linearly dependent in $V_3 \Leftrightarrow$ they are coplanar;

(iv) four vectors are always linearly dependent in V_3.

(d) If K is a field and $n \in \mathbb{N}^*$, then the vectors

$$(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1)$$

from K^n are linearly independent in the K-vector space K^n.
(e) Let K be a field and $n \in \mathbb{N}$. Then the vectors $1, X, X^2, \dots, X^n$ are linearly independent in the vector space $K_n[X] = \{f \in K[X] \mid \deg f \leq n\}$ over K and, more generally, the vectors $1, X, X^2, \dots, X^n, \dots$ are linearly independent in the K-vector space $K[X]$.

We are going to define a key notion concerning vector spaces, namely *basis*, which will perfectly determine a vector space. We will discuss *only the case of finitely generated vector spaces*. This is why, till the end of the chapter, *by a vector space we will understand a finitely generated vector space*. However, many results from the next part hold for arbitrary vector spaces.

Definition 17. Let V be a vector space over K. By a **list of vectors** in V we understand an n-tuple $(v_1, \dots, v_n) \in V^n$ for some $n \in \mathbb{N}^*$.

Definition 18. Let V be a vector space over K. An n-tuple $B = (v_1, \dots, v_n) \in V^n$ is called a **basis** of V if:

(1) B is a system of generators for V, that is, $\langle B \rangle = V$;

(2) B is linearly independent in V.

Theorem 19. Let V be a vector space over K. A list $B = (v_1, \dots, v_n)$ of vectors in V is a basis of V if and only if each vector $v \in V$ can be uniquely written as a linear combination of the vectors $v_1, \dots, v_n$, i.e.

$$\forall v \in V,\ \exists k_1, \dots, k_n \in K \text{ uniquely determined}:\ v = k_1 v_1 + \dots + k_n v_n.$$

Proof. See the seminar. □

Definition 20. Let V be a vector space over K, $B = (v_1, \dots, v_n)$ a basis of V and $v \in V$. Then the scalars $k_1, \dots, k_n \in K$ from the unique writing of v as a linear combination

$$v = k_1 v_1 + \dots + k_n v_n$$

of the vectors of B are called the **coordinates of v in the basis** B.

Examples 21. (a) $\emptyset$ is basis for the zero vector space.
(b) If K is a field and $n \in \mathbb{N}^*$, then the list $E = (e_1, \ldots, e_n)$ of vectors of K^n, where

$$e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, 0, \ldots, 1)$$

is a basis of the canonical vector space K^n over K, called the **standard basis**. Indeed, we saw that E is linearly independent and each vector $(x_1, \ldots, x_n) \in K^n$ can be written as a linear combination of the vectors of E,

$$(x_1, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n.$$

Notice that the coordinates of a vector in the standard basis are just the components of the vector, fact that is not true in general.

In particular, if $n = 1$, the set $\{1\}$ is a basis of the canonical vector space K over K. For instance, $\{1\}$ is a basis of the vector space $\mathbb{C}$ over $\mathbb{C}$.
(c) Consider the canonical real vector space $\mathbb{R}^2$. We already know a basis of $\mathbb{R}^2$, namely the standard basis $((1, 0), (0, 1))$. But it is easy to show that the list $((1, 0), (1, 1))$ is also a basis of $\mathbb{R}^2$. Therefore, a vector space may have more than one basis.
(d) Let V_3 be the real vector space of all vectors (in the classical sense) in the space with a fixed origin O. Any 3 vectors which are not coplanar form a basis of V_3; e.g. the three pairwise orthogonal *unit vectors* $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$.
(e) The sets $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ and $T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$ are subspaces of ${}_{\mathbb{R}}\mathbb{R}^3$. As a matter of fact, $S = \langle (1, 0, -1), (0, 1, -1) \rangle$ and $T = \langle (1, 1, 1) \rangle$. Since the two generators of S are linearly independent, they form a basis of S. The only generator of T is clearly linearly independent, hence it forms a basis of T.
(f) Since for any $z \in \mathbb{C}$, there exist the uniquely determined real numbers $x, y \in \mathbb{R}$ such that $z = x \cdot 1 + y \cdot i$, the list $B = (1, i)$ is a basis of the vector space $\mathbb{C}$ over $\mathbb{R}$ (see Theorem 19). The coordinates of a vector $z \in \mathbb{C}$ in the basis B are just its real and its imaginary part.
(g) Let K be a field and $n \in \mathbb{N}$. Then the list $B = (1, X, X^2, \ldots, X^n)$ is a basis of the vector space $K_n[X] = \{f \in K[X] \mid \deg f \leq n\}$ over K, because each vector (polynomial) $f \in K_n[X]$ can be uniquely written as a linear combination

$$f = a_0 \cdot 1 + a_1 \cdot X + \cdots + a_n \cdot X^n$$

$(a_0, \ldots, a_n \in K)$ of the vectors of B (see Theorem 19). In this case, the coordinates of a vector $f \in K_n[X]$ in the basis B are just its coefficients as a polynomial.
(h) If V_1 and V_2 are K-vector spaces and $B_1 = (x_1, \ldots, x_m)$ and $B_2 = (y_1, \ldots, y_n)$ are bases for V_1 and V_2, respectively, then $((x_1, 0), \ldots, (x_m, 0), (0, y_1), \ldots, (0, y_n))$ is a basis for the direct product $V_1 \times V_2$.

Theorem 22. Every vector space has a basis.

Proof. Let V be a vector space over K. If $V = \{0\}$, then it has the basis $\emptyset$.

Now let $\{0\} \neq V = \langle B \rangle$, where $B = (v_1, \ldots, v_n)$. If B is linearly independent, then B is a basis and we are done. Suppose that the list B is linearly dependent. Then by Theorem 15, there exists $j_1 \in \{1, \ldots, n\}$ such that

$$v_{j_1} = \sum_{\substack{i=1 \\ i \neq j_1}}^{n} k_i v_i$$

for some $k_i \in K$. It follows that $V = \langle B \setminus \{v_{j_1}\}\rangle$, because every vector of V can be written as a linear combination of the vectors of $B \setminus \{v_{j_1}\}$. If $B \setminus \{v_{j_1}\}$ is linearly independent, it is a basis and we are done. Otherwise, there exists $j_2 \in \{1, \ldots, n\} \setminus \{j_1\}$ such that

$$v_{j_2} = \sum_{\substack{i=1\\ i\neq j_1,j_2}}^{n} k'_i v_i$$

for some $k'_i \in K$. It follows that $V = \langle B \setminus \{v_{j_1}, v_{j_2}\}\rangle$, because every vector of V can be written as a linear combination of the vectors of $B \setminus \{v_{j_1}, v_{j_2}\}$. If $B \setminus \{v_{j_1}, v_{j_2}\}$ is linearly independent, then it is a basis and we are done. Otherwise, we continue the procedure. If all the previous intermediate subsets are linearly dependent, we get to the step $V = \langle B \setminus \{v_{j_1}, \ldots, v_{j_{n-1}}\}\rangle = \langle v_{j_n}\rangle$. If v_{j_n} were linearly dependent, then $v_{j_n} = 0$, hence $V = \langle v_{j_n}\rangle = \{0\}$, contradiction. Hence v_{j_n} is linearly independent and thus forms a single element basis of V. $\square$

Remarks 23. (1) We have proved the existence of a basis of a vector space. As we saw in Example 21 (c) such a basis not necessarily unique.
(2) In the proof of Theorem 22 we saw that if B is an n-elements set which generates V one can successively eliminate elements from B in order to find a basis for V. It follows that any basis of V has at most n vectors. Later we will prove even a stronger result, namely if a vector space has a basis of n elements, then all its bases have n elements.

Theorem 24. i) Let $f : V \to V'$ be a K-linear map and let $B = (v_1, \ldots, v_n)$ be a basis of V. Then f is determined by its values on the vectors of the basis B.
ii) Let $f, g : V \to V'$ be K-linear maps and let $B = (v_1, \ldots, v_n)$ be a basis of V. If $f(v_i) = g(v_i)$, for any $i \in \{1, \ldots, n\}$, then $f = g$.

Proof. i) Let $v \in V$. Since B is a basis of V, there exists $k_1, \ldots, k_n \in K$ uniquely determined such that $v = k_1v_1 + \cdots + k_nv_n$. Then

$$f(v) = f(k_1v_1 + \cdots + k_nv_n) = k_1f(v_1) + \cdots + k_nf(v_n)\,,$$

that is, f is determined by $f(v_1)$, $\ldots$, $f(v_n)$.
ii) Let $v \in V$. Then $v = k_1v_1 + \cdots + k_nv_n$ for some $k_1, \ldots, k_n \in K$, hence

$$f(v) = f(k_1v_1 + \cdots + k_nv_n) = k_1f(v_1) + \cdots + k_nf(v_n) = k_1g(v_1) + \cdots + k_ng(v_n) = g(v)\,.$$

Therefore, $f = g$. $\square$

Remark 25. From the previous theorem one deduces that *given two K-vector spaces V, V', a basis B of V and a function $f' : B \to V'$, there exists a unique linear map $f : V \to V'$ which extends f' (i.e. $f|_B = f'$ or, equivalently, $f(x_i) = f'(x_i)$, $i = 1, \ldots, n$)*, result also known as **universal property of vector spaces**.

Theorem 26. Let $f : V \to V'$ be a K-linear map. Then:
(i) f is injective if and only if for any X linearly independent in V, $f(X)$ is linearly independent in V'.
(ii) f is surjective if and only if for any X system of generators for V, $f(X)$ is a system of generators for V'.
(iii) f is bijective if and only if for any X basis of V, $f(X)$ is a basis of V'.

Proof. (i) Let $X = (v_1, \ldots, v_n)$ be a linearly independent list of vectors in V and let $k_1, \ldots, k_n \in K$ be such that $k_1 f(v_1) + \cdots + k_n f(v_n) = 0$. Since f is a K-linear map, we deduce $f(k_1 v_1 + \cdots + k_n v_n) = f(0)$. By the injectivity of f we get $k_1 v_1 + \cdots + k_n v_n = 0$. But since X is linearly independent in V, we have $k_1 = \cdots = k_n = 0$. Therefore, $f(X)$ is linearly independent in V'.

Conversely, let $x, y \in V$ with $x \neq y$. Then the non-zero vector $x - y$ is linearly independent, hence $f(x - y)$ is linearly independent by hypothesis. So, $f(x - y) \neq 0$ and thus, $f(x) \neq f(y)$. Thus f is injective.

(ii) Let X be a system of generators for V. Then $\langle X \rangle = V$. By the surjectivity of f we have:

$$\langle f(X) \rangle = f(\langle X \rangle) = f(V) = V' ,$$

that is, $f(X)$ is a system of generators for V'.

Conversely, V is, clearly, a system of generators for V. By hypothesis, it follows that $f(V)$ is a system of generators for V'. Hence $f(\langle V \rangle) = \langle f(V) \rangle = V'$, that is, $f(V) = V'$. Hence f is surjective.

(iii) It follows by (i) and (ii). □

Recall that we consider only finitely generated vector spaces. Let us begin with a very useful lemma, that will be often implicitly used.

Lemma 27. Let V be a K-vector space and let $Y = \langle y_1, \ldots, y_n, z \rangle$. If $z \in \langle y_1, \ldots, y_n \rangle$, then $Y = \langle y_1, \ldots, y_n \rangle$.

Proof. The generated subspace Y is the set of all linear combinations of the vectors $y_1, \ldots, y_n, z$. Since $z \in \langle y_1, \ldots, y_n \rangle$, z is a linear combination of the vectors $y_1, \ldots, y_n$. It follows that every vector in Y can be written as a linear combination only of the vectors $y_1, \ldots, y_n$. Consequently, $Y = \langle y_1, \ldots, y_n \rangle$. □

Let us now discuss a key theorem for proving that any two bases of a vector space have the same number of elements. But it is worth mentioning that it has a much broader importance in Linear Algebra.

Theorem 28. (Steinitz, The Exchange Theorem) Let V be a vector space over K, let $X = (x_1, \ldots, x_m)$ be a linearly independent list of vectors of V and $Y = (y_1, \ldots, y_n)$ a system of generators of V $(m, n \in \mathbb{N}^*)$. Then $m \leq n$ and m vectors of Y can be replaced by the vectors of X in order to obtain a system of generators for V.

Proof. We prove this result by way of induction on m. Let us take $m = 1$. Then clearly $m \leq n$. Since Y is a system of generators for V, we have $x_1 = \sum_{i=1}^n k_i y_i$ for some $k_1, \ldots, k_n \in K$. The list $X = \{x_1\}$ is linearly independent, hence $x_1 \neq 0$. It follows that there exists $j \in \{1, \ldots, n\}$ such that $k_j \neq 0$. Then

$$y_j = k_j^{-1} x_1 - \sum_{\substack{i=1 \\ i \neq j}}^{n} k_j^{-1} k_i y_i ,$$

that is, y_j is a linear combination of the vectors $y_1, \ldots, y_{j-1}, x_1, y_{j+1}, \ldots, y_n$. Hence, in any linear combination of $y_1, \ldots, y_n$, the vector y_j can be expressed as a linear combination of the other vectors and x_1. Therefore, we have

$$V = \langle y_1, \ldots, y_n \rangle = \langle y_1, \ldots, y_{j-1}, x_1, y_{j+1}, \ldots, y_n \rangle .$$

Thus, we have obtained again a system of n generators for V containing x_1.

Let us assume that the statement holds for a list with $m-1$ linearly independent vectors of V ($m \in \mathbb{N}$, $m \geq 2$) and let us prove it for the linearly independent list $X = (x_1, \dots, x_m)$. Then $(x_1, \dots, x_{m-1})$ is also linearly independent in V. By the induction step hypothesis, we have $m-1 \leq n$. If necessary, we can reindex the elements of Y and we have

$$V = \langle x_1, \dots, x_{m-1}, y_m, \dots, y_n \rangle .$$

Assume by contradiction that $m-1 = n$. Then from $V = \langle x_1, \dots, x_{m-1} \rangle$ it follows that $x_m \in \langle x_1, \dots, x_{m-1} \rangle$, which is absurd since X is linearly independent in V. Thus $m-1 < n$ or, equivalently, $m \leq n$.

We have $x_m \in V = \langle x_1, \dots, x_{m-1}, y_m, \dots, y_n \rangle$, hence

$$x_m = \sum_{i=1}^{m-1} k_i x_i + \sum_{i=m}^{n} k_i y_i$$

for some $k_1, \dots, k_n \in K$. The list X being linearly independent in V, it follows that there exists $m \leq j \leq n$ such that $k_j \neq 0$ (otherwise, $x_m = \sum_{i=1}^{m-1} k_i x_i$ and the list X would be linearly dependent in V). For simplicity of writing, assume that $j = m$. It follows that

$$y_m = k_m^{-1} x_m - \sum_{i=1}^{m-1} k_m^{-1} k_i x_i - \sum_{i=m+1}^{n} k_m^{-1} k_i y_i .$$

Thus, $y_m \in \langle x_1, \dots, x_m, y_{m+1}, \dots, y_n \rangle$. Therefore, we have

$$V = \langle x_1, \dots, x_{m-1}, y_m, \dots, y_n \rangle = \langle x_1, \dots, x_m, y_{m+1}, \dots, y_n \rangle .$$

Thus, we have obtained again a system of generators for V, where m vectors of the list Y have been replaced by the vectors of the list X. This completes the proof. □

Theorem 29. Any two bases of a vector space have the same number of elements.

Proof. Let V be a vector space over K and let $B = (v_1, \dots, v_m)$ and $B' = (v'_1, \dots, v'_n)$ be bases of V. Since B is linearly independent in V and B' is a system of generators for V, we have $m \leq n$ by Theorem 28. Since B is a system of generators for V and B' is linearly independent in V, we have $n \leq m$ by the same Theorem 28. Hence $m = n$. □

Definition 30. Let V be a vector space over K. Then the number of elements of any of its bases is called the **dimension of** V and is denoted by $\dim_K V$ or simply by $\dim V$.

Examples 31. Using the bases given in Examples 21, one can easily determine the dimension of those vector spaces.
(a) If $V = \{0\}$, V has the basis $\emptyset$ and $\dim V = 0$.
(b) Let K be a field and $n \in \mathbb{N}^*$. Then $\dim_K K^n = n$. In particular, $\dim_{\mathbb{C}} \mathbb{C} = 1$.
(c) $\dim_{\mathbb{R}} \mathbb{C} = 2$.
(d) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ and $T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$ are subspaces of ${}_{\mathbb{R}}\mathbb{R}^3$ with $\dim S = 2$ and $\dim T = 1$. More general, the subspaces of $\mathbb{R}^3$ are $\{(0, 0, 0)\}$, any line containing the origin, any plane containing the origin and ${}_{\mathbb{R}}\mathbb{R}^3$. Their dimensions are 0, 1, 2 and 3, respectively.

(e) Let K be a field and $n \in \mathbb{N}$. Then $\dim K_n[X] = n+1$.
(f) If V_1 and V_2 are K-vector spaces and $B_1 = (x_1, \ldots, x_m)$ and $B_2 = (y_1, \ldots, y_n)$ are bases for V_1 and V_2, respectively, then $\dim(V_1 \times V_2) = m + n = \dim V_1 + \dim V_2$.

Theorem 32. Let V be a vector space over K. Then the following statements are equivalent:

(i) $\dim V = n$;

(ii) The maximum number of linearly independent vectors in V is n;

(iii) The minimum number of generators for V is n.

Proof. (i)$\Rightarrow$(ii) Assume $\dim V = n$. Let $B = (v_1, \ldots, v_n)$ be a basis of V. Since B is a system of generators for V, any linearly independent list in V must have at most n elements by Theorem 28.
(ii)$\Rightarrow$(i) Let $B = (v_1, \ldots, v_m)$ be a basis of V and let $(u_1, \ldots, u_n)$ be a linearly independent list in V. Since B is linearly independent, we have $m \leq n$ by hypothesis. Since B is a system of generators for V, we have $n \leq m$ by Theorem 28. Hence $m = n$ and consequently $\dim V = n$.
(i)$\Rightarrow$(iii) Assume $\dim V = n$. Let $B = (v_1, \ldots, v_n)$ be a basis of V. Since B is a linearly independent list in V, any system of generators for V must have at least n elements by Theorem 28.
(iii)$\Rightarrow$(i) Let $B = (v_1, \ldots, v_m)$ be a basis of V and let $(u_1, \ldots, u_n)$ be a system of generators for V. Since B is a system of generators for V, we have $n \leq m$ by hypothesis. Since B is linearly independent, we have $m \leq n$ by Theorem 28. Hence $m = n$ and consequently $\dim V = n$. $\square$

Theorem 33. Let V be a vector space over K with $\dim V = n$ and $X = (u_1, \ldots, u_n)$ a list of vectors in V. Then X is linearly independent in V if and only if X is a system of generators for V.

Proof. Let $B = (v_1, \ldots, v_n)$ be a basis of V.

Let us assume that X is linearly independent. Since B is a system of generators for V, we know by Theorem 28 that n vectors of B, i.e., all the vectors of B, can be replaced by the vectors of X and we get another system of generators for V. Hence $\langle X \rangle = V$. Thus, X is a system of generators for V.

Conversely, let us suppose that X is a system of generators for V. Assume by contradiction that X is linearly dependent. Then there exists $j \in \{1, \ldots, n\}$ such that

$$u_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} k_i u_i$$

for some $k_i \in K$. It follows that $V = \langle X \rangle = \langle u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n \rangle$. This contradicts the fact that the minimum number of generators for V is n (see Theorem 32). Thus our assumption must have been false. So X is linearly independent. $\square$

Theorem 34. Any linearly independent list of vectors in a vector space can be completed to a basis of the vector space.

Proof. Let V be a K-vector space, let $B = (v_1, \ldots, v_n)$ be a basis of V and $(u_1, \ldots, u_m)$ be a linearly independent list in V. Since B is a system of generators for V, we know by Theorem 28 that $m \leq n$ and m vectors of B can be replaced by the vectors $(u_1, \ldots, u_m)$ obtaining again a system of generators for V, say $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$. But by Theorem 33, this is also linearly independent in V and consequently a basis of V. $\square$

Remark 35. The completion of a linearly independent list to a basis of the vector space is not unique.

Example 36. The list (e_1, e_2), where $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$, is linearly independent in the canonical real vector space $\mathbb{R}^3$. It can be completed to the standard basis of the space, namely (e_1, e_2, e_3), where $e_3 = (0,0,1)$. On the other hand, since $\dim_{\mathbb{R}} \mathbb{R}^3 = 3$, in order to obtain a basis of the space it is enough to add to our list any vector v_3 for which (e_1, e_2, v_3) is linearly independent (see Theorem 33). For instance, we may take $v_3 = (1,1,1)$.

Corollary 37. Let V be a vector space over K and $S \leq_K V$. Then:
(i) Any basis of S is a part of a basis of V.
(ii) $\dim S \leq \dim V$.
(iii) $\dim S = \dim V \Leftrightarrow S = V$.

Proof. (i) Let $(u_1, \dots, u_m)$ be a basis of S. Since the list is linearly independent, it can be completed to a basis $(u_1, \dots, u_m, v_{m+1}, \dots, v_n)$ of V by Theorem 34.
(ii) follows from (i).
(iii) Assume that $\dim S = \dim V = n$. Let $(u_1, \dots, u_n)$ be a basis of S. Then it is linearly independent in V, hence it is a basis of V by Theorem 33. Thus, if $v \in V$, then $v = k_1 u_1 + \cdots + k_n u_n$ for some $k_1, \dots, k_n \in K$, hence $v \in S$. Therefore, $S = V$. □

Remark 38. For the equivalence (iii) from the previous corollary the fact that we are working in a finitely generated space is essential.

Theorem 39. Let V and V' be vector spaces over K. Then

$$V \simeq V' \Leftrightarrow \dim V = \dim V' .$$

Proof.

□

Corollary 40. Any vector space V over K with $\dim V = n (\in \mathbb{N}^*)$ is isomorphic to the canonical vector space K^n over K.

Remark 41. Corollary 40 is a very important structure result, saying that, up to an isomorphism, any finite dimensional vector space over K is, actually, the canonical vector space K^n over K. Thus, we have an explanation why we have used so often this kind of vector spaces: not only because the operations are very nice and easily defined, but they are, up to an isomorphism, the only types of finite dimensional vector spaces.

We end this section with some important formulas involving vector space dimension.

Theorem 42. Let $f : V \to V'$ be a K-linear map. Then

$$\dim V = \dim(\mathrm{Ker} f) + \dim(\mathrm{Im} f) .$$

Proof.

□

Corollary 43. a) Let V be a K-vector space and let S, T be subspaces of V. Then

$$\dim S + \dim T = \dim(S \cap T) + \dim(S + T).$$

Indeed, $f : S \times T \to S + T$, $f(x, y) = x - y$ is a surjective linear map with the kernel $\operatorname{Ker} f = \{(x, x) \mid x \in S \cap T\}$. Hence,

$$\dim(S \times T) = \dim(\operatorname{Ker} f) + \dim(S + T).$$

Since $g : S \cap T \to \operatorname{Ker} f$, $g(x) = (x, x)$ is an isomorphism, we have

$$\dim(\operatorname{Ker} f) = \dim(S \cap T),$$

and by Example 31 g) we have $\dim(S \times T) = \dim S + \dim T$, which completes the proof of the statement.
b) If V is a K-vector space and $S, T \leq_K V$, then

$$\dim(S + T) = \dim S + \dim T \Leftrightarrow S + T = S \oplus T.$$

c) Let V be a K-vector space and $f \in End_K(V)$. The following statements are equivalent:
(i) f is injective;
(ii) f is surjective;
(iii) f is bijective.