

SEMINAR 12

1) a) Let $\varphi \in \mathbb{R}$. Show that the plane rotation

$$h : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad h(x, y) = (x \cos \varphi - y \sin \varphi, x \sin \varphi + y \cos \varphi),$$

is an automorphism of \mathbb{R}^2 . Write the matrix of h of \mathbb{R}^2 in the standard basis (i.e. the basis $E = (e_1, e_2)$, with $e_1 = (1, 0)$, $e_2 = (0, 1)$).

b) Show that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f(x, y) = (x, -y)$ (the symmetry with respect to Ox) and $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $g(x, y) = (-x, y)$ (the symmetry with respect to Oy) are automorphisms of \mathbb{R}^2 . Find the matrices of f , g , $f - g$, $f + 2g$ and $g \circ f$ in the standard basis.

2) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $f(x, y) = (x + y, 2x - y, 3x + 2y)$. Show that f is an \mathbb{R} -linear map, that $B = ((1, 2), (-2, 1))$ and $B' = ((1, -1, 0), (-1, 0, 1), (1, 1, 1))$ are bases for \mathbb{R}^2 and \mathbb{R}^3 , respectively, then determine the matrix of f in the pair of bases (B, B') .

3) Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 5 & -1 \\ 2 & 7 & -3 \end{pmatrix}$, $B = (v_1, v_2, v_3)$ be a basis of the \mathbb{R} -vector space \mathbb{R}^3 , the vectors

$$u_1 = v_1 + 2v_2 + v_3, \quad u_2 = v_1 + v_2 + 2v_3, \quad u_3 = v_1 + v_2$$

and $f \in \text{End}_{\mathbb{R}}(\mathbb{R}^3)$.

a) Show that $B' = (u_1, u_2, u_3)$ is a basis of \mathbb{R}^3 .

b) Provided that $[f]_B = A$, find the matrix $[f]_{B'}$.

c) Provided that $[f]_{B'} = A$, find the matrix $[f]_B$.

4) Let V, V' be \mathbb{R} -vector spaces, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ bases in V and V' , respectively and $f : V \rightarrow V'$ a \mathbb{R} -linear map with

$$[f]_{a,b} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Determine:

i) $f(v)$ for an arbitrary $v \in V$;

ii) the dimensions of $\text{Im } f$ and $\text{Ker } f$;

iii) the matrix $[f]_{a',b'}$ when $a' = (a_1, a_1 + a_2, a_1 + a_2 + a_3)$ and $b' = (b_1, b_1 + b_2, b_1 + b_2 + b_3)$.

5) Let V, V' be \mathbb{R} -vector spaces, $B = (v_1, v_2, v_3)$ a basis for V , $B' = (v'_1, v'_2, v'_3)$ a basis for V' and $f : V \rightarrow V'$ a \mathbb{R} -linear map with

$$[f]_{B,B'} = \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & -5 \end{pmatrix}.$$

Determine:

i) the dimension and a basis for each of the spaces $\text{Im } f$ and $\text{Ker } f$;

ii) $[f]_{B,E'}$ when $V' = \mathbb{R}^3$, $v'_1 = (1, 0, 0)$, $v'_2 = (0, 1, 1)$, $v'_3 = (0, 0, 1)$ and E' is the standard basis of \mathbb{R}^3 ;

iii) $f(x)$ for $x = 2v_1 - v_2 + 3v_3$, under the circumstances of ii).

6) Let $f \in \text{End}_{\mathbb{Q}}(\mathbb{Q}^4)$ with the matrix in the standard basis

$$a) \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 \\ -1 & -3 & 0 & 4 \\ 0 & 4 & -1 & -3 \end{pmatrix}; \quad b) \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 2 & 1 & 0 \\ 3 & 0 & -1 & -2 \\ 5 & -3 & -1 & 1 \end{pmatrix}.$$

Find a basis for each of the \mathbb{Q} -vector spaces $\text{Ker } f$, $\text{Im } f$, $\text{Ker } f + \text{Im } f$ and $\text{Ker } f \cap \text{Im } f$.