Linear and Differential Geometry for Computer Scientists

Paul A. Blaga

Contents

Part I

Elements of Analytic Geometry in the Plane and in Space

CHAPTER 1

Vectors, Points, and Coordinates

1.1 The Concept of a Vector

1.1.1 Directed Segments (Bound Vectors)

A line segment with a specified origin and endpoint is called a *directed segment*. A directed segment with origin at point A and endpoint at point B is usually denoted by $\overline{AB}$. Graphically, a directed segment is represented as an arrow, with its tail at the origin and its tip at the endpoint (see Figure 1.1).

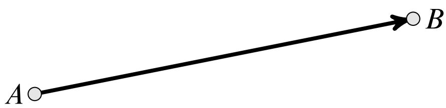


Figure 1.1: Directed segment

A directed segment is uniquely defined by its endpoints and their order. In other words, a directed segment is uniquely determined if the origin and endpoint are specified. If the two points coincide, we say that the segment is a *null directed segment*, and it is written as[1] $\overline{AA} = \bar{0}$.

[1]The notation for the null vector is incomplete, as it does not emphasise that it is the null vector at point A. Practically, a null vector at a point reduces to the point itself.

If a unit of length is chosen, we can define the length of the directed segment $\overline{AB}$ as the length of the undirected segment AB, denoted as:

$$\left\|\overline{AB}\right\| = |AB|$$

or simply,

$$\left\|\overline{AB}\right\| = AB.$$

The length of a directed segment is also called its *magnitude* or *norm*.

Two directed segments $\overline{AB}$ and $\overline{CD}$ are said to be *equal* if $A = C$ and $B = D$, i.e., if they have the same origin and endpoint.

A directed segment $\overline{AB}$ is also referred to as a *bound vector* with origin at point A and endpoint at point B. By fixing point A, we can define the operations of addition and scalar multiplication for all bound vectors originating from A, and it can be shown that this set forms a real vector space. However, in geometry, the concept of bound vectors is of limited interest since vectors often originate from different points, requiring rules to operate on such vectors. To this end, we slightly modify the concept of a vector so that its origin (or application point) no longer plays a role.

1.1.2 Free Vectors

As mentioned earlier, we aim to develop a vector theory that allows comparison of vectors with different origins.

We start with some definitions. First, we say that two *non-null* directed segments $\overline{AB}$ and $\overline{CD}$ have the *same direction* if the lines AB and CD are parallel. By convention, a null segment is considered to have the same direction as any other directed segment.

If the two directed segments (non-null) have the same direction but their supporting lines do not coincide, we say they have *the same sense* if the undirected segments AC and BD do not intersect (see Figure 1.2). However, if these two segments do intersect, we say that the directed segments $\overline{AB}$ and $\overline{CD}$ have *opposite senses* (see Figure 1.3).

If the non-null directed segments $\overline{AB}$ and $\overline{CD}$ have the same supporting line: $AB = CD$ (as lines), we say they have the same sense if there exists a third directed segment, $\overline{EF}$, having the same direction (but not the same supporting line) as $\overline{AB}$ and $\overline{CD}$, and which has the same sense as both segments. Otherwise, we say that the segments $\overline{AB}$ and $\overline{CD}$ have opposite senses.

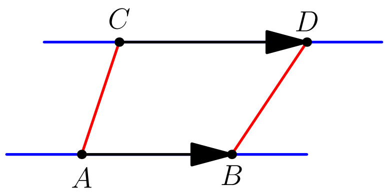


Figure 1.2: Directed segments with the same sense

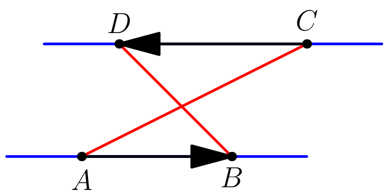


Figure 1.3: Directed segments with opposite senses

By convention, the null vector is considered to have the same sense as any other vector[2].

Whenever we say that two directed segments have the same sense, we implicitly assume, even if not explicitly stated, that the segments have the same direction. The relation "same sense" is not defined for pairs of directed segments that do not have the same direction. We sometimes say that two directed segments that have the same direction and the same sense have *the same orientation*.

Definition 1.1. Two directed segments $\overline{AB}$ and $\overline{CD}$ are said to be *equipollent*, written as $\overline{AB} \sim \overline{CD}$, if either both are null, or both are non-null and they have the same direction, the same sense, and the same magnitude.

It is easy to see that the relation of equipollence is an equivalence relation (i.e., it is reflexive, symmetric, and transitive).

Definition 1.2. A *free vector* is an equivalence class of directed segments under the relation of equipollence. The free vector determined by the directed segment $\overline{AB}$ is denoted by $\overrightarrow{AB}$. Thus,

$$\overrightarrow{AB} = \{\overline{CD} \mid \overline{CD} - \text{directed segment such that } \overline{CD} \sim \overline{AB}\}$$

[2] This ultimately means that the concept of *sense* is not well-defined for the null vector.

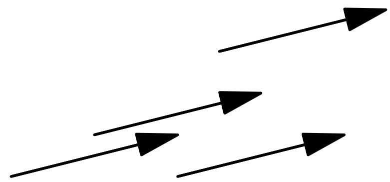


Figure 1.4: Free vector

Thus, a free vector is, in fact, a family of bound vectors, all of which are equivalent to each other. There is a kinematic interpretation of the free vector, which actually suggests its name: a free vector can be viewed as a directed segment whose origin has not been fixed. It can be moved to any point in space, provided that its magnitude, direction, and sense remain unchanged. Naturally, the rigorous meaning of this statement is that *if $\overrightarrow{AB}$ is a free vector, then, for any point C in space, there exists a bound vector $\overline{CD}$ with origin at C such that $\overline{AB} \sim \overline{CD}$* (see Figure 1.4).

Two free vectors are called *equal* if they are equal as an equivalence class, i.e., they consist of the same directed segments. In other words,

$$\overrightarrow{AB} = \overrightarrow{CD} \quad \Longleftrightarrow \quad \overline{AB} \sim \overline{CD}.$$

Usually, if we do not wish to emphasise a representative of a free vector, we use bold lowercase letters, typically from the beginning of the alphabet, $\mathbf{a}, \mathbf{b}, \ldots$. The null vector is denoted by $\mathbf{0}$[3]. A free vector is represented by one of the directed segments that form it.

Assume that a free vector $\mathbf{a}$ and a point A are given. Clearly, there exists a unique point B in space such that

$$\overrightarrow{AB} = \mathbf{a}.$$

We will say that, by constructing the point B for which the above relation holds, we have *attached* the free vector $\mathbf{a}$ to the point A.

The *magnitude* of the free vector $\mathbf{a}$ is defined as the magnitude of any of the directed segments that constitute it. The magnitude of $\mathbf{a}$ is denoted by $\|\mathbf{a}\|$.

Suppose two vectors $\mathbf{a}$ and $\mathbf{b}$ are given. We attach them to a point O (we construct the points A and B such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$). Then the *angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$ is, by definition, the angle (ranging between 0 and π) between the directed segments $\overline{OA}$ and $\overline{OB}$. Clearly, this angle does not depend on the choice of the point O.

[3] An alternative notation is to place an arrow above the letter without making it bold: $\vec{a}, \vec{b}, \ldots$.

We say that a directed segment $\overline{AB}$ is parallel to a line Δ (or to a plane Π) if its supporting line is parallel to the line Δ (or to the plane Π). The null segment is conventionally considered to be parallel to any line or plane. Free vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are called *collinear (coplanar)* if the segments that constitute them are parallel to the same line (or lie in the same plane).

Finally, we give another interpretation of free vectors. Consider the free vector $\overrightarrow{AB}$ (i.e., the set of all directed segments equivalent to the directed segment $\overline{AB}$). We consider the transformation of space that maps any point C to a point D such that $\overline{CD} \sim \overline{AB}$. Such a transformation is called a *parallel transport* or a translation by the vector $\overrightarrow{AB}$. This establishes a bijection between the set of all free vectors and the set of all translations. Due to this identification, translations are sometimes also referred to as *vectors*.

If a plane Π is fixed in space and only the points belonging to this plane are considered, then by a (free) vector, we mean an equivalence class of directed segments lying in that plane. Similarly, vectors on a line are defined.

1.2 Vector Addition

Let us consider two vectors $\mathbf{a}$ and $\mathbf{b}$. We choose an arbitrary point O in space and construct a point A such that $\overrightarrow{OA} = \mathbf{a}$ and a point B such that $\overrightarrow{AB} = \mathbf{b}$.

Definition 1.3. The vector $\overrightarrow{OB}$ is called the *sum of the vectors* $\mathbf{a}$ and $\mathbf{b}$ and is denoted by $\mathbf{a} + \mathbf{b}$.

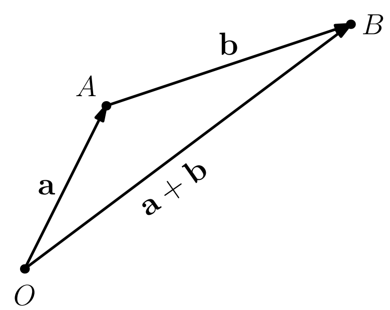


Figure 1.5: Triangle Rule

It is clear, from basic geometric reasoning, that the sum $\mathbf{a}+\mathbf{b}$ does not depend on the choice of point O. The method of constructing the sum of two vectors described above is called the *triangle rule* (or the *closure rule*, because the sum of the two vectors is determined by the directed segment that closes the triangle formed by the two directed segments representing the two free vectors being added).

If the vectors **a** and **b** are not collinear, then there is another method to determine the sum of two vectors, which, naturally, gives the same result as the triangle rule. Let **a** and **b** be two non-collinear vectors. We choose a point O and attach the two vectors to point O, that is, we determine points A and B such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Since the vectors **a** and **b** are not collinear, it follows that the points O, A, and B are not collinear, so they determine a plane. In this plane, we construct the parallelogram $OACB$. As it is easily observed that $\overrightarrow{BC} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$, it follows, based on the triangle rule mentioned above, that the following equalities hold:

$$\overrightarrow{OC} = \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}. \tag{1.2.1}$$

There are two equalities because there are two situations in which the triangle rule can be applied, and in each case, the vector that closes the triangle is $\overrightarrow{OC}$.

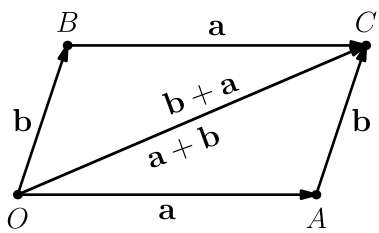


Figure 1.6: Parallelogram Rule

Thus, a new rule for calculating the sum of two vectors is established (*parallelogram rule*): To find the sum of two non-collinear vectors, the two vectors are attached to a point O, and a parallelogram is constructed with the resulting directed segments as sides. The diagonal of the parallelogram that starts at point O will then be the directed segment that determines the sum of the two vectors. The parallelogram rule allows for a very simple demonstration of the *commutativity* of vector addition (see formula (1.2.1)) for the case of *non-collinear* vectors. For the case of collinear vectors, commutativity can be easily verified using the closure rule, both for vectors oriented in the same direction and for those oriented in opposite directions. Thus, the operation of free vector addition is commutative.

Let us now consider three vectors **a**, **b**, **c**. We attach vector **a** to a point O, thereby constructing point A such that $\overrightarrow{OA} = \mathbf{a}$. Next, we construct point B such that $\overrightarrow{AB} =$

$\mathbf{b}$. According to the definition of addition, $\overrightarrow{OB} = \mathbf{a} + \mathbf{b}$. We now add vector $\mathbf{c}$ to this result. To do so, we construct point C such that $\overrightarrow{BC} = \mathbf{c}$. We then have

$$\overrightarrow{OC} = (\mathbf{a} + \mathbf{b}) + \mathbf{c}. \tag{1.2.2}$$

On the other hand, $\overrightarrow{AC} = \mathbf{b} + \mathbf{c}$, so

$$\overrightarrow{OC} = \mathbf{a} + (\mathbf{b} + \mathbf{c}). \tag{1.2.3}$$

Combining (1.2.2) and (1.2.3), we obtain

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}),$$

which means that vector addition is *associative*.

The operation of free vector addition also has a *neutral element*, which is, naturally, the zero vector $\mathbf{0}$, because it is evident that for any vector $\mathbf{a}$ we have:

$$\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a}.$$

Finally, we observe that every vector has an inverse with respect to the addition operation. Thus, if the free vector $\mathbf{a}$ is represented by the directed segment $\overline{AB}$, we denote by $-\mathbf{a}$ the free vector represented by the directed segment $\overline{BA}$, and it is immediately evident that:

$$\mathbf{a} + (-\mathbf{a}) = 0.$$

With this in mind, we can state that *the set of all free vectors in space forms an abelian group with respect to the operation of vector addition*.

As in any abelian (additive) group, together with vector addition we can also define vector subtraction, which is given by:

$$\mathbf{a} - \mathbf{b} := \mathbf{a} + (-\mathbf{b}).$$

If we attach vector $\mathbf{a}$ to a point O and choose A and B such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, then, as is easily observed, $\mathbf{a} - \mathbf{b} = \overrightarrow{BA}$ or

$$\overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{BA}.$$

The parallelogram rule can also be applied without difficulty to determine the difference of two vectors, not just their sum. As seen above, to determine the sum of two vectors, we consider a representative of each vector with origins at the same point. A parallelogram is completed by drawing parallels to the supporting lines of the two directed segments through their endpoints. Then the sum of the two vectors is the vector represented by the directed segment associated with the diagonal that originates at the application point of the two vectors. The difference between the two vectors, on the other hand, is determined by the other diagonal, whose orientation is chosen such that its origin lies at the tip of the subtracted vector, and its end lies at the tip of the minuend.

1.3 Multiplication of a Vector by a Real Number (Scalar Multiplication)

Our goal is to equip the set of free vectors in space with the structure of a vector space. So far, we have seen that this set, together with vector addition, forms an abelian group. What remains is to define scalar multiplication (external multiplication) and verify the compatibility of this operation with vector addition.

Definition 1.4. Let $\mathbf{a}$ be a vector and $\lambda \in \mathbb{R}$ a real number. The *product of the vector* $\mathbf{a}$ *with the scalar* λ is, by definition, a vector denoted by $\lambda\mathbf{a}$, characterised as follows:

(i) The magnitude of $\lambda\mathbf{a}$ is given by

$$\|\lambda\mathbf{a}\| := |\lambda| \cdot \|\mathbf{a}\|,$$

where the product on the right-hand side is the product of real numbers;

(ii) The direction of $\lambda\mathbf{a}$ coincides with the direction of $\mathbf{a}$;

(iii) The sense of $\lambda\mathbf{a}$ coincides with the sense of $\mathbf{a}$ if $\lambda > 0$, or with the opposite sense of $\mathbf{a}$ if $\lambda < 0$.

We now list a series of fundamental properties of scalar multiplication of vectors.

1) $1\mathbf{a} = \mathbf{a}$.

2) $(-1)\mathbf{a} = -\mathbf{a}$.

3) $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$ for any scalars $\lambda, \mu \in \mathbb{R}$ and any vector $\mathbf{a}$.

These three properties are evident, following directly from the definition of scalar multiplication.

4) $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$, for any $\lambda \in \mathbb{R}$ and for any two free vectors $\mathbf{a}, \mathbf{b}$.

Property 4) is evident in certain special cases: if the scalar λ is zero, if the sum $\mathbf{a} + \mathbf{b}$ is zero, or if at least one of the vectors is zero.

Leaving aside these situations, let us assume $\lambda > 0$, and the vectors $\mathbf{a}$ and $\mathbf{b}$ are not collinear. We choose an arbitrary point O and construct points A and B such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$, and hence $\overrightarrow{OB} = \mathbf{a} + \mathbf{b}$. We further construct points A' and B' such that

$$\overrightarrow{OA'} = \lambda\mathbf{a}, \quad \overrightarrow{OB'} = \lambda(\mathbf{a} + \mathbf{b}). \tag{1.3.1}$$

The triangles OAB and $OA'B'$ are similar, as they share a common angle and the sides corresponding to this common angle are proportional. It follows that $|A'B'| = |\lambda| \cdot |AB| = \lambda \cdot |AB|$. Moreover, since the vectors $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ also have the same direction and sense, it follows that

$$\overrightarrow{A'B'} = \lambda\mathbf{b}. \tag{1.3.2}$$

Property 4) now follows from relations (1.3.1) and (1.3.2).

Now let us assume $\lambda > 0$, but the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear. We choose an arbitrary point O and construct points A and B such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$.

We choose a point S, not on the line OAB, and construct the rays SO, SA, and SB. On the ray SO, we choose a point O' such that $|SO'| = \lambda \cdot |SO|$. Through O', we draw a line δ, parallel to the line OAB, and denote by A' and B' the intersections of this line with the rays SA and SB, respectively. In this way, we obtain three pairs of similar triangles:

$$\Delta OAS \simeq \Delta O'A'S, \quad \Delta ABS \simeq \Delta A'B'S, \quad \Delta OBS \simeq \Delta O'B'S.$$

It immediately follows that

$$\overrightarrow{O'A'} = \lambda\mathbf{a}, \quad \overrightarrow{A'B'} = \lambda\mathbf{b}, \quad \overrightarrow{O'B'} = \lambda(\mathbf{a} + \mathbf{b}),$$

from which property 4) follows. The case $\lambda < 0$ is treated similarly.

5) $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$ for any scalars λ and μ and any vector $\mathbf{a}$.

As with property 4), this equality is evident in certain specific cases: if $\mathbf{a} = 0$, if $\lambda + \mu = 0$, or if at least one of the scalars is zero.

Leaving aside these situations, let us assume first that λ and μ have the same sign. Clearly, the vectors in both members of relation 5) have the same direction and sense. To prove the relation, it suffices to show that they also have the same magnitude. Given our assumptions:

$$\begin{aligned}\|\lambda\mathbf{a} + \mu\mathbf{a}\| &= \|\lambda\mathbf{a}\| + \|\mu\mathbf{a}\| = |\lambda| \cdot \|\mathbf{a}\| + |\mu| \cdot \|\mathbf{a}\| = (|\lambda| + |\mu|) \cdot \|\mathbf{a}\| = \\ &= |\lambda + \mu| \cdot \|\mathbf{a}\| = \|(\lambda + \mu)\mathbf{a}\|.\end{aligned}$$

Now suppose λ and μ have opposite signs, and to fix ideas, let us assume $|\lambda| > \mu$. Then the numbers $\lambda + \mu$ and $-\mu$ have the same sign, and based on the above, we have

$$(\lambda + \mu)\mathbf{a} + (-\mu)\mathbf{a} = (\lambda + \mu - \mu)\mathbf{a} = \lambda\mathbf{a},$$

a relation equivalent to property 5).

Properties 1)–5), together with the fact that the set $\mathcal{V}$ is an abelian group (as demonstrated in the previous section), imply that this set forms a *vector space* over the set of real numbers.

Remark. Properties 4) and 5) can be extended inductively to any finite number of summands. In other words, it can be easily shown that:

$$\lambda(\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_k) = \lambda\mathbf{a}_1 + \lambda\mathbf{a}_2 + \cdots + \lambda\mathbf{a}_k,$$
$$(\lambda_1 + \lambda_2 + \cdots + \lambda_k)\mathbf{a} = \lambda_1\mathbf{a} + \lambda_2\mathbf{a} + \cdots + \lambda_k\mathbf{a},$$

for any natural $k \geq 2$, any real numbers $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_k$, and any vectors $\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$.

1.4 Projections of Vectors onto Axes or Planes

1.4.1 Axes

Let us choose an arbitrary line in space. We designate one of the two senses on this line as *positive* and indicate it in the diagram with an arrow. The opposite sense will be designated as *negative*. A line with a designated positive sense is called an *axis* or an *oriented line*.

Now we choose an axis Δ and select a non-zero segment on it as the unit of length. The *magnitude* of a directed segment $\overline{AB}$ on the axis, denoted by (AB), is defined as the number:

$$(AB) = \begin{cases} \|\overline{AB}\| & \text{if } \overline{AB} \text{ has the same sense as } \Delta, \\ -\|\overline{AB}\| & \text{if } \overline{AB} \text{ and } \Delta \text{ have opposite senses.} \end{cases} \tag{1.4.1}$$

The number (AB) is also called the *signed length* or *oriented length* of the directed segment $\overline{AB}$. Evidently, $(AB) = -(BA)$.

Theorem 1.1 (Chasles)**.** *For any three points A, B, C lying on an axis with a chosen unit of length, the following relation holds:*

$$(AB) + (BC) = (AC). \tag{1.4.2}$$

Proof This can be directly verified by considering the different relative positions of points A, B, and C. □

1.4.2 Projection onto an Axis in Space

Let Δ be an axis in space and Π a plane that is not parallel to Δ. Through an arbitrary point A in space, we draw a plane Π_1 parallel to the plane Π. This plane intersects the axis Δ at a point A'. The point A' is called the *projection of the point A onto the axis Δ, parallel to the plane* Π. If the plane Π is perpendicular to the axis Δ, the projection is called *orthogonal.* In this case, A' is the foot of the perpendicular dropped from point A to the axis Δ.

Now consider an arbitrary directed segment $\overline{AB}$. If we project points A and B onto the axis Δ, parallel to the plane Π, we obtain a directed segment $\overline{A'B'}$, which is called the *projection of the directed segment $\overline{AB}$ onto the axis Δ, parallel to the plane* Π. Now suppose a unit of length (a scale) has been chosen on the axis Δ. Then we can also refer to the magnitude of the projection of a segment onto the axis, denoted by $\mathrm{pr}_\Delta \overline{AB}$ ($\parallel \Pi$).

It is clear that two equivalent directed segments will have equivalent projections onto any axis, and the magnitudes of these projections will be equal.

Now consider a free vector $\mathbf{a}$, i.e., an equivalence class of directed segments. The projections of these segments onto the axis Δ, parallel to the plane Π, form, as mentioned above, a family of equivalent directed segments, i.e., they form *a free vector on the line*. This vector is called the *projection of the vector* $\mathbf{a}$ *onto the axis Δ, parallel to the plane* Π, and it is denoted by $\mathrm{pr}_\Delta\, \mathbf{a}$ ($\parallel \Pi$).

1.4.3 Projection onto an Axis in a Plane

Now suppose both the axis Δ and the figure being projected lie in the same plane Π. We reformulate the definition of projection as follows. Let Δ_1 be a line in the plane Π, which is not parallel to the axis Δ. Through a point A in the plane, we draw a line parallel to the line Δ_1, which intersects the axis at a point A'. This point is called the *projection of the point A onto the axis Δ, parallel to the line* Δ_1. The other notions in the previous paragraph are defined analogously and share the same properties.

1.4.4 Projection onto a Plane

Let Π be a plane and Δ a line that is not parallel to the plane. Through a point A in space, we draw a line Δ_1, parallel to the line Δ. The line Δ_1 intersects the plane Π at a point A', which is called the *projection of the point A onto the plane Π, parallel to the line* Δ. If the line Δ is perpendicular to the plane Π, the projection is called *orthogonal.*

1.4.5 Projection of the Sum of Vectors

Suppose two vectors $\mathbf{a}$ and $\mathbf{b}$ are projected onto the axis Δ. The projection is either parallel to a plane Π or parallel to a line Δ_1, if both the vectors and the axis lie in the same plane.

We choose a point O and construct points A and B such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$, and hence $\overrightarrow{OB} = \mathbf{a} + \mathbf{b}$.

If O', A', B' are the projections of the points O, A, B onto the axis Δ, then the vectors $\overrightarrow{O'A'}$, $\overrightarrow{A'B'}$, and $\overrightarrow{O'B'}$ are, respectively, the projections of the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{a} + \mathbf{b}$. It follows that *the projection of the sum of the vectors equals the sum of the projections of the terms*. It is clear that this property can be extended, without difficulty, to sums of more than two vectors. Moreover, if a unit of length is chosen on the axis, then, by virtue of equality (1.4.2), we also have:

$$(O'B') = (O'A') + (A'B')$$

or, using the notation introduced earlier,

$$\mathrm{pr}_\Delta(\mathbf{a} + \mathbf{b}) = \mathrm{pr}_\Delta\, \mathbf{a} + \mathrm{pr}_\Delta\, \mathbf{b}, \tag{1.4.3}$$

i.e., the magnitude of the projection of the sum of the vectors onto an axis is equal to the sum of the magnitudes of the projections of the terms.

1.4.6 Projection of the Product of a Vector by a Scalar

We will demonstrate that by multiplying a vector $\mathbf{a}$ by a scalar λ, the projection of this vector onto any axis Δ, as well as the magnitude of this projection, are multiplied by the same scalar.

If $\mathbf{a} = 0$ or $\lambda = 0$, there is nothing to demonstrate. We therefore assume that $\mathbf{a} \neq 0$ and $\lambda \neq 0$. We choose an axis Δ, fix a point O on it, and determine points A and B in space such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \lambda\mathbf{a}$. As is known, the two vectors have the same direction, and their senses coincide for $\lambda > 0$ and are opposite for $\lambda < 0$.

Projecting points A and B onto the axis Δ at points A' and B', we obtain two similar triangles, OAA' and OBB'. From the properties of similarity, our claim follows immediately, and we have:

$$\mathrm{pr}_\Delta(\lambda\mathbf{a}) = \lambda\, \mathrm{pr}_\Delta\, \mathbf{a}. \tag{1.4.4}$$

1.4.7 Projection of a Linear Combination of Vectors

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ be an arbitrary system of vectors (not necessarily distinct), and $\lambda_1, \lambda_2, \ldots, \lambda_k$ an arbitrary system of k real numbers. Then the vector

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k$$

is called a *linear combination* of the given vectors.

From equalities (1.4.3) and (1.4.4), we obtain the following equality:

$$\operatorname{pr}_\Delta(\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k) = \lambda_1 \operatorname{pr}_\Delta \mathbf{a}_1 + \lambda_2 \operatorname{pr}_\Delta \mathbf{a}_2 + \cdots + \lambda_k \operatorname{pr}_\Delta \mathbf{a}_k,$$

i.e., *the magnitude of the projection of a linear combination of vectors onto an axis is equal to the linear combination of the projections of the vectors (with the same coefficients).*

1.5 Linear Dependence of Vectors

Definition 1.5. The vectors

$$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \tag{1.5.1}$$

are called *linearly dependent* if there exist real numbers

$$\lambda_1, \ldots, \lambda_k, \tag{1.5.2}$$

not all zero, such that

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k = \mathbf{0}. \tag{1.5.3}$$

Otherwise, the vectors are called *linearly independent*.

It is clear that the vectors are linearly independent if and only if from the equality (1.5.3) it follows that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0.$$

It is also said that the vectors (1.5.1) form a *linearly dependent system*, or a *linearly independent system*.

If a vector $\mathbf{a}$ can be expressed in terms of the vectors (1.5.1) as

$$\mathbf{a} = \mu_1 \mathbf{a}_1 + \mu_2 \mathbf{a}_2 + \cdots + \mu_k \mathbf{a}_k,$$

then we say that $\mathbf{a}$ is *a linear combination of these vectors*.

Theorem 1.2. *For the vectors (1.5.1) (with $k > 1$) to be linearly dependent, it is necessary and sufficient that at least one of these vectors can be expressed as a linear combination of the others.*

Proof Let us assume that the vectors (1.5.1) satisfy a relation of the form (1.5.3), in which at least one of the coefficients is non-zero. It is clear that we do not lose generality by assuming that the last coefficient is non-zero: $\lambda_k \neq 0$. Then, from the equality (1.5.3), we obtain

$$\mathbf{a}_k = -\frac{\lambda_1}{\lambda_k}\mathbf{a}_1 - \frac{\lambda_2}{\lambda_k}\mathbf{a}_2 \cdots - \frac{\lambda_{k-1}}{\lambda_k}\mathbf{a}_{k-1},$$

which means that $\mathbf{a}_k$ is a linear combination of the other $k-1$ vectors.

Conversely, let us assume that one of the vectors (1.5.1), for instance, again the last one, is a linear combination of the other $k-1$ vectors:

$$\mathbf{a}_k = \beta_1\mathbf{a}_1 + \beta_2\mathbf{a}_2 + \cdots + \beta_{k-1}\mathbf{a}_{k-1}.$$

If we bring all terms to the left-hand side, we obtain

$$-\beta_1\mathbf{a}_1 - \beta_2\mathbf{a}_2 - \cdots - \beta_{k-1} + 1 \cdot \mathbf{a}_k = 0,$$

an equality of the form (1.5.3) with at least one non-zero coefficient (since the coefficient of $\mathbf{a}_k$ is 1), meaning that the vectors are indeed linearly dependent. □

Corollary 1.1. *If the vectors (1.5.1) are linearly independent, then none of them can be expressed as a linear combination of the others. In particular, none of the vectors can be equal to zero.*

In this course, we will typically deal with systems of at most three vectors. Therefore, it is interesting to highlight the geometric meaning of linear dependence or independence for systems of 1, 2, or 3 vectors.

It is evident that a system consisting of a single vector is linearly dependent if and only if the vector is zero.

For the case of two vectors, we have the following result:

Theorem 1.3. *Two vectors are linearly dependent if and only if they are collinear.*

Proof Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be two linearly dependent vectors. By the previous theorem, there exists a non-zero scalar λ such that $\mathbf{a}_2 = \lambda\mathbf{a}_1$, meaning that the vectors are collinear.

Conversely, assume now that the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are collinear. If at least one of the vectors is zero, for instance $\mathbf{a}_2 = 0$, then we can write

$$0\mathbf{a}_1 + 1\mathbf{a}_2 = 0,$$

meaning that the vectors are linearly dependent. Now assume that both vectors are non-zero. We choose a point O and construct the points A_1 and A_2 such that $\overrightarrow{OA_1} = \mathbf{a}_1$ and $\overrightarrow{OA_2} = \mathbf{a}_2$. Due to the collinearity of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, the points O, A_1, and A_2 lie on the same line Δ. If $A_1 = A_2$, then, clearly, the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are equal, and hence they are linearly dependent, as one is a linear combination of the other. If $A_1 \neq A_2$, let $\lambda = \|\mathbf{a}_2\|/\|\mathbf{a}_1\|$. If the directed segments $\overline{OA_1}$ and $\overline{OA_2}$ have the same direction, then $\mathbf{a}_2 = \lambda\mathbf{a}_1$, while if they have opposite directions, then $\mathbf{a}_2 = -\lambda\mathbf{a}_1$. $\square$

Corollary 1.2. *Two vectors are linearly independent if and only if they* are not collinear.

Linearly independent vectors play a crucial role. In particular, they allow us to decompose other vectors. A prototype of such a decomposition is given by the following theorem:

Theorem 1.4. *Suppose that in a plane* Π*, two non-collinear vectors* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *are given. Then any other vector* $\mathbf{a}$ *in the plane can be decomposed in terms of the vectors* $\mathbf{e}_1$ *and* $\mathbf{e}_2$*, meaning there exist two real numbers (uniquely determined)* x *and* y *such that*

$$\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2. \tag{1.5.4}$$

The remainder of the theorem and related results follow similarly.

Proof The fact that the vectors belong to the plane Π implies that their directions are parallel to the plane or, equivalently, that if we attach them to a point in the plane, the resulting directed segments lie entirely within the plane. Let O be an arbitrary point in the plane. Then there exist (and are unique) three points E_1, E_2, and M in the plane such that

$$\overrightarrow{OE_1} = \mathbf{e}_1, \quad \overrightarrow{OE_2} = \mathbf{e}_2, \quad \overrightarrow{OM} = \mathbf{a}.$$

Project the point M onto the line OE_1, parallel to the line OE_2, obtaining a point M_1. Similarly, project M onto the line OE_2, parallel to OE_1, obtaining a point M_2. Since the vectors $\overrightarrow{OE_1}$ and $\overrightarrow{OM_1}$ are collinear, and $\overrightarrow{OE_1} \neq 0$, it follows that there exists a real number x such that $\overrightarrow{OM_1} = x\overrightarrow{OE_1}$. Similarly, there exists a real number y such that $\overrightarrow{OM_2} = y\overrightarrow{OE_2}$. Since $\overrightarrow{OM} = \overrightarrow{OM_1} + \overrightarrow{OM_2}$, the equality (1.5.4) is satisfied.

It remains to demonstrate the uniqueness of the real numbers x and y. Suppose there exist other real numbers, x' and y', such that

$$\mathbf{a} = x'\mathbf{e}_1 + y'\mathbf{e}_2. \tag{1.5.5}$$

Subtracting (1.5.5) from (1.5.4) gives

$$(x - x')\mathbf{e}_1 + (y - y')\mathbf{e}_2 = 0. \tag{1.5.6}$$

Since the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent, it follows that $x - x' = 0$ and $y - y' = 0$, which implies $x = x'$ and $y = y'$. □

We now examine what happens in the case where we have *three* vectors. The following result holds:

Theorem 1.5. *For three vectors to be linearly dependent, it is necessary and sufficient that they are coplanar.*

Proof Assume that the vectors

$$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \tag{1.5.7}$$

are linearly dependent. Then at least one of them can be expressed as a linear combination of the other two. Without loss of generality, let this be the third vector. Thus, there exist two real numbers λ_1 and λ_2 such that

$$\mathbf{a}_3 = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2. \tag{1.5.8}$$

If we attach the vectors to a point O, we obtain three points M_1, M_2, M_3 such that

$$\overrightarrow{OM_1} = \mathbf{a}_1, \quad \overrightarrow{OM_2} = \mathbf{a}_2, \quad \overrightarrow{OM_3} = \mathbf{a}_3.$$

If the vectors $\overrightarrow{OM_1}$ and $\overrightarrow{OM_2}$ are non-collinear, then the points O, M_1, M_2 are non-collinear, and thus they define a plane Π. Due to the relation (1.5.8), the vector $\overrightarrow{OM_3}$ also lies in the plane Π, so the three vectors are coplanar. If the vectors $\overrightarrow{OM_1}$ and $\overrightarrow{OM_2}$ are collinear, then from (1.5.8), the vector $\overrightarrow{OM_3}$ is also collinear with the other two, and thus the three vectors are trivially coplanar.

Conversely, suppose the vectors (1.5.7) are coplanar. Assume first that two of the vectors, say $\mathbf{a}_1$ and $\mathbf{a}_2$, are non-collinear. Then, by virtue of Theorem 1.4, there exist constants λ_1 and λ_2 such that

$$\mathbf{a}_3 = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2,$$

and thus the vectors (1.5.7) are linearly dependent.

If all three vectors are collinear, then, for example, $\mathbf{a}_1 = \lambda \mathbf{a}_2$, which can be rewritten as

$$\mathbf{a}_1 = \lambda \mathbf{a}_2 + 0\mathbf{a}_3,$$

and again we conclude that the three vectors are coplanar. $\square$

As a consequence of this theorem, we can conclude that *in space, there exist triples of linearly independent vectors*.

In space, we also have a result similar to Theorem 1.4, namely:

Theorem 1.6. *If the vectors*

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \tag{1.5.9}$$

are linearly independent, and $\mathbf{a}$ *is any vector, then there exist three real numbers* x, y, z *such that*

$$\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3. \tag{1.5.10}$$

This decomposition of $\mathbf{a}$ *is unique.*

Proof Choose an arbitrary point O in space, and determine points E_1, E_2, E_3, and M such that

$$\overrightarrow{OE_1} = \mathbf{e}_1, \quad \overrightarrow{OE_2} = \mathbf{e}_2, \quad \overrightarrow{OE_3} = \mathbf{e}_3, \quad \overrightarrow{OM} = \mathbf{a}.$$

Denote by M_1, M_2, M_3 the projections of point M onto the lines OE_1, OE_2, OE_3, parallel to the planes OE_2E_3, OE_1E_3, and OE_1E_2, respectively. It is easily observed that

$$\overrightarrow{OM} = \overrightarrow{OM_1} + \overrightarrow{OM_2} + \overrightarrow{OM_3}. \tag{1.5.11}$$

Since the vectors $\overrightarrow{OE_1}$ and $\overrightarrow{OM_1}$ are collinear and $\overrightarrow{OE_1} \neq 0$, there exists a real number x such that $\overrightarrow{OM_1} = x\overrightarrow{OE_1}$. Similarly, there exist real numbers y and z such that $\overrightarrow{OM_2} = y\overrightarrow{OE_2}$ and $\overrightarrow{OM_3} = z\overrightarrow{OE_3}$.

Unicity of the numbers x, y, z can be proved in the same way as in Theorem 1.4. $\square$

Finally, we consider the case where we have *more than three vectors*. The following theorem provides the answer:

Theorem 1.7. *Any four vectors are linearly dependent.*

Proof Suppose that among the four vectors

$$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}, \tag{1.5.12}$$

three are linearly independent, for instance,

$$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3. \tag{1.5.13}$$

Then, by Theorem 1.6, there exist three real numbers $\lambda_1, \lambda_2, \lambda_3$ such that

$$\mathbf{a} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \lambda_3 \mathbf{a}_3,$$

which means that the four vectors are indeed linearly dependent.

If the vectors (1.5.13) are linearly dependent, i.e., there exists a relation of the form

$$\mu_1 \mathbf{a}_1 + \mu_2 \mathbf{a}_2 + \mu_3 \mathbf{a}_3 = 0, \tag{1.5.14}$$

where not all coefficients are zero, this relation can be rewritten as

$$\mu_1 \mathbf{a}_1 + \mu_2 \mathbf{a}_2 + \mu_3 \mathbf{a}_3 + 0\mathbf{a} = 0,$$

which again shows that the vectors (1.5.12) are linearly dependent. □

Adding a Point and a Vector. Operations with Points

Let $O \in \mathbb{R}^3$ be a given point. Consider two other points, P and Q. Then, the following relation holds:

$$\overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{PQ}$$

or

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}. \tag{1.5.15}$$

We rewrite equation (1.5.15) as

$$Q = P + \overrightarrow{PQ} \tag{1.5.16}$$

("removing" the point O). This notation is allowed because the vector $\overrightarrow{PQ}$ does *not* depend on the choice of the point O.

We now give the following general definition:

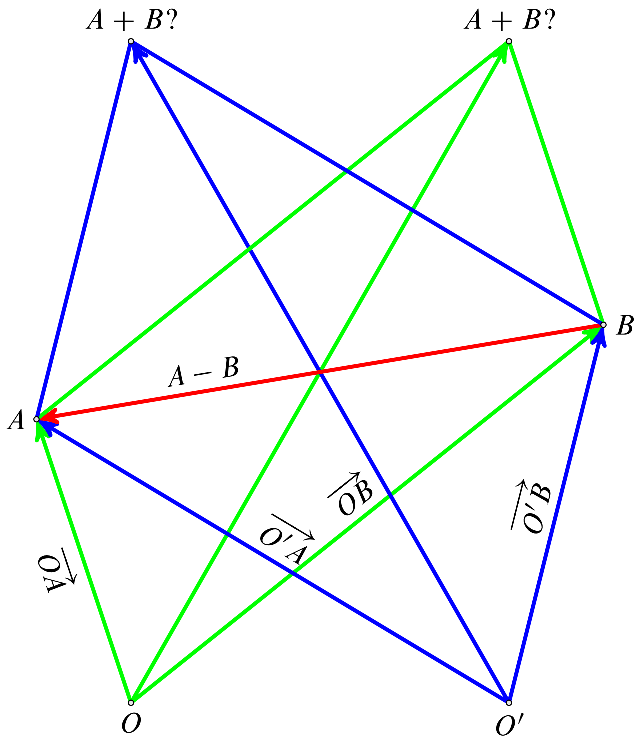


Figure 1.7: "Addition" of points

Definition 1.6. The sum of a point P and a vector $\mathbf{v}$ is a *point* Q (uniquely determined!) such that $\overrightarrow{PQ} = \mathbf{v}$. We write this as

$$Q = P + \mathbf{v}. \tag{1.5.17}$$

This relation can also be interpreted by saying that the vector $\mathbf{v}$ is the *difference* $Q - P$ of the points Q and P.

It is natural to ask whether more general combinations of points make sense, beyond just the difference of two points. An answer is given by the following result:

Proposition 1.1. *Let $A_1, \ldots, A_n$ be n points in space, and let $\alpha_1, \ldots, \alpha_n$ be n real numbers such that*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0.$$

Then, the vector

$$\mathbf{v} = \alpha_1 \overrightarrow{OA_1} + \alpha_2 \overrightarrow{OA_2} + \cdots + \alpha_n \overrightarrow{OA_n}$$

does not depend on the choice of the point O.

Proof We have

$$\begin{aligned}\mathbf{v} &= -(\alpha_2 + \cdots + \alpha_n)\overrightarrow{OA_1} + \alpha_2\overrightarrow{OA_2} + \cdots + \alpha_n\overrightarrow{OA_n} = \\ &= \alpha_2\left(\overrightarrow{OA_2} - \overrightarrow{OA_1}\right) + \alpha_3\left(\overrightarrow{OA_3} - \overrightarrow{OA_1}\right) + \cdots + \alpha_n\left(\overrightarrow{OA_n} - \overrightarrow{OA_1}\right) = \\ &= \alpha_2\overrightarrow{A_1A_2} + \alpha_3\overrightarrow{A_1A_3} + \cdots + \alpha_n\overrightarrow{A_1A_n}.\end{aligned}$$

□

The vector $\mathbf{v}$ above is written as a combination of the points $A_1, \ldots, A_n$:

$$\mathbf{v} = \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n, \quad \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0.$$

A second way of combining points is suggested by the following result:

Proposition 1.2. *Let $A_1, \ldots, A_n$ be n points in space, and let $\alpha_1, \ldots, \alpha_n$ be n real numbers such that*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1.$$

Then the point P, given by

$$\overrightarrow{OP} = \alpha_1\overrightarrow{OA_1} + \alpha_2\overrightarrow{OA_2} + \cdots + \alpha_n\overrightarrow{OA_n}$$

does not depend on the choice of the point O.

Proof We have

$$\begin{aligned}\overrightarrow{OP} &= (1 - \alpha_2 - \alpha_3 - \cdots - \alpha_n)\overrightarrow{OA_1} + \alpha_2\overrightarrow{OA_2} + \cdots + \alpha_n\overrightarrow{OA_n} = \\ &= \overrightarrow{OA_1} + \alpha_2\left(\overrightarrow{OA_2} - \overrightarrow{OA_1}\right) + \cdots + \alpha_n\left(\overrightarrow{OA_n} - \overrightarrow{OA_1}\right) = \\ &= \overrightarrow{OA_1} + \alpha_2\overrightarrow{A_1A_2} + \cdots + \alpha_n\overrightarrow{A_1A_n}.\end{aligned}$$

This means that the point P can be written, using the rule for adding a point and a vector, as

$$\begin{aligned}P &= A_1 + \alpha_2\overrightarrow{A_1A_2} + \cdots + \alpha_n\overrightarrow{A_1A_n} = \\ &= A_1 + \alpha_2(A_2 - A_1) + \alpha_3(A_3 - A_1) + \ldots \alpha_n(A_n - A_1) = \\ &= (1 - \alpha_2 - \alpha_3 - \cdots - \alpha_n)A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n = \\ &= \alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n.\end{aligned}$$

□

Definition 1.7. A combination of points of the form

$$\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n,$$

with $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, is called an *affine combination* or *barycentric combination* of the points $A_1, \ldots, A_n$.

An affine combination in which all coefficients are positive is called a *convex combination*.

Definition 1.8. Let $\mathcal{S}$ be a set of points in space.

The set of all affine combinations of a finite number of points in $\mathcal{S}$ is called the *affine hull* of the set $\mathcal{S}$ and is denoted by $\mathrm{aff}(\mathcal{S})$.

The set of all convex combinations of a finite number of points in $\mathcal{S}$ is called the *convex hull* of the set $\mathcal{S}$ and is denoted by $\mathrm{conv}(\mathcal{S})$.

Definition 1.9. A subset $S \subset \mathbb{R}^3$ is called an *affine subspace of* $\mathbb{R}^3$ if there exists a point P and a vector subspace $\mathcal{D}$ of the space of free vectors such that

$$S = P + \mathcal{D} = \{P + \mathbf{v} \mid \mathbf{v} \in \mathcal{D}\}.$$

The *dimension* of the affine subspace S is, by definition, the dimension of the vector subspace $\mathcal{D}$. If the dimension is zero, the affine subspace reduces to the point P. If the dimension is 1, then the subspace is a line passing through P and having the direction given by any non-zero vector in $\mathcal{D}$. If the dimension is 2, then the subspace is a plane passing through P and parallel to two non-collinear vectors in $\mathcal{D}$. Finally, if the dimension is 3, the affine subspace coincides with the entire space $\mathbb{R}^3$.

Another fundamental concept, related to convex combinations, is that of a *convex set*, which plays a significant role in mathematics.

Definition 1.10. A set $S \subset \mathbb{R}^3$ is called *convex* if, whenever two distinct points A and B belong to S, the segment $[AB]$ is also contained in S, i.e., *if two distinct points belong to S, then the entire segment joining them is included in S.*

Examples of convex sets abound in mathematics. Any convex polygon is a convex set (here, by a polygon, we mean the closed polygonal line along with its interior, also known as the *solid polygon*). Similarly, any convex polyhedron (solid) is a convex set. It is evident that any affine subspace is also a convex set.

Examples. 1. The affine hull of a pair of distinct points A, B is the line AB.

Indeed, let M be a point in aff(A, B). Then there exist two real numbers α and β with $\alpha + \beta = 1$ such that

$$M = \alpha A + \beta B$$

or

$$M = \alpha A + (1 - \alpha)B,$$

meaning that

$$\overrightarrow{OM} = \alpha\overrightarrow{OA} + (1 - \alpha)\overrightarrow{OB},$$

from which

$$\overrightarrow{OM} - \overrightarrow{OB} = \alpha(\overrightarrow{OA} - \overrightarrow{OB}),$$

i.e.,

$$\overrightarrow{BM} = \alpha\overrightarrow{BA},$$

indicating that the vectors $\overrightarrow{BM}$ and $\overrightarrow{BA}$ are collinear, so the point M lies on the line AB. Conversely, suppose that the point M lies on the line AB. This means that the vectors $\overrightarrow{AM}$ and $\overrightarrow{AB}$ are collinear, i.e., there exists a real number α such that

$$\overrightarrow{AM} = \alpha\overrightarrow{AB},$$

or equivalently,

$$\overrightarrow{OM} - \overrightarrow{OA} = \alpha\left(\overrightarrow{OB} - \overrightarrow{OA}\right),$$

which simplifies to

$$\overrightarrow{OM} = (1 - \alpha)\overrightarrow{OA} + \alpha\overrightarrow{OB},$$

or

$$M = (1 - \alpha)A + \alpha B,$$

proving that M is an affine combination of the points A and B, or equivalently, $M \in$ aff(A, B).

2. The affine hull of three non-collinear points A, B, C is the plane determined by these three points.

   Indeed, let $M \in$ aff(A, B, C). Then there exist three real numbers α, β, and γ such that $\alpha + \beta + \gamma = 1$ and

$$M = \alpha A + \beta B + \gamma C,$$

or equivalently,

$$\overrightarrow{OM} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB} + \gamma\overrightarrow{OC}.$$

Since $\alpha + \beta + \gamma = 1$, we can rewrite this as

$$\overrightarrow{OM} = (1 - \beta - \gamma)\overrightarrow{OA} + \beta\overrightarrow{OB} + \gamma\overrightarrow{OC},$$

or equivalently,

$$\overrightarrow{AM} = \beta\overrightarrow{AB} + \gamma\overrightarrow{AC}.$$

This shows that the point M lies in the plane determined by the points A, B, and C.

Conversely, suppose M is a point in the plane determined by the points A, B, and C. Then, clearly, the vector $\overrightarrow{AM}$ lies in this plane. Since A, B, and C are non-collinear, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ form a basis for the space of vectors parallel to this plane. Therefore, there exist unique real numbers β and γ such that

$$\overrightarrow{AM} = \beta\overrightarrow{AB} + \gamma\overrightarrow{AC}.$$

Substituting, we have

$$\overrightarrow{OM} = \overrightarrow{OA} + \beta\left(\overrightarrow{OB} - \overrightarrow{OA}\right) + \gamma\left(\overrightarrow{OC} - \overrightarrow{OA}\right),$$

which simplifies to

$$\overrightarrow{OM} = (1 - \beta - \gamma)\overrightarrow{OA} + \beta\overrightarrow{OB} + \gamma\overrightarrow{OC},$$

or equivalently,

$$M = (1 - \beta - \gamma)A + \beta B + \gamma C,$$

showing that $M \in \operatorname{aff}(A, B, C)$.

3. The affine hull of four non-coplanar points A, B, C, D is the entire space $\mathbb{R}^3$.

   The proof is analogous to the case of three points, extended to three-dimensional space.

4. The convex hull of a pair of distinct points A, B is the segment $[AB]$.

   Indeed, let $M \in \operatorname{conv}(A, B)$. This means that there exists a real number $\alpha \in [0, 1]$ such that

$$M = (1 - \alpha)A + \alpha B,$$

or equivalently,

$$\overrightarrow{OM} = (1-\alpha)\overrightarrow{OA} + \alpha\overrightarrow{OB}.$$

Simplifying, we find

$$\overrightarrow{AM} = \alpha\overrightarrow{AB}.$$

Since $\alpha \in [0, 1]$, this means that M lies on the segment $[AB]$.

Conversely, suppose M lies on the segment $[AB]$. Then there exists $\alpha \in [0, 1]$ such that

$$\overrightarrow{AM} = \alpha\overrightarrow{AB}.$$

Substituting, we find

$$M = (1-\alpha)A + \alpha B,$$

proving that $M \in \mathrm{conv}(A, B)$.

5. The convex hull of three non-collinear points A, B, C is the (solid) triangle determined by these points.

6. The convex hull of four non-coplanar points A, B, C, D is the (solid) tetrahedron $ABCD$.

1.6 Coordinates on a Line

Let Δ be an arbitrary line. Choose an arbitrary non-zero vector $\mathbf{e}$ on this line, which we will call the *unit vector* or *direction vector*.

If $\mathbf{a}$ is any vector on the line, then, according to the previous section, there exists a unique real number x such that $\mathbf{a} = x\mathbf{e}$. The number x is called the *component* of the vector $\mathbf{a}$ relative to the line Δ, equipped with the direction vector $\mathbf{e}$.

On the line Δ, equipped with the direction vector $\mathbf{e}$, we select a point O, which we call the *origin of coordinates*. The line Δ will now be referred to as the *coordinate axis*. If M is an arbitrary point on the line, the vector $\overrightarrow{OM}$ is called the *position vector* or *radius vector* of the point M, and the component of this vector is called the *coordinate* of the point M.

We further choose a point E on the line such that $\overrightarrow{OE} = \mathbf{e}$. The segment OE will serve as the length scale on the line Δ. Consequently, the coordinate of a point M on the line is nothing but the magnitude (OM) of the directed segment $\overline{OM}$. To emphasise that the real number x is the coordinate of the point M, we will usually write $M(x)$.

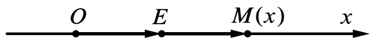


Figure 1.8:

It should be noted that there are infinitely many ways to assign coordinates to the points on a line. The coordinate of a point is uniquely determined only when:

- the direction vector of the line is chosen;
- the origin of the line is chosen.

By introducing coordinates, each point M on the coordinate axis Δ is associated with a unique real number—its coordinate x. Conversely, for every real number x, there exists a unique point M on the axis Δ whose coordinate is x. Thus, the position of every point on the coordinate axis is uniquely determined by prescribing its coordinate.

Denote by $\rho(M_1, M_2)$ the distance between the points M_1 and M_2, that is, the length of the segment M_1M_2. This distance can be expressed in terms of coordinates. Specifically, we have the following theorem:

Theorem 1.8. *For any points $M_1(x_1)$ and $M_2(x_2)$ on the coordinate axis, the following equalities hold:*

$$(M_1M_2) = x_2 - x_1, \tag{1.6.1}$$

$$\rho(M_1, M_2) = |x_2 - x_1|. \tag{1.6.2}$$

Proof From Chasles' theorem, it follows that

$$(OM_1) + (M_1M_2) = (OM_2) \quad \Longrightarrow \quad (M_1M_2) = (OM_2) - (OM_1).$$

Using the definition of coordinates, we obtain equality (1.6.1). Formula (1.6.2) follows immediately from formula (1.6.1). □

1.7 Coordinates in the Plane

1.7.1 Affine Coordinates

Throughout this section, all points and vectors are assumed to lie in a plane Π.

Definition 1.11. Let O be a point, and let $\mathbf{e}_1$ and $\mathbf{e}_2$ be two linearly independent (non-collinear) vectors in the plane Π. The triplet $(O, \mathbf{e}_1, \mathbf{e}_2)$ is called an *affine frame* or *affine coordinate system* in the plane Π.

Attach the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ to the point O, constructing the points E_1 and E_2 such that $\overrightarrow{OE_1} = \mathbf{e}_1$ and $\overrightarrow{OE_2} = \mathbf{e}_2$. The directed segments $\overline{OE_1}$ and $\overline{OE_2}$ define two coordinate axes, Ox and Oy. The point O is called the *origin of coordinates*, and the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are called the *basis vectors*.

Now let $\mathbf{a}$ be an arbitrary vector in the plane Π. From Theorem 1.4, it follows that $\mathbf{a}$ can be uniquely represented in the form

$$\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2. \tag{1.7.1}$$

Definition 1.12. The coefficients x and y in the decomposition (1.7.1) are called the *components* of the vector $\mathbf{a}$ relative to the coordinate system $(O, \mathbf{e}_1, \mathbf{e}_2)$.

As seen in Section 1.4, x and y are, in fact, the magnitudes of the projections of the vector $\mathbf{a}$ onto the axes Ox and Oy, parallel to the axes OY and Ox, respectively. To highlight that x and y are the components of the vector $\mathbf{a}$, we will write $\mathbf{a} = \mathbf{a}(x, y)$ or, simply, $\mathbf{a}(x, y)$.

Now consider a point M in the plane Π, where an affine coordinate system $(O, \mathbf{e}_1, \mathbf{e}_2)$ has been fixed. The vector $\overrightarrow{OM}$ is called the *radius vector* or *position vector* of the point M.

Definition 1.13. The components x and y of the vector $\overrightarrow{OM}$ are called the *affine coordinates* of the point M relative to the frame $(O, \mathbf{e}_1, \mathbf{e}_2)$. Typically, x is called the *abscissa*, while y is called the *ordinate*.

An affine coordinate system is also denoted by Oxy, if the basis vectors are understood. If x and y are the coordinates of a point M, we frequently use the notation $M(x, y)$.

Introducing the components of vectors allows replacing various relationships between vectors with relationships between their components. For example:

Theorem 1.9. *The components of a linear combination of vectors are equal to the same linear combination of the components of those vectors. Specifically, if*

$$\mathbf{a}(X, Y) = \sum_{i=1}^{k} \lambda_i \mathbf{a}_i(X_i, Y_i),$$

then

$$X = \sum_{i=1}^{k} \lambda_i X_i, \quad Y = \sum_{j=1}^{k} \lambda_j Y_j.$$

Proof Since the components of vectors are nothing but the magnitudes of their projections onto the coordinate axes, the theorem follows directly from the properties of projections. □

Corollary 1.3. *If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in the plane, then*

$$\overrightarrow{AB} = \overrightarrow{AB}(x_2 - x_1, y_2 - y_1),$$

meaning that to obtain the components of the vector defined by the directed segment $\overline{AB}$, one must subtract the coordinates of its origin from the coordinates of its endpoint.

Proof This follows immediately from the previous theorem and the relation

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}.$$

□

Corollary 1.4. *For two vectors $\mathbf{a}(x_1, y_1)$ and $\mathbf{b}(x_2, y_2)$ to be collinear, it is necessary and sufficient for their corresponding components to be proportional.*

Proof As shown in Section 1.5, vectors **a** and **b** are collinear if and only if there exists a relation of the form

$$\mathbf{b} = \lambda \mathbf{a}, \tag{1.7.2}$$

where λ is a real number. From Theorem 1.9, it follows that equality (1.7.2) is equivalent to two numerical equalities:

$$x_2 = \lambda x_1, \quad y_2 = \lambda y_1, \tag{1.7.3}$$

thus completing the proof. Note that equalities (1.7.3) also imply the proportionality condition

$$\frac{x_2}{x_1} = \frac{y_2}{y_1},$$

provided that both denominators are non-zero. Alternatively, we may adopt the convention that whenever a denominator is zero, the corresponding numerator is also zero, which allows the proportionality condition to be formally written even when one of the denominators vanishes. □

Corollary 1.5. *The coordinates of the midpoint A of a line segment with endpoints at $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ are given by:*

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}.$$

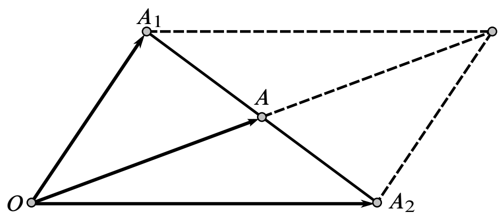


Figure 1.9:

Proof This follows directly from the equality

$$\overrightarrow{OA} = \frac{1}{2}\left(\overrightarrow{OA_1} + \overrightarrow{OA_2}\right),$$

(see Figure 1.9) and the theorem. □

1.7.2 Dividing a Line Segment in a Given Ratio

Let Δ be an arbitrary line, and A and B two points on it. Let k be a real number. We choose an arbitrary orientation for Δ. A point M on the line is said to *divide the segment AB in the ratio k* if

$$\frac{(MA)}{(MB)} = k. \tag{1.7.4}$$

It is clear that this definition does not depend on the chosen orientation of the line Δ. Moreover, the ratio k is negative if the point M lies inside the segment AB and positive if M lies outside it. Additionally, relation (1.7.4) is equivalent to the vectorial relation

$$\overrightarrow{MA} = k\overrightarrow{MB}. \tag{1.7.5}$$

Now, let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of points A and B, respectively, and let $\mathbf{r}$ be the position vector of point M. Relation (1.7.5) can be rewritten as:

$$\mathbf{r}_1 - \mathbf{r} = k(\mathbf{r}_2 - \mathbf{r}),$$

or

$$(1-k)\mathbf{r} = \mathbf{r}_1 - k\mathbf{r}_2,$$

which leads to:

$$\mathbf{r} = \frac{\mathbf{r}_1 - k\mathbf{r}_2}{1-k}. \tag{1.7.6}$$

Remarks. (1) In formula (1.7.6), $k = 1$ is not allowed, as it would imply $(MA) = (MB)$, which leads to $A = B$, contradicting the assumption that the points are distinct.

(2) Relation (1.7.6) can also be expressed as:

$$\mathbf{r} = \frac{1}{1-k}\mathbf{r}_1 + \left(1 - \frac{1}{1-k}\right)\mathbf{r}_2. \tag{1.7.7}$$

This implies that point M is an affine combination of points A and B. When k takes all real values (except 1), M traverses the entire real axis.

(3) If $k \leq 0$, then clearly, M belongs to the segment AB. Additionally, if k is negative, the coefficients in relation (1.7.7) are both positive, meaning M is a *convex combination* of points A and B, and the segment AB is the set of all convex combinations of these two points (the *convex hull* of A and B). Note that point A corresponds to $k = 0$, while point B is obtained as $k \to -\infty$.

(4) For $k = -1$, the expression for the position vector of the midpoint of segment AB is recovered:

$$\mathbf{r} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2).$$

1.7.3 Rectangular Coordinates

Let us assume that a unit of measurement for length has been chosen in the plane Π. We select a point O and two vectors of length 1, perpendicular to each other, $\mathbf{i}$ and $\mathbf{j}$. The affine coordinate system $(O, \mathbf{i}, \mathbf{j})$ is called a *rectangular coordinate system* or *Cartesian coordinate system*. The basis $\{\mathbf{i}, \mathbf{j}\}$ is referred to as an *orthonormal basis* (which means the vectors are *orthogonal*, i.e., perpendicular, and *normalized*, i.e., of length 1).

All properties valid in an arbitrary affine coordinate system remain true in a rectangular coordinate system; however, expressions involved are generally much simpler when written in Cartesian coordinates.

1.7.4 Polar Coordinates

We select a point O in the plane Π, referred to as the *pole*, and a half-line OA, referred to as the *polar axis*. Additionally, we choose a unit of measurement for length and a positive direction of rotation around the pole O. Typically, the positive direction is counterclockwise. Angles are measured in radians.

Now, let M be any point in the plane. Denote by ρ the distance from M to the pole O, and by φ the angle between the half-lines OM and OA. The quantities ρ and φ are called the *polar coordinates* of the point M. Specifically, ρ is referred to as the *polar radius*, while φ is the *polar angle*.

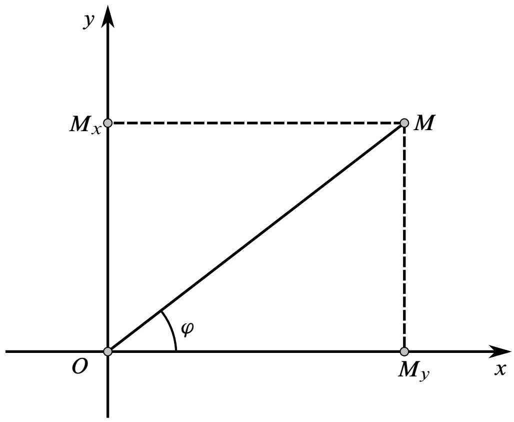


Figure 1.10:

Each point in the plane can uniquely be assigned a polar radius $\rho \geq 0$. The values of φ, however, for points other than the pole, are defined up to a term $2k\pi$, where k is an integer. To associate a single polar angle with each point in the plane, distinct from the pole, it suffices to consider $-\pi < \varphi \leq \pi$. These values of φ are called *principal values*. We now say that a *polar coordinate system* has been introduced in the plane.

We consider in the plane, simultaneously, a Cartesian coordinate system Oxy and a polar coordinate system such that the pole coincides with the origin of the Cartesian coordinates, and the polar axis coincides with the positive direction of the Ox axis. Finally, we consider the positive direction of rotation around the pole as the direction in which the positive direction of the Ox axis must be rotated, via the shortest path, to align with the positive direction of the Oy axis.

Let M be any point in the plane (distinct from the origin), with x, y as its Cartesian coordinates and ρ, φ as its polar coordinates. Clearly, we have:

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi. \tag{1.7.8}$$

Formulas (1.7.8) express the Cartesian coordinates of the point M in terms of its polar coordinates. To express the polar coordinates in terms of the Cartesian coordinates, again assuming that M does not coincide with the pole, we can use the following formulas derived immediately from (1.7.8):

$$\rho = \sqrt{x^2 + y^2}, \quad \cos\varphi = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\varphi = \frac{y}{\sqrt{x^2+y^2}}.$$

If $x \neq 0$ (i.e., if M is not located on the Oy axis), we can write

$$\tan\varphi = \frac{y}{x}.$$

From this equation, we must determine the angle φ. As is known from trigonometry, the general solution of the above equation is

$$\varphi = \arctan\frac{y}{x} + k\pi, \ k \in \mathbb{Z},$$

where $\arctan\frac{y}{x} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We choose the integer k such that the angle φ lies within the interval $(-\pi, \pi)$. We distinguish four cases based on the quadrant in which the point M is located:

(1) If M is in the first quadrant, then $x > 0, y \geq 0$, so $\arctan(y/x) \in \left[0, \frac{\pi}{2}\right)$. Since φ must lie in the same interval, we can choose $k = 0$, hence

$$\varphi = \arctan\frac{y}{x}.$$

(2) If M is in the second quadrant, i.e., $x < 0, y \geq 0$, then $\arctan(y/x) \in \left(-\frac{\pi}{2}, 0\right]$. Since φ must lie in the interval $\left(\frac{\pi}{2}, \pi\right]$, k must equal 1, thus

$$\varphi = \arctan\frac{y}{x} + \pi.$$

(3) If M is in the third quadrant, i.e., $x < 0, y < 0$, then $\arctan(y/x) \in \left[0, \frac{\pi}{2}\right)$. Since φ must lie in the interval $\left[-\pi, -\frac{\pi}{2}\right)$, k must equal -1, hence

$$\varphi = \arctan\frac{y}{x} - \pi.$$

(4) If M is in the fourth quadrant, i.e., $x > 0, y < 0$, then $\arctan(y/x) \in \left(-\frac{\pi}{2}, 0\right]$. Since φ must lie in the interval $\left(-\frac{\pi}{2}, 0\right]$, k must equal 0, thus

$$\varphi = \arctan\frac{y}{x}.$$

1.8 Coordinates in Space

1.8.1 Affine and Rectangular Coordinates

Let O be an arbitrary point in space, and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be three linearly independent vectors (i.e., non-coplanar vectors).

Definition 1.14. The quadruplet $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is called an *affine frame* or *affine coordinate system* in space. The point O is referred to as the *origin of coordinates*, while the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are called the *basis vectors*.

Definition 1.15. The *components* of a vector $\mathbf{a}$ relative to the frame $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the coefficients x, y, z in the decomposition:

$$\mathbf{a} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

The *coordinates* of a point M, relative to the same frame, are defined as the components x, y, z of its position vector $\overrightarrow{OM}$. The coordinate x is called the *abscissa*, y the *ordinate*, and z the *elevation*.

An affine coordinate system is often denoted by $Oxyz$ when the basis vectors are understood. Let us construct the points E_1, E_2, E_3 such that

$$\overrightarrow{OE_1} = \mathbf{e}_1, \ \overrightarrow{OE_2} = \mathbf{e}_2, \ \overrightarrow{OE_3} = \mathbf{e}_3. \tag{1.8.1}$$

The oriented segments $\overline{OE_1}, \overline{OE_2}, \overline{OE_3}$ determine the three *coordinate axes*, Ox, Oy, and Oz. The three planes determined by any two of the coordinate axes are called *coordinate planes*. These planes divide space into eight regions, called *coordinate octants*.

As in the case of planar frames, we distinguish between *right-handed* and *left-handed* coordinate systems. Consider a triplet of non-coplanar vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Attach these vectors to a point O, thereby determining points E_1, E_2, E_3 such that relations (1.8.1) hold. Rotate the oriented segment $\overline{OE_1}$ in the plane OE_1E_2, around O, via the shortest path, until it coincides in direction and sense with the oriented segment $\overline{OE_2}$. If this rotation, viewed from the end of the oriented segment $\overline{OE_3}$ (i.e., from the point E_3), occurs in the counterclockwise direction, we say the triplet of vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is *right-handed*; otherwise, it is *left-handed*.

A coordinate system $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is called *right-handed* or *left-handed*, depending on whether the triplet $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is right-handed or left-handed. Unless otherwise stated, all coordinate systems in the following will be assumed to be right-handed.

The simplest affine coordinate system in space is the rectangular or Cartesian coordinate system. Assume that a unit of measurement for length has been chosen in space. Then a rectangular or Cartesian coordinate system in space is determined by selecting a point O and three vectors of length 1, $\mathbf{i}, \mathbf{j}, \mathbf{k}$, mutually perpendicular.

Theorem 1.9 and its consequences can be adapted to the case of space without difficulty; the only difference is the addition of one more coordinate.

1.8.2 Cylindrical Coordinates

We select a unit of measurement for length in space. We choose a plane Π, and in this plane, we select a point O and a half-line Ox starting from this point. We also choose a positive direction of rotation in the plane Π around the point O. In this way, a polar coordinate system is defined in the plane Π, where the pole is O and the polar axis is Ox. Finally, we choose an axis Oz, perpendicular to the plane Π and oriented such that positive rotation in the plane Π, viewed from the positive direction of the Oz axis, occurs counterclockwise.

Now, let M be any point in space, M_1 its orthogonal projection onto the plane Π, and M_z its orthogonal projection onto the Oz axis.

The *cylindrical coordinates* of the point M are the three numbers ρ, φ, z, where ρ, φ are the polar coordinates of the point M_1 in the plane Π, and $z = OM_z$. The term "cylindrical coordinates" is associated with the fact that the surface containing all points corresponding to the same value of the coordinate ρ is a cylinder. Each point in space can be uniquely assigned ρ and z coordinates, with ρ always positive. The value of φ is defined only for points not located on the Oz axis, and as in the case of polar coordinates, it is defined up to an integer multiple of π. If a Cartesian coordinate system is also given in space, with its origin at O and such that the axes Ox and Oz coincide with those of the cylindrical system, then the Cartesian coordinates can be expressed in terms of the cylindrical ones as follows:

$$x = \rho \cos \varphi, \; y = \rho \sin \varphi, \; z = z.$$

1.8.3 Spherical Coordinates

To introduce spherical coordinates, as in the case of cylindrical coordinates, we must choose a unit of measurement for length, a plane Π with a point O and an axis Ox,

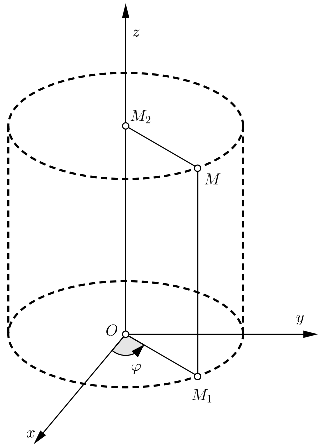


Figure 1.11:

as well as an axis Oz.

Let M be any point in space, and let M_1 be its orthogonal projection onto the plane Π. Further, let ρ be the distance from the point M to the point O, θ the angle between the axis Oz and the oriented segment $\overline{OM}$, and finally, φ the angle by which the axis Ox must be rotated so as to coincide with the half-line OM_1. The numbers ρ, θ, φ are called the *spherical coordinates* of the point M. The angle φ is referred to as the *longitude*, while the angle θ is referred to as the *latitude*.

The term ”spherical coordinates” arises from the fact that the surface containing all points corresponding to the same value of the coordinate ρ is a sphere. Each point in space, distinct from O, has well-defined values for the coordinates ρ and θ. ρ is always strictly positive, and the angle θ is considered to vary within the interval $[0, \pi]$. The value of φ is not defined for points on the Oz axis. Where defined, as in the case of polar coordinates, this value is defined up to an integer multiple of π. As with cylindrical coordinates, if a Cartesian coordinate system is chosen such that the origins of both systems coincide and the axes Ox and Oz also coincide, then the Cartesian coordinates can be expressed in terms of the spherical ones as follows:

$$x = \rho \sin\theta \cos\varphi,\ y = \rho \sin\theta \sin\varphi,\ z = \rho \cos\theta.$$

1.9 Coordinate Transformations

1.9.1 Affine Coordinates

Consider, in the plane, two affine coordinate systems, $(O, \mathbf{e}_1, \mathbf{e}_2)$ and $(O', \mathbf{e}'_1, \mathbf{e}'_2)$. The first system will be referred to as the *old system*, and the second as the *new system*. We assume that the coordinates of the point O', as well as the components of the vectors $\mathbf{e}'_1$ and $\mathbf{e}'_2$ relative to the old system, are known:

$$O'(\alpha_1, \alpha_2), \ \mathbf{e}'_1(\alpha_{11}, \alpha_{21}), \ \mathbf{e}'_2(\alpha_{12}, \alpha_{22}).$$

Now suppose that an arbitrary point M in the plane has old coordinates x and y and new coordinates x' and y'. We seek a relationship between the two pairs of coordinates of this point. We have, first of all,

$$\begin{cases} \mathbf{e}'_1 = \alpha_{11}\mathbf{e}_1 + \alpha_{21}\mathbf{e}_2, & \mathbf{e}'_2 = \alpha_{12}\mathbf{e}_1 + \alpha_{22}\mathbf{e}_2, \\ \overrightarrow{OO'} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2, & \overrightarrow{OM} = x\mathbf{e}_1 + y\mathbf{e}_2, \ \overrightarrow{O'M} = x'\mathbf{e}'_1 + y'\mathbf{e}'_2. \end{cases} \tag{1.9.1}$$

Next,

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}.$$

From this equality, using formulas (1.9.1), we obtain

$$\begin{aligned} x\mathbf{e}_1 + y\mathbf{e}_2 &= \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + x'(\alpha_{11}\mathbf{e}_1 + \alpha_{21}\mathbf{e}_2) + y'(\alpha_{12}\mathbf{e}_1 + \alpha_{22}\mathbf{e}_2) \\ &= (\alpha_{11}x' + \alpha_{12}y' + \alpha_1)\mathbf{e}_1 + (\alpha_{21}x' + \alpha_{22}y' + \alpha_2)\mathbf{e}_2. \end{aligned}$$

By virtue of Theorem 1.4, we then obtain the transformation formulas:

$$\begin{cases} x = \alpha_{11}x' + \alpha_{12}y' + \alpha_1, \\ y = \alpha_{21}x' + \alpha_{22}y' + \alpha_2. \end{cases} \tag{1.9.2}$$

If, in the preceding reasoning, we interchange the old and new coordinates, we obtain the transformation formulas:

$$\begin{cases} x' = \beta_{11}x + \beta_{12}y + \beta_1, \\ y' = \beta_{21}x + \beta_{22}y + \beta_2, \end{cases} \tag{1.9.3}$$

where

$$O(\beta_1, \beta_2), \ \mathbf{e}_1(\beta_{11}, \beta_{21}), \ \mathbf{e}_2(\beta_{12}, \beta_{22}).$$

It can be easily verified that $\alpha_i = -\beta_i$, $i = 1, 2$, and that the matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

is the inverse of the matrix

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}.$$

Now consider another point N, alongside the point M, whose old and new coordinates will be denoted, respectively, by x_1, y_1 and x_1', y_1'. Then, from formulas (1.9.2), we obtain

$$\begin{cases} x_1 = \alpha_{11}x_1' + \alpha_{12}y_1' + \alpha_1, \\ y_1 = \alpha_{21}x_1' + \alpha_{22}y_1' + \alpha_2. \end{cases} \tag{1.9.4}$$

As is known, the components (old and new) of the vector $\overrightarrow{MN}$ are given by

$$X = x_1 - x,\ Y = y_1 - y,$$
$$X' = x_1' - x',\ Y' = y_1' - y'.$$

Using formulas (1.9.2) and (1.9.4), we obtain the transformation formulas for the components of a vector:

$$X = \alpha_{11}X' + \alpha_{12}Y',$$
$$Y = \alpha_{21}X' + \alpha_{22}Y'.$$

Coordinate transformations in space are written in an entirely analogous manner. Let $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(O', \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$ be two affine coordinate systems in space (the old and the new systems). Assume also, as before, that the coordinates of the new origin and the components of the vectors of the new basis relative to the old basis are known:

$$O'(\alpha_1, \alpha_2, \alpha_3),\ \mathbf{e}_1'(\alpha_{11}, \alpha_{21}, \alpha_{31}),$$
$$\mathbf{e}_2'(\alpha_{12}, \alpha_{22}, \alpha_{32}),\ \mathbf{e}_3'(\alpha_{13}, \alpha_{23}, \alpha_{33}).$$

Then the old coordinates are expressed in terms of the new ones by the formulas:

$$\begin{cases} x = \alpha_{11}x' + \alpha_{12}y' + \alpha_{13}z' + \alpha_1, \\ y = \alpha_{21}x' + \alpha_{22}y' + \alpha_{23}z' + \alpha_2, \\ z = \alpha_{31}x' + \alpha_{32}y' + \alpha_{33}z' + \alpha_3. \end{cases}$$

Analogous formulas express the new coordinates in terms of the old ones. Regarding the rules for transforming vector components, we obtain:

$$X = \alpha_{11}X' + \alpha_{12}Y' + \alpha_{13}Z',$$
$$Y = \alpha_{21}X' + \alpha_{22}Y' + \alpha_{23}Z',$$
$$Z = \alpha_{31}X' + \alpha_{32}Y' + \alpha_{33}Z'.$$

1.9.2 Rectangular Coordinates in the Plane

Now consider the particular case of coordinate transformations in the plane, where both the old and new coordinate systems are orthogonal. We denote the old coordinate system by $(O, \mathbf{i}, \mathbf{j})$ and the new one by $(O', \mathbf{i}', \mathbf{j}')$.

We distinguish two cases, depending on whether the shortest rotation from $\mathbf{i}$ to $\mathbf{j}$ and from $\mathbf{i}'$ to $\mathbf{j}'$ occurs:

a) in the same direction (either both clockwise or both counterclockwise);

b) in opposite directions.

In both cases, let φ denote the angle between the vectors $\mathbf{i}$ and $\mathbf{i}'$, measured in the direction of the shortest rotation from $\mathbf{i}$ to $\mathbf{i}'$. If we denote by ψ the angle between the vectors $\mathbf{i}$ and $\mathbf{j}'$, then in the first case we have

$$\psi = \varphi + \frac{\pi}{2} + 2k\pi,$$

while in the second case we have

$$\psi = \varphi - \frac{\pi}{2} + 2k\pi.$$

In both cases, the following expressions hold for the components of the vectors $\mathbf{i}'$ and $\mathbf{j}'$:

$$\alpha_{11} = \cos\varphi, \ \alpha_{21} = \sin\varphi, \ \alpha_{12} = \cos\psi, \ \alpha_{22} = \sin\psi.$$

In the first case, the formulas (1.9.2) take the form:

$$\begin{cases} x = x'\cos\varphi - y'\sin\varphi + \alpha_1, \\ y = x'\sin\varphi + y'\cos\varphi + \alpha_2, \end{cases} \tag{1.9.5}$$

while in the second case, we have:

$$\begin{cases} x = x'\cos\varphi + y'\sin\varphi + \alpha_1, \\ y = x'\sin\varphi - y'\cos\varphi + \alpha_2. \end{cases}$$

The following two important particular cases of formulas (1.9.5) are noteworthy:

1. Assume that $\mathbf{i} = \mathbf{i}'$ and $\mathbf{j} = \mathbf{j}'$. Then formulas (1.9.5) become:

$$\begin{cases} x = x' + \alpha_1, \\ y = y' + \alpha_2, \end{cases}$$

and are called the *transformation formulas for coordinates under a parallel translation (translation) of the coordinate axes by the vector* $\mathbf{a}(\alpha_1, \alpha_2)$.

2. If $O' = O$, then formulas (1.9.5) take the form:

$$\begin{cases} x = x' \cos\varphi - y' \sin\varphi, \\ y = x' \sin\varphi + y' \cos\varphi, \end{cases}$$

and are called the *transformation formulas for coordinates under a rotation of the system about the origin by the angle* φ.

1.10 Applications of Elementary Operations with Vectors in Geometry

1.10.1 The Vector Equation of a Line Determined by Two Distinct Points

Let A and B be two distinct points, and let $\mathbf{r}_A$ and $\mathbf{r}_B$ be their position vectors relative to an arbitrary origin. Clearly, the vector

$$\overrightarrow{AB} = \mathbf{r}_B - \mathbf{r}_A \tag{1.10.1}$$

is collinear with the line AB. Let M be a point on the line, and let $\mathbf{r}$ be its position vector. Then the vector

$$\overrightarrow{AM} = \mathbf{r} - \mathbf{r}_A \tag{1.10.2}$$

is also collinear with the line and therefore collinear with $\overrightarrow{AB}$. Hence, there exists a real number λ such that

$$\overrightarrow{AM} = \lambda\overrightarrow{AB}, \tag{1.10.3}$$

or equivalently,

$$\mathbf{r} - \mathbf{r}_A = \lambda(\mathbf{r}_B - \mathbf{r}_A),$$

or finally,

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_A + \lambda\mathbf{r}_B. \tag{1.10.4}$$

Equation (1.10.4) is called the *vector equation of a line* when two distinct points on it are given.

1.10.2 The Condition for Collinearity of Three Points

Consider three distinct points A, B, and C, represented by their position vectors $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$. As seen in the previous subsection, the vector equation of the line AB is given by

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_A + \lambda\mathbf{r}_B. \tag{1.10.5}$$

The points A, B, and C are collinear if and only if point C lies on the line AB, which means there exists a real λ such that

$$\mathbf{r}_C = (1 - \lambda)\mathbf{r}_A + \lambda\mathbf{r}_B, \tag{1.10.6}$$

or equivalently,

$$(1 - \lambda)\mathbf{r}_A + \lambda\mathbf{r}_B - \mathbf{r}_C = 0. \tag{1.10.7}$$

Multiplying this equation by a non-zero scalar $-\gamma$, we get:

$$-\gamma(1 - \lambda)\mathbf{r}_A - \gamma\lambda\mathbf{r}_B + \gamma\mathbf{r}_C = 0,$$

or, if we let $\alpha = -\gamma(1 - \lambda)$ and $\beta = -\gamma\lambda$:

$$\alpha\mathbf{r}_A + \beta\mathbf{r}_B + \gamma\mathbf{r}_C = 0, \tag{1.10.8}$$

where the coefficients satisfy the relation:

$$\alpha + \beta + \gamma = -\gamma(1 - \lambda) - \gamma\lambda + \gamma = 0.$$

Thus, we have the following result:

Theorem 1.10. *Three points A, B, and C are collinear if and only if their position vectors satisfy a relation of the form:*

$$\alpha\mathbf{r}_A + \beta\mathbf{r}_B + \gamma\mathbf{r}_C = 0,$$

where the coefficients satisfy:

$$\alpha + \beta + \gamma = 0.$$

1.10.3 The Vector Equation of a Plane Determined by Three Non-Collinear Points

Consider three non-collinear points A, B, and C, and let O be a point outside the plane. Let $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$ be the position vectors of the points A, B, and C relative to O.

Since the points A, B, and C are non-collinear, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly independent and lie in the plane determined by the three points.

Let M be any point in the plane. Then the vector $\overrightarrow{AM}$ also lies in the plane and can be written uniquely as a linear combination of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. That is, there exist two real numbers λ and μ such that:

$$\overrightarrow{AM} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}, \tag{1.10.9}$$

or equivalently:

$$\mathbf{r} - \mathbf{r}_A = \lambda(\mathbf{r}_B - \mathbf{r}_A) + \mu(\mathbf{r}_C - \mathbf{r}_A),$$

which simplifies to:

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{r}_A + \lambda\mathbf{r}_B + \mu\mathbf{r}_C. \tag{1.10.10}$$

Equation (1.10.10) is called the *vector equation of the plane determined by three non-collinear points given by their position vectors relative to a point outside the plane*.

1.10.4 The Condition for Coplanarity of Four Points

Consider four distinct points A, B, C, and D, represented by their position vectors $\mathbf{r}_A$, $\mathbf{r}_B$, $\mathbf{r}_C$, and $\mathbf{r}_D$. If the points are collinear, they are obviously coplanar. Assume that A, B, and C are non-collinear and determine a plane Π, with the origin chosen such that it is outside the plane Π.

As shown in the previous subsection, the equation of the plane Π can be written as:

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{r}_A + \lambda\mathbf{r}_B + \mu\mathbf{r}_C. \tag{1.10.11}$$

The point D lies in the plane Π if and only if its position vector satisfies the plane equation:

$$\mathbf{r}_D = (1 - \lambda - \mu)\mathbf{r}_A + \lambda\mathbf{r}_B + \mu\mathbf{r}_C,$$

or equivalently:

$$(1 - \lambda - \mu)\mathbf{r}_A + \lambda\mathbf{r}_B + \mu\mathbf{r}_C - \mathbf{r}_D = 0.$$

Multiplying this equation by a non-zero scalar $-\delta$, we get:

$$-\delta(1-\lambda-\mu)\mathbf{r}_A - \delta\lambda\mathbf{r}_B - \delta\mu\mathbf{r}_C + \delta\mathbf{r}_D = 0,$$

or, letting $\alpha = -\delta(1-\lambda-\mu)$, $\beta = -\delta\lambda$, and $\gamma = -\delta\mu$:

$$\alpha\mathbf{r}_A + \beta\mathbf{r}_B + \gamma\mathbf{r}_C + \delta\mathbf{r}_D = 0, \tag{1.10.12}$$

where the coefficients satisfy:

$$\alpha + \beta + \gamma + \delta = 0. \tag{1.10.13}$$

Thus, we have:

Theorem 1.11. *Four points A, B, C, and D are coplanar if and only if their position vectors satisfy a relation of the form:*

$$\alpha\mathbf{r}_A + \beta\mathbf{r}_B + \gamma\mathbf{r}_C + \delta\mathbf{r}_D = 0,$$

where the coefficients are not all zero and satisfy:

$$\alpha + \beta + \gamma + \delta = 0.$$

1.10.5 Concurrence of the Medians of a Triangle

We will demonstrate that the three medians of any triangle intersect at a single point G, and this point divides each median in the same ratio.

First Proof

Consider a triangle ABC, and let A', B', and C' denote the midpoints of the sides BC, CA, and AB, respectively. Choose a point O located outside the plane of the triangle, and let $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ be the position vectors of the points A, B, and C relative to O. Similarly, let $\mathbf{r}_{11}$, $\mathbf{r}_{21}$, and $\mathbf{r}_{31}$ be the position vectors of the midpoints A', B', and C' relative to O.

As shown earlier, we have:

$$\mathbf{r}_{11} = \frac{1}{2}(\mathbf{r}_2 + \mathbf{r}_3), \quad \mathbf{r}_{21} = \frac{1}{2}(\mathbf{r}_3 + \mathbf{r}_1), \quad \mathbf{r}_{31} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2).$$

The position vector of any point on the median AA' is an affine combination of the position vectors of the points A and A'. Thus, we have:

$$AA' : \ \mathbf{r} = (1-\lambda)\mathbf{r}_1 + \lambda\mathbf{r}_{11} = (1-\lambda)\mathbf{r}_1 + \frac{\lambda}{2}(\mathbf{r}_2 + \mathbf{r}_3).$$

Similarly, the position vector of any point on the median BB' is:

$$BB' : \ \mathbf{r} = (1-\mu)\mathbf{r}_2 + \mu\mathbf{r}_{21} = (1-\mu)\mathbf{r}_2 + \frac{\mu}{2}(\mathbf{r}_3 + \mathbf{r}_1).$$

The lines AA' and BB' intersect at a single point, denoted G. Since G lies on both lines, there exist real numbers λ and μ such that:

$$(1-\lambda)\mathbf{r}_1 + \frac{\lambda}{2}(\mathbf{r}_2 + \mathbf{r}_3) = (1-\mu)\mathbf{r}_2 + \frac{\mu}{2}(\mathbf{r}_3 + \mathbf{r}_1).$$

Expanding and simplifying, this equation becomes:

$$\left(1 - \lambda - \frac{\mu}{2}\right)\mathbf{r}_1 + \left(\frac{\lambda}{2} + \mu - 1\right)\mathbf{r}_2 + \frac{\lambda - \mu}{2}\mathbf{r}_3 = \mathbf{0}. \tag{1.10.14}$$

Since the points A, B, and C are non-collinear, the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are linearly independent. Thus, the coefficients of each vector in (1.10.14) must vanish, leading to the system of equations:

$$1 - \lambda - \frac{\mu}{2} = 0, \quad \frac{\lambda}{2} + \mu - 1 = 0, \quad \frac{\lambda - \mu}{2} = 0. \tag{1.10.15}$$

Solving this system, we find:

$$\lambda = \mu = \frac{2}{3}.$$

Now consider the third median, CC'. By analogous reasoning, the position vector of a point on CC' is:

$$CC' : \ \mathbf{r} = (1-\nu)\mathbf{r}_3 + \nu\mathbf{r}_{31} = (1-\nu)\mathbf{r}_3 + \frac{\nu}{2}(\mathbf{r}_1 + \mathbf{r}_2).$$

To find the intersection of AA' and CC', we solve a system similar to (1.10.15), yielding:

$$\lambda = \nu = \frac{2}{3}.$$

Thus, the intersection point of AA' and CC' coincides with the intersection of AA' and BB'. Therefore, the medians AA', BB', and CC' are concurrent.

Substituting $\lambda = 2/3$ into the equation for AA', the position vector of the centroid G (the intersection of the medians) is:

$$\mathbf{r} = \frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3). \tag{1.10.16}$$

Finally, we determine the ratio in which the centroid divides a median. For instance, on AA', let:

$$k = \frac{GA}{GA'}.$$

The position vector of G in terms of k is:

$$\mathbf{r} = \frac{1}{1-k}\mathbf{r}_1 - \frac{k}{1-k}\mathbf{r}_{11}.$$

From earlier, we also have:

$$\mathbf{r} = \frac{1}{3}\mathbf{r}_1 + \frac{2}{3}\mathbf{r}_{11}.$$

Equating the two expressions, we find $k = -2$, so:

$$\frac{GA}{GA'} = -2. \tag{1.10.17}$$

Similar results hold for the other medians. Thus, the centroid divides each median into two segments, with the segment between the centroid and a vertex being twice as long as the segment between the centroid and the midpoint of the opposite side.

Second Proof

Using the same notation as above, we now choose the origin at G, the intersection of AA' and BB'. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ denote the position vectors of A, B, and C relative to this origin.

The midpoints of the sides are then given by:

$$\begin{aligned}
\mathbf{a}' &= \frac{1}{2}(\mathbf{b} + \mathbf{c}),\\
\mathbf{b}' &= \frac{1}{2}(\mathbf{c} + \mathbf{a}),\\
\mathbf{c}' &= \frac{1}{2}(\mathbf{a} + \mathbf{b}).
\end{aligned}$$

Since A' lies on GA, the vectors $\mathbf{a}'$ and $\mathbf{a}$ are collinear, as are $\mathbf{b}'$ and $\mathbf{b}$. Hence, there exist real numbers α and β such that:

$$\begin{cases} \frac{1}{2}(\mathbf{b} + \mathbf{c}) = \alpha\mathbf{a},\\ \frac{1}{2}(\mathbf{c} + \mathbf{a}) = \beta\mathbf{b}. \end{cases} \tag{1.10.18}$$

Eliminating $\mathbf{c}$, we find:

$$(2\alpha + 1)\mathbf{a} - (2\beta + 1)\mathbf{b} = 0.$$

Since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, we conclude:

$$\alpha = \beta = -\frac{1}{2}.$$

Substituting back into (1.10.18) and adding, we obtain:

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0.$$

Thus:

$$\frac{1}{2}(\mathbf{a} + \mathbf{b}) = -\frac{1}{2}\mathbf{c}. \tag{1.10.19}$$

Equation (1.10.19) confirms that G lies on CC', completing the proof.

1.10.6 Concurrence of the Interior Angle Bisectors of a Triangle

Consider a triangle ABC and choose an origin O located outside the plane of the triangle. Let A', B', and C' denote the feet of the interior angle bisectors of the angles A, B, and C, respectively.

As in the previous cases, let $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ be the position vectors of the vertices A, B, and C relative to O. Let $\mathbf{r}_{11}$, $\mathbf{r}_{21}$, and $\mathbf{r}_{31}$ be the position vectors of the points A', B', and C', respectively. Finally, let a, b, and c denote the lengths of the sides opposite the vertices A, B, and C, respectively.

The position vector of any point on the angle bisector AA' is an affine combination of the position vectors of A and A'. Thus, there exists a real λ such that:

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_{11}. \tag{1.10.20}$$

According to the angle bisector theorem, applied to angle A, the point A' divides the side BC in the ratio:

$$k_1 = \frac{(A'B)}{(A'C)} = -\frac{c}{b}.$$

Hence, the position vector of A' is:

$$\mathbf{r}_{11} = \frac{\mathbf{r}_2 - k_1\mathbf{r}_3}{1 - k_1} = \frac{\mathbf{r}_2 + \frac{c}{b}\mathbf{r}_3}{1 + \frac{c}{b}},$$

or equivalently:

$$\mathbf{r}_{11} = \frac{b\mathbf{r}_2 + c\mathbf{r}_3}{b+c}. \tag{1.10.21}$$

Thus, the vector equation of the line AA' is:

$$AA' : \ \mathbf{r} = (1-\lambda)\mathbf{r}_1 + \lambda\frac{b\mathbf{r}_2 + c\mathbf{r}_3}{b+c}. \tag{1.10.22}$$

Similarly, the vector equations of the other two angle bisectors are:

$$BB' : \ \mathbf{r} = (1-\mu)\mathbf{r}_2 + \mu\frac{c\mathbf{r}_3 + a\mathbf{r}_1}{c+a}, \tag{1.10.23}$$

and:

$$CC' : \ \mathbf{r} = (1-\nu)\mathbf{r}_3 + \nu\frac{a\mathbf{r}_1 + b\mathbf{r}_2}{a+b}. \tag{1.10.24}$$

Let I be the intersection point of the lines AA' and BB'. Since I lies on both lines, its position vector as a point on the first line must coincide with its position vector as a point on the second line. Thus, from equations (1.10.22) and (1.10.23), we have:

$$(1-\lambda)\mathbf{r}_1 + \lambda\frac{b\mathbf{r}_2 + c\mathbf{r}_3}{b+c} = (1-\mu)\mathbf{r}_2 + \mu\frac{c\mathbf{r}_3 + a\mathbf{r}_1}{c+a}.$$

Expanding and simplifying, this equation becomes:

$$\left(1-\lambda-\frac{\mu a}{c+a}\right)\mathbf{r}_1 + \left(-1+\mu+\frac{\lambda b}{b+c}\right)\mathbf{r}_2 + \left(\frac{\lambda c}{b+c}-\frac{\mu c}{c+a}\right)\mathbf{r}_3 = 0. \tag{1.10.25}$$

Since the vertices A, B, and C are not collinear, and the origin O is outside the plane of the triangle, the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are linearly independent. Therefore, the coefficients of each vector in equation (1.10.25) must vanish, yielding the linear system of equations (in λ and μ):

$$1-\lambda-\frac{\mu a}{c+a} = 0, \quad -1+\mu+\frac{\lambda b}{b+c} = 0, \quad \frac{\lambda c}{b+c}-\frac{\mu c}{c+a} = 0. \tag{1.10.26}$$

This system is consistent and determined, with the solution:

$$\lambda = \frac{b+c}{a+b+c}, \quad \mu = \frac{c+a}{a+b+c}. \tag{1.10.27}$$

Now let I' be the intersection point of the bisectors AA' and CC'. Using the same reasoning as above, we find that I' corresponds to the parameter values:

$$\lambda = \frac{b+c}{a+b+c}, \quad \nu = \frac{a+b}{a+b+c}. \tag{1.10.28}$$

Since I and I' correspond to the same value of λ on the line AA', the two points coincide. Thus, the interior angle bisectors of triangle ABC are concurrent.

Substituting the value of λ found in equation (1.10.27) into the vector equation of AA', we obtain the position vector of I (the *incentre* of the triangle ABC):

$$\mathbf{r} = \frac{1}{a+b+c}(a\mathbf{r}_1 + b\mathbf{r}_2 + c\mathbf{r}_3). \tag{1.10.29}$$

1.10.7 The Excentres of a Triangle

In this section, we will demonstrate that the exterior bisectors of two angles of a triangle and the interior bisector of the third angle intersect at a point. There are three such points, which are called the *excentres* of the given triangle.

We maintain the notation from the previous section and introduce new ones. Let A'', B'', and C'' denote the feet of the exterior bisectors of the angles A, B, and C of the triangle ABC, respectively. Let $\mathbf{r}_{12}$, $\mathbf{r}_{22}$, and $\mathbf{r}_{32}$ be the position vectors of these points.

According to the angle bisector theorem, the point A'' divides the segment BC in the ratio:

$$k_1 = \frac{(A''B)}{(A''C)} = \frac{c}{b},$$

where the ratio is now positive because the point A'' lies outside the segment BC.

Hence, the position vector of A'' is:

$$\mathbf{r}_{12} = \frac{\mathbf{r}_2 - k_1\mathbf{r}_3}{1-k_1} = \frac{\mathbf{r}_2 - \frac{c}{b}\mathbf{r}_3}{1-\frac{c}{b}},$$

or equivalently:

$$\mathbf{r}_{12} = \frac{b\mathbf{r}_2 - c\mathbf{r}_3}{b-c}. \tag{1.10.30}$$

Thus, the vector equation of the line AA'' is:

$$AA'': \ \mathbf{r} = (1-\alpha)\mathbf{r}_1 + \alpha\frac{b\mathbf{r}_2 - c\mathbf{r}_3}{b-c}. \tag{1.10.31}$$

Similarly, the vector equations of the other two exterior bisectors are:

$$BB'' : \ \mathbf{r} = (1-\beta)\mathbf{r}_2 + \beta \frac{c\mathbf{r}_3 - a\mathbf{r}_1}{c-a}, \tag{1.10.32}$$

and:

$$CC'' : \ \mathbf{r} = (1-\gamma)\mathbf{r}_3 + \gamma \frac{a\mathbf{r}_1 - b\mathbf{r}_2}{a-b}. \tag{1.10.33}$$

Let I_a denote the intersection point of the exterior bisectors of the angles B and C. According to equations (1.10.32) and (1.10.33), we have:

$$(1-\beta)\mathbf{r}_2 + \beta \frac{c\mathbf{r}_3 - a\mathbf{r}_1}{c-a} = (1-\gamma)\mathbf{r}_3 + \gamma \frac{a\mathbf{r}_1 - b\mathbf{r}_2}{a-b}.$$

Expanding and simplifying, this equation becomes:

$$\left(\frac{\beta a}{c-a} + \frac{\gamma a}{a-b}\right)\mathbf{r}_1 + \left(-1 + \beta - \frac{\gamma b}{a-b}\right)\mathbf{r}_2 + \left(1 - \gamma - \frac{\beta c}{c-a}\right)\mathbf{r}_3 = 0. \tag{1.10.34}$$

Since the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are linearly independent, their coefficients in equation (1.10.34) must vanish, resulting in the linear system:

$$\frac{\beta a}{c-a} + \frac{\gamma a}{a-b} = 0, \quad -1 + \beta - \frac{\gamma b}{a-b} = 0, \quad 1 - \gamma - \frac{\beta c}{c-a} = 0. \tag{1.10.35}$$

This system is consistent and determined, with the solution:

$$\beta = \frac{c-a}{-a+b+c}, \quad \gamma = \frac{b-a}{-a+b+c}. \tag{1.10.36}$$

Now let I'_a denote the intersection point of the interior bisector AA' and the exterior bisector BB''. Using the same reasoning as above, from equations (1.10.22) and (1.10.32), we find:

$$(1-\lambda)\mathbf{r}_1 + \lambda \frac{b\mathbf{r}_2 + c\mathbf{r}_3}{b+c} = (1-\beta)\mathbf{r}_2 + \beta \frac{c\mathbf{r}_3 - a\mathbf{r}_1}{c-a}.$$

Expanding and simplifying, this equation becomes:

$$\left(1 - \lambda + \frac{\beta a}{c-a}\right)\mathbf{r}_1 + \left(-1 + \beta + \frac{\lambda b}{b+c}\right)\mathbf{r}_2 + \left(\frac{\lambda c}{b+c} - \frac{\beta c}{c-a}\right)\mathbf{r}_3 = 0. \tag{1.10.37}$$

The coefficients must vanish, leading to the linear system:

$$1-\lambda+\frac{\beta a}{c-a}=0, \quad -1+\beta+\frac{\lambda b}{b+c}=0, \quad \frac{\lambda c}{b+c}-\frac{\beta c}{c-a}=0. \tag{1.10.38}$$

This system is consistent and determined, with the solution:

$$\lambda=\frac{b+c}{-a+b+c}, \quad \beta=\frac{c-a}{-a+b+c}. \tag{1.10.39}$$

Since I_a and I'_a correspond to the same value of λ along the line AA', the two points coincide. Thus, the exterior bisectors of the angles B and C and the interior bisector of angle A intersect. The position vector of the intersection point I_a is obtained by substituting the value of λ from equation (1.10.39) into the vector equation of AA'':

$$(I_a) \quad \mathbf{r}=\frac{1}{-a+b+c}(-a\mathbf{r}_1+b\mathbf{r}_2+c\mathbf{r}_3). \tag{1.10.40}$$

Similarly, the other excentres are:

$$(I_b) \quad \mathbf{r}=\frac{1}{a-b+c}(a\mathbf{r}_1-b\mathbf{r}_2+c\mathbf{r}_3), \tag{1.10.41}$$

and:

$$(I_c) \quad \mathbf{r}=\frac{1}{a+b-c}(a\mathbf{r}_1+b\mathbf{r}_2-c\mathbf{r}_3). \tag{1.10.42}$$

1.10.8 Ceva's Theorem

The result presented in this section generalises the conclusions established earlier in this subsection. Instead of medians or angle bisectors, we consider arbitrary lines passing through the vertices of a triangle, known as *cevians*. The question we address is: in what ratios must the cevians divide the opposite sides such that they are concurrent?

Consider a triangle ABC defined by the position vectors $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$ of its vertices relative to a point O outside the plane of the triangle. Let AM, BN, and CP be three cevians, where the points M, N, and P lie on the sides BC, CA, and AB, respectively, or on their extensions. Assume that these points divide the sides BC, CA, and AB in the ratios λ, μ, and ν:

$$\frac{(MB)}{(MC)}=\lambda, \quad \frac{(NC)}{(NA)}=\mu, \quad \frac{(PA)}{(PB)}=\nu. \tag{1.10.43}$$

It is important to note that although the quantities (MB), etc., represent signed lengths and depend on the chosen orientation along the sides, the ratios themselves are independent of orientation and need no further specification.

From the formula for dividing a segment into a given ratio (see (1.7.6)), the position vectors of the feet of the cevians are:

$$\mathbf{r}_M = \frac{\mathbf{r}_B - \lambda \mathbf{r}_C}{1-\lambda}, \quad \mathbf{r}_N = \frac{\mathbf{r}_C - \mu \mathbf{r}_A}{1-\mu}, \quad \mathbf{r}_P = \frac{\mathbf{r}_A - \nu \mathbf{r}_B}{1-\nu}. \tag{1.10.44}$$

Thus, the vector equations of the three cevians are:

$$AM : \mathbf{r} = (1-k)\mathbf{r}_A + \frac{k}{1-\lambda}(\mathbf{r}_B - \lambda \mathbf{r}_C), \tag{1.10.45}$$

$$BN : \mathbf{r} = (1-\ell)\mathbf{r}_B + \frac{\ell}{1-\mu}(\mathbf{r}_C - \mu \mathbf{r}_A), \tag{1.10.46}$$

$$CP : \mathbf{r} = (1-m)\mathbf{r}_C + \frac{m}{1-\nu}(\mathbf{r}_A - \nu \mathbf{r}_B). \tag{1.10.47}$$

Assume that the three cevians are concurrent at a single point Q. To determine the relationship between λ, μ, and ν, it is necessary that the point of intersection of AM and BN coincides with the point of intersection of AM and CP. We first compute the parameter values k, ℓ, and m for these intersection points.

The condition for the intersection of AM and BN is:

$$(1-k)\mathbf{r}_A + \frac{k}{1-\lambda}(\mathbf{r}_B - \lambda \mathbf{r}_C) = (1-\ell)\mathbf{r}_B + \frac{\ell}{1-\mu}(\mathbf{r}_C - \mu \mathbf{r}_A),$$

which simplifies to:

$$\left(1 - k + \frac{\ell\mu}{1-\mu}\right)\mathbf{r}_A + \left(-1 + \ell + \frac{k}{1-\lambda}\right)\mathbf{r}_B + \left(-\frac{k\lambda}{1-\lambda} - \frac{\ell}{1-\mu}\right)\mathbf{r}_C = 0. \tag{1.10.48}$$

Since $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$ are linearly independent, their coefficients in (1.10.48) must vanish, resulting in the system:

$$1 - k + \frac{\ell\mu}{1-\mu} = 0, \quad -1 + \ell + \frac{k}{1-\lambda} = 0, \quad -\frac{k\lambda}{1-\lambda} - \frac{\ell}{1-\mu} = 0. \tag{1.10.49}$$

This linear system is consistent and determined, yielding:

$$k = \frac{1-\lambda}{\lambda\mu - \lambda + 1}, \quad \ell = \frac{\lambda(\mu - 1)}{\lambda\mu - \lambda + 1}. \tag{1.10.50}$$

The condition for the intersection of AM and CP is:

$$(1-k)\mathbf{r}_A + \frac{k}{1-\lambda}(\mathbf{r}_B - \lambda\mathbf{r}_C) = (1-m)\mathbf{r}_C + \frac{m}{1-\nu}(\mathbf{r}_A - \nu\mathbf{r}_B),$$

which simplifies to:

$$\left(1-k-\frac{m}{1-\nu}\right)\mathbf{r}_A + \left(\frac{k}{1-\lambda}+\frac{m\nu}{1-\nu}\right)\mathbf{r}_B + \left(-1+m-\frac{k\lambda}{1-\lambda}\right)\mathbf{r}_C = 0. \tag{1.10.51}$$

Again, the coefficients of $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$ must vanish, leading to:

$$1-k-\frac{m}{1-\nu}=0, \quad \frac{k}{1-\lambda}+\frac{m\nu}{1-\nu}=0, \quad -1+m-\frac{k\lambda}{1-\lambda}=0. \tag{1.10.52}$$

This system is consistent and determined, yielding:

$$k = \frac{\nu(\lambda-1)}{\lambda\nu-\nu+1}, \quad m = \frac{1-\nu}{\lambda\nu-\nu+1}. \tag{1.10.53}$$

For AM, BN, and CP to be concurrent, the values of k obtained from (1.10.50) and (1.10.53) must agree. Setting these equal, we obtain:

$$\frac{1-\lambda}{\lambda\mu-\lambda+1} = \frac{\nu(\lambda-1)}{\lambda\nu-\nu+1}. \tag{1.10.54}$$

Dividing through by $1-\lambda$ (noting that $\lambda \neq 1$), this simplifies to:

$$\frac{1}{\lambda\mu-\lambda+1} = \frac{-\nu}{\lambda\nu-\nu+1}. \tag{1.10.55}$$

Cross-multiplying and simplifying, we find:

$$\lambda\mu\nu = -1. \tag{1.10.56}$$

Conversely, if $\lambda\mu\nu = -1$, then the values of k for the intersection points of AM with BN and CP coincide, and the three cevians are concurrent. Thus, we have the following theorem:

Theorem 1.12 (Ceva's Theorem)**.** *Let ABC be a triangle, and let M, N, and P be points on the sides BC, CA, and AB (or their extensions), respectively. If the cevians AM, BN, and CP divide the sides in the ratios:*

$$\frac{(MB)}{(MC)} = \lambda, \quad \frac{(NC)}{(NA)} = \mu, \quad \frac{(PA)}{(PB)} = \nu, \tag{1.10.57}$$

then the cevians are concurrent if and only if:

$$\lambda\mu\nu = -1, \tag{1.10.58}$$

or equivalently:

$$\frac{(MB)}{(MC)} \cdot \frac{(NC)}{(NA)} \cdot \frac{(PA)}{(PB)} = -1. \tag{1.10.59}$$

Remark. Ceva's theorem implies, in particular, that if AM, BN, and CP are three concurrent cevians, the feet of these cevians (points M, N, and P) cannot all lie on the extensions of the sides of the triangle. The only possible cases are those where two cevians have their feet on extensions of the sides, while the third has its foot inside the side, or where all feet lie inside the sides.

1.11 Scalar Product of Vectors

1.11.1 Definition and Fundamental Properties

Definition 1.16. Let **a** and **b** be two vectors. The *scalar product* of these two vectors is the real number, denoted by $\mathbf{a} \cdot \mathbf{b}$, equal to the product of their magnitudes and the cosine of the angle between them, i.e.,

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\varphi, \tag{1.11.1}$$

where φ is the angle between the two vectors.

We choose an arbitrary point O in space and construct an oriented segment $\overline{OA}$ such that

$$\overrightarrow{OA} = \frac{\mathbf{a}}{\|\mathbf{a}\|}.$$

Let Δ denote the axis defined by the oriented segment $\overline{OA}$. Then,

$$\|\mathbf{b}\| \cos\varphi = \mathrm{pr}_\Delta\, \mathbf{b},$$

where the projection is orthogonal. Consequently, formula (1.11.1) can be rewritten as:

$$\mathbf{a} \cdot \mathbf{b} = \|a\|\, \mathrm{pr}_\Delta\, \mathbf{b}. \tag{1.11.2}$$

The term "scalar product" is used both because the result of the product is a scalar and because this type of product shares certain properties with the multiplication of real numbers, although, as we will see below, there are essential differences between the two.

We list below the main properties of the scalar product and will prove those that are not immediately obvious.

1) Commutativity:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}. \tag{1.11.3}$$

This property follows directly from the definition of the scalar product.

2) Compatibility with scalar multiplication of vectors:

$$(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}), \tag{1.11.4}$$

$$\mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b}). \tag{1.11.5}$$

It is clear that due to the commutativity of the scalar product, if one of these relations holds, the other also holds, so it suffices to prove one of them, for instance, (1.11.5). Using formula (1.11.2) and the properties of projection, we successively obtain:

$$\mathbf{a} \cdot (\lambda \mathbf{b}) = \|\mathbf{a}\| \operatorname{pr}_\Delta(\lambda \mathbf{b}) = \lambda \|\mathbf{a}\| \operatorname{pr}_\Delta \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}).$$

3) Distributivity with respect to vector addition:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}, \tag{1.11.6}$$

$$(\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}. \tag{1.11.7}$$

Again, due to commutativity, it suffices to prove the first statement. We will again use the representation (1.11.2) of the scalar product, as well as the properties of projection. Thus,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \|\mathbf{a}\| \operatorname{pr}_\Delta(\mathbf{b} + \mathbf{c}) = \|\mathbf{a}\| \operatorname{pr}_\Delta \mathbf{b} + \|\mathbf{a}\| \operatorname{pr}_\Delta \mathbf{c} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

Based on these three properties, we can conclude that the multiplication of linear combinations of vectors can be performed term by term, as in the following example:

$$(2\mathbf{a} + 3\mathbf{b})(4\mathbf{c} - 5\mathbf{d}) = 8\mathbf{a} \cdot \mathbf{c} - 10\mathbf{a} \cdot \mathbf{d} + 12\mathbf{b} \cdot \mathbf{c} - 15\mathbf{b} \cdot \mathbf{d}.$$

We can also easily demonstrate this property of the scalar product directly from its definition. Let us choose a point O and points A, B, and C such that $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{BC}$.

Then, clearly, according to the triangle rule, $\mathbf{b} + \mathbf{c} = \overrightarrow{OC}$. Let θ denote the angle between $\mathbf{a}$ and $\mathbf{b} + \mathbf{c}$, ϕ the angle between $\mathbf{a}$ and $\mathbf{b}$, and ψ the angle between $\mathbf{a}$ and $\mathbf{c}$.

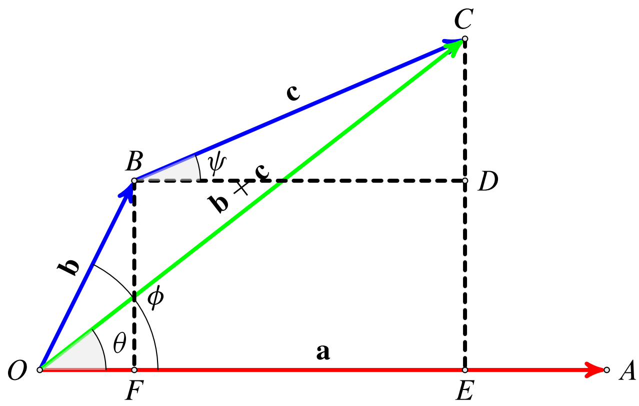


Figure 1.12: Distributivity of the scalar product

Let E and F be the projections of C and B, respectively, on the line OA, and D the projection of B on the line CE. Clearly, θ is the measure of $\widehat{AOC}$, ϕ is the measure of $\widehat{AOB}$, and ψ is the measure of $\widehat{DOC}$.

We have:

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\cdot\overrightarrow{OC} = \|\mathbf{a}\|\|\overrightarrow{OC}\|\cos\theta = \|\mathbf{a}\|\cdot(OE)\\ &= \|\mathbf{a}\|\cdot[(OF)+(FE)] = \|\mathbf{a}\|\cdot(OF)+\|\mathbf{a}\|\cdot(BD)\\ &= \|\mathbf{a}\|\cdot\|\overrightarrow{OB}\|\cos\phi+\|\mathbf{a}\|\cdot\|\overrightarrow{BC}\|\cos\psi\\ &= \|\mathbf{a}\|\cdot\|\mathbf{b}\|\cos\phi+\|\mathbf{a}\|\cdot\|\mathbf{c}\|\cos\psi\\ &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c},\end{aligned}$$

which is what we needed to prove.

The last two properties have a somewhat more "geometric" nature.

4) Two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if their scalar product is zero:

$$\mathbf{a}\cdot\mathbf{b}=0. \tag{1.11.8}$$

Indeed, we begin by writing relation (1.11.8) in the form:

$$\|\mathbf{a}\|\|\mathbf{b}\|\cos\varphi=0. \tag{1.11.9}$$

If the vectors are perpendicular, then $\varphi=\pi/2$, so $\cos\varphi=0$, which implies $\mathbf{a}\cdot\mathbf{b}=0$. Conversely, suppose now that the vectors satisfy relation (1.11.9). If either vector is zero, it can be considered perpendicular to any other vector, so the condition is fulfilled. If both vectors are nonzero, their magnitudes are also nonzero, so relation (1.11.9) can only hold if $\cos\varphi = 0$, i.e., if the two vectors are perpendicular.

We note that property 4) highlights a fundamental difference between the scalar product of vectors and the product of real numbers. The product of two real numbers is zero if and only if at least one of the numbers is zero. In contrast, the scalar product of two vectors can be zero even if both vectors are nonzero.

5) The scalar product of a vector with itself equals the square of the vector's magnitude:

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2. \tag{1.11.10}$$

This result follows immediately from the definition of the scalar product (in this case, naturally, $\varphi = 0$).

Remark. To avoid confusion, we must note that it does not make sense to discuss scalar products involving more than two factors. The scalar product of two vectors is a scalar, which cannot be multiplied by a third vector to form another scalar product. This is why it does not make sense to consider the associativity of the scalar product of free vectors.

1.11.2 Expressing the Scalar Product in Coordinates

Let us choose a rectangular coordinate system in space with the origin at a point O. Let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ denote the orthonormal basis that generates this coordinate system. Recall that the basis is orthonormal, meaning all vectors have length 1, and they are mutually perpendicular. From the properties of the scalar product, described above, it follows immediately that the scalar products between the basis vectors are given by the following multiplication table:

$$\begin{array}{c|ccc} \cdot & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \hline \mathbf{i} & 1 & 0 & 0 \\ \mathbf{j} & 0 & 1 & 0 \\ \mathbf{k} & 0 & 0 & 1 \end{array} \tag{1.11.11}$$

Suppose two vectors $\mathbf{a}$ and $\mathbf{b}$ are given by their expressions in terms of the coordinate basis:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$

Using the scalar multiplication table (1.11.11) for the basis vectors, the scalar product of **a** and **b** is:

$$
\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\\
&= a_1b_1\mathbf{i}^2 + a_1b_2\mathbf{i}\cdot\mathbf{j} + a_1b_3'\mathbf{i}\cdot\mathbf{k}\\
&+ a_2b_1\mathbf{j}\cdot\mathbf{i} + a_2b_2\mathbf{j}^2 + a_2b_3\mathbf{j}\cdot\mathbf{k}\\
&+ a_3b_1\mathbf{k}\cdot\mathbf{i} + a_3b_3\mathbf{k}^2\\
&= a_1b_1 + a_2b_2 + a_3b_3.
\end{aligned}
$$

Thus, the scalar product of two vectors, given by their components relative to a rectangular coordinate system $Oxyz$, is expressed by the formula:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3. \tag{1.11.12}$$

Combining formulas (1.11.10) and (1.11.12), the magnitude of the vector **a** can be computed as:

$$\|\mathbf{a}\| = \sqrt{a_1^2+a_2^2+a_3^2}. \tag{1.11.13}$$

Suppose two points in space are given by their Cartesian orthogonal coordinates, $M(x,y,z)$ and $M'(x',y',z')$. As is well known, the distance $d(M,M')$ between the two points is equal to the length of the vector $\overrightarrow{MM'}(x'-x,y'-y,z'-z)$, i.e., it is given by the formula:

$$d(M,M') = \sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}.$$

Finally, using formulas (1.11.1), (1.11.12), and (1.11.13), we can establish a formula for calculating the cosine of the angle formed by the vectors $\mathbf{a}(X,Y,Z)$ and $\mathbf{b}(X',Y',Z')$, given their components relative to an orthonormal basis:

$$\cos\varphi = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}}.$$

1.11.3 Orthonormal Bases and the Gram-Schmidt Orthogonalisation Process

1.11.4 Solved Problems

Problem 1.11.1. Find the angle ω between the vectors $\mathbf{a} = 2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i}-3\mathbf{k}$.

Solution As we know:

$$\cos\omega = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\cdot\|\mathbf{b}\|} = \frac{2\cdot 4 + (-1)\cdot 0 + 2\cdot(-3)}{\sqrt{2^2 + (-1)^2 + 2^2}\cdot\sqrt{4^2 + 0^2 + (-3)^2}} = \frac{2}{15}.$$

□

Problem 1.11.2. Derive the generalised Pythagorean theorem using the scalar product.

Solution Consider the triangle ABC. Introduce the notations:

$$\mathbf{b} = \overrightarrow{AB},\ \mathbf{c} = \overrightarrow{AC},\ \mathbf{a} = \overrightarrow{BC} = \mathbf{c} - \mathbf{b},\ a = \|\mathbf{a}\|,\ b = \|\mathbf{b}\|,\ c = \|\mathbf{c}\|,$$

(see Figure 1.13).

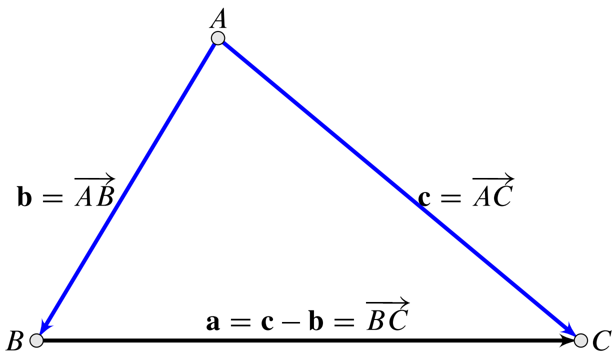


Figure 1.13: The generalised Pythagorean theorem

Then, we have:

$$\begin{aligned} a^2 &= \mathbf{a}\cdot\mathbf{a} = (\mathbf{c}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b}) = \mathbf{c}^2 - 2\mathbf{b}\cdot\mathbf{c} + \mathbf{b}^2 \\ &= b^2 + c^2 - 2bc\cos A. \end{aligned}$$

□

Problem 1.11.3. A vector $\mathbf{a}$ forms equal angles with the vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ and has a magnitude equal to 3. Determine the possible values of the angle the vector forms with the basis unit vectors.

Solution Let $\mathbf{a} = (a_1, a_2, a_3)$ represent the decomposition of $\mathbf{a}$ relative to the canonical basis. Then, if θ is the angle the vector makes with each of the three unit vectors of the basis, we have:

$$\mathbf{a} \cdot \mathbf{i} = a_1 = \mathbf{a} \cdot \mathbf{j} = a_2 = \mathbf{a} \cdot \mathbf{k} = 3\cos\theta.$$

On the other hand, since $\mathbf{a}$ is a vector of magnitude 3:

$$3 = \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{27\cos^2\theta} = \pm 3\sqrt{3}\cos\theta.$$

Thus:

$$\cos\theta = \pm\frac{1}{\sqrt{3}}.$$

□

Problem 1.11.4. Determine the components relative to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the unit vectors perpendicular to the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Solution Let $\mathbf{u} = (u_1, u_2, u_3)$ be a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Since $\mathbf{u}$ is a unit vector, we have:

$$u_1^2 + u_2^2 + u_3^2 = 1. \tag{1.11.14}$$

Since $\mathbf{u}$ is perpendicular to $\mathbf{a}$, we have $\mathbf{a} \cdot \mathbf{u} = 0$, which gives:

$$u_1 + 2u_2 - u_3 = 0. \tag{1.11.15}$$

Similarly, since $\mathbf{u}$ is perpendicular to $\mathbf{b}$, we have $\mathbf{b} \cdot \mathbf{u} = 0$, which gives:

$$3u_1 - u_2 + 2u_3 = 0. \tag{1.11.16}$$

Thus, the components of a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ must satisfy the system formed by equations (1.11.14)–(1.11.16).

Solving this system, we find, as expected from geometric considerations, two solutions (opposite unit vectors):

$$\mathbf{u}_1 = \left(\frac{3}{\sqrt{83}}, -\frac{5}{\sqrt{83}}, -\frac{7}{\sqrt{83}}\right) \quad \text{and} \quad \mathbf{u}_2 = \left(-\frac{3}{\sqrt{83}}, \frac{5}{\sqrt{83}}, \frac{7}{\sqrt{83}}\right).$$

□

We will revisit this problem after studying the vector product, which will allow for a direct solution without requiring the system of equations.

Problem 1.11.5. A unit vector $\mathbf{a}$ forms angles of $\pi/4$ with $\mathbf{i}$ and $\pi/3$ with $\mathbf{j}$. If the angle θ between $\mathbf{a}$ and $\mathbf{k}$ is acute, determine the components of $\mathbf{a}$ relative to the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the value of the angle θ.

Solution Suppose that, relative to the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, the given vector can be written as:

$$\mathbf{a} = (a_1, a_2, a_3).$$

Since both $\mathbf{a}$ and $\mathbf{i}$ are unit vectors, we have:

$$a_1 = \mathbf{a} \cdot \mathbf{i} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad a_2 = \mathbf{a} \cdot \mathbf{j} = \cos\frac{\pi}{3} = \frac{1}{2}.$$

Additionally, since $\mathbf{a}$ is a unit vector:

$$a_3^2 = 1 - a_1^2 - a_2^2 = \frac{1}{4}.$$

Thus:

$$a_3 = \pm\frac{1}{2}.$$

On the other hand:

$$a_3 = \mathbf{a} \cdot \mathbf{k} = \cos\theta.$$

Hence:

$$\cos\theta = \pm\frac{1}{2}.$$

Since the angle θ is acute, $\cos\theta > 0$, and therefore:

$$\cos\theta = \frac{1}{2}, \quad \text{so } \theta = \frac{\pi}{3}.$$

□

Problem 1.11.6. Let $\mathbf{p}, \mathbf{q}, \mathbf{r}$ be three vectors of lengths 1, 2, and 3, respectively. Assume that $\mathbf{q}$ is perpendicular to $\mathbf{r}$, the angle between $\mathbf{r}$ and $\mathbf{p}$ is $\pi/3$, and the angle between $\mathbf{p}$ and $\mathbf{q}$ is also $\pi/3$.

(i) Show that the vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}$ form a basis.

(ii) If $\mathbf{a}$ is a unit vector coplanar with $\mathbf{q}$ and $\mathbf{r}$ and forms an angle of $\pi/4$ with each of them, show that:

$$\mathbf{a} = \frac{\sqrt{2}}{12}(3\mathbf{q} + 2\mathbf{r}).$$

(iii) Determine the components relative to the basis $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ of the unit vectors perpendicular to both $\mathbf{q}$ and $\mathbf{r}$.

Solution (i) To show that the three vectors form a basis, it suffices to prove that they are linearly independent. Starting with $\mathbf{q}$ and $\mathbf{r}$, it is evident that they are not collinear (being perpendicular and non-zero), so they span a plane. If $\mathbf{p}$ were coplanar with the other two, the angle between $\mathbf{p}$ and $\mathbf{q}$ or $\mathbf{p}$ and $\mathbf{r}$ would differ from the given angles, leading to a contradiction. Hence, $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are linearly independent and form a basis.

(ii) Since $\mathbf{a}$ is coplanar with $\mathbf{q}$ and $\mathbf{r}$, it can be expressed as:

$$\mathbf{a} = \alpha\mathbf{q} + \beta\mathbf{r}.$$

The rest of the proof involves using the given angles and magnitudes to find α and β, which results in the desired expression for $\mathbf{a}$.

(iii) The unit vectors perpendicular to both $\mathbf{q}$ and $\mathbf{r}$ can be obtained by solving the system of equations formed by their orthogonality conditions, leading to two solutions:

$$\mathbf{u}_1 = \dots, \quad \mathbf{u}_2 = \dots \quad \text{(explicit components depending on the calculations).}$$

□

Problem 1.11.7. Prove that the altitudes of a triangle are concurrent.

Solution Let H be the point of intersection of the altitudes AK and BL. Define $\mathbf{a} = \overrightarrow{AH}$, $\mathbf{b} = \overrightarrow{BH}$, and $\mathbf{c} = \overrightarrow{CH}$. Since AK and BL are altitudes, we have:

$$\mathbf{a} \perp \overrightarrow{BC} \quad \text{and} \quad \mathbf{b} \perp \overrightarrow{AC}. \tag{1.11.17}$$

Moreover:

$$\overrightarrow{BC} = \overrightarrow{HC} - \overrightarrow{HB} = \mathbf{c} - \mathbf{b}, \quad \text{and} \quad \overrightarrow{AC} = \overrightarrow{HA} - \overrightarrow{HC} = \mathbf{a} - \mathbf{c}.$$

Substituting into equations (1.11.17), we obtain:

$$\begin{cases} \mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0, \\ \mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = 0, \end{cases}$$

which simplifies to:

$$\begin{cases} \mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}, \\ \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}. \end{cases}$$

From this, we deduce that:

$$\mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) = 0,$$

which means that $\overrightarrow{CH} \perp \overrightarrow{AB}$, i.e., CH is the altitude perpendicular to the side AB. Thus, all three altitudes of the triangle intersect at point H, known as the *orthocentre* of triangle ABC. □

Problem 1.11.8. Let $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ be three vectors of lengths 2, 2, and 3, respectively. Assume that the angle between $\mathbf{q}$ and $\mathbf{r}$ is $\pi/3$, while the angles between $\mathbf{r}$ and $\mathbf{p}$, and $\mathbf{p}$ and $\mathbf{q}$ are both $2\pi/3$.

(i) Calculate the scalar products $\mathbf{p} \cdot \mathbf{p}$, $\mathbf{q} \cdot \mathbf{q}$, $\mathbf{r} \cdot \mathbf{r}$, $\mathbf{p} \cdot \mathbf{q}$, $\mathbf{p} \cdot \mathbf{r}$, and $\mathbf{q} \cdot \mathbf{r}$.

(ii) Determine the unit vectors coplanar with $\mathbf{p} + \mathbf{q}$ and $\mathbf{r}$ that form an angle of $\pi/4$ with $\mathbf{p} + \mathbf{q}$.

Solution (i) Using the given magnitudes and angles, the scalar products are calculated as follows:

$$\mathbf{p} \cdot \mathbf{p} = \|\mathbf{p}\|^2 = 4, \quad \mathbf{q} \cdot \mathbf{q} = \|\mathbf{q}\|^2 = 4, \quad \mathbf{r} \cdot \mathbf{r} = 9,$$
$$\mathbf{p} \cdot \mathbf{q} = 2 \cdot 2 \cdot \cos\frac{2\pi}{3} = -2, \quad \mathbf{p} \cdot \mathbf{r} = 2 \cdot 3 \cdot \cos\frac{2\pi}{3} = -3, \quad \mathbf{q} \cdot \mathbf{r} = 2 \cdot 3 \cdot \cos\frac{\pi}{3} = 3.$$

(ii) Let $\mathbf{u}$ be a unit vector coplanar with $\mathbf{p} + \mathbf{q}$ and $\mathbf{r}$ that forms an angle of $\pi/4$ with $\mathbf{p} + \mathbf{q}$. Then:

$$\mathbf{u} = \alpha(\mathbf{p} + \mathbf{q}) + \beta\mathbf{r},$$

where α and β are scalars. Since $\mathbf{u}$ is a unit vector, we have:

$$\|\mathbf{u}\|^2 = 1 \quad \Rightarrow \quad \alpha^2\|\mathbf{p} + \mathbf{q}\|^2 + \beta^2\|\mathbf{r}\|^2 + 2\alpha\beta(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} = 1.$$

We calculate $\|\mathbf{p} + \mathbf{q}\|^2$ and $(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r}$:

$$\|\mathbf{p} + \mathbf{q}\|^2 = \mathbf{p} \cdot \mathbf{p} + 2\mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{q} = 4 - 4 + 4 = 4,$$
$$(\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} = \mathbf{p} \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{r} = -3 + 3 = 0.$$

Thus, the equation simplifies to:

$$4\alpha^2 + 9\beta^2 = 1.$$

Additionally, since $\mathbf{u}$ forms an angle of $\pi/4$ with $\mathbf{p} + \mathbf{q}$:

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{\mathbf{u} \cdot (\mathbf{p} + \mathbf{q})}{\|\mathbf{u}\|\,\|\mathbf{p} + \mathbf{q}\|} = \frac{4\alpha}{2} = 2\alpha,$$

hence:

$$\alpha = \frac{\sqrt{2}}{4}.$$

Substituting α into $4\alpha^2 + 9\beta^2 = 1$, we find:

$$\beta = \pm\frac{\sqrt{2}}{6}.$$

Thus, the unit vectors are:

$$\mathbf{u}_1 = \frac{\sqrt{2}}{4}(\mathbf{p} + \mathbf{q}) + \frac{\sqrt{2}}{6}\mathbf{r}, \quad \mathbf{u}_2 = \frac{\sqrt{2}}{4}(\mathbf{p} + \mathbf{q}) - \frac{\sqrt{2}}{6}\mathbf{r}.$$

□

1.12 Vector Product of Vectors

1.12.1 Definition and Fundamental Properties

Definition 1.17. The *vector product* of the vector $\mathbf{a}$ and the vector $\mathbf{b}$ is, by definition, the vector, denoted by $\mathbf{a} \times \mathbf{b}$, determined by the following conditions:

1) If the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, then, by definition, their vector product $\mathbf{a} \times \mathbf{b}$ is zero.

2) If the two vectors are not collinear, i.e., they form an angle φ, where $0 < \varphi < \pi$, their vector product is defined by the following three conditions:

   (i) The magnitude of the vector $\mathbf{a} \times \mathbf{b}$ is equal to $\|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \sin\varphi$;

   (ii) The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both vectors $\mathbf{a}$ and $\mathbf{b}$;

   (iii) The triplet of vectors $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ forms a right-handed system.

We will first enumerate the fundamental properties of the vector product of two vectors.

1) The first property expresses a geometric fact: if the vectors $\mathbf{a}$ and $\mathbf{b}$ are not collinear, then the norm of the vector $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram formed by the segments OA and OB, where O is an arbitrary point in space, and $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Furthermore, it is evident that the area of the triangle OAB is equal to half the magnitude of the vector product of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

This property follows immediately from the very definition of the vector product.

2) The vector product is *anticommutative*:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}. \tag{1.12.1}$$

This property follows directly from the definition.

3) The vector product is compatible with scalar multiplication of vectors:

$$(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}), \tag{1.12.2}$$

$$\mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b}). \tag{1.12.3}$$

Due to the anticommutativity of the vector product, it is sufficient to demonstrate one of these properties. We will therefore prove equation (1.12.2). If $\lambda = 0$, or if the two vectors are collinear, then both sides of the equation are zero, and the property holds. Now assume that neither of these cases applies. The magnitude of the vector $\lambda(\mathbf{a} \times \mathbf{b})$ is equal to $|\lambda| \|\mathbf{a}\| \|\mathbf{b}\| \sin \varphi$, where φ is the angle between the two vectors. Let us evaluate the magnitude of the vector in the right-hand side. If $\lambda > 0$, then the vectors $\mathbf{a}$ and $\lambda \mathbf{a}$ have the same direction, so the angle between $\lambda \mathbf{a}$ and $\mathbf{b}$ is still φ. Thus,

$$\|(\lambda \mathbf{a}) \times \mathbf{b}\| = \|\lambda \mathbf{a}\| \|\mathbf{b}\| \sin \varphi = |\lambda| \|\mathbf{a}\| \|\mathbf{b}\| \sin \varphi.$$

If $\lambda < 0$, then the vectors $\lambda \mathbf{a}$ and $\mathbf{a}$ have opposite directions, so the angle between $\lambda \mathbf{a}$ and $\mathbf{b}$ is $\pi - \varphi$. Hence, in this case:

$$\|(\lambda \mathbf{a}) \times \mathbf{b}\| = \|\lambda \mathbf{a}\| \|\mathbf{b}\| \sin(\pi - \varphi) = |\lambda| \|\mathbf{a}\| \|\mathbf{b}\| \sin \varphi.$$

Thus, the vectors on both sides of equation (1.12.2) have the same magnitude for both positive and negative values of the scalar λ. Clearly, these two vectors are collinear, as they are both perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$. Finally, we need to demonstrate that these vectors also have the same direction. It is immediately apparent that for $\lambda > 0$, both vectors have the same direction as $\mathbf{a} \times \mathbf{b}$, whereas for $\lambda < 0$, they both have the opposite direction to $\mathbf{a} \times \mathbf{b}$.

4) The vector product is distributive with respect to vector addition:

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}, \tag{1.12.4}$$

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}. \tag{1.12.5}$$

Once again, it is sufficient to demonstrate the first equality, as the second follows from anticommutativity.

This equality is clearly valid if any of the three vectors is zero or if the sum $\mathbf{a}+\mathbf{b}$ is zero. Assume now that none of these cases applies.

We will first demonstrate the equality

$$(\mathbf{a}+\mathbf{b})\times\mathbf{c}_0=\mathbf{a}\times\mathbf{c}_0+\mathbf{b}\times\mathbf{c}_0, \tag{1.12.6}$$

where $\mathbf{c}_0=\mathbf{c}/\|\mathbf{c}\|$ is a unit vector.

Let us first show how to construct the vector product of an arbitrary vector $\mathbf{a}$ with a unit vector $\mathbf{c}_0$. Attach the vectors $\mathbf{a}$ and $\mathbf{c}_0$ to an arbitrary point O, thus constructing points A and C such that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OC}=\mathbf{c}_0$. Draw a plane Π passing through O and perpendicular to the line OC, and project the directed segment $\overline{OA}$ orthogonally onto this plane. The directed segment $\overline{OA'}$, the orthogonal projection of $\overline{OA}$ onto Π, is rotated within the plane Π about O by an angle of $\pi/2$ in the clockwise direction, as viewed from point C. The directed segment $\overline{OA''}$, obtained from the rotation, will represent the vector product of $\mathbf{a}$ and $\mathbf{c}_0$.

Indeed, if φ denotes the angle between the vectors $\mathbf{a}$ and $\mathbf{c}_0$, we can write:

$$\|\overline{OA''}\|=\|\overline{OA'}\|=\|\overline{OA}\|\cos\left(\frac{\pi}{2}-\varphi\right)=\|\mathbf{a}\|\|\mathbf{c}_0\|\sin\varphi.$$

Additionally, it is straightforward to verify that the vector $\overrightarrow{OA''}$ has the same direction and sense as $\mathbf{a}\times\mathbf{c}_0$.

Returning to the proof of relation (1.12.6), fix a point O and construct points A, B, C such that $\overrightarrow{OC}=\mathbf{c}_0$, $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{AB}=\mathbf{b}$, and $\overrightarrow{OB}=\mathbf{a}+\mathbf{b}$. Next, construct the plane Π passing through O and perpendicular to the directed segment $\overline{OC}$. Denote by A' and B' the orthogonal projections of points A and B onto plane Π. Rotate triangle $OA'B'$ within the plane Π about point O by an angle of $\pi/2$ in the clockwise direction as viewed from point C. This rotation produces triangle $OA''B''$. We then have:

$$\begin{aligned}&\overrightarrow{OB''}=\overrightarrow{OA''}+\overrightarrow{A''B''},\\&\overrightarrow{OB''}=(\mathbf{a}+\mathbf{b})\times\mathbf{c}_0,\quad \overrightarrow{OA''}=\mathbf{a}\times\mathbf{c}_0,\quad \overrightarrow{A''B''}=\mathbf{b}\times\mathbf{c}_0.\end{aligned} \tag{1.12.7}$$

Thus, from (1.12.7), it follows that equality (1.12.6) holds. To derive (1.12.4), it suffices to multiply relation (1.12.6) by the scalar $\|\mathbf{c}\|$.

The properties of the vector product described above allow us to formulate a rule for calculating the vector product of two linear combinations of free vectors: simply calculate the product of each term in the first combination with each term in the second combination, and then sum the results. For example,

$$(\mathbf{a}+2\mathbf{b})\times(2\mathbf{c}-3\mathbf{d})=2\mathbf{a}\times\mathbf{c}-3\mathbf{a}\times\mathbf{d}+4\mathbf{b}\times\mathbf{c}-6\mathbf{b}\times\mathbf{d}.$$

Remark. The vector product shares certain similarities with the scalar product of vectors. However, there are several key differences to bear in mind:

1) The vector product is *not* commutative—the order of the factors matters.

2) The vector product of two vectors is a vector, not a scalar. Consequently, it makes sense to consider products involving more than two factors. Nevertheless, as we will see later, the vector product is *not* associative.

1.12.2 Expression of the Vector Product in Terms of the Components of the Factors

Consider an orthogonal coordinate system $Oxyz$ with the orthonormal basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. It is easy to verify that the basis vectors multiply vectorially according to the rules described in the following table:

$\times$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{0}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{0}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$\mathbf{0}$

Now consider two vectors $\mathbf{a}$ and $\mathbf{b}$ given by their components:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.$$

Using the distributivity of the vector product with respect to vector addition, we obtain:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}, \end{aligned}$$

so that

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \tag{1.12.8}$$

Taking into account the rule for expanding a determinant of order three along its first row, the above formula can also be written in a more compact and memorable form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{1.12.9}$$

Remark. From the analytical expression (1.12.9), one can immediately derive analytical formulas for the area of the parallelogram and the area of the triangle determined by the two vectors. Thus, from the mentioned formula, we can write:

$$\mathbf{a}\times\mathbf{b}=\mathbf{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix},$$

that is,

$$\text{Aria}_{par}\equiv\|\mathbf{a}\times\mathbf{b}\|=\sqrt{\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}^2+\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}^2+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}^2}. \qquad (1.12.10)$$

Thus, the area of the triangle determined by the two vectors is:

$$\text{Aria}_{triun}=\frac{1}{2}\sqrt{\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}^2+\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}^2+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}^2}. \qquad (1.12.11)$$

Now consider the case where we have three arbitrary points in the xOy plane: $A(x_A, y_A, 0)$, $B(x_B, y_B, 0)$, and $C(x_C, y_C, 0)$. These points determine two vectors: $\mathbf{a}=\overrightarrow{AB}$ and $\mathbf{b}=\overrightarrow{AC}$. Clearly, $\mathbf{a}=(x_B-x_A, y_B-y_A, 0)$ and $\mathbf{b}=(x_C-x_A, y_C-y_A, 0)$. Therefore,

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_B-x_A & y_B-y_A & 0\\ x_C-x_A & y_C-y_A & 0\end{vmatrix}=\mathbf{k}\begin{vmatrix}x_B-x_A & y_B-y_A\\ x_C-x_A & y_C-y_A\end{vmatrix}.$$

Thus,

$$\|\mathbf{a}\times\mathbf{b}\|=\pm\begin{vmatrix}x_A & y_A & 1\\ x_B & y_B & 1\\ x_C & y_C & 1\end{vmatrix}.$$

Therefore, the area of triangle ABC in the xOy plane is given by:

$$\text{Aria}_{ABC}=\pm\frac{1}{2}\begin{vmatrix}x_A & y_A & 1\\ x_B & y_B & 1\\ x_C & y_C & 1\end{vmatrix}. \qquad (1.12.12)$$

Naturally, the sign is chosen so that the right-hand side is positive. It is also noteworthy that the above formula provides a criterion for the collinearity of points A, B, C. Specifically, they are collinear if and only if the determinant on the right-hand side of equation (1.12.12) is zero.

1.12.3 The Double Vector Product

As observed earlier, the vector product of two vectors is, again, a vector, so it makes sense to compute the product of this vector with a third vector. The result of this operation is called the *double vector product*. Note that the *vector product is not associative*, which means we cannot omit parentheses as we do, for instance, with the product of real or complex numbers, or even with the product of matrices. This fact can be easily verified by studying the products of the canonical basis elements in three-dimensional space. For example:

$$(\mathbf{i}\times\mathbf{j})\times\mathbf{j}=\mathbf{k}\times\mathbf{j}=-\mathbf{i},$$

while

$$\mathbf{i}\times(\mathbf{j}\times\mathbf{j})=0.$$

Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three vectors in space. As stated earlier, the *double vector product* of these three vectors is, by definition, the vector $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. We will demonstrate that the following relation holds:

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}. \tag{1.12.13}$$

If one of the vectors is null, then, clearly, both sides of the above equality are zero, so there is nothing to prove. The same is true if the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear.

Let us assume, therefore, that all three vectors are non-zero and that $\mathbf{a}$ and $\mathbf{b}$ are not collinear. Consider the vector $\mathbf{x}=(\mathbf{a}\times\mathbf{b})\times\mathbf{a}$.

Choose an arbitrary origin O in space, and let X, A, B, C be four points in space such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$, and $\overrightarrow{OX}=\mathbf{x}$. Then, we have:

$$\overrightarrow{OX}=\left(\overrightarrow{OA}\times\overrightarrow{OB}\right)\times\overrightarrow{OA}.$$

From the definition of the vector product, it follows immediately that $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{OA}$ and lies in the plane OAB. Moreover, since the triple of vectors

$$\{(\overrightarrow{OA}\times\overrightarrow{OB}),\overrightarrow{OA},\overrightarrow{OX}\}$$

is right-handed, it follows that the vectors $\overrightarrow{OB}$ and $\overrightarrow{OX}$ are on the same side of the plane containing the vectors $\overrightarrow{OA}$ and $\overrightarrow{OA}\times\overrightarrow{OB}$. Additionally, we have:

$$\|\overrightarrow{OX}\|=\|\overrightarrow{OA}\|^2\|\overrightarrow{OB}\|\sin\theta,$$

where θ is the angle $\angle AOB$.

These properties uniquely define the vector $\overrightarrow{OX}$. On the other hand, it is easy to verify that these properties are satisfied by the vector:

$$(\overrightarrow{OA} \cdot \overrightarrow{OA})\overrightarrow{OB} - (\overrightarrow{OA} \cdot \overrightarrow{OB})\overrightarrow{OA}.$$

Therefore:

$$\overrightarrow{OX} = \left(\overrightarrow{OA} \times \overrightarrow{OB}\right) \times \overrightarrow{OA} = (\overrightarrow{OA} \cdot \overrightarrow{OA})\overrightarrow{OB} - (\overrightarrow{OA} \cdot \overrightarrow{OB})\overrightarrow{OA}.$$

Similarly, it can be shown that:

$$\left(\overrightarrow{OA} \times \overrightarrow{OB}\right) \times \overrightarrow{OB} = (\overrightarrow{OA} \cdot \overrightarrow{OB})\overrightarrow{OB} - (\overrightarrow{OB} \cdot \overrightarrow{OB})\overrightarrow{OA}.$$

The frame $\{\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OA} \times \overrightarrow{OB}\}$ is a basis of $\mathbb{R}^3$, so there exist three real numbers u, v, w such that:

$$\overrightarrow{OC} = u\overrightarrow{OA} + v\overrightarrow{OB} + w(\overrightarrow{OA} \times \overrightarrow{OB}).$$

Expanding $(\overrightarrow{OA} \times \overrightarrow{OB}) \times \overrightarrow{OC}$ and using the fact that $\overrightarrow{OA} \times \overrightarrow{OB}$ is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$, we obtain:

$$(\overrightarrow{OA} \times \overrightarrow{OB}) \times \overrightarrow{OC} = (\overrightarrow{OA} \cdot \overrightarrow{OC})\overrightarrow{OB} - (\overrightarrow{OB} \cdot \overrightarrow{OC})\overrightarrow{OA}.$$

Thus, relation (1.12.13) is proven.

Remark. Naturally, it makes sense to calculate the product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Using the anticommutativity of the vector product and relation (1.12.13), we find:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}.$$

Comparing the vectors $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, we conclude that they can only be equal if:

$$-(\mathbf{b} \cdot \mathbf{c})\mathbf{a} + 2(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} = 0.$$

Thus, *a necessary condition for the two double vector products to be equal is that the three vectors are coplanar.* This condition, however, is not sufficient, as the coefficients of the three vectors in the above equality are not arbitrary.

It can also be easily shown, using relation (1.12.13), that for any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the following identity (*Jacobi's identity*) holds:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = 0. \tag{1.12.14}$$

1.12.4 Solved Problems

Problem 1.12.1. Consider the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution As discussed in the section on the scalar product, this type of problem can be solved in several ways. Equipped with the vector product, we can now use an alternative approach. The two unit vectors we seek can be obtained by first computing the vector product of the two given vectors (in either order) and then dividing the resulting vector by its magnitude to obtain vectors of unit length. Since the given vectors are not collinear, all vectors perpendicular to both of them will be collinear with the result of their vector product. Consequently, there will be only two unit vectors among these, which can be constructed as described above.

Let us first compute the vector product of **a** and **b**. We have:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix} = -\mathbf{i} - 3\mathbf{j} + 5\mathbf{k},$$

which gives:

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{(-1)^2 + (-3)^2 + 5^2} = \sqrt{35}.$$

Thus, the two unit vectors we are looking for are:

$$\mathbf{u}_{1,2} = \pm \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} = \pm \frac{1}{\sqrt{35}}(-\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}).$$

□

Problem 1.12.2. Let $\mathbf{a} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Find a vector **d** that is perpendicular to both **a** and **b**, such that $\mathbf{c} \cdot \mathbf{d} = 15$.

Solution Any vector perpendicular to both **a** and **b** must be collinear with their vector product. Let us first compute this vector product. We have:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix} = 32\mathbf{i} - \mathbf{j} - 14\mathbf{k}.$$

Therefore, **d** must be of the form:

$$\mathbf{d} = \lambda(32\mathbf{i} - \mathbf{j} - 14\mathbf{k}),$$

where λ will be determined using the second condition given in the problem, namely:

$$15 = \mathbf{c} \cdot \mathbf{d} = \lambda[(2 \cdot 32) + (-1 \cdot -1) + (4 \cdot -14)] = \lambda[64 + 1 - 56] = 9\lambda.$$

This gives $\lambda = \frac{15}{9} = \frac{5}{3}$, and therefore:

$$\mathbf{d} = \frac{5}{3}(32\mathbf{i} - \mathbf{j} - 14\mathbf{k}) = \frac{1}{3}(160\mathbf{i} - 5\mathbf{j} - 70\mathbf{k}).$$

□

Problem 1.12.3. Let ABC be a triangle with sides $BC = a$, $CA = b$, and $AB = c$. Prove, using the vector product, that:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \tag{1.12.15}$$

(the *law of sines in triangle* ABC).

Proof Consider the vectors $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$, and $\overrightarrow{AB} = \mathbf{c}$. Note that:

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0. \tag{1.12.16}$$

Taking the vector product of both sides of (1.12.16) with $\mathbf{a}$ gives:

$$\mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = 0,$$

which simplifies to:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}. \tag{1.12.17}$$

The angle between $\mathbf{a} = \overrightarrow{BC}$ and $\mathbf{b} = \overrightarrow{CA}$ is $\pi - C$, and the angle between $\mathbf{c} = \overrightarrow{AB}$ and $\mathbf{a}$ is $\pi - B$. Passing to magnitudes in (1.12.17), we get:

$$ab \sin(\pi - C) = ca \sin(\pi - B),$$

or equivalently:

$$ab \sin C = ca \sin B.$$

Dividing both sides by abc, we obtain:

$$\frac{\sin C}{c} = \frac{\sin B}{b}.$$

By similarly taking the vector product of (1.12.16) with $\mathbf{b}$, it can be shown that:

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Combining these results, we arrive at (1.12.15), as required. □

Problem 1.12.4. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three vectors such that $\mathbf{a} = \mathbf{b} \times \mathbf{c}$, $\mathbf{b} = \mathbf{c} \times \mathbf{a}$, and $\mathbf{a}$ is a unit vector, prove that the three vectors form an orthonormal basis.

Proof First, observe that none of the vectors $\mathbf{a}$, $\mathbf{b}$, or $\mathbf{c}$ can be zero since $\mathbf{a}$ is a unit vector.

Given that $\mathbf{a} = \mathbf{b} \times \mathbf{c}$ and $\mathbf{b} = \mathbf{c} \times \mathbf{a}$, we deduce:

$$\mathbf{a} = (\mathbf{b} \times \mathbf{c}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{c}.$$

Since $\mathbf{a}$ is perpendicular to $\mathbf{c}$, it follows that $\mathbf{a} \cdot \mathbf{c} = 0$. Therefore:

$$\mathbf{a} = (\mathbf{c} \cdot \mathbf{c})\mathbf{b}.$$

This implies that $\mathbf{c} \cdot \mathbf{c} = 1$, meaning $\mathbf{c}$ is a unit vector.

Additionally, since $\mathbf{b} = \mathbf{c} \times \mathbf{a}$, and the vectors $\mathbf{c}$ and $\mathbf{a}$ are perpendicular, we have:

$$\|\mathbf{b}\| = \|\mathbf{c}\| \cdot \|\mathbf{a}\| \sin \frac{\pi}{2} = 1.$$

Thus, the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ all have unit length and are mutually perpendicular. Hence, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ forms an orthonormal basis. Furthermore, this basis is positively oriented since $\mathbf{b} = \mathbf{c} \times \mathbf{a}$. $\square$

Problem 1.12.5. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three vectors in $\mathbb{R}^3$, prove the Jacobi identity:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = 0. \tag{1.12.18}$$

Proof Let $\mathbf{d} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b}$. Using the vector triple product identity:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a},$$

we expand $\mathbf{d}$:

$$\mathbf{d} = [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}] + [(\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}] + [(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}].$$

Collecting terms for each vector $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:

$$\mathbf{d} = \underbrace{[-(\mathbf{b} \cdot \mathbf{c}) + (\mathbf{c} \cdot \mathbf{b})]\mathbf{a}}_{=0} + \underbrace{[(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{c} \cdot \mathbf{a})]\mathbf{b}}_{=0} + \underbrace{[(\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{b})]\mathbf{c}}_{=0}.$$

Since all coefficients vanish, it follows that $\mathbf{d} = 0$, which proves (1.12.18). $\square$

1.13 The Mixed Product of Vectors

1.13.1 Definition and Fundamental Properties

Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ be three vectors. The *mixed product* of the three vectors is defined as the number

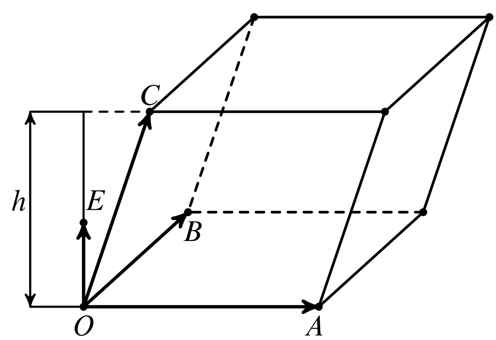
$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) := (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \tag{1.13.1}$$

The mixed product of vectors has a remarkable geometric interpretation, expressed in the following theorem.

Theorem 1.13. *Let* $\mathbf{a}, \mathbf{b}$*, and* $\mathbf{c}$ *be three non-coplanar vectors. Associate them with a point* O*, and let* A, B, C *be the points for which*

$$\overrightarrow{OA} = \mathbf{a}, \ \overrightarrow{OB} = \mathbf{b}, \ \overrightarrow{OC} = \mathbf{c}.$$

Then the mixed product $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is equal to the volume of the parallelepiped constructed on the segments* OA*,* OB*, and* OC*, taken with a positive sign if the triplet* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *is right-handed and with a negative sign if the triplet is left-handed.*



Proof Let V be the volume of the parallelepiped constructed on the segments OA, OB, and OC, S the area of the parallelogram constructed on the segments OA and OB, and h the height of the parallelepiped. Then $V = Sh$.

Now, assign to point O a unit vector $\overrightarrow{OE}$, perpendicular to the segments OA and OB and oriented so that the triplet of vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\mathbf{e} = \overrightarrow{OE}$ is right-handed. Evidently, $\mathbf{a} \times \mathbf{b} = S\mathbf{e}$. Therefore,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = S(\mathbf{e} \cdot \mathbf{c}) = S \operatorname{pr}_{\mathbf{e}} \mathbf{c} = \pm Sh = \pm V,$$

where the positive sign is taken if the triplet $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is right-handed and the negative sign is taken if the triplet is left-handed. $\square$

Corollary 1.6. *The volume of the tetrahedron $OABC$ is given by the formula*

$$Vol_{OABC} = \pm\frac{1}{6}(\mathbf{a}, \mathbf{b}, \mathbf{c}),$$

where $\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB}, \mathbf{c} = \overrightarrow{OC}$.

Corollary 1.7. *A system of three linearly independent vectors* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *is right-handed if* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) > 0$ *and left-handed if* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) < 0$.

Corollary 1.8. *An* orthonormal *system of three linearly independent vectors* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *is right-handed if* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 1$ *and left-handed if* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -1$.

The mixed product of vectors also allows us to establish a criterion for the coplanarity of three vectors, stated in the following theorem.

Theorem 1.14. *For three vectors* $\mathbf{a}, \mathbf{b}$, *and* $\mathbf{c}$ *to be coplanar, it is necessary and sufficient that their mixed product equals zero:*

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0. \tag{1.13.2}$$

Proof If the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar, then the vector $\mathbf{a} \times \mathbf{b}$ is either zero (if $\mathbf{a}$ and $\mathbf{b}$ are collinear) or perpendicular to $\mathbf{c}$. In both cases, equality (1.13.2) holds.

Conversely, suppose equality (1.13.2) holds. If the vectors were not coplanar, they would determine, as in the previous theorem, a parallelepiped with a volume

$$0 \neq V = \pm(\mathbf{a}, \mathbf{b}, \mathbf{c}),$$

contradicting equality (1.13.2). $\square$

1.13.2 Linearity of the Mixed Product

From the properties of the vector product and the scalar product, the following theorem results:

Theorem 1.15. *The mixed product of vectors is multilinear, i.e., it is variable in each component:*

(i) For any vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}$*, and* $\mathbf{c}$ *and any scalars* $\lambda_1, \lambda_2 \in \mathbb{R}$*, we have*

$$\big(\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2, \mathbf{b}, \mathbf{c}\big) = \lambda_1\big(\mathbf{a}_1, \mathbf{b}, \mathbf{c}\big) + \lambda_2\big(\mathbf{a}_2, \mathbf{b}, \mathbf{c}\big); \tag{1.13.3}$$

(ii) For any vectors $\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2$*, and* $\mathbf{c}$ *and any scalars* $\lambda_1, \lambda_2 \in \mathbb{R}$*, we have*

$$\big(\mathbf{a}, \lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2, \mathbf{c}\big) = \lambda_1\big(\mathbf{a}, \mathbf{b}_1, \mathbf{c}\big) + \lambda_2\big(\mathbf{a}, \mathbf{b}_2, \mathbf{c}\big); \tag{1.13.4}$$

(iii) For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}_1$*, and* $\mathbf{c}_2$ *and any scalars* $\lambda_1, \lambda_2 \in \mathbb{R}$*, we have*

$$\big(\mathbf{a}, \mathbf{b}, \lambda_1\mathbf{c}_1 + \lambda_2\mathbf{c}_2\big) = \lambda_1\big(\mathbf{a}, \mathbf{b}, \mathbf{c}_1\big) + \lambda_2\big(\mathbf{a}, \mathbf{b}, \mathbf{c}_2\big). \tag{1.13.5}$$

Proof (i) From the definition, we have:

$$\begin{aligned}\big(\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2, \mathbf{b}, \mathbf{c}\big) &= \big(\,(\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2) \times \mathbf{b}\big) \cdot \mathbf{c} = \\ &= \big(\lambda_1\,(\mathbf{a}_1 \times \mathbf{b}) + \lambda_2\,(\mathbf{a}_2 \times \mathbf{b})\,\big) \cdot \mathbf{c} = \\ &= \lambda_1\,(\mathbf{a}_1 \times \mathbf{b}) \cdot \mathbf{c} + \lambda_2\,(\mathbf{a}_2 \times \mathbf{b}) \cdot \mathbf{c} = \lambda_1\big(\mathbf{a}_1, \mathbf{b}, \mathbf{c}\big) + \\ &+ \lambda_2\big(\mathbf{a}_2, \mathbf{b}, \mathbf{c}\big).\end{aligned}$$

(ii) This property follows immediately from the previous one, using the anticommutativity of the vector product.
(iii) Again, from the definition of the mixed product, we have:

$$\begin{aligned}\big(\mathbf{a}, \mathbf{b}, \lambda_1\mathbf{c}_1 + \lambda_2\mathbf{c}_2\big) &= (\mathbf{a} \times \mathbf{b}) \cdot (\lambda_1\mathbf{c}_1 + \lambda_2\mathbf{c}_2) = \lambda_1\big(\,(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}_1\big) + \\ &+ \lambda_2\big(\,(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}_2\big) = \lambda_1\big(\mathbf{a}, \mathbf{b}, \mathbf{c}_1\big) + \lambda_2\big(\mathbf{a}, \mathbf{b}, \mathbf{c}_2\big).\end{aligned}$$

□

1.13.3 The Expression of the Mixed Product in Coordinates

Assume that, relative to an orthonormal basis, the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are given by their components:

$$\mathbf{a}(a_1, a_2, a_3),\ \mathbf{b}(b_1, b_2, b_3),\ \mathbf{c}(c_1, c_2, c_3). \tag{1.13.6}$$

Using the coordinate expressions for the vector product and scalar product, we obtain:

$$\begin{aligned}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + \\ &+ (a_1b_2 - a_2b_1)c_3 = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - \\ &- a_2b_1c_3 - a_3b_2c_1.\end{aligned}$$

It is easy to see that this relation can be rewritten using a third-order determinant:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{1.13.7}$$

From the properties of determinants, the following relationships between the mixed products of three vectors, taken in different orders, are immediately obtained:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{c}, \mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{c}, \mathbf{a}) = -(\mathbf{b}, \mathbf{a}, \mathbf{c}) = -(\mathbf{c}, \mathbf{b}, \mathbf{a}) = -(\mathbf{a}, \mathbf{c}, \mathbf{b}).$$

Therefore:

- If a circular permutation of the factors in a mixed product is made, the value of the product does not change;
- If the order of two factors (not necessarily adjacent) is swapped, the *sign* of the product changes (but not the absolute value).

It is also evident, either from the definition or from the properties of determinants, that *if two factors in a mixed product are linearly dependent, the product vanishes*. In particular, from (1.13.7), it follows that the necessary and sufficient condition (1.13.2) for the vectors (1.13.6) to be coplanar can be rewritten as:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0. \tag{1.13.8}$$

1.13.4 Solved Problems

Problem 1.13.1. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three non-coplanar vectors. Determine the values of the real parameter λ for which the vectors $\mathbf{u} = \mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\mathbf{v} = \lambda\mathbf{b} + 4\mathbf{c}$, and $\mathbf{w} = (2\lambda - 1)\mathbf{c}$ are coplanar.

Solution. We know that three vectors are coplanar if and only if their mixed product equals zero. Thus, we compute the mixed product of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, using the multilinearity of the mixed product:

$$\begin{aligned}(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \big(\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, \lambda\mathbf{b} + 4\mathbf{c}, (2\lambda - 1)\mathbf{c}\big) = \lambda(2\lambda - 1)(\mathbf{a}, \mathbf{b}, \mathbf{c}) + \\ &+ 4(2\lambda - 1)\underbrace{(\mathbf{a}, \mathbf{c}, \mathbf{c})}_{=0} + 2\lambda(2\lambda - 1)\underbrace{(\mathbf{b}, \mathbf{b}, \mathbf{c})}_{=0} + 8(2\lambda - 1)\underbrace{(\mathbf{b}, \mathbf{c}, \mathbf{c})}_{=0} + \\ &+ 3\lambda(2\lambda - 1)\underbrace{(\mathbf{c}, \mathbf{b}, \mathbf{c})}_{=0} + 12(2\lambda - 1)\underbrace{(\mathbf{c}, \mathbf{c}, \mathbf{c})}_{=0} = \lambda(2\lambda - 1)(\mathbf{a}, \mathbf{b}, \mathbf{c}),\end{aligned}$$

where the indicated mixed products vanish because (at least) two of the factors are collinear (in fact, equal).

Hence, the mixed product $\big(\mathbf{u}, \mathbf{v}, \mathbf{w}\big)$ equals zero if and only if

$$\lambda(2\lambda - 1)\big(\mathbf{a}, \mathbf{b}, \mathbf{c}\big) = 0.$$

Since the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are linearly independent, their mixed product does not vanish, leading to the equation

$$\lambda(2\lambda - 1) = 0,$$

thus $\lambda = 0$ or $\lambda = \dfrac{1}{2}$. □

Problem 1.13.2. Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ be three vectors. Prove that the vectors $\mathbf{a} - \mathbf{b}$, $\mathbf{b} - \mathbf{c}$, and $\mathbf{c} - \mathbf{a}$ are coplanar if and only if the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are coplanar.

Proof We will prove that the mixed product of these three vectors is zero if and only if the mixed product of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ vanishes. Indeed, we have:

$$\begin{aligned}\big(\mathbf{a} - \mathbf{b}, \mathbf{b} - \mathbf{c}, \mathbf{c} - \mathbf{a}\big) = \big(\mathbf{a}, \mathbf{b}, \mathbf{c}\big) - \big(\mathbf{a}, \mathbf{b}, \mathbf{a}\big) - \big(\mathbf{a}, \mathbf{c}, \mathbf{c}\big) + \big(\mathbf{a}, \mathbf{c}, \mathbf{a}\big) - \\ - \big(\mathbf{b}, \mathbf{b}, \mathbf{c}\big) + \big(\mathbf{b}, \mathbf{b}, \mathbf{a}\big) - \big(\mathbf{b}, \mathbf{c}, \mathbf{c}\big) + \big(\mathbf{b}, \mathbf{c}, \mathbf{a}\big) = 2\big(\mathbf{a}, \mathbf{b}, \mathbf{c}\big),\end{aligned}$$

where we used the multilinearity of the mixed product, the fact that it vanishes when two factors are collinear, and that the product does not change under a cyclic permutation of factors. Thus,

$$\big(\mathbf{a} - \mathbf{b}, \mathbf{b} - \mathbf{c}, \mathbf{c} - \mathbf{a}\big) = 0 \quad \text{if and only if} \quad \big(\mathbf{a}, \mathbf{b}, \mathbf{c}\big) = 0,$$

which is what we needed to prove. □

Problem 1.13.3. Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ be three vectors. Prove that the following formula holds:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix}.$$

Solution. Starting from the definition of the mixed product, we have:

$$(\mathbf{a},\mathbf{b},\mathbf{c})^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$
$$= \det \left(\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right) =$$
$$= \det \left(\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \right) =$$
$$= \begin{vmatrix} a_1a_1 + a_2a_2 + a_3a_3 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1b_1 + b_2b_2 + b_3b_3 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1c_1 + c_2c_2 + c_3c_3 \end{vmatrix} =$$
$$= \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}.$$

□

Problem 1.13.4. Let $\mathbf{b} = -\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - 7\mathbf{j} + 10\mathbf{k}$. Determine a unit vector $\mathbf{a}$ such that the product $(\mathbf{a},\mathbf{b},\mathbf{c})$ is maximised.

1.14 Probleme

Problem 1.1. Un vector **v** face un unghi de 45° cu axa Ox, un unghi de 60° cu axa Oy, iar componenta sa pe Ox este $v_x = 3\sqrt{2}$. Să se determine modulul vectorului, unghiul pe care îl face cu axa Oz şi celelalte două componente.

Soluţie Fie v modulul vectorului. Atunci componentele vectorului sunt:

$$v_x = v \cdot \cos 45^\circ = \frac{\sqrt{2}}{2}v,$$
$$v_y = v \cdot \cos 60^\circ = \frac{1}{2}v,$$
$$v_z = v \cdot \cos \gamma.$$

Dar pe v_x îl cunoaştem şi obţinem, prin urmare,

$$\frac{\sqrt{2}}{2}v = 3\sqrt{2},$$

de unde obţinem că $v = 6$, aşadar $v_y = 3$. Pe de altă parte, $v_x^2 + v_y^2 + v_z^2 = v^2 = 36$, adică

$$18 + 9 + 36\cos^2\gamma = 36,$$

de unde

$$\cos^2\gamma = \frac{1}{4}$$

sau

$$\cos\gamma = \pm\frac{1}{2}.$$

□

Problem 1.2. Să se determine un vector care face unghiuri egale cu cele trei axe ale unui reper ortonormat direct, ştiind că modulul vectorului este egal cu unitatea.

Problem 1.3. Ce relaţii trebuie să îndeplinească vectorii $\mathbf{a}, \mathbf{b}, \mathbf{c}$ astfel încât ei să formeze un triunghi?

Problem 1.4. Punctele $A'(1,2,1)$, $B'(2,0,0)$, $C'(0,1,3)$ sunt mijloacele laturilor unui triunghi. Să se determine coordonatele vârfurilor triunghiului.

Problem 1.5. Pe laturile triunghiului ABC se construiesc paralelogramele arbitrare $ABB'A''$, $BCC'B''$, $CAA'C''$. Să se arate că se poate construi un triunghi având laturile egale şi paralele cu $\overrightarrow{A'A''}, \overrightarrow{B'B''}, \overrightarrow{C'C''}$.

Problem 1.6. Se dă o piramidă cu vârful în S şi baza un paralelogram $ABCD$ ale cărui diagonale se intersectează în punctul O. Să se demonstreze egalitatea vectorială:

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}.$$

Problem 1.7. Verificaţi că vectorii $\mathbf{a}(4,1,-1), \mathbf{b}(1,2,-5), \mathbf{c}(-1,1,1)$ formează o bază în spaţiu. Determinaţi componentele vectorilor $\mathbf{l}(4,4,-5), \mathbf{m}(2,4,-10), \mathbf{n}(0,3,-4)$ în această bază.

Problem 1.8. Se dau vectorii necoplanari $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Stabiliţi dacă vectorii $\mathbf{l}, \mathbf{m}, \mathbf{n}$ sunt coplanari în fiecare dintre cazurile de mai jos. În caz afirmativ, indicaţi o relaţie de dependenţă liniară dintre ei.

1) $\mathbf{l} = 2\mathbf{a} - \mathbf{b} - \mathbf{c},\ \mathbf{m} = 2\mathbf{b} - \mathbf{c} - \mathbf{a},\ \mathbf{n} = 2\mathbf{c} - \mathbf{a} - \mathbf{b}$;

2) $\mathbf{l} = \mathbf{a} + \mathbf{b} + \mathbf{c},\ \mathbf{m} = \mathbf{b} + \mathbf{c},\ \mathbf{n} = -\mathbf{a} + \mathbf{c}$;

3) $\mathbf{l} = \mathbf{c},\ \mathbf{m} = \mathbf{a} - \mathbf{b} - \mathbf{c},\ \mathbf{n} = \mathbf{a} - \mathbf{b} + \mathbf{c}$.

Problem 1.9. În paralelogramul $ABCD$ punctul K este mijlocul segmentului CD, iar punctul O este punctul de intersecţie a diagonalelor. Considerând baza formată din vectorii $\overrightarrow{AB}$ şi $\overrightarrow{AD}$, determinaţi, în această bază, componentele vectorilor $\overrightarrow{AM}, \overrightarrow{AO}, \overrightarrow{MO}$.

Problem 1.10. În trapezul $ABCD$ lungimile bazelor AD şi BC sunt în raportul $3 : 2$. Luând ca bază vectorii $\overrightarrow{AC}$ şi $\overrightarrow{BD}$, să se determine,în această bază, componentele vectorilor $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$.

Problem 1.11. În trapezul $ABCD$, lungimile bazelor AD şi BC sunt în raportul $3 : 1$. O este punctul de intersecţie a diagonalelor trapezului, în timp ce S este punctul de intersecţie al prelungirilor laturilor neparalele. Luând ca bază vectorii $\overrightarrow{AD}$ şi $\overrightarrow{AB}$, determinaţi componentele vectorilor $\overrightarrow{AC}, \overrightarrow{AO}, \overrightarrow{AS}$.

Problem 1.12. Trei puncte, $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, necoliniare, sunt vârfuri consecutive ale unui paralelogram. Determinaţi coordonatele celui de-al patrulea punct, D, al paralelogramului.

Problem 1.13. Se dau două puncte distincte $A(x_1, y_1, z_1)$ şi $B(x_2, y_2, z_2)$. Determinaţi coordonatele:

1) punctului M, situat pe segmentul AB, astfel încât $AM : BM = m : n$;

2) punctului M, situat în exteriorul segmentului AB, astfel încât $AM : BM = m : n$.

Problem 1.14. Se dau punctele $A(3, -2)$ şi $B(1, 4)$. Punctul M se află pe dreapta AB, iar $AM = 3AB$. Determinaţi coordonatele punctului M dacă:

1) M se află de aceeaşi parte a punctelor A şi B;

2) punctele M şi B se află de-o parte şi de cealaltă a punctului A.

Problem 1.15. Se dau vectorii necoliniar $\mathbf{a}$ şi $\mathbf{b}$. Demonstraţi că sistemul de vectori $\mathbf{m} = 3\mathbf{a} - \mathbf{b}$, $\mathbf{n} = 2\mathbf{a} + \mathbf{b}$, $\mathbf{p} = \mathbf{a} + 3\mathbf{b}$ este liniar dependent, iar vectorii $\mathbf{n}, \mathbf{p}$ sunt necoliniari. Exprimaţi vectorul $\mathbf{m}$ în funcţie de vectorii $\mathbf{n}, \mathbf{p}$.

Problem 1.16. Punctul M este centrul de greutate al triunghiului ABC. Exprimaţi:

1) vectorul $\overrightarrow{MA}$ în funcţie de vectorii $\overrightarrow{BC}, \overrightarrow{CA}$;

2) vectorul $\overrightarrow{AB}$ în funcţie de vectorii $\overrightarrow{MB}, \overrightarrow{MC}$;

3) vectorul $\overrightarrow{OA}$ în funcţie de vectorii $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OM}$, unde O este un punct oarecare din spaţiu.

Problem 1.17. Determinaţi coordonatele vârfurilor unui tetraedru $OABC$ în sistemul de coordonate cu originea în vârful O, în care baza de coordonate este formată din medianele $\overrightarrow{OD}$, $\overrightarrow{OE}$, $\overrightarrow{OF}$ ale feţelor BOC, COA, AOB.

Problem 1.18. Să se determine coordonatele vârfurilor tetraedrului $ABCD$ într-un sistem de coordonate în care originea este centrul de greutate P al feţei BCD, iar baza este formată din vectorii $\overrightarrow{BQ}$, $\overrightarrow{CR}$, $\overrightarrow{DS}$, unde Q, R, S sunt, respectiv, centrele de greutate ale feţelor ACD, ABD şi ABC.

Problem 1.19. a Determinaţi unghiurile dintre vectorii **a** şi **b**, daţi prin intermediul componentelor lor, faţă de un reper ortonormat.

1) $\mathbf{a}(1, -1, 1), \mathbf{b}(5, 1, 1)$;

2) $\mathbf{a}(1, -1, 1), \mathbf{b}(-2, 2, -2)$;

3) $\mathbf{a}(1, -1, 1), \mathbf{b}(3, 1, -2)$.

Problem 1.20. Să se determine distanţa dintre punctele A şi B, date prin intermediul coordonatelor lor faţă de o bază ortonormată.

1) $A(4, -2, 3)$, $B(4, 5, 2)$;

2) $A(-3, 1, -1)$, $B(-1, 1, -1)$;

3) $A(3, -3, -7)$, $B(1, -4, -5)$.

Problem 1.21. Se dau trei vectori: $\mathbf{a}(-1, 2), \mathbf{b}(5, 1), \mathbf{c}(4, -2)$. Calculaţi:

1) $\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$;

2) $\|\mathbf{a}\|^2 - \mathbf{b} \cdot \mathbf{c}$;

3) $\|\mathbf{b}\|^2 + \mathbf{b} \cdot (\mathbf{a} + 3\mathbf{c})$;

Problem 1.22. Se dau trei vectori: $\mathbf{a}(1, -1, 1), \mathbf{b}(5, 1, 1), \mathbf{c}(0, 3, -2)$. Calculaţi:

1) $\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$;

2) $\|\mathbf{a}\|^2 + \|\mathbf{c}\|^2 - (\mathbf{a} \cdot \mathbf{b}) \cdot (\mathbf{b} \cdot \mathbf{c})$;

3) $(\mathbf{a} \cdot \mathbf{c}) \cdot (\mathbf{a} \cdot \mathbf{b}) - \|\mathbf{a}\|^2(\mathbf{b} \cdot \mathbf{c})$.

Problem 1.23. Demonstraţi că vectorii $\mathbf{a}$ şi $\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ sunt perpendiculari.

Problem 1.24. Este adevărat că, pentru orice vectori $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ este îndeplinită egalitatea

$$(\mathbf{a} \cdot \mathbf{b}) \cdot (\mathbf{c} \cdot \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c}) \cdot (\mathbf{b} \cdot \mathbf{d})?$$

Problem 1.25. Se ştie că $\mathbf{a} = \mathbf{b} \times \mathbf{c}$, $\mathbf{b} = \mathbf{c} \times \mathbf{a}$, $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. Determinaţi lungimile vectorilor $\mathbf{a}, \mathbf{b}, \mathbf{c}$ şi unghiurile dintre ei.

Problem 1.26. Pe vectorii $\mathbf{a}(2, 3, 1)$ şi $\mathbf{b}(-1, 1, 2)$ (ataşaţi unui punct) se construieşte un triunghi. Determinaţi aria acestui triunghi, precum şi lungimile celor trei înălţimi ale sale.

Problem 1.27. Unui punct i se ataşează patru vectori $\mathbf{a}(-1, 1, -1), \mathbf{b}(-1, 1, 1), \mathbf{c}(5, -1, -1)$ şi $\mathbf{d}$. Vectorul $\mathbf{d}$ are lungimea 1 şi formează cu vectorii $\mathbf{a}, \mathbf{b}, \mathbf{c}$ unghiuri ascuţite egale. Determinaţi componentele vectorului $\mathbf{d}$.

Problem 1.28. Stabiliţi dacă următoarele triplete de vectori sunt formate din vectori coplanari:

1) $\mathbf{a}(2, 3, 5), \mathbf{b}(7, 1, -1), \mathbf{c}(3, -5, -11)$;

2) $\mathbf{a}(2, 0, 1), \mathbf{b}(5, 3, -3), \mathbf{c}(3, 3, 10)$.

Problem 1.29. Vectorii $\mathbf{a}, \mathbf{b}, \mathbf{c}$ sunt necoplanari. Pentru ce valori ale lui λ sunt coplanari vectorii $\mathbf{a} + 2\mathbf{b} + \lambda\mathbf{c}$, $4\mathbf{a} + 5\mathbf{b} + 6\mathbf{c}$, $7\mathbf{a} + 8\mathbf{b} + \lambda^2\mathbf{c}$?

Problem 1.30. Se dau vectorii necoliniari $\mathbf{a}, \mathbf{b}$ şi scalarul p. Găsiţi un vector $\mathbf{x}$, care verifică egalitatea $(\mathbf{x}, \mathbf{a}, \mathbf{b}) = p$.

Problem 1.31. Diagonalele unui trapez isoscel sunt perpendiculare. Determinaţi aria trapezului, ştiind că înălţimea lui este egală cu h.

Problem 1.32. Aria unui trapez $ABCD$ este egală cu S, iar raportul bazelor este $AD : BC = 3 : 1$. Segmentul MN este paralel cu latura BC şi intersectează latura AB, astfel încât $AM : BN = 3 : 2$, $MN : CD = 1 : 3$. Segmentul AM este paralel cu segmentul BN. Determinaţi aria triunghiului BNC.

Problem 1.33. Măsura, în radiani, a unghiului dintre vectorii $\mathbf{u}$ şi $\mathbf{v}$ este de $\dfrac{\pi}{6}$. Dacă $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 7$, calculaţi $\mathbf{u} \times \mathbf{v}$ şi $\left\|\dfrac{1}{3}\mathbf{u} \times \dfrac{3}{4}\mathbf{v}\right\|$.

Problem 1.34. Dacă $ABCD$ este un tetraedru regulat de latură 1, calculaţi $\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$.

Problem 1.35. Calculaţi aria paralelogramului $ABCD$, dacă $\overrightarrow{AB} = (1, 1, -1)$, în timp ce $\overrightarrow{AD} = (2, 1, 4)$.

Problem 1.36. Calculaţi aria triunghiului ABC dacă $\overrightarrow{AB} = (-1, 1, 0)$, iar $\overrightarrow{AC} = (0, 1, 3)$.

Problem 1.37. Determinaţi un vector de lungime 1 care să fie perpendicular pe vectorii $\mathbf{u} = (1, -3, 1)$ şi $\mathbf{v} = (-3, 3, 3)$.

Problem 1.38. Se dau vectorii $\mathbf{u}(1, 1, 1)$ şi $\mathbf{v}(0, 1, 2)$. Să se determine o bază ortonormată pozitivă $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ a lui $\mathbb{R}^3$ astfel încât:

(i) $\mathbf{a}$ să aibă aceeaşi direcţie şi sens cu $\mathbf{u}$;

(ii) $\mathbf{b}$ să fie o combinaţie liniară a lui $\mathbf{u}$ şi $\mathbf{v}$, iar prima sa componentă să fie pozitivă.

Problem 1.39. Demonstraţi că dacă $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$, atunci

(a) $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$;

(b) $\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u} = 3(\mathbf{u} \times \mathbf{v})$.

CHAPTER 2

The Line in the Plane

2.1 The Equation of the Line Using the Slope (Gradient)

Let Δ be a line in a plane. A *direction vector* of the line is any nonzero vector whose direction coincides with the direction of the line. It is understood that a given line has infinitely many direction vectors, but all of them are collinear.

Assume a coordinate system Oxy is chosen. We will now derive the equation of the line Δ. First, assume that the line is parallel to the Oy-axis and intersects the Ox-axis at a point $P(a, 0)$. Then it is clear that for all points $M(x, y)$ on the line Δ, and only for them, we have

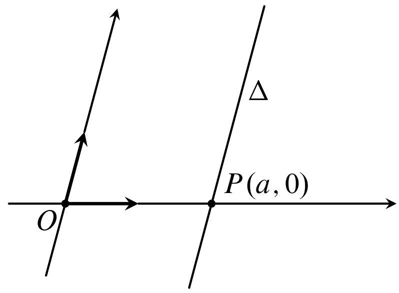
$$x = a. \tag{2.1.1}$$

Thus, (2.1.1) *is the equation of a line parallel to the Oy-axis* (i.e., a *vertical line*) and which intersects the Ox-axis at the point $P(a, 0)$. Clearly, all such lines have direction vectors with components $(0, m)$, where m is any nonzero real number.

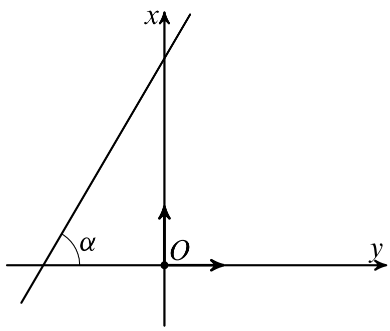
Now assume the line Δ is not parallel to the Oy-axis. Then for any direction vector $\mathbf{a}(l, m)$ of this line, we have $l \neq 0$, and the ratio $m : l$ has the same constant value k, called the *slope* (or *gradient*) of the line Δ relative to the chosen coordinate system.

If, in particular, we consider an orthogonal coordinate system $(O, \mathbf{i}, \mathbf{j})$, then for the slope we have, evidently,

$$k = \tan\alpha,$$



where α is the angle between $\mathbf{i}$ and any direction vector of the line Δ. The angle α is called the *angle of inclination* or *slope* of the line Δ relative to the Ox-axis. Since we will exclusively use inclination relative to the Ox-axis in what follows, we will not explicitly mention this fact further and will simply refer to the slope of the line.



From now on, unless otherwise specified, we will exclusively use a rectangular coordinate system that is fixed once and for all and will no longer refer to it explicitly.

We now show how the equation of a line can be obtained if its slope and a point on the line are known. Let Δ be a line with slope k. Let $P(a, b)$ be a point on the line. Let $M(x, y)$ be another point on the line, distinct from P. Then the vector $\overrightarrow{PM}(x - a, y - b)$ is a direction vector of the line Δ. Thus,

$$\frac{y - b}{x - a} = k. \tag{2.1.2}$$

It follows that

$$y - b = k(x - a). \tag{2.1.3}$$

This equation is satisfied by every point on the line, including the point P. Conversely, we prove that if a point satisfies this equation, then it lies on the line. Let

$M_1(x_1, y_1) \neq P$ be a point that satisfies equation (2.1.3), i.e.,

$$y_1 - b = k(x_1 - a). \tag{2.1.4}$$

Since $M_1 \neq P$, $x_1 - a \neq 0$, and from (2.1.2) and (2.1.4), we obtain

$$\frac{y - b}{x - a} = \frac{y_1 - b}{x_1 - a}.$$

Thus, the direction vectors of the lines Δ and PM_1 are collinear. Since both lines pass through P, they coincide, and therefore $M_1 \in \Delta$. Thus, equation (2.1.3) describes a line that passes through P and has slope k.

If, in particular, the point P lies on the Oy-axis (which is possible since we assumed the line is not parallel to this axis), i.e., if P has coordinates $(0, b)$, then equation (2.1.3) simplifies to:

$$y = kx + b.$$

If the line is parallel to the Oy-axis, then its slope is zero, and if it passes through the point $P(0, b)$, its equation is:

$$y = b.$$

2.2 The General Equation of the Line. The Intercept Form

Definition 2.1. A *first-degree equation* or *linear equation* in the unknowns x and y is an equation of the form

$$Ax + By + C = 0, \tag{2.2.1}$$

where $A, B, C \in \mathbb{R}$, and the coefficients A and B do not vanish simultaneously.

Theorem 2.1. *Any line in the plane can be described by an equation of the form (2.2.1). Conversely, any equation of the form (2.2.1) represents a line.*

Proof If we have a line Δ that is not parallel to the Oy-axis, then, as we saw in the previous section, it can be described by the linear equation

$$y - kx - b = 0. \tag{2.2.2}$$

If the line Δ is parallel to the Oy-axis, it is again given by a linear equation

$$x - a = 0. \tag{2.2.3}$$

Now consider any equation of the form (2.2.1). If $B \neq 0$, then by dividing both sides of the equation by B and introducing the notations $k = -A/B$ and $b = -C/B$, we can rewrite the equation in the form (2.2.2). But equation (2.2.2) represents a line with slope k passing through the point $P(0, b)$. If $B = 0$ in equation (2.2.1), then this equation can be rewritten in the form (2.2.3) and, consequently, represents a line parallel to the Oy-axis. □

Equation (2.2.1) is called the *general equation of the line in the plane*. We now highlight a few special cases where one or two coefficients of the general equation vanish.

1. $C = 0$. In this case, equation (2.2.1) reduces to

$$Ax + By = 0. \tag{2.2.4}$$

   The resulting line passes through the origin, as can easily be verified by substituting $x = 0$ and $y = 0$ into the equation above. Conversely, if a line passes through the origin, then substituting $x = 0$ and $y = 0$ into equation (2.2.1) gives $C = 0$. Thus, a *necessary and sufficient condition for a line given by its general equation to pass through the origin is that* $C = 0$.

2. $B = 0, C \neq 0$. In this case, equation (2.2.1) becomes

$$Ax + C = 0. \tag{2.2.5}$$

   The line described by this equation is parallel to the Oy-axis but does not intersect it. It can be easily observed that this line intersects the Ox-axis at the point with coordinates $\left(-\frac{C}{A}, 0\right)$, and all its points have the x-coordinate equal to $-C/A$. Thus, a direction vector for the line is, for instance, $(0, 1)$, confirming that the line is indeed parallel to the Oy-axis. Conversely, any line parallel to the Oy-axis and not passing through the origin can be written in the form (2.2.5).

3. $B = 0, C = 0$. In this case, the equation reduces to

$$x = 0,$$

   and the line is the Oy-axis.

4. $A = 0, C \neq 0$. This case is analogous to case 2) and leads to a line parallel to the Ox-axis but not coinciding with it.

5. $A = 0, C = 0$. This case is analogous to case 3), and the line in question is the Ox-axis.

We now highlight the following facts: *Let Δ be the line given by its general equation (2.2.1). Then the vector* $\mathbf{n}(A, B)$ *is perpendicular to the line, while the vector* $\mathbf{a}(-B, A)$ *is a direction vector of the line.*

Indeed, consider two distinct points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ on the line Δ. We have, therefore,

$$\begin{aligned} Ax_1 + By_1 + C &= 0, \\ Ax_2 + By_2 + C &= 0. \end{aligned}$$

Subtracting these equations, we obtain

$$A(x_2 - x_1) + B(y_2 - y_1) = 0.$$

This equality implies that the vector $\mathbf{n}(A, B)$ is perpendicular to the vector $\overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1)$, and therefore it is also perpendicular to the line Δ. Since the vector $\mathbf{a}(-B, A)$ is evidently also perpendicular to $\mathbf{n}$, it follows that $\mathbf{a}$ is a direction vector of the line Δ.

Remark. Let $\mathbf{r}$ be the position vector of a point M on the line, $\mathbf{n}$ the normal vector to the line, and $\mathbf{r}_0$ the position vector of a given point M_0 through which the line passes. Then, as shown above, the equation of the line can be written in the form:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0. \tag{2.2.6}$$

This is a form of the line's equation that we will frequently use whenever the line is given by a point it passes through and a normal vector.

Assume now that in equation (2.2.1), all coefficients A, B, C are nonzero. Divide the equation by $-C$ and denote $a = -C/A$ and $b = -C/B$. The equation then becomes:

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{2.2.7}$$

Evidently, a and b are the signed lengths of the segments cut by the line Δ on the coordinate axes Ox and Oy (these are the segments between the origin and the intersection points of the line with the axes). These lengths are called the *intercepts* of the line, and equation (2.2.7) is called the *intercept form of the line* Δ.

Example 2.1. Let Δ be the line given by its general equation:

$$\Delta\colon 2x + 3y - 5 = 0.$$

Divide the equation by 5 so that the constant term becomes -1. We obtain:

$$\frac{2}{5}x + \frac{3}{5}y - 1 = 0,$$

which can be rewritten as:

$$\frac{x}{\frac{5}{2}} + \frac{y}{\frac{5}{3}} - 1 = 0,$$

which is the intercept form of the line Δ, with intercepts:

$$a = \frac{5}{2} \quad \text{and} \quad b = \frac{5}{3}.$$

2.3 The Vector Equation and the Parametric Equations of a Line. The Line Passing Through Two Points

Any point M in the plane is uniquely identified by its position vector $\overrightarrow{OM}$, relative to the origin of coordinates. Let Δ be a line in the plane, $\mathbf{r}_0 = \overrightarrow{OM_0}$ the position vector of a point on the line, and $\mathbf{a}$ the direction vector of the line. Denote by $\mathbf{r}$ the position vector of an arbitrary point M in the plane. If M lies on the line, then:

$$\mathbf{r} - \mathbf{r}_0 = \overrightarrow{M_0M},$$

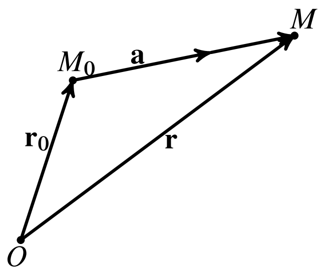
so $\mathbf{r} - \mathbf{r}_0$ is a direction vector of the line, and thus it is collinear with the vector $\mathbf{a}$. From this, it follows that there exists a real number t such that:

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{a}. \tag{2.3.1}$$

Conversely, if t is any real number, it is clear that the point M in the plane, whose position vector $\mathbf{r}$ satisfies equation (2.3.1), lies on the line. Equation (2.3.1), or its equivalent form:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}, \tag{2.3.2}$$

is called the *vector equation of the line*.



Now, suppose that the vectors are given by their components, $\mathbf{a}(l, m)$, $\mathbf{r}_0(x_0, y_0)$, and $\mathbf{r}(x, y)$. Then the vector equation (2.3.2) is equivalent to the system of equations:

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt \end{cases} . \tag{2.3.3}$$

The equations (2.3.3) are called the *parametric equations of the line* Δ.

If the line Δ is not parallel to any of the coordinate axes, then clearly $l \neq 0$ and $m \neq 0$, and the system (2.3.3) is equivalent to the equation:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m}, \tag{2.3.4}$$

which is called the *canonical equation of the line in the plane*.

Note that, in general, analytic geometry uses a certain convention that allows the canonical equation to be written even in cases where the line is parallel to one of the coordinate axes. The convention is as follows: *Whenever one of the denominators in the canonical equation of the line is zero, the numerator of that fraction is considered identically zero*. For example, consider the equation:

$$\frac{x - 1}{1} = \frac{y - 2}{0}.$$

According to the convention, this equation is equivalent to $y = 2$, i.e., it represents the equation of a line parallel to the Ox-axis.

Now, consider two points $M_0(x_0, y_0)$ and $M_1(x_1, y_1)$ on the line Δ. Then $\overrightarrow{M_0M_1}(x_1 - x_0, y_1 - y_0)$ is a direction vector of the line, and thus:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} \tag{2.3.5}$$

is the *equation of the line passing through the points* M_0 *and* M_1. Note that, according to the above convention, the points M_0 and M_1 may lie on a line parallel to one of the coordinate axes (in which case one of the coordinates of both points will be the same).

Examples. 1. Consider the point $M_0(1,2)$ and the vector $\mathbf{a}(1,4)$. The vector equation of the line passing through M_0 and in the direction of $\mathbf{a}$ is:

$$\mathbf{r} = \mathbf{r}_{M_0} + t\mathbf{a}$$

or:

$$(x, y) = (1, 2) + t(1, 4).$$

Projecting this equation onto components, we obtain the parametric equations of the line:

$$\begin{cases} x = 1 + t, \\ y = 2 + 4t. \end{cases}$$

Eliminating the parameter t between the two parametric equations, we obtain the canonical equation of the line:

$$\frac{x-1}{1} = \frac{y-2}{4}.$$

From this, if desired, we can deduce the general equation:

$$4x - y - 2 = 0$$

or, since the slope of the line is evidently nonzero, the explicit equation:

$$y = 4x - 2.$$

2. Determine the canonical equation of the line passing through the points $M_1(2,3)$ and $M_2(5,7)$. We immediately find the direction vector of the line, $\overrightarrow{M_1M_2}(3,4)$. Considering the line as passing through M_1 and directed by $\overrightarrow{M_1M_2}$, we obtain the equation:

$$\frac{x-2}{3} = \frac{y-3}{4}.$$

Notably, the same line can be viewed as passing through M_2 and directed by $\overrightarrow{M_1M_2}$. The canonical equation will then be:

$$\frac{x-5}{3} = \frac{y-7}{4},$$

but it is easy to verify that both canonical equations lead to the same general equation:

$$4x - 3y + 1 = 0.$$

2.4 The Relative Position of Two Lines in the Plane

Consider two lines Δ_1 and Δ_2, given by their general equations:

$$\begin{cases} A_1x + B_1y + C_1 = 0, \\ A_2x + B_2y + C_2 = 0. \end{cases} \tag{2.4.1}$$

Studying the relative position of these two lines means determining the number of common points they share. It is evident that we may exclusively encounter one of the following three situations:

(i) The lines intersect at a single point.

(ii) The lines coincide (which means they have an infinite number of common points).

(iii) The lines are parallel (and hence have no common points).

It is clear that studying the relative position of the lines Δ_1 and Δ_2 amounts to investigating the system of linear equations (2.4.1), formed by the general equations of the lines. Thus, the three cases above correspond (in the same order) to the following possible cases in the analysis of the system of equations:

(i) The system of equations has a unique solution. As is known from linear algebra, this condition is equivalent to the system being a Cramer system, i.e.,

$$\det \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \neq 0, \tag{2.4.2}$$

(ii) The system of equations is compatible but undetermined. This means that:

$$\det \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0, \tag{2.4.3}$$

but the rank of the coefficient matrix coincides with the rank of the augmented matrix, or, in our case, since there are only two equations, this is equivalent to:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}, \tag{2.4.4}$$

meaning that the two equations (2.4.1) describe the same line.

(iii) The system of equations is incompatible, which means that equation (2.4.3) is again satisfied, but this time:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}, \tag{2.4.5}$$

in other words, the rank of the coefficient matrix is equal to 1, while the rank of the augmented matrix is equal to 2. The equality in (2.4.5) means that the two lines have collinear direction vectors, while the inequality implies that the two lines do not coincide, and hence they are parallel.

Example 2.2. 1. For the lines:

$$\Delta_1 : 2x + 3y + 7 = 0$$

and:

$$\Delta_2 : 3x - 5y + 3 = 0,$$

we compute:

$$\begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix} = -24 \neq 0,$$

so the lines intersect.

2. For the lines:

$$\Delta_1 : 2x + 3y + 7 = 0$$

and:

$$\Delta_2 : 4x + 6y + 14 = 0,$$

we compute:

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0,$$

and:

$$\frac{2}{4} = \frac{3}{6} = \frac{7}{14},$$

so the lines coincide.

3. For the lines:

$$\Delta_1 : 2x + 3y + 7 = 0$$

and:

$$\Delta_2 : 4x + 6y + 10 = 0,$$

we compute:

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0,$$

and:

$$\frac{2}{4} = \frac{3}{6} \neq \frac{7}{10},$$

so the lines are parallel but do not coincide.

Remark. If:

$$\Delta_1\colon A_1x + B_1y + C_1 = 0$$

and:

$$\Delta_2\colon A_2x + B_2y + C_2 = 0$$

are parallel (possibly identical), then their equations can be modified such that they differ only by the constant term.

Indeed, since the lines are parallel, their normal vectors are collinear, which means there exists a nonzero real number λ such that:

$$(A_2, B_2) = \lambda(A_1, B_1).$$

Thus, the equation of the second line can be written as:

$$\Delta_2\colon \lambda A_1x + \lambda B_1y + C_2 = 0,$$

or, after dividing by λ:

$$\Delta_2\colon A_1x + B_1y + \frac{C_2}{\lambda} = 0.$$

2.5 Bundles of Lines

Definition 2.2. A *bundle of lines* is defined as the set of all lines in a plane passing through a point S, called the *centre of the bundle*.

To specify a bundle of lines in the plane, it is sufficient to specify the centre of the bundle and two of its lines.

Let there be two distinct lines in the plane passing through the point $S(x_0, y_0)$, given by their general equations:

$$A_1x + B_1y + C_1 = 0, \tag{2.5.1}$$

$$A_2x + B_2y + C_2 = 0. \tag{2.5.2}$$

Now consider the equation:

$$\alpha(A_1x + B_1y + C_1) + \beta(A_2x + B_2y + C_2) = 0, \tag{2.5.3}$$

where α and β are arbitrary real numbers that do not vanish simultaneously. We will prove that this equation defines a line passing through the point S.

Rewriting the equation:

$$(\alpha A_1 + \beta A_2)x + (\alpha B_1 + \beta B_2)y + \alpha C_1 + \beta C_2 = 0. \tag{2.5.4}$$

Here, the coefficients of the unknowns cannot vanish simultaneously. Indeed, suppose:

$$\alpha A_1 + \beta A_2 = 0, \quad \alpha B_1 + \beta B_2 = 0, \tag{2.5.5}$$

and, for instance, $\alpha \neq 0$. Then $A_2 \neq 0$, because if $A_2 = 0$, we would also have $A_1 = 0$, contradicting the assumption that the lines (2.5.1) and (2.5.2) intersect at a point. Similarly, $B_2 \neq 0$, and the equations (2.5.5) can be rewritten as:

$$\frac{A_1}{A_2} = -\frac{\beta}{\alpha}, \quad \frac{B_1}{B_2} = -\frac{\beta}{\alpha} \quad \Rightarrow \frac{A_1}{A_2} = \frac{B_1}{B_2},$$

which is not possible since the lines (2.5.1) and (2.5.2) are not parallel but intersect at a point. Therefore, the coefficients of the unknowns in equation (2.5.4) cannot vanish simultaneously, and for any α and β not both zero, this equation represents a line. It is evident that the line (2.5.3) indeed passes through the point $S(x_0, y_0)$.

We now show the converse: any line in the bundle has an equation of the form (2.5.3). In other words, we will demonstrate that for any line in the bundle of lines passing through $S(x_0, y_0)$, we can choose constants α and β, at least one of which is nonzero, so that equation (2.5.3) is the equation of the chosen line.

Let $M_1(X_1, y_1)$ be an arbitrary point in the plane such that $M_1 \neq S$. It suffices to demonstrate that we can choose α, β such that the line (2.5.3) coincides with the line SM_1. This reduces to requiring that x_1 and y_1, the coordinates of M_1, satisfy the equality:

$$\alpha(A_1x_1 + B_1y_1 + C_1) + \beta(A_2x_1 + B_2y_1 + C_2) = 0. \tag{2.5.6}$$

Since $M_1 \neq S$, at least one of the terms in parentheses is nonzero. If:

$$A_1x_1 + B_1y_1 + C_1 \neq 0,$$

then equation (2.5.6) can be rewritten as:

$$\alpha = -\frac{A_2x_1 + B_2y_1 + C_2}{A_1x_1 + B_1y_1 + C_1}\beta.$$

By assigning an arbitrary nonzero value to β, we obtain the corresponding value of α.

Thus, for any α and β that do not vanish simultaneously, equation (2.5.3) represents a line in the bundle determined by lines (2.5.1) and (2.5.2). Conversely, any line of the bundle can be written in the form (2.5.3). Equation (2.5.3) is called the *equation of the bundle of lines* determined by lines (2.5.1) and (2.5.2). Note that the equation of line (2.5.1) is obtained from (2.5.3) by setting $\beta = 0$ and choosing any nonzero α, while the equation of line (2.5.2) is obtained by setting $\alpha = 0$ and choosing any nonzero β.

Dividing both sides of equation (2.5.3) by α and letting $\beta/\alpha = \lambda$, the resulting equation becomes:

$$A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0. \tag{2.5.7}$$

For any λ, this equation corresponds to a line in the bundle of lines determined by (2.5.1) and (2.5.2). Conversely, any line of this bundle, except for line (2.5.2), can be written in the form (2.5.7) for some λ.

If the coordinates of the centre $S(x_0, y_0)$ of the bundle are known, the equation of the bundle can be written in the very simple form:

$$\alpha(x - x_0) + \beta(y - y_0) = 0. \tag{2.5.8}$$

Examples. We will provide several examples of problems that can be conveniently solved using the concept of a bundle of lines.

(a) Consider a triangle ABC, whose sides have the equations:

$$\begin{aligned} AB&{:}2x - 3y + 1 = 0,\\ BC&{:}3x + y - 5 = 0,\\ CA&{:}x + y - 2 = 0. \end{aligned}$$

We aim to write the equation of the altitude from A without determining the coordinates of point A.

First, we write the equation of an arbitrary line passing through vertex A, that is, a line belonging to the bundle of lines determined by the concurrent lines AB and CA. Let AA' denote this line. Its equation will therefore have the form:

$$\lambda(2x - 3y + 1) + \mu(x + y - 2) = 0,$$

or equivalently:

$$AA' \colon (2\lambda + \mu)x + (-3\lambda + \mu)y + \lambda - 2\mu = 0.$$

A normal vector to this line is $\mathbf{n}(2\lambda + \mu, -3\lambda + \mu)$. This line must be perpendicular to side BC, which has the normal vector $\mathbf{n}_a(3, 1)$. For the lines to be perpendicular, it is necessary and sufficient for their normal vectors to be perpendicular, which means:

$$0 = \mathbf{n}_a \cdot \mathbf{n} = 3(2\lambda + \mu) + 1(-3\lambda + \mu) = 3\lambda + 4\mu.$$

Thus, we must have:

$$\mu = -\frac{3}{4}\lambda.$$

Choosing $\lambda = 4$, we find $\mu = -3$, and the equation of the altitude AA' becomes:

$$AA' \colon 5x - 15y + 10 = 0,$$

or:

$$AA' \colon x - 3y + 2 = 0.$$

(b) Let A be the intersection point of the lines:

$$\Delta_1 \colon x + 2y - 3 = 0$$

and:

$$\Delta_2 \colon 2x - 3y + 1 = 0.$$

Through A, draw a line parallel to the line:

$$\Delta_3 \colon 4x + 3y - 5 = 0.$$

Let Δ denote the desired line. It must belong to the bundle of lines determined by Δ_1 and Δ_2, which means its equation must have the form:

$$\Delta \colon \alpha(x + 2y - 3) + \beta(2x - 3y + 1) = 0,$$

or equivalently:

$$\Delta\colon (\alpha + 2\beta)x + (2\alpha - 3\beta)y - 3\alpha + \beta = 0.$$

For Δ to be parallel to Δ_3, its normal vector $\mathbf{n}(\alpha+2\beta, 2\alpha-3\beta)$ must be collinear with the normal vector of Δ_3, $\mathbf{n}(4, 3)$. Thus, we must have:

$$\frac{\alpha + 2\beta}{4} = \frac{2\alpha - 3\beta}{3}.$$

Solving this, we find $\beta = \frac{5}{18}\alpha$. Setting $\alpha = 18$, we obtain $\beta = 5$. Substituting these values, the equation of Δ becomes:

$$\Delta\colon 28x + 21y - 49 = 0,$$

or:

$$\Delta\colon 4x + 3y - 7 = 0.$$

(c) Let A be the intersection point of the lines:

$$\Delta_1\colon x + y - 11 = 0$$

and:

$$\Delta_2\colon x - 5y + 2 = 0.$$

Write the equation of the line AM, where M is the point with coordinates $(10, 5)$.

As before, we write the equation of the line AM as a line belonging to the bundle of lines determined by Δ_1 and Δ_2:

$$AM\colon \alpha(x + y - 11) + \beta(x - 5y + 2) = 0,$$

or equivalently:

$$AM\colon (\alpha + \beta)x + (\alpha - 5\beta)y - 11\alpha + 2\beta = 0.$$

The coordinates of point M must satisfy this equation, and from this condition, we find:

$$\beta = \frac{4}{13}\alpha.$$

Setting $\alpha = 13$, we find $\beta = 4$. Substituting these values, the equation becomes:

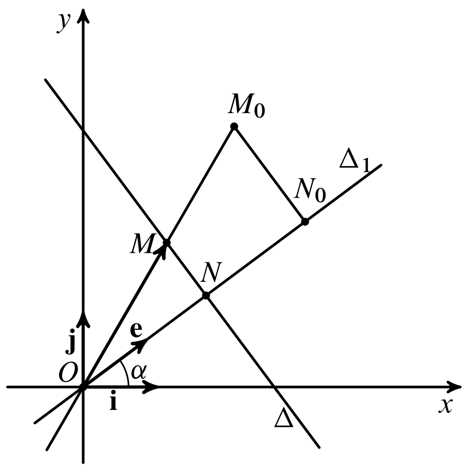
$$AM\colon 17x - 7y - 135 = 0.$$

2.6 Distance from a Point to a Line

Let Δ be an arbitrary line in the plane.

Definition 2.3. The *distance* from a point M_0 in the plane to the line Δ is defined as the length of the perpendicular dropped from M_0 to Δ.

Consider now a unit vector $\mathbf{e}$ perpendicular to the line Δ. If Δ passes through the origin, then we can take as $\mathbf{e}$ either of the two (opposite) unit vectors perpendicular to the line. If the line does not pass through the origin, we choose the unit vector $\mathbf{e}$, perpendicular to Δ, oriented from the origin toward the line.



Let α denote the angle between the vectors $\mathbf{i}$ and $\mathbf{e}$. Then:

$$\mathbf{e} = \mathbf{e}(\cos\alpha, \sin\alpha).$$

Let Δ_1 be the line passing through the origin and perpendicular to the line Δ. Denote by N the intersection of the two lines. Also, let p be the distance from the origin to the line Δ, that is, the length of the segment ON. Of course, if the line Δ passes through the origin, then $N = O$ and $p = 0$.

A point $M(x, y)$ in the plane belongs to the line Δ if and only if its orthogonal projection onto the line Δ_1 coincides with N. This condition is equivalent to:

$$\overrightarrow{OM} \cdot \mathbf{e} = p.$$

Expressing the scalar product in terms of the components of the vectors, we obtain:

$$x\cos\alpha + y\sin\alpha - p = 0. \tag{2.6.1}$$

This equation is called the *normal equation* or *Hesse normal form* of the line.

The line Δ divides the set of all points in the plane that do not belong to it into two subsets, called *open half-planes*. The half-plane containing the unit vector $\mathbf{e}$, when it is attached to the point N, is called the *positive half-plane*, while the other half-plane is called the *negative half-plane*. Note that the origin is always either in the negative half-plane or on the line Δ.

Definition 2.4. Let d be the distance from point M_0 to the line Δ. The *deviation* of point M_0 from the line Δ is defined as the number δ, determined by the following conditions:

1) $\delta = d$ if M_0 lies in the positive half-plane;

2) $\delta = -d$ if M_0 lies in the negative half-plane;

3) $\delta = d = 0$ if M_0 lies on the line Δ.

Theorem 2.2. *Suppose a line Δ is given in the plane by its normal equation* (2.6.1). *Then, the deviation δ of an arbitrary point $M_0(x_0, y_0)$ from the line Δ, and the distance d from the point to the line, are given by the formulas:*

$$\delta = x_0 \cos\alpha + y_0 \sin\alpha - p, \tag{2.6.2}$$

$$d = |x_0 \cos\alpha + y_0 \sin\alpha - p|. \tag{2.6.3}$$

Proof Let N_0 be the foot of the perpendicular dropped from M_0 onto the line Δ_1. Using Chasles' relation, we have:

$$\delta = (NN_0) = (ON_0) - (ON) = \overrightarrow{OM} \cdot \mathbf{e} - p = x_0 \cos\alpha + y_0 \sin\alpha - p.$$

Formula (2.6.3) follows directly from formula (2.6.2), since $d = |\delta|$. □

Formula (2.6.2) leads to the following rule: *To compute the deviation of an arbitrary point M_0 from a line, it suffices to substitute the coordinates of the point into the left-hand side of the normal equation of the line. The resulting number is the desired deviation.*

Now suppose the line is given by its general equation:

$$Ax + By + C = 0, \tag{2.6.4}$$

and we want to find its normal equation (2.6.1). Since equations (2.6.1) and (2.6.4) represent the same line, their coefficients must be proportional. Thus:

$$\cos\alpha = \lambda A, \quad \sin\alpha = \lambda B, \quad -p = \lambda C. \tag{2.6.5}$$

From the first two relations in (2.6.5), we find:

$$\lambda = \pm\frac{1}{\sqrt{A^2 + B^2}}.$$

According to the third equality in (2.6.5), the sign of λ must be opposite to the sign of the constant term C in equation (2.6.4), if $C \neq 0$. If $C = 0$, λ can have any sign. A change of sign in λ swaps the positive and negative half-planes. The number λ is called the *normalising factor* for equation (2.6.4), because multiplying by it converts the equation into the normal form.

Using these observations, the formulas for the deviation and the distance of a point $M_0(x_0, y_0)$ from the line (2.6.4) can be written as:

$$\delta = \frac{Ax_0 + By_0 + C}{\pm\sqrt{A^2 + B^2}}, \quad d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}. \tag{2.6.6}$$

Suppose the line is given by equation (2.6.4). Let:

$$\delta' = \delta'(x_0, y_0) = Ax_0 + By_0 + C.$$

Theorem 2.3. *For all points in the same half-plane determined by the line (2.6.4), δ' has the same sign. For points in the opposite half-plane, δ' has the opposite sign.*

Proof This statement for the normal equation:

$$\frac{1}{\pm\sqrt{A^2 + B^2}}(Ax + By + C),$$

or equivalently for:

$$\delta(x_0, y_0) = \frac{1}{\pm\sqrt{A^2 + B^2}}\delta'(x_0, y_0),$$

follows from Theorem 2.2. Since $\delta(x_0, y_0)$ and $\delta'(x_0, y_0)$ differ only by a strictly positive constant factor independent of the point $M_0(x_0, y_0)$, the statement remains true for $\delta'(x_0, y_0)$. □

Theorem 2.3 provides the geometric interpretation of the inequalities:

$$Ax + By + C > 0, \tag{2.6.7}$$
$$Ax + By + C < 0, \tag{2.6.8}$$

which relate the variables x and y. If x and y are the Cartesian coordinates of a point in the plane, then inequality (2.6.7) is satisfied only by the coordinates of points in one of the open half-planes determined by the line:

$$Ax + By + C = 0.$$

Inequality (2.6.8) is satisfied only by points in the other open half-plane. Similarly, the inequalities:

$$Ax + By + C \geq 0,$$
$$Ax + By + C \leq 0,$$

describe one closed half-plane together with the boundary line (or, as it is sometimes called, a *closed half-plane*).

Remark. If we have two lines:

$$\Delta_1 \colon A_1x + B_1y + C_1 = 0$$

and:

$$\Delta_2 \colon A_2x + B_2y + C_2 = 0,$$

then these two lines divide the plane into four angular regions (four ”angles”), described by the inequalities:

1° $\delta'_{\Delta_1} \geq 0$ and $\delta'_{\Delta_2} \geq 0$;

2° $\delta'_{\Delta_1} \leq 0$ and $\delta'_{\Delta_2} \geq 0$;

3° $\delta'_{\Delta_1} \leq 0$ and $\delta'_{\Delta_2} \leq 0$;

4° $\delta'_{\Delta_1} \geq 0$ and $\delta'_{\Delta_2} \leq 0$.

If the inequalities are strict, then we obtain the interiors of these regions (open angles).

Remark. Let $\Delta_1, \Delta_2, \Delta_3$ be three lines in the plane that do not pass through the same point but are pairwise concurrent, forming a triangle ABC, where $A = \Delta_2 \cap \Delta_3$, $B = \Delta_3 \cap \Delta_1$, and $C = \Delta_1 \cap \Delta_2$. We can describe the interior of triangle ABC.

Indeed, suppose $\delta'_{\Delta_1}(A) > 0$. This means that A lies in the positive open half-plane determined by Δ_1. Consequently, the entire interior of the triangle lies in this half-plane. Similarly, if $\delta'_{\Delta_2}(B) > 0$, then B lies in the positive open half-plane determined by Δ_2, and so on. Combining these conditions, the interior of the triangle ABC can be described by the system of inequalities:

$$\delta'_{\Delta_1}(x, y) > 0, \quad \delta'_{\Delta_2}(x, y) > 0, \quad \delta'_{\Delta_3}(x, y) > 0.$$

Remark. If the line Δ is given in its general form:

$$Ax + By + C = 0,$$

then the expression:

$$\delta'(x_0, y_0) = Ax_0 + By_0 + C$$

is often called the *signed distance function* of point $M_0(x_0, y_0)$ from the line Δ, as it provides information about both the distance and the relative position of M_0 with respect to Δ.

Examples. (a) Find the distance from the point $M(3, -1)$ to the line:

$$\Delta\colon 4x - 3y + 10 = 0.$$

Using formula (2.6.6) for the distance, we compute:

$$d = \frac{|4(3) - 3(-1) + 10|}{\sqrt{4^2 + (-3)^2}} = \frac{|12 + 3 + 10|}{\sqrt{16 + 9}} = \frac{25}{5} = 5.$$

(b) Determine whether the point $M(2, 5)$ lies in the positive or negative half-plane with respect to the line:

$$\Delta\colon x - 2y + 3 = 0.$$

We compute:

$$\delta'(2, 5) = 2(1) + (-2)(5) + 3 = 2 - 10 + 3 = -5.$$

Since $\delta'(2, 5) < 0$, the point M lies in the negative half-plane determined by Δ.

(c) Find the equation of the line passing through the point $P(1, -2)$ and parallel to the line:

$$\Delta\colon 2x - 3y + 5 = 0.$$

A line parallel to Δ has the form:

$$2x - 3y + C = 0.$$

To find C, substitute the coordinates of P:

$$2(1) - 3(-2) + C = 0 \quad \Rightarrow \quad 2 + 6 + C = 0 \quad \Rightarrow \quad C = -8.$$

Thus, the equation of the desired line is:

$$2x - 3y - 8 = 0.$$

2.7 The Distance Between Two Parallel Lines

Consider two parallel lines given by their general equations:

$$\Delta_1 \colon Ax + By + C_1 = 0, \tag{2.7.1}$$

and

$$\Delta_2 \colon Ax + By + C_2 = 0. \tag{2.7.2}$$

Since the lines are parallel, we can assume that they share the same normal vector, so their equations differ only by the constant term.

The *distance between the lines* Δ_1 *and* Δ_2 is, by definition, the distance from any point on Δ_1 to Δ_2. Since the lines are parallel, this distance does not depend on the choice of the point.

Let $M(x_M, y_M) \in \Delta_1$. Then the coordinates of M satisfy the equation of the line Δ_1, which means:

$$Ax_M + By_M + C_1 = 0.$$

From this, we deduce:

$$Ax_M + By_M = -C_1.$$

Therefore:

$$d(\Delta_1, \Delta_2) = d(M, \Delta_2) = \frac{|Ax_M + By_M + C_2|}{\sqrt{A^2 + B^2}} = \frac{|-C_1 + C_2|}{\sqrt{A^2 + B^2}}.$$

Thus, the *distance between the parallel lines* (2.7.1) and (2.7.2) is given by:

$$d(\Delta_1, \Delta_2) = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}. \tag{2.7.3}$$

Now, suppose the first line is given in its normal form:

$$\Delta_1 \colon x \cos\alpha + y \sin\alpha - p_1 = 0. \tag{2.7.4}$$

Then a line parallel to it will have the equation:

$$\Delta_2 \colon x \cos\alpha + y \sin\alpha - p_2 = 0, \tag{2.7.5}$$

if it is on the same side of the origin, or:

$$\Delta_2 \colon -x \cos\alpha - y \sin\alpha - p_2 = 0, \tag{2.7.6}$$

if the origin is between the two lines.

Using formula (2.7.3), we can state that the *distance between two parallel lines given by their normal equations* is:

$$d(\Delta_1, \Delta_2) = |p_1 - p_2|, \tag{2.7.7}$$

if the lines are on the same side of the origin, or:

$$d(\Delta_1, \Delta_2) = p_1 + p_2, \tag{2.7.8}$$

if the origin is between the two lines.

Finally, suppose the two lines are given by their explicit equations:

$$y = mx + n_1, \tag{2.7.9}$$

and:

$$y = mx + n_2. \tag{2.7.10}$$

Rewriting these equations in their general form and using formula (2.7.3), we deduce that the *distance between two parallel lines given by their explicit equations* is:

$$d(\Delta_1, \Delta_2) = \frac{|n_1 - n_2|}{\sqrt{1 + m^2}}. \tag{2.7.11}$$

2.8 The Angle Between Two Lines

Suppose two lines Δ_1 and Δ_2 are given in the plane by their general equations:

$$A_1 x + B_1 y + C_1 = 0, \tag{2.8.1}$$

$$A_2 x + B_2 y + C_2 = 0. \tag{2.8.2}$$

As seen earlier, the direction vectors of these lines can be taken as $\mathbf{a}_1(-B_1, A_1)$ and $\mathbf{a}_2(-B_2, A_2)$. Therefore:

$$\cos\varphi = \frac{A_1 A_2 + B_1 B_2}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}}, \tag{2.8.3}$$

where φ is one of the angles formed by the two lines. If the lines are parallel, by convention, the angle between them is considered to be zero.

From formula (2.8.3), we also obtain the necessary and sufficient condition for the lines (2.8.1) and (2.8.2) to be perpendicular:

$$A_1 A_2 + B_1 B_2 = 0. \tag{2.8.4}$$

Now suppose the lines Δ_1 and Δ_2 (not vertical) are given by their slope-intercept forms:

$$y = k_1 x + b_1, \tag{2.8.5}$$

$$y = k_2 x + b_2. \tag{2.8.6}$$

Let φ be the angle of rotation needed to align Δ_1 with Δ_2. If the lines are parallel, we assume $\varphi = 0$. Let α_1 and α_2 be the angles the lines make with the Ox-axis, where $k_1 = \tan\alpha_1$ and $k_2 = \tan\alpha_2$. Then:

$$\tan\varphi = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{k_2 - k_1}{1 + k_1 k_2}.$$

Thus:

$$\tan\varphi = \frac{k_2 - k_1}{1 + k_1 k_2}. \tag{2.8.7}$$

From formula (2.8.7), the condition for perpendicularity of the lines (2.8.5) and (2.8.6) is:

$$1 + k_1 k_2 = 0,$$

or equivalently:

$$k_2 = -\frac{1}{k_1}. \tag{2.8.8}$$

2.9 The Bisectors of Angles Formed by Two Lines

2.9.1 The Case of Lines Given by General Equations

Consider two lines given by the equations:

$$L_1 \colon a_1 x + b_1 y + c_1 = 0, \tag{2.9.1}$$

and:

$$L_2 \colon a_2 x + b_2 y + c_2 = 0. \tag{2.9.2}$$

We assume that the two lines are concurrent. This means that the system formed by their equations is compatible and determined, i.e.:

$$a_1 b_2 - a_2 b_1 \neq 0. \tag{2.9.3}$$

As known from elementary geometry, the bisectors of the angles formed by two lines in the plane are defined as the *locus of points in the plane that are equidistant from the sides of the angle.*

The equations of the two bisectors follow directly from the second formula in (2.6.6):

$$L_{\pm}\colon \frac{|a_1 x + b_1 y + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2 x + b_2 y + c_2|}{\sqrt{a_2^2 + b_2^2}}, \tag{2.9.4}$$

or, explicitly removing the modulus:

$$L_{\pm}\colon \frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}. \tag{2.9.5}$$

The Bisector of the Acute Angle

If the two lines are not perpendicular, then one of the angles they form is acute, and the other is obtuse. In many applications, it is important to identify the bisector of the acute angle using the coefficients of the equations of the two lines.

Suppose we want L_+ to be the bisector of the acute angle. This means that it is the bisector of the angle between L_1 and L_2 that is less than $\pi/2$. Therefore, the acute angle that L_1 makes with L_+, say θ, must be less than $\pi/4$. θ is the acute angle between the normals to the two lines. The normal vector to the first line is:

$$\mathbf{n}_1 = (a_1, b_1).$$

To determine the normal vector to L_+, we rewrite the equation of this line in the form:

$$L_+\colon \left(\frac{a_1}{\sqrt{a_1^2 + b_1^2}} - \frac{a_2}{\sqrt{a_2^2 + b_2^2}}\right) x + \left(\frac{b_1}{\sqrt{a_1^2 + b_1^2}} - \frac{b_2}{\sqrt{a_2^2 + b_2^2}}\right) y + \frac{c_1}{\sqrt{a_1^2 + b_1^2}} - \frac{c_2}{\sqrt{a_2^2 + b_2^2}} = 0$$

Thus, the normal vector to L_+ is:

$$\mathbf{n}_+ = \left(\frac{a_1}{\sqrt{a_1^2 + b_1^2}} - \frac{a_2}{\sqrt{a_2^2 + b_2^2}}, \frac{b_1}{\sqrt{a_1^2 + b_1^2}} - \frac{b_2}{\sqrt{a_2^2 + b_2^2}}\right).$$

The cosine of the acute angle formed by the two normals (and hence the acute angle formed by L_1 and L_+) is given by:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_+|}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_+\|}.$$

After computation, we find:

$$\cos\theta = \frac{\sqrt{2}}{2}\sqrt{1 - \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}}}. \tag{2.9.6}$$

L_+ is the bisector of the acute angle if $\theta < \pi/4$. Since the cosine function is decreasing on the interval $[0, \pi/2]$, we conclude:

- If $a_1a_2 + b_1b_2 > 0$, then L_+ is the bisector of the acute angle (and L_- is the bisector of the obtuse angle).
- If $a_1a_2 + b_1b_2 = 0$, then the lines are perpendicular, and the bisectors play identical roles.
- If $a_1a_2 + b_1b_2 < 0$, then L_+ is the bisector of the obtuse angle (and L_- is the bisector of the acute angle).

2.9.2 The Case of Lines Given by Normal Equations

Suppose two non-parallel lines are given by their normal equations:

$$L_1\colon x\cos\alpha_1 + y\sin\alpha_1 - p_1 = 0, \tag{2.9.7}$$

$$L_2\colon x\cos\alpha_2 + y\sin\alpha_2 - p_2 = 0. \tag{2.9.8}$$

We assume $\alpha_1 \neq \alpha_2$ to ensure the lines are not parallel.

Applying the definition of the bisector as the locus of points equidistant from the two given lines, we obtain:

$$L_\pm\colon |x\cos\alpha_1 + y\sin\alpha_1 - p_1| = |x\cos\alpha_2 + y\sin\alpha_2 - p_2|, \tag{2.9.9}$$

or:

$$L_\pm\colon x\cos\alpha_1 + y\sin\alpha_1 - p_1 = \pm(x\cos\alpha_2 + y\sin\alpha_2 - p_2). \tag{2.9.10}$$

The equations of the two bisectors are:

$$L_+\colon x(\cos\alpha_1 - \cos\alpha_2) + y(\sin\alpha_1 - \sin\alpha_2) - p_1 + p_2 = 0, \tag{2.9.11}$$

and:

$$L_-\colon x(\cos\alpha_1 + \cos\alpha_2) + y(\sin\alpha_1 + \sin\alpha_2) - p_1 - p_2 = 0. \tag{2.9.12}$$

The Bisector of the Acute Angle

To determine which of the two bisectors is the bisector of the acute angle formed by the two lines and which one is the bisector of the obtuse angle, we proceed as above and first calculate the cosine of the acute angle formed by L_1 and the bisector L_+.

The normal vector to the line L_1 is $\mathbf{n}_1(\cos\alpha_1, \sin\alpha_1)$, while the normal vector to the line L_+ is

$$\mathbf{n}_+(\cos\alpha_1 - \cos\alpha_2, \sin\alpha_1 - \sin\alpha_2).$$

Thus, we have

$$\begin{aligned}
\cos\theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_+|}{\|\mathbf{n}_1\| \cdot \|\mathbf{n}_+\|} = \\
&= \frac{|\cos\alpha_1 \cdot (\cos\alpha_1 - \cos\alpha_2) + \sin\alpha_1 \cdot (\sin\alpha_1 - \sin\alpha_2)|}{1 \cdot \sqrt{(\cos\alpha_1 - \cos\alpha_2)^2 + (\sin\alpha_1 - \sin\alpha_2)^2}} = \\
&= \frac{1 - \cos(\alpha_1 - \alpha_2)}{\sqrt{2(1 - \cos(\alpha_1 - \alpha_2))}} = \frac{\sqrt{2}}{2}\sqrt{1 - \cos(\alpha_1 - \alpha_2)}.
\end{aligned}$$

Repeating the reasoning from the case of general equations, we conclude the following:

- If $\cos(\alpha_1 - \alpha_2) > 0$ (i.e., the angle $\alpha_1 - \alpha_2$ is acute), then L_+ is the bisector of the acute angle, and L_- is the bisector of the obtuse angle.

- If $\cos(\alpha_1 - \alpha_2) = 0$, the two lines are perpendicular, and both bisectors play the same role.

- If $\cos(\alpha_1 - \alpha_2) < 0$ (i.e., the angle $\alpha_1 - \alpha_2$ is obtuse), then L_+ is the bisector of the obtuse angle, and L_- is the bisector of the acute angle.

2.9.3 The Case of Lines Given by Explicit Equations

Consider the lines

$$L_1 : \; y = m_1 x + n_1 \tag{2.9.13}$$

and

$$L_2 : \; y = m_2 x + n_2, \tag{2.9.14}$$

where we assume $m_1 \neq m_2$, i.e., the lines are not parallel.

We will reduce this case to the case of lines given by general equations. To this end, we rewrite the equations of the lines in the form

$$L_1 : \; m_1 x - y + n_1 = 0 \tag{2.9.15}$$

and

$$L_2 : \; m_2 x - y + n_2 = 0, \tag{2.9.16}$$

and apply the discussion from the general case, where this time $a_1 = m_1, b_1 = -1, c_1 = n_1$, and $a_2 = m_2, b_2 = -1, c_2 = n_2$. Then the equations of the bisectors are

$$L_+ : \; \frac{m_1 x - y + n_1}{m_1^2 + 1} = \frac{m_2 x - y + n_2}{m_2^2 + 1}, \tag{2.9.17}$$

and

$$L_- : \; \frac{m_1 x - y + n_1}{m_1^2 + 1} = -\frac{m_2 x - y + n_2}{m_2^2 + 1}, \tag{2.9.18}$$

while

$$a_1 a_2 + b_1 b_2 = m_1 m_2 + 1. \tag{2.9.19}$$

Thus, adapting the conclusions from the general case, we can assert the following:

1. If $m_1 m_2 + 1 > 0$, then L_+ is the bisector of the acute angle, and L_- is the bisector of the obtuse angle.
2. If $m_1 m_2 + 1 = 0$, the two lines are perpendicular, and the bisectors play the same role.
3. If $m_1 m_2 + 1 < 0$, then L_+ is the bisector of the obtuse angle, and L_- is the bisector of the acute angle.

2.10 The Reflection of a Point About a Line

We limit the discussion to the case where the line is given by the general equation, as other cases can be easily adapted.

Definition 2.5. Let

$$L : \; ax + by + c = 0 \tag{2.10.1}$$

be a line and $M(x_0, y_0)$ a point in the plane. The *reflection* of M about the line L is either the point M itself if $M \in L$, or a point $M'(x_0', y_0')$ in the plane such that the line L is the perpendicular bisector of the segment MM'. The midpoint M_1 of the segment MM' is the *orthogonal projection* of M onto L.

The definition also suggests the algorithm for determining the reflection:

1. Check if $M \in L$. If so, then $M' = M$.
2. If $M \notin L$, construct the line L' passing through M and perpendicular to L.
3. Determine the point $M_1 = L \cap L'$.
4. Determine the point M' such that

$$M_1 = \frac{1}{2}M + \frac{1}{2}M'. \tag{2.10.2}$$

Assuming $M \notin L$, the direction vector of the line L', which passes through M and is perpendicular to L, is the normal vector of L, $\mathbf{n}(a, b)$. Hence, the equation of this line is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}. \tag{2.10.3}$$

The coordinates of M_1 are obtained by solving the system formed by equations (2.10.1) and (2.10.3). We find

$$\begin{cases} x_1 = \dfrac{b^2}{a^2 + b^2}x_0 - \dfrac{ab}{a^2 + b^2}y_0 - \dfrac{ac}{a^2 + b^2}, \\ y_1 = -\dfrac{ab}{a^2 + b^2}x_0 + \dfrac{a^2}{a^2 + b^2}y_0 - \dfrac{bc}{a^2 + b^2}. \end{cases} \tag{2.10.4}$$

Relation (2.10.2), written in coordinates, means

$$(x_1, y_1) = \frac{1}{2}(x_0, y_0) + \frac{1}{2}(x_0', y_0'),$$

from which we immediately deduce the relations

$$\begin{cases} x_0' = 2x_1 - x_0, \\ y_0' = 2y_1 - y_0. \end{cases} \tag{2.10.5}$$

Substituting relations (2.10.4) into (2.10.5), we obtain the coordinates of the reflection:

$$\begin{cases} x_0' = -\dfrac{a^2 - b^2}{a^2 + b^2}x_0 - \dfrac{2ab}{a^2 + b^2}y_0 - \dfrac{2ac}{a^2 + b^2}, \\ y_0' = -\dfrac{2ab}{a^2 + b^2}x_0 + \dfrac{a^2 - b^2}{a^2 + b^2}y_0 - \dfrac{2bc}{a^2 + b^2}. \end{cases} \tag{2.10.6}$$

2.11 Probleme

Problem 2.1. Determinaţi ecuaţia dreptei care trece prin $(2, -3)$ şi este paralelă cu dreapta care trece prin $(4, 1)$ şi $(-2, 2)$.

Problem 2.2. Determinaţi ecuaţia dreptei care trece prin $(-2, 3)$ şi este pe perpendiculară pe dreapta $2x - 3y + 6 = 0$.

Problem 2.3. Determinaţi ecuaţia mediatoarei segmentului care uneşte punctele $(7, 4)$ şi $(-1, -2)$.

Problem 2.4. Stabiliţi ecuaţia dreptei care trece prin $(2, -3)$ şi face un unghi de $60°$ cu direcţia pozitivă a axei Ox.

Problem 2.5. Stabiliţi ecuaţiile dreptelor care au panta $-3/4$ şi formează cu axele de coordonate un triunghi de arie 24.

Problem 2.6. Determinaţi forma normală Hesse a dreptei $3x - 4y - 6 = 0$.

Problem 2.7. Stabiliţi ecuaţia dreptelor care trec prin $(4, -2)$ şi sunt la o distanţă 2 faţă de origine.

Problem 2.8. Determinaţi ecuaţiile bisectoarelor unghiurilor formate de dreptele

$$(L_1)\ 3x - 4y + 8 = 0$$

şi

$$(L_2)\ 5x + 12y - 15 = 0.$$

Problem 2.9. Determinaţi ecuaţiile dreptelor paralele cu dreapta $12x - 5y - 15 = 0$, situate la distanţa 4 faţă de aceasta.

Problem 2.10. Determinaţi valoarea lui k astfel încât distanţa de la punctul $(2, 3)$ la dreapta $8x + 15y + k = 0$ să fie egală cu 5.

Problem 2.11. Două mediane ale unui triunghi sunt situate pe dreptele $x + y = 2$ şi $2x + 3y = 1$, iar punctul $A(1, 1)$ este un vârf al triunghiului. Scrieţi ecuaţiile laturilor acestui triunghi.

Problem 2.12. Punctele $K(1, -1)$, $L(3, 4)$ şi $M(5, 0)$ sunt, respectiv, mijloacele laturilor AD, AB şi CD ale patrulaterului $ABCD$, ale cărui diagonale se intersectează în $O(2, 2)$. Determinaţi coordonatele vârfurilor patrulaterului.

Problem 2.13. Stabiliţi ecuaţiile dreptelor care trec prin punctul $A(-1, 5)$ şi sunt egal depărtate de punctele $B(3, 7)$ şi $C(1, -1)$.

Problem 2.14. Stabiliţi ecuaţiile dreptelor egal depărtate de punctele $A(3, -1)$, $B(9, 1)$ şi $C(-5, 5)$.

Problem 2.15. Punctele $K(1, -1)$, $L(3, 4)$ şi $M(5, 0)$ sunt, respectiv, mijloacele laturilor AD, AB şi CD ale patrulaterului $ABCD$, ale cărui diagonale se intersectează în $O(2, 2)$. Determinaţi coordonatele vârfurilor patrulaterului.

Problem 2.16. Stabiliţi ecuaţiile dreptelor care trec prin punctul $A(-1, 5)$ şi sunt egal depărtate de punctele $B(3, 7)$ şi $C(1, -1)$.

Problem 2.17. Stabiliţi ecuaţiile dreptelor egal depărtate de punctele $A(3, -1)$, $B(9, 1)$ şi $C(-5, 5)$.

Problem 2.18. Punctul $A(3, -2)$ este un vârf al unui pătrat, iar $M(1, 1)$ este punctul de intersecţie a diagonalelor sale. Stabiliţi ecuaţiile laturilor pătratului.

Problem 2.19. Lungimea laturii unui romb cu unghiul ascuţit de $60°$ este egală cu 2. Diagonalele rombului se intersectează în punctul $M(1, 2)$, iar diagonala cea mai lungă este paralelă cu axa Ox. Stabiliţi ecuaţiile laturilor rombului.

Problem 2.20. Determinaţi un punct de pe dreapta $5x - 4y - 4 = 0$ egal depărtat de punctele $A(1, 0)$ şi $B(-2, 1)$.

Problem 2.21. Determinaţi coordonatele unui punct A de pe dreapta $x + y = 8$, care este egal depărtat de punctul $B(2, 8)$ şi de dreapta $x - 3y + 2 = 0$.

Problem 2.22. Determinaţi coordonatele tuturor punctelor egal depărtate de punctul $A(-1, 1)$ şi de dreptele $y = -x$ şi $y = x + 1$.

Problem 2.23. În triunghiul ABC punctele $M_1(2, 3)$, $M_2(0, 7)$ şi $M_3(-2, 5)$ sunt mijloacele laturilor BC, CA şi AB. Stabiliţi ecuaţia dreptei AB. Găsiţi unghiul dintre medianele AM_1 şi BM_2.

Problem 2.24. În paralelogramul $ABCD$ vârfurile A şi C au coordonatele $(1, 2)$, respectiv $(7, 10)$, iar $H(3, 0)$ este piciorul perpendicularei coborâte din vârful B pe latura AD. Stabiliţi ecuaţia dreptei AD. Găsiţi unghiul dintre dreptele AD şi AB.

Problem 2.25. În paralelogramul $ABCD$ punctele $K(-1, 2)$, $L(3, 4)$ şi $M(5, 6)$ sunt mijloacele laturilor AB, BC, respectiv CD. Stabiliţi ecuaţia dreptei BC. Găsiţi unghiul dintre dreptele AL şi AM.

Problem 2.26. În trapezul $ABCD$ cu bazele AD şi BC latura CD este perpendiculară pe baze, punctele A şi C au coordonatele $(5, 2)$, respectiv $(-2, 3)$, iar prelungirile laturilor neparalelele se intersectează în punctul $P(-3, 6)$. Stabiliţi ecuaţia dreptei AD. Găsiţi unghiul dintre dreptele AD şi AB.

Problem 2.27. Punctele $K(1, 3)$ şi $L(-1, 1)$ sunt mijloacele bazelor unui trapez isoscel, iar punctele $P(3, 0)$ şi $Q(-3, 5)$ se află pe laturile neparalele. Stabiliţi ecuaţiile laturilor trapezului.

Problem 2.28. Stabiliţi ecuaţia dreptei care trece prin $A(3, 1)$ şi face un unghi de 45° cu dreapta $3x - y - 2 = 0$.

Problem 2.29. Punctul $A(2, 0)$ este un vârf al unui triunghi echilateral, iar latura opusă se află pe dreapta $x + y - 1 = 0$. Stabiliţi ecuaţiile celorlalte două laturi ale triunghiului.

Problem 2.30. Baza unui triunghi isoscel se află pe dreapta $x + 2y = 2$, iar una dintre laturile egale se află pe dreapta $y + 2x = 1$. Stabiliţi ecuaţia celei de-a treia laturi, ştiind că distanţa de la punctul de intersecţie a dreptelor date până la această latură este egală cu $1/\sqrt{5}$.

Problem 2.31. Se consideră acel unghi format de dreptele $y = x + 1$ şi $y = 7x + 1$ în interiorul căruia se află punctul $A(1, 3)$. Determinaţi coordonatele punctului B, situat în interiorul aceluiaşi unghi, situat la distanţa $4\sqrt{2}$ fată de prima dreaptă şi $\sqrt{2}$ faţă de cea de-a doua dreaptă.

Problem 2.32. Stabiliţi ecuaţia bisectoarei acelui unghi format de dreptele $x - 7y = 1$ şi $x + y = -7$, înăuntru căruia se află punctul $A(1, 1)$.

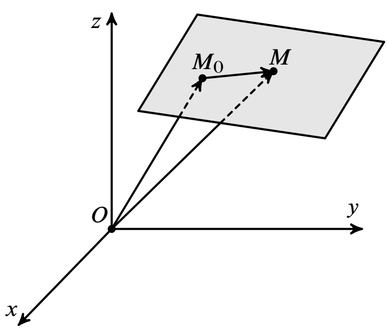
CHAPTER 3

The Line and the Plane in Space

3.1 The Plane

3.1.1 The Vector Equation of a Plane

Let $\mathbf{v}$ and $\mathbf{w}$ be two non-collinear vectors in space and M_0 any point. If we attach the vectors to the point M_0, there are two uniquely determined points, P and Q, such that $\mathbf{v} = \overrightarrow{M_0P}$ and $\mathbf{w} = \overrightarrow{M_0Q}$. Since the vectors $\mathbf{v}$ and $\mathbf{w}$ are non-collinear, the points M_0, P, and Q are also non-collinear and thus determine a plane Π. We aim to describe the points of this plane using the point M_0 and the vectors $\mathbf{v}$ and $\mathbf{w}$.



Let M be a point in space. Denote by $\mathbf{r}_0$ the position vector of M_0 and by $\mathbf{r}$

the position vector of M. Clearly, the point M belongs to the plane if and only if the vector $\overrightarrow{M_0M}$ is coplanar with the vectors $\overrightarrow{M_0P}$ and $\overrightarrow{M_0Q}$, i.e., with the vectors $\mathbf{v}$ and $\mathbf{w}$. Assume that M *belongs* to the plane Π. Since the vectors $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, $\overrightarrow{M_0M}$ has a (unique) decomposition as a linear combination of the vectors $\mathbf{v}$ and $\mathbf{w}$, meaning there exist (and are unique) two real numbers s and t such that

$$\overrightarrow{M_0M} = s\mathbf{v} + t\mathbf{w}. \tag{3.1.1}$$

On the other hand, $\overrightarrow{M_0M} = \mathbf{r} - \mathbf{r}_0$, so the previous equation can be rewritten as

$$\mathbf{r} = \mathbf{r}_0 + s\mathbf{v} + t\mathbf{w}, \tag{3.1.2}$$

which is called the *vector equation of the plane* Π.

Now assume that the point M has coordinates (x, y, z), M_0 has coordinates (x_0, y_0, z_0), and the vectors $\mathbf{v}$ and $\mathbf{w}$ have components (v_x, v_y, v_z) and (w_x, w_y, w_z), respectively. Then the vector equation (3.1.2) is equivalent to the system of scalar equations

$$\begin{cases} x = x_0 + sv_x + tw_x \\ y = y_0 + sv_y + tw_y \\ z = z_0 + sv_z + tw_z \end{cases}, \tag{3.1.3}$$

which are called the *parametric equations of the plane* Π.

The equation of the plane can also be written in vector form without using parameters. Indeed, we have the following result:

Theorem 3.1. *The vector equation of a plane passing through a point M_0 and perpendicular to a given vector $\mathbf{n}$ is*

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0. \tag{3.1.4}$$

Proof Let Π be the plane determined by the point and the normal vector. If M is any point on the plane, then $\overrightarrow{M_0M} \perp \mathbf{n}$, which implies that

$$\overrightarrow{M_0M} \cdot \mathbf{n} = 0 \tag{3.1.5}$$

or

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0.$$

□

3.1.2 The General Equation of a Plane

Definition 3.1. A *linear (general) equation* in the unknowns x, y, z is an equation of the form

$$Ax + By + Cz + D = 0, \tag{3.1.6}$$

where at least one of the coefficients A, B, C of the unknowns is non-zero.

Theorem 3.2. *In a Cartesian coordinate system, a plane is defined by a general linear equation of the form (3.1.6).*

Proof Consider a plane passing through the point M_0 and having the normal vector $\mathbf{n}(A, B, C)$. Then, for any point $M(x, y, z)$ in the plane, we have

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0,$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which simplifies to

$$Ax + By + Cz - (Ax_0 + By_0 + Cz_0) = 0,$$

a general linear equation in x, y, z.

Conversely, let $M(x, y, z)$ be a point in space satisfying a linear equation of the form

$$Ax + By + Cz + D = 0,$$

with $A^2 + B^2 + C^2 \neq 0$.

Suppose, for instance, that $A \neq 0$ in the above equation. Then, it is evident that the point $M_0(-D/A, 0, 0)$ also satisfies this equation. Denote by $\mathbf{n}$ the vector with components (A, B, C). Since

$$\overrightarrow{M_0M} \equiv \mathbf{r} - \mathbf{r}_0 = (x + D/A, y, z),$$

the equation of the plane passing through M_0 and having the normal vector $\mathbf{n}$ is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0,$$

or

$$A(x + D/A) + By + Cz = 0 \text{ or } Ax + By + Cz + D = 0,$$

hence the point M lies on the plane passing through M_0 with $\mathbf{n}$ as the normal vector. □

Special Cases of the General Equation of a Plane

a) The equation of a plane passing through the origin is:

$$Ax + By + Cz = 0.$$

Indeed, it is immediately apparent that the above equation is satisfied by the origin $O(0, 0, 0)$.

b) The equations of planes parallel to the coordinate axes are:

$$\begin{aligned} Ax + By + D &= 0 \quad \text{(parallel to the } Oz \text{ axis)},\\ Ax + Cz + D &= 0 \quad \text{(parallel to the } Oy \text{ axis)},\\ By + Cz + D &= 0 \quad \text{(parallel to the } Ox \text{ axis)}. \end{aligned}$$

Indeed, if we set $C = 0$ in the general equation of the plane, it reduces to

$$Ax + By + D = 0.$$

In this case, the normal vector to the plane, $\mathbf{n}(A, B, 0)$, has a zero orthogonal projection onto the Oz axis, meaning the vector is perpendicular to the axis, so the plane is parallel to the Oz axis. Similarly, this reasoning applies to the other two cases. If, in particular, $D = 0$ as well, the planes not only parallel the axes but also pass through them.

c) The equations of planes parallel to the coordinate planes are:

$$\begin{aligned} Ax + D &= 0 \quad \text{(parallel to the } yOz \text{ plane)},\\ By + D &= 0 \quad \text{(parallel to the } xOz \text{ plane)},\\ Cz + D &= 0 \quad \text{(parallel to the } xOy \text{ plane)}. \end{aligned}$$

Indeed, if, for instance, we set $B = C = 0$ in the equation of the plane, it transforms into

$$Ax + D = 0.$$

The normal vector to this plane is $\mathbf{n}(A, 0, 0)$. This vector is perpendicular to the yOz plane, so the plane with this normal vector is *parallel* to the yOz plane. The same reasoning applies to the other two coordinate planes.

As above, if we also set $D = 0$, we obtain planes that are parallel to the coordinate planes and pass through the origin, i.e., we obtain the equations of the *coordinate planes*, $x = 0$, $y = 0$, and $z = 0$.

3.1.3 Another Form of the Vector Equation of a Plane

We start from the vector equation of a plane passing through a point and parallel to two non-collinear vectors:

$$\mathbf{r} - \mathbf{r}_0 = s\mathbf{v} + t\mathbf{w}.$$

This equation is equivalent to requiring that the vectors $\mathbf{r} - \mathbf{r}_0$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent, which is also equivalent to the condition that

$$(\mathbf{r} - \mathbf{r}_0, \mathbf{v}, \mathbf{w}) = 0. \tag{3.1.7}$$

This equation is commonly referred to as *the equation of the plane passing through the point M_0 and parallel to the vectors* $\mathbf{v}$ *and* $\mathbf{w}$. Expanding the mixed product in (3.1.7), it becomes evident that this equation is equivalent to

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0 \tag{3.1.8}$$

or to

$$\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} (x - x_0) + \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix} (y - y_0) + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} (z - z_0) = 0. \tag{3.1.9}$$

3.1.4 Equation of a Plane Determined by Three Non-Collinear Points

Let $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, and $M_3(x_3, y_3, z_3)$ be three non-collinear points in space. These three points determine a plane. To find its equation, we use the method outlined in the previous section. Specifically, let

$$\begin{aligned} \mathbf{v} &\equiv \overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1), \\ \mathbf{w} &\equiv \overrightarrow{M_1M_3}(x_3 - x_1, y_3 - y_1, z_3 - z_1). \end{aligned}$$

These two vectors are evidently non-collinear and parallel to the plane. The plane we seek passes through M_1 and is parallel to the vectors $\mathbf{v}$ and $\mathbf{w}$. Therefore, its equation is given by (see 3.1.8):

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0. \tag{3.1.10}$$

Equation (3.1.10) can also be rewritten in the following, easier-to-remember form:

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \tag{3.1.11}$$

3.1.5 Coplanarity Condition for Four Points

From formula (3.1.11), we immediately obtain the *coplanarity condition for four points*:

Four points M_1, M_2, M_3, M_4 are coplanar if and only if:

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0. \tag{3.1.12}$$

3.1.6 Plane Equation by Intercepts

Let Π be a plane that does not pass through the origin or any of the coordinate axes. As shown earlier, its general equation is

$$Ax + By + Cz + D = 0,$$

where none of the four coefficients is zero. Let P, Q, R be the three points where the plane intersects the coordinate axes. The intersection with Ox, determined by $y = 0, z = 0$, is $P(-D/A, 0, 0)$; the intersection with Oy, given by $x = 0, z = 0$, is $Q(0, -D/B, 0)$; and the intersection with Oz, given by $x = 0, y = 0$, is $R(0, 0, -D/C)$. Using the notations

$$a = -\frac{D}{A},\ b = -\frac{D}{B},\ c = -\frac{D}{C},$$

the equation of the plane passing through the points P, Q, R is

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 0.$$

Expanding this determinant yields

$$bcx + cay + abz - abc = 0$$

or, dividing by abc,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0. \tag{3.1.13}$$

Equation (3.1.13) is called the *intercept form of the plane equation*. The lengths a, b, c, referred to as *intercepts*, are the signed distances from the origin to the points where the plane intersects the coordinate axes.

3.1.7 Normal Equation of a Plane

Let Π be a plane, and let OP be the perpendicular from the origin to the plane. If the plane passes through the origin, then P coincides with the origin, and the length of the vector $\overrightarrow{OP}$ is zero. In the general case, let

$$p \equiv \left\| \overrightarrow{OP} \right\|$$

be the length of this vector (equal to the distance from the origin to the plane Π).

Let $\mathbf{n}(\cos\alpha, \cos\beta, \cos\gamma)$ be the unit vector along $\overrightarrow{OP}$ (also the unit normal vector of the plane). Then the coordinates of P, the foot of the perpendicular from the origin to the plane, are

$$P(p\cos\alpha, p\cos\beta, p\cos\gamma).$$

Thus, if $M(x, y, z)$ is any point in the plane Π, its components are

$$\overrightarrow{PM}(x - p\cos\alpha, y - p\cos\beta, z - p\cos\gamma).$$

Since the vectors $\overrightarrow{PM}$ and $\mathbf{n}$ are perpendicular, we have

$$\overrightarrow{PM} \cdot \mathbf{n} = 0$$

or

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p\underbrace{\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right)}_{=1} = 0,$$

or, finally,

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0. \tag{3.1.14}$$

Equation (3.1.14) is called the *Hesse normal form* or simply the *normal form* of the plane equation.

The normal form of a plane equation is useful in certain situations, so we will show how it can be derived. Starting with the general equation of a plane,

$$Ax + By + Cz + D = 0,$$

this plane also has a normal equation,

$$x \cos\alpha + y \cos\beta + z \cos\gamma - p = 0.$$

Since both equations represent the same plane, their coefficients must be proportional:

$$\cos\alpha = \lambda A, \ \cos\beta = \lambda B, \ \cos\gamma = \lambda C, \ -p = \lambda D.$$

If we square the first three equalities and sum them, we get

$$\lambda^2 \left(A^2 + B^2 + C^2\right) = 1,$$

using the fact that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Therefore,

$$\lambda = \pm \frac{1}{\sqrt{A^2 + B^2 + C^2}}. \tag{3.1.15}$$

The sign in (3.1.15) is chosen to be opposite to the sign of the constant term D in the general equation. If $D = 0$, the sign of λ can be chosen arbitrarily. For obvious reasons, λ is called the *normalising factor* of the general plane equation.

The plane Π divides the set of all points in space that do not belong to it into two subsets called *open half-spaces*. The *positive half-space* is the one towards which the vector $\mathbf{n}$ is directed. The other is called the *negative half-space*. Note that the origin of the coordinate system is always either in the plane Π or in the negative half-space.

3.1.8 Distance from a Point to a Plane

Definition 3.2. The *distance* from a point $M_0(x_0, y_0, z_0)$ to a plane Π is defined as the length d of the perpendicular dropped from M_0 to the plane Π. The *deviation* (or *offset*) of the point M_0 relative to the plane Π is the number δ defined as follows:

a) $\delta = d$ if M_0 is in the positive half-space determined by Π;

b) $\delta = 0$ if $M_0 \in \Pi$;

c) $\delta = -d$ if M_0 is in the negative half-space.

Theorem 3.3. *If the plane is given by its normal equation*

$$x \cos\alpha + y \cos\beta + z \cos\gamma - p = 0,$$

then the following formulas hold:

$$\delta = x_0 \cos\alpha + y_0 \cos\beta + z_0 \cos\gamma - p\,; \tag{3.1.16}$$

$$d = |x_0 \cos\alpha + y_0 \cos\beta + z_0 \cos\gamma - p\,|\,. \tag{3.1.17}$$

If the plane is given by its general equation,

$$Ax + By + Cz + D = 0,$$

then the following formulas hold:

$$\delta = \frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2 + B^2 + C^2}}, \tag{3.1.18}$$

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D\,|}{\sqrt{A^2 + B^2 + C^2}}\,. \tag{3.1.19}$$

Proof Let P_0 be the orthogonal projection of M_0 onto the line OP. Then

$$\delta = (PP_0) = (OP_0) - (OP) = \mathbf{n} \cdot \overrightarrow{OM_0} - p = x_0 \cos\alpha + y_0 \cos\beta + z_0 \cos\gamma - p\,.$$

Thus, formula (3.1.16) is proven. Formula (3.1.17) follows from (3.1.16), since, evidently, $d = |\delta|$. $\square$

3.1.9 Angle Between Two Planes

The *angle between two planes* is defined as the measure of the planar angle associated with the dihedral angle formed by the two planes. This is equivalent to the angle between the normal directions of the two planes.

It is worth noting that two planes actually form not one but *four* angles: two pairs of opposite and equal angles, which are supplementary when adjacent.

Consider two planes

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \tag{3.1.20}$$

and

$$A_2x + B_2y + C_2z + D_2 = 0. \tag{3.1.21}$$

The normal vectors to these planes are $\mathbf{n}_1(A_1, B_1, C_1)$ and $\mathbf{n}_2(A_2, B_2, C_2)$, so the angles between the planes are given by

$$\cos\alpha_{1,2} = \pm\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}. \tag{3.1.22}$$

If the right-hand side is positive, the acute angles are obtained; if it is negative, the obtuse angles are obtained.

From formula (3.1.22), it follows that the two planes are perpendicular if and only if

$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \tag{3.1.23}$$

On the other hand, the planes are parallel if and only if their normal vectors are parallel, which occurs if and only if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}. \tag{3.1.24}$$

3.2 The Line in Space

3.2.1 Vector Equation and Parametric Equations of a Line

Let Δ be a line in space. A non-zero vector $\mathbf{a}$ is called a *direction vector* of the line Δ if any oriented segment parallel to $\mathbf{a}$ lies along Δ. If $\mathbf{a}(l, m, n)$ is a direction vector of Δ, and $M_0(x_0, y_0, z_0)$ is any point on the line, then an arbitrary point $M(x, y, z)$ in space lies on the line if and only if the vector $\overrightarrow{M_0M}(x - x_0, y - y_0, z - z_0)$ is collinear with $\mathbf{a}$. Denoting the position vectors of M_0 and M as $\mathbf{r}_0$ and $\mathbf{r}$ respectively, we have

$$\overrightarrow{M_0M} = \mathbf{r} - \mathbf{r}_0,$$

so the vectors $\overrightarrow{M_0M}$ and $\mathbf{a}$ are collinear if and only if there exists a real number t such that

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{a},$$

or equivalently,

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}. \tag{3.2.1}$$

Equation (3.2.1) is called the *vector equation* of the line Δ that passes through the point M_0 and has $\mathbf{a}$ as a direction vector. This equation is equivalent to the system of three scalar equations:

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \\ z = z_0 + nt \end{cases}, \tag{3.2.2}$$

which are called the *parametric equations* of the line passing through the point $M_0(x_0, y_0, z_0)$ and having the direction vector $\mathbf{a}(l, m, n)$. Note that when switching to a different coordinate system, the form of the parametric equations may change (as the coordinates of M_0 and the components of $\mathbf{a}$ change), but the vector equation retains the same form in any affine coordinate system, even if it is not orthonormal.

3.2.2 Canonical Equations of a Line in Space

If each of the components l, m, n of the direction vector $\mathbf{a}$ is non-zero, the parametric equations (3.2.2) are equivalent to the system:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}, \tag{3.2.3}$$

which is commonly written as

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}. \tag{3.2.4}$$

Equations (3.2.4) are called the *canonical equations* of the line passing through the point $M_0(x_0, y_0, z_0)$ and having $\mathbf{a}(l, m, n)$ as its direction vector.

Remark. Since the direction vector $\mathbf{a}$ is non-zero, it is always possible to choose a coordinate system in which all its components are non-zero. However, in certain coordinate systems, one or two of its components may be zero. There is no reason not to write the canonical equations of the line in such coordinate systems. As in the case of the canonical equation of a line in the plane, we adopt the convention that $0/0 = 0$. For clarity, with this convention, a system of canonical equations of the form

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{0}$$

is equivalent to the system

$$\frac{x - x_0}{l} = \frac{y - y_0}{m}, \ z = z_0, \quad l \neq 0, \, m \neq 0,$$

while a system of equations of the form

$$\frac{x - x_0}{l} = \frac{y - y_0}{0} = \frac{z - z_0}{0}, \quad l \neq 0,$$

is equivalent to the system

$$y = y_0, \ z = z_0.$$

3.2.3 The Line as the Intersection of Two Planes

A line in space can be represented as the intersection of two distinct planes that both contain the line. Thus, it can be described by a system of two linear equations:

$$\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0, \\ A_2 x + B_2 y + C_2 z + D_2 = 0. \end{cases} \tag{3.2.5}$$

Since the planes defining the line are not parallel, the coefficients of the unknowns in the two equations of system (3.2.5) are not proportional. In other words, the rank of the matrix of this system is maximal (equal to two).

The equations of system (3.2.5) that define a given line are not unique. Clearly, each of the equations can be replaced by an equation of the form

$$\alpha(A_1 x + B_1 y + C_1 z + D_1) + \beta(A_2 x + B_2 y + C_2 z + D_2) = 0,$$

where α and β are real numbers not both zero, so that the system still has maximal rank.

It is evident that the converse is also true: any system of equations of the form (3.2.5) with rank two describes a line in space.

Often, we need to find the direction vector of a line given as the intersection of two planes. Consider the line (3.2.5), and let $\mathbf{n}_1(A_1, B_1, C_1)$ and $\mathbf{n}_2(A_2, B_2, C_2)$ be the normal vectors to the two planes that define the line. Then their vector product,

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2,$$

is clearly a direction vector of the line. Therefore,

$$\mathbf{v} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \mathbf{i} + \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \mathbf{k}.$$

3.2.4 Equations of a Line Passing Through Two Points

Let $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ be two distinct points on a line Δ. Then, the vector $\overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ is a direction vector of the line. Therefore, the line Δ is the one passing through the point M_1 with $\overrightarrow{M_1M_2}$ as its direction vector.

Thus, the parametric equations of the line Δ (passing through M_1 and having $\overrightarrow{M_1M_2}$ as its direction vector) are:

$$\begin{cases} x = x_1 + (x_2 - x_1)t, \\ y = y_1 + (y_2 - y_1)t, \\ z = z_1 + (z_2 - z_1)t. \end{cases} \tag{3.2.6}$$

These equations can also be rewritten in canonical form:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{3.2.7}$$

3.2.5 Angle Between Two Lines in Space

By definition, the angle between two lines in space is the angle formed by their *direction vectors*. Note that it is not necessary for the two lines to be coplanar. Since direction vectors are free vectors, they can be positioned with their origins at the same point. It is also worth noting that using direction vectors does not uniquely determine the angle between the lines. Specifically, if the direction of one of the vectors is reversed, the angle becomes its supplement. Therefore, to determine the *acute* angle between the two lines, we must ensure the cosine of the angle is positive.

Consider the two lines:

$$(D_1) : \frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}, \tag{3.2.8}$$

and

$$(D_2) : \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}, \tag{3.2.9}$$

with direction vectors $\mathbf{v}_1(l_1, m_1, n_1)$ and $\mathbf{v}_2(l_2, m_2, n_2)$ respectively. The angle between the two lines is given by:

$$\cos\varphi_{1,2} = \pm\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|} = \pm\frac{l_1l_2 + m_1m_2 + n_1n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}}. \tag{3.2.10}$$

The *acute* angle between the two lines is given by:

$$\cos\varphi = \frac{|l_1 l_2 + m_1 m_2 + n_1 n_2|}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}}. \tag{3.2.11}$$

The lines (3.2.8) and (3.2.9) are *perpendicular* if their direction vectors are perpendicular, i.e., if:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 \equiv l_1 l_2 + m_1 m_2 + n_1 n_2 = 0. \tag{3.2.12}$$

The lines (3.2.8) and (3.2.9) are *parallel* if their direction vectors are collinear, i.e., if there exists a scalar $\lambda \neq 0$ such that:

$$\mathbf{v}_1 = \lambda \mathbf{v}_2, \tag{3.2.13}$$

or (with the same convention as for the line equations):

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}. \tag{3.2.14}$$

3.3 Various Problems Related to Lines and Planes in Space

3.3.1 Relative Positions of Two Planes

Let us fix an affine coordinate system $Oxyz$ and consider two planes given by their general equations:

$$A_1 x + B_1 y + C_1 z + D_1 = 0, \tag{3.3.1}$$

$$A_2 x + B_2 y + C_2 z + D_2 = 0. \tag{3.3.2}$$

From geometric considerations, the two planes can be in one of the following situations:

1) They intersect along a line.

2) They are parallel but do not coincide.

3) They coincide.

Our goal is to establish the relationships between the coefficients of the two equations in each case.

The *trace* of the plane (3.3.1) on the xOy coordinate plane is the intersection of the plane with the xOy plane. If plane (3.3.1) is not parallel to the xOy plane, this intersection is a line whose equation in the xOy plane is:

$$A_1x + B_1y + D_1 = 0.$$

Similarly, the traces of the plane on the xOz and yOz coordinate planes can be determined, provided the plane is not parallel to these coordinate planes. Clearly, the plane (3.3.2) coincides with plane (3.3.1) if and only if their traces on the coordinate planes coincide. From the study of the relative position of two lines in the plane, we know this occurs if and only if all coefficients of the two planes are proportional, i.e., if and only if there exists a non-zero scalar λ such that:

$$A_1 = \lambda A_2,\ B_1 = \lambda B_2,\ C_1 = \lambda C_2,\ D_1 = \lambda D_2, \tag{3.3.3}$$

or equivalently:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}.$$

This notation can be used if none of the coefficients of the second plane is zero, or if we adopt the convention that whenever a coefficient of the second plane is zero, the corresponding coefficient of the first plane must also be zero.

From an algebraic point of view, the same conclusion can be reached differently. For the planes (3.3.1) and (3.3.2) to coincide, it is necessary and sufficient for the system of their equations to be compatible and doubly undetermined. This condition is precisely expressed by (3.3.3).

Now assume, for example, that the first plane is parallel to the xOy plane. This means, clearly, that $A_1 = B_1 = 0$. The algebraic reasoning above leads to the same conclusion as in the general position of the planes.

If the system of equations (3.3.1)–(3.3.2) is incompatible, it means that the rank of the system is 1, while the rank of the augmented matrix is 2. Therefore, the planes are parallel if and only if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2}, \tag{3.3.4}$$

with the same convention regarding zero denominators as before.

The last possible situation is that the system formed by the equations of the planes has maximum rank, which means the intersection is a line. In this case, the first three coefficients cannot be proportional.

3.3.2 Relative Positions of Three Planes

Consider three planes given by their general equations:

$$\begin{cases} (P_1)\ A_1x + B_1y + C_1z + D_1 = 0, \\ (P_2)\ A_2x + B_2y + C_2z + D_2 = 0, \\ (P_3)\ A_3x + B_3y + C_3z + D_3 = 0. \end{cases} \tag{3.3.5}$$

To determine the relative positions of the three planes, we must analyse the system of equations (3.3.5). Let Δ be the determinant of the system:

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix},$$

m the coefficient matrix:

$$m = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix},$$

and M the augmented matrix:

$$M = \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{pmatrix}.$$

Let $\mathbf{n}_1(A_1, B_1, C_1)$, $\mathbf{n}_2(A_2, B_2, C_2)$, and $\mathbf{n}_3(A_3, B_3, C_3)$ be the normal vectors to the three planes. The following cases arise:

(a) If $\Delta \neq 0$, then the system (3.3.5) is uniquely compatible, meaning it has a single solution: *the planes intersect at a single point.*

(b) Assume $\Delta = 0$, $\operatorname{rg} m = 2$, $\operatorname{rg} M = 3$, and the normal vectors of the three planes are pairwise non-collinear. Since the rank of the coefficient matrix is strictly less than the rank of the augmented matrix, the system is incompatible, meaning the three planes have no common point. Since the normal vectors are pairwise non-collinear, the planes intersect pairwise along lines, and the three lines are parallel.

(c) Again, assume $\operatorname{rg} m = 2$, $\operatorname{rg} M = 3$, but now two of the three normal vectors are collinear[1]. Two of the planes (those with collinear normal vectors) are parallel to each other, while the third plane intersects both.

[1] They cannot all be collinear since $\operatorname{rg} m = 2$!

(d) Suppose $\operatorname{rg} m = 2$, $\operatorname{rg} M = 2$ (so the system is compatible), and the normal vectors are pairwise non-collinear. In this case, the planes are pairwise distinct and intersect along a common line.

(e) If $\operatorname{rg} m = 2$, $\operatorname{rg} M = 2$, and two of the normal vectors are collinear, then again, the system is compatible. Two of the planes coincide (those with collinear normal vectors), and the third plane intersects them along a line.

(f) If $\operatorname{rg} m = 1$, $\operatorname{rg} M = 3$, then the system is incompatible. Thus, the planes do not intersect, and they are parallel to each other.

(g) If $\operatorname{rg} m = 1$, $\operatorname{rg} M = 2$, then two of the planes coincide, and the third is parallel to them.

(h) If $\operatorname{rg} m = 1$, $\operatorname{rg} M = 1$, then the system is doubly compatible, and all three planes coincide.

3.3.3 Plane Bundles. Pencils of Planes

Definition 3.3. A *plane bundle* is the set of all planes passing through a specific straight line, called the *axis of the bundle*.

Assume two distinct intersecting planes are given:

$$A_1x + B_1y + C_1z + D_1 = 0, \tag{3.3.6}$$

$$A_2x + B_2y + C_2z + D_2 = 0. \tag{3.3.7}$$

Theorem 3.4. *If α and β are two real numbers that do not simultaneously vanish, then the equation*

$$\alpha(A_1x + B_1y + C_1z + D_1) + \beta(A_2x + B_2y + C_2z + D_2) = 0 \tag{3.3.8}$$

is the equation of a plane belonging to the plane bundle determined by the planes (3.3.6) and (3.3.7). Conversely, any plane in this bundle can be represented by an equation of the form (3.3.8) with a specific choice of constants α and β, where neither is zero simultaneously.

The proof of this theorem is perfectly analogous to the proof of the similar theorem for bundles of lines in a plane and will therefore not be reproduced here.

Unlike the case of lines in a plane, where only families of lines passing through a point (i.e., line bundles) are considered, in the case of planes in space, besides plane bundles (which pass through a line), we can also consider other notable families of planes, namely those passing through a point. We begin with the following definition:

Definition 3.4. A *pencil of planes* is the set of all planes passing through a given point, called the *centre of the pencil of planes*.

Assume the centre of the pencil of planes is given by its coordinates $S(x_0, y_0, z_0)$. Then, it is evident that any plane passing through the centre (and thus belonging to the pencil) can be written in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0, \tag{3.3.9}$$

where the real constants A, B, C are not all equal to zero. Conversely, for any constants A, B, C that are not all zero, the equation (3.3.9) represents the equation of a plane passing through the centre of the pencil.

As with bundles, however, the centre of the pencil of planes is often not explicitly given but described by the equations of planes passing through this point. It is useful to describe the planes of the pencil using a reduced number of planes (precisely three) that uniquely determine the centre of this pencil. We have the following result:

Theorem 3.5. *Let*

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases} \tag{3.3.10}$$

be the equations of three planes passing through the point $S(x_0, y_0, z_0)$ such that the condition

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0 \tag{3.3.11}$$

is satisfied. Then for any real numbers α, β, γ that do not simultaneously vanish, the equation

$$\alpha(A_1x+B_1y+C_1z+D_1)+\beta(A_2x+B_2y+C_2z+D_2)+\gamma(A_3x+B_3y+C_3z+D_3) = 0 \tag{3.3.12}$$

describes a plane in the pencil of planes centred at S. Conversely, any plane in this pencil can be described by an equation of this type for a specific choice of constants α, β, γ.

Proof First, rewrite equation (3.3.12) in the form

$$(A_1\alpha+A_2\beta+A_3\gamma)x+(B_1\alpha+B_2\beta+B_3\gamma)y+(C_1\alpha+C_2\beta+C_3\gamma)z+D_1\alpha+D_2\beta+D_3\gamma = 0. \tag{3.3.13}$$

For this equation to indeed represent a plane, its coefficients must not all vanish simultaneously.

Assume the coefficients of x, y, z in this equation are all zero:

$$\begin{cases} A_1\alpha + A_2\beta + A_3\gamma = 0, \\ B_1\alpha + B_2\beta + B_3\gamma = 0, \\ C_1\alpha + C_2\beta + C_3\gamma = 0. \end{cases} \tag{3.3.14}$$

By virtue of equation (3.3.11), the system of equations (3.3.14) with unknowns α, β, γ only admits the trivial solution $\alpha = \beta = \gamma = 0$, contradicting the assumption on the parameters α, β, γ. Therefore, equation (3.3.13) (and hence the equivalent equation (3.3.12)) is linear in x, y, z and thus represents a plane. Since the point S lies on each of the planes (3.3.10), it also satisfies equation (3.3.12), meaning the plane belongs to the pencil.

Now let

$$Ax + By + Cz + D = 0 \tag{3.3.15}$$

be an arbitrary plane in the pencil.

Consider the system of equations

$$\begin{cases} A_1\alpha + A_2\beta + A_3\gamma = A, \\ B_1\alpha + B_2\beta + B_3\gamma = B, \\ C_1\alpha + C_2\beta + C_3\gamma = C, \end{cases} \tag{3.3.16}$$

with unknowns α, β, γ. By virtue of relation (3.3.11), this system admits a unique solution. Since each of the planes (3.3.10) and the plane (3.3.15) passes through the point $S(x_0, y_0, z_0)$, we have

$$\begin{cases} D_1 = -A_1x_0 - B_1y_0 - C_1z_0, \\ D_2 = -A_2x_0 - B_2y_0 - C_2z_0, \\ D_3 = -A_3x_0 - B_3y_0 - C_3z_0, \\ D = -Ax_0 - By_0 - Cz_0. \end{cases}$$

From this and the equalities (3.3.16), we obtain

$$D_1\alpha + D_2\beta + D_3\gamma = D.$$

Thus, for the chosen values of α, β, γ, equations (3.3.12) and (3.3.15) coincide. □

3.3.4 Relative Position of a Line to a Plane

Consider a plane Π given by the general equation

$$Ax + By + Cz + D = 0 \tag{3.3.17}$$

and a line Δ given by its parametric equations

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \\ z = z_0 + nt. \end{cases} \tag{3.3.18}$$

We need to determine the position of the line Δ relative to the plane Π.

It is clear from geometric considerations that the following cases are possible:

(i) The line intersects the plane at a point.

(ii) The line is parallel to the plane but not contained in it.

(iii) The line is contained in the plane.

We will determine the relationship between the coefficients of the plane and those of the line for each of the three cases.

If we substitute the expressions for x, y, z from the parametric equations of the line Δ into the equation of the plane Π, we obtain:

$$(Al + Bm + Cn)t + Ax_0 + By_0 + Cz_0 + D = 0. \tag{3.3.19}$$

The solution of this equation in t represents the parameter value on the line corresponding to the point(s) of intersection between the line and the plane. It is easy to see that the equation admits a unique solution if and only if the coefficient of t is nonzero, i.e.,

$$Al + Bm + Cn \neq 0.$$

The geometric significance of this condition is clear: the direction vector of the line is not perpendicular to the normal vector of the plane, meaning the line is not parallel to the plane. Therefore, this condition is the requirement for the *line and plane to intersect at a point*.

If the condition

$$Al + Bm + Cn = 0, \quad Ax_0 + By_0 + Cz_0 + D \neq 0,$$

is satisfied, the line is parallel to the plane but does not intersect it. Indeed, the first condition indicates that the line is parallel to the plane, while the second condition indicates that the equation has no solution.

Finally, if the condition

$$Al + Bm + Cn = 0, \quad Ax_0 + By_0 + Cz_0 + D = 0,$$

is satisfied, the line is contained in the plane because, in this case, the intersection equation reduces to an identity, which is satisfied for any real t.

3.3.5 Equation of a Plane Determined by Two Intersecting Lines

Consider the lines

$$(D_1): \frac{x - x_0}{l_1} = \frac{y - y_0}{m_1} = \frac{z - z_0}{n_1} \tag{3.3.20}$$

and

$$(D_2): \frac{x - x_0}{l_2} = \frac{y - y_0}{m_2} = \frac{z - z_0}{n_2}, \tag{3.3.21}$$

which pass through the point $M_0(x_0, y_0, z_0)$.

The plane passing through these two lines is, in fact, the plane passing through the point M_0 and parallel to the vectors $\mathbf{v}_1(l_1, m_1, n_1)$ and $\mathbf{v}_2(l_2, m_2, n_2)$, so its equation is

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0. \tag{3.3.22}$$

3.3.6 Equation of a Plane Determined by a Line and a Point

Consider the line

$$(D): \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \tag{3.3.23}$$

and the point $M_2(x_2, y_2, z_2)$, which does not lie on the line. The plane we seek is the one that passes through the point $M_1(x_1, y_1, z_1)$ and is parallel to the vectors $\mathbf{v}(l, m, n)$ and $\overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Hence, its equation is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0. \tag{3.3.24}$$

3.3.7 Equation of a Plane Determined by Two Parallel Lines

Consider the parallel (and distinct) lines

$$(D_1): \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \tag{3.3.25}$$

and

$$(D_2): \frac{x - x_2}{l} = \frac{y - y_2}{m} = \frac{z - z_2}{n}, \tag{3.3.26}$$

which pass through the points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$, respectively. The plane we seek is the one that passes through the point $M_1(x_1, y_1, z_1)$ and is parallel to the vectors $\mathbf{v}(l, m, n)$ and $\overrightarrow{M_1M_2}(x_2-x_1, y_2-y_1, z_2-z_1)$. Therefore, its equation is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0. \tag{3.3.27}$$

3.3.8 Projection of a Line onto a Plane

Consider the line

$$(D): \frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} \tag{3.3.28}$$

and the plane

$$(P): Ax + By + Cz + D = 0. \tag{3.3.29}$$

It is straightforward to verify that if we orthogonally project all points of the line (D) onto the plane (P), we obtain a line situated in the plane, which we shall call the *projection of the line* (D) *onto the plane* (P). If the line is *perpendicular* to the plane, then this line reduces to a single point, the point where the line intersects the plane. Hence, in what follows, we assume that the line is *not* perpendicular to the plane.

The line we seek is written as the intersection of two planes: the plane (P) and the plane (P'), which passes through the line (D) and is perpendicular to the plane (P). In practice, this plane is the one that passes through the point $M_0(x_0, y_0, z_0)$ on the line and is parallel to the direction vector of the line, $\mathbf{v}(l, m, n)$, and the normal vector to the plane (P), $\mathbf{n}(A, B, C)$. Due to the above assumption, these two vectors are not collinear, so the point and the two vectors uniquely determine the plane (P').

As we have seen, the equation of the plane (P') is

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ l & m & n \\ A & B & C \end{vmatrix} = 0. \tag{3.3.30}$$

Thus, the equations of the projection of the line onto the plane are

$$\begin{cases} \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ l & m & n \\ A & B & C \end{vmatrix} = 0, \\ Ax + By + Cz + D = 0. \end{cases} \tag{3.3.31}$$

3.3.9 Projection of a Point onto a Line in Space

Let

$$\Delta : \frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$$

be a line and $M_0(x_0, y_0, z_0)$ a point in space. If $M_0 \in \Delta$, then the *projection* of M_0 onto Δ is simply M_0. Otherwise, the projection is the foot of the perpendicular dropped from the point M_0 onto the line Δ.

Let us assume the latter case. We first determine the equations of the perpendicular from M_0 to the line Δ.

3.3.10 Relative Position of Two Lines in Space

Assume two lines in space are given by their parametric equations:

$$x = x_1 + l_1 t,\ y = y_1 + m_1 t,\ z = z_1 + n_1 t, \tag{3.3.32}$$
$$x = x_2 + l_2 s,\ y = y_2 + m_2 s,\ z = z_2 + n_2 s, \tag{3.3.33}$$

and we wish to determine their relative position.

From geometric considerations, it is clear that the following cases are possible:

(a) The lines intersect.

(b) The lines coincide.

(c) The lines are parallel but do not coincide.

(d) The lines are skew (non-coplanar).

We now determine the relationships between the coefficients of the two lines for each case.

Consider the direction vectors of the two lines:

$$\mathbf{a}_1(l_1, m_1, n_1),\ \mathbf{a}_2(l_2, m_2, n_2).$$

Assume these vectors are collinear, i.e.,

$$l_1 = \lambda l_2,\ m_1 = \lambda m_2,\ n_1 = \lambda n_2. \tag{3.3.34}$$

Then the lines are parallel, i.e., they either coincide or are parallel without any common points. The lines coincide if and only if the vector $\overrightarrow{M_1M_2}$, where $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$, is parallel to the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, i.e.,

$$x_2 - x_1 = \mu l_1,\ y_2 - y_1 = \mu m_1,\ z_2 - z_1 = \mu n_1. \tag{3.3.35}$$

Thus, the equalities (3.3.34) and (3.3.35) represent the necessary and sufficient conditions for the lines (3.3.32) and (3.3.33) to coincide. For the lines to be parallel but not coincident, condition (3.3.34) must be satisfied, while condition (3.3.35) must not.

Now suppose the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are not collinear, i.e., condition (3.3.34) is not satisfied. Then the lines (3.3.32) and (3.3.33) either intersect at a point or are skew. If they intersect and are thus in the same plane Π, then the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\overrightarrow{M_1M_2}$ are coplanar. Therefore:

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0. \tag{3.3.36}$$

Conversely, suppose the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are not collinear, and condition (3.3.36) is satisfied. Choosing points A_1 and A_2 such that $\overrightarrow{M_1A_1} = \mathbf{a}_1$ and $\overrightarrow{M_2A_2} = \mathbf{a}_2$, the segments M_1M_2, M_1A_1, and M_2A_2 define a plane in which the lines (3.3.32) and (3.3.33) lie. Since the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are not collinear, the lines intersect. Thus, the lines (3.3.32) and (3.3.33) intersect if and only if their direction vectors are not collinear and condition (3.3.36) is satisfied. Note that this equality also holds if the lines are parallel, as in this case the second and third rows of the determinant are proportional. Therefore, the necessary condition for the lines to be *skew* is

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \neq 0.$$

In the remainder of this chapter, we assume the reference frame is orthonormal.

3.3.11 Distance from a Point to a Line in Space

We now derive a formula for the distance d from a point M_1, with position vector $\mathbf{r}_1$, in space to a line Δ given by the vector equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}$. First, we choose a point M_2 on the line Δ such that $\overrightarrow{M_0M_2} = \mathbf{a}$.

We construct a parallelogram based on the vectors $\overrightarrow{M_0M_1}$ and $\overrightarrow{M_0M_2}$. The distance d is equal to the length of the perpendicular from M_1 to the opposite side of the parallelogram. Since the area of the parallelogram is

$$\|(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{a}\| = \|\mathbf{a}\| d,$$

the formula

$$d = \frac{\|(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{a}\|}{\|\mathbf{a}\|}$$

gives the distance from the point M_1, with position vector $\mathbf{r}_1$, to the line Δ.

3.3.12 Common Perpendicular of Two Skew Lines

Consider two lines in space:

$$(\Delta_1) : \frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \tag{3.3.37}$$

and

$$(\Delta_2) : \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}, \tag{3.3.38}$$

such that the two lines are skew, i.e.,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \neq 0.$$

As is known from elementary geometry, there exists a unique line that intersects both given lines and is perpendicular to each of them. This line is called the *common perpendicular* of the two lines.

The method for constructing the common perpendicular is straightforward. First, we determine the direction vector of this perpendicular. Let $\mathbf{a}_1(l_1, m_1, n_1)$ and $\mathbf{a}_2(l_2, m_2, n_2)$ be the direction vectors of the lines Δ_1 and Δ_2, respectively. Then the vector $\mathbf{a} =$

$\mathbf{a}_1 \times \mathbf{a}_2$ is, by definition, perpendicular to both $\mathbf{a}_1$ and $\mathbf{a}_2$ and is therefore a direction vector of the common perpendicular. Since

$$\mathbf{a}_1 \times \mathbf{a}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix},$$

the components of this vector are

$$\beta_1 \equiv \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}, \quad \beta_2 \equiv \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}, \quad \beta_3 \equiv \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}. \tag{3.3.39}$$

Now we write the equations of the common perpendicular as the intersection of two planes: one that passes through the first line and is perpendicular to the second, and another that passes through the second line and is perpendicular to the first.

The first plane can thus be written as:

$$\pi_1 : \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = 0.$$

Indeed, this plane passes through the line Δ_1 and is parallel to the common perpendicular, and therefore, in particular, is perpendicular to the line Δ_2. Similarly, the second plane is given by:

$$\pi_2 : \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = 0,$$

and thus the equations of the common perpendicular are:

$$\begin{cases} \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = 0, \\ \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = 0. \end{cases}$$

3.3.13 Length of the Common Perpendicular of Two Skew Lines

Consider again the skew lines (3.3.37) and (3.3.38). Also, consider the plane π that passes through the first line and is parallel to the second, i.e., the plane with equation

$$\pi : \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

The normal vector to this plane is clearly $\mathbf{n}(\beta_1, \beta_2, \beta_3)$, where $\beta_1, \beta_2, \beta_3$ are given by (3.3.39).

The length of the common perpendicular (i.e., the distance between the given skew lines) is equal to the distance from any point on the line Δ_2 (e.g., the point $M_2(x_2, y_2, z_2)$) to the plane π. Using the formula for the distance from a point to a plane, we obtain:

$$d(\Delta_1, \Delta_2) = d(M_2, \pi) = \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}} \right| .$$

3.3.14 Angle Between a Line and a Plane

Suppose the line is given by

$$x = x_0 + lt, \ y = y_0 + mt, \ z = z_0 + nt \tag{3.3.40}$$

and the plane is given by

$$Ax + By + Cz + D = 0. \tag{3.3.41}$$

Let φ denote the angle between the line and the plane (specifically, the angle between the line and its projection onto the plane). If the line is perpendicular to the plane, we set $\varphi = \pi/2$. We assume $0 \leq \varphi \leq \pi/2$. Since the vector $\mathbf{n}(A, B, C)$ is perpendicular to the plane (3.3.41), the angle formed by the direction vector $\mathbf{a}(l, m, n)$ of the line (3.3.40) with the vector $\mathbf{n}$ is either $\psi = \pi/2 - \varphi$ or $\psi = \pi/2 + \varphi$. Therefore,

$$\sin \varphi = |\cos \psi| = \frac{|Al + Bm + Cn|}{\sqrt{A^2 + B^2 + C^2}\sqrt{l^2 + m^2 + n^2}}.$$

The condition for the line to be parallel to the plane is that the normal vector to the plane is perpendicular to the direction vector of the line, i.e.,

$$Al + Bm + Cn = 0,$$

while the condition for the line to be perpendicular to the plane is that the normal vector to the plane is parallel to the direction vector of the line, i.e.,

$$\frac{A}{l} = \frac{B}{m} = \frac{C}{n}.$$

3.4 Probleme

Problem 3.1. Stabiliţi ecuaţia unui plan care trece prin $A(1,3,0)$ şi este paralel cu dreptele

$$\begin{cases} x + y - z + 3 = 0 \\ 2x - y + 5z + 1 = 0 \end{cases}$$

şi

$$\begin{cases} -x + y = 1 \\ 5x + y - z + 2 = 0 \end{cases}.$$

Problem 3.2. Stabiliţi ecuaţia unui plan care trece prin dreapta

$$\frac{x-1}{3} = \frac{y+2}{4} = \frac{z-1}{2}$$

şi este paralel cu dreapta

$$\frac{x}{5} = \frac{y-1}{4} = \frac{z+1}{3}.$$

Problem 3.3. Stabiliţi ecuaţia unui plan care trece prin dreapta

$$\begin{cases} x = 3 + t, \\ y = 2 + 5t, \\ z = -1 + 3t, \end{cases}$$

şi este paralel cu dreapta

$$\begin{cases} x = 4 - 2t, \\ y = -8 + t, \\ z = 5 + 2t. \end{cases}$$

Problem 3.4. Determinaţi ecuaţia unui plan care trece prin punctul $A(-1,1,2)$ şi prin dreapta de ecuaţii

$$\begin{cases} x = 1 + 5t, \\ y = -1 + t, \\ z = 2t. \end{cases}$$

Problem 3.5. Determinaţi ecuaţia unui plan care trece prin punctul $A(-1,1,2)$ şi prin dreapta de ecuaţii

$$\begin{cases} x + 5y - 7z + 1 = 0, \\ 3x - y + 2z + 3 = 0. \end{cases}$$

Problem 3.6. Stabiliţi ecuaţia planului care trece prin dreptele paralele

$$\frac{x-1}{5} = \frac{y+2}{3} = \frac{z-1}{1} \quad \text{şi} \quad \frac{x-2}{5} = \frac{y}{3} = \frac{z+3}{1}.$$

Problem 3.7. Demonstraţi că următoarele două drepte sunt concurente şi stabiliţi ecuaţia planului determinat de ele:

$$\frac{x+1}{-2} = \frac{y-2}{1} = \frac{z-5}{4} \quad \text{şi} \quad \frac{x+5}{2} = \frac{y+8}{3} = \frac{z-4}{-1}.$$

Problem 3.8. Demonstraţi că următoarele două drepte sunt concurente şi stabiliţi ecuaţia planului determinat de ele:

$$\begin{cases} x = 1 + 3t, \\ y = -1 + 4t, \\ z = 2 + 5t \end{cases} \qquad \text{şi} \qquad \begin{cases} x = 1 - t, \\ y = -1 + 2t, \\ z = 2 + 4t. \end{cases}$$

Problem 3.9. O dreaptă se proiectează pe planul yOz, paralel cu axa Ox. Stabiliţi ecuaţia proiecţiei dacă dreapta este dată de ecuaţiile:

$$\begin{cases} x = 1 + 2t, \\ y = 3t, \\ z = 1 - t. \end{cases}$$

Problem 3.10. Stabiliţi ecuaţiile unei drepte care trece prin $O(0,0,0)$ şi intersectează dreptele

$$\begin{cases} x - y + z + 2 = 0, \\ x - 2y + 3z - 8 = 0 \end{cases} \qquad \text{şi} \qquad \begin{cases} y - z + 1 = 0, \\ x + y - 2z + 4 = 0. \end{cases}$$

Problem 3.11. Stabiliţi ecuaţiile unei drepte care trece prin $O(0,0,0)$ şi intersectează dreptele

$$\begin{cases} x = 1 + 2t, \\ y = 2 + 3t, \\ z = -t \end{cases} \quad \text{şi} \quad \begin{cases} x = 4t, \\ y = 5 - 5t, \\ z = 3 + 2t. \end{cases}$$

Problem 3.12. Stabiliţi ecuaţiile unei drepte care trece prin $A(-1,1,-1)$ şi intersectează dreptele

$$\begin{cases} x - y + z + 2 = 0, \\ x - 2y + 3z - 8 = 0 \end{cases} \quad \text{şi} \quad \begin{cases} y - z = 0, \\ x + y - 2z + 4 = 0. \end{cases}$$

Problem 3.13. În fascicolul de plane determinat de planele $x + 2y - 3z + 5 = 0$ şi $4x - y + 3z + 5 = 0$ să se determine două plane perpendiculare, dintre care unul dintre care să treacă prin punctul $M(1,3,1)$.

Problem 3.14. Să se determine coordonatele unui punct A situat pe dreapta

$$\frac{x-3}{2} = \frac{y}{3} = \frac{z-1}{-1}$$

şi este egal de părtat de punctele $B(3,0,-2)$ şi $C(-1,1,5)$.

Problem 3.15. Determinaţi coordonatele unui punct A, situat pe dreapta

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{1},$$

aflat la distanţa $\sqrt{3}$ faţă de planul $x + y + z + 3 = 0$.

Problem 3.16. Determinaţi coordonatele unui punct A, situat pe dreapta

$$\frac{x-1}{1} = \frac{y-3}{3} = \frac{z+4}{-5},$$

egalo depărtat de punctul $B(0,1,1)$ şi de planul $2x - y + 2z + 1 = 0$.

Problem 3.17. Punctele $A(1,-1,2)$ şi $B(3,0,4)$ sunt vârfuri ale cubului $ABCDA_1B_1C_1D_1$. Vectorul $\overrightarrow{AD}$ este perpendicular pe dreapta

$$\begin{cases} x = 0, \\ y - z = 0, \end{cases}$$

iar orientarea bazei $\left\{\overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AA_1}\right\}$ este directă, în timp ce suma coordonatelor vectorului $\overrightarrow{AA_1}$ este negativă. Stabiliţi ecuaţiile feţelor cubului.

Problem 3.18. Stabiliţi ecuaţiile simetricei dreptei

$$\frac{x-2}{3} = \frac{y+1}{1} = \frac{z-2}{4}$$

faţă de planul $5x - y + z - 4 = 0$.

Problem 3.19. Stabiliţi ecuaţiile proiecţiei ortogonale pe planul $x + 5y - z - 25 = 0$ ale dreptei

$$\begin{cases} x - y + 2z - 1 = 0, \\ 3x - y + 2z = 0. \end{cases}$$

Problem 3.20. Stabiliţi ecuaţiile proiecţiei ortogonale pe planul $x + 5y - z - 25 = 0$ ale dreptei

$$\frac{x+1}{1} = \frac{y}{5} = \frac{z-1}{-1}.$$

Problem 3.21. Determinaţi unghiul dintre planul $4x + 4y - 7z + 1 = 0$ şi dreapta

$$\begin{cases} x + y + z + 1 = 0, \\ 2x + y + 3z + 2 = 0. \end{cases}$$

Problem 3.22. Stabiliţi ecuaţiile dreptei care trece prin $A(1, 3, 2)$, este paralelă cu planul xOy şi formează un unghi de $45°$ cu dreapta

$$\begin{cases} x = z, \\ z = 0. \end{cases}$$

Problem 3.23. Stabiliţi ecuaţiile dreptei care trece prin $A(1, 3, 2)$, este paralelă cu planul xOy şi formează un unghi de $\arcsin(1\sqrt{10})$ cu planul $x - y = 1$.

Problem 3.24. Stabiliţi ecuaţia planului care trece prin $A(-1, 2, 1)$, este paralel cu dreapta

$$\frac{x}{2} = \frac{y}{-3} = \frac{z}{-1}$$

şi formează un unghi de $60°$ cu dreapta

$$\begin{cases} x = z, \\ z = 0. \end{cases}$$

Problem 3.25. Laturile egale ale unui triunghi isoscel se intersectează în vârful $A(3,4,5)$. Celelalte două vârfuri sunt situate pe axele Ox şi Oy, iar planul triunghiului este paralel cu axa Oz. Determinaţi unghiurile triunghiului şi scrieţi ecuaţia planului său.

Problem 3.26. Determinaţi coordonatele unui punct A de pe dreapta

$$\begin{cases} x - y - 3 = 0, \\ 2y + z = 0, \end{cases}$$

situat la distanţa $\sqrt{6}$ faţă de dreapta $x = y = z$.

Problem 3.27. Determinaţi distanţa dintre dreptele

$$\frac{x-4}{3} = \frac{y+1}{6} = \frac{z-1}{-2} \quad \text{şi} \quad \frac{x-5}{-6} = \frac{y}{-12} = \frac{z}{4}.$$

Problem 3.28. Punctele $A(-1,-3,1)$, $B(5,3,8)$, $C(-1,-3,5)$ şi $D(2,1,-4)$ sunt vârfurile unui tetraedru. Determinaţi înălţimea tetraedrului coborâtă din vârful D pe faţa ABC.

Problem 3.29. Punctele $A(-1,-3,1)$, $B(5,3,8)$, $C(-1,-3,5)$ şi $D(2,1,-4)$ sunt vârfurile unui tetraedru. Determinaţi înălţimea feţei ABC, coborâte din C pe latura AB.

Problem 3.30. Punctele $A(-1,-3,1)$, $B(5,3,8)$, $C(-1,-3,5)$ şi $D(2,1,-4)$ sunt vârfurile unui tetraedru. Determinaţi distanţa dintre muchiile (strâmbe) AD şi BC.

Problem 3.31. Lungimea muchiei cubului $ABCDA_1B_1C_1D_1$ este egală cu 1. Determinaţi distanţa dintre vârful A şi planul B_1CD_1.

Problem 3.32. Feţele $ABCD$, ABB_1A_1 şi ADD_1A_1 ale paralelipipedului $ABCDA_1B_1C_1D_1$ sunt situate în planele $2x + 3y + 4z + 8 = 0$, $x + 3y - 6 = 0$, respectiv $z + 5 = 0$. Vârful C_1 are coordonatele $(6,-5,1)$. Determinaţi distanţa de la vârful A_1 la planul B_1BD.

Problem 3.33. Feţele $ABCD$, ABB_1A_1 şi ADD_1A_1 ale paralelipipedului $ABCDA_1B_1C_1D_1$ sunt situate în planele $2x + 3y + 4z + 8 = 0$, $x + 3y - 6 = 0$, respectiv $z + 5 = 0$. Vârful C_1 are coordonatele $(6,-5,1)$. Determinaţi distanţa de la vârful D la dreapta AB.

Problem 3.34. Feţele $ABCD$, ABB_1A_1 şi ADD_1A_1 ale paralelipipedului $ABCDA_1B_1C_1D_1$ sunt situate în planele $2x + 3y + 4z + 8 = 0$, $x + 3y - 6 = 0$, respectiv $z + 5 = 0$. Vârful C_1 are coordonatele $(6,-5,1)$. Determinaţi distanţa dintre dreptele AC şi A_1C_1.

CHAPTER 4

Conics in Canonical Form

4.1 Ellipse

Definition and Canonical Equation

Definition 4.1. An *ellipse* is defined as the geometric locus of points in a plane such that the sum of their distances to two fixed points F_1 and F_2, called *foci*, is constant, equal to $2a$, where the distance between the two foci is $2c$, and c is a positive or zero real number, satisfying the inequality $c < a$.

Obviously, if the foci coincide, the ellipse is a circle.

To derive the equation of the ellipse, we construct an orthonormal coordinate system in the plane as follows. We choose as the origin the midpoint of the segment F_1F_2, and we choose the Ox axis as the line F_1F_2, oriented such that F_2 has a positive abscissa. Hence, the coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$. The Oy axis is chosen such that it completes a right-handed orthonormal system (so, in particular, as a line, this axis is simply the perpendicular bisector of the segment F_1F_2). If the ellipse happens to be a circle (that is, $c = 0$), then any right-handed orthonormal coordinate system with the origin at the circle's centre will satisfy our needs.

Let $M(x, y)$ be an arbitrary point on the ellipse. Then, on the one hand,

$$F_1M = \sqrt{(x+c)^2 + y^2}, \quad F_2M = \sqrt{(x-c)^2 + y^2}. \tag{4.1.1}$$

On the other hand, from the definition of the ellipse, we must have

$$F_1M + F_2M = 2a. \tag{4.1.2}$$

Combining these two equations, we obtain:

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a. \tag{4.1.3}$$

This is, in fact, the equation of the ellipse, since the points of the ellipse, and only they, satisfy this equation.

We shall now establish a different, more appealing form for equation (4.1.3) of the ellipse. Transposing the last term to the right-hand side and squaring both sides, we obtain:

$$a\sqrt{(x-x)^2 + y^2} = a^2 - cx.$$

Squaring again and regrouping terms, we get:

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

or, equivalently,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{4.1.4}$$

We now introduce a new quantity,

$$b = \sqrt{a^2 - c^2}.$$

According to our assumptions, this quantity is real. Thus,

$$b^2 = a^2 - c^2, \tag{4.1.5}$$

and consequently, equation (4.1.4) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4.1.6}$$

We have shown that every point on the ellipse satisfies equation (4.1.6). We will now demonstrate that the converse is also true, meaning that any point $M(x, y)$ whose coordinates satisfy equation (4.1.6) is a point on the ellipse, i.e., satisfies equation (4.1.2). From equation (4.1.6), we get

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right).$$

Using this relationship and equality (4.1.5), we find

$$F_1M = \sqrt{(x+c)^2 + y^2} = \sqrt{x^2 + 2cx + c^2 + b^2 - \frac{b^2}{a^2}x^2} =$$
$$= \sqrt{\frac{c^2}{a^2}x^2 + 2cx + a^2} = \left|a + \frac{c}{a}x\right|.$$

Since, by virtue of (4.1.6), $|x| \leq a$ and, moreover, $c < a$, we have

$$F_1M = a + \frac{c}{a}x. \tag{4.1.7}$$

Similarly, we find

$$F_2M = a - \frac{c}{a}x. \tag{4.1.8}$$

Adding these two equalities yields equation (4.1.2). Thus, relation (4.1.6) is the equation of the ellipse. It is called the *canonical equation of the ellipse*.

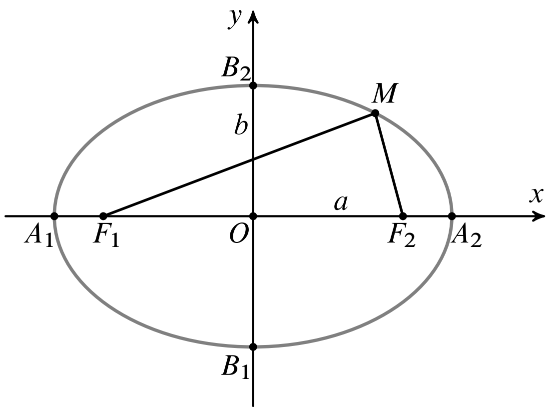


Figure 4.1: The Ellipse

Shape Analysis. Starting from equation (4.1.6), we shall study the shape of the ellipse. The coordinates of points on the ellipse are subject to the restrictions $|x| \leq a$ and $|y| \leq b$. Therefore, the ellipse is bounded by a rectangle with sides $2a$ and $2b$, its sides parallel to the axes, and its centre at the origin. Furthermore, note that only even powers of the coordinates appear in equation (4.1.6), so the ellipse is symmetric with respect to both axes and hence also with respect to the origin. This means that if the point $M(x, y)$ lies on the ellipse, then so do the points $M(-x, y)$, $M(x, -y)$, and $M(-x, -y)$. Thus, to determine the shape of the ellipse, it suffices to consider its portion within the first quadrant, while in the other quadrants, its shape can be obtained by symmetry. For the first quadrant, from equation (4.1.6), we find:

$$y = \frac{b}{a}\sqrt{a^2 - x^2}. \tag{4.1.9}$$

Thus, we study the graph of the function

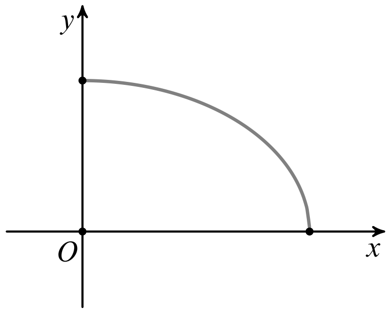


Figure 4.2: The Portion of the Ellipse in Quadrant I

$$f : [0, a] \to \mathbb{R}, \quad f(x) = \frac{b}{a}\sqrt{a^2 - x^2}.$$

It is clear that we do not need to study limits at $\pm\infty$ or asymptotes, so we begin by calculating derivatives. We have:

$$f'(x) = \frac{-bx}{a\sqrt{a^2 - x^2}}.$$

For the second derivative, we obtain:

$$f''(x) = \frac{ab}{(x-a)\,(x+a)\,\sqrt{a^2 - x^2}}.$$

It is immediately observed that both derivatives are negative, so the function is strictly decreasing and concave over its entire domain of definition. This means that its graph is as shown in Figure 4.2.

The symmetry axes of the ellipse (the axes Ox and Oy) are simply called *axes*, and the centre of symmetry (the origin of coordinates) is called the *centre of the ellipse*. The points A_1, A_2, B_1, and B_2, where the axes intersect the ellipse, are called the *vertices* of the ellipse. The term *semi-axes* is used both for the segments OA_1, OA_2, OB_1, and OB_2 and for their lengths, a and b. Under our assumptions, when the foci of the ellipse lie on the Ox axis, from relation (4.1.5), it follows that $a > b$. In this case, a is called the *major semi-axis*, and b the *minor semi-axis*. However, equation (4.1.6) also holds in the case where $a < b$; in this case, it represents an ellipse whose foci lie on the Oy axis instead of Ox, and the major semi-axis equals b. A third possible case is when $a = b$. In this case, equation (4.1.6) reduces to

$$x^2 + y^2 = a^2. \tag{4.1.10}$$

From now on, we will consider the circle as a special case of the ellipse, in which the two semi-axes are equal, and the foci coincide with the centre of the circle.

Eccentricity. The *eccentricity* of an ellipse is defined as the real number

$$\varepsilon = \frac{c}{a}. \tag{4.1.11}$$

Since, by our initial assumption, $c < a$, it follows that $\varepsilon < 1$. In the case of a circle, the foci coincide, so $c = 0$, and the eccentricity is $\varepsilon = 0$.

We can rewrite equality (4.1.11) in the form

$$\varepsilon = \sqrt{1 - \left(\frac{b}{a}\right)^2}.$$

From this, it is evident that the eccentricity determines the shape of the ellipse: the closer ε is to zero, the more the ellipse resembles a circle; as the eccentricity increases, the ellipse becomes progressively more flattened.

Focal Rays. The *focal rays* of a point M on the ellipse are the line segments that connect this point to the foci F_1 and F_2 of the ellipse. Their lengths, r_1 and r_2, are given by formulas (4.1.7) and (4.1.8), respectively, which we can rewrite as

$$r_1 = a + \varepsilon x,$$
$$r_2 = a - \varepsilon x.$$

Intersection with a Line. Tangent to an Ellipse. We will study the number of points of intersection a line can have with an ellipse. We will assume, first, that the line is not parallel to the Oy axis. Then its equation can be written using the slope:

$$y = kx + m. \tag{4.1.12}$$

To find the intersection points of this line with the ellipse (4.1.6), we substitute the expression of y from (4.1.12) into equation (4.1.6). We obtain:

$$\frac{x^2}{a^2} + \frac{(kx+m)^2}{b^2} = 1$$

or

$$(a^2k^2 + b^2)x^2 + 2a^2kmx + a^2(m^2 - b^2) = 0.$$

This equation gives us the abscissas of the intersection points. Since it is a quadratic equation, there will always be two intersection points (distinct, coincident, or imaginary). The discriminant of the equation is:

$$\Delta = 4a^4k^2m^2 - 4a^2(m^2 - b^2)(a^2k^2 + b^2) = 4a^4k^2m^2 - 4a^4k^2m^2 -$$
$$- 4a^2m^2b^2 + 4a^2b^2k^2 + 4a^2b^4 = 4a^2b^2(a^2k^2 + b^2 - m^2).$$

The sign of the discriminant is given by the factor $s = a^2k^2 + b^2 - m^2$. We can therefore have the following cases:

1. if $-\sqrt{a^2k^2+b^2} < m < \sqrt{a^2k^2+b^2}$, then $\Delta > 0$, and the line and the ellipse have two common points;
2. if $m = \pm\sqrt{a^2k^2+b^2}$, then $\Delta = 0$, meaning the line has a single point in common with the ellipse (the line is tangent to the ellipse);
3. if $m \in \left(-\infty, -\sqrt{a^2k^2+b^2}\right) \cup \left(\sqrt{a^2k^2+b^2}, \infty\right)$, then $\Delta < 0$, and the line and the ellipse have no points in common.

Therefore, for any slope k, there exist two tangents to the ellipse having this slope, namely:

$$y = kx \pm \sqrt{a^2k^2 + b^2}. \tag{4.1.13}$$

If the line is parallel to the Oy axis, then its equation is of the form $x = h$. If we substitute into the equation of the ellipse, we obtain:

$$\frac{h^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Figure 4.3: Tangents to an ellipse parallel to a given line

from which

$$y^2 = b^2\left(1 - \frac{h^2}{a^2}\right).$$

It is immediately evident that the line $x = h$ intersects the ellipse in two distinct points if $h \in (-a, a)$, is tangent to the ellipse if $h = \pm a$, and does not intersect the ellipse if $h \in (-\infty, -a) \cup (a, \infty)$.

The problem of the tangent to an ellipse can also be approached in another way. Suppose we want to determine the equation of the tangent at a given point (x_0, y_0) on the ellipse. To be specific, we assume that x_0 and y_0 are both positive. Then they lie on a branch of the ellipse described by the equation:

$$y = b\sqrt{1 - \frac{x^2}{a^2}}.$$

To write the equation of the tangent, we first need the derivative of y at x_0:

$$y'(x_0) = b\frac{-\frac{x_0}{a^2}}{\sqrt{1 - \frac{x_0^2}{a^2}}} = -\frac{b^2}{a^2}\frac{x_0}{y_0},$$

where we have used the fact that the point (x_0, y_0) satisfies the equation of the ellipse. Therefore, as is known from calculus, the equation of the tangent is:

$$y - y_0 = y'(x_0)(x - x_0) = -\frac{b^2}{a^2}\frac{x_0}{y_0}x + \frac{b^2}{y_0}\frac{x_0^2}{a^2}.$$

If we multiply this equation by $\frac{y_0}{b^2}$ we get:

$$\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = -\frac{xx_0}{a^2} + \frac{x_0^2}{a^2}$$

or, again using the equation of the ellipse (for (x_0, y_0)),

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1,$$

that is, the equation of the tangent at a point on an ellipse can be written using the duplication method. The same result is obtained, without difficulty, even if the point on the ellipse lies on one of the other branches.

We now describe another method to obtain the equation of the tangent at a point on an ellipse. Let $M_0(x_0, y_0)$ be a point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The parametric equations of a line passing through M_0 are:

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt. \end{cases}$$

Substituting into the equation of the ellipse, we obtain:

$$b^2(x_0^2 + 2tlx_0 + l^2t^2) + a^2(y_0^2 + 2tmy_0 + m^2t^2) - a^2b^2 = 0$$

or, grouping by powers of t,

$$t^2\left(b^2l^2 + a^2m^2\right) + 2t(a^2lx_0 + b^2my_0) + b^2x_0^2 + a^2y_0^2 - a^2b^2 = 0.$$

Since the point M_0 is on the ellipse, the free term of the equation must be zero, hence the equation reduces to:

$$t^2\left(b^2l^2 + a^2m^2\right) + 2t(b^2lx_0 + a^2my_0) = 0.$$

For the line to be tangent to the ellipse, this equation must have a double root. This happens if and only if the linear term vanishes, that is, if:

$$b^2lx_0 + a^2my_0 = 0.$$

If the direction vector of the line is $\mathbf{v}(l, m)$, the above equation means that the vector $\mathbf{n}(b^2x_0, a^2y_0)$ is perpendicular to the line, i.e., this vector is the normal to the tangent. It follows that the equation of the tangent can be written as:

$$b^2x_0(x - x_0) + a^2y_0(y - y_0) = 0$$

or

$$b^2x_0x + a^2y_0y - b^2x_0^2 - a^2y_0^2 = 0.$$

Again, since the point M_0 is on the ellipse, the constant term is $-a^2b^2$, so the tangent equation becomes:

$$b^2x_0x + a^2y_0y - a^2b^2 = 0$$

or

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1. \tag{4.1.14}$$

Finally, we discuss the case where we are asked to determine the tangents drawn from a point to an ellipse. We start again from the canonical equation of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

As we saw earlier, the equation of a non-vertical tangent to this ellipse can be written as:

$$y = kx + \sqrt{a^2k^2 + b^2}.$$

We require that this tangent passes through a point $M(x_1, y_1)$. This means:

$$y_1 = kx_1 + \sqrt{a^2k^2 + b^2}$$

or

$$(y_1 - kx_1)^2 - a^2k^2 - b^2 = 0$$

or, again,

$$k^2(x_1^2 - a^2) - 2kx_1y_1 + y_1^2 - b^2 = 0. \tag{4.1.15}$$

The discriminant of equation (4.1.15) is

$$\Delta = 4\left(b^2x_1^2 + a^2y_1^2 - a^2b^2\right). \tag{4.1.16}$$

To have two tangents through point M we must have $\Delta > 0$, that is:

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0. \tag{4.1.17}$$

This means that point M lies outside the ellipse. The slopes of the two tangents will be:

$$k_{1,2} = \frac{-x_1y_1 \pm \sqrt{b^2x_1^2 + a^2y_1^2 - a^2b^2}}{x_1^2 - a^2}.$$

To have a single tangent, we must have $\Delta = 0$, that is:

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0. \tag{4.1.18}$$

Figure 4.4: Non-vertical tangents to the ellipse from an exterior point

In this case, the point lies on the ellipse, and the slope of the unique tangent is:

$$k = \frac{-x_1 y_1}{x_1^2 - a^2}.$$

It is easy to verify that in this case the equation of the tangent is precisely the one obtained by duplication, namely:

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0.$$

Finally, we have no tangents if $\Delta < 0$, that is, if:

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0 \tag{4.1.19}$$

or, equivalently, if point M lies inside the ellipse.

We now examine the case where one of the tangents from point M is vertical. It is clear then that this tangent must have the equation:

$$x = \pm a.$$

Hence, we must have $M = M(\pm a, y_1)$. The second tangent from M cannot also be vertical, so its equation must be of the form:

$$y = kx \pm \sqrt{a^2k^2 + b^2}. \tag{4.1.20}$$

For the tangent to pass through M, we must have:

$$y_1 = \pm ka \pm \sqrt{a^2k^2 + b^2} \tag{4.1.21}$$

or, squaring:

$$(y_1 \mp ka)^2 = a^2k^2 + b^2 \tag{4.1.22}$$

that is,

$$y_1^2 \mp 2y_1ka + a^2k^2 = a^2k^2 + b^2$$

or

$$\pm 2y_1ak = y_1^2 - b^2. \tag{4.1.23}$$

Figure 4.5: Tangents to the ellipse, with one vertical tangent

If $y_1 = 0$, then the above equation in k has no solution, which is normal, because in this case M is one of the ellipse's vertices on the Ox axis, so there is only one tangent, the vertical one. Otherwise, from equation (4.1.23) we get:

$$k = \pm \frac{y_1^2 - b^2}{2ay_1}, \tag{4.1.24}$$

the sign being chosen according to the vertex through which the vertical tangent passes.

4.2 Hyperbola

Definition and Derivation of the Canonical Form Consider two points F_1 and F_2 in a plane, separated by a distance of $2c$. Additionally, let a be a real number satisfying the inequality:

$$0 < a < c. \tag{4.2.1}$$

Definition 4.2. A *hyperbola* is the geometric figure consisting of all points in the plane such that the absolute value of the difference of their distances to the fixed points F_1 and F_2 is constant, equal to $2a$. The points F_1 and F_2 are called the *foci* of the hyperbola.

The inequality (4.2.1) is necessary: if $a = 0$, the figure degenerates into a line (the perpendicular bisector of the segment F_1F_2), whereas if $a > c$, the figure becomes the empty set.

We now derive the equation of the hyperbola. To do so, we choose a Cartesian coordinate system where the Ox-axis coincides with the line F_1F_2, oriented from F_1 to F_2, and the origin is at the midpoint of the segment F_1F_2. Thus, the coordinates of the foci are $F_1(-c, 0)$ and $F_2(c, 0)$. For a point $M(x, y)$ on the hyperbola, the definition implies:

$$|F_1M - F_2M| = 2a,$$

or equivalently:

$$F_1M - F_2M = \pm 2a. \tag{4.2.2}$$

Substituting the expressions:

$$F_1M = \sqrt{(x+c)^2 + y^2}, \quad F_2M = \sqrt{(x-c)^2 + y^2},$$

into equation (4.2.2), we obtain:

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a. \tag{4.2.3}$$

This is the equation of the hyperbola. To simplify it, we move the second square root to the right-hand side and square both sides:

$$cx-a^2=\pm a\sqrt{(x-c)^2+y^2}.$$

Squaring again and simplifying, we find:

$$(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2),$$

or:

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1. \tag{4.2.4}$$

Introducing:

$$b=\sqrt{c^2-a^2},$$

which is real due to inequality (4.2.1), we have:

$$b^2=c^2-a^2, \tag{4.2.5}$$

and equation (4.2.4) becomes:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{4.2.6}$$

We have now demonstrated that all points of the hyperbola, which satisfy equation (4.2.3), also satisfy equation (4.2.6). We will now show that the converse is also true. Let $M(x,y)$ be a point satisfying equation (4.2.6). Then:

$$y^2=b^2\left(\frac{x^2}{a^2}-1\right).$$

Using this relationship and equality (4.2.5), we obtain:

$$\begin{aligned} F_1M &= \sqrt{(x+c)^2+y^2}=\sqrt{x^2+2cx+c^2+\frac{b^2}{a^2}x^2-b^2}= \\ &= \sqrt{\frac{c^2}{a^2}x^2+2cx+a^2}=\left|\frac{c}{a}x+a\right|. \end{aligned} \tag{4.2.7}$$

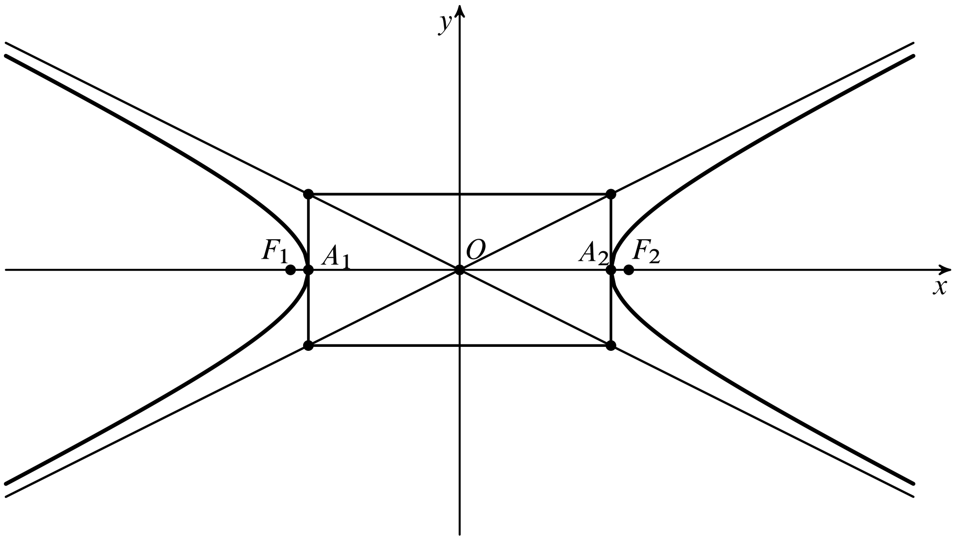


Figure 4.6: Hyperbola

Similarly, we find:

$$F_2M = \left|\frac{c}{a}x - a\right|. \tag{4.2.8}$$

Since $|x| \geq a$ follows from equation (4.2.6), and given the inequality (4.2.1), $c > a$, we conclude that for $x \geq a$ the formulas (4.2.7) and (4.2.8) yield:

$$F_1M = \frac{c}{a}x + a, \quad F_2M = \frac{c}{a}x - a. \tag{4.2.9}$$

Thus:

$$F_1M - F_2M = 2a.$$

For $x \leq -a$, we have:

$$F_1M = -\frac{c}{a}x - a, \quad F_2M = -\frac{c}{a}x + a. \tag{4.2.10}$$

Therefore:

$$F_1M - F_2M = -2a.$$

Thus, any point that satisfies equation (4.2.6) also satisfies equation (4.2.2) and, consequently, equation (4.2.3). Therefore, equation (4.2.6) is equivalent to the equation of the hyperbola. It is called the *canonical equation of the hyperbola*.

Study of the Hyperbola's Shape. Asymptotes. From equation (4.2.6), it is immediately apparent that $|x| \geq a$. This means that the hyperbola is entirely located outside the vertical strip bounded by the lines $x = -a$ and $x = a$.

As in the case of the ellipse, the canonical equation of the hyperbola contains only even powers of the variables x and y. Consequently, the hyperbola has two axes of symmetry (the coordinate axes) and a centre of symmetry (the origin). It is therefore sufficient to study the hyperbola's shape in the first quadrant of the coordinate axes, as it can be deduced in the other three quadrants by symmetry. In the first quadrant, we obtain from equation (4.2.6):

$$y = \frac{b}{a}\sqrt{x^2 - a^2}, \quad x \geq a. \tag{4.2.11}$$

The graph of this function, which starts from the point $A(a, 0)$, extends infinitely to the right and upwards. It is easy to see that the line:

$$y = \frac{b}{a}x \tag{4.2.12}$$

is an oblique asymptote to this graph as $x \to +\infty$. By symmetry, it is also clear that the lines:

$$y = \pm\frac{b}{a}x$$

are both oblique asymptotes to the hyperbola's graph, both as $x \to +\infty$ and as $x \to -\infty$.

In the case of the ellipse, we noted the existence of a rectangle centred at the origin, with sides $2a$ and $2b$ parallel to the coordinate axes, which contains the entire ellipse and is tangent to it. In the case of the hyperbola, this same rectangle plays a similar role, with the following differences:

- The hyperbola lies outside the rectangle.
- Only two of the rectangle's sides are tangent to the hyperbola.
- The diagonals of the rectangle (or their supporting lines) are the asymptotes of the hyperbola.

The hyperbola consists of two branches. The "+" sign in equation (4.2.2) corresponds to the right branch, while the "−" sign corresponds to the left branch. The centre of symmetry of the hyperbola is simply called the *centre* of the hyperbola. Its axes of symmetry are called the *axes of the hyperbola*. More precisely, the axis that intersects the hyperbola is called the *real axis*, while the axis that does not intersect it is called the *imaginary axis*. The points A_1 and A_2, where the real axis intersects the hyperbola, are called the *vertices* of the hyperbola. Additionally, as in the case of the ellipse, the numbers a and b are called the *semi-axes* of the hyperbola. If $a = b$, the hyperbola is called *rectangular*. It is easy to see that in the case of a rectangular hyperbola, the asymptotes form a 45° angle with the Ox-axis.

In addition to the hyperbola (4.2.6), one can also consider the curve with the equation:

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4.2.13}$$

It is easily verified that this curve is also a hyperbola, with foci located on the Oy-axis. These two hyperbolas (sharing the same axes and asymptotes) are called *conjugate hyperbolas*.

Eccentricity. The *eccentricity* of a hyperbola is defined as the number

$$\varepsilon = \frac{c}{a} = \sqrt{1 + \left(\frac{b}{a}\right)^2}.$$

It is clear, from the very definition of the hyperbola, that $\varepsilon > 1$. The eccentricity determines the shape of the fundamental rectangle and, ultimately, the shape of the hyperbola. Thus, the greater the eccentricity, the closer the two branches of the hyperbola approach the Oy axis, and the closer the eccentricity is to 1, the nearer the hyperbola approaches the Ox axis.

Intersection of the hyperbola with a line. Tangent to a hyperbola. Analogously to the case of the ellipse, we will now address the problem of the intersection between a hyperbola given by the implicit equation (4.2.6) and a line.

Assume, first, that the line is not vertical. Then its equation can be expressed in slope-intercept form as

$$y = kx + m. \tag{4.2.14}$$

By substituting y from the above formula into the hyperbola's equation (4.2.6), we obtain the equation that gives the abscissa (or abscissae) of the point(s) of intersection:

$$(b^2 - a^2k^2)x^2 - 2a^2kmx - a^2(b^2 + m^2) = 0. \tag{4.2.15}$$

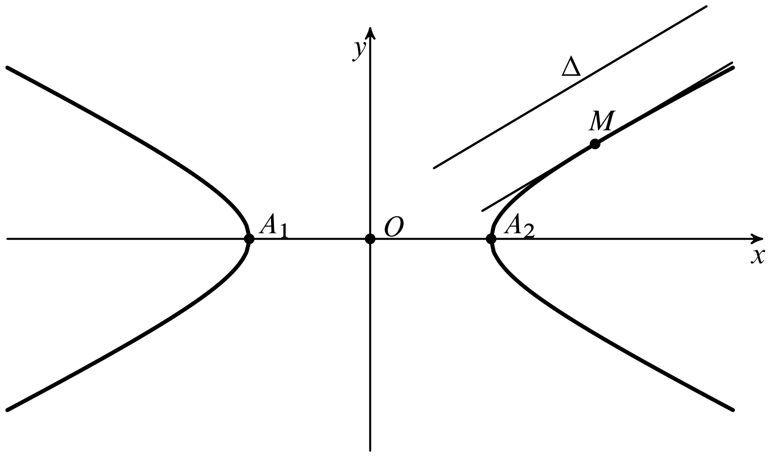


Figure 4.7: Tangent to a hyperbola with a given direction

This equation is called the *intersection equation*. If $b^2 - a^2k^2 \neq 0$, the equation (4.2.15) is quadratic, so to determine the number of its real roots, we must consider its discriminant. We have

$$\Delta = -4\,a^2b^2\left(-b^2 + a^2k^2 - m^2\right). \tag{4.2.16}$$

For the line to be tangent to the hyperbola, we must have $\Delta = 0$, that is,

$$a^2k^2 = b^2 + m^2.$$

Clearly, since $b \neq 0$, there will always be two tangents for a given m. If $\Delta > 0$, that is, if

$$a^2k^2 < b^2 + m^2,$$

then the line and the hyperbola will have two points in common.

If $\Delta < 0$, that is, if

$$a^2k^2 > b^2 + m^2,$$

then the line and the hyperbola will not intersect.

If, on the other hand, $b^2 - a^2k^2 = 0$, there are two possible scenarios:

1. $m = 0$;

2. $m \neq 0$.

In the first case, we have two lines with slopes $k = \pm b/a$ passing through the origin. These lines are, in fact, the asymptotes of the hyperbola, which, as we know, do not intersect the hyperbola.

In the second case, we have lines parallel to the asymptotes, which intersect the hyperbola exactly at one point.

Now consider the intersection of the hyperbola with a vertical line. Such a line has the equation $x = h$. We then solve the system

$$\begin{cases} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, \\ x = h, \end{cases}$$

leading to

$$\frac{y^2}{b^2} = \frac{h^2}{a^2} - 1$$

or

$$y^2 = \frac{b^2(h^2 - a^2)}{a^2}.$$

Clearly, there are three possible cases:

1. If $|h| > a$, the hyperbola and the line intersect at two points with abscissa $x = h$ and ordinates
$$y = \pm \frac{b}{a}\sqrt{h^2 - a^2}.$$

2. If $h = \pm a$, the hyperbola and the line have two coincident points in common, so the line is tangent to the hyperbola at one of its two vertices.

3. If $|h| < a$, the hyperbola and the line have no common points.

The equation of the tangent to the hyperbola by duplication. Using exactly the same method as in the case of the ellipse, we obtain

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$$

as the equation of the tangent at the point $M_0(x_0, y_0)$ on the hyperbola.

Tangents to the hyperbola drawn from an external point. We proceed exactly as in the case of the ellipse. We begin with the situation where neither of the tangents drawn from $M(x_1, y_1)$ is vertical. Then, as we have seen, the equation of the tangent is of the form

$$y = kx \pm \sqrt{a^2k^2 - b^2}.$$

For the tangent to pass through M, we must have

$$y_1 - kx_1 = \pm\sqrt{a^2k^2 - b^2},$$

or, squaring both sides,

$$(y_1 - kx_1)^2 = a^2k^2 - b^2.$$

Thus, we obtain the equation

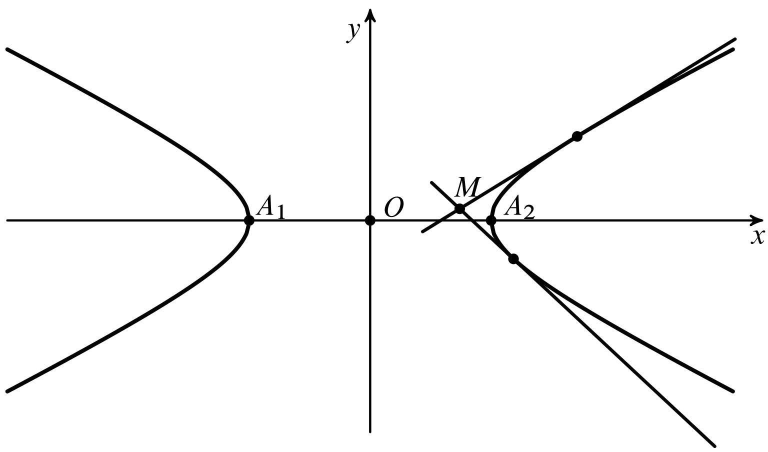


Figure 4.8: Tangents to a hyperbola from an external point (non-vertical tangents)

$$(x_1^2 - a^2)k^2 - 2x_1y_1k + (y_1^2 + b^2) = 0.$$

This is the equation that provides the slopes of the two tangents. Its discriminant is:

$$\Delta = 4(a^2y_1^2 - b^2x_1^2 + a^2b^2) = -4a^2b^2\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right).$$

It is immediately apparent that:

- there are two tangents if
$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0,$$
i.e., the point lies between the two branches of the hyperbola;

- there is one tangent if
$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 = 0,$$
i.e., the point lies on the hyperbola;

- there are no tangents if
$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0,$$
i.e., the point lies inside one of the branches of the hyperbola.

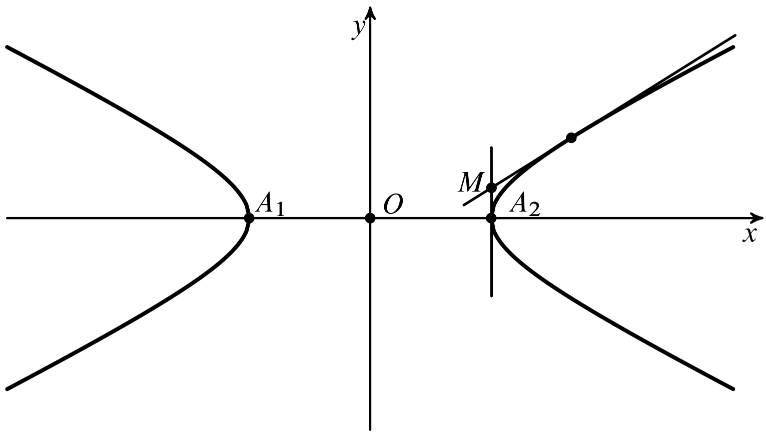


Figure 4.9: Tangents to a hyperbola from an external point (one vertical tangent)

We now examine the case where one of the tangents is vertical, which means it has the equation
$$x = \pm a.$$

Thus, the point M has coordinates $M(\pm a, y_1)$. The second tangent from M cannot also be vertical, so its equation must take the form
$$y = kx \pm \sqrt{a^2k^2 - b^2}. \tag{4.2.17}$$

For this tangent to pass through M, we must have

$$y_1 = \pm ka \pm \sqrt{a^2k^2 - b^2}, \tag{4.2.18}$$

or, squaring both sides,

$$(y_1 \mp ka)^2 = a^2k^2 - b^2, \tag{4.2.19}$$

which simplifies to

$$y_1^2 \mp 2y_1ka + a^2k^2 = a^2k^2 - b^2,$$

or

$$\pm 2y_1ak = y_1^2 + b^2. \tag{4.2.20}$$

If $y_1 = 0$, then the equation above (in k) has no solution, which is expected since, in this case, M is one of the vertices of the hyperbola on the Ox axis, and there is only one tangent, the vertical one. Otherwise, from equation (4.1.23), it follows that

$$k = \pm \frac{y_1^2 + b^2}{2ay_1}, \tag{4.2.21}$$

where the sign is chosen depending on the vertex through which the vertical tangent passes.

4.3 The Parabola

Definition and Canonical Equation

Definition 4.3. A *parabola* is the geometric locus of points in a plane that are equidistant from a fixed line Δ, called the *directrix*, and a fixed point F, called the *focus*.

Let p denote the distance from the point F to the line Δ. We construct a rectangular coordinate system in the plane as follows: we choose as the Ox axis the perpendicular dropped from F to Δ, oriented from the directrix toward the focus, while the Oy axis is the perpendicular bisector of the segment determined by F and the foot of the perpendicular from F to Δ. The orientation is chosen so that the coordinate system xOy is right-handed, where O is the intersection of the two coordinate axes. This means that the coordinates of F are $(p/2, 0)$, while the equation of the line Δ is $x = -p/2$.

Let $M(x, y)$ be an arbitrary point on the parabola. Then, the distance from M to F is

$$d(M, F) = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2},$$

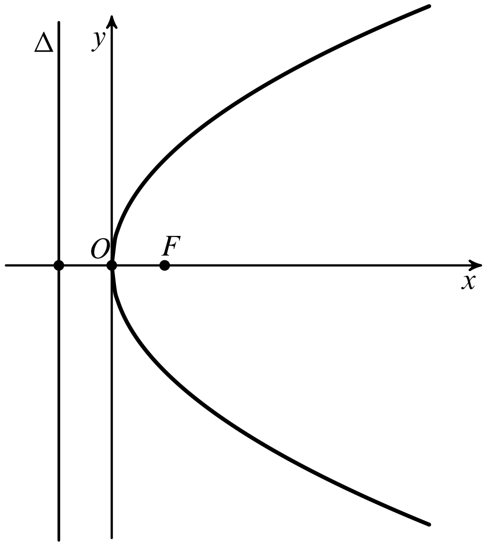


Figure 4.10: The Parabola

while the distance from M to Δ is

$$d(M,\Delta) = \left|x + \frac{p}{2}\right|.$$

Thus, the equation of the parabola, by definition, is

$$\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = \left|x + \frac{p}{2}\right|. \tag{4.3.1}$$

It is straightforward to verify that equality (4.3.1) holds only if

$$\left|x + \frac{p}{2}\right| = x + \frac{p}{2},$$

that is,

$$x \geq -\frac{p}{2}.$$

This means that the equation of the parabola, (4.3.1), can be rewritten as

$$\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = x + \frac{p}{2}. \tag{4.3.2}$$

Squaring both sides, this equation is equivalent (in this specific case!) to

$$\left(x - \frac{p}{2}\right)^2 + y^2 = \left(x + \frac{p}{2}\right)^2,$$

or, after simplification, to

$$y^2 = 2px. \tag{4.3.3}$$

The equation (4.3.3) is called the *canonical equation of the parabola with parameter* p.

Shape of the Parabola. First, note that from the canonical equation (4.3.3), it immediately follows that x can only take non-negative values. Therefore, the parabola described by this equation lies entirely on the right-hand side of the Oy axis.

Since the equation (4.3.3) contains the variable y only to the second power, it follows that the parabola is symmetric with respect to the Ox axis. Thus, to study its shape, it suffices to examine the part of the parabola lying in the first quadrant. In this quadrant, the equation of the parabola can be written as

$$y = \sqrt{2px}. \tag{4.3.4}$$

Calculating the first two derivatives of y, we find

$$y' = \frac{1}{\sqrt{2px}}, \tag{4.3.5}$$

and

$$y'' = -\frac{\sqrt{2}p^2}{4(px)^{3/2}}. \tag{4.3.6}$$

Thus, over the interval $(0, \infty)$, we have $y' > 0$ and $y'' < 0$, indicating that the function y is strictly increasing and concave on this interval. The shape of the curve in the fourth quadrant is obtained by reflecting its first-quadrant portion across the Ox axis.

The symmetry axis of the parabola (4.3.3), that is, the Ox axis, is called the *axis of the parabola*, while the point where the parabola intersects the axis (the origin, in this case) is called the *vertex of the parabola*.

The Parameter of the Parabola. The quantity p appearing in the canonical equation (4.3.3) is called the *focal parameter*, or simply the *parameter*, of the parabola. Next, we provide another geometric interpretation of this parameter. Consider the line passing through the focus of the parabola and perpendicular to the parabola's axis. It is easy to observe that the equation of this line is

$$x = \frac{p}{2}. \tag{4.3.7}$$

Let M_1 and M_2 be the points of intersection of this line with the parabola. Solving the system of equations formed by (4.3.3) and (4.3.7), we find $y = \pm p$, so

$$p = FM_1. \tag{4.3.8}$$

Thus, *the parameter p of the parabola is equal to the length of the perpendicular dropped from the focus of the parabola to the point where it intersects the parabola.*

The parameter defines the shape and dimensions of the parabola.

Remark. In addition to equation (4.3.3), the following three (canonical) equations also describe parabolas:

$$y^2 = -2px, \quad x^2 = 2py, \quad x^2 = -2py. \tag{4.3.9}$$

The first equation gives a parabola symmetric with respect to the Oy axis in comparison to parabola (4.3.3), while the other two parabolas are obtained by rotating parabola (4.3.3) by 90° and −90°, respectively, around the origin.

Intersection of the Parabola with a Line. Tangent to a Parabola. Consider the parabola (4.3.3) and a line, initially assumed to be non-vertical. To determine the points of intersection, we solve the system

$$\begin{cases} y^2 = 2px, \\ y = kx + b. \end{cases}$$

Substituting the second equation into the first, we obtain the equation in x:

$$k^2x^2 + 2(kb - p)x + b^2 = 0. \tag{4.3.10}$$

This equation is called the *intersection equation* of the parabola and the line. Since it is a quadratic equation in x, it follows that the parabola and the line can have two distinct points, one point (or two coincident points), or no points in common, depending on the discriminant of equation (4.3.10).

A straightforward calculation shows that the discriminant is given by

$$\Delta = 4p(p - 2bk). \tag{4.3.11}$$

Thus:

(i) The parabola and the line have two distinct points in common if

$$\Delta > 0, \quad \text{that is,} \quad kb < \frac{p}{2}.$$

(ii) The parabola and the line have two coincident points (that is, only one point) in common if

$$\Delta = 0, \quad \text{that is,} \quad kb = \frac{p}{2}.$$

In this case, the parabola and the line are tangent. Note that since the parameter p of the parabola is strictly positive, it follows (since $kb \neq 0$) that:

- the tangent to a parabola cannot be horizontal (specifically, it cannot be parallel to the parabola's axis);
- there are no non-vertical tangents to a parabola passing through its vertex.

(iii) The parabola and the line have no points in common if

$$\Delta < 0, \quad \text{that is,} \quad kb > \frac{p}{2}.$$

In particular, any line parallel to the parabola's axis (i.e., horizontal) intersects the parabola at two distinct points; likewise, any non-vertical line passing through the origin intersects the parabola at two distinct points.

Now, consider the intersection between parabola (4.3.3) and a vertical line. In this case, we solve the system of equations:

$$\begin{cases} y^2 = 2px, \\ x = a, \end{cases}$$

which leads to the intersection equation (now in y):

$$y^2 - 2ap = 0. \tag{4.3.12}$$

Since $p > 0$, the intersection is:

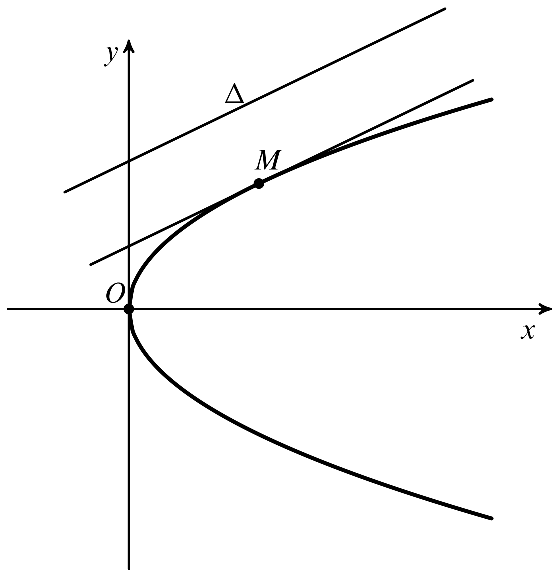


Figure 4.11: Tangent to a parabola, parallel to a given direction

(i) a pair of distinct points, if $a > 0$;

(ii) a pair of coincident points (that is, a single point) if $a = 0$. In this case, the line is tangent to the parabola and coincides with the Oy axis;

(iii) the empty set, if $a < 0$.

Equation of the Tangent at a Point on the Parabola. We again start from the canonical equation of the parabola and consider a line intersecting the parabola at a point $M_0(x_0, y_0)$. The parametric equations of the line will then be:

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt. \end{cases}$$

Substituting into the equation of the parabola, we get:

$$(y_0 + mt)^2 = 2p(x_0 + lt),$$

or, after expanding,

$$m^2t^2 + 2t(-pl + my_0) + y_0^2 - 2px_0 = 0.$$

The constant term vanishes because M_0 lies on the parabola, so the equation reduces to:

$$m^2t^2 + 2t(-pl + my_0) = 0.$$

The tangency condition, as in the case of other conics, means that the intersection equation must have a double root. In this case, the coefficient of t must vanish:

$$-pl + my_0 \equiv \mathbf{v}(l, m) \cdot \mathbf{n}(-p, y_0) = 0,$$

indicating that the vector $\mathbf{n}(-p, y_0)$ is the normal vector of the tangent. Therefore, the equation of the tangent is:

$$-p(x - x_0) + y_0(y - y_0) = 0,$$

or equivalently:

$$yy_0 = p(x + x_0),$$

where we again use the fact that the point M_0 lies on the parabola. As in the case of centred conics, this form of the parabola's equation is said to have been obtained by *duplication*. There is a slight difference compared to the other two conics, as x appears only to the first power here. In this case, the duplication rule means that:

- x is replaced by $(x + x_0)/2$;
- y^2 is replaced by yy_0.

Tangents Drawn from a Point Outside the Parabola. Consider a point $M(x_1, y_1)$. We begin, as usual, with the case of non-vertical tangents. We have seen that such tangents have the equation:

$$y = kx + \frac{p}{2k}.$$

For M to lie on the tangent, we must have:

$$y_1 = kx_1 + \frac{p}{2k},$$

which simplifies to:

$$2x_1k^2 - 2y_1k + p = 0.$$

The discriminant is:

$$\Delta = 4(y_1^2 - 2px_1).$$

Thus:

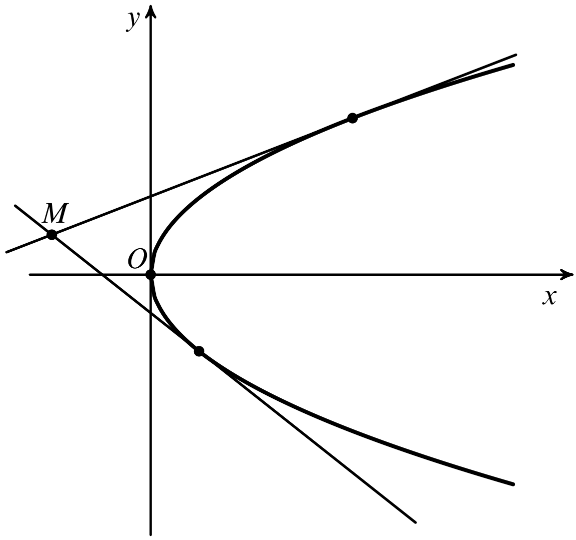


Figure 4.12: Non-vertical tangents to a parabola from an external point

- there are two tangents if $y_1^2 - 2px_1 > 0$ (the point lies outside the parabola);
- there is one tangent if $y_1^2 - 2px_1 = 0$ (the point lies on the parabola);
- there are no tangents if $y_1^2 - 2px_1 < 0$ (the point lies inside the parabola).

The equation of the tangent is of the form:

$$y - y_1 = k(x - x_1),$$

with k given by the quadratic equation above.

Now consider the case where one of the tangents is vertical. The vertical tangent must have the equation:

$$x = x_1.$$

To find the slope of the second tangent, note that $x_1 = 0$. The equation of the tangent must then be:

$$y = kx + \frac{p}{2k}.$$

For the tangent to pass through M, we must have:

$$y_1 = \frac{p}{2k}.$$

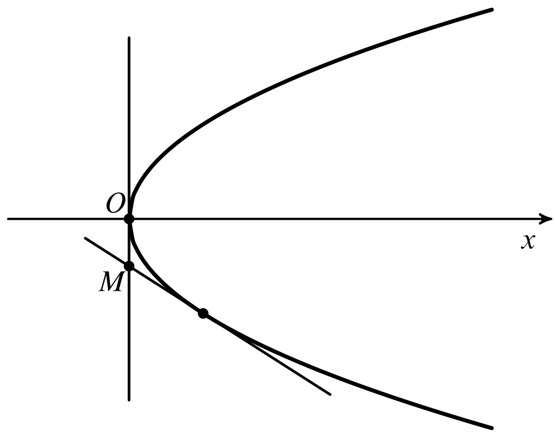


Figure 4.13: Tangents to a parabola from an external point (one vertical tangent)

If $y_1 = 0$, then there is only one tangent (the vertical one). Otherwise:

$$k = \frac{p}{2y_1},$$

and the equation of the tangent is:

$$y - y_1 = \frac{p}{2y_1}x.$$

4.4 Probleme

Problem 4.1. Determinaţi locul geometric al mijloacelor coardelor elipsei $\frac{x^2}{25} + \frac{y^2}{9} = 1$, care sunt paralele cu dreapta $x + 2y = 1$.

Problem 4.2. Se consideră elipsa $x^2 + 4y^2 = 25$. Să se determine coardele care trec prin $A(7/2, 7/4)$ pentru care punctul A este mijlocul lor.

Problem 4.3. Prin punctul $M(0, 3)$ să se ducă o dreaptă care să intersecteze elipsa $x^2 + 4y^2 = 20$ în două puncte A şi B astfel încât $MA = 2MB$.

Problem 4.4. Se consideră elipsa $\frac{x^2}{4} + y^2 = 1$. Să se găsească punctele M de pe elipsă pentru care unghiul $\widehat{F_1MF_2}$ este drept.

Problem 4.5. Se consideră elipsa $\frac{x^2}{4} + y^2 = 1$. Să se găsească punctele M de pe elipsă pentru care unghiul $\widehat{F_1MF_2}$ este de $60°$.

Problem 4.6. Se consideră elipsa $\frac{x^2}{4} + y^2 = 1$. Să se găsească punctele M de pe elipsă pentru care unghiul $\widehat{F_1MF_2}$ este maxim.

Problem 4.7. Determinaţi tangentele la elipsa $\frac{x^2}{10} + \frac{2y^2}{5} = 1$ care sunt paralele cu dreapta

$$3x + 2y + 7 = 0.$$

Problem 4.8. Determinaţi tangentele la elipsa $\frac{x^2}{30} + \frac{y^2}{24} = 1$ care sunt paralele cu dreapta

$$4x - 2y + 23 = 0$$

şi calculaţi distanţa dintre ele.

Problem 4.9. Determinaţi locul geometric al mijloacelor coardelor hiperbolei $x^2 - 2y^2 = 1$, care sunt paralele cu dreapta $2x - y = 0$.

Problem 4.10. Se consideră hiperbola $x^2 - \frac{y^2}{4} = 1$. Să se găsească punctele M de pe hiperbolă pentru care unghiul $\widehat{F_1MF_2}$ este drept.

Problem 4.11. Se consideră hiperbola $x^2 - \frac{y^2}{4} = 1$. Să se găsească punctele M de pe hiperbolă pentru care unghiul $\widehat{F_1MF_2}$ este de $60°$.

Problem 4.12. Se consideră hiperbola $x^2 - \frac{y^2}{4} = 1$. Să se găsească punctele M de pe hiperbolă pentru care unghiul $\widehat{F_1MF_2}$ este maxim.

Problem 4.13. Calculaţi aria formată de asimptotele hiperbolei $\frac{x^2}{4} - \frac{y^2}{9} = 1$ şi de dreapta

$$3x + 2y - 7 = 0.$$

Problem 4.14. Determinaţi tangentele la hiperbola $\frac{x^2}{16} - \frac{y^2}{8} = 1$ care sunt paralele cu dreapta

$$4x + 2y - 5 = 0.$$

Problem 4.15. Determinaţi tangenta la parabola $y^2 = 16x$ care trece prin punctul $(-2, 2)$.

Problem 4.16. Determinaţi tangentele comune la elipsele

$$\frac{x^2}{45} + \frac{y^2}{9} = 1 \quad \text{şi} \quad \frac{x^2}{9} + \frac{y^2}{18} = 1.$$

Problem 4.17. Determinaţi relaţia dintre coordonatele punctului (x_0, y_0) astfel încât din el să nu se poată duce nici o tangentă la hiperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1.$$

Problem 4.18. Pe hiperbola $\dfrac{x^2}{24} - \dfrac{y^2}{18} = 1$ determinaţi punctul M cel mai apropiat de dreapta

$$3x + 2y + 1 = 0$$

şi determinaţi distanţa de la acest punct la dreaptă.

Problem 4.19. Determinaţi ecuaţiile tangentelor duse din punctul $A(-1, -7)$ la hiperbola $x^2 - y^2 = 16$.

Problem 4.20. Din punctul $P(1, -5)$ se duc tangente la hiperbola $\dfrac{x^2}{3} - \dfrac{y^2}{5} = 1$. Determinaţi distanţa de la punctul P la coarda hiperbolei care uneşte punctele de contact ale tangentelor cu hiperbola.

Problem 4.21. Determinaţi valorile pantei k pentru care dreapta $y = kx + 2$ este tangentă la parabola $y^2 = 4x$.

Problem 4.22. Scrieţi ecuaţia tangentei la parabola $y^2 = 8x$ care este paralelă cu dreapta

$$3x + 2y - 3 = 0.$$

Problem 4.23. Scrieţi ecuaţia tangentei la parabola $y^2 = 16x$ care este perpendiculară pe dreapta

$$4x + 2y + 7 = 0.$$

Problem 4.24. Scrieţi ecuaţia tangentei la parabola $y^2 = 12x$ care este paralelă cu dreapta

$$3x - 2y + 30 = 0$$

şi determinaţi distanţa dintre tangentă şi această dreaptă.

Problem 4.25. Scrieţi ecuaţiile tangentelor la parabola $y^2 = 36x$ duse din punctul $A(2, 9)$.

Problem 4.26. Din punctul $A(5, 12)$ se duc tangente la parabola $y^2 = 5x$. Scrieţi ecuaţia dreptei care uneşte punctele de contact.

Problem 4.27. Din punctul $P(-3, 12)$ se duc tangente la parabola $y^2 = 10x$. Calculaţi distanţa de la punctul P la coarda parabolei care uneşte punctele de contact.

CHAPTER 5

Quadrics in Reduced Equations

5.1 Quadrics in Reduced Equations

A *quadric* in $\mathbb{R}^3$ is defined as the set of points $P(x, y, z)$ whose coordinates satisfy a second-degree equation, i.e., an equation of the form

$$a_{11}x^2+2a_{12}xy+2a_{13}xz+a_{22}y^2+a_{23}yz+a_{33}z^2+a_{10}x+a_{20}y+a_{30}z+a_{00}=0, \tag{5.1.1}$$

where all coefficients are real numbers, and not all coefficients of the second-degree terms are zero (that is, in other words, the equation is genuinely of second degree). In this section, we will not address the general theory of quadrics but will focus on studying those quadrics written in canonical form, meaning, essentially, that the mixed second-degree terms are absent, and, if possible, the first-degree terms are also absent. We will see that this is not always the case. Later, we will demonstrate that, through a change of coordinates, any quadric can be reduced to one of the quadrics studied below.

5.2 The Ellipsoid

A subset $S \subset \mathbb{R}^3$ is called an *ellipsoid* if there exists a Cartesian coordinate system and three strictly positive numbers a, b, c such that

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}. \tag{5.2.1}$$

In other words, an ellipsoid is a surface described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{5.2.2}$$

The numbers a, b, c are called the *semi-axes* of the ellipsoid. If all three are distinct, the ellipsoid is referred to as a *triaxial ellipsoid*. If two of them are equal (e.g., $a = b$), the ellipsoid is called a *rotational ellipsoid*:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$$

and can be obtained by rotating an ellipse about an axis (the Oz axis, in this case). Finally, if all semi-axes are equal ($a = b = c$), the ellipsoid is a sphere with radius a.

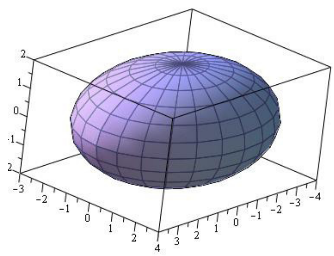


Figure 5.1: The Ellipsoid

We now enumerate a series of properties of the ellipsoid that will allow us to determine its shape.

Property 1. *The ellipsoid (5.2.2) is bounded by a rectangular parallelepiped with faces parallel to the coordinate planes, centred at the origin, and edges of length $2a$, $2b$, $2c$. Hence, in particular,* the ellipsoid, as a set of points, is a bounded set.

Proof Indeed, the equation (5.2.2) can be rewritten as

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}.$$

Since the left-hand side of this equation is evidently non-negative, the same must hold for the right-hand side, leading to the inequality

$$\frac{x^2}{a^2} \le 1,$$

which implies $x \in [-a, a]$. Similarly, it follows that $y \in [-b, b]$ and $z \in [-c, c]$, which is exactly what we wanted to show: $(x, y, z) \in [-a, a] \times [-b, b] \times [-c, c]$. □

The ellipsoid is a highly symmetric shape:

Property 2. *The ellipsoid has three planes of symmetry: xOy, yOz, zOx, three axes of symmetry: Ox, Oy, Oz, and a centre of symmetry, the origin $O(0, 0, 0)$ of the coordinate system. Moreover, if the ellipsoid is not triaxial, it may have additional planes and axes of symmetry (but not additional centres of symmetry).*

Proof Let $M_0(x_0, y_0, z_0)$ be a point on the ellipsoid. Then its coordinates satisfy the equation

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1.$$

To prove, for instance, that the xOy plane is a plane of symmetry, it suffices to show that the reflection of M_0 with respect to this plane also belongs to the ellipsoid. The reflection of M_0 is the point with coordinates $(x_0, y_0, -z_0)$, whose coordinates clearly satisfy the ellipsoid equation, hence it lies on the ellipsoid. The reasoning is similar for the other coordinate planes. These coordinate planes are also called the *principal planes* of the ellipsoid because they are planes of symmetry.

That the coordinate axes are axes of symmetry follows immediately from the above observations, as they are intersections of planes of symmetry. Similarly, the origin is a centre of symmetry because it is the intersection of two (in fact, three) axes of symmetry. The coordinate axes, as axes of symmetry of the ellipsoid, are also called its *principal axes*.

Now suppose the ellipsoid is a rotational ellipsoid, for instance, about the Oz axis. We claim that any plane passing through the axis of rotation is a plane of symmetry.

We have seen that the equation of a plane passing through the Oz axis is of the form $\Pi : Ax + By = 0$. Suppose, as before, that $M_0(x_0, y_0, z_0)$ is a point on the ellipsoid, now rotationally symmetric about the Oz axis, so its coordinates satisfy the equation:

$$\frac{x_0^2 + y_0^2}{a^2} + \frac{z_0^2}{c^2} = 1.$$

We determine the coordinates of the reflection M_0' of M_0 with respect to the plane Π. The equations of the normal to the plane Π passing through M_0 are:

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{0}.$$

To find the intersection M_1 of this normal with the plane Π, the coordinates of M_1 are given by solving the system:

$$\begin{cases} Bx - Ay = Bx_0 - Ay_0 \\ z = z_0 \\ Ax + By = 0 \end{cases} .$$

This gives $x_1 = \frac{B^2x_0 - ABy_0}{A^2+B^2}$, $y_1 = \frac{A^2y_0 - ABx_0}{A^2+B^2}$, $z_1 = z_0$. The point M_1 must be the midpoint of segment M_0M_0', so:

$$\begin{cases} x_0' = 2x_1 - x_0 \\ y_0' = 2y_1 - y_0 \\ z_0' = 2z_1 - z_0. \end{cases}$$

Thus, we have:

$$x_0' = \frac{2B^2x_0 - 2ABy_0}{A^2 + B^2} - x_0 = \frac{(B^2 - A^2)x_0 - 2ABy_0}{A^2 + B^2},$$

$$y_0' = \frac{2A^2y_0 - 2ABx_0}{A^2 + B^2} - y_0 = \frac{-2ABx_0 + (A^2 - B^2)y_0}{A^2 + B^2},$$

$$z_0' = z_0.$$

We now calculate:

$$\frac{x_0'^2 + y_0'^2}{a^2} + \frac{z_0'^2}{c^2} = \frac{(B^2 - A^2)^2 x_0^2 + 4A^2B^2y_0^2 - 4AB(B^2 - A^2)x_0y_0}{a^2(A^2 + B^2)^2}$$
$$+ \frac{(A^2 - B^2)^2 y_0^2 + 4A^2B^2x_0^2 + 4AB(B^2 - A^2)x_0y_0}{a^2(A^2 + B^2)^2} + \frac{z_0^2}{c^2} =$$
$$= \frac{(A^2 + B^2)^2 x_0^2 + (A^2 + B^2)^2 y_0^2}{a^2(A^2 + B^2)^2} + \frac{z_0^2}{c^2} = \frac{x_0^2 + y_0^2}{a^2} + \frac{z_0^2}{c^2} = 1,$$

which shows that the point M_0' lies on the ellipsoid, and thus the plane Π is a plane of symmetry of the ellipsoid.

Similarly, in the case of a sphere, it can be shown that any plane passing through the origin of the coordinates is a plane of symmetry, and implicitly, any line passing through the origin is an axis of symmetry. □

To determine the shape of the ellipsoid, we start by examining the curves formed by its intersection with the planes of symmetry:

Property 3. *The intersections of the planes of symmetry of an ellipsoid with the ellipsoid are three real ellipses.*

Proof The intersection of the ellipsoid with the xOy plane is given by the system of equations:

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \\ z = 0. \end{cases}$$

Or equivalently:

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \\ z = 0. \end{cases}$$

These are clearly the equations of an ellipse lying in the xOy plane, with semi-axes a and b.

Similarly, the intersection of the ellipsoid with the yOz plane is given by:

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \\ x = 0, \end{cases}$$

or:

$$\begin{cases} \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \\ x = 0. \end{cases}$$

These are the equations of an ellipse lying in the yOz plane, with semi-axes b and c.

Finally, the intersection of the ellipsoid with the zOx plane is given by:

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \\ y = 0, \end{cases}$$

or:

$$\begin{cases} \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \\ y = 0. \end{cases}$$

These are the equations of an ellipse lying in the zOx plane, with semi-axes a and c. $\square$

(1) We will now study the intersections of the ellipsoid with planes of equations $z = k$, where k is a real number (planes parallel to the xOy plane). This intersection is given by the equations

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \\ z = k \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - \dfrac{k^2}{c^2} \\ z = k \end{cases}.$$

For the same reason as in point (1), the second system above has a non-empty solution if and only if $1 - \frac{k^2}{c^2} \leq 0$, that is, if and only if $-c \leq k \leq c$. If $k = \pm c$, then the intersection reduces to a point. This is the point $(0, 0, c)$ when $k = c$, respectively the point $(0, 0, -c)$ when $k = -c$. We note that these points are, in fact, the points where the Oz axis, given by the equations $x = 0$, $y = 0$, intersects the ellipsoid. We will see that there are also four similar points, on the other two coordinate axes. These are called the *vertices* of the ellipsoid.

The truly interesting situation, which gives us an initial idea of the shape of the ellipsoid (and justifies the name), is the one where $-c < k < c$. In this case, the intersection is given by the system of equations

$$\begin{cases} \dfrac{x^2}{a^2\left(1-\frac{k^2}{c^2}\right)} + \dfrac{y^2}{b^2\left(1-\frac{k^2}{c^2}\right)} = 1 \\ z = k \end{cases},$$

which, since the denominators are strictly positive, is clearly an ellipse, situated in the plane $z = k$, with semi-axes equal to $a\sqrt{\left(1-\frac{k^2}{c^2}\right)}$ and $b\sqrt{\left(1-\frac{k^2}{c^2}\right)}$. It is clear that the lengths of the semi-axes decrease as $|k|$ increases. In particular, they are maximal when $k = 0$, that is, when the intersection plane is the coordinate plane xOy. In this case, the equations of the intersection ellipse are

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\ z = 0 \end{cases}.$$

(2) The intersections with planes parallel to the coordinate planes xOz and yOz are analogous and lead to analogous results.

The Tangent Plane at a Point on an Ellipsoid

Consider the ellipsoid (5.2.2) and a point $M(x_0, y_0, z_0)$ on it. We will study the intersection of an arbitrary line passing through M_0 with the ellipsoid. The parametric equations of such a line are:

$$(\Delta) : \begin{cases} x = x_0 + l \cdot t, \\ y = y_0 + m \cdot t, \\ z = z_0 + n \cdot t, \end{cases} \qquad t \in \mathbb{R}. \tag{5.2.3}$$

$\mathbf{v}(l, m, n)$ is, of course, the direction vector of the line Δ. To determine the intersection points between the ellipsoid and the line Δ, we substitute x, y, z from the line equations (5.2.3) into the ellipsoid equation. We obtain:

$$\frac{(x_0 + lt)^2}{a^2} + \frac{(y_0 + mt)^2}{b^2} + \frac{(z_0 + nt)^2}{c^2} - 1 = 0$$

or, after performing the calculations and grouping the powers of t,

$$t^2\left(b^2c^2l^2 + a^2c^2m^2 + a^2b^2n^2\right) + 2t\left(b^2c^2x_0l + a^2c^2y_0m + a^2b^2z_0n\right) + \\ + b^2c^2x_0^2 + a^2c^2y_0^2 + a^2b^2z_0^2 - a^2b^2c^2 = 0.$$

The constant term in the above equation is zero because the coordinates of the point M_0 satisfy the ellipsoid equation. Therefore, the equation becomes

$$t^2\left(b^2c^2l^2 + a^2c^2m^2 + a^2b^2n^2\right) + 2t\left(b^2c^2x_0l + a^2c^2y_0m + a^2b^2z_0n\right) = 0. \tag{5.2.4}$$

This equation will be called the *intersection equation* between the ellipsoid and the line Δ. It is clear that the intersection equation is always of degree two and it will have two real solutions, corresponding to the point M_0 and the second intersection point. For the line to be *tangent* to the ellipsoid, it is necessary (and sufficient) for the intersection equation to have a double solution (clearly, $t = 0$). For this to happen, the coefficient of the linear term in t in equation (5.2.4) must be zero, that is,

$$b^2c^2x_0l + a^2c^2y_0m + a^2b^2z_0n = 0. \tag{5.2.5}$$

Equation (5.2.5) has a remarkable geometric interpretation. Consider the vector $\mathbf{n}\left(b^2c^2x_0, a^2c^2y_0, a^2b^2z_0\right)$. Then, equation (5.2.5) can be written as

$$\mathbf{n}\cdot\mathbf{v} = 0. \tag{5.2.6}$$

The significance of this equation is that *every line passing through M_0 and whose direction vector satisfies equation (5.2.5) is perpendicular to the vector* $\mathbf{n}$. Therefore, the set of these lines through M_0, tangent to the ellipsoid, forms a plane, the *tangent plane to the ellipsoid at the point M_0*, which has the normal vector $\mathbf{n}$. Hence, the equation of the tangent plane at M_0 is

$$b^2c^2x_0(x - x_0) + a^2c^2y_0(y - y_0) + a^2b^2z_0(z - z_0) = 0$$

or

$$b^2c^2x_0x + a^2c^2y_0y + a^2b^2z_0z - b^2c^2x_0^2 - a^2c^2y_0^2 - a^2b^2z_0^2 = 0.$$

From the equation of the ellipse, it follows that the constant term in the above equation is equal to $-a^2b^2c^2$, so the equation becomes

$$b^2c^2x_0x + a^2c^2y_0y + a^2b^2z_0z - a^2b^2c^2 = 0$$

or, after dividing by $a^2b^2c^2$,

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1. \tag{5.2.7}$$

Equation (5.2.7) is called the *equation of the tangent plane to the ellipsoid at the point M_0 on the ellipsoid, obtained by differentiation*, because it is obtained from the ellipsoid equation by replacing x^2 with xx_0, y^2 with yy_0, and z^2 with zz_0.

5.3 Second-Degree Cone

Definition 5.1. The *second-degree cone* is defined as the set of points in space whose coordinates, relative to an orthonormal system, satisfy an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \tag{5.3.1}$$

where a, b, c are strictly positive real numbers.

The second-degree cone shares the same symmetries as the ellipsoid, which stem directly from the fact that in its equation all coordinates appear exclusively squared:

(1) Three planes of symmetry (the coordinate planes);

(2) Three axes of symmetry (the coordinate axes);

(3) A centre of symmetry (the origin).

A remarkable property of the second-degree cone is that *it is a ruled surface*: through each of its points passes a straight line (called a *generator* of the cone).

More precisely, *if $M_0(x_0, y_0, z_0)$ is an arbitrary point on the cone and O is the origin of the coordinate system, then every point $M(x, y, z)$ of the line OM_0 lies on the cone.*

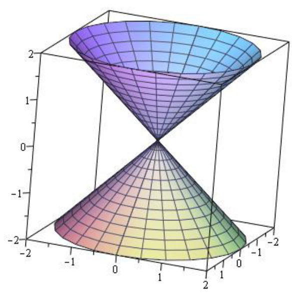


Figure 5.2: Second-degree cone

The proof of this statement is very simple. Indeed, it is easy to see that the parametric equations of the line OM_0 are:

$$\begin{cases} x = x_0 \cdot t, \\ y = y_0 \cdot t, \\ z = z_0 \cdot t. \end{cases}$$

Substituting into the equation of the cone yields:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = t^2 \underbrace{\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} \right)}_{=0} = 0,$$

hence the points on the line satisfy the equation of the cone.

Due to this property, O is called the *vertex of the cone*.

Intersections with Planes Parallel to the Coordinate Planes

We use this method, even for the second-degree cone, to identify the shape of the surface.

(1) *Planes parallel to xOy*. Such a plane clearly has an equation of the form $z = k$, where k is a real constant. Such an intersection is given by the system of equations

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0 \\ z = k \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{k^2}{c^2} \\ z = k \end{cases} .$$

If $k \neq 0$, the second system of equations can be rewritten as

$$\begin{cases} \dfrac{x^2}{a^2k^2/c^2} + \dfrac{y^2}{b^2k^2/c^2} = 1 \\ z = k \end{cases} .$$

These equations describe an ellipse with semi-axes $\dfrac{a|k|}{c}$ and $\dfrac{b|k|}{c}$, located in the plane $z = k$.

If, on the other hand, $k = 0$ (i.e., the intersection is with the plane xOy), the system of intersection equations reduces to

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \\ z = 0 \end{cases},$$

which is satisfied by a single point (the origin, i.e., the vertex of the cone).

(2) *Intersections with planes parallel to xOz.* In this case, the system of equations giving the intersection points is

$$\begin{cases} y = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0, \end{cases}$$

which leads to

$$\begin{cases} y = h, \\ \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} = \dfrac{h^2}{b^2}. \end{cases} \tag{5.3.2}$$

The equations (5.3.2) represent, if $h \neq 0$, the equations of a hyperbola located in the plane $y = h$, with semi-axes $\dfrac{a|h|}{b}$ (along the axis parallel to Ox) and $\dfrac{c|h|}{b}$ (along the axis parallel to Oz). Note that the axis parallel to Oz intersects the hyperbola, while the axis parallel to Ox does not.

On the other hand, if $h = 0$, the same equations represent a pair of straight lines (generators of the cone) with equations

$$\begin{cases} y = 0, \\ \dfrac{z}{c} - \dfrac{x}{a} = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} y = 0, \\ \dfrac{z}{c} + \dfrac{x}{a} = 0. \end{cases}$$

(3) *Intersections with planes parallel to yOz.* – This is perfectly analogous to the previous case.

Remark. It can be shown that by using planes that are not necessarily parallel to the coordinate planes, all conic sections can be obtained as plane sections of the second-degree cone. In fact, this is the reason why conic sections are also called *conical sections*.

The Tangent Plane at a Point of the Second-Degree Cone. The equation of the tangent plane at a point $M_0(x_0, y_0, z_0)$ of the second-degree cone is obtained, as in the case of the ellipsoid, through duplication of the equation, so we will not repeat the reasoning. Consequently, the equation of the tangent plane at M_0 is

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = 0. \tag{5.3.3}$$

A remarkable property of the tangent plane at a point of the cone is that it contains the generator passing through that point. Indeed, the generator passing through $M_0(x_0, y_0, z_0)$ has the parametric equations

$$\begin{cases} x = x_0 t, \\ y = y_0 t, \\ z = z_0 t. \end{cases}$$

Substituting into the left-hand side of the tangent plane equation yields

$$\frac{x_0^2 t}{a^2} + \frac{y_0^2 t}{b^2} - \frac{z_0^2 t}{c^2} = t\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) = 0,$$

which means that the tangent plane indeed contains the straight-line generator of the cone passing through the point of tangency.

Right Circular Cone. If $a = b$, the equation of the cone becomes

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 0.$$

In this case, the sections with planes parallel to the xOy plane are circles. Cones of this type are called *right circular cones*. We will see later that surfaces of this type can be obtained by rotating a straight line passing through the origin about the Oz axis.

5.4 The One-Sheeted Hyperboloid

5.4.1 Shape and Symmetries

Definition 5.2. The *one-sheeted hyperboloid* is defined as the locus of points in space whose coordinates, relative to a rectangular coordinate system, satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \tag{5.4.1}$$

where a, b, c are strictly positive real numbers, called the *semi-axes* of the hyperboloid.

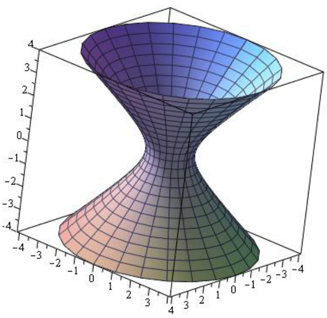


Figure 5.3: The one-sheeted hyperboloid

The symmetries of the one-sheeted hyperboloid are the same as those of the ellipsoid; therefore, we will not describe them here. Instead, we focus on its intersections with planes parallel to the coordinate planes.

(1) *Planes parallel to the xOy plane.* In this case, we study the system of equations:

$$\begin{cases} z = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1, \end{cases}$$

which leads to

$$\begin{cases} z = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{h^2}{c^2} + 1. \end{cases}$$

Since the right-hand side is always strictly positive, the equations can be rewritten as

$$\begin{cases} z = h, \\ \dfrac{x^2}{\left(a\sqrt{\frac{h^2}{c^2}+1}\right)^2} + \dfrac{y^2}{\left(b\sqrt{\frac{h^2}{c^2}+1}\right)^2} = 1. \end{cases}$$

These are the equations of an ellipse with semi-axes $a\sqrt{\frac{h^2}{c^2}+1}$ and $b\sqrt{\frac{h^2}{c^2}+1}$, for any value of h. An important special case is when $h = 0$ (i.e., in the xOy

coordinate plane). The resulting ellipse (with minimal semi-axes) is called the *waist ellipse* or the *throat ellipse*.

(2) *Planes parallel to the xOz plane.* In this case, the curve of intersection has the equations

$$\begin{cases} y = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1, \end{cases}$$

which means

$$\begin{cases} y = h, \\ \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{h^2}{b^2}. \end{cases} \tag{5.4.2}$$

Here, we analyse three cases:

(a) If $1 - \dfrac{h^2}{b^2} < 0$, i.e., $h^2 > b^2$, the system (5.4.2) becomes

$$\begin{cases} y = h, \\ \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} = \dfrac{h^2}{b^2} - 1, \end{cases}$$

or

$$\begin{cases} y = h, \\ \dfrac{z^2}{\left(c\sqrt{\frac{h^2}{b^2} - 1}\right)^2} - \dfrac{x^2}{\left(a\sqrt{\frac{h^2}{b^2} - 1}\right)^2} = 1, \end{cases}$$

which represents a hyperbola with semi-axes $c\sqrt{\frac{h^2}{b^2} - 1}$ and $a\sqrt{\frac{h^2}{b^2} - 1}$, lying in a plane parallel to the xOz plane. The axis intersecting the hyperbola is parallel to the Oz axis, while the other axis is parallel to the Ox axis.

(b) If $1 - \dfrac{h^2}{b^2} = 0$, i.e., $h = \pm b$, the system (5.4.2) becomes

$$\begin{cases} y = \pm b, \\ \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} = 0, \end{cases}$$

or

$$\begin{cases} y = \pm b, \\ \left(\dfrac{z}{c} - \dfrac{x}{a}\right)\left(\dfrac{z}{c} + \dfrac{x}{a}\right) = 0. \end{cases} \tag{5.4.3}$$

For each value of h (b or $-b$), the above equation represents a pair of straight lines. For $h = b$, we obtain

$$\begin{cases} y = b, \\ \dfrac{z}{c} - \dfrac{x}{a} = 0 \end{cases} \quad \text{or} \quad \begin{cases} y = b, \\ \dfrac{z}{c} + \dfrac{x}{a} = 0. \end{cases}$$

While for $h = -b$, we obtain

$$\begin{cases} y = -b, \\ \dfrac{z}{c} - \dfrac{x}{a} = 0 \end{cases} \quad \text{or} \quad \begin{cases} y = -b, \\ \dfrac{z}{c} + \dfrac{x}{a} = 0. \end{cases}$$

(c) If $1 - \dfrac{h^2}{b^2} > 0$, i.e., $h^2 < b^2$, the system (5.4.2) becomes

$$\begin{cases} y = h, \\ \dfrac{x^2}{\left(a\sqrt{1 - \frac{h^2}{b^2}}\right)^2} - \dfrac{z^2}{\left(c\sqrt{1 - \frac{h^2}{b^2}}\right)^2} = 1, \end{cases}$$

representing a hyperbola with semi-axes $a\sqrt{1 - \frac{h^2}{b^2}}$ and $c\sqrt{1 - \frac{h^2}{b^2}}$, lying in a plane parallel to the xOz plane. The axis intersecting the hyperbola is parallel to the Ox axis, while the other axis is parallel to the Oz axis.

(3) *Planes parallel to the yOz plane.* This case is perfectly analogous to the previous one.

5.4.2 Straight-Line Generators of the One-Sheeted Hyperboloid

As seen above, the one-sheeted hyperboloid contains straight lines. Four of these lines were found earlier as the intersections of the xOz and yOz planes with the surface. However, the surface contains many more such lines. Practically, through every point of the surface passes a pair of straight lines entirely contained in the surface. These lines are called *straight-line generators* of the one-sheeted hyperboloid.

To demonstrate this fact, we rewrite the equation of the one-sheeted hyperboloid as

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2},$$

or equivalently

$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right).$$

Now consider the system of equations

$$\begin{cases}\lambda\left(\dfrac{x}{a}+\dfrac{z}{c}\right)=\mu\left(1+\dfrac{y}{b}\right),\\[2mm] \mu\left(\dfrac{x}{a}-\dfrac{z}{c}\right)=\lambda\left(1-\dfrac{y}{b}\right),\end{cases} \tag{5.4.4}$$

where λ and μ are real numbers not simultaneously zero.

As stated, since the two parameters are not simultaneously zero, the system (5.4.4) represents a straight line. By multiplying the two equations, we obtain either $0 = 0$ (if one parameter is zero) or the equation of the one-sheeted hyperboloid. This implies that the straight line (5.4.4) lies on the hyperboloid. Letting the two parameters vary generates a family of lines forming the *first family of straight-line generators of the hyperboloid*. The system of equations (5.4.4) is, in fact, formed by the equations of two pencil planes. If we divide the first equation by λ and the second by μ, we obtain the system

$$\begin{cases}\dfrac{x}{a}+\dfrac{z}{c}=\dfrac{\mu}{\lambda}\left(1+\dfrac{y}{b}\right),\\[2mm] \dfrac{x}{a}-\dfrac{z}{c}=\dfrac{\lambda}{\mu}\left(1-\dfrac{y}{b}\right),\end{cases} \tag{5.4.5}$$

or, by renaming the parameter,

$$\begin{cases}\dfrac{x}{a}+\dfrac{z}{c}=\nu\left(1+\dfrac{y}{b}\right),\\[2mm] \dfrac{x}{a}-\dfrac{z}{c}=\dfrac{1}{\nu}\left(1-\dfrac{y}{b}\right),\end{cases} \tag{5.4.6}$$

where, naturally, $\nu \neq 0$.

The system of equations (5.4.6) is not entirely equivalent to system (5.4.4). Indeed, the first pencil of planes lacks the plane $y + b = 0$, while the second pencil lacks the plane $y - b = 0$. However, these two planes do not contain straight-line generators of the surface (the intersections of these planes with the hyperboloid are non-degenerate hyperbolas). Thus, the system of equations (5.4.6) describes the same family of straight-line generators as system (5.4.4), with the advantage of using a single parameter. The generators in this family will be called *ν-generators*.

The second family of straight-line generators is obtained in a similar way, by identifying the first-degree factors differently. Its equations are

$$\begin{cases} \alpha\left(\dfrac{x}{a}+\dfrac{z}{c}\right)=\beta\left(1-\dfrac{y}{b}\right), \\ \beta\left(\dfrac{x}{a}-\dfrac{z}{c}\right)=\alpha\left(1+\dfrac{y}{b}\right), \end{cases} \tag{5.4.7}$$

where α and β are again real parameters not simultaneously zero.

Reasoning as above, this system of equations can also be simplified to the form

$$\begin{cases} \dfrac{x}{a}+\dfrac{z}{c}=\gamma\left(1-\dfrac{y}{b}\right), \\ \dfrac{x}{a}-\dfrac{z}{c}=\dfrac{1}{\gamma}\left(1+\dfrac{y}{b}\right), \end{cases} \tag{5.4.8}$$

with $\gamma \neq 0$. The generators in this second family will be called *γ-generators*.

Examples. 1. Consider the one-sheeted hyperboloid

$$x^2+y^2-\frac{z^2}{4}=1.$$

We will determine the straight-line generators of the hyperboloid passing through the point $A(1,4,8)$ and calculate the angle between them.

First, note that point A is indeed on the hyperboloid (its coordinates satisfy the hyperboloid equation). Moving the term containing y^2 to the right-hand side and factoring both sides into first-degree factors, we get

$$\left(x+\frac{z}{2}\right)\left(x-\frac{z}{2}\right)=(1+y)(1-y).$$

We first determine the generator from the first family. Its equations are

$$\begin{cases} \lambda\left(x+\dfrac{z}{2}\right)=\mu(1+y), \\ \mu\left(x-\dfrac{z}{2}\right)=\lambda(1-y), \end{cases}$$

or, after calculation,

$$\begin{cases} 2\lambda x-2\mu y+\lambda z-2\mu=0, \\ 2\mu x+2\lambda y-\mu z-2\lambda=0. \end{cases}$$

Imposing the condition that the generator passes through point A, we obtain the system

$$\begin{cases} 2\lambda - 8\mu + 8\lambda - 2\mu = 0, \\ 2\mu + 8\lambda - 8\mu - 2\lambda = 0, \end{cases}$$

which reduces to a single equation, $\lambda - \mu = 0$. Assigning an arbitrary non-zero value to one parameter, we set $\lambda = 1$ and consequently $\mu = 1$, so the equations of the generator from the first family become

$$(\Delta_1)\ \begin{cases} 2x - 2y + z - 2 = 0, \\ 2x + 2y - z - 2 = 0. \end{cases}$$

Next, we determine the generator from the second family. The equations of an arbitrary generator in this family are:

$$\begin{cases} \alpha\left(x + \dfrac{z}{2}\right) = \beta(1 - y), \\ \beta\left(x - \dfrac{z}{2}\right) = \alpha(1 + y), \end{cases}$$

or, after calculation,

$$\begin{cases} 2\alpha x + 2\beta y + \alpha z - 2\beta = 0, \\ 2\beta x - 2\alpha y - \beta z - 2\alpha = 0. \end{cases}$$

Imposing the condition that the generator passes through point A, we obtain

$$\begin{cases} 2\alpha + 8\beta + 8\alpha - 2\beta = 0, \\ 2\beta - 8\alpha - 8\beta - 2\alpha = 0, \end{cases}$$

which leads to the relation $\beta = -\frac{5}{3}\alpha$. Setting $\alpha = 3$, we find $\beta = -5$, so the equations of the generator from the second family passing through A are

$$(\Delta_2)\ \begin{cases} 6x - 10y + 3z + 10 = 0, \\ 10x + 6y - 5z + 6 = 0. \end{cases}$$

To calculate the angle between the two generators, we first find their direction vectors. For the first generator, the normal vectors to the two planes defining it are $\mathbf{n}_1(2, -2, 1)$ and $\mathbf{n}_2(2, 2, -1)$. Thus,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 2 & 2 & -1 \end{vmatrix} = (0, 4, 8),$$

so we can take $\mathbf{v}_1(0,1,2)$ as the direction vector of the first generator.

Similarly, for the second generator, the normal vectors to the two planes defining it are $\mathbf{n}_3(6,-10,3)$ and $\mathbf{n}_4(10,6,-5)$. Hence,

$$\mathbf{n}_3 \times \mathbf{n}_4 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -10 & 3 \\ 10 & 6 & -5 \end{vmatrix} = (32, 60, 136),$$

so we can take $\mathbf{v}_2(8,15,34)$ as the direction vector of the second generator.

Thus, the cosine of the angle between the two generators is

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|} = \frac{83}{85}.$$

5.5 The Two-Sheeted Hyperboloid

Definition 5.3. The *two-sheeted hyperboloid* is defined as the locus of points in space whose coordinates, relative to an orthonormal coordinate system, satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \tag{5.5.1}$$

where a, b, c are strictly positive constants.

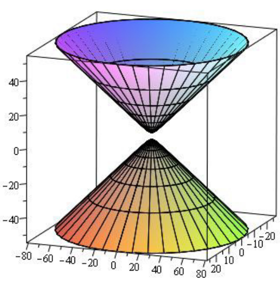


Figure 5.4: The two-sheeted hyperboloid

The Shape of the Two-Sheeted Hyperboloid. The *symmetries* are the same as in the case of the ellipsoid, so we proceed directly to the study of intersections with planes parallel to the coordinate planes.

(1) *Intersections with planes parallel to the xOy plane.* We study the solutions of the system of equations

$$\begin{cases} z = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1, \end{cases}$$

or equivalently

$$\begin{cases} z = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{h^2}{c^2} - 1. \end{cases} \tag{5.5.2}$$

We analyse three cases:

(a) If $\dfrac{h^2}{c^2} - 1 < 0$, i.e., $-c < h < c$, the system (5.5.2) has no solutions, so the plane and the surface do not intersect.

(b) If $\dfrac{h^2}{c^2} - 1 = 0$, i.e., $h = \pm c$, the system has a unique solution for each value of h (c or $-c$). In this case, the plane is tangent to the surface at the points $(0, 0, c)$ and $(0, 0, -c)$, respectively.

(c) If $\dfrac{h^2}{c^2} - 1 > 0$, i.e., $|h| > c$, the system (5.5.2) is equivalent to the system

$$\begin{cases} z = h, \\ \dfrac{x^2}{\left(a\sqrt{\frac{h^2}{c^2} - 1}\right)^2} + \dfrac{y^2}{\left(b\sqrt{\frac{h^2}{c^2} - 1}\right)^2} = 1, \end{cases}$$

which represents the equations of an ellipse with semi-axes $a\sqrt{\frac{h^2}{c^2} - 1}$ and $b\sqrt{\frac{h^2}{c^2} - 1}$, located in the plane $z = h$.

(2) *Intersections with planes parallel to the xOz plane.* This time, we study the solutions of the system of equations

$$\begin{cases} y = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1, \end{cases}$$

or equivalently

$$\begin{cases} y = h, \\ \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = -1 - \dfrac{h^2}{b^2}. \end{cases} \tag{5.5.3}$$

This system is equivalent to

$$\begin{cases} y = h, \\ \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} = 1 + \dfrac{h^2}{b^2}, \end{cases}$$

or

$$\begin{cases} y = h, \\ \dfrac{z^2}{\left(c\sqrt{\frac{h^2}{b^2}+1}\right)^2} - \dfrac{x^2}{\left(a\sqrt{\frac{h^2}{b^2}+1}\right)^2} = 1, \end{cases}$$

which represents, for any value of h, the equations of a hyperbola with semi-axes $c\sqrt{\frac{h^2}{b^2}+1}$ and $a\sqrt{\frac{h^2}{b^2}+1}$, located in the plane $y = h$, where the axis of the hyperbola intersecting the curve is parallel to the Oz axis, and the other axis is parallel to the Ox axis.

(3) *Intersections with planes parallel to the yOz plane.* This case is perfectly analogous to the one discussed above.

The Two-Sheeted Hyperboloid of Revolution. If $a = b$, the equation of the two-sheeted hyperboloid becomes

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = -1.$$

This particular type of hyperboloid is called a *two-sheeted hyperboloid of revolution* because, as we will see in the next chapter, it can be obtained by rotating a hyperbola around the Oz axis. It is worth noting that, for two-sheeted hyperboloids of revolution, *any plane passing through the Oz axis is a plane of symmetry for the hyperboloid.*

The Tangent Plane at a Point of the Two-Sheeted Hyperboloid. Let $M_0(x_0, y_0, z_0)$ be an arbitrary point on the two-sheeted hyperboloid. It can be shown, exactly as in the case of the ellipsoid, that the equation of the tangent plane at M_0 on the hyperboloid is

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = -1,$$

which can be obtained by duplicating the equation of the two-sheeted hyperboloid.

5.6 The Elliptic Paraboloid

Definition 5.4. The *elliptic paraboloid* is defined as the set of points in space whose coordinates, relative to a Cartesian coordinate system, satisfy an equation of the form

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z, \tag{5.6.1}$$

where p and q are strictly positive real numbers, referred to as the *parameters of the elliptic paraboloid.*

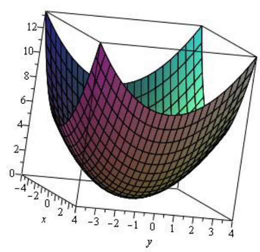


Figure 5.5: The elliptic paraboloid

The Shape of the Elliptic Paraboloid. The *symmetries* of the elliptic paraboloid are not as numerous as those of the quadric surfaces studied so far. Specifically, it has:

(1) Two planes of symmetry (yOz and xOz, because the coordinates x and y appear only squared);

(2) One axis of symmetry, the Oz axis, which is the intersection of the two planes of symmetry.

Next, as before, we study the intersection of the elliptic paraboloid with planes parallel to the three coordinate planes.

(1) *Intersections with planes parallel to the xOy plane.* We study the solutions of the system

$$\begin{cases} z = h, \\ \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2z, \end{cases}$$

or equivalently

$$\begin{cases} z = h, \\ \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2h. \end{cases} \tag{5.6.2}$$

We examine three cases:

(a) If $h < 0$, the system (5.6.2) has no solutions, meaning the plane and the surface have no common points.

(b) If $h = 0$, the system (5.6.2) has a unique solution, $(0, 0, 0)$, i.e., the origin. This means that the xOy coordinate plane is tangent to the elliptic paraboloid at the origin.

(c) If $h > 0$, the system (5.6.2) can be rewritten as

$$\begin{cases} z = h, \\ \dfrac{x^2}{\left(\sqrt{2ph}\right)^2} + \dfrac{y^2}{\left(\sqrt{2qh}\right)^2} = 1, \end{cases}$$

which represents the equations of an ellipse located in the plane $z = h$, with semi-axes $\sqrt{2ph}$ and $\sqrt{2qh}$.

(2) *Intersections with planes parallel to the xOz plane.* In this case, we study the system

$$\begin{cases} y = h, \\ \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2z, \end{cases}$$

or equivalently

$$\begin{cases} y = h, \\ x^2 = 2pz - \dfrac{ph^2}{q}, \end{cases}$$

which are the equations of a parabola with parameter p, located in the plane $y = h$.

(3) *Intersections with planes parallel to the yOz plane.* We study the solutions of the system

$$\begin{cases} x = h, \\ \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2z, \end{cases}$$

or equivalently

$$\begin{cases} x = h, \\ y^2 = 2qz - \dfrac{qh^2}{p}, \end{cases}$$

which are the equations of a parabola with parameter q, located in the plane $x = h$.

The Elliptic Paraboloid of Revolution. If the two parameters of the paraboloid are equal, $p = q$, the equation of the surface becomes

$$\frac{x^2 + y^2}{p} = 2z,$$

or

$$x^2 + y^2 = 2pz.$$

This particular paraboloid is called an *elliptic paraboloid of revolution*. The surface can indeed be obtained by rotating a parabola around the Oz axis, as we will see later.

The Tangent Plane at a Point of the Elliptic Paraboloid. Let $M_0(x_0, y_0, z_0)$ be an arbitrary point on the elliptic paraboloid (5.6.1). We first examine the intersection between the paraboloid and an arbitrary line passing through M_0. The parametric equations of such a line can be written as:

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \\ z = z_0 + nt. \end{cases}$$

Substituting into the equation of the paraboloid, we obtain:

$$\frac{(x_0+lt)^2}{p}+\frac{(y_0+mt)^2}{q}=2(z_0+nt),$$

or equivalently

$$q(x_0+lt)^2+p(y_0+mt)^2-2pq(z_0+nt)=0.$$

After expanding and grouping terms by powers of t, the equation becomes:

$$t^2(ql^2+pm^2)+2t(qx_0l+py_0m-pqn)+qx_0^2+py_0^2-2pqz_0=0.$$

The constant term equals zero because M_0 lies on the paraboloid, so the intersection equation reduces to

$$t^2(ql^2+pm^2)+2t(qx_0l+py_0m-pqn)=0. \tag{5.6.3}$$

For the line and paraboloid to intersect at a single (double) point, the intersection equation (5.6.3) must have a double root. Since $t=0$ is always a root, the other root must also be zero, which occurs if the linear term vanishes:

$$qx_0l+py_0m-pqn=0. \tag{5.6.4}$$

Let $\mathbf{n}$ be the vector with components $(qx_0, py_0, -pq)$, and let $\mathbf{v}(l,m,n)$ be the direction vector of the line. Then, equation (5.6.4) is equivalent to $\mathbf{n}\cdot\mathbf{v}=0$, which means *any line passing through M_0 whose direction vector satisfies equation (5.6.4) is perpendicular to the vector* $\mathbf{n}$. This implies that $\mathbf{n}$ is the normal vector to the tangent plane of the elliptic paraboloid at M_0. Therefore, the equation of the tangent plane at M_0 is:

$$qx_0(x-x_0)+py_0(y-y_0)-pq(z-z_0)=0,$$

or

$$qx_0x+py_0y-pqz-qx_0^2-py_0^2+pqz_0=0,$$

or equivalently

$$qx_0x+py_0y-pqz-pqz_0-(qx_0^2+py_0^2-2pqz_0)=0.$$

The term in parentheses is zero because M_0 lies on the paraboloid, so the equation becomes:

$$qx_0x+py_0y=pq(z+z_0),$$

or, dividing through by pq,

$$\frac{xx_0}{p} + \frac{yy_0}{q} = z + z_0. \tag{5.6.5}$$

It is worth noting that, for the elliptic paraboloid, as with the other quadric surfaces studied so far, the equation of the tangent plane can be obtained from the equation of the paraboloid (5.6.1) by *duplication*. The duplication rules are as follows:

- Replace x^2 and y^2 with xx_0 (respectively yy_0);
- Replace z with $(z + z_0)/2$.

5.7 The Hyperbolic Paraboloid

Definition 5.5. The set of points in space whose coordinates, relative to a Cartesian coordinate system, satisfy an equation of the form

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z, \tag{5.7.1}$$

where p and q are strictly positive real numbers, called the *parameters of the hyperbolic paraboloid*, is referred to as a *hyperbolic paraboloid*.

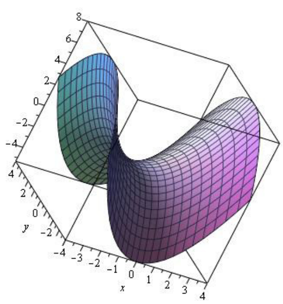


Figure 5.6: The hyperbolic paraboloid

Shape of the hyperbolic paraboloid. The *symmetries* of the hyperbolic paraboloid are the same as those of the elliptic paraboloid:

(1) Two planes of symmetry (yOz and xOz, as the coordinates x and z appear only squared);

(2) An axis of symmetry, the Oz axis, as the intersection of the two planes of symmetry.

Next, we will study the intersection of the hyperbolic paraboloid with planes parallel to the three coordinate planes.

(1) *Intersections with planes parallel to the xOy plane.* We examine the solutions of the system

$$\begin{cases} z = h \\ \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2z, \end{cases}$$

or

$$\begin{cases} z = h \\ \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2h. \end{cases} \tag{5.7.2}$$

There are three cases to consider:

(a) If $h < 0$, then $-h > 0$, and the system (5.7.2) can be rewritten as

$$\begin{cases} z = h \\ \dfrac{y^2}{\left(\sqrt{-2qh}\right)^2} - \dfrac{x^2}{\left(\sqrt{-2ph}\right)^2} = 1, \end{cases}$$

which represents the equations of a hyperbola with semi-axes $\sqrt{-2qh}$ and $\sqrt{-2ph}$, lying in the plane $z = h$, such that the semi-axis intersecting the hyperbola is parallel to the Oy axis, and the other semi-axis is parallel to the Ox axis.

(b) If $h = 0$, then the system (5.7.2) becomes

$$\begin{cases} z = 0 \\ \dfrac{x^2}{p} - \dfrac{y^2}{q} = 0. \end{cases}$$

These are the equations of a pair of concurrent lines lying in the xOy plane, passing through the origin:

$$\begin{cases} z = 0, \\ \dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} z = 0, \\ \dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = 0. \end{cases}$$

(c) If $h > 0$, the system (5.7.2) can be written as

$$\begin{cases} z = h \\ \dfrac{x^2}{\left(\sqrt{2ph}\right)^2} - \dfrac{y^2}{\left(\sqrt{2qh}\right)^2} = 1, \end{cases}$$

which represents the equations of a hyperbola with semi-axes $\sqrt{2ph}$ and $\sqrt{2qh}$, lying in the plane $z = h$, such that the semi-axis intersecting the hyperbola is parallel to the Ox axis, and the other semi-axis is parallel to the Oy axis.

(2) *Intersections with planes parallel to the xOz plane.* In this case, we examine the system

$$\begin{cases} y = h, \\ \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2z \end{cases}$$

or

$$\begin{cases} y = h, \\ x^2 = 2pz + \dfrac{ph^2}{q}, \end{cases}$$

which are the equations of a parabola with parameter p, lying in the plane $y = h$.

(3) *Intersections with planes parallel to the yOz plane.* We examine the solutions of the system

$$\begin{cases} x = h, \\ \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2z \end{cases}$$

or

$$\begin{cases} x = h, \\ y^2 = -2qz + \dfrac{qh^2}{p}, \end{cases}$$

which are the equations of a parabola with parameter q, lying in the plane $x = h$.

Rectilinear generators of the hyperbolic paraboloid. The hyperbolic paraboloid shares an important property with the one-sheeted hyperboloid: both contain two families of straight lines, with each point on the paraboloid lying on a pair of lines, one from each family. To find the equations of these families of lines, called *rectilinear generators of the hyperbolic paraboloid*, we proceed as for the one-sheeted hyperboloid.

First, rewrite the equation of the hyperbolic paraboloid as

$$\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) \cdot \left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z \cdot 1.$$

From this equation, we can derive a family of lines:

$$\begin{cases} \lambda\left(\dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}}\right) = 2\mu z, \\ \mu\left(\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}}\right) = \lambda, \end{cases} \tag{5.7.3}$$

where λ and μ are real parameters that do not vanish simultaneously. Multiplying the two equations in system (5.7.3), we obtain either $0 = 0$, if one of the parameters vanishes, or the equation of the hyperbolic paraboloid, meaning the line lies on the paraboloid for any admissible values of the two parameters[1].

Similarly, it can be shown that the lines

$$\begin{cases} \alpha\left(\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}}\right) = 2\beta z, \\ \beta\left(\dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}}\right) = \alpha, \end{cases} \tag{5.7.4}$$

where α and β are real parameters that do not vanish simultaneously, are also located on the hyperbolic paraboloid (5.7.1).

It can be demonstrated that through each point on the hyperboloid passes exactly one pair of rectilinear generators, one from each family.

The tangent plane at a point on the hyperbolic paraboloid. It can be easily shown, as in the case of the elliptic paraboloid, that the equation of the tangent plane

[1]"Admissible" means that $\lambda^2 + \mu^2 \neq 0$.

to the paraboloid at a point $M_0(x_0, y_0, z_0)$ can be derived by duplication from the equation of the surface. Thus, the equation of the tangent plane is

$$\frac{xx_0}{p} - \frac{yy_0}{q} = z + z_0. \tag{5.7.5}$$

5.8 The Elliptic Cylinder

Definition 5.6. The *elliptic cylinder* is defined as the locus of points in space whose coordinates, relative to an orthogonal coordinate system, satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{5.8.1}$$

where a, b are two strictly positive real numbers, called the *semi-axes of the elliptic cylinder*.

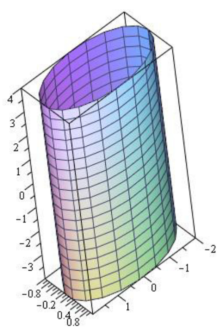


Figure 5.7: The elliptic cylinder

Shape of the elliptic cylinder. We begin by examining the symmetries of the cylinder. Clearly, the cylinder exhibits all the symmetries of the ellipsoid and hyperboloids:

- three planes of symmetry (the coordinate planes);
- three axes of symmetry (the coordinate axes);
- a centre of symmetry (the origin).

Moreover, since the cylinder equation does not involve the z coordinate, the elliptic cylinder has an additional family of planes of symmetry (all planes parallel to the xOy plane) and two families of axes of symmetry:

- any line parallel to the Ox axis and intersecting the Oz axis;
- any line parallel to the Oy axis and intersecting the Oz axis.

Furthermore, any point on the Oz axis is a centre of symmetry.

Now, let us study the intersections with planes parallel to the coordinate planes.

(1) *Planes parallel to the xOy plane.* The system to analyse is

$$\begin{cases} z = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \end{cases}$$

which represents, for any real h, the equations of an ellipse lying in the plane $z = h$, with semi-axes equal to a and b.

(2) *Planes parallel to the xOz plane.* In this case, the system to study is

$$\begin{cases} y = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \end{cases}$$

or

$$\begin{cases} y = h, \\ x^2 = a^2\left(1 - \dfrac{h^2}{b^2}\right). \end{cases} \tag{5.8.2}$$

Here, three cases must be examined:

(a) If $1 - \dfrac{h^2}{b^2} < 0$, i.e., $h^2 > b^2$, then system (5.8.2) has no solutions, meaning the plane and the cylinder do not intersect.

(b) If $1 - \dfrac{h^2}{b^2} = 0$, i.e., $h = \pm b$, then system (5.8.2) reduces to

$$\begin{cases} y = \pm b, \\ x = 0, \end{cases}$$

which are the equations of a line parallel to the Oz axis (one line for each value of h, b or $-b$).

(c) If $1 - \dfrac{h^2}{b^2} > 0$, i.e., $h^2 < b^2$, then system (5.8.2) reduces to

$$\begin{cases} y = h, \\ x = \pm a \sqrt{\left(1 - \dfrac{h^2}{b^2}\right)}, \end{cases}$$

which represents a pair of lines (parallel to the Oz axis) for each admissible value of h.

(3) *Planes parallel to the yOz plane.* The analysis is perfectly analogous to the previous case.

Remark. The elliptic cylinder is a so-called *cylindrical surface*, generated by a family of lines parallel to a given line (the Oz axis, in our case), called *generators*, which intersect a given curve. In our case, that given curve can be chosen as any one of the (equal) ellipses obtained by intersections with the xOy plane.

Elliptic cylinder of rotation (circular cylinder). If the two semi-axes of the cylinder are equal, $a = b$, then the cylinder equation can be written as

$$x^2 + y^2 = a^2.$$

This surface is called a *cylinder of rotation or circular cylinder* with radius a, and it can be obtained by rotating any of its generators about the Oz axis.

The tangent plane at a point on the elliptic cylinder. Let $M_0(x_0, y_0, z_0)$ be an arbitrary point on the elliptic cylinder (5.8.1). We will study, as usual, the condition for a line passing through M_0 to be tangent to the cylinder. Recall that the equations of an arbitrary line through M_0 are:

$$(\Delta) \begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \\ z = z_0 + nt. \end{cases}$$

If we substitute into the cylinder equation, we obtain:

$$\frac{(x_0 + lt)^2}{a^2} + \frac{(y_0 + mt)^2}{b^2} = 1$$

or

$$b^2 (x_0 + lt)^2 + a^2 (y_0 + mt)^2 - a^2 b^2 = 0.$$

After performing the calculations, we obtain the equation

$$t^2(b^2 l^2 + a^2 m^2) + 2t(b^2 x_0 l + a^2 y_0 m) + b^2 x_0^2 + a^2 y_0^2 - a^2 b^2 = 0.$$

Since M_0 belongs to the cylinder, the constant term equals zero, so the intersection equation reduces to

$$t^2(b^2 l^2 + a^2 m^2) + 2t(b^2 x_0 l + a^2 y_0 m) = 0. \tag{5.8.3}$$

For the line and the cylinder to have a single (double) point in common, the intersection equation (5.8.3) must have a double root. But one root is always $t = 0$; thus, the second root must also be zero, which is possible only if the first-degree term in t vanishes, i.e., if

$$b^2 x_0 l + a^2 y_0 m = 0. \tag{5.8.4}$$

If $\mathbf{n}$ is the vector with components $(b^2 x_0, a^2 y_0, 0)$, and $\mathbf{v}(l, m, n)$ is the direction vector of the line, then equation (5.8.4) is equivalent to $\mathbf{n} \cdot \mathbf{v} = 0$, i.e., *any line passing through M_0, whose direction vector satisfies equation (5.8.4), is perpendicular to vector* $\mathbf{n}$. But this simply means that $\mathbf{n}$ is the normal vector to the tangent plane at the elliptic cylinder at point M_0. Thus, the equation of the tangent plane at M_0 is:

$$b^2 x_0 (x - x_0) + a^2 y_0 (y - y_0) = 0$$

or, after simplifying and noting once again that M_0 belongs to the cylinder,

$$\frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1, \tag{5.8.5}$$

i.e., once again, the equation of the tangent plane can be derived by duplication, starting from the elliptic cylinder equation.

5.9 The Hyperbolic Cylinder

Definition 5.7. The *hyperbolic cylinder* is defined as the locus of points in space whose coordinates, relative to an orthogonal coordinate system, satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{5.9.1}$$

where a, b are two strictly positive real numbers, called the *semi-axes of the hyperbolic cylinder*.

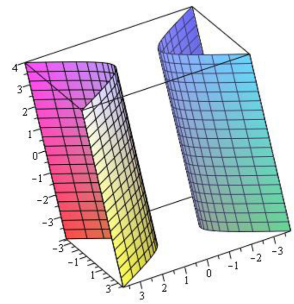


Figure 5.8: The hyperbolic cylinder

Shape of the cylinder. The symmetries of the hyperbolic cylinder are the same as those of the elliptic cylinder, so they will not be discussed again.

Now, let us study the intersections with planes parallel to the coordinate planes.

(1) *Planes parallel to the xOy plane.* The system to analyse is

$$\begin{cases} z = h, \\ \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, \end{cases}$$

which represents, for any real h, the equations of a hyperbola lying in the plane $z = h$, with semi-axes equal to a and b.

(2) *Planes parallel to the xOz plane.* In this case, the system to study is

$$\begin{cases} y = h, \\ \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \end{cases}$$

or

$$\begin{cases} y = h, \\ x^2 = a^2 \left(1 + \dfrac{h^2}{b^2}\right). \end{cases} \tag{5.9.2}$$

This represents, for each real h, a pair of distinct lines:

$$\begin{cases} y = h, \\ x = \pm a \sqrt{1 + \dfrac{h^2}{b^2}}. \end{cases}$$

(3) *Planes parallel to the yOz plane.* The system determining the intersection is now

$$\begin{cases} x = h, \\ \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \end{cases}$$

or

$$\begin{cases} x = h, \\ y^2 = b^2\left(\dfrac{h^2}{a^2} - 1\right). \end{cases} \tag{5.9.3}$$

Here, three cases must be examined:

(a) If $\dfrac{h^2}{a^2} - 1 < 0$, i.e., $h^2 < a^2$, then system (5.9.3) has no solutions, meaning the plane and the cylinder do not intersect.

(b) If $\dfrac{h^2}{a^2} = 1$, i.e., $h = \pm a$, then system (5.9.3) reduces to

$$\begin{cases} x = \pm a, \\ y = 0, \end{cases}$$

which are the equations of a line parallel to the Oz axis (one line for each value of h, a or $-a$).

(c) If $\dfrac{h^2}{a^2} - 1 > 0$, i.e., $h^2 > a^2$, then system (5.9.3) reduces to

$$\begin{cases} x = h, \\ y = \pm b\sqrt{\left(\dfrac{h^2}{a^2} - 1\right)}, \end{cases}$$

which represents a pair of lines (parallel to the Oz axis) for each admissible value of h.

Remark. The hyperbolic cylinder, like the elliptic cylinder, is a *cylindrical surface*, generated by a family of lines parallel to a given line (the Oz axis, in our case), called *generators*, which intersect a given curve. In our case, that given curve can be chosen as any one of the (equal) hyperbolas obtained by intersections with the xOy plane.

The tangent plane at a point on the hyperbolic cylinder. As in the case of the elliptic cylinder, it can be shown that the tangent plane at a point $M_0(x_0, y_0, z_0)$ of the hyperbolic cylinder can be derived from the surface equation by duplication. Thus, the equation of the tangent plane is

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1. \tag{5.9.4}$$

5.10 The Parabolic Cylinder

Definition 5.8. The *parabolic cylinder* is defined as the locus of points in space whose coordinates, relative to an orthogonal coordinate system, satisfy the equation

$$y^2 = 2px, \tag{5.10.1}$$

where p is a strictly positive real number, called the *parameter of the parabolic cylinder*.

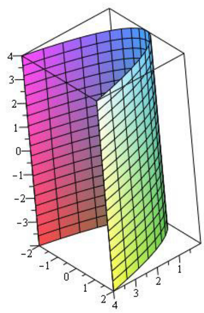


Figure 5.9: The parabolic cylinder

Shape of the parabolic cylinder. The parabolic cylinder (5.10.1) is symmetric with respect to:

- the yOz plane;
- the xOy plane and any plane parallel to it;
- the Oy axis and any line parallel to it that intersects the Oz axis.

Now, let us study the intersections with planes parallel to the coordinate planes.

(1) *Planes parallel to the xOy plane.* The system to analyse is

$$\begin{cases} z = h, \\ y^2 = 2px, \end{cases}$$

which represents, for any real h, the equations of a parabola with parameter p, lying in the plane $z = h$.

(2) *Planes parallel to the xOz plane.* In this case, the system to study is

$$\begin{cases} y = h, \\ y^2 = 2px \end{cases}$$

or

$$\begin{cases} y = h, \\ x = \dfrac{h^2}{2p}. \end{cases} \tag{5.10.2}$$

Equation (5.10.2) represents a line parallel to the Oz axis, for any value of h.

(3) *Planes parallel to the yOz plane.* The system to investigate is

$$\begin{cases} x = h, \\ y^2 = 2px \end{cases}$$

or

$$\begin{cases} x = h, \\ y^2 = 2ph. \end{cases} \tag{5.10.3}$$

Here, three cases must be examined:

(a) If $h < 0$, then system (5.10.3) has no solutions, meaning the plane and the cylinder do not intersect.

(b) If $h = 0$, then system (5.10.3) reduces to

$$\begin{cases} x = 0, \\ y = 0, \end{cases}$$

which are the equations of the Oz axis.

(c) If $h > 0$, then system (5.10.3) reduces to

$$\begin{cases} x = h, \\ y = \pm\sqrt{2ph}, \end{cases}$$

which represents a pair of lines (parallel to the Oz axis) for each admissible value of h.

Remark. The parabolic cylinder is also a *cylindrical surface*, generated by a family of lines parallel to a given line (the Oz axis, in our case), called *generators*, which intersect a given curve. In our case, that given curve can be chosen as any one of the (equal) parabolas obtained by intersections with the xOy plane.

The tangent plane at a point on the parabolic cylinder. Let $M_0(x_0, y_0, z_0)$ be an arbitrary point on the parabolic cylinder (5.10.1). We will study, as usual, the condition for a line passing through M_0 to be tangent to the cylinder. Recall that the equations of an arbitrary line through M_0 are:

$$(\Delta)\ \begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \\ z = z_0 + nt. \end{cases}$$

If we substitute into the cylinder equation, we obtain:

$$(y_0 + mt)^2 = 2p(x_0 + lt).$$

After performing the calculations, we obtain the equation

$$m^2t^2 + 2t(-pl + y_0m) + y_0^2 - 2px_0 = 0.$$

Since M_0 belongs to the cylinder, the constant term equals zero, so the intersection equation reduces to

$$m^2t^2 + 2t(-pl + y_0m) = 0. \tag{5.10.4}$$

For the line and the cylinder to have a single (double) point in common, the intersection equation (5.10.4) must have a double root. But one root is always $t = 0$; thus, the second root must also be zero, which is possible only if the first-degree term in t vanishes, i.e., if

$$-pl + y_0m = 0. \tag{5.10.5}$$

If **n** is the vector with components $(-p, y_0, 0)$, and $\mathbf{v}(l, m, n)$ is the direction vector of the line, then equation (5.10.5) is equivalent to $\mathbf{n} \cdot \mathbf{v} = 0$, i.e., *any line passing through M_0, whose direction vector satisfies equation (5.10.5), is perpendicular to vector* **n**. But this simply means that **n** is the normal vector to the tangent plane at the parabolic cylinder at point M_0. Thus, the equation of the tangent plane at M_0 is:

$$-p(x - x_0) + y_0(y - y_0) = 0$$

or, after simplifying and noting once again that M_0 belongs to the cylinder,

$$yy_0 = p(x + x_0), \tag{5.10.6}$$

i.e., once again, the equation of the tangent plane can be derived by duplication, starting from the parabolic cylinder equation and applying the rules of duplication:

- y^2 is replaced by yy_0;
- x is replaced by $(x + x_0)/2$.

5.11 Probleme

Problem 5.1. Să se găsească punctele de intersecţie ale elipsoidului

$$\frac{x^2}{12} + \frac{y^2}{8} + \frac{z^2}{4} = 1$$

cu dreapta

$$x = 4 + 2t, \ y = -6 + 3t, \ z = -2 - 2t.$$

Problem 5.2. Să se determine curbele de intersecţie ale elipsoidului

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

cu planele de coordonate.

Problem 5.3. Să se scrie ecuaţia planului tangent la elipsoidul

$$\frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1$$

în punctele lui de intersecţie cu planul $x = y = z$.

Problem 5.4. Să se scrie ecuaţiile planelor tangente la elipsoidul

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{8} = 1,$$

paralele cu planul

$$3x - 2y + 5z + 1 = 0.$$

Problem 5.5. Determinaţi unghiul pe care îl formează generatoarele conului

$$x^2 + y^2 - \frac{z^2}{6} = 0$$

cu axa Oz.

Problem 5.6. Determinaţi punctele de intersecţie ale elipsoidului

$$\frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} = 1$$

cu dreapta

$$\frac{x-4}{2} = \frac{y+6}{-3} = \frac{z+2}{-2}.$$

Problem 5.7. Determinaţi punctele de intersecţie ale hiperboloidului cu două pânze

$$\frac{x^2}{4} + \frac{y^2}{1} - \frac{z^2}{9} = -1$$

cu dreapta

$$\frac{x-3}{1} = \frac{y-1}{1} = \frac{z-6}{3}.$$

Problem 5.8. Determinaţi punctele de intersecţie ale hiperboloidului cu o pânză

$$\frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

cu dreapta

$$\frac{x-4}{4} = \frac{y+2}{0} = \frac{z-1}{1}.$$

Problem 5.9. Determinaţi punctele de intersecţie ale paraboloidului hiperbolic

$$x^2 - 4y^2 = 4z$$

cu dreapta

$$\frac{x-2}{2} = \frac{y}{1} = \frac{z-3}{-2}.$$

Problem 5.10. Determinaţi o dreaptă care să treacă prin punctul $M(5,1,2)$ şi care să aibă un singur punct comun cu suprafaţa

$$\frac{x^2}{9}+\frac{y^2}{4}-\frac{z^2}{1}=1.$$

Problem 5.11. Determinaţi generatoarele rectilinii ale suprafeţei

$$\frac{x^2}{9}+\frac{y^2}{4}-\frac{z^2}{16}=1$$

care trec prin punctul $M(6,2,8)$.

Problem 5.12. Determinaţi generatoarele rectilinii ale suprafeţei

$$\frac{x^2}{16}-\frac{y^2}{4}=z$$

care sunt paralele cu planul $3x+2y-4z=0$.

Problem 5.13. Stabiliţi ecuaţia planului tangent la suprafaţa

$$\frac{x^2}{9}+\frac{y^2}{1}-\frac{z^2}{4}=-1$$

în punctul $M(-6,2,6)$.

Problem 5.14. Să se scrie ecuaţia planului tangent la hiperboloidul

$$-\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{17}-1=0$$

în punctul $M\left(2,-1,\frac{17}{3}\right)$.

Problem 5.15. Să se scrie ecuaţia planului tangent la hiperboloidul

$$\frac{x^2}{4}+\frac{y^2}{1}-\frac{z^2}{5}+1=0$$

în punctul $M\left(4,-\sqrt{15},10\right)$.

Problem 5.16. Să se scrie ecuaţia planului tangent la hiperboloidul

$$\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{3} - 1 = 0,$$

paralel cu planul

$$2x + 3y - z + 11 = 0.$$

Problem 5.17. Să se scrie ecuaţia planului tangent la hiperboloidul

$$3x^2 - 12y^2 + z^2 - 3 = 0,$$

paralel cu planul

$$2x + 3y - z + 11 = 0.$$

Problem 5.18. Să se scrie ecuaţiile dreptelor care trec prin punctul $M(6, 2, 8)$ şi se află pe hiperboloidul

$$16x^2 + 36y^2 - 9z^2 - 144 = 0.$$

Problem 5.19. Să se găsească punctele de intersecţie ale dreptei

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{1}$$

cu paraboloidul eliptic

$$\frac{x^2}{4} + \frac{y^2}{9} = 2z.$$

Problem 5.20. Să se găsească punctele de intersecţie ale dreptei

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{1}$$

cu paraboloidul hiperbolic

$$\frac{x^2}{4} - \frac{y^2}{9} = 2z.$$

Problem 5.21. Să se scrie ecuaţiile planelor tangente la paraboloidul eliptic

$$\frac{x^2}{2} + \frac{y^2}{4} = 2z$$

în punctele de intersecţie cu dreapta

$$x = y = z.$$

Problem 5.22. Să se scrie ecuaţiile planelor tangente la paraboloidul hiperbolic

$$\frac{x^2}{2} - \frac{y^2}{4} = 2z$$

în punctele de intersecţie cu dreapta

$$x = y = z.$$

Problem 5.23. Să se scrie ecuaţia planului tangent la paraboloidul eliptic

$$\frac{x^2}{5} + \frac{y^2}{3} = z,$$

paralel cu planul

$$x - 3y + 2z - 1 = 0.$$

Problem 5.24. Să se scrie ecuaţia planului tangent la paraboloidul hiperbolic

$$x^2 - \frac{y^2}{4} = 3z,$$

paralel cu planul

$$x - 3y + 2z - 1 = 0.$$

Problem 5.25. Să se scrie ecuaţiile generatoarelor rectilinii ale paraboloidului hiperbolic

$$\frac{x^2}{16} - \frac{y^2}{4} = z$$

care sunt paralele cu planul

$$3x + 2y - 4z = 0.$$

Problem 5.26. Să se scrie ecuaţiile generatoarelor rectilinii ale paraboloidului hiperbolic

$$4x^2 - 9y^2 = 36z$$

care trec prin punctul $M(3\sqrt{2}, 2, 1)$.

Problem 5.27. Se dă paraboloidul hiperbolic

$$x^2 - \frac{y^2}{4} = z$$

şi unul dintre planele sale tangente,

$$10x - 2y - z - 21 = 0.$$

Determinaţi ecuaţiile celor două drepte de intersecţie dintre paraboloid şi plan.

Problem 5.28. Determinaţi planele tangente la paraboloidul

$$\frac{x^2}{12} + \frac{y^2}{4} = z$$

care sunt paralele cu planul

$$x - y - 2z = 0.$$

CHAPTER 6

Generation of Surfaces

6.1 Cylindrical Surfaces

A *cylindrical surface* is a surface generated by a line moving parallel to a fixed direction and satisfying an additional condition. Usually, this additional condition is expressed by requiring the moving line to always intersect a given curve, called the *directrix of the cylindrical surface*. However, in many practical problems, this condition is naturally replaced by others arising from the problem itself: for instance, the line may need to remain tangent to a surface or at a fixed distance from a specific fixed line.

Assume a fixed line is given:

$$(\Delta)\begin{cases} P_1(x,y,z)=0 \\ P_2(x,y,z)=0 \end{cases},$$

where P_1 and P_2 are linear functions in the variables x, y, z. This line is called the *directrix* of the cylindrical surface. Additionally, consider a curve given by the equations:

$$(C)\begin{cases} F_1(x,y,z)=0 \\ F_2(x,y,z)=0 \end{cases},$$

where, this time, the only restriction on the functions F_1 and F_2 is their smoothness. This curve C is called the *directrix curve* of the cylindrical surface. To determine the equation of the surface, we first establish the equations of an arbitrary line parallel to the directrix Δ. Such a line will be called a *generator*. Since the directrix is given as the intersection of two planes, a generator will be written as the intersection of two planes parallel to those defining the directrix, that is:

$$(G_{\lambda,\mu}) \begin{cases} P_1(x, y, z) = \lambda \\ P_2(x, y, z) = \mu, \end{cases} \tag{6.1.1}$$

where λ and μ are two real parameters, currently arbitrary. For the generators to intersect the directrix curve, the following system of equations must be compatible:

$$\begin{cases} P_1(x, y, z) = \lambda \\ P_2(x, y, z) = \mu \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases} . \tag{6.1.2}$$

The above system contains four equations with three unknowns, x, y, z. In general, such a system is not compatible. If the functions F_1 and F_2 were also linear, general algebraic methods could be applied to study compatibility. However, this is not usually the case. In practice, the process proceeds as follows:

i) Three of the four equations are selected. Obviously, the simplest equations are chosen. Usually, the equations of the generators and one of the equations of the directrix curve are selected. If this curve is planar, one of its equations might be linear, and this equation is naturally selected.

ii) Solve the system from the previous step to find x, y, z as functions of the parameters of the generators, λ and μ.

iii) For the system of four equations to be compatible, the solution obtained in the previous step must also satisfy the fourth equation. Imposing this condition yields an equation of the form:

$$\varphi(\lambda, \mu) = 0, \tag{6.1.3}$$

where $\varphi(\lambda, \mu)$ is the left-hand side of the fourth equation after substituting x, y, z with their expressions in terms of λ and μ.

iv) Express λ and μ as functions of x, y, z from the equations of the generators and substitute them into the compatibility condition (6.1.3). The resulting equation:

$$\varphi\left(P_1(x, y, z), P_2(x, y, z)\right) = 0, \tag{6.1.4}$$

is the equation of the required cylindrical surface.

Example 6.1. We will write the equation of a cylindrical surface whose generators are parallel to the line given by:

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{-1},$$

and supported on the equilateral hyperbola:

$$xy = a^2, \ z = 0.$$

We begin by writing the directrix as the intersection of two planes. A simple calculation yields:

$$\begin{cases} P_1(x, y, z) \equiv 3x - 2y - 3 = 0 \\ P_2(x, y, z) \equiv y + 3z + 3 = 0 \end{cases}.$$

Therefore, the equations of the generators will be:

$$\begin{cases} 3x - 2y - 3 = \lambda \\ y + 3z + 3 = \mu \end{cases}.$$

The condition that the generators intersect the directrix curve translates into the system of equations:

$$\begin{cases} 3x - 2y - 3 = \lambda \\ y + 3z + 3 = \mu \\ z = 0 \\ xy = a^2 \end{cases}.$$

From the first three equations, we immediately find:

$$\begin{cases} x = \dfrac{\lambda + 2\mu - 3}{3} \\ y = \mu - 3 \\ z = 0 \end{cases}.$$

Substituting into the fourth equation, we obtain:

$$\varphi(\lambda, \mu) \equiv \frac{1}{3}(\lambda + 2\mu - 3)(\mu - 3) - a^2 = 0. \tag{6.1.5}$$

From the equations of the generators, we also have:

$$\begin{cases} \lambda = 3x - 2y - 3 \\ \mu = y + 3z + 3 \end{cases}.$$

Substituting these into the compatibility condition above, we obtain:

$$(x + 2z)(y + 3z) - a^2 = 0,$$

which is the equation of the cylindrical surface.

Example 6.2. To illustrate other ways of describing a cylindrical surface, we will determine the equation of a cylindrical surface circumscribed to the sphere:

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 25,$$

with generators parallel to a line Δ with a direction vector $(-2, 4, 5)$.

We will solve this problem using two different methods. The first solution involves reducing the problem to one similar to the previous example. To this end, we first need to determine the directrix curve of the surface. Geometrically, it is clear that this curve is a great circle of the sphere, lying in a plane perpendicular to the generators. Since the sphere's center is at the point $C(1, 2, 3)$, the equation of this plane is:

$$-2(x - 1) + 4(y - 2) + 5(z - 3) = 0,$$

or:

$$-2x + 4y + 5z - 21 = 0.$$

We can assume, without loss of generality, that the directrix Δ passes through the origin, so its equations are:

$$\frac{x}{-2} = \frac{y}{4} = \frac{z}{5},$$

or equivalently:

$$\begin{cases} 2x + y = 0 \\ 5x + 2z = 0 \end{cases}.$$

Thus, the equations of the generators are:

$$\begin{cases} 2x + y = \lambda \\ 5x + 2z = \mu \end{cases}. \tag{6.1.6}$$

The condition for the generators to intersect the directrix curve translates into the system:

$$\begin{cases} 2x + y = \lambda \\ 5x + 2z = \mu \\ -2x + 4y + 5z - 21 = 0 \\ (x-1)^2 + (y-2)^2 + (z-3)^2 = 25. \end{cases}.$$

Solving the system formed by the first three equations, we immediately find:

$$\begin{cases} x = \dfrac{8\lambda + 5\mu - 42}{45} \\ y = \dfrac{29\lambda - 10\mu + 84}{45} \\ z = \dfrac{-4\lambda + 2\mu + 21}{9} \end{cases}.$$

Substituting into the last equation, we derive the compatibility condition:

$$(8\lambda + 5\mu - 87)^2 + (29\lambda - 10\mu - 6)^2 + 25(-4\lambda + 2\mu - 6)^2 = 50625.$$

Finally, substituting $\lambda = 2x + y$ and $\mu = 5x + 2z$ from the equations of the generators into this condition yields:

$$(41x+8y+10z-87)^2+(58x+29y-20z-6)^2+25(2x-4y+4z-6)^2 = 50625.$$

Expanding this equation, it is straightforward to verify that it is equivalent to:

$$936+174x+12y+60z-41x^2-16xy-20xz-29y^2+40yz-20z^2 = 0. \tag{6.1.7}$$

For the second method, we again consider the equations (6.1.6) of the generators obtained earlier. The tangency condition mentioned in the problem implies that the sphere and the generators must share double points. In other words, the system:

$$\begin{cases} 2x + y = \lambda \\ 5x + 2z = \mu \\ (x-1)^2 + (y-2)^2 + (z-3)^2 = 25. \end{cases}.$$

must have a double solution. The idea is to express two unknowns in terms of the third and the parameters from the first two equations, substitute into the sphere equation to obtain a quadratic equation in the third variable, and then impose the tangency condition, i.e., that the discriminant vanishes. Finally, substituting the parameters from the equations of the generators, we derive the equation of the cylindrical surface.

From the first two equations, we immediately find:

$$\begin{cases} y = \lambda - 2x \\ z = \frac{\mu - 5x}{2} \end{cases}.$$

Substituting into the sphere equation yields:

$$45x^2 + (-16\lambda + 84 - 10\mu)\,x - 44 + 4\lambda^2 - 16\lambda + \mu^2 - 12\mu = 0.$$

Imposing the tangency condition, i.e., setting the discriminant of this equation to zero, we obtain:

$$-3744 - 48\lambda - 120\mu + 116\lambda^2 - 80\lambda\mu + 20\mu^2 = 0.$$

Finally, substituting $\lambda = 2x + y$ and $\mu = 5x + 2z$ from the equations of the generators into this condition and simplifying, we again arrive at equation (6.1.7).

6.2 Conical Surfaces

A *conical surface* is a surface generated by a family of lines (called *generators*) that share a common point (called the *vertex*) and satisfy an additional condition. This additional condition is usually that the generators intersect a given curve, referred to as the *generating curve* of the conical surface. As in the case of cylindrical surfaces, this condition can be replaced by another (e.g., requiring the generators to be tangent to a surface).

The method for describing conical surfaces is, in principle, the same as that used for cylindrical surfaces:

- First, the equations of lines that can serve as generators are written down (in this case, lines passing through the vertex). These equations will depend on two, initially arbitrary, parameters.

- The condition is imposed that these lines satisfy the additional condition, thus obtaining a relationship between the two parameters.

- In the relationship obtained in the previous step, the parameters are replaced with their expressions in terms of x, y, z, derived from the equations of the generators. The resulting equation is the equation of the conical surface.

Typically, the vertex is given either by its coordinates or as the intersection of three planes. We will consider the second case, as the first can easily be reduced to it. Assume, therefore, that the cone's vertex is given by the intersection of three planes, represented by the system:

$$(V)\begin{cases} P_1 \equiv a_{11}x + a_{12}y + a_{13}z + a_{14} = 0, \\ P_2 \equiv a_{21}x + a_{22}y + a_{23}z + a_{24} = 0, \\ P_3 \equiv a_{31}x + a_{32}y + a_{33}z + a_{34} = 0, \end{cases} \tag{6.2.1}$$

where, naturally:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0,$$

as otherwise, the system would not have a unique solution.

The condition for a line to pass through the cone's vertex is easily described geometrically, based on the description of the vertex as the intersection of three planes: *A line passes through the cone's vertex if and only if it simultaneously belongs to the pencil of planes determined by planes P_1 and P_3 and the pencil determined by planes P_2 and P_3*. Therefore, the equations of the generators can be written in the form:

$$(G_{\lambda,\mu})\begin{cases} P_1 = \lambda P_3, \\ P_2 = \mu P_3, \end{cases} \tag{6.2.2}$$

where λ and μ are two real parameters, initially arbitrary.

Now assume there is an additional condition requiring the generators to intersect a directrix curve, given as the intersection of two surfaces by the equations:

$$(C)\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{6.2.3}$$

The condition for the generators to intersect the directrix is equivalent to requiring that the system consisting of the generator equations and the directrix equations, namely:

$$\begin{cases} P_1 = \lambda P_3, \\ P_2 = \mu P_3, \\ F(x, y, z) = 0, \\ G(x, y, z) = 0, \end{cases} \tag{6.2.4}$$

be compatible.

The strategy is similar to that used for cylindrical surfaces: first, add the simplest of the directrix equations to the generator equations. Solving the resulting system yields x, y, z as functions of the parameters λ and μ. Substituting into the fourth equation provides a relationship of the form:

$$\varphi(\lambda, \mu) = 0, \tag{6.2.5}$$

which we will call the *compatibility condition*. On the other hand, from the generator equations, we obtain expressions for the parameters λ and μ in terms of x, y, z:

$$\begin{cases} \lambda = \dfrac{P_1}{P_3}, \\ \mu = \dfrac{P_2}{P_3}. \end{cases}$$

Substituting these expressions into the compatibility condition yields the equation of the conical surface:

$$\varphi\left(\frac{P_1(x, y, z)}{P_3(x, y, z)}, \frac{P_2(x, y, z)}{P_3(x, y, z)}\right) = 0, \tag{6.2.6}$$

or more explicitly:

$$\varphi\left(\frac{a_{11}x + a_{12}y + a_{13}z + a_{14}}{a_{31}x + a_{32}y + a_{33}z + a_{34}}, \frac{a_{21}x + a_{22}y + a_{23}z + a_{24}}{a_{31}x + a_{32}y + a_{33}z + a_{34}}\right) = 0. \tag{6.2.7}$$

Example 6.3. As a first example, we will determine the equation of a conical surface with vertex at $V(0, 0, 0)$ and whose generators intersect the curve:

$$\begin{cases} x + y + z - 1 = 0, \\ x^2 - y = 0. \end{cases}$$

The equations of the vertex (as the intersection of three planes) are evidently:

$$\begin{cases} P_1(x, y, z) \equiv x = 0, \\ P_2(x, y, z) \equiv y = 0, \\ P_3(x, y, z) \equiv z = 0, \end{cases}$$

so the equations of the generators are:

$$\begin{cases} x = \lambda z, \\ y = \mu z. \end{cases}$$

To derive the compatibility condition, first add the simplest of the directrix equations to the generator equations, obtaining the system:

$$\begin{cases} x = \lambda z, \\ y = \mu z, \\ x + y + z - 1 = 0. \end{cases}$$

From here, we immediately find:

$$x = \frac{\lambda}{\lambda + \mu + 1}, \ y = \frac{\mu}{\lambda + \mu + 1}, \ z = \frac{1}{\lambda + \mu + 1}.$$

Substituting into the second directrix equation, we obtain the compatibility relation:

$$\frac{\lambda^2}{(\lambda + \mu + 1)^2} - \frac{\mu}{\lambda + \mu + 1} = 0,$$

or:

$$\varphi(\lambda, \mu) = \lambda^2 - \mu(\lambda + \mu + 1) = 0.$$

Substituting into this relation the expressions $\lambda = x/z$, $\mu = y/z$ from the generator equations, we find:

$$\frac{x^2}{z^2} - \frac{y}{z}\left(\frac{x}{z} + \frac{y}{z} + 1\right) = 0,$$

or equivalently:

$$x^2 - y(x + y + z) = 0.$$

6.3 Conoidal Surfaces (The Right Conoid with a Director Plane)

Conoidal surfaces share certain characteristics with conical surfaces (hence their name). They form, in fact, a broader class of surfaces. Here, we will focus only on a special subclass. Since we will not refer to more general conoidal surfaces, we will retain the term.

A *conoidal surface* (*right conoid with a director plane*) is a surface generated by a family of lines (called *generators*) that rest on a given line, remain parallel to a given plane, and satisfy an additional condition (usually, as before, this condition is that the generators intersect a given curve, the *directrix curve* of the surface).

The method for determining the equation of a conoidal surface is, in principle, the same as before: the generators, which will form a family of lines depending on two parameters, are first written down. These lines intersect the given line and are parallel to the given plane. Once the equations of the generators are written, the remainder of the process is identical to that used for cylindrical and conical surfaces, so we will not repeat it.

The first problem to address is determining the equations of the generators. As mentioned, these intersect the given line and are parallel to the given plane. Consequently, their equations will be given as the intersection of a plane parallel to the given plane and a plane passing through the given line.

Suppose the fixed line (the directrix) is given by the equations:

$$\begin{cases} P_1(x, y, z) \equiv a_{11}x + a_{12}y + a_{13}z + a_{14} = 0, \\ P_2(x, y, z) \equiv a_{21}x + a_{22}y + a_{23}z + a_{24} = 0, \end{cases} \tag{6.3.1}$$

while the director plane is given by the equation:

$$P(x, y, z) \equiv ax + by + cz + d = 0. \tag{6.3.2}$$

Then, the equations of the generators will take the form:

$$\begin{cases} P_1 = \lambda P_2, \\ P = \mu. \end{cases} \tag{6.3.3}$$

Indeed, the first plane passes through the directrix (since it belongs to the pencil of planes determined by this line), while the second plane is parallel to the director plane.

Therefore, if the equations of the directrix curve are:

$$\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0, \end{cases}$$

then a system of equations is formed from the generator equations and one of the equations of this curve. Solving this system provides the unknowns in terms of the parameters λ and μ. Substituting into the second equation of the curve yields the compatibility condition, again in the form of a relationship between the parameters:

$$\varphi(\lambda, \mu) = 0.$$

Substituting the parameters from the generator equations yields the equation of the conoidal surface in the form:

$$\varphi\left(\frac{P_1(x, y, z)}{P_2(x, y, z)}, P(x, y, z)\right) = 0, \tag{6.3.4}$$

or more explicitly:

$$\varphi\left(\frac{a_{11}x + a_{12}y + a_{13}z + a_{14}}{a_{21}x + a_{22}y + a_{23}z + a_{24}}, ax + by + cz + d\right) = 0. \tag{6.3.5}$$

Example 6.4. Write the equation of the conoidal surface with a director plane whose generators are parallel to the plane xOy,

$$z = 0, \tag{P}$$

rest on the Oz axis,

$$\begin{cases} x = 0, \\ y = 0, \end{cases} \tag{D}$$

and intersect the curve:

$$\begin{cases} y^2 - 2z + 1 = 0, \\ x^2 - 2z + 1 = 0. \end{cases} \tag{C}$$

Proof The equations of the generators are:

$$\begin{cases} x = \lambda y, \\ z = \mu. \end{cases} \tag{G}$$

Since they must rest on the directrix curve (C), the system:

$$\begin{cases} x = \lambda y, \\ z = \mu, \\ y^2 - 2z + 1 = 0, \\ x^2 - 2z + 1 = 0, \end{cases}$$

must be compatible. The compatibility relation between the parameters is obtained by eliminating x, y, z from the equations of the system. This yields:

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0.$$

To find the equation of the surface, λ and μ must be eliminated from the system formed by the generator equations and the compatibility condition:

$$\begin{cases} x = \lambda y, \\ z = \mu, \\ 2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0. \end{cases}$$

Substituting $\lambda = \frac{x}{y}$ and $\mu = z$ into the third equation and clearing the denominator, we find:

$$2x^2z - 2y^2z - 2x^2 + y^2 = 0.$$

□

6.4 Surfaces of Revolution

Definition 6.1. *Surfaces of revolution* are surfaces generated by a curve C that rotates, without slipping, around a fixed axis D.

During the rotation, any point on the curve C describes a circle with its centre on the axis of rotation D, located in a plane perpendicular to the axis. Consequently, the surface itself can be viewed as being generated by these circles, called *parallels*. More precisely, we have the following theorem:

Theorem 6.1. *Let:*

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}, \tag{D}$$

be the equations of the axis D, and let:

$$\begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0, \end{cases} \tag{C}$$

be the equations of the curve C. The equation of the surface of revolution is:

$$F(\sigma, P) = 0,$$

where:

$$\sigma = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2},$$
$$P = lx + my + nz.$$

Proof Assume, as stated, that the curve C is given as the intersection of two surfaces:

$$(C)\begin{cases} F_1(x,y,z) = 0, \\ F_2(x,y,z) = 0. \end{cases} \tag{6.4.1}$$

On the other hand, the axis of rotation is given by the equations:

$$(D)\ \frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}. \tag{6.4.2}$$

The generating circle Γ is obtained as the intersection of a sphere with its centre on the axis of rotation and a variable radius, and a variable plane perpendicular to the axis. Consequently, its equations are:

$$(\Gamma)\ \begin{cases} (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \lambda^2, \\ lx + my + nz = \mu. \end{cases} \tag{6.4.3}$$

For the circle Γ to rest on the curve C, the two curves must have at least one point in common, i.e., the system of equations consisting of their equations:

$$\begin{cases} F_1(x,y,z) = 0, \\ F_2(x,y,z) = 0, \\ (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \lambda^2, \\ lx + my + nz = \mu, \end{cases} \tag{6.4.4}$$

must be compatible. The compatibility condition is obtained by eliminating x, y, z from these four equations. Assume that the resulting relationship is:

$$F(\lambda, \mu) = 0. \tag{6.4.5}$$

Now, as in the case of other generated surfaces, the equation of the surface is obtained by eliminating the parameters λ and μ from the system formed by the equations of the generating circle and the compatibility condition:

$$\begin{cases} (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \lambda^2, \\ lx + my + nz = \mu, \\ F(\lambda, \mu) = 0. \end{cases} \tag{6.4.6}$$

The parameters λ and μ can obviously be expressed from the first two equations, and we obtain:

$$F\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}, lx+my+nz\right)=0. \tag{6.4.7}$$

□

Example 6.5. Find the equation of the surface of revolution generated by the curve:

$$(C)\ \begin{cases} x^2-2y^2+z^2-5=0, \\ x+z+3=0, \end{cases}$$

rotating around the axis:

$$x=y=z.$$

Proof The equations of the generating circle are:

$$(\Gamma)\ \begin{cases} x^2+y^2+z^2=\lambda^2, \\ x+y+z=\mu. \end{cases}$$

Thus, the compatibility condition is obtained by eliminating x, y, z from the system:

$$\begin{cases} x^2-2y^2+z^2-5=0, \\ x+z+3=0, \\ x^2+y^2+z^2=\lambda^2, \\ x+y+z=\mu. \end{cases}$$

After straightforward computation, we obtain:

$$\lambda^2-3(\mu-3)^2-5=0.$$

Finally, the equation of the surface is derived by eliminating the parameters λ and μ from the system consisting of the equations of the generating circle and the compatibility relation:

$$\begin{cases} x^2+y^2+z^2=\lambda^2, \\ x+y+z=\mu, \\ \lambda^2-3(\mu-3)^2-5=0. \end{cases}$$

We find, after simplification:

$$x^2+y^2+z^2-3(x+y+z-3)^2-5=0.$$

□

Part II

The differential geometry of curves and surfaces

CHAPTER 7

Space curves

7.1 Introduction

Intuitively, the curves are just deformations of straight lines. They can be thought of, therefore, as "one-dimensional" objects. We are familiar with some of them already from elementary mathematics, since, obviously, the graphs of functions can be considered as curves, from this point of view. On the other hand, clearly, usually the curves *are not* graphs of functions (well, at least not globally); it is enough to think of a conical section (for instance an ellipse or a circle, in particular). Thus, in general we cannot represent a curve by an equation of the form $y = f(x)$, as it would be the case for a graph. On the other hand, we can represent a conical section also by an implicit equation of the form $f(x, y) = 0$, where, in this particular case, of course, as it is known, f is a second degree polynomial in x and y. Finally, we can represent the coordinates of each point of the curve as functions of one parameter. As we shall see, this is, usually, the most convenient way to represent, locally, an arbitrary curve.

An important issue that has to be dealt with is related to the smoothness of the functions used to describe a curve. Of course, we are interested to apply the tools of differential calculus. We shall assume, therefore, that all the functions are at least once continuous differentiable and, if higher order derivatives are involved, we shall assume, without stating it in clear, that all the derivatives exists and are continuous. We shall use for the functions which are satisfying these conditions the generic term "smooth". Apart from the computational aspects, there are other, deeper reasons for considering functions at least once continuously differentiable. Suppose, for instance, that a curve is described by a set of equations of the form

$$\begin{cases} x = f(t), \\ y = g(t) \end{cases}.$$

Now, it can be shown that if the functions f and g are only continuous, the curve can fill an entire square (or the entire plane). The first example of such an anomalous curve (which, obviously, contradicts the common sense image of a curve as an one-dimensional object) has been constructed by the Italian mathematician Giuseppe Peano, at the end of the XIX-th century. Moreover, this strange phenomenon does not disappear not even if the functions f and g are differentiable, but not *continuously* differentiable. In the figure **??** we indicate an iterative process that defines a square-filling (Peano) curve. The curve itself is the limit of the curves obtained by this iterative process. It is possible, actually, to describe analytically this curve (i.e. we can find an expression for each iteration), but, as these "curves" are not the subject of our investigation, we prefer to let the reader satisfy the curiosity by himself. We have to say, however, that the functions we use to describe a curve do not have to be, necessarily, continuous differentiable to avoid aforementioned anomalies. What it is required is a little bit less, namely the functions have to be of *bounded variation*. It is a well-known fact that the continuous differentiable functions do verify this condition and, moreover, as we already emphasized, they also endow us with the necessary computational tools, which are not available for arbitrary functions with bounded variation.

7.2 Parameterized curves (paths)

Let I be an interval on the real axis $\mathbb{R}$. We shall not assume always that the interval is open. Sometimes it is even important that the interval be closed. In particular, the interval can be unbounded and can coincide with the entire real axis.

Definition 7.1. A C^k $(k > 0)$ *parameterized curve* (or *path*) in the Euclidean space $\mathbb{R}^3$ is a C^k mapping

$$\mathbf{r} : I \to \mathbb{R}^3 : t \to (x(t), y(t), z(t)). \tag{7.2.1}$$

A parameterized curve is, typically, denoted by $(I, \mathbf{r})$, $(I, \mathbf{r} = \mathbf{r}(t))$ or, when the interval is implicit, just $\mathbf{r} = \mathbf{r}(t)$. We note that, in fact, a path is C^k iff the (real valued) functions x, y, z are C^k. If the interval is not open, we shall assume, first of all, that the functions we are considering are of class C^k in the interior of the interval and all the derivatives, up to the k-th order, have a finite lateral (left, or right) limit at the extremities of the interval, if these extremities belong to the interval.

The path is called *compact, half-open* or *open*, if the interval I is, respectively, compact, half-open or open.

If the interval I is bounded from below, from above or from both parts, then the imagine of any extremity of I is called an *endpoint* of the path. If, in particular, the curve is compact and the two endpoints coincide, the path is called *closed*. An alternative denomination used for a closed path is that of a *loop*.

Occasionally (for instance in the theory of the line integral) we might need to consider paths which are of class C^k at all the points of an interval, with the exception of a finite number of points. The following definition will make this more precise.

Definition 7.2. We shall say that a compact parameterized curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is *piecewise* C^k if there exists a finite subdivision $(a = a_0, a_1, \dots, a_n = b)$ of the interval $[a, b]$ such that the restriction of $\mathbf{r}$ to each compact interval $[a_{i-1}, a_i]$ is of class C^k, where $i \in \{1, \dots, n\}$.

Remark. It is not difficult to show that a parameterized curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is piecewise C^k iff the following conditions are simultaneously fulfilled:

(i) The set
$$S = \left\{ t \in [a, b] \mid f^{(k)} \text{ does not exist} \right\}$$
is finite.

(ii) $f^{(k)}$ is continuous on $[a, b] \setminus S$.

(iii) $f^{(k)}$ has a finite left and right lateral limit at each point of S.

Hereafter we shall suppose, all the time, that the order of smoothness k is high enough and we will not mention it any longer (with some exceptions, however), using the generic term of *smooth parameterized curve* (meaning, thus, at least C^1 and, in any particular case, if kth order derivatives are involved, at least C^k).

The image $\mathbf{r}(I) \subset \mathbb{R}^3$ of the interval I through the mapping (7.2.1) is termed the *support* of the path $(I, \mathbf{r})$.

If $\mathbf{r}(t_0) = a$, we shall say that the parameterized curve *passes* through the point a for $t = t_0$. Sometimes, for short, we shall refer to this point as *the point* t_0 of the parameterized curve.

Examples

1. Let $\mathbf{r}_0 \in \mathbb{R}^3$ be an arbitrary point, and $\mathbf{a} \in \mathbb{R}^3$ a vector, $\mathbf{a} \neq 0$, while $I = \mathbb{R}$. The parameterized curve $\mathbb{R} \to \mathbb{R}^3$, $t \to \mathbf{r}_0 + t\mathbf{a}$ is called *a straight line*. Its support is the straight line passing through $\mathbf{r}_0$ (for $t = 0$) and having the direction given by the vector $\mathbf{a}$.

2. $I = \mathbb{R}$, $\mathbf{r}(t) = \mathbf{r}_0 + t^3\mathbf{a}$. The support of this path is the same straight line.

3. $I = \mathbb{R}$, $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, $a, b \in \mathbb{R}$. The support of this parameterized curve is called *a circular cylindrical helix* (see figure **??**).

4. $I = [0, 2\pi]$, $\mathbf{r}(t) = (\cos t, \sin t, 0)$. The support of the path is the unit circle, lying in the xOy-plane, having the center at the origin of the coordinates.

5. $I = [0, 2\pi]$, $\mathbf{r}(t) = (\cos 2t, \sin 2t, 0)$. The support is the same from the previous example.

6. $I = \mathbb{R}$, $\mathbf{r}(t) = (t^2, t^3, 0)$. This curve has a cusp at the point $t = 0$.

Definition 7.3. A parameterized curve (7.2.1) is called *regular for* $t = t_0$ if $\mathbf{r}'(t_0) \neq 0$ and *regular* if it is regular for each $t \in I$.

As we shall see a little bit later, the notion of regularity at a point of a parameterized curve is related to the existence of a well defined tangent to the curve in that point.

The curves from the previous example are regular, with the exception of those from the points 2 and 6, which are not regular from $t = 0$.

Remark. The fact that the *same* support can correspond both to a regular and a nonregular parameterized curve suggests that the absence of the regularity at a point doesn't necessarily mean that the corresponding point of the support has some geometric peculiarities. It's only that the regularity *guarantees* the absence of these peculiarities. Indeed, if we look, again, at the curves 2 and 6 from the previous example, we notice immediately that, although they are, both, nonregular for $t = 0$, only for the second curve this analytic singularity implies a geometric singularity (a cusp), while for the first one the support is just a straight line, without any special points.

To each path corresponds a subset of $\mathbb{R}^3$, its support. Nevertheless, as the examples 1 and 2 show, different parameterized curves may have the same support. A parameterized curve can be thought of as a subset of $\mathbb{R}^3$, with a parameterization.[1]

[1] Of course, the information is redundant, since the parametric representation defines the support.

The support of a parameterized curve corresponds to our intuitive image of a curve as a one dimensional geometrical object. As we shall see later, the support of a parameterized curve may have self-intersections or cusps, which, for many reasons, are not desirable in applications. The regularity conditions rules out the cusps, but not the self-intersections. To eliminate them, we have to impose some further conditions.

We have seen that different parameterized curves may have the same support. In the end, it is the support, as a set of points, we are interested in. It is, therefore, necessary to identify some relations between the parameterized curves that define the same support. For reasons that will become clear later, for the moment, at least, we are interested only in regular curves. Therefore, for instance, passing from a parameterized representation of the support to another should not change the regularity of the curve.

Definition 7.4. Let $(I, \mathbf{r} = \mathbf{r}(t))$, $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ two parameterized curves. A diffeomorphism $\lambda : I \to J : t \to s = \lambda(t)$ such that $\mathbf{r} = \boldsymbol{\rho} \circ \lambda$, i.e. $\mathbf{r}(t) \equiv \boldsymbol{\rho}(\lambda(t))$, is called *a parameter change* or *a reparameterization*. Two parameterized curves for which there is a parameter change are called *equivalent*, while the points t and $s = \lambda(t)$ are called *correspondent*.

Remarks. 1) The relation just defined is an equivalence relation on the set of all parameterized curves.

2) The reparameterization has a simple kinematical interpretation. If we interpret the parametric equations of the path as being the equations of motion of a particle, then the support is just the trajectory, while the vector $\mathbf{r}'(t)$ is the velocity of the particle. The effect of a reparameterization is the modification of the speed with which the trajectory is traversed. Also, if $\lambda'(t) < 0$, then the trajectory is traversed in the opposite sense. Note that the two velocity vectors of equivalent parameterized curves in correspondent points have the same direction. They may have different lengths (the speed) and sense.

Example 7.1. The parameterized curves from the examples 1 and 2 are not equivalent, although, as mentioned, they have the same support.

Remark. Sometimes, the equivalence classes determined by the relation defined before between parameterized curves are called *curves*. We shall not follow this line here, because we would like the curves to be objects slightly more general that just supports of parameterized curves. In particular, as we shall see in a moment, usually the curves cannot be represented globally through the same parametric equations. It is enough to think about the circle. One of the most common parametric representation of the circle is

$$\begin{cases} x = \cos\theta, \\ y = \sin\theta \end{cases}.$$

Now, if we let the parameter vary only in the interval $(0, 2\pi)$, then one of the points of the circle is not represented. Of course, we can extend the interval, but then the same point corresponds to more than a value of the parameter, which is, again, not acceptable.

Among all parameterized curves equivalent to a given parameterized curve, there is one which has a special theoretical meaning and which simplifies many proofs in the theory of curve, although, in most cases, it is very difficult to find it analytically and, thus, its practical value is limited.

Definition 7.5. We shall say that a parameterized curve is *naturally parameterized* if $\|\mathbf{r}'(s)\| = 1$ for any $s \in I$. Usually, the natural parameter is denoted by s.

Remark. One can see immediately that any smooth, naturally parameterized curve $(I, \mathbf{r} = \mathbf{r}(s))$ is *regular*, since, clearly, $\mathbf{r}'(s)$ cannot vanish, as its norm is nowhere vanishing.

It is by no means obvious that for any parameterized curve there is another one, equivalent to it and naturally parameterized. To construct such a curve, we need first another notion.

The *arc length* of a path $(I, \mathbf{r} = \mathbf{r}(t))$ between the points t_1 and t_2 is the number[2]

$$l_{t_1,t_2} = \left| \int_{t_1}^{t_2} \|\mathbf{r}'(t)\| dt \right|.$$

Remark. There is a good motivation for defining in this way the length of an arc. We can consider an arbitrary division $t_1 = a_0 < a_1 < \cdots < a_n = t_2$ of the interval $[t_1, t_2]$ and examine the polygonal line $\gamma_n = \mathbf{r}(a_0)\mathbf{r}(a_1)\cdots\mathbf{r}(a_n)$. The length of this polygonal line is the sum of the lengths of its segments. It can be shown that if the parameterized curve $(I, \mathbf{r})$ is "good enough" (for instance at least once continuously differentiable), then the limit of length of the polygonal line γ_n, when the norm of the division goes to zero, exists and it is equal to the arc length of the path. It is to be mentioned, also, that the definition of the arc length makes sense also for piecewise smooth curves, because in this case the tangent vector has only a finite number of discontinuity points and, therefore, its norm is integrable.

[2] Although the integrand is positive we did not assume that $t_1 < t_2$, therefore the integral might be negative and the absolute value is necessary, if we want to obtain something positive.

We are going to show that the arc lengths of two equivalent paths between correspondents points are equal, therefore the arc length is, in a way, a characteristic of the support[3].

Indeed, let $\mathbf{r}(t) = \boldsymbol{\rho}(\lambda(t))$, then $\mathbf{r}'(t) = \lambda'(t)\boldsymbol{\rho}'(\lambda(t))$. Therefore,

$$\left|\int_{t_1}^{t_2} \|\mathbf{r}'(t)\| dt\right| = \left|\int_{t_1}^{t_2} \|\boldsymbol{\rho}'(\lambda(t))\| \cdot |\lambda'| dt\right| =$$

$$= \left|\int_{t_1}^{t_2} \|\boldsymbol{\rho}'(\lambda)\| \underbrace{\lambda'(t)dt}_{d\lambda}\right| = \left|\int_{\lambda_1}^{\lambda_2} \|\boldsymbol{\rho}'(\lambda)\| d\lambda\right|.$$

For naturally parameterized curves, $(I, \mathbf{r} = \mathbf{r}(s))$,

$$l_{s_1,s_2} = |s_2 - s_1|.$$

In particular, if $0 \in I$ (which can always be assumed, since the translation is a diffeomorphism), then $l_{0,s} = |s|$, i.e., up to sign, the natural parameter is the arc length.

Proposition 1. *For any regular parameterized curve there is a naturally parameterized curve equivalent to it.*

Proof Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a regular parameterized curve, $t_0 \in I$, and

$$\lambda : I \to \mathbb{R}, \ \ t \to \int_{t_0}^{t} \|\mathbf{r}'(\tau)\| d\tau.$$

The function λ is smooth and strictly increasing, since $\lambda'(t) = \|\mathbf{r}'(t)\| > 0$. Therefore, its image will be an open interval J, while the function $\lambda : I \to J$ will be a diffeomorphism. The parameterized curve $(J, \boldsymbol{\rho}(s) = \mathbf{r}(\lambda^{-1}(s))$ is equivalent to $(I, \mathbf{r})$ and it is naturally parameterized, since $\boldsymbol{\rho}'(s) = \mathbf{r}'(\lambda^{-1}(s))(\lambda^{-1})'(s)$, while

$$(\lambda^{-1})'(s) = \frac{1}{\lambda'(\lambda^{-1}(s))} = \frac{1}{\|\mathbf{r}'(\lambda^{-1}(s))\|}$$

and, hence,

$$\|\boldsymbol{\rho}'(s)\| = \|\mathbf{r}'(\lambda^{-1}(s))\| \cdot |(\lambda^{-1})'(s)| = 1.$$

□

[3] In a way, because we may represent the same set of points as support of another parameterized curve, which is not equivalent to the initial one. The new path might have a different arc length between the same points of the support.

Remark. In the proof of the previous proposition we used, in an essential way, the fact that all the points of the curve are regular. On an interval on which the curve has singular points, there is no naturally parameterized curve equivalent to it.

Example 7.2. For the circular helix

$$\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt, \end{cases}$$

we get, through a parameter change,

$$s(t) = \int_0^t \|\mathbf{r}'(\tau)\| d\tau = \int_0^t \|\{-a\sin\tau, a\cos\tau, b\}\| d\tau = \sqrt{a^2+b^2},$$

therefore,

$$t = \frac{s}{\sqrt{a^2+b^2}}.$$

Thus, the natural parameterization of the helix is given by the equations

$$\begin{cases} x = a\cos\frac{s}{\sqrt{a^2+b^2}} \\ y = a\sin\frac{s}{\sqrt{a^2+b^2}} \\ z = b\frac{s}{\sqrt{a^2+b^2}}. \end{cases}$$

Exercise 7.2.1. Find a natural parameterization of the curve

$$x = e^t\cos t, \quad , y = e^t\sin t, \quad z = e^t.$$

Exercise 7.2.2. Show that the parameter along the curve

$$x = \frac{s}{2}\cos\left(\ln\frac{s}{2}\right), \quad y = \frac{s}{2}\sin\left(\ln\frac{s}{2}\right), \quad z = \frac{s}{\sqrt{2}}$$

is a natural parameter.

Remark. It should be noticed that, usually, the natural parameter along a parameterized curve cannot be expressed in finite terms (i.e. using only elementary functions) with respect to the parameter along the curve. This is, actually, impossible even for very simple curves, such that the ellipse

$$\begin{cases} x = a\cos t \\ y = b\sin t, \end{cases}$$

with $a \neq b$, for which the arc length can be expressed only in terms of elliptic functions (this is, actually, the origin of their name!). Therefore, although the natural parameter is very important for theoretical consideration and for performing the proofs, as the reader will have more than once the opportunity to see in this book, for concrete examples of parameterized curves we will hardly ever use it.

7.3 The definition of the curve

As we mentioned before, we can imagine, intuitively, a curve as being just a deformation of a straight line, without thinking, necessary, at an analytical representation. We expect the curve to have a well defined tangent at each point. In particular, this condition should rule out both cusps and self intersections.

Definition 7.6. A subset $M \subset \mathbb{R}^3$ is called a *regular curve* (or *a 1-dimensional smooth submanifold of* $\mathbb{R}^3$) if, for each point $a \in M$ there is a regular parameterized curve $(I, \mathbf{r})$, whose support, $\mathbf{r}(I)$, is an open neighbourhood in M of the point a (i.e. is a set of the form $M \cap U$, where U is an open neighborhood of a in $\mathbb{R}^3$), while the map $\mathbf{r} : I \to \mathbf{r}(I)$ is a homeomorphism, with respect to the topology of subspace of $\mathbf{r}(I)$. A parameterized curve with these properties is called a *local parameterization* of the curve M around the point a. If for a curve M there is a local parameterization $(I, \mathbf{r})$ which is *global*, i.e. for which $\mathbf{r}(I) = M$, the curve is called *simple*.

Remark. In some books in the definition it is required that the map $\mathbf{r} : I \to \mathbf{r}(I)$ be smooth, which is not completely rigorous, as $\mathbf{r}(I)$ is not an open subset of $\mathbb{R}^3$. What is meant, is, however, exactly the same, i.e. the map $\mathbf{r} : I \to \mathbb{R}^3$ is smooth.

Examples.

1. Any straight line in $\mathbb{R}^3$ is a simple curve, because it has a global parameterization, given by a function of the form $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, $\mathbf{r}(t) = \mathbf{a} + \mathbf{b} \cdot t$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, $\mathbf{b} \neq 0$.

2. The circular helix is a simple regular curve, with the global parameterization $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, given by $\mathbf{r}(t) = (a \cos t, b \sin t, bt)$.

3. A circle in $\mathbb{R}^3$ is a curve, but it is not simple, since no open interval can be homeomorphic to the circle, which is a compact subset of $\mathbb{R}^3$.

Thus, a regular curve is just a subset of $\mathbb{R}^3$ obtained "gluing together smoothly" supports of parameterized curves. If we look carefully at the definition of the curve, we see that not any regular parameterized curve can be used as a local parameterization of a curve. For an arbitrary parameterized curve $(I, \mathbf{r})$, the map $\mathbf{r} : I \to \mathbb{R}^3$ is not injective and, thus, it cannot be a local parameterization. Let us, also, mention that even if the function is injective, $\mathbf{r} : I \to \mathbf{r}(I)$ might fail to be a homeomorphism (even if the map is continuous and bijective, the inverse might not be continuous).

If, for instance, we consider the parameterized curve $(I, \mathbf{r})$, with $I = \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$,

$$\mathbf{r}(t) = (\cos t, \sin t, 0),$$

then the support of this parameterized curve is the unit circle in the coordinate plane xOy, centred at the origin. We should not conclude, however, that the circle is a simple curve, since $\mathbf{r}$ is *not* a homeomorphism onto (in fact, it is not even injective, because it is periodic).

Now let us assume that $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(\tau))$ are two local parameterizations of a regular curve M around the same point $a \in M$. Then, as one could expect, the two parameterized curves are equivalent, if we restrict the definition intervals such that they have the same support. More specifically, the following theorem holds:

Theorem 7.1. *Let $M \subset \mathbb{R}^3$ be a regular curve and $(I, \mathbf{r} = \mathbf{r}(t))$, $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(\tau))$ – two local parameterizations of M such that $W \equiv \mathbf{r}(I) \cap \boldsymbol{\rho}(J) \neq \emptyset$. Then $(\mathbf{r}^{-1}(W), \mathbf{r}|_{\mathbf{r}^{-1}(W)})$ and $(\boldsymbol{\rho}^{-1}(W), \boldsymbol{\rho}|_{\boldsymbol{\rho}^{-1}(W)})$ are equivalent parameterized curves.*

Proof

$$(I, \mathbf{r}(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$$

and

$$(J, \boldsymbol{\rho}(\tau) = (x(\tau), y(\tau), z(\tau))$$

be two local parameterizations of the curve M. To simplify the notations, we shall assume, from the very beginning, that $\mathbf{r}(I) = \boldsymbol{\rho}(J)$. Clearly, the generality is not reduced by this assumption. We claim that the map $\lambda : I \to J$, $\lambda = \boldsymbol{\rho}^{-1} \circ \mathbf{r}$, is a diffeomorphism, providing, thus, a parameter change between the two parameterized curves.

λ is, clearly, a homeomorphism, as a composition of the homeomorphisms

$$\mathbf{r} : I \to \mathbf{r}(I)$$

and

$$\boldsymbol{\rho}^{-1} : \boldsymbol{\rho}(J) \to J.$$

Moreover $\mathbf{r} = \boldsymbol{\rho} \circ \lambda$. Therefore, all we have to prove is that the maps λ and λ^{-1} are both smooth. One might be tempted, at this point, to write λ as

$$\lambda = \boldsymbol{\rho}^{-1} \circ \mathbf{r}$$

and conclude that λ is smooth, as composition of two smooth functions. While this representation is legitimate, as both $\mathbf{r}$ and $\boldsymbol{\rho}$ are homeomorphism onto, $\boldsymbol{\rho}^{-1}$ *is not* a differentiable function and, for the time being, at least, it doesn't make sense to speak about its differentiability, as its domain of definition is not an open subset of the ambient Euclidean space. We will prove, instead that, locally, $\boldsymbol{\rho}^{-1}$ is the restriction of a differentiable mapping, defined, this time, on an open subset of $\mathbb{R}^3$.

Since the notion of differentiability is a local notion, it is enough to prove that λ is smooth in a neighbourhood of each point of the interval I. This could be achieved, for instance, by representing λ, locally, as a composition of smooth functions. Let $t_0 \in I,\ \tau_0 = \lambda(t_0)$. Due to the regularity of the map $\boldsymbol{\rho}$, we have $\boldsymbol{\rho}'(\tau_0) \neq 0$. We can assume, without restricting the generality, that the first component of this vector is different from zero, i.e. $x'(\tau_0) \neq 0$. From the inverse function theorem, applied to the function x, there is an inverse function $\tau = f(x)$, defined and smooth in an open neighbourhood $V \subset \mathbb{R}$ of the point $x_0 = x(\tau_0)$. Then, in the open neighbourhood $\boldsymbol{\rho}(f(V))$ of the point $(x(\tau_0), y(\tau_0), z(\tau_0))$ from M we will have $\boldsymbol{\rho}^{-1}(x, y, z) = f(x)$, which means that, in fact, we have

$$\boldsymbol{\rho}^{-1}\big|_{\boldsymbol{\rho}(f(V))} = f \circ pr_1|_{\boldsymbol{\rho}(f(V))},$$

where $pr_1 : \mathbb{R}^3 \to \mathbb{R}$ is the projection of $\mathbb{R}^3$ on the first factor.

With this local expression of $\boldsymbol{\rho}^{-1}$ in hand, we can write λ on the open neighbourhood $\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))$ of t_0 as

$$\lambda|_{\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))} = \boldsymbol{\rho}^{-1}\big|_{\boldsymbol{\rho}(f(V))} \circ \mathbf{r}|_{\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))} = f \circ pr_1|_{\boldsymbol{\rho}(f(V))} \circ \mathbf{r}|_{\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))}.$$

As the functions f, pr_1 and $\mathbf{r}$ are, all of them, smooth on the indicated domains, it follows that λ is smooth on the open neighbourhood $\mathbf{r}^{-1}(\boldsymbol{\rho}(f(V)))$ of t_0. As t_0 was arbitrary, λ is smooth on the entire I. The smoothness of λ^{-1} is proven in the same way, changing the roles of $\mathbf{r}$ and $\boldsymbol{\rho}$. □

It follows from the definition that any regular curve is, locally, the support of a parameterized curve. Globally, this is not true, unless the curve is simple. Also, in general, the support of an arbitrary parameterized curve is not a regular curve. Take, for instance, the lemniscate of Bernoulli $(\mathbb{R}, \mathbf{r}(t) = (x(t), y(t), z(t)))$, where

$$\begin{cases} x(t) = \frac{t(1+t^2)}{1+t^4} \\ y(t) = \frac{t(1-t^2)}{1+t^4} \\ z = 0 \end{cases}.$$

r is continuous, even bijective, but the inverse is not continuous. In fact, the support has a self intersection, because $\lim_{t\to-\infty}\mathbf{r} = \lim_{t\to\infty}\mathbf{r} = \mathbf{r}(0)$ (see the figure **??**). However, we can always restrict the domain of definition of a parameterized curve such that the support of the restriction is a regular curve.

Theorem 7.2. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ a regular parameterized curve. Then each point $t_0 \in I$ has a neighbourhood $W \subset I$ such that $\mathbf{r}(W)$ is a simple regular curve.*

Proof The regularity of $\mathbf{r}$ at each point means, in particular, that $\mathbf{r}'(t_0) \neq 0$. Without restricting the generality, we may assume that $x'(t_0) \neq 0$. Let us consider the mapping $\psi : I \times \mathbb{R}^2 \to \mathbb{R}^3$, given by

$$\psi(t, u, v) = \mathbf{r}(t) + (0, u, v),$$

where $(u, v) \in \mathbb{R}^2$. ψ is, clearly, smooth and its Jacobi matrix at the point $(t_0, 0, 0)$ is given by

$$J(\psi)(t_0, 0, 0) = \begin{bmatrix} x'(t_0) & 0 & 0 \\ y'(t_0) & 1 & 0 \\ z'(t_0) & 0 & 1 \end{bmatrix}$$

Its determinant is

$$\det J(\psi)(t_0, 0, 0) = x'(t_0),$$

therefore ψ is a local diffeomorphism around the point $(t_0, 0, 0)$. Accordingly, there exist open neighborhoods $U \subset \mathbb{R}^3$ of $(t_0, 0, 0)$ and $V \subset \mathbb{R}^3$ of $\psi(t_0, 0, 0)$ such that $\psi|_V$ is a diffeomorphism from U to V. Let us denote by $\varphi : V \to U$ its inverse (which, of course, is, equally, a diffeomorphism from V to U, this time). If we put

$$W := \{t \in I : (t, 0, 0) \in U\},$$

then, clearly, W is an open neighborhood of t_0 in I such that

$$\varphi(V \cap \mathbf{r}(W)) = \varphi(\psi(W \times \{(0, 0)\}) = W \times \{(0, 0)\}.$$

$\square$

Remark. The previous theorem plays a very important conceptual role. It just tells us that any *local* property of regular parameterized curves is valid, also, for regular curves, if it is invariant under parameter changes, *without* the assumptions of the parameterized curves being homeomorphisms onto. Of course, all the precautions should be taken when we investigate the *global* properties of regular curves.

7.4 Analytical representations of curves

7.4.1 Plane curves

A regular curve $M \subset \mathbb{R}^3$ is called *plane* if it is contained into a plane π. We shall, usually, assume that the plane π coincides with the coordinate plane xOy and we shall use, therefore, only the coordinates x and y to describe such curves.

Parametric representation. We choose an arbitrary local parameterization $(I, \mathbf{r}(t) = (x(t), y(t), z(t)))$ of the curve. Then the support $\mathbf{r}(I)$ of this local parameterization is an open subset of the curve. For a global parameterization of a simple curve, $\mathbf{r}(I)$ is the entire curve. Thus, any point a of the curve has an open neighbourhood which is the support of the parameterized curve

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} . \tag{7.4.1}$$

The equations (7.4.1) are called *the parametric equations* of the curve in the neighbourhood of the point a. Usually, unless the curve is simple, we cannot use the same set of equations to describe the points of an entire curve.

Explicit representation. Let $f : I \to \mathbb{R}$ be a smooth function, defined on an open interval from the real axis. Then its graph

$$C = \{(x, f(x)) \mid x \in I\}, \tag{7.4.2}$$

is a simple curve, which has the global parameterization

$$\begin{cases} x = t \\ y = f(t) \end{cases} . \tag{7.4.3}$$

The equation

$$y = f(x) \tag{7.4.4}$$

is called *the explicit equation* of the curve (7.4.2).

In the literature for the explicit representation of a curve it is, also, used the term *non-parametric form*, which we do not find particularly appealing since, in fact, an explicit representation can be thought of as a particular case of parameter representation (the parameter being the coordinate x).

Implicit representation. Let $F : D \to \mathbb{R}$ be a smooth function, defined on a domain $D \subset \mathbb{R}^2$, and let

$$C = \{(x, y) \in D \mid F(x, y) = 0\} \tag{7.4.5}$$

be the 0-level set of the function F. Generally speaking, C is not a regular curve (we can only say that it is a closed subset of the plane). Nevertheless, if at the point $(x_0, y_0) \in C$ the vector grad $F = \{\partial_x F, \partial_y F\}$ is non vanishing, for instance $\partial_y F(x_0, y_0) \neq 0$, then, by the implicit functions theorem there exist:

- an open neighborhood U of the point (x_0, y_0) in $\mathbb{R}^2$;
- a smooth function $y = f(x)$, defined on an open neighborhood $I \subset \mathbb{R}$ of the point x_0,

such that

$$C \cap U = \{(x, f(x)|x \in I\}.$$

If grad $F \neq 0$ in all the points of C, then C is a regular curve (although, in general, not a simple one).

Figure 7.1: The bisectors of the coordinate axes

Examples.

1. $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x^2 + y^2 - 1$. Let

   $$(x_0, y_0) \in C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 = 0\}.$$

   Then we have

   $$grad\ F(x_0, y_0) = \{2x_0, 2y_0\}.$$

   Obviously, since $x_0^2 + y_0^2 = 1$, the vector grad F cannot vanish on C and, thus, C is a curve (the unit circle, centred at the origin).

2. $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x^2 - y^2$. C is not a curve in this case (the gradient is vanishing at the origin). In fact, the set C has a self intersection at the origin (C is just the union of the two bissectors of the coordinate axes, see figure 7.1). It might not be obvious why we encounter problems in the neighbourhood of the origin for this "curve". The point is that there is no neighbourhood of the origin (on C), homeomorphic to an open interval on the real axis. An neighbourhood of the origin of C is the intersection of an open neighbourhood

of the origin in the plane and the set C. Now, if we restrict the neighbourhood of the origin in the plane, its intersection with the set C will be a cross. If we remove the origin from the cross, the remaining set will have four connected components. On the other hand, suppose there is a homeomorphism f from the cross to an open interval from the real axis. If we remove from the interval the image of the origin through the homeomorphism f, we will get, clearly, only two connected components, or it can be proved that the number of connected components resulted by removing a point is homeomorphisms-invariant.

Remark. It should be clear that the condition of nonsingularity of the gradient of F is only a *sufficient* condition for the equation $F(x, y) = 0$ to represent a curve. If the gradient of F is zero at a point, we cannot claim that the equation represent a curve in the neighborhood of that point, but we cannot claim the opposite, either. Consider, as a trivial example, the equation

$$F(x, y) \equiv (x - y)^2 = 0.$$

Then we have

$$\operatorname{grad} F(x, y) = 2\{x - y, -(x - y)\}$$

and if we denote

$$C = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\},$$

then grad $F = 0$ at all the points of C. But, clearly, C *is* a curve (it is easy to see that it is the first bissector of the coordinate axes, i.e. a straight line).

7.4.2 Space curves

Parametric representation. As in the case of plane curves, with a local parameterization

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \tag{7.4.6}$$

we can represent either the entire curve, or only a neighbourhood of one of its points.

Explicit representation. If $f, g : I \to \mathbb{R}$ are two smooth functions, defined on an open interval from the real axis, then the set

$$C = \{(x, f(x), g(x)) \in \mathbb{R}^3 \mid x \in I\} \tag{7.4.7}$$

is a simple curve, with a global parameterization given by

$$\begin{cases} x = t \\ y = f(t) \\ z = g(t) \end{cases}. \tag{7.4.8}$$

The equations

$$\begin{cases} y = f(x) \\ z = g(x) \end{cases} \tag{7.4.9}$$

are called *the explicit equations* of the curve. Let us note that, in fact, each equation of the system (7.4.9) is the equation of a cylindrical surface, with the generators parallel to one of the coordinate axis. Therefore, representing explicitly a curve actually means representing it as an intersection of two cylindrical surfaces, with the two families of generators having orthogonal directions.

Implicit representation. Let $F, G : D \to \mathbb{R}$, defined on a domain $D \subset \mathbb{R}^3$. We consider the set

$$C = \{(x, y, z) \in D \mid F(x, y, z) = 0,\ G(x, y, z) = 0\},$$

in other words, the solutions' set for the system

$$\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{7.4.10}$$

In the general case, the set C is not a regular curve. Nevertheless, if at the point $a = (x_0, y_0, z_0) \in C$ the rank of the Jacobi matrix

$$\begin{pmatrix} \partial_x F & \partial_y F & \partial_z F \\ \partial_x G & \partial_y G & \partial_z G \end{pmatrix} \tag{7.4.11}$$

is equal to two, then there is an open neighborhood $U \subset D$ of the point (x_0, y_0, z_0) such that $C \cap U$ — the set of solutions of the system (7.4.10) in U — is a curve. Indeed, suppose, for instance, that

$$\det \begin{pmatrix} \partial_y F(a) & \partial_z F(a) \\ \partial_y G(a) & \partial_z G(a) \end{pmatrix} \neq 0.$$

Then, from the implicit functions theorem, there is an open neighborhood $U \subset D$ such that the set $C \cap U$ can be written as

$$C \cap U = \{(x, f(x), g(x)) | x \in W\},$$

where W is an open neighborhood in $\mathbb{R}$ of the point x_0, while $y = f(x)$, $z = g(x)$ are smooth functions, defined on W. Clearly, $C \cap U$ is a simple curve, while the pair $(W, \mathbf{r}(t) = (t, f(t), g(t))$ is a global parameterization of it.

If the rank of the matrix (7.4.11) is equal to two everywhere, then C is a curve (although, generally, not a simple one).

Example 7.3 (The Viviani's temple)**.** An important example of space curve given by implicit equations is the so-called *temple of Viviani*[4]. This curve is obtained as the intersection between the sphere with the center at the origin and radius $2a$ and the circular cylinder with radius a and axis parallel to the z-axis, situated at a distance a from this axis. In other words, the equations of the Viviani's temple are

$$\begin{cases} x^2 + y^2 + z^2 = 4a^2, \\ (x - a)^2 + y^2 = a^2. \end{cases}$$

It is instructive to make some computations for the case of the Viviani's temple. As we shall see, it is not, globally, a curve. We will have to eliminate a point to get, indeed, a regular curve. In fact, the shape of Viviani's temple is easy to understand. The curve has a shape which is similar to a Bernoulli's lemniscate, lying on the surface of a sphere. So, let

$$\begin{cases} F(x, y, z) = x^2 + y^2 + z^2 - 4a^2, \\ G(x, y, z) = (x - a)^2 + y^2 - a^2. \end{cases}$$

Then the equations of the curve reads

$$\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases}$$

We have, now,

$$\begin{pmatrix} F'_x & F'_y & F'_z \\ G'_x & G'_y & G'_z \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ 2(x - a) & 2y & 0 \end{pmatrix} = 2 \begin{pmatrix} x & y & z \\ x - a & y & 0 \end{pmatrix}.$$

[4] Vincenzo Viviani (1622–1703) was a Italian mathematician and architect, who was in contact with Galileo Galilei in the last years of the great scientist, and who liked to recommend himself as "Galileo's last student".

To get a singular point, the following system of equations has to be fulfilled

$$\begin{cases} y = 0 \\ yz = 0 \\ (x-a)z = 0 \end{cases}.$$

Clearly, the only solution of the system that verifies, also, the equations of the curve is $x = 2a, y = z = 0$. Thus, the Viviani's temple (see figure **??**) is a regular curve everywhere except the point which has these coordinates. It is not difficult to show that Viviani's temple is, in fact, the support of the parameterized curve

$$\mathbf{r}(t) = (a(1+\cos t), a\sin t, 2a\sin\frac{t}{2}),$$

with $t \in (-2\pi, 2\pi)$. If we compute $\mathbf{r}'(t)$ we get

$$\mathbf{r}'(t) = \{-a\sin t, a\cos t, a\cos\frac{t}{2}\},$$

which means that this parameterized curve is regular. In particular, the existence of this regular parametric representation of the Viviani's temple shows that the point of coordinates $x = 2a, y = z = 0$ is, in fact, a point of self-intersection, rather then a singular point.

7.5 The tangent and the normal plane. The normal at a plane curve

Definition 7.7. For a parameterized curve $\mathbf{r} = \mathbf{r}(t)$ the vector $\mathbf{r}'(t_0)$ is called the *tangent vector* or the *velocity vector* of the curve at the point t_0. If the point t_0 is regular, then the straight line passing through $\mathbf{r}(t_0)$ and having the direction of the vector $\mathbf{r}'(t_0)$ is called the *tangent to the curve at the point* $\mathbf{r}(t_0)$ (or at the point t_0).

The vectorial equation of the tangent line reads, thus:

$$R(\tau) = \mathbf{r}(t_0) + \tau\mathbf{r}'(t_0). \tag{7.5.1}$$

Example 7.4. The cylindrical helix has the parameterization

$$\mathbf{r}(t) = (a\cos t, a\sin t, bt),$$

therefore, for a point t_0,

$$\mathbf{r}'(t_0) = \{-a\cos t_0, a\sin t_0, b\}.$$

Thus, the equation of the tangent is

$$\begin{aligned} R(\tau) =&(a\cos t_0 - \tau a\sin t_0, a\sin t_0 + \tau a\cos t_0, bt_0 + \tau b) = \\ =&(a(cost_0 - \tau\sin t_0), a(sint_0 + \tau\cos t_0), b(t_0 + \tau)). \end{aligned}$$

Property 4. *The tangent vectors of two equivalent parameterized curves at corresponding points are colinear, while the tangent lines coincide.*

Proof Let $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ the two equivalent parameterized curves and $\lambda : I \to J$ the parameter change, i.e. $\mathbf{r} = \boldsymbol{\rho}(\lambda(t))$. Then, according to the chain rule,

$$\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t),$$

with $\lambda'(t) \neq 0$. □

Remarks.

1. Clearly, $\mathbf{r}'$ and $\boldsymbol{\rho}'$ have the same sense when $\lambda' > 0$ (the parameter change does not modify the sense in which the support of the parameterized curve is traversed) and they have opposite sense when $\lambda' < 0$.
2. Since the parameter change modifies the tangent vector, it doesn't make sense to define the tangent vector at a point of a regular curve, through a local parameterization. Nevertheless, as we have seen, only the orientation and the length of the tangent vector can vary, but not the direction. Thus, it makes sense to speak about the tangent line at a point of a regular curve, defined through *any* local parameterization of the curve around that point.

We can use a more "geometric" way to define the tangent to a parameterized curve. Let $\mathbf{r}(t_0 + \Delta t)$ be a point of the curve close to the point $\mathbf{r}(t_0)$. Then, according to the Taylor's formula,

$$\mathbf{r}(t_0 + \Delta t) = \mathbf{r}(t_0) + \Delta t \cdot \mathbf{r}'(t_0) + \Delta t \cdot \boldsymbol{\epsilon}, \tag{7.5.2}$$

with $\lim\limits_{\Delta t \to 0} \boldsymbol{\epsilon} = 0$. We consider an arbitrary straight line π, passing through $\mathbf{r}(t_0)$ and having the direction given by the unit vector $\mathbf{m}$. Let

$$d(\Delta t) \stackrel{\text{def}}{=} d((\mathbf{r}(t_0 + \Delta t), \pi).$$

Theorem 7.3. *The straight line π is the tangent line to the parameterized curve $\mathbf{r} = \mathbf{r}(t)$ at the point t_0 iff*

$$\lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|} = 0. \tag{7.5.3}$$

Proof From the Taylor's formula (7.5.2), we have

$$\Delta \mathbf{r} \equiv \mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0) = \Delta t \cdot \mathbf{r}'(t_0) + \Delta t \cdot \boldsymbol{\epsilon}.$$

The distance $d(\Delta t)$ is equal to

$$\|\Delta \mathbf{r} \times \mathbf{m}\| = |\Delta t| \|\mathbf{r}'(t_0) \times \mathbf{m} + \boldsymbol{\epsilon} \times \mathbf{m}\|.$$

Thus,

$$\lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|} = \lim_{\Delta t \to 0} \|\mathbf{r}'(t_0) \times \mathbf{m} + \underbrace{\boldsymbol{\epsilon} \times \mathbf{m}}_{\to 0}\| = \|\mathbf{r}'(t_0) \times \mathbf{m}\|.$$

Now, if the straight line π is the tangent line at t_0, then the vectors $\mathbf{r}'(t_0)$ and $\mathbf{m}$ are colinear, therefore $\mathbf{r}'(t_0) \times \mathbf{m} = 0$.

Conversely, if the condition (7.5.3) is fulfilled, then $\|\mathbf{r}'(t_0) \times \mathbf{m}\| = 0$, therefore either $\mathbf{r}'(t_0) = 0$ (which cannot happen, since the parameterized curve is regular), either the vectors $\mathbf{r}'(t_0)$ and $\mathbf{m}$ are colinear, i.e. π is the tangent line at t_0. $\square$

Remark. The condition (7.5.3) is expressed by saying that the tangent line and the curve have *a first order contact* (or tangency contact). Another way of interpreting this formula is that the tangent line is the limit position of the straight line determined by the chosen point and a neighbouring point on the curve, when the neighbouring point is approaching indefinitely to the given one.

Hereafter, if not specified otherwise, all the parameterized curves considered will be regular.

Definition 7.8. Let $\mathbf{r} = \mathbf{r}(t)$ be a parameterized curve and $t_0 \in I$. The *normal plane* at the point $\mathbf{r}(t_0)$ of the curve $\mathbf{r} = \mathbf{r}(t)$ is the plane which passes through $\mathbf{r}(t_0)$ and it is perpendicular to the tangent line to the curve at $\mathbf{r}(t_0)$.

If $\mathbf{r} = \mathbf{r}(t)$ is a *plane* parameterized curve (i.e. its support is contained into a plane, which we will assume to be identical to the coordinate plane xOy), then the *normal* to the curve at the point $\mathbf{r}(t_0)$ will be the straight line through $\mathbf{r}(t_0)$, which is perpendicular to the tangent line to the curve at the point $\mathbf{r}(t_0)$.

Remark. Because it makes sense to define the tangent line at a point of a regular curve, by using an arbitrary local parameterization around that point, the same is true for the normal plane (or the normal line, in the case of plane curves).

The vectorial equation of the normal plane (line) follows immediately from the definition:

$$(R - \mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = 0. \tag{7.5.4}$$

7.5.1 The equations of the tangent line and normal plane (line) for different representations of curves

Parametric representation

If we start from the vectorial equation (7.5.1) of the tangent line and project it on the coordinate axes, we obtain the parametric equations of the tangent line, i.e., for space curves,

$$\begin{cases} X(\tau) = x(t_0) + \tau x'(t_0), \\ Y(\tau) = y(t_0) + \tau y'(t_0), \\ Z(\tau) = z(t_0) + \tau z'(t_0), \end{cases} \tag{7.5.5}$$

and, for plane curves,

$$\begin{cases} X(\tau) = x(t_0) + \tau x'(t_0), \\ Y(\tau) = y(t_0) + \tau y'(t_0). \end{cases} \tag{7.5.6}$$

If we eliminate the parameter τ, we get the canonical equations:

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'}, \tag{7.5.7}$$

for space curves, respectively

$$\frac{X - x}{x'} = \frac{Y - y}{y'}, \tag{7.5.8}$$

for plane curves.

As for the equation of the normal plane (line), we can obtain it from (7.5.4), expressing it as

$$\{X - x, Y - y, Z - z\} \cdot \{x', y', z'\} = 0,$$

for space curves and

$$\{X - x, Y - y\} \cdot \{x', y'\} = 0,$$

for plane curves. Expanding the scalar products, we get:

$$(X - x)x' + (Y - y)y' + (Z - z)z' = 0, \tag{7.5.9}$$

for the equation of the normal plane to a space curve and, for the normal line to a plane curve,

$$(X - x)x' + (Y - y)y' = 0. \tag{7.5.10}$$

Explicit representation

If we have a space curve given by the equations

$$\begin{cases} y = f(x) \\ z = g(x) \end{cases},$$

then we can construct a parameterization

$$\begin{cases} x = t \\ y = f(t) \\ z = g(t). \end{cases}$$

For the derivatives we obtain immediately the expressions

$$\begin{cases} x' = 1 \\ y' = f' \\ z' = g', \end{cases}$$

which, when substituted into the equations (7.5.7), give

$$X - x = \frac{Y - f(x)}{f'(x)} = \frac{Z - g(x)}{g'(x)}, \tag{7.5.11}$$

while for the equation of the normal plane, after substituting the derivatives into the equation (7.5.9), we obtain

$$X - x + (Y - f(x))f'(x) + (Z - g(x))g'(x) = 0. \tag{7.5.12}$$

For a plane curve given explicitly

$$y = f(x),$$

we have the parametric representation

$$\begin{cases} x = t \\ y = f(t), \end{cases}$$

and, thus, the equation of the tangent line is

$$X - x = \frac{Y - f(x)}{f'(x)} \tag{7.5.13}$$

or, in a more familiar form,

$$Y - f(x) = f'(x)(X - x), \tag{7.5.14}$$

while for the normal we get

$$X - x + (Y - f(x))f'(x) = 0 \tag{7.5.15}$$

ore

$$Y - f(x) = -\frac{1}{f'(x)}(X - x). \tag{7.5.16}$$

Implicit representation

Let us consider a curve given by the implicit equations

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0. \end{cases} \tag{7.5.17}$$

Let us suppose that, at a point (x_0, y_0, z_0)

$$\det \begin{pmatrix} F'_y & F'_z \\ G'_y & G'_z \end{pmatrix} \neq 0$$

Then, as we saw before, around this point, the curve can be represented as

$$\begin{cases} y = f(x) \\ z = g(x), \end{cases} \tag{7.5.18}$$

i.e. the system (7.5.17) can be written as

$$\begin{cases} F(x, f(x), g(x)) = 0 \\ G(x, f(x), g(x)) = 0. \end{cases}$$

By computing the derivatives with respect to x of F and G, we get the system

$$\begin{cases} F'_x + f'(x)F'_y + g'(x)F'_z = 0 \\ G'_x + f'(x)G'_y + g'(x)G'_z = 0, \end{cases}$$

therefore

$$\begin{cases} f'F'_y + g'F'_z = -F'_x \\ f'G'_y + g'G'_z = -G'_x. \end{cases}$$

From this system, one can obtain f' and g', through the Cramer method:

$$\Delta = \begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix} \overset{not}{=} \frac{D(F,G)}{D(y,z)} \overset{hyp}{\neq} 0,$$

$$\Delta_{f'} = \begin{vmatrix} -F'_x & F'_z \\ -G'_x & G'_z \end{vmatrix} = \begin{vmatrix} F'_z & F'_x \\ G'_z & G'_x \end{vmatrix} \overset{not}{=} \frac{D(F,G)}{D(z,x)}$$

$$\Delta_{g'} = \begin{vmatrix} F'_y & -F'_x \\ G'_y & -G'_x \end{vmatrix} = \begin{vmatrix} F'_x & F'_y \\ G'_x & G'_y \end{vmatrix} \overset{not}{=} \frac{D(F,G)}{D(x,y)},$$

therefore

$$\begin{cases} f' = \dfrac{\frac{D(F,G)}{D(z,x)}}{\frac{D(F,G)}{D(y,z)}} \\ g' = \dfrac{\frac{D(F,G)}{D(x,y)}}{\frac{D(F,G)}{D(y,z)}} \end{cases} \tag{7.5.19}$$

As we saw before, for the curve (7.5.18) the equations of the tangent are

$$X - x_0 = \frac{Y - f(x_0)}{f'(x_0)} = \frac{Z - g(x_0)}{g'(x_0)}$$

or, using (7.5.19),

$$X - x_0 = \frac{Y - f(x_0)}{\frac{\frac{D(F,G)}{D(z,x)}}{\frac{D(F,G)}{D(y,z)}}} = \frac{Z - g(x_0)}{\frac{\frac{D(F,G)}{D(x,y)}}{\frac{D(F,G)}{D(y,z)}}},$$

from where, having in mind that $f(x_0) = y_0$ and $g(x_0) = z_0$, we have

$$\frac{X - x_0}{\frac{D(F,G)}{D(y,z)}} = \frac{Y - y_0}{\frac{D(F,G)}{D(z,x)}} = \frac{Z - z_0}{\frac{D(F,G)}{D(x,y)}}.$$

For a plane curve

$$F(x, y) = 0,$$

if at the point (x_0, y_0) there is fulfilled the condition $F'_y \neq 0$, then, from the implicit functions theorem, locally, $y = f(x)$, hence the equation of the curve can be written as

$$F(x, f(x)) = 0.$$

By differentiating this relation with respect to x, one obtains

$$F'_x + f'F'_y = 0 \Longrightarrow f' = -\frac{F'_x}{F'_y}.$$

Thus, from the equation of the tangent:

$$Y - y_0 = f'(x_0)(X - x_0),$$

we deduce

$$Y - y_0 = -\frac{F'_x}{F'_y}(X - x_0)$$

or

$$(X - x_0)F'_x + (Y - y_0)F'_y = 0,$$

while for the normal we obtain the equation

$$(X - x_0)F'_y - (Y - y_0)F'_x = 0.$$

7.6 The osculating plane

Definition 7.9. A parameterized curve $\mathbf{r} = \mathbf{r}(t)$ is called *biregular* (or *in general position*) at the point t_0 if the vectors $\mathbf{r}'(t_0)$ and $\mathbf{r}''(t_0)$ are not colinear, i.e.

$$\mathbf{r}'(t_0) \times \mathbf{r}''(t_0) \neq 0.$$

The parameterized curve is called *biregular* if it is biregular at each point[5].

Remark. It is not difficult to check that the notion of a biregular point is independent of the parameterization: if a point is biregular for a given parameterized curve, then its corresponding point through any parameter change is, also, a biregular point.

Definition 7.10. Let $(I, \mathbf{r})$ be a parameterized curve and $t_0 \in I$ – a biregular point. The *osculating plane* of the curve at $\mathbf{r}(t_0)$ is the plane through $\mathbf{r}(t_0)$, parallel to the vectors $\mathbf{r}'(t_0)$ and $\mathbf{r}''(t_0)$, i.e. the equation of the plane is

$$(R - \mathbf{r}(t_0), \mathbf{r}'(t_0), \mathbf{r}''(t_0)) = 0, \tag{7.6.1}$$

[5] A biregular curve is also called in some books a *complete* curve. We find this term a little bit misleading, since this terms has, usually, other meaning in the global theory of curves (and, especially, surfaces).

or, expanding the mixed product,

$$\begin{vmatrix} X - x_0 & Y - y_0 & Z - z_0 \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0. \tag{7.6.2}$$

Theorem 7.4. *The osculating planes to two equivalent parameterized curves at corresponding biregular points coincides.*

Proof Let $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ be two equivalent parameterized curves and $\lambda : I \to J$ – the parameter change. Then

$$\begin{aligned} \mathbf{r}(t) &= \boldsymbol{\rho}(\lambda(t)), \\ \mathbf{r}'(t) &= \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t), \\ \mathbf{r}''(t) &= \boldsymbol{\rho}''(\lambda(t)) \cdot (\lambda'(t))^2 + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t). \end{aligned}$$

Since $\lambda'(t) \neq 0$, from these relations it follows that the system of vectors $\{\mathbf{r}'(t), \mathbf{r}''(t)\}$ and $\{\boldsymbol{\rho}'(\lambda(t)), \boldsymbol{\rho}''(\lambda(t))\}$ are equivalent, i.e. they generate the same vector subspace $\mathbb{R}^3$, hence the osculating planes to the two parameterized curves at the corresponding points t_0 and $\lambda(t_0)$ have the same directing subspace, therefore, they are parallel. As they have a common point (since $\mathbf{r}(t_0) = \boldsymbol{\rho}(\lambda(t_0))$), they have to coincide. □

Remark. From the previous theorem it follows that the notion of an osculating plane makes sense also for regular curves.

As in the case of the tangent line, there is a more geometric way of defining the osculating plane which is, at the same time, more general, since it can be applied also for the case of the points which are not biregular.

Let $\mathbf{r}(t_0)$ şi $\mathbf{r}(t_0 + \Delta t)$ two neighboring points on a parameterized curve, with $\mathbf{r}(t_0)$ biregular. We consider a plane α, of normal versor $\mathbf{e}$, passing through $\mathbf{r}(t_0)$, and denote $d(\Delta t) = d(\mathbf{r}(t_0 + \Delta t), \alpha)$.

Theorem 7.5. *α is the osculating plane to the parameterized curve $\mathbf{r} = \mathbf{r}(t)$ at the biregular point $\mathbf{r}(t_0)$ iff*

$$\lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|^2} = 0,$$

i.e. the curve and the plane have a second order contact.

Proof From the Taylor formula we have

$$\mathbf{r}(t_0 + \Delta t) = \mathbf{r}(t_0) + \Delta t \cdot \mathbf{r}'(t_0) + \frac{1}{2}(\Delta t)^2 \cdot \mathbf{r}''(t_0) + (\Delta t)^2 \cdot \boldsymbol{\epsilon},$$

with $\lim_{\Delta t \to 0} \boldsymbol{\epsilon} = 0$.

On the other hand,

$$\begin{aligned} d(\Delta t) &= |\mathbf{e} \cdot (\mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0))| = \\ &= |(\mathbf{e} \cdot \mathbf{r}'(t_0)) \cdot \Delta t + \frac{1}{2}(\mathbf{e} \cdot \mathbf{r}''(t_0)) \cdot (\Delta t)^2 + (\mathbf{e} \cdot \boldsymbol{\epsilon}) \cdot (\Delta t)^2|. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|^2} &= \lim_{\Delta t \to 0} \left| \frac{\mathbf{e} \cdot \mathbf{r}'(t_0)}{\Delta t} + \frac{1}{2} \cdot (\mathbf{e} \cdot \mathbf{r}''(t_0)) + \underbrace{\mathbf{e} \cdot \boldsymbol{\epsilon}}_{\to 0} \right| = \\ &= \lim_{\Delta t \to 0} \left| \frac{\mathbf{e} \cdot \mathbf{r}'(t_0)}{\Delta t} + \frac{1}{2} \cdot (\mathbf{e} \cdot \mathbf{r}''(t_0)) \right|. \end{aligned}$$

If $\lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|^2} = 0$, then $\mathbf{e} \cdot \mathbf{r}'(t_0) = 0$ and $\mathbf{e} \cdot \mathbf{r}''(t_0) = 0$, i.e. $\mathbf{e} \parallel \mathbf{r}'(t_0) \times \mathbf{r}''(t_0)$, meaning that α is the osculating plane.

The converse is obvious. □

Remarks. (i) The previous theorem justifies the name of the osculating plane. Actually, the name (coined by Johann Bernoulli), comes from the Latin verb *osculare*, which means *to kiss* end emphasizes the fact that, among all the planes that are passing through a given point of a curve, the osculating plane has the higher order ("the closer") contact.

(ii) Defining the osculating plane through the contact, one can define the notion of an osculating plane also for the points which are not biregular, but in that case any plane passing through the tangent is an osculating plane, in the sense that it has a second order contact with the curve. Saying that the osculating plane at a biregular point is the only plane which has at that point a second order contact with the curve is the same with saying that the osculating plane is the limit position of a plane determined by the considered point and two neighbouring points when this ones are approaching indefinitely to the given one. Also, on can define the osculating plane at biregular point of a parameterized curve as being the limit position of a plane passing through the tangent at the given point and a neighboring point from the curve, when this point is approaching indefinitely the given one.

A natural question that one may ask is what happens with the osculating plane in the particular case of a plane parameterized curve. The answer is given by the following proposition, whose prove is left to the reader:

Property 5. *If a biregular parameterized curve is plane, i.e. its support is contained into a plane π, then the osculating plane to this curve at each point coincides to the plane of the curve. Conversely, if a given biregular parameterized curve has the same osculating plane at each point, then the curve is plane and its support is contained into the osculating plane.*

7.7 The curvature of a curve

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a regular parameterized curve. Let $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ be a naturally parameterized curve, equivalent to it. Then $\|\boldsymbol{\rho}'(s)\| = 1$, while the vector $\boldsymbol{\rho}''(s)$ is orthogonal to $\boldsymbol{\rho}'(s)$[6].

One can show that $\boldsymbol{\rho}''(s)$ does not depend on the choice of the naturally parameterized curve equivalent to the given curve $\mathbf{r} = \mathbf{r}(t)$. Indeed, if $(J_1, \boldsymbol{\rho}_1 = \boldsymbol{\rho}_1(\tilde{s}))$ is another equivalent naturally parameterized curve with the parameter change $\tilde{s} = \lambda(s)$, then, from the condition

$$\|\boldsymbol{\rho}'(s)\| = \|\boldsymbol{\rho}_1'(\lambda(s))\| = 1$$

we get $|\lambda'(s)| = 1$ for any $s \in J$. Thus, $\lambda' = \pm 1$ and, therefore, $\tilde{s} = \pm s + s_0$, where s_0 is a constant. It follows that

$$\boldsymbol{\rho}''(s) = \boldsymbol{\rho}''{}_1(\tilde{s}) \underbrace{(\lambda'(s))^2}_{=1} + \boldsymbol{\rho}'{}_1 \cdot \underbrace{\lambda''(s)}_{=0} = \boldsymbol{\rho}''{}_1(\tilde{s}).$$

Definition 7.11. The vector $\mathbf{k} = \boldsymbol{\rho}''(s(t))$ is called the *curvature vector* of the parameterized curve $\mathbf{r} = \mathbf{r}(t)$ at the point t, while its norm, $k(t) = \|\boldsymbol{\rho}''(s(t))\|$ – the *curvature* of the parameterized curve at the point t.

We shall express now the curvature vector $\mathbf{k}(t)$ as a function of $\mathbf{r}(t)$ and its derivatives. We choose as natural parameter the arc length of the curve. Then we have

$$\begin{aligned}
\mathbf{r}(t) &= \boldsymbol{\rho}(s(t)) \Rightarrow \\
\mathbf{r}'(t) &= \boldsymbol{\rho}'(s(t)) \cdot s'(t) \\
\mathbf{r}''(t) &= \boldsymbol{\rho}''(s(t)) \cdot (s'(t))^2 + \boldsymbol{\rho}'(s(t)) \cdot s''(t),
\end{aligned}$$

[6]Indeed, since $\boldsymbol{\rho}'$ is a unit vector, we have $\boldsymbol{\rho}'^2 = 1$. Differentiating this relation, we obtain that $\boldsymbol{\rho}' \cdot \boldsymbol{\rho}'' = 0$, which expresses the fact that the two vectors are orthogonal.

where $s'(t) = \|\mathbf{r}'(t)\|$, $s''(t) = \|\mathbf{r}'(t)\|' = \dfrac{d}{dt}\left(\sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}\right) = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}$.
Thus, we have

$$\mathbf{k}(t) = \boldsymbol{\rho}''(s(t)) = \frac{\mathbf{r}''}{\|\mathbf{r}'\|^2} - \frac{\mathbf{r}'\cdot\mathbf{r}''}{\|\mathbf{r}'\|^4}\cdot\mathbf{r}'. \tag{7.7.1}$$

Now, since the vectors $\boldsymbol{\rho}'$ and $\boldsymbol{\rho}''$ are orthogonal, while $\boldsymbol{\rho}'$ has unit length, we have

$$k(t) = \|\mathbf{k}(t)\| = \|\boldsymbol{\rho}''\| = \|\boldsymbol{\rho}'\times\boldsymbol{\rho}''\|.$$

Substituting $\boldsymbol{\rho}' = \dfrac{\mathbf{r}'}{\|\boldsymbol{\rho}'\|}$, and $\boldsymbol{\rho}''$ by the formula (7.7.1) we obtain

$$k(t) = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3}. \tag{7.7.2}$$

Remarks. 1. From the formula (7.7.2) it follows that a parameterized curve $\mathbf{r} = \mathbf{r}(t)$ is biregular at a point t_0 iff $k(t_0) \neq 0$.

2. Since for equivalent parameterized curve the naturally parameterized curve to each of them are equivalent between them, the notion of curvature makes sense also for regular curves.

Examples. 1. For the straight line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}$ the curvature vector (and, therefore, also the curvature) is identically zero.

2. For the circle $S^1_R = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = R^2,\ z = 0\}$ we choose the parameterization

$$\begin{cases} x = R\cos t \\ y = R\sin t \qquad 0 < t < 2\pi. \\ z = 0 \end{cases}$$

Then

$$\mathbf{r}'(t) = \{-R\sin t, R\cos t, 0\}, \quad \mathbf{r}''(t) = \{-R\cos t, R\sin t, 0\}$$

and, thus, $\|\mathbf{r}'(t)\| = R$, $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = 0$. Therefore, for the curvature vector we get

$$\mathbf{k}(t) = \left\{-\frac{1}{R}\cos t, -\frac{1}{R}\sin t, 0\right\} = -\frac{1}{R}\{x, y, z\}, \quad k(t) = \frac{1}{R}.$$

Remarks. 1. The computation we just made explains why the inverse of the curvature is called the *curvature radius* of the curve.

2. We saw that the curvature of a straight line is identically zero. The converse is also true, in some sense, as the following proposition shows.

Property 6. *If the curvature of a regular parameterized curve is identically zero, then the support of the curve lies on a straight line.*

Proof Suppose, for the very beginning, that we are dealing with a naturally parameterized curve $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$. From the hypothesis, $\boldsymbol{\rho}''(s) = 0$, therefore $\boldsymbol{\rho}'(s) = \mathbf{a} = \text{const}$, $\boldsymbol{\rho}(s) = \boldsymbol{\rho}_0 + s\mathbf{a}$, i.e. the support $\boldsymbol{\rho}(I)$ lies on a straight line. As two equivalent parameterized curves have the same support, the result still holds also for non-naturally parameterized curves. $\square$

Remark. Be careful, the fact that a parameterized curve has zero curvature simply means that the *support* of the curve lies on a straight line, but it doesn't necessarily means that the parameterized curve is (the restriction of) an affine map from $\mathbb{R}$ to $\mathbb{R}^3$, nor that it is equivalent to such a particular parameterized curve. We can consider, as we did before, the parameterized curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t^3$, where $\mathbf{a}$ is a constant, nonvanishing, vector from $\mathbb{R}^3$. Then we have, immediately, $\mathbf{r}'(t) = 2\mathbf{a}t^2$ and $\mathbf{r}''(t) = 3\mathbf{a}t$, which means that the velocity and the acceleration of the curve are parallel, hence the curve has zero curvature but, as we also saw earlier, this parameterized curve is not equivalent to an affine parameterized curve.

7.7.1 The geometrical meaning of curvature

Let us consider a naturally parameterized curve $(I, \mathbf{r} = \mathbf{r}(s))$. We denote by $\Delta\varphi(s)$ the measure of the angle between the versors $\mathbf{r}(s)$ and $\mathbf{r}(s + \Delta s)$. Then

$$\|\mathbf{r}(s + \Delta s) - \mathbf{r}(s)\| = 2\left|\sin\frac{\Delta\varphi(s)}{2}\right|.$$

Therefore,

$$k(s) = \|\mathbf{r}''(s)\| = \left\|\lim_{\Delta s \to 0} \frac{\mathbf{r}(s + \Delta s) - \mathbf{r}(s)}{\Delta s}\right\| = \lim_{\Delta s \to 0} \frac{2\left|\sin\frac{\Delta\varphi(s)}{2}\right|}{|\Delta s|} =$$

$$= \lim_{\Delta s \to 0}\left|\frac{\Delta\varphi(s)}{\Delta s}\right| \cdot \frac{\left|\sin\frac{\Delta\varphi(s)}{2}\right|}{\left|\frac{\Delta\varphi(s)}{2}\right|} = \lim_{\Delta s \to 0}\left|\frac{\Delta\varphi(s)}{\Delta s}\right| = \left|\frac{d\varphi}{ds}\right|.$$

Thus, if we have in mind that $\Delta\varphi(s)$ is the measure of the angle between the tangents to the curve at s and $s + \Delta s$, the last formula gives us:

Property 7. *The curvature of a curve is the speed of rotation of the tangent line to the curve, when the tangency point is moving along the curve with unit speed.*

7.8 The Frenet frame (the moving frame) of a parameterized curve

At each point of the support of a biregular parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ one can construct a frame of the space $\mathbb{R}^3$. The idea is that, if we want to investigate the local properties of a parameterized curve around a given point of the curve, that it might be easier to do that if we don't use the standard coordinate system of $\mathbb{R}^3$, but a coordinate system with the origin at the given point of the curve, while the coordinate axes have some connection with the local properties of the curve. Such a coordinate system was constructed, independently, at the middle of the XIX-th century, by the French mathematicians Frenet and Serret.

Definition 7.12. The *Frenet frame* (or the *moving frame*) of a biregular parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ at the point $t_0 \in I$ is an orthonormal frame of the space $\mathbb{R}^3$, with the origin at the point $\mathbf{r}(t_0)$, the coordinate versors being the vectors $\{\boldsymbol{\tau}(t_0), \boldsymbol{\nu}(t_0), \boldsymbol{\beta}(t_0)\}$, where:

- $\boldsymbol{\tau}(t_0)$ is the versor of the tangent to the curve at t_0, i.e.

$$\boldsymbol{\tau}(t_0) = \frac{\mathbf{r}'(t_0)}{\|\mathbf{r}'(t_0)\|}.$$

  $\boldsymbol{\tau}(t_0)$ is also called the *unit tangent* at the point t_0;

- $\boldsymbol{\nu}(t_0) = \mathbf{k}(t_0)/k(t_0)$ is the versor of the curvature vector:

$$\boldsymbol{\nu}(t_0) = \frac{\mathbf{k}(t_0)}{k(t_0)}$$

  and it is called the *unit principal normal* at the point t_0.

- $\boldsymbol{\beta}(t_0) = \boldsymbol{\tau}(t_0) \times \boldsymbol{\nu}(t_0)$ it is called the *unit binormal* at the point t_0.

- The axis of direction $\boldsymbol{\tau}(t_0)$ is, obviously, the tangent to the curve at t_0.

- The axis of direction $\boldsymbol{\nu}(t_0)$ is called the *principal normal*. In fact, this straight line is contained into the normal plane (since it is perpendicular on the tangent), but it is also contained into the osculating plane. Thus, *the principal normal is the normal contained into the osculating plane*.

- The axis of direction $\boldsymbol{\beta}(t_0)$ is called the *binormal*. *The binormal is the normal perpendicular on the osculating plane*.

- The plane determined by the vectors $\{\boldsymbol{\tau}(t_0), \boldsymbol{\nu}(t_0\}$ is the osculating plane (O in the figure **??**).

- The plane determined by the vectors $\{\boldsymbol{\nu}(t_0), \boldsymbol{\beta}(t_0)\}$ is the normal plane (N in the figure **??**).

- The plane determined by the vectors $\{\boldsymbol{\tau}(t_0), \boldsymbol{\beta}(t_0)\}$ is called the *rectifying plane* , for reasons that will become clear later (R in the figure **??**).

For a naturally parameterized curve $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$, the expressions of the vectors of the Frenet frame are quite simple:

$$\begin{cases} \boldsymbol{\tau}(s) & = \boldsymbol{\rho}'(s) \\ \boldsymbol{\nu}(s) & = \dfrac{\boldsymbol{\rho}''(s)}{\|\boldsymbol{\rho}''(s)\|} \\ \boldsymbol{\beta}(s) & \equiv \boldsymbol{\tau}(s) \times \boldsymbol{\nu}(s) = \dfrac{\boldsymbol{\rho}'(s) \times \boldsymbol{\rho}''(s)}{\|\boldsymbol{\rho}''(s)\|} \end{cases}. \tag{7.8.1}$$

For an arbitrary biregular parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ the situation is a little bit more complicated. Thus, obviously, from the definition, at an arbitrary point $t \in I$,

$$\boldsymbol{\tau}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}. \tag{7.8.2}$$

Then, having in mind that

$$\mathbf{k}(t) = \frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^2} - \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^4} \cdot \mathbf{r}'(t)$$

and

$$k(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3},$$

we get

$$\boldsymbol{\nu}(t) \equiv \frac{\mathbf{k}(t)}{k(t)} = \frac{\|\mathbf{r}'(t)\|}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \cdot \mathbf{r}''(t) - \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\| \cdot \|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \cdot \mathbf{r}'(t), \quad (7.8.3)$$

while

$$\boldsymbol{\beta}(t) \equiv \boldsymbol{\tau}(t) \times \boldsymbol{\nu}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}. \quad (7.8.4)$$

Remark. The above computations show that, in practice, for an arbitrary parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ it is easier to compute directly $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ and then compute $\boldsymbol{\nu}$ by the formula

$$\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}.$$

7.8.1 The behaviour of the Frenet frame at a parameter change

A notion defined for parameterized curves makes sense for regular curves iff it is invariant at a parameter change, in other words if it doesn't change when we replace a parameterized curve by another one, equivalent to it. The Frenet frame is "almost" invariant, i.e. we have

Theorem 7.6. *Let* $(I, \mathbf{r} = \mathbf{r}(t))$ *and* $\boldsymbol{\rho} = \boldsymbol{\rho}(u)$ *be two equivalent parameterized curves with the parameter change* $\lambda : I \to J$, $u = \lambda(t)$. *Then, at the corresponding points* t *and* $u = \lambda(t)$, *their Frenet frame coincide if* $\lambda'(t) > 0$. *If* $\lambda'(t) < 0$, *then the origins and the unit principal normals coincide, while the other two pairs of versors have the same direction, but opposite senses.*

Proof Since $\mathbf{r}(t) = \boldsymbol{\rho}(\lambda(t))$, the origins of the Frenet frames coincide in any case. As seen before, the curvature vectors of two equivalent curves coincide and, thus, the same is true for the unit principal normals. From $\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t)$ it follows the coincidence of the Frenet frames for $\lambda'(t) > 0$. If $\lambda'(t) < 0$, the tangent vectors $\boldsymbol{\rho}'(u)$ and $\mathbf{r}'(t)$ have opposite senses, and the same is true for their versors. From $\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$ it follows that, in this case, the unit binormals also have opposite senses. □

7.9 Oriented curves. The Frenet frame of an oriented curve

As we saw before, the Frenet frame of a parameterized curve is not invariant at a parameter change (well, at least not at *any* parameter change). Therefore, in order to be able to use this apparatus also for regular curve, we need to make it invariant. The idea is to modify a little bit the definition of the regular curve, imposing some further condition on the local parameterization, to make sure that the parameter changes will not modify the Frenet frames.

Definition 7.13. Two parameterized curves are $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(u))$ are called *positively equivalent* if there is a parameter change $\lambda : I \to J$, $u = \lambda(t)$, with $\lambda'(t) > 0$, $\forall t \in I$.

Definition 7.14. An *orientation* of a regular curve $C \subset \mathbb{R}^3$ is a family of local parameterizations $\{(I_\alpha, \mathbf{r}_\alpha = \mathbf{r}_\alpha(t))\}_{\alpha\in A}$ such that

a) $C = \bigcup\limits_{\alpha\in A} \mathbf{r}_\alpha(I_\alpha)$,

b) For any connected component $C^b_{\alpha\beta}$ of the intersection $C_{\alpha\beta} = \mathbf{r}_\alpha(I_\alpha) \cap \mathbf{r}_\beta(I_\beta)$ with $\alpha, \beta \in A$ the parameterized curves $(I^b_\alpha, \mathbf{r}^b_\alpha)$ and $(I^b_\beta, \mathbf{r}^b_\beta)$ with $I^b_\alpha = \mathbf{r}_\alpha^{-1}(C^b_{\alpha\beta})$, $\mathbf{r}^b_\alpha = \mathbf{r}_\alpha|_{I^b_\alpha}$, $I^b_\beta = \mathbf{r}_\beta^{-1}(C^0_{\alpha\beta})$, $\mathbf{r}^b_\beta = \mathbf{r}_\beta|_{I^b_\beta}$ are positively equivalent.

Example 7.5. For the unit circle S^1 the following parameterizations:

$$(I_1 = (0, 2\pi), \mathbf{r}_1(t) = (\cos t, \sin t, 0))$$

and

$$(I_2 = (-\pi, \pi), \mathbf{r}_2(t) = (\cos t, \sin t, 0))$$

give an orientation of S^1. $C_{12} = \mathbf{r}_1(I_1) \cap \mathbf{r}_2(I_2)$ has two connected components (the upper and the lower half circles).

Starting with the upper component, C^1_{12}, we have

$$\begin{aligned} I^1_1 &= \mathbf{r}_1^{-1}(C^1_{12}) = (0, \pi), \\ I^1_2 &= \mathbf{r}_2^{-1}(C^1_{12}) = (0, \pi) \end{aligned}$$

and the parameter change is the identity, $\lambda : (0, \pi) \to (0, \pi)$, $\lambda(t) = t$, $\forall t \in (0, \pi)$, therefore the two parameterized curves are, clearly, positively equivalent.

As for the lower connected component, C^2_{12}, we get

$$\begin{aligned} I^2_1 &= \mathbf{r}_1^{-1}(C^2_{12}) = (\pi, 2\pi), \\ I^2_2 &= \mathbf{r}_2^{-1}(C^2_{12}) = (-\pi, 0) \end{aligned}$$

and the parameter change is $\lambda : I^2_1 \to I^2_2$, $\lambda(t) = t - 2\pi$, therefore, since $\lambda'(t) = 1 > 0$, also this time the two local parameterizations are positively equivalent.

Definition 7.15. A regular curve $C \subset \mathbb{R}^3$, with an orientation, is called an *oriented regular curve*.

Example 7.6. If C is a simple regular curve, it can be turned into an oriented curve by using an orientation given by any *global* parameterization $(I, \mathbf{r})$.

Remark. If C is a connected regular curve, it has only two distinct orientations, corresponding to the two possible senses of moving along the curve.

Definition 7.16. A local parameterization $(I, \mathbf{r})$ of an oriented regular curve C is called *compatible with the orientation* defined by the family $\{(I_\alpha, \mathbf{r}_\alpha)\}_{\alpha \in A}$ if on the intersections $\mathbf{r}(I) \cap \mathbf{r}_\alpha(I_\alpha)$ the parameterized curves $(I, \mathbf{r})$ and $(I_\alpha, \mathbf{r}_\alpha)$ are positively equivalent.

$$I_1^1 = \mathbf{r}_1^{-1}(C_{12}^1) = (0, \pi),$$
$$I_2^1 = \mathbf{r}_2^{-1}(C_{12}^1) = (0, \pi)$$

Remark. For an oriented regular curve C, with the orientation given by the family of local parameterizations $\{(I_\alpha, \mathbf{r}_\alpha)\}_{\alpha \in A}$, one can define, by using the vectors $\mathbf{r}'_\alpha(t)$, a sense on each tangent line, since, passing to another local parameterization $\mathbf{r}_\beta(t)$, the vectors $\mathbf{r}'_\alpha$ and $\mathbf{r}'_\beta$ have the same direction and the same sense (only their norms may differ).

The orientation itself can be given through the choice of a sense on each tangent line. Thus, if the direction of the tangent vector at $x \in C$ is given by the vector $\mathbf{a}(x)$, then wave to impose the continuity of the map $C \to \mathbb{R}^3$, $x \to \mathbf{a}(x)$. For this definition of the orientation, a local parameterization $(I, \mathbf{r})$ is compatible with the orientation if, for each point $x \in C$, $x = \mathbf{r}(t)$, the vectors $\mathbf{a}(x)$ and $\mathbf{r}'(t)$ have the same sense.

Definition 7.17. The Frenet frame of an oriented biregular curve C at a point $x \in C$ is the Frenet frame of a biregular parameterized curve $\mathbf{r} = \mathbf{r}(t)$ at t_0, where $\mathbf{r} = \mathbf{r}(t)$ is a local parameterization of the curve C, compatible with the orientation, such that $\mathbf{r}(t_0) = x$.

Remark. Clearly, this definition does not depend on the choice of the local parameterization, compatible with the orientation of the curve.

7.10 The Frenet formulae. The torsion

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a biregular parameterized curve. Then the vectors $\boldsymbol{\tau}(t)$, $\boldsymbol{\nu}(t)$, $\boldsymbol{\beta}(t)$ are, in fact, smooth vector functions with respect to the parameter t. We want to find their derivatives with respect to t, more precisely, the decomposition of these derivatives with respect to the vectors $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$. These derivatives show, in fact, the way the vectors of the Frenet frame vary along the curve. From the definition, we have,

$$\boldsymbol{\tau} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} = \frac{\mathbf{r}'}{\sqrt{\mathbf{r}'^2}}.$$

Therefore,

$$\begin{aligned}\boldsymbol{\tau}' &= \frac{\mathbf{r}'' \cdot \|\mathbf{r}'\| - \mathbf{r}' \cdot \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|}}{\|\mathbf{r}'\|^2} = \frac{\mathbf{r}'' \cdot \|\mathbf{r}'\|^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')}{\|\mathbf{r}'\|^3} = \\ &= \|\mathbf{r}'\| \cdot \underbrace{\frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}}_{k} \left[\underbrace{\frac{\|\mathbf{r}'\|}{\|\mathbf{r}' \times \mathbf{r}''\|} \cdot \mathbf{r}'' - \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\| \cdot \|\mathbf{r}' \times \mathbf{r}''\|} \cdot \mathbf{r}'}_{\boldsymbol{\nu}} \right].\end{aligned}$$

Thus,

$$\boldsymbol{\tau}' = \|\mathbf{r}'\| k \cdot \boldsymbol{\nu}. \tag{7.10.1}$$

Further,

$$\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu} \ \Rightarrow\ \boldsymbol{\beta}' = \boldsymbol{\tau}' \times + \boldsymbol{\tau} \times \boldsymbol{\nu}' = k(\underbrace{\boldsymbol{\nu} \times \boldsymbol{\nu}}_{=0}) + \boldsymbol{\tau} \times \boldsymbol{\nu}' \ \Rightarrow \boldsymbol{\beta}' = \boldsymbol{\tau} \times \boldsymbol{\nu}',$$

hence $\boldsymbol{\beta}' \perp \boldsymbol{\tau}$. On the other hand,

$$\boldsymbol{\beta} \cdot \boldsymbol{\beta} = 1 \ \Rightarrow\ \boldsymbol{\beta}' \cdot \boldsymbol{\beta} = 0 \ \Rightarrow\ \boldsymbol{\beta}' \perp \boldsymbol{\beta}.$$

Thus, the vector $\boldsymbol{\beta}'$ is colinear to the vector $\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}$ and we can write

$$\boldsymbol{\beta}' = -\|\mathbf{r}'\| \cdot \chi \boldsymbol{\nu},$$

where χ is a proportionality factor, the meaning of which will be made clear later.

We differentiate now the equality

$$\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}.$$

We have

$$\boldsymbol{\nu}' = \boldsymbol{\beta}' \times \boldsymbol{\tau} + \beta \times \boldsymbol{\tau}' = -\|\mathbf{r}'\| \cdot \chi(\boldsymbol{\nu} \times \boldsymbol{\tau}) + \|\mathbf{r}'\| \cdot k(\boldsymbol{\beta} \times \boldsymbol{\nu}),$$

therefore,

$$\boldsymbol{\nu}' = \|\mathbf{r}'\|(-k\boldsymbol{\tau} + \chi\boldsymbol{\beta}). \tag{7.10.2}$$

We obtained, thus, the equations

$$\begin{cases} \boldsymbol{\tau}'(t) = \|\mathbf{r}'\|k(t)\boldsymbol{\nu}(t) \\ \boldsymbol{\nu}'(t) = \|\mathbf{r}'\|(-k(t)\boldsymbol{\tau}(t) + \chi(t)\boldsymbol{\beta}(t)) \\ \boldsymbol{\beta}'(t) = -\|\mathbf{r}'\|\chi(t)\boldsymbol{\nu}(t). \end{cases} \tag{7.10.3}$$

These equations are called the *Frenet formulae* for the parameterized curve $\mathbf{r} = \mathbf{r}(t)$. If we are dealing win a naturally parameterized curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$, then the Frenet equations are a little bit simpler:

$$\begin{cases} \boldsymbol{\tau}'(t) = k(t)\boldsymbol{\nu}(t) \\ \boldsymbol{\nu}'(t) = -k(t)\boldsymbol{\tau}(t) + \chi(t)\boldsymbol{\beta}(t) \\ \boldsymbol{\beta}'(t) = -\chi(t)\boldsymbol{\nu}(t). \end{cases} \tag{7.10.3$'$}$$

Definition 7.18. The quantity $\chi(t)$ is called the *torsion* (or *second curvature*) of the biregular parameterized curve $\mathbf{r} = \mathbf{r}(t)$ at the point t.

We shall compute first the torsion for a naturally parameterized curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$. For such a curve, the versors of the Frenet frame are given by the expressions:

$$\begin{cases} \boldsymbol{\tau} = \boldsymbol{\rho}' \\ \boldsymbol{\nu} = \frac{1}{k}\boldsymbol{\rho}'' \\ \boldsymbol{\beta} = \frac{1}{k}\boldsymbol{\rho}' \times \boldsymbol{\rho}''. \end{cases}$$

From the third Frenet formula, we have:

$$\boldsymbol{\beta}' \cdot \boldsymbol{\nu} = -\chi(s) \cdot (\underbrace{\boldsymbol{\nu} \times \boldsymbol{\nu}}_{=1}) = -\chi(t).$$

But, on the other hand, from the definition,

$$\boldsymbol{\beta}' = \left(\frac{1}{k}\right)' + \frac{1}{k}\underbrace{\boldsymbol{\rho}'' \times \boldsymbol{\rho}''}_{=0} + \frac{1}{k}\boldsymbol{\rho}' \times \boldsymbol{\rho}''',$$

hence

$$\chi = -\boldsymbol{\beta}' \cdot \boldsymbol{\nu} = -\underbrace{\left(\frac{1}{k}\right)' (\boldsymbol{\rho}' \times \boldsymbol{\rho}'') \cdot \frac{1}{k}\boldsymbol{\rho}''}_{=0} - \frac{1}{k}\left(\boldsymbol{\rho}' \times \boldsymbol{\rho}'''\right) \cdot \frac{1}{k}\boldsymbol{\rho}'',$$

therefore,

$$\chi = \frac{1}{k^2}\left(\boldsymbol{\rho}', \boldsymbol{\rho}'', \boldsymbol{\rho}'''\right). \tag{7.10.4}$$

Now, the following theorem gives us the way to compute the torsion for an arbitrary (biregular) parameterized curve:

Theorem. *If $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(u))$ are two positively equivalent parameterized curves, with the parameter change $\lambda : I \to J$, $\lambda' > 0$, then they have the same torsion at the corresponding points t and $u = \lambda(t)$.*

Proof Let $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$, respectively $\{\boldsymbol{\tau_1}, \boldsymbol{\nu_1}, \boldsymbol{\beta_1}\}$ be the Frenet frames of the two parameterized curves at the corresponding points t and $u\lambda(t)$, respectively, and χ and χ_1 their – torsions at these points. Then

$$\begin{aligned}\boldsymbol{\beta_1}(\lambda(t)) &= \boldsymbol{\beta}(t)\\ \boldsymbol{\nu_1}(\lambda(t)) &= \boldsymbol{\nu}(t),\end{aligned}$$

$$\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t) \;\Rightarrow\; \mathbf{r}'(t) = \frac{d}{du}(\boldsymbol{\beta_1}(u))\lambda'(t).$$

From the last Frenet equation for the curve $\mathbf{r}$, we get

$$\boldsymbol{\beta}'(t) \cdot \boldsymbol{\nu}(t) = -\|\mathbf{r}'(t)\| \cdot \chi(t),$$

i.e.

$$\begin{aligned}\chi(t) &= -\frac{1}{\|\mathbf{r}'(t)\|} \cdot \boldsymbol{\beta}'(t) \cdot \boldsymbol{\nu}(t) = -\frac{1}{\|\boldsymbol{\rho}'(\lambda(t))\| \cdot \lambda'(t)} \cdot \boldsymbol{\beta_1}'(\lambda(t)) \cdot \lambda'(t) \cdot \boldsymbol{\nu_1}(\lambda(t)) =\\ &= -\frac{1}{\|\boldsymbol{\rho}'(\lambda(t))\|} \cdot \left(-\|\boldsymbol{\rho}'(\lambda(t))\| \cdot \chi_1(\lambda(t))\right) = \chi_1(\lambda(t)),\end{aligned}$$

where we used once more the last Frenet equation, but this time for the curve $\boldsymbol{\rho}$, as well as the fact that the vector $\boldsymbol{\nu_1}(\lambda(t))$ has unit length. □

Let now $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ be a naturally parameterized curve, positively equivalent to the parameterized curve $\mathbf{r} = \mathbf{r}(t)$, where $s = \lambda(t)$ is the parameter change. Then

$\mathbf{r}$ and its derivatives up to the third order can be expressed as functions of $\boldsymbol{\rho}$ and its derivatives as:

$$\begin{aligned}
\mathbf{r}(t) &= \boldsymbol{\rho}(\lambda(t)) \\
\mathbf{r}'(t) &= \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t) \\
\mathbf{r}''(t) &= \boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'^2(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t) \\
\mathbf{r}'''(t) &= \boldsymbol{\rho}'''(\lambda(t)) \cdot \lambda'^3(t) + 3\boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'(t) \cdot \lambda''(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'''(t),
\end{aligned}$$

therefore the mixed product of the first three derivatives of $\mathbf{r}$ reads

$$\begin{aligned}
\big(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)\big) = \Big(&\boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t), \boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'^2(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t), \\
&\boldsymbol{\rho}'''(\lambda(t)) \cdot \lambda'^3(t) + 3\boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'(t) \cdot \lambda''(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'''(t)\Big) = \\
&= \lambda'^6(t) \left(\boldsymbol{\rho}'(\lambda(t)), \boldsymbol{\rho}''(\lambda(t)), \boldsymbol{\rho}'''(\lambda(t))\right),
\end{aligned}$$

all the other mixed products from the right hand side vanishing, because two of the factors are colinear. Hence

$$\left(\boldsymbol{\rho}'(\lambda(t)), \boldsymbol{\rho}''(\lambda(t)), \boldsymbol{\rho}'''(\lambda(t))\right) = \frac{1}{\lambda'^6(t)} \left(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)\right).$$

But, since $\boldsymbol{\rho}$ is naturally parameterized and the two curves are positively equivalent, we have $\lambda' = \|\mathbf{r}'\|$, therefore the previous formula becomes

$$\left(\boldsymbol{\rho}', \boldsymbol{\rho}'', \boldsymbol{\rho}'''\right) = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{\|\mathbf{r}'\|^6}.$$

Moreover (see (7.7.2)), the curvature can be expressed with respect to the derivatives of $\mathbf{r}$ through the formula

$$k = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3},$$

hence, from the expression of the torsion (7.10.4) and the previous relation, we get

$$\chi(t) = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{\|\mathbf{r}' \times \mathbf{r}''\|^2}. \tag{7.10.5}$$

Exercise 7.10.1. Let $\boldsymbol{\omega} = \chi\boldsymbol{\tau} + k\boldsymbol{\beta}$. Show that Frenet's formulae can be written as

$$\begin{cases} \boldsymbol{\tau}' = \boldsymbol{\omega} \times \boldsymbol{\tau} \\ \boldsymbol{\nu}' = \boldsymbol{\omega} \times \boldsymbol{\nu} \\ \boldsymbol{\beta}' = \boldsymbol{\omega} \times \boldsymbol{\beta} \end{cases} \tag{7.10.6}$$

The vector $\boldsymbol{\omega}$ is called the *Darboux vector*.

7.10.1 The geometrical meaning of the torsion

The torsion is, in a way, an analogue of the curvature (this is the reason why in the old-fashioned books the torsion is called the *second curvature*). What we mean is that the torsion can be also interpreted as being the speed of rotation of a straight line, this time the binormal. In other words, we have

Property 8. *If* $(I, \mathbf{r} = \mathbf{r}(s))$ *is a naturally parameterized curve and* $\Delta\alpha$ *is the angle of the osculating planes of the curve at* $\mathbf{r}(s)$ *and* $\mathbf{r}(s + \Delta s)$ *(in other words, the angle of the binormals of the curve at that points), then we have*

$$\chi(s) = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s}$$

Notice that, this time, unlike the case of the curvature, the torsion is the *algebraic value* of the limit, not the absolute value. We have to say, however, that the curvature of space curve is *defined* to be positive because one could find no geometrical meaning of a signed curvature. As we shall see in the sequel, for plane curve we can define a *signed curvature*, whose absolute value is the curvature and which will help us to get some more information about the curve.

As we said before, the torsion is analogue to the curvature. Thus, the curvature is a measure of the deviation of a curve from a straight line. On the other hand, the torsion is a measure of the deviation of the curve from a plane curve. More precisely, we have

Theorem 7.7. *The support of a biregular parameterized curve lies in a plane iff the torsion of the curve vanishes identically.*

Proof Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a biregular parameterized curve such that $\mathbf{r}(I) \subset \pi$, where π is a plane. Then, obviously, the vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are parallel to this plane, which, as we know, is the osculating plane of the curve. Therefore, $\boldsymbol{\beta}(t) =$ const, hence we have

$$0 = \boldsymbol{\beta}'(t) = -\underbrace{\|\mathbf{r}'\|}_{\neq 0} \cdot \chi(t) \cdot \underbrace{\boldsymbol{\nu}(t)}_{\neq 0} \Rightarrow \chi(t) \equiv 0.$$

Conversely, if $\chi(t) \equiv 0$, then the unit binormal, $\boldsymbol{\beta}(t)$, is always equal to a constant vector, $\boldsymbol{\beta}_0$. But $\boldsymbol{\beta}(t) = \mathbf{r}'(t) \times \mathbf{r}''(t)$, therefore the vector $\mathbf{r}'(t)$ is always perpendicular to the constant vector $\boldsymbol{\beta}_0$. Thus, we have the series of implications

$$\mathbf{r}'(t) \cdot \boldsymbol{\beta}_0 = (\mathbf{r} \cdot \boldsymbol{\beta}_0)' = 0 \Rightarrow \mathbf{r} \cdot \boldsymbol{\beta}_0 = const = \mathbf{r}_0 \cdot \boldsymbol{\beta}_0 \Rightarrow (\mathbf{r} - \mathbf{r}_0) \times \boldsymbol{\beta}_0 = 0,$$

i.e. the support of $\mathbf{r}(I)$ the curve is contained in a plane perpendicular to the constant vector $\boldsymbol{\beta}_0$, and passing through the point $\mathbf{r}_0$. □

7.10.2 Some further applications of the Frenet formulae

We saw that the curvature of a parameterized curve vanishes identically if and only if the support of the curve lies on a straight line. On the other hand, we know that the curvature of a circle is constant and equal to the inverse of the radius of the circle. We could expect that the converse is also through, in other words that if the curvature of a parameterized curve is constant, then its support lies on a circle. Unfortunately, this claim is not true. It is enough to think about the circular cylindrical helix, which has constant curvature (and, also, constant torsion). We have, however, the weaker result:

Property 9. *If* $(I, \mathbf{r} = \mathbf{r}(s))$ *is a naturally parameterized curve with a curvature* k *equal to a positive constant* k_0*, while the torsion vanishes identically, then the support of the curve lies on a circle of radius* $1/k_0$.

Proof Since the torsion is identically zero, the curve is plane. Let us consider the parameterized curve $(I, \mathbf{r}_1 = \mathbf{r}_1(s))$, where

$$\mathbf{r}_1 = \mathbf{r} + \frac{1}{k_0}\boldsymbol{\nu}. \tag{*}$$

We differentiate with respect to s and we get, using the second Frenet formula for $\mathbf{r}$,

$$\mathbf{r}_1^{\prime} = \mathbf{r}' + \frac{1}{k_0}\boldsymbol{\nu}' = \boldsymbol{\tau} + \frac{1}{k_0}(-k_0\boldsymbol{\tau}) = \boldsymbol{\tau} - \boldsymbol{\tau} = 0.$$

Thus, the curve $\mathbf{r}_1$ reduces to a point, say $\mathbf{r}_1(s) \equiv \mathrm{c} = \text{const}$. But, from (*), we get

$$\|\mathbf{r} - c\| = \left\|\frac{1}{k_0}\boldsymbol{\nu}\cdot\boldsymbol{\nu}\right\| = \frac{1}{k_0},$$

which means that any point of the support of the curve $\mathbf{r}$ is at a (constant) distance $1/k_0$ from the fixed point c, i.e. the support $\mathbf{r}(I)$ lies on the circle of radius $1/k_0$, centred at c. □

Another interesting situation is when the support of the curve lies not in a plane, but on a sphere. In this case we have

Property 10. *If a naturally parameterized curve* $(I, \mathbf{r} = \mathbf{r}(s))$ *has the support on a sphere centred at the origin and radius equal to a, then its curvature is subject to*

$$k \geq \frac{1}{a}.$$

Proof The distance from a point of the curve to the origin is equal to $\|\mathbf{r}\|$, i.e. we have $\mathbf{r}^2 = a^2$. Differentiating, we get $\mathbf{r} \cdot \mathbf{r}' = 0$ or $\mathbf{r} \cdot \boldsymbol{\tau}$. Differentiating once more, we obtain

$$\mathbf{r}' \cdot \boldsymbol{\tau} + \mathbf{r} \cdot \boldsymbol{\tau}' = 0,$$

or

$$1 + \mathbf{r} \cdot \boldsymbol{\tau}' = 0 \quad \Longleftrightarrow \quad 1 + k\mathbf{r} \cdot \boldsymbol{\nu} = 0 \quad \Longrightarrow k\mathbf{r} \cdot \boldsymbol{\nu} = -1.$$

From the properties of the scalar product, we have

$$|\mathbf{r} \cdot \boldsymbol{\nu}| \leq \|\mathbf{r}\| \cdot \|\boldsymbol{\nu}\| = \|\mathbf{r}\| = a,$$

therefore,

$$k = |k| = \frac{1}{|\mathbf{r} \cdot \boldsymbol{\nu}|} \geq \frac{1}{\|\mathbf{r}\| \cdot \|\boldsymbol{\nu}\|} = \frac{1}{a}.$$

□

In the figure **??** we give an example of a curve which lies on a sphere, it is a so-called *spherical helix*, because it is both a general helix (see the next section) and a spherical curve.

7.10.3 General helices. Lancret's theorem

Definition 7.19. A parameterized curve curve $(I, \mathbf{r})$ is called a *general helix* if its tangents make a constant angle with a fixed direction in space.

The following theorem was formulated in 1802 by the French mathematician Paul Lancret, but the first known proof belongs to another celebrated French mathematician, known especially for his contributions to mechanics, A. de Saint Venant (1845).

Theorem 7.8 (Lancret, 1802)**.** *A space curve with the curvature $k > 0$ is a general helix if and only if the ratio between its torsion and its curvature is constant.*

Proof Let us assume, to begin with, that the curve is parameterized by the arc length.

To prove the first implication, let us suppose that $\mathbf{r}$ is a general helix and let $\mathbf{c}$ be the versor of the fixed direction:

$$\boldsymbol{\tau} \cdot \mathbf{c} = \cos\alpha_0 = const.$$

By differentiating, we get

$$\boldsymbol{\tau}' \cdot \mathbf{c} = 0,$$

hence

$$k \cdot \mathbf{c} \cdot \boldsymbol{\nu} = 0.$$

As, by hypothesis, $k > 0$, it follows that

$$\mathbf{c} \cdot \boldsymbol{\nu} = 0,$$

i.e., at each point of the curve, $\mathbf{c} \perp \boldsymbol{\nu}$. This means that $\mathbf{c}$ is in the rectifying plane and therefore

$$\boldsymbol{\beta} \cdot \mathbf{c} = \sin \alpha_0.$$

By differentiating the relation $\boldsymbol{\nu} \cdot \mathbf{c} = 0$ we get, having in mind that $\mathbf{c}$ is constant and using the second formula of Frenet:

$$(-k\boldsymbol{\tau} + \chi\boldsymbol{\beta}) \cdot \mathbf{c} = 0$$

which leads to

$$-k \cdot \cos \alpha_0 + \chi \cdot \sin \alpha_0 = 0,$$

i.e.

$$\frac{\chi}{k} = \cot \alpha_0 = const.$$

Conversely, let us assume that

$$\frac{\chi(s)}{k(s)} = c_0 = const,$$

or

$$c_0 \cdot k - \chi = 0.$$

On the other hand, from the first and the third of the Frenet's equations, we get,

$$(c_0 \cdot k - \chi)\boldsymbol{\nu} = c_0\boldsymbol{\tau}' + \boldsymbol{\beta}' = 0.$$

We integrate once and we obtain

$$c_0\boldsymbol{\tau} + \boldsymbol{\beta} = \mathbf{c}^*,$$

where $\mathbf{c}^* \neq 0$ is a constant vector.

We define

$$\mathbf{c} := \frac{\mathbf{c}^*}{\|\mathbf{c}^*\|} = \frac{c_0\boldsymbol{\tau} + \boldsymbol{\beta}}{\|c_0\boldsymbol{\tau} + \boldsymbol{\beta}\|} = \frac{c_0\boldsymbol{\tau} + \boldsymbol{\beta}}{\sqrt{1 + c_0^2}},$$

whence

$$\mathbf{c} \cdot \boldsymbol{\tau} = \frac{c_0}{\sqrt{1 + c_0^2}} = const \leq 1.$$

Hence the vectors $\mathbf{c}$ şi $\boldsymbol{\tau}$ make a constant angle and the curve is a general helix. □

We end this paragraph by noting, as a historical curiosity, that, although in most of the books this theorem is credited to Lancret, in fact he (and, with him, Saint-Venant), only gave one implication, the first one: on a general helix, the ratio between the torsion and the curvature is constant. The other implication was given and proved later by Joseph Bertrand (see, for instance, the book of Eisenhart [**?**]).

7.10.4 Bertrand curves

An interesting problem in the theory of curves is whether it is possible for several curves to share the same family of tangents, principal normal or binormals. For the tangents, the answer is easily seen to be negative: the family of tangents uniquely determines the curve. For the principal normals, the problem, raised by the same Saint-Venant, was answered by Joseph Bertrand, who discovered that, for an arbitrary curve, the answer is negative, however, there are special curves for which there might be, also, other curves with the same family of principal normals. These curves are called *Bertrand curves*. Usually, for a Bertrand curve, there is only one curve having the same principal normals. We will say that the two curves are *Bertrand mates*, or that they are *associated, or conjugated Bertrand curves*. It turns out that if a Bertrand curve has more than one Bertrand mate, then it has an infinity and the curve (and all of its mates) is a circular cylindrical helix.

We shall prove, in what follows, a series of interesting results related to Bertrand curves, without respecting the historical order. Let $\mathbf{r}$ and $\mathbf{r}^*$ be two Bertrand mates. We assume that the first curve is naturally parameterized. Then the second curve is, in fact, also dependent on the arc length s of the first and we assume that $\mathbf{r}^*(s)$ is the point on the Bertrand mate having the same principal normal as the first one at $\mathbf{s}$. The two points are called *corresponding*. We have the following result:

Theorem 7.9 (Schell)**.** *The angle of the tangents of two associated Bertrand curves at corresponding points is constant.*

Proof Clearly, if $\boldsymbol{\nu}(s)$ is the versor of the principal normal of the first curve, then we have

$$\mathbf{r}^*(s) = \mathbf{r}(s) + a(s)\boldsymbol{\nu}(s). \tag{7.10.7}$$

As the two curves have the same principal normals, the second curve ought to have the versor of the principal normal

$$\boldsymbol{\nu}^*(s) = \pm\boldsymbol{\nu}(s), \tag{7.10.8}$$

We differentiate the relation (7.10.7) with respect to s and we get

$$\frac{d\mathbf{r}^*}{ds} = \frac{d\mathbf{r}}{ds} + a\frac{d\boldsymbol{\nu}}{ds} + \frac{da}{ds}\mathbf{r} \tag{7.10.9}$$

or, using the second Frenet formula,

$$\frac{d\mathbf{r}^*}{ds} = (1 - ak)\,\boldsymbol{\tau} + \frac{da}{ds}\boldsymbol{\nu} + a\chi\boldsymbol{\beta}. \tag{7.10.10}$$

The vector $\frac{d\mathbf{r}^*}{ds}$ is tangent to the second curve, therefore it is perpendicular both on $\boldsymbol{\nu}^*$ and on $\boldsymbol{\nu}$. We deduce, by multiplying both sides of (7.10.10) by $\boldsymbol{\nu}$, that $\frac{da}{ds} = 0$, i.e. a is a constant. As such, the relation (7.10.10) turns into

$$\frac{d\mathbf{r}^*}{ds} = (1 - ak)\,\boldsymbol{\tau} + a\chi\boldsymbol{\beta}. \tag{7.10.11}$$

We denote by s^* the arc length of the second curve. Then

$$\boldsymbol{\tau}^* = \frac{d\mathbf{r}^*}{ds^*} = \frac{dr^*}{ds}\frac{ds}{ds^*} \tag{7.10.12}$$

or, using (7.10.11)

$$\boldsymbol{\tau}^* = (1 - ak)\,\frac{ds}{ds^*}\boldsymbol{\tau} + a\chi\frac{ds}{ds^*}\boldsymbol{\beta}. \tag{7.10.13}$$

Let ω be the angle of the tangents of the two curvs at the corresponding points. Then this angle is given by

$$\cos\omega = \boldsymbol{\tau}\boldsymbol{\tau}^*, \tag{7.10.14}$$

where, of course, $\boldsymbol{\tau}^*$ is the versor of the second curve at the point $\mathbf{r}^*(s)$. In terms of ω, $\boldsymbol{\tau}^*$ and the versor of the binormal of the second curve,, $\boldsymbol{\beta}^*$, can be written as

$$\boldsymbol{\tau}^* = \cos\omega\boldsymbol{\tau} + \sin\omega\boldsymbol{\beta}, \tag{7.10.15}$$

$$\boldsymbol{\beta}^* = \varepsilon\,(-\sin\omega\boldsymbol{\tau} + \cos\omega\boldsymbol{\beta})\,, \tag{7.10.16}$$

where $\varepsilon = \pm 1$.

We differentiate the relation (7.10.15) with respect to s and we get:

$$\frac{d\boldsymbol{\tau}^*}{ds} = -\sin\omega\frac{d\omega}{ds}\boldsymbol{\tau} + k\cos\omega\boldsymbol{\nu} + \cos\omega\frac{d\omega}{ds}\boldsymbol{\beta} - \chi\sin\omega\boldsymbol{\nu}. \tag{7.10.17}$$

If we multiply scalarly both sides of the relation (7.10.17) by $\boldsymbol{\tau}$ and then by $\boldsymbol{\beta}$ and use the fact that, on the ground of the first Frenet formula, $\frac{d\boldsymbol{\tau}^*}{ds}$ is colinear to $\boldsymbol{\nu}^*$, and, thsu, also to $\boldsymbol{\nu}$, we obtain the relations:

$$\sin\omega \frac{d\omega}{ds} = 0, \quad \cos\omega \frac{d\omega}{ds} = 0, \tag{7.10.18}$$

i.e. $\frac{d\omega}{ds} = 0$, hence ω is constant. □

The following theorem was proved by Joseph Bertrand and it is considered to be the central result of the entire theory of Bertrand curves.

Theorem 7.10 (Bertrand)**.** *A curve* $\mathbf{r}$ *is a Bertrand curve if and only if its torsion and curvature verify a relation of the form*

$$a \cdot k + b \cdot \chi = 1, \tag{7.10.19}$$

with constant a and b.

Proof Let us assume that $\mathbf{r}$ is a Bertrand curve. Comparing the relations (7.10.13) and (7.10.15), we obtain that

$$\cos\omega = (1 - ak)\frac{ds}{ds^*}, \tag{7.10.20}$$

$$\sin\omega = a\chi \frac{ds}{ds^*}. \tag{7.10.21}$$

Dividing side by side these two equalities, we obtain

$$\operatorname{ctg}\omega = \frac{1 - ak}{a\chi} \tag{7.10.22}$$

or

$$1 - ak = a\chi \operatorname{ctg}\omega. \tag{7.10.23}$$

If we denote

$$b = a \operatorname{ctg}\omega, \tag{7.10.24}$$

we obtain the relation (7.10.19).

Let us suppose now, conversely, that the relation (7.10.19) holds. Let $\mathbf{r}^*$ be the curve given by (7.10.7), where a is the constant from the equation (7.10.19). Differentiating the relation (7.10.7), we get, as we saw earlier, the relation (7.10.11). As the basis $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$ is orthonormal we conclude from (7.10.11) that

$$\left(\frac{ds^*}{ds}\right)^2 = (1 - ak)^2 + a^2\chi^2 \tag{7.10.25}$$

or, using (7.10.25),

$$\left(\frac{ds^*}{ds}\right)^2 = (a^2 + b^2)\chi^2, \tag{7.10.26}$$

whence

$$\chi \frac{ds}{ds^*} = \text{const.} \tag{7.10.27}$$

In the same way follows that

$$(1 - ak)\frac{ds}{ds^*} = \text{const.} \tag{7.10.28}$$

Taking into account these two relations, we differntiate the relation (7.10.13) and we obtain, by using the first and the third Frenet formulae for the curve $\mathbf{r}$:

$$\frac{d\boldsymbol{\tau}^*}{ds} = \left[k(1 - ak) - a\chi^2\right] \frac{ds}{ds^*}\boldsymbol{\nu} \tag{7.10.29}$$

or, using the first Frenet formula for the curve $\mathbf{r}^*$:

$$k^*\boldsymbol{\nu}^* = \left[k(1 - ak) - a\chi^2\right]\left(\frac{ds}{ds^*}\right)^2 \boldsymbol{\nu}, \tag{7.10.30}$$

whence it follows that $\boldsymbol{\nu}^* = \pm\boldsymbol{\nu}$, i.e., indeed, $\mathbf{r}$ is a Bertrand curve. □

Corollary 7.10.1. *The circular cylindrical helices, the curves of constant torsion (in particular the plane curves), and the curves of constant curvature are Bertrand curves.*

Remark. For a plane curve, the principal normals are, in fact, the normals, therefore, as one can see easily, any plane curve has an infinity of Bertrand mates, all of them congruent (they are, all of them, parallel to the given one).

Corollary 7.10.2. *If a Bertrand curve has more than one mate, then it has an infinity and it is a circular cylindrical helix.*

Proof As we saw from the proof of the theorem of Bertrand, the constant a from the relation (7.10.19) is the one that identifies the Bertrand mate of our curve. Thus, if a curve $\mathbf{r}$ has more then one Bertrand mate, this means that there are (at least) two distinct pairs of real numbers (a, b), (a_1, b_1) such that

$$a \cdot k + b \cdot \chi = 1,$$
$$a_1 \cdot k + b_1 \cdot \chi = 1.$$

Subtracting side by side the two relations, we obtain

$$(a - a_1) \cdot k = (b_1 - b) \cdot \chi.$$

At least one of the coefficients $(a - a_1)$ and $(b_1 - b)$ is different from zero, therefore the ratio between the curvature and torsion is a constant. This means, via the Lancret's theorem, that the curve is a general helix. However, as, say, $\chi = c \cdot k$, with c – constant, from (7.10.19) follows that

$$(a + b \cdot c) \cdot k = 1,$$

which means that k (and, hence, also χ) is a constant, therefore the curve is a *circular* cylindrical helix. Clearly, for the case of a cylindrical helix, when both the curvature and the torsion are constants, we can get an infinity of pairs of real numbers that verify (7.10.19), so our curve has an infinity of Bertrand mates. They lie on circular cylinders which have the same axis of symmetry with the cylinder associated to the given curve. □

7.11 The local behaviour of a parameterized curve around a biregular point

Let $(I, \mathbf{r})$ be a naturally parameterized curve. We shall assume that $0 \in I$, as an interior point and $M_0 \equiv \mathbf{r}(0)$ is a biregular point of the curve. We shall make use of the following notations: $\boldsymbol{\tau}_0 = \boldsymbol{\tau}(0)$, $\boldsymbol{\nu}_0 = \boldsymbol{\nu}(0)$, $\boldsymbol{\beta}_0 = \boldsymbol{\beta}(0)$.

We expand $\mathbf{r}$ in a Taylor series around the origin. Up to the third order, the Taylor expansion is

$$\mathbf{r}(s) = \mathbf{r}(0) + s\mathbf{r}'(0) + \frac{1}{2}s^2\mathbf{r}''(0) + \frac{1}{6}s^3\mathbf{r}'''(0) + o(s^3). \tag{7.11.1}$$

We want to express the derivatives of $\mathbf{r}$ as functions of the Frenet vectors $\boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0$. We have, obviously, since $\mathbf{r}$ is naturally parameterized,

$$\mathbf{r}'(0) = \boldsymbol{\tau}_0. \tag{7.11.2}$$

On the other hand, from the definition of the curvature vector, we have

$$\mathbf{r}''(0) = \mathbf{k}(0) = k(0) \cdot \boldsymbol{\nu}_0. \tag{7.11.3}$$

Moreover, if we differentiate the relation

$$\mathbf{r}''(s) = \mathbf{k}(s) = k(s) \cdot \boldsymbol{\nu} \tag{7.11.4}$$

we get

$$\begin{aligned} \mathbf{r}'''(s) &= k'(s)\boldsymbol{\nu} + k(s)\boldsymbol{\nu}' = k'(s)\boldsymbol{\nu} + k(s)(-k(s)\boldsymbol{\tau} + \chi(s)\boldsymbol{\beta}) = \\ &= -k^2(s)\boldsymbol{\tau} + k'(s)\boldsymbol{\nu} + k(s)\chi(s)\boldsymbol{\beta}, \end{aligned} \tag{7.11.5}$$

where we have made use of the second Frenet equation.

Substituting $s = 0$ in (7.11.5), we get

$$\mathbf{r}'''(0) = -k^2(0)\boldsymbol{\tau}_0 + k'(0)\boldsymbol{\nu}_0 + k(0)\chi(0)\boldsymbol{\beta}_0. \tag{7.11.6}$$

Thus, the relation (7.11.1) becomes

$$\mathbf{r}(s) - \mathbf{r}(0) = s\boldsymbol{\tau}_0 + \frac{1}{2}s^2 k(0)\boldsymbol{\nu}_0 + \frac{1}{6}s^3\left(-k^2(0)\boldsymbol{\tau}_0 + k'(0)\boldsymbol{\nu}_0 + k(0)\chi(0)\boldsymbol{\beta}_0\right) + o(s^3) \tag{7.11.7}$$

or

$$\begin{aligned} \mathbf{r}(s) - \mathbf{r}(0) = &\left(s - \frac{1}{6}k^2(0)s^3 + o(s^3)\right)\boldsymbol{\tau}_0 + \left(\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)\right)\boldsymbol{\nu}_0 + \\ &+ \left(\frac{1}{6}k(0)\chi(0)s^3 + o(s^3)\right)\boldsymbol{\beta}_0. \end{aligned} \tag{7.11.8}$$

We consider now a coordinate frame with the origin at M_0 and having as axes the Frenet axes. The position vector of a point of the curve with respect to this frame is exactly the vector $\mathbf{r}(s) - \mathbf{r}(0) \equiv \overrightarrow{M_0M}$, where $M = \mathbf{r}(s)$. Therefore, projecting (7.11.8) on the axes, we get

$$\begin{cases} x(s) &= s - \frac{1}{6}k^2(0)s^3 + o(s^3) \\ y(s) &= \frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3) \\ z(s) &= \frac{1}{6}k(0)\chi(0)s^3 + o(s^3) \end{cases}. \tag{7.11.9}$$

It is not at all difficult to see that locally, close enough to 0, we have the following relations between coordinates, giving, in fact, the equations of the projections of the curve on the coordinate planes of the Frenet frame:

$$\begin{cases} y = \frac{1}{2}k(0)x^2, \\ z = \frac{1}{6}k(0)\chi(0)x^3, \\ z^2 = \frac{2}{9}\frac{\chi^2(0)}{k(0)}y^3. \end{cases} \tag{7.11.10}$$

Thus, the projection of the curve on the xOy-plane (the osculating plane) is a parabola, the projection on the xOz-plane (the rectifying plane) is a cubic, while the projection on the yOz-plane (the normal plane) is a semicubic parabola.

7.12 The contact between a space curve and a plane

We consider a naturally parameterized space curve $(I, \mathbf{r})$. We assume, as we did before, that 0 is an interior point of the interval I and that the point $M_0 = \mathbf{r}(0)$ of the curve is biregular. We consider a plane Π passing through the point M_0. We choose as coordinate frame the Frenet frame of the curve $\mathbf{r}$ at the point M_0, $\mathcal{R}_0 = \{M_0; \boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0\}$. With respect to this coordinate frame, he cartesian equation of the plane Π will be of the form

$$F(x, y, z) \equiv ax + by + cz = 0. \tag{7.12.1}$$

Now, if $x = x(s), y = y(s), z = z(s)$ are the local equations of the curve with respect to the Frenet frame (see (7.11.9)), the intersection condition between the the plane and the curve is $F(x(s), y(s), z(s)) = 0$, i.e.

$$a\left(s - \frac{1}{6}k^2(0)s^3 + o(s^3)\right) + b\left(\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)\right) + c\left(\frac{1}{6}k(0)\chi(0)s^3 + o(s^3)\right) = 0 \tag{7.12.2}$$

or

$$as + \frac{1}{2}bk(0)s^2 + \frac{1}{6}\left(-ak^2(0) + bk'(0) + ck(0)\chi(0)\right)s^3 + o(s^3) = 0. \tag{7.12.3}$$

We have, now, several possibilities,

a) If $a \neq 0$ (i.e. the plane does not contain the tangent), that the plane has a contact of order zero with the curve (intersection, they have a single point in common).

b) If $a = 0$, then the intersection equation has $s = 0$ as a double solution, which means that the curve and the plane have a contact of order 1 (tangency contact). It can be observed immediately that in this case the plane Π passes through the tangent at the curve at the point M_0 (which is, in fact, the straight line of equations $y = 0, z = 0$ in the coordinate frame considered by us).

c) If we want a contact of order at least 2 (osculation contact), then the coefficient of s^2 in the intersection equation should also vanish. This may happen only if

$b = 0$ (as the point M_0 is biregular, the curvature is not zero at this point). Thus, we have osculation contact if we impose $a = 0$ and $b = 0$. But this choice leads us to the equation $z = 0$ for the plane Π, i.e. the plane is already completely determined (the osculating plane).

d) As the plane Π is completely determined by the condition of having a second order contact with the curve, we cannot specialize more, in order to have higher order contact (superosculation). Anyway, looking at the intersection equation, we conclude that the osculating plane has a contact or superosculation with the curve at the planar points of the curve, where the torsion vanishes. At all other points, the contact is just of second order (osculation).

7.13 The contact between a space curve and a sphere. The osculating sphere

As in the previous paragraph, we consider a naturally parameterized curve $(I, \mathbf{r})$, we assume that 0 is an interior point of the interval I and that $\mathbf{r}(0)$ is a biregular point of the curve. We choose as coordinate frame the Frenet frame of the curve $\mathbf{r}$ at the point M_0, $\mathcal{R}_0 = \{M_0; \boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0\}$. Then an arbitrary sphere passing through the point M_0 will have, with respect to this coordinate system, the equation

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0, \tag{7.13.1}$$

where $\Omega(a, b, c)$ is the center of the sphere. Then the condition of the intersection will be, again, $F(x(s), y(s), z(s)) = 0$, where $x = x(s), y = y(s), z = z(s)$ are the local equations of the curve with respect to the Frenet frame at M_0. Thus, in our case, we have

$$\begin{aligned} F(x(s), y(s), z(s)) &= \left(s - \frac{1}{6}k^2(0)s^3 + o(s^3)\right)^2 + \left(\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)\right)^2 + \\ &+ \left(\frac{1}{6}k(0)\chi(0)s^3 + o(s^3)\right)^2 - 2a\left(s - \frac{1}{6}k^2(0)s^3 + o(s^3)\right) - \\ &- 2b\left(\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)\right) - 2c\left(\frac{1}{6}k(0)\chi(0)s^3 + o(s^3)\right) = 0. \end{aligned} \tag{7.13.2}$$

or

$$-2as + (1 - bk(0))s^2 + \frac{1}{3}(ak^2(0) - bk'(0) - ck(0)\chi(0))s^3 + o(s^3). \tag{7.13.3}$$

Now, the discussion that follows is, also, similar to that from the case of the contact between a curve and a plane. Still, here we have more cases to take into consideration.

a) If $a \neq 0$, then $s = 0$ is a simple solution of the intersection equation. Thus, in this case the sphere and the curve have a contact of order zero (intersection contact).

b) The sphere and the curve have a first order contact (tangency contact) if and only if the intersection equation has a double zero at the origin. This happens, obviously, iff $a = 0$. This means that the first coordinate of the center of the sphere is zero, i.e. this center lies in the normal plane of the curve at M_0. It is easy to see that in this case the tangent line at M_0 of the curve lies in the tangent plane of the sphere at the same point, which justifies once more the denomination of *tangency contact*.

c) For an osculation (second order) contact, the coefficient of s^2 in the intersection equation also has to vanish and we get

$$b = \frac{1}{k(0)} = R(0), \tag{7.13.4}$$

where $R(0)$ is the curvature radius of the curve at the point M_0. Thus, in this case, the center of the sphere lies on the intersection line of the planes $x = 0$ (the normal plane) and $y = R(0)$. This line, which, as one can easily see, is parallel to the binormal of the curve at M_0, is called *curvature axis* or *polar axis* of the curve. It intersects the osculating plane at M_0 of the curve at the center of curvature at M_0 of the curve. Thus, any sphere with the center on the curvature axis of the curve has an osculation contact with this one.

d) We shall say that the sphere has a contact of *superosculation* with the curve if they have a contact of order at least three, which means that in the intersection equation the coefficients of the of the powers up to three of s have to vanish simultaneously. It is usually claimed that there is a single sphere which has a contact of superosculation with the curve and this sphere is called the *osculating sphere*. In reality, the things are a little bit more subtle, as we shall see in a moment.

   1) Let us assume, first, that the torsion of the curve at M_0 does not vanish, i.e. $\chi(0) \neq 0$. In this case, as one sees immediately, between the sphere and the

curve there is a superosculation contact if and only if we have

$$\begin{cases} a = 0, \\ b = \frac{1}{k(0)}, \\ c = \frac{k'(0)}{k^2(0)\chi(0)}, \end{cases} \tag{7.13.5}$$

i.e. in this case the sphere is uniquely determined and we will call it *osculating sphere* [7].

2) If $\chi(0) = 0$, while $k'(0) \neq 0$, as one can see easily, the coefficient of s^3 is never zero, therefore there is no sphere which has a superosculation contact with the curve. However, as we have seen in the previous paragraph, in this case the osculating plane has a superosculation contact with the curve and we can think of this plane as playing also the role of the osculating sphere, of infinite radius, though.
3) If both $\chi(0)$ and $k'(0)$ vanish, then the coefficient of s^3 in the intersection equation vanishes for any c, in other words in this case any sphere which has an osculation contact with the curve also has a superosculation contact. Thus, now we have an *infinity* of osculating spheres.

7.14 Existence and uniqueness theorems for parameterized curves

7.14.1 The behaviour of the Frenet frame under a rigid motion

Definition 7.20. A *rigid motion* of $\mathbb{R}^3$ is a map $D : \mathbb{R}^3 \to \mathbb{R}^3$, $D(x) = \mathcal{A} \cdot x + \mathbf{b}$, where $\mathcal{A} \in M_{3\times 3}(\mathbb{R})$ is an orthogonal matrix, $\mathcal{A}^t \cdot \mathcal{A} = I_3$, with the determinant equal to one: $\det \mathcal{A} = 1$, while $\mathbf{b} \in \mathbb{R}^3$ is a constant vector. The linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$, $A(x) = \mathcal{A} \cdot x$ is called the *homogeneous part* of the rigid motion.

Remarks. (i) A rigid motion in $\mathbb{R}^3$ is just a rotation around an arbitrary axis followed by a translation.

(ii) If we don't ask $\det \mathcal{A} = 1$, we get, equally, an isometry of $\mathbb{R}^3$. However, in this case (if $\det \mathcal{A} = -1$), a transformation $D(x) = \mathcal{A} \cdot x + b$ does not reduce

[7] In some books all the spheres which have an osculation contact with the curve are called osculating spheres, while the one which has a superosculation contact is called, accordingly, *superosculating sphere*

anymore to a rotation and a translation, we have to add, also, a reflection with respect to a plane. In many books the term "motion" implies only that the matrix $\mathcal{A}$ is orthogonal and for what we call "rigid motion" it is used the term "proper motion".

Definition 7.21. Let $D : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion, with the homogeneous part $A : \mathbb{R}^3 \to \mathbb{R}^3$. The *image* of a parameterized curve $(I, \mathbf{r} = \mathbf{r}(t))$ through D is, by definition, the parameterized curve $(I, \mathbf{r}_1 = (D \circ \mathbf{r})(t))$.

Remark. Since A is a nondegenerate linear map, the image of a regular parameterized curve is, also, a regular parameterized curve.

Theorem 7.11. *Let D be a rigid motion of $\mathbb{R}^3$, with the homogeneous part A, $(I, \mathbf{r} = \mathbf{r}(t))$ – a biregular parameterized curve, $(I, \mathbf{r}_1 = D \circ \mathbf{r})$ – its image through D and $\{\mathbf{r}(t); \boldsymbol{\tau}(t), \boldsymbol{\nu}(t), \boldsymbol{\beta}(t)\}$ – the Frenet frame of the parameterized curve $\mathbf{r}$ at t. Then the frame $\{\mathbf{r}_1(t); A(\boldsymbol{\tau}(t)), A(\boldsymbol{\nu}(t)), A(\boldsymbol{\beta}(t))\}$ is the Frenet frame of $\mathbf{r}_1$ at t.*

Proof Let $\mathbf{r}(t) = (x(t), y(t), z(t))$, $D(x, y, z) = (x_1, y_1, z_1)$, where

$$\begin{cases} x_1 = \alpha_{11}x + \alpha_{12}y + \alpha_{13}z + b_1 \\ y_1 = \alpha_{21}x + \alpha_{22}y + \alpha_{23}z + b_2 \\ z_1 = \alpha_{31}x + \alpha_{32}y + \alpha_{33}z + b_3 \end{cases} \tag{7.14.1}$$

Then

$$\mathbf{r}_1(t) = (x_1(t), y_1(t), z_1(t)),$$

hence

$$\mathbf{r}'_1(t) = A(\mathbf{r}'(t)), \quad \mathbf{r}''_1(t) = A(\mathbf{r}''(t)). \tag{7.14.2}$$

Since A is a linear isometry, preserving the orientation of $\mathbb{R}^3$, we have

$$\|\mathbf{r}'_1\| = \|A(\mathbf{r}')\| = \|\mathbf{r}'\|, \ \mathbf{r}'_1 \cdot \mathbf{r}''_1 = \mathbf{r}' \cdot \mathbf{r}'',$$
$$\mathbf{r}'_1 \times \mathbf{r}''_1 = A(\mathbf{r}' \times \mathbf{r}''), \ \|\mathbf{r}'_1 \times \mathbf{r}''_1\| = \|\mathbf{r}' \times \mathbf{r}''\|.$$

Then:

$$\boldsymbol{\tau}_1 = \frac{\mathbf{r}_1'}{\|\mathbf{r}'\|} = \frac{A(\mathbf{r}')}{\|A(\mathbf{r}')\|} = \frac{A(\mathbf{r}')}{\|\mathbf{r}'\|} = A\left(\frac{\mathbf{r}'}{\|\mathbf{r}'\|}\right) = A(\boldsymbol{\tau})$$

$$\boldsymbol{\nu}_1 = \frac{\|\mathbf{r}_1'\|}{\|\mathbf{r}_1' \times \mathbf{r}_1''\|}\mathbf{r}_1'' - \frac{\mathbf{r}_1' \cdot \mathbf{r}_1''}{\|\mathbf{r}_1'\| \cdot \|\mathbf{r}_1' \times \mathbf{r}_1''\|}\mathbf{r}_1' = \frac{\|\mathbf{r}'\|}{\mathbf{r}' \times \mathbf{r}''} \cdot A(\mathbf{r}'') -$$
$$- \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\| \cdot \|\mathbf{r}' \times \mathbf{r}''\|} \cdot A(\mathbf{r}') = A(\boldsymbol{\nu})$$

$$\boldsymbol{\beta}_1 = \frac{\mathbf{r}_1' \times \mathbf{r}_1''}{\|\mathbf{r}_1' \times \mathbf{r}_1''\|} = A(\boldsymbol{\beta}).$$

□

Corollary 7.1. *The parameterized curves* $(I, \mathbf{r})$ *and* $(I, \mathbf{r}_1 = D \circ \mathbf{r})$ *have the same curvature and torsion.*

Proof We have

$$k_1 = \frac{\|\mathbf{r}_1' \times \mathbf{r}_1''\|}{\|\mathbf{r}_1'\|} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|} = k.$$

For the torsion, the situation is slightly more complicated. We have, from the theorem,

$$\boldsymbol{\beta}_1(t) = A(\boldsymbol{\beta}(t)) \quad \text{and} \quad \boldsymbol{\nu}_1(t) = A(\boldsymbol{\nu}(t)).$$

A being a linear operator, a similar result holds for the derivatives of the two Frenet vectors, i.e. we have

$$\boldsymbol{\beta}_1'(t) = A(\boldsymbol{\beta}'(t)) \quad \text{and} \quad \boldsymbol{\nu}_1'(t) = A(\boldsymbol{\nu}'(t)).$$

Using the last Frenet equations for the two curves, we have the equalities

$$-\chi_1(t)\boldsymbol{\nu}_1(t) = A(-\chi_(t)\boldsymbol{\nu}(t)) = -\chi(t)A(\boldsymbol{\nu}(t)) = -\chi(t)\boldsymbol{\nu}_1(t),$$

and, thus, the two torsions are equal, as claimed. □

7.14.2 The uniqueness theorem

Theorem 7.12. *Let* $(I, \mathbf{r} = \mathbf{r}(t))$ *and* $(I, \mathbf{r}_1 = \mathbf{r}_1(t))$ *be two biregular parameterized curves. If* $k(t) = k_1(t)$, $\chi(t) = \chi_1(t)$ *and* $\|\mathbf{r}'(t)\| = \|\mathbf{r}_1'(t)\|$ $\forall t \in I$, *then there is a single rigid motion* D *of* $\mathbb{R}^3$ *such that* $\mathbf{r}_1 = D \circ \mathbf{r}$.

Proof Let $t_0 \in I$ be an arbitrary point and D – the rigid motion of $\mathbb{R}^3$ sending the Frenet frame $\{\mathbf{r}(t_0); \boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0\}$ of the curve $\mathbf{r}$ at t_0 into the Frenet frame $\{\mathbf{r}_1(t_0); \boldsymbol{\tau}_{10}, \boldsymbol{\nu}_{10}, \boldsymbol{\beta}_{10}\}$ of the curve $\mathbf{r}_1$ at the same point. Obviously, there is a single rigid motion with this property. Let $(I, \mathbf{r}_2(t = D \circ \mathbf{r}(t))$ – the image of the curve $\mathbf{r}$ through D, and k_2, χ_2 – the curvature and the torsion, respectively, of the parameterized curve $\mathbf{r}_2$. Then

$$k_2(t) \equiv k(t) \equiv k_1(t)$$

$$\chi_2(t) \equiv \chi(t) \equiv \chi_1(t)$$

and, moreover,

$$\|\mathbf{r}_2'(t)\| \equiv \|\mathbf{r}_1'(t)\|.$$

Therefore, the vector functions $\boldsymbol{\tau}_1(t)$, $\boldsymbol{\nu}_1(t)$, $\boldsymbol{\beta}_1(t)$ and $\boldsymbol{\tau}_2(t)$, $\boldsymbol{\nu}_2(t)$, $\boldsymbol{\beta}_2(t)$ giving the Frenet frame are solutions of the same system of Frenet equations

$$\begin{cases} \boldsymbol{\tau}' = \|\mathbf{r}_1'\| k_1 \boldsymbol{\nu} \\ \boldsymbol{\nu}' = -\|\mathbf{r}_1'\| k_1 \boldsymbol{\tau} + \|\mathbf{r}_1'\| \chi_1 \boldsymbol{\beta} \\ \boldsymbol{\beta}' = -\|\mathbf{r}_1'\| \chi_1 \boldsymbol{\nu}. \end{cases}$$

Since for $t = t_0$ the solutions coincide, due to the uniqueness theorem for the solution of a Cauchy problem they have to coincide globally. In particular, we have

$$\boldsymbol{\tau}_1(t) \equiv \boldsymbol{\tau}_2(t) \quad \text{or} \quad \frac{\mathbf{r}_1'(t)}{\|\mathbf{r}_1'(t)\|} = \frac{\mathbf{r}_2'(t)}{\|\mathbf{r}_2'(t)\|},$$

hence

$$\mathbf{r}_1'(t) - \mathbf{r}_2'(t) = 0 \;\Rightarrow\; \mathbf{r}_1(t) - \mathbf{r}_2(t) = \text{const.}$$

But for $t = t_0$, $\mathbf{r}_1(t_0) - \mathbf{r}_2(t_0) = 0$, hence the two functions coincide for all t, thus $\mathbf{r}_1(t) \equiv \mathbf{r}_2(t) = D \circ \mathbf{r}(t)$.

As for the uniqueness of D, we notice that, for any other point $t_1 \in I$, since $\mathbf{r}_1 \equiv \mathbf{r}_2$, D sends the Frenet frame of the curve $\mathbf{r}$ at t_1 into the Frenet frame of the curve $\mathbf{r}_1$ at t_1. □

Remark. For naturally parameterized curves the condition $\|\mathbf{r}'(t)\| = \|\mathbf{r}_1'(t)\|$ is always fulfilled.

7.14.3 The existence theorem

Theorem 7.13. *Let $f(s)$ and $g(s)$ be two smooth functions, defined on an interval I, such that $f(s) > 0$, $\forall t \in I$. Then there is a single naturally parameterized curve $(I, \mathbf{r} = \mathbf{r}(s))$ for which $f(s) = k(s)$ $\forall s \in I$ and $g(s) = \chi(s)$ $\forall s \in I$. This curve is uniquely defined, up to a rigid motion of $\mathbb{R}^3$.*

Proof Let $\{\mathbf{r_0}; \mathbf{T_0}, \mathbf{N_0}, \mathbf{B_0}\}$ be a direct orthonormal frame in $mathbbR^3$. We consider the system of linear ordinary differential equations

$$\begin{cases} \mathbf{T}'(s) = f(s)\mathbf{N}(s) \\ \mathbf{N}'(s) = -f(s)\mathbf{T}(s) + g(s)\mathbf{B}(s) \\ \mathbf{B}'(s) = -g(s)\mathbf{N}(s) \end{cases} \tag{7.14.3}$$

with respect to the vector functions $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$.

If we denote

$$X(s) = (\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)), \tag{7.14.4}$$

then the system (7.14.3) can be written as

$$X'(s) = A(s) \cdot X(s), \tag{7.14.5}$$

with

$$A(s) = \begin{pmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{pmatrix}.$$

In the theory of ordinary differential equations one proves that the system (7.14.5) has a single solution subject to

$$X(s_0) = (\mathbf{T_0}, \mathbf{N_0}, \mathbf{B_0}),$$

where $s_0 \in I$, while the columns of the matrix $X(s_0)$ are the vectors $\mathbf{T_0}, \mathbf{N_0}, \mathbf{B_0}$ of the initial orthonormal basis.

We are going to show first that, for any $s \in I$ the vectors din $(\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s))$ form an orthonormal basis. It is enough to show for any $s \in I$, $X(s)$ is orthogonal, i.e. $X^t(s) \cdot X(s) = I_3$, for any $s \in I$. We have

$$\frac{d}{dt}\left(X^t \cdot X\right) = \frac{d}{dt}\left(X^t(s)\right) \cdot X + X^t \cdot \frac{d}{dt}(X(s)) = X^t(A^t X + AX) = X^t(A^t + A)X.$$

but, as A is skew symmetrical, $A^t + A = 0$, therefore

$$\frac{d}{dt}\left(X^t \cdot X\right) = 0 \;\Rightarrow\; X^t \cdot X = const.$$

On the other hand, from the initial condition, $\left(X^t \cdot X\right)(s_0) = I_3$, hence $X^t(s) \cdot X(s) = I_3$ for any $s \in I$.

Let us define now

$$\mathbf{r} = \mathbf{r_0} + \int_{s_0}^{s} \mathbf{T}(s)ds, \tag{sol}$$

where $\mathbf{r_0}$ is the origin of the original frame, while $\mathbf{T}(s)$ is the first column of $X(s)$. We are going to show that $(I, \mathbf{r}(s))$ is the curved searched for. We have, clearly:

$$\begin{aligned}
\mathbf{r}'(s) &= \mathbf{T}(s), \\
\|\mathbf{r}'(s)\| &= \|\mathbf{T}(s)\| = 1, \\
\mathbf{r}''(s) &= \mathbf{T}^{'}(s) = f(s)\mathbf{N}(s).
\end{aligned}$$

We note immediately that $\mathbf{r}'(s) \times \mathbf{r}''(s) \neq 0$, therefore $\mathbf{r}(s)$ is a biregular naturally parameterized curve. On the other hand,

$$\mathbf{r}'''(s) = (f(s)\mathbf{N})' = f'\mathbf{N} + f\mathbf{N}^{'} = f'\mathbf{N} + f(-f\mathbf{T} + g\mathbf{B}) = -f^2\mathbf{T} + f'\mathbf{N} + fg\mathbf{B},$$

hence

$$(\mathbf{r}', \mathbf{r}'', \mathbf{r}''') = (\mathbf{T}, f\mathbf{N}, -f^2\mathbf{T} + f'\mathbf{N} + fg\mathbf{B}) = (\mathbf{T}, f\mathbf{N}, fg\mathbf{B}) = f^2 g \underbrace{(\mathbf{T}, \mathbf{N}, \mathbf{B})}_{=1} = f^2 g.$$

Now we have all we need to compute the curvature and the torsion:

$$\begin{aligned}
k(s) &= \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = |f(s)| = f(s), \\
\chi(s) &= \frac{f^2(s)g(s)}{f^2(s)} = g(s),
\end{aligned}$$

hence the curve $\mathbf{r}$ fulfils the conditions of the theorem.

The uniqueness of $\mathbf{r}$, up to rigid motions, follows from the previous theorem. $\square$

CHAPTER 8

Plane curves

8.1 Introduction

After the exposition of the theory of space curves in all the generality, we shall focus in this chapter on topics which are specific to the theory of plane curves. In particular, we shall discuss here notions which cannot be defined for space curves (such as the signed curvature), or which are easier to investigate and richer in contents for plane curves.

8.2 Envelopes of plane curves

In this section, if not mentioned otherwise, all the curves are *parameterized curves*, if not specified otherwise. Let

$$\mathbf{r} = \mathbf{r}(t, \lambda) \tag{8.2.1}$$

be a family of plane parameterized curves, depending smoothly on a real parameter λ.

Definition. The *envelope* of the family (8.2.1) is a parameterized curve (J, Γ) which, at each point, is tangent to a curve from the family.

Theorem. *The points of the envelope of the family* $\mathbf{r}(t,\lambda)$ *are subject to*

$$\mathbf{r} = \mathbf{r}(t,\lambda) \tag{8.2.2}$$

$$\mathbf{r}'_{\boldsymbol{\lambda}} \times \mathbf{r}'_{\boldsymbol{t}} = 0. \tag{8.2.3}$$

Proof If Γ is the envelope of the family (γ_λ) and P is a point of Γ, then P is a tangency point between Γ and a curve from the family, corresponding to some value of the parameter λ. Thus, the equation of Γ will be of the form

$$\mathbf{r}_1 = \mathbf{r}_1(\lambda).$$

On the other hand, P is on a curve γ_λ and, therefore, it verifies

$$\mathbf{r}_1 = \mathbf{r}(t(\lambda),\lambda).$$

The tangency condition between Γ and γ_λ reads

$$\mathbf{r}'_{\mathbf{1}\boldsymbol{\lambda}} \parallel \mathbf{r}'_{\boldsymbol{t}}$$

or

$$\mathbf{r}'_{\mathbf{1}\boldsymbol{\lambda}} \times \mathbf{r}'_{\boldsymbol{t}} = 0$$

or, also, since $\mathbf{r}'_{\mathbf{1}\boldsymbol{\lambda}} = \mathbf{r}'_{\boldsymbol{t}} \cdot t'_\lambda + \mathbf{r}'_{\boldsymbol{\lambda}}$,

$$(\mathbf{r}'_{\boldsymbol{t}} \cdot t'_\lambda + \mathbf{r}'_{\boldsymbol{\lambda}}) \times \mathbf{r}'_{\boldsymbol{t}} = 0,$$

and the theorem is proven, because $\mathbf{r}'_{\boldsymbol{t}} \times \mathbf{r}'_{\boldsymbol{t}} = 0$. □

Remarks. 1. The set of points described by the equations (8.2.2) and (8.2.3) is called the *discriminant set* of the family γ_λ. It includes not only the support of the envelope, but, also, the singular points of the curves from the family, for which $\mathbf{r}'_{\boldsymbol{t}} = 0$.

2. The equation $\mathbf{r}'_{\boldsymbol{\lambda}} \times \mathbf{r}'_{\boldsymbol{t}} = 0$ can be written also as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'_\lambda & y'_\lambda & 0 \\ x'_t & y'_t & 0 \end{vmatrix} = 0 \;\Leftrightarrow\; x'_\lambda y'_t - x'_t y'_\lambda = 0 \;\Leftrightarrow$$

$$\Leftrightarrow\; \frac{x'_\lambda}{x'_t} = \frac{y'_\lambda}{y'_t}. \tag{8.2.4}$$

Example 8.1. Let us consider the family of curves

$$\mathbf{r}(t,\lambda) = (\lambda + a\cos t, \lambda + a\sin t), \qquad \lambda, t \in \mathbb{R}, a > 0.$$

Clearly, we are dealing with a family of circles of radius a, with the centres on the first bisector of the coordinate axes. Then we have

$$\begin{aligned} \mathbf{r}'_\lambda &= \{1, 1\}, \\ \mathbf{r}'_t &= \{-a\sin t, a\cos t\}, \end{aligned}$$

therefore, the points of the envelope (and only them, since the circles have no singular points) verify

$$\begin{cases} x = \lambda + a\cos t \\ y = \lambda + a\sin t \\ x'_\lambda \cdot y'_t = x'_t \cdot y'_\lambda \end{cases}$$

or

$$\begin{cases} x = \lambda + a\cos t \\ y = \lambda + a\sin t \\ a\cos t = -a\sin t. \end{cases}$$

Eliminating t, we get the parametric equations of the envelope:

$$\begin{cases} x(\lambda) = \lambda \pm \frac{a}{\sqrt{2}} \\ y(\lambda) = \lambda \mp \frac{a}{\sqrt{2}}, \end{cases}$$

i.e. the envelope is, in fact, a pair of straight lines, parallel to the first bissector of the coordinate axes.

8.2.1 Curves given through an implicit equation

Property 11. *The points of the envelope of a family of plane curves given through the implicit equation*

$$F(x, y, \lambda) = 0 \tag{8.2.5}$$

verify the system of equations

$$\begin{cases} F(x, y, \lambda) = 0 \\ F'_\lambda(x, y, \lambda) = 0 \end{cases} \tag{8.2.6}$$

Proof Locally, around each point of a the curve of the family, we can parameterize the curve, i.e. we can represent it as

$$\begin{cases} x = x(t,\lambda) \\ y = y(t,\lambda) \end{cases}.$$

By substituting into the equation of the family, we get

$$F(x(t,\lambda), y(t,\lambda), \lambda) = 0,$$

whence, differentiating with respect to t and λ, respectively, we obtain the system:

$$\begin{cases} F'_x x'_t + F'_y y'_t = 0 \\ F'_x x'_\lambda + F'_y y'_\lambda + F'_\lambda = 0 \end{cases}.$$

But, from (8.2.4),

$$x'_\lambda = K x'_t, \quad y'_\lambda = K y'_t,$$

with $K =$ const., therefore, the second equation from above becomes

$$K(\underbrace{F'_x x'_t + F'_y y'_t}_{=0}) + F'_\lambda = 0$$

or

$$F'_\lambda = 0.$$

□

Example 8.2. We consider again the family of circles from the previous paragraph, this time given through their implicit equation

$$F(x,y,\lambda) \equiv (x-\lambda)^2 + (y-\lambda)^2 - a^2 = 0.$$

Then the second equation of the discriminant set will be

$$F'_\lambda(x,y,\lambda) = -2(x+y-2\lambda) = 0,$$

whence we get

$$\lambda = \frac{x+y}{2},$$

which, when substituted into the equation of the family, gives

$$(x-y)^2 = 2a^2,$$

i.e. we get, again, the same equations of the envelope as before, namely

$$y = x \pm a\sqrt{2}.$$

8.2.2 Families of curves depending on two parameters

Property 12. *Suppose we are given a family of curves depending smoothly on two parameters, λ and μ*

$$F(x, y, \lambda, \mu) = 0, \tag{8.2.7}$$

where the parameters λ and μ are connected through a relation

$$\varphi(\lambda, \mu) = 0, \tag{8.2.8}$$

then the points of the envelope verify the system

$$\begin{cases} F(x, y, \lambda, \mu) = 0 \\ \varphi(\lambda, \mu) = 0 \\ \dfrac{D(F, \varphi)}{D(\lambda, \mu)} = 0 \end{cases} . \tag{8.2.9}$$

Proof From the equation

$$\varphi(\lambda, \mu) = 0,$$

we can assume, if, for instance, that

$$\mu = \mu(\lambda),$$

therefore, substituting in F, and φ,

$$\begin{cases} F(x, y, \lambda, \mu(\lambda)) = 0, \\ \varphi(\lambda, \mu(\lambda)) = 0. \end{cases}$$

Differentiating with respect to λ these two equations, we get:

$$\begin{cases} F'_\lambda + F'_\mu \mu'_\lambda = 0 \\ \varphi'_\lambda + \varphi'_\mu \mu'_\lambda = 0. \end{cases}$$

Eliminating the derivative μ'_λ between the two equations, we get the third equation from (8.2.9), as required. □

8.2.3 Applications: the evolute of a plane curve

Definition. Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve. The *evolute* of $\mathbf{r}$ is, by definition, the envelope of the family of normals of the curve.

The following result holds:

Property 13. *The parametric equations of the evolute of the curve* $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t))$ *are*

$$\begin{cases} X = x - \dfrac{y'(x'^2 + y'^2)}{x'y'' - x''y'} \\ Y = y + \dfrac{x'(x'^2 + y'^2)}{x'y'' - x''y'} \end{cases} \tag{8.2.10}$$

Proof As we know, the equation of the normal of a plane curve is

$$F(X, Y, t) = (X - x(t)) \cdot x'(t) + (Y - y(t)) \cdot y'(t) = 0.$$

The relations verified by the points of the envelope of the family of normals (and *only* by them, since in this case the curves of the family are straight line and they have no singular points) are (see (8.2.6)):

$$\begin{cases} F(X, Y, t) = 0 \\ F_t'(X, Y, t) = 0 \end{cases},$$

i.e.

$$\begin{cases} x'(t)X + y'(t)Y = x(t) \cdot x'(t) + y(t) \cdot y'(t) \\ x''(t)X + y''(t)Y = x'^2(t) + x''(t) \cdot x(t) + y'^2(t) + y(t) \cdot y''(t) \end{cases}.$$

The equations (8.2.10) follow now instantly, after solving this system of linear equations with respect to X and Y. □

Example 8.3. For the ellipse

$$\begin{cases} x = a \cos t \\ y = b \sin t \end{cases}$$

one gets, after the computations,

$$\begin{cases} X = \frac{a^2 - b^2}{a} \cos^3 t \\ Y = \frac{b^2 - a^2}{b} \sin^3 t \end{cases}$$

or, eliminating the parameter t,

$$a^{\frac{2}{3}} X^{\frac{2}{3}} + b^{\frac{2}{3}} Y^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

The curved described by this equation is called a *lengthened astroid* (see figure 8.2.3).

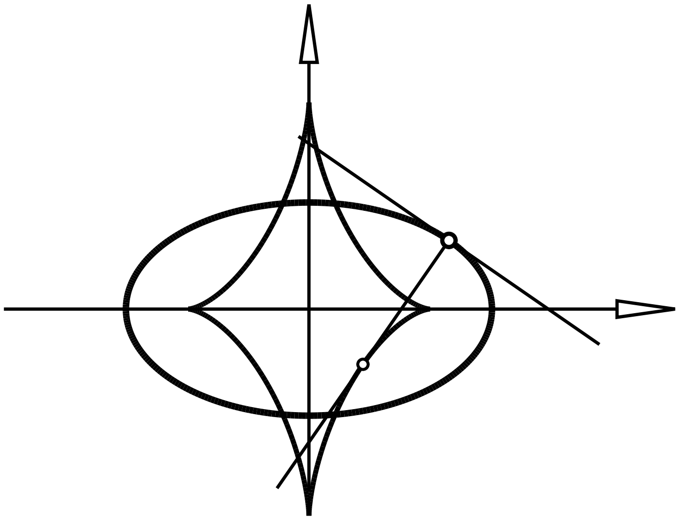


Figure 8.1: The evolute of an ellipse

8.3 The curvature of a plane curve

As we saw, in the case of an arbitrary space curve, the curvature is always a positive scalar. Of course, this concept of curvature can be equally applied for plane curves. It turns out, however that, in this particular case, we can obtain more information about the curve if we use a slightly different concept, allowing the curvature to have a sign. To define the curvature of a plane curve we shall use a little technical trick, which will allow us to give the definition in a coordinate independent way.

Definition. The *complex structure* on $\mathbb{R}^2$ is the map $J : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$J(x, y) = (-y, x).$$

Remark. Applying J simply means to rotate the vector $\{x, y\}$ by $\frac{\pi}{2}$ or multiplying the complex number $x + iy$ by the complex unit i (this is, actually, the origin of the name).

Some obvious properties of the complex structure are collected into the following proposition:

Property 14. *a)* $J\mathbf{v} \cdot J\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$.

b) $(J\mathbf{v}) \cdot \mathbf{v} = 0$.

c) $J(J\mathbf{v}) = -\mathbf{v}$ *(i.e.* $J^2 = -id$*).*

All these properties follow immediately from the geometrical interpretation of the complex structure.

Anticipating a little bit, we would like to say a few words about the kind of curvature we are going to define. We remember that the curvature of an arbitrary parameterized space curve $\mathbf{r} = \mathbf{r}(t)$ can be computed by

$$k(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Now, if $\mathbf{r}$ is a *plane* curve, with the support lying into the coordinate plane xOy, then the vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are situated, also, into this plane. Therefore, their vector product is a vector which is directed along the z axis, and, therefore, the norm of this vector is just the absolute value of the z-component. Now, the idea of the definition of the signed curvature is just to replace this absolute value with the component itself. To do that, the following characterization of the vector product of two vectors in the plane will prove very useful.

Property 15. *Let* $\mathbf{u}(x_1, y_1), \mathbf{v}(x_2, y_2) \in \mathbb{R}^2$*. Then*

$$\mathbf{u} \times \mathbf{v} = [\mathbf{v} \cdot J\mathbf{u}] \cdot \mathbf{k}.$$

Proof As it is known, the cross product of the vectors $\mathbf{u}$ and $\mathbf{v}$ can be computed by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix} = (x_1 y_2 - x_2 y_1) \cdot \mathbf{k}.$$

On the other hand,

$$[\mathbf{v} \cdot J\mathbf{u}] = \{x_2, y_2\} \cdot \{-y_1, x_1\} = -x_2 y_1 + x_1 y_2,$$

whence the equality announced. $\square$

We are now ready to define the curvature of a plane curve.

Definition. Let $\mathbf{r} = \mathbf{r}(t)$ be a plane parameterized curve. The *signed curvature* of $\mathbf{r}$ is, by definition, the quantity

$$k_{\pm} = \frac{\mathbf{r}'' \cdot J\mathbf{r}'}{\|\mathbf{r}'\|^3}. \tag{8.3.1}$$

Remark. According to the proposition 15, the signed curvature is just the projection of the curvature vector on the z-axis. Now, as the curvature vector is parallel to the z-axis, we have, thus

$$|k_{\pm}| = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = k.$$

Another immediate, but important, for computational reasons, result, is the following:

Property 16. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve. If $\mathbf{r}(t) = (x(t), y(t))$, then the signed curvature of $\mathbf{r}$ can be expressed as*

$$k_{\pm}(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\left(x'^2(t) + y'^2(t)\right)^{3/2}}.$$

Corollary 8.1. *If $y = f(x)$ is the explicit equation of a plane curve, then its signed curvature is given by*

$$k_{\pm}(x) = \frac{f''(x)}{(1 + f'^2)^{3/2}}.$$

Remark. The previous consequence show that, for an explicitly given curve (i.e. the graph of a real function of a single real variable) the sign of the signed curvature is, in fact, the sign of the second derivative of the function f, i.e., as it known from calculus, the signed of the signed curvature is a n indication of the convexity or concavity of the function.

Exactly as happens for the torsion of the space curves, the signed curvature of a plane curve is "almost" invariant at a parameter change, i.e. we have

Theorem. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(u))$ an equivalent parameterized curve, with the parameter change $\lambda : I \to J$, $u = \lambda(t)$. Then*

$$k_{\pm}[\boldsymbol{\rho}](u) = \operatorname{sgn}(\lambda') \cdot k_{\pm}[\mathbf{r}](t).$$

Proof We have

$$
\begin{aligned}
\mathbf{r}(t) &= \boldsymbol{\rho}(\lambda(t)) \\
\mathbf{r}'(t) &= \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t) \\
\mathbf{r}''(t) &= \boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'^2(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t) \\
\mathbf{r}'' \cdot J\boldsymbol{\rho}' &= (\boldsymbol{\rho}''\lambda'^2 + \boldsymbol{\rho}'\lambda'' J(\lambda'\boldsymbol{\rho}') = \\
&= \lambda'^3 \boldsymbol{\rho}'' \cdot J\boldsymbol{\rho}' + \lambda'\lambda'' \underbrace{\boldsymbol{\rho}' \cdot J\boldsymbol{\rho}'} = 0 = \lambda'^3 \boldsymbol{\rho}'' \cdot J\boldsymbol{\rho}' \\
\|\mathbf{r}'\|^3 &= |\lambda'^3| \|\boldsymbol{\rho}'\|^3
\end{aligned}
$$

therefore

$$
k_\pm[\mathbf{r}](t) = \frac{\mathbf{r}'' \cdot J\mathbf{r}'}{\|\mathbf{r}'\|^3} = \frac{\lambda'^3}{|\lambda'^3|} \cdot \frac{\boldsymbol{\rho}''(u) \cdot J\boldsymbol{\rho}'(u)}{\|\boldsymbol{\rho}'(u)\|^3} = \operatorname{sgn}(\lambda') \cdot k_\pm[\boldsymbol{\rho}](u),
$$

whence

$$
k_\pm[\boldsymbol{\rho}](u) = \operatorname{sgn}(\lambda') \cdot k_\pm[\mathbf{r}](t).
$$

□

Remark. The previous theorem shows that the signed curvature is invariant under any *positive* parameter change, therefore it makes sense to define it also for regular *oriented* plane curves.

The curvature vector of a plane, naturally parameterized curve can be expressed easily as a function of the signed curvature:

Lemma. *Let* $(I, \mathbf{r} = \mathbf{r}(s))$ *be a plane naturally parameterized curve. Then*

$$
\mathbf{r}''(s) = k_\pm(s) \cdot J\mathbf{r}'(s).
$$

Proof We have $\mathbf{r}'^2(s) = 1$ (the curve is *naturally* parameterized), therefore $\mathbf{r}' \cdot \mathbf{r}'' = 0$, whence it follows that $\mathbf{r}'' \perp \mathbf{r}'$ or, which is the same, $\mathbf{r}'' \parallel J\mathbf{r}'$.

On the other hand, from the definition of the signed curvature, $k_\pm(s) = \mathbf{r}''(s) \cdot J\mathbf{r}'(s)$. If we put $\mathbf{r}''(s) = \alpha(s) \cdot J\mathbf{r}'(s)$, then we should have $\mathbf{r}''(s) \cdot J\mathbf{r}'(s) = \alpha(s) \cdot [J\mathbf{r}'(s)]^2 = \alpha(s)$, whence $\alpha(s) = k_\pm(s)$. □

8.3.1 The geometrical interpretation of the signed curvature

For the signed curvature of plane curve we have a geometrical interpretation which is similar to the geometrical interpretation of the curvature of a space curve, except that, this time the sign is also taken into account. We will need first the following definition.

Definition. Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve. The *rotation angle* of $\mathbf{r}$ is the function $\theta[\mathbf{r}] : I \to \mathbb{R}$, defined through:

$$\boldsymbol{\tau}(t) = \{\cos\theta[\mathbf{r}](t), \sin\theta[\mathbf{r}](t)\} = \exp(i\theta[\mathbf{r}](t)), \tag{8.3.2}$$

where $\boldsymbol{\tau}(t)$ is the unit tangent, i.e. $\theta[\mathbf{r}]$ is the angle made by the unit tangent with the positive direction of the x axis.

Remark. This definition is very innocent looking and natural. After all, is should be clear for anyone that θ is just the angle made by the unit tangent vector with the positive direction of the x-axis. In reality, however, for an arbitrary plane curve, it is by no mean obvious that we can find a *continuous* angle function, let alone a smooth one. Such functions *do* exist (see, for instance, for a modern proof, the book of Bär, [**?**]) and any two such functions differ by an integer multiple of 2π.

The following lemma provides the connection between the signed curvature and the variation of the rotation angle. The contents of this lemma is quite similar to the geometrical interpretation of the curvature of a space curve. In fact, when the plane curve is regarded as a particular case of space curve, then the *variation* of the rotation angle is equal (as absolute value) with the variation of the contingency angle, therefore, in fact, the geometrical interpretation of the *absolute* curvature of a plane curve (regarded as a curve in space) is an immediate consequence of this lemma.

Lemma. *If $(I, \mathbf{r} = \mathbf{r}(t))$ is a plane regular parameterized curve, θ is its rotation angle and $k_\pm$ – its signed curvature, then:*

$$\frac{d\theta}{dt} = \|\mathbf{r}'(t)\| k_\pm(t).$$

Proof From the definition of the unit tangent vector we have $\boldsymbol{\tau}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, whence

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} + \mathbf{r}'(t)\frac{d}{dt}\left(\frac{1}{\|\mathbf{r}'(t)\|}\right).$$

On the other hand, if we use the expression of $\boldsymbol{\tau}(t)$ as a function of the angle θ, given by (8.3.2), we get for $\frac{d\boldsymbol{\tau}}{dt}$ the formula

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{d\theta}{dt}\{-\sin\theta(t), \cos\theta(t)\} = \frac{d\theta}{dt}J\boldsymbol{\tau}(t).$$

Combining the two relations, we get the equality

$$\frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} + \mathbf{r}'(t)\frac{d}{dt}\left(\frac{1}{\|\mathbf{r}'(t)\|}\right) = \frac{d\theta}{dt}J\boldsymbol{\tau}(t) \equiv \frac{d\theta}{dt}\cdot\frac{J\mathbf{r}''(t)}{\|J\mathbf{r}'(t)\|}.$$

Multiplying both sides by $J\mathbf{r}'(t)$ and having in mind that $J\mathbf{r}'(t)\cdot\mathbf{r}'(t) = 0$ and $J\mathbf{r}'(t)\cdot J\mathbf{r}'(t) = \|\mathbf{r}'(t)\|^2$, we obtain:

$$\frac{\mathbf{r}''(t)\cdot J\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{d\theta}{dt}\cdot\|\mathbf{r}'(t)\|,$$

whence, using the definition of the signed curvature, it follows that

$$\frac{d\theta}{dt}\cdot\|\mathbf{r}'(t)\| = k_{\pm}(t)\cdot\|\mathbf{r}'(t)\|^2$$

or, after simplification,

$$\frac{d\theta}{dt} = k_{\pm}(t)\cdot\|\mathbf{r}'(t)\|,$$

which is exactly what we had to prove. □

Corollary. *For a naturally parameterized curve, $(I, \mathbf{r} = \mathbf{r}(s))$, we have*

$$k_{\pm}(s) = \frac{d\theta}{ds}.$$

Remark. From the previous formula we get

$$k \equiv \|k_{\pm}\| = \left|\frac{d\theta}{ds}\right|,$$

which is exactly the formula for the curvature of arbitrary space curves, which, thus, remains valid, as expected, for the particular case of plane curves.

8.4 The curvature center. The evolute and the involute of a plane curve

Definition. A point $\Omega \in \mathbb{R}^2$ is called the *curvature center* at $\mathbf{r}_0 = \mathbf{r}(t_0)$ of a plane parameterized curve $\mathbf{r} : I \to \mathbb{R}^2$ if there is a circle (γ), centred at Ω, which is tangent to the curve at $\mathbf{r}_0 = \mathbf{r}(t_0)$, with $t_0 \in I$, such that the signed curvatures $\mathbf{r}$ and γ at $\mathbf{r}_0$ coincide, whence the position of the point Ω for an arbitrary $t \in I$:

$$\Omega(t) = \mathbf{r}(t) + \frac{1}{k_\pm(t)} \frac{J\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Remark. The notion of curvature center is invariant at a parameter change: if $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(u))$ is equivalent to $\mathbf{r}$, with the parameter change $\lambda : I \to J$, then $\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t))\lambda'(t)$ and $k_\pm[\mathbf{r}](t) = \operatorname{sgn}(\lambda')k_\pm[\boldsymbol{\rho}](\lambda(t))$. Obviously, we could have problems only when $\lambda' < 0$, but in this case $J\mathbf{r}'$ changes the sense and $k_\pm$ changes the sign and, overall, the situation remains unchanged.

We defined the evolute of a plane curve as being the envelope of the normals of the curve. The following proposition provides a different approach.

Property 17. *The evolute of a plane curve is the geometrical locus of the curvature centers of the curve.*

Proof The curvature center of the curve at an arbitrary value of the parameter is

$$\Omega(t) = \mathbf{r}(t) + \frac{\|\mathbf{r}'(t)\|^2}{\mathbf{r}''(t) \cdot J\mathbf{r}'(t)} \cdot J\mathbf{r}'(t) = (x(t), y(t)) + \frac{x'^2 + y'^2}{x'y'' - x''y'}\{-y', x'\}.$$

Thus, if $\Omega(t) = (X(t), Y(t))$, projecting the previous equation on the coordinate axes, we get the parametric equations of the locus of curvature centers:

$$\begin{cases} X(t) = x(t) - \frac{y'(x'^2+y'^2)}{x'y''-x''y'} \\ Y(t) = y(t) + \frac{x'(x'^2+y'^2)}{x'y''-x''y'} \end{cases},$$

which are exactly the equations of the evolute. $\square$

From the previous remark, we get immediately:

Corollary. *The definition of the evolute makes sense also for regular curves (i.e. two equivalent parameterized curves have the same evolute).*

Exercise 8.4.1. Find the evolute of the astroid

$$\begin{cases} x(t) = a\cos^3 t, \\ y = a\sin^3 t \end{cases}.$$

Show that the evolute is also an astroid (see the figure **??**.

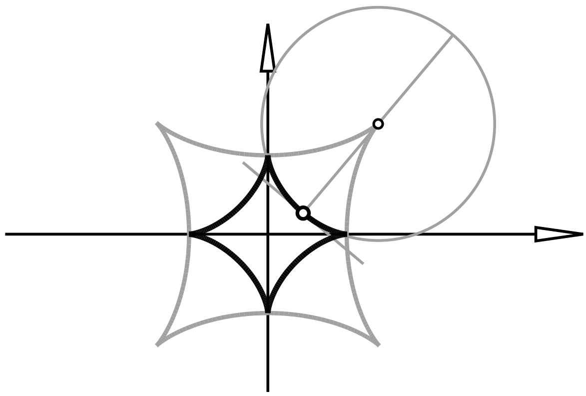


Figure 8.2: The evolute of an astroid

Exercise 8.4.2. Find the evolute of the cycloid

$$\begin{cases} x(t) = a(t - \sin t), \\ y = a(1 - \cos t) \end{cases}.$$

Show that the evolute is also an cycloid (see the figure **??**.

Another interesting plane curve associated to a given one is the so-called *involute*, which,a s we shall see immediately, is, in a sense, the inverse of the evolute.

Definition. Let $(I, \mathbf{r} = \mathbf{r}(s))$ be a naturally parameterized curve $c \in I$. The *involute* of $\mathbf{r}$ with the origin at $\mathbf{r}(c)$ (or at c, for short) is the parameterized curve $(I, \boldsymbol{\rho}[\mathbf{r}, c] = \boldsymbol{\rho}[\mathbf{r}, c](s))$ where

$$\boldsymbol{\rho}[\mathbf{r}, c](s) = \mathbf{r}(s) + (c - s)\mathbf{r}'(s).$$

Remark. Generally speaking, s is *not* a natural parameter along $\boldsymbol{\rho}$.

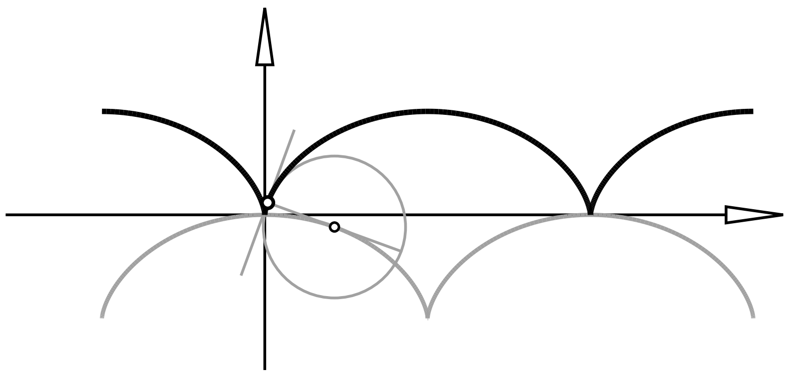


Figure 8.3: The evolute of an cycloid

If $(I, \mathbf{r} = \mathbf{r}(t))$ is an arbitrary parameterized curve, the we can replace the parameter t by the arc length $s = \int\limits_0^t \|\mathbf{r}'(\tau)\| d\tau$ and define the involute of $\mathbf{r}$ as being the involute of the naturally parameterized curve equivalent to it, the natural parameter being the arc length. Is is easy to see that the following proposition holds true:

Property 18. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a parameterized curve. The the involute of $\mathbf{r}$ with the origin at $c \in I$ is given by*

$$\boldsymbol{\rho}(t) = \mathbf{r}(t) + (c - s(t))\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

where $s = s(t)$ is the arc length of $\mathbf{r}$.

Example 8.4. Let $\mathbf{r}(t) = (a\cos t, a\sin t)$ be a circle. Then

$$\begin{cases} \mathbf{r}'(t) = \{-a\sin t, a\cos t\} \\ x'^2 + y'^2 = a^2 \\ s(t) = \int\limits_0^t a dt = at, \end{cases}$$

hence the equation of the involute is

$$\boldsymbol{\rho}(t) = (a\cos t, a\sin t) + \frac{(c - at)}{a}\{-a\sin t, a\cos t\}$$

or, in projection,

$$\begin{cases} X(t) = a\cos t - (c - at)\sin t \\ Y(t) = a\sin t + (c - at)\cos t \end{cases} .$$

We represented in the figure 8.4 an involute of a circle of radius 1.5, with the origin at the point of parameter 0.

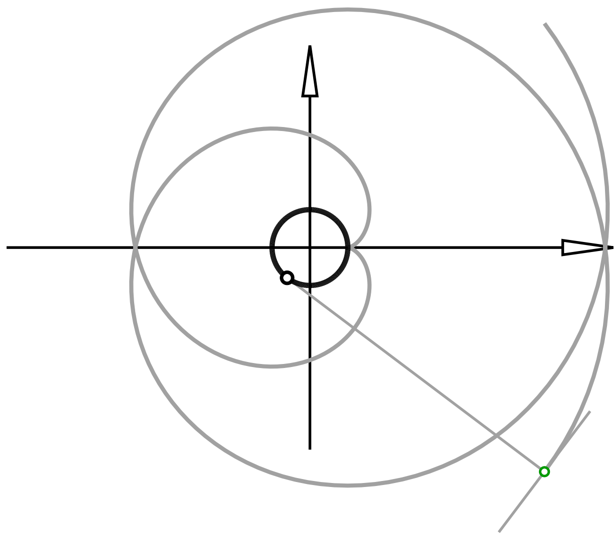


Figure 8.4: An involute of a circle

The following lemma establishes the connection between the signed curvature of a parameterized curve and that of one of its involutes and will serve as a tool to establish the relation between evolute and involute.

Lemma. *Let $(I, \mathbf{r} = \mathbf{r}(s))$ be a naturally parameterized curve and $\boldsymbol{\rho}$ the involute of $\mathbf{r}$ with the origin at $c \in I$. Then the signed curvature of $\boldsymbol{\rho}$ is given by*

$$k_{\pm}[\boldsymbol{\rho}](s) = \frac{\operatorname{sgn}(k_{\pm}[\mathbf{r}](s))}{|c - s|}.$$

Proof We have

$$\begin{aligned}
\boldsymbol{\rho}'(s) &= \mathbf{r}'(s) + (c - s)\mathbf{r}''(s) - \mathbf{r}'(s) = (c - s)\mathbf{r}''(s) = (c - s)k_{\pm}[\mathbf{r}](s) \cdot J\mathbf{r}'(s) \\
\boldsymbol{\rho}''(s) &= -k_{\pm}[\mathbf{r}](s.J\mathbf{r}'(s) + (c - s)(k_{\pm}[\mathbf{r}](s))' \cdot J\mathbf{r}'(s) + (c - s)k_{\pm}[\mathbf{r}](s) \cdot J\mathbf{r}''(s) \\
&= [-k_{\pm}[\mathbf{r}](s) + (c - s)(k_{\pm}[\mathbf{r}](s))'] \cdot J\mathbf{r}'(s) - (c - s)(k_{\pm}[\mathbf{r}](s))^2 \cdot \mathbf{r}'(s),
\end{aligned}$$

whence

$$J\boldsymbol{\rho}' = -(c-s)k_{\pm}[\mathbf{r}](s)\cdot\mathbf{r}'(s),$$

while

$$\boldsymbol{\rho}''(s)\cdot J\boldsymbol{\rho}'(s) = (c-s)^2\cdot(k_{\pm}[\mathbf{r}](s))^3.$$

The conclusion follows now from the definition of the signed curvature. $\square$

The following theorem provides a connection between the involute and the evolute. In many textbooks this connection is taken, in fact, as the definition of the involute.

Theorem. *Let $(I, \mathbf{r} = \mathbf{r}(s))$ be a naturally parameterized curve and $\boldsymbol{\rho}$ – its involute with the origin at $c \in I$. The the evolute of $\boldsymbol{\rho}$ is $\mathbf{r}$.*

Proof The evolute of $\boldsymbol{\rho}$ is given, as known, by the equation

$$\boldsymbol{\rho}_1(s) = \boldsymbol{\rho}(s) + \frac{1}{k_{\pm}[\boldsymbol{\rho}](s)}\cdot\frac{J\boldsymbol{\rho}'(s)}{\|\boldsymbol{\rho}'(s)\|}.$$

Using the previous lemma to express the signed curvature of $\boldsymbol{\rho}$ as a function of the signed curvature of $\mathbf{r}$, we get

$$\boldsymbol{\rho}_1(s) = \mathbf{r}(s) + (c-s)\mathbf{r}'(s) + \frac{|c-s|}{\operatorname{sgn}(k_{\pm}[\mathbf{r}](s))}\cdot\frac{(c-s)k_{\pm}[\mathbf{r}](s)\cdot J^2\mathbf{r}'(s)}{\|(c-s)k_{\pm}[\mathbf{r}](s)\cdot J\mathbf{r}'(s)\|} = \mathbf{r}(s).$$

$\square$

8.5 The osculating circle of a curve

Definition. Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a parameterized curve. The *osculating circle* of $\mathbf{r}$ at a point $t \in I$ is the circle centered at the curvature center $\Omega(t)$, with the radius equal to the curvature radius $\frac{1}{k(t)}$ of the curve at that point.

Exactly as the osculating plane at a point of a space curve can be regarded as a limit position of a plane determined by three neighboring points, when these points approach indefinitely the given one, the osculating circle is the limit position of a *circle* determined by three neighboring points, when these points approach indefinitely the given one. More precisely, we have:

Theorem. *Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a plane parameterized curve and $t_1 < t_2 < t_3 \in I$ Let $C(t_1, t_2, t_3)$ be the circle passing through $\mathbf{r}(t_1)$, $\mathbf{r}(t)2)$, $\mathbf{r}(t_3)$. We assume that, for a value $t \in I$ of the parameter we have $k_\pm(t) \neq 0$. Then the osculating circle of $\mathbf{r}$ at the point $\mathbf{r}(t)$ is the circle*

$$C = \lim_{\substack{t_1 \to t \\ t_2 \to t \\ t_3 \to t}} C(t_1, t_2, t_3).$$

Proof Let $A(t_1, t_2, t_3)$ be the centre of the circle $C(t_1, t_2, t_3)$ and $f : I \to \mathbb{R}$ – the function defined through $f(t) = \|\mathbf{r}(t) - A\|^2$. Then, clearly, f is smooth and we have:

$$\begin{cases} f'(t) = 2\mathbf{r}' \cdot (\mathbf{r}(t) - A) \\ f''(t) = 2\mathbf{r}''(t) \cdot (\mathbf{r}(t) - A) + 2\|\mathbf{r}'(t)\|^2 \end{cases}.$$

Since f is differentiable and $f(t_1) = f(t_2) = f(t_3)$, from the mean value theorem it follows that there exist two points $u_1, u_2 \in I$, with $t_1 < u_1 < t_2 < u_2 < t_3$ such that

$$f'(u_1) = f(u_2) = 0.$$

On the other hand, applying once more the mean value theorem, this time to the derivative f', which is also differentiable , it follows that there is a $v \in (u_1, u_2)$ such that

$$f''(v) = 0.$$

Now, if we let $t_1, t_2, t_3 \to t$, then we will have, equally, $u_1, u_2, v \to t$, therefore, at the limit, we should get:

$$\begin{cases} \mathbf{r}'(t) \cdot (\mathbf{r}(t) - A(t)) = 0 \\ \mathbf{r}''(t) \cdot (\mathbf{r}(t) - A(t)) = -\|\mathbf{r}'(t)\|^2 \end{cases}, \tag{*}$$

where

$$A(t) = \lim_{\substack{t_1 \to t \\ t_2 \to t \\ t_3 \to t}} A(t_1, t_2, t_3).$$

From (*) and the definition of the signed curvature, it follows that

$$\mathbf{r}(t) - A(t) = \frac{-1}{k_\pm(t)} \cdot \frac{J\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

hence C is the osculating circle of the curve $\mathbf{r}$ at the point $\mathbf{r}(t)$. □

8.6 The existence and uniqueness theorem for plane curves

The existence and uniqueness theorem for plane parameterized curves is similar to the analogue theorem for space curves and can be proved in the same manner, therefore we shall skip the proof here.

Theorem 8.1. *Let $f : I \to \mathbb{R}$ be a continuous function. Then there is a regular, naturally parameterized curve $(I, \mathbf{r} = \mathbf{r}(s))$ such that $\forall s \in I$, $k_{\pm}(s) = f(s)$. $\mathbf{r}$ is unique, up to a proper motion of $\mathbb{R}^2$.*

To give an example, we will find the curve $\mathbf{r}$ for the particular case when the function f is a constant α, for any real value of the parameter s.

Starting from the geometrical interpretation of the signed curvature, we get

$$\alpha = k_{\pm}(s) = \frac{d\theta}{ds},$$

therefore θ (the rotation angle) will be a linear function of s:

$$\theta = \alpha s + \theta_0,$$

where θ_0 is a constant. On the other hand, from the definition of the rotation angle, we obtain:

$$\boldsymbol{\tau}(s) \equiv \left\{ \frac{dx}{ds}, \frac{dy}{ds} \right\} = \{\cos\theta(s), \sin\theta(s)\} = \{\cos(\alpha s + \theta_0), \sin(\alpha s + \theta_0)\}$$

whence the system of differential equations:

$$\begin{cases} \frac{dx}{ds} = \cos(\alpha s + \theta_0) \\ \frac{dy}{ds} = \sin(\alpha s + \theta_0) \end{cases}.$$

Since the equations are separated, the system can be integrated very easy and we get the solution:

$$\begin{cases} x = \frac{1}{\alpha}[\sin\alpha s \sin\theta_0 + \cos\alpha s \sin\theta_0] + x_0, \\ y = \frac{1}{\alpha}[-\cos\alpha s \cos\theta_0 + \sin\alpha s \sin\theta_0] + y_0 \end{cases}, \tag{*}$$

where x_0 and y_0 are two integration constant. The solution (*) can be written in the matrix form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha}\sin\alpha s \sin\theta_0 + \frac{1}{\alpha}\cos\alpha s \sin\theta_0 \\ -\frac{1}{\alpha}\cos\alpha s \cos\theta_0 + \frac{1}{\alpha}\sin\alpha s \sin\theta_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

or, also,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{2} - \theta_0\right) & \sin\left(\frac{\pi}{2} - \theta_0\right) \\ -\sin\left(\frac{\pi}{2} - \theta_0\right) & \cos\left(\frac{\pi}{2} - \theta_0\right) \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha}\cos\alpha s \\ \frac{1}{\alpha}\sin\alpha s \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

which shows that any plane curve of constant signed curvature, equal to α, can be obtained from the curve

$$\begin{cases} x = \frac{1}{\alpha}\cos\alpha s \\ y = \frac{1}{\alpha}\sin\alpha s \end{cases}$$

by applying a rotation followed by a translation, i.e. a *rigid motion*. But this is the circle of radius $1/\alpha$, centered at the origin. The conclusion is that *the only plane curve of constant positive curvature* α *is the circle of radius* $1/\alpha$.

CHAPTER 9

The integration of the natural equations of a curve

9.1 The Riccati equation associated to the natural equations of a curve

The general theory of differential equations guarantees the existence and uniqueness (up to a rigid motion) for the Frenet equations. However, finding *analytically* a curve when the curvature and torsion are given is quite another matter. This, of course, amounts to finding an analytical solution for the Frenet equations. Although they look quite innocently, it turns out that, in the general situation, it is not possible to integrate the system.

Clearly, if we manage to solve the Frenet system, the, in particular, we can find the tangent versor and, by another quadrature, we can find the curve. Apparently, the Frenet system should be equivalent to a third order vectorial differential equation or to a system of three third order scalar equations. Nevertheless, it turns out that the system (made of none scalar equations) actually contains three identical sets of three equations, therefore it should be equivalent to a single third order scalar equations. Moreover, the three vectors of the solution are subject to the orthonormality conditions, which should allow further reductions. In fact, we shall prove this indirectly, by showing that the Frenet system is equivalent to a second order Ricatti differential equation. As it is known from the theory of ordinary differential equations, we can find the general solution of a Ricatti equation if and only if a particular solution is already available (and there is no general procedure for finding such a solution).

Notice, first of all, that the Frenet system contains three copies of the scalar system:

$$\begin{cases} X' = k(s)Y, \\ Y' = -k(s)X + \chi(s)Z \\ Z' = -\chi(s)Y \end{cases} \tag{9.1.1}$$

and the solution of the system should verify, also, the condition

$$X^2 + Y^2 + Z^2 = 1. \tag{9.1.2}$$

The idea of the method we are going to describe (and which goes back to Sophus Lie and Gaston Darboux), relies exactly on the supplementary condition (9.1.2). Namely, it was observed that this equation can be decomposed (over the complex numbers) as

$$(X + iY)(X - iY) = (1 - Z)(1 + Z).$$

We introduce now the complex functions u and v by letting

$$u = \frac{X + iY}{1 - Z}; \quad -\frac{1}{v} = \frac{X - iY}{1 - Z}. \tag{9.1.3}$$

Clearly, w and $-1/v$ are conjugated to each other. It is now possible to express X, Y, Z in terms of u and v. One can check easily that

$$X = \frac{1 - uv}{u - v}; \quad Y = i\frac{1 + uv}{u - v}; \quad Z = \frac{u + v}{u - v}. \tag{9.1.4}$$

Thus, the solution of the Frenet system amounts to the finding of the complex functions u and v. Now, an easy manipulation of the formulas show that both u and v are solution of the Riccati equation

$$\frac{dw}{ds} = -\frac{i}{2}\chi(s) - ik(s)w + \frac{i\chi(s)}{2}w^2. \tag{9.1.5}$$

This is, as we mentioned earlier, an indirect prove of the fact that the natural equations of a space curve cannot be integrated through quadratures, in the general situation.

9.2 Examples for the integration of the natural equation of a plane curve

We saw, in the previous section, that the plane curves of constant curvature are the circles. We shall give, here, other interesting examples of natural equation for plane curves that can be integrated. We remind that in the case of plane curves the problem is not the *integrability* of the natural equation, but, rather, the possibility of expressing the solution in terms of elementary functions. Usually this is not the case, and this makes even more interesting the (few) cases in which this possibility does exist. For convenience, we shall use, for the rest of this paragraph, the *curvature radius* $R = 1/k$ instead of the curvature.

The logarithmic spiral. In this case the curvature radius is given by

$$R = a \cdot s, \tag{9.2.1}$$

where a is a constant. Let α be the contingency angle. Then, as we know, we have

$$\frac{d\alpha}{ds} = \frac{1}{R(s)} = \frac{1}{as},$$

therefore, by integrating and neglecting the integration constant, we get

$$\alpha = \frac{1}{a} \ln s, \quad \text{whence} \quad s = e^{a\alpha}.$$

To get the coordinates, we have to integrate the system of differential equations

$$\frac{dx}{ds} = \cos\alpha, \quad \frac{dy}{ds} = \sin\alpha.$$

We have (neglecting, again, the integration constants)

$$x = \int \cos\alpha ds = a \int \cos\alpha e^{a\alpha} d\alpha = \frac{ae^{a\alpha}}{a^2+1}(a\cos\alpha + \sin\alpha)$$

and, analogously,

$$y = \int \sin\alpha \, ds = \frac{ae^{a\alpha}}{a^2+1}(a\sin\alpha - \cos\alpha).$$

Hence, the parametric equations of the logarithmic spiral are

$$\begin{cases} x = \dfrac{ae^{a\alpha}}{a^2+1}(a\cos\alpha + \sin\alpha) \\[2ex] y = \dfrac{ae^{a\alpha}}{a^2+1}(a\sin\alpha - \cos\alpha) \end{cases} . \tag{9.2.2}$$

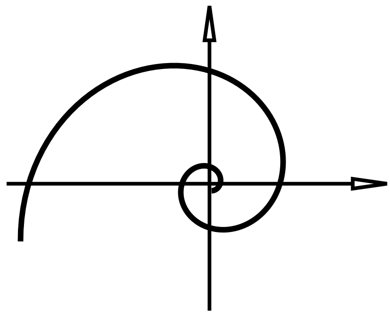


Figure 9.1: The logarithmic spiral

The logarithmic spiral (see figure 9.1) is most conveniently described through its *polar* equation, $\mathbf{r} = \mathbf{r}(\varphi)$. We shall indicate now how can one obtain that equation starting from the parametric equations. First of all, we notice that

$$x^2 + y^2 = \frac{a^2 e^{2a\alpha}}{a^2+1},$$

whence

$$\mathbf{r} = \sqrt{x^2+y^2} = \frac{ae^{a\alpha}}{\sqrt{a^2+1}}.$$

We put $a = \tan\psi$. Then, for the polar angle we get

$$\tan\varphi = \frac{y}{x} = \frac{a\sin\alpha - \cos\alpha}{\sin\alpha + a\cos\alpha} = -\cot(\alpha+\psi),$$

therefore $\varphi = \alpha + \psi + \frac{\pi}{2}$, hence $\alpha = \varphi - \psi - \frac{\pi}{2}$. Thus

$$\mathbf{r} = \sin\psi e^{\varphi-\psi-\frac{\pi}{2}},$$

i.e. the polar equation of the logarithmic spiral is of the form

$$r = C \cdot e^{\varphi}, \tag{9.2.3}$$

where C is a constant.

Cycloidal curves. They correspond to the natural equation

$$\frac{s^2}{a^2} + \frac{R^2}{b^2} = 1, \tag{9.2.4}$$

where a and b are nonvanishing constants. A possibility would be to express R in terms of s and then integrate. However, this would lead to complications, due to the the presence of the square root and the sign ambiguity. We prefer, instead, to introduce a new parameter t, through the relations

$$s = a \sin t, \quad R = b \cos t.$$

It is, then, very easy to find the contingency angle in terms of this new parameter. Indeed, we have

$$\alpha = \int \frac{1}{R} ds = \int \frac{1}{b \cos t} a \cos t \, dt = \frac{at}{b}.$$

We can proceed now with the determination of the coordinates x and y, in terms of the parameter t:

$$x = \int \cos \alpha \, ds = \int \cos \frac{at}{b} \cdot a \cdot \cos t \, dt = \frac{a}{2} \left(\frac{b}{a-b} \sin \frac{(a-b)t}{b} + \frac{b}{a+b} \sin \frac{(a+b)t}{b} \right),$$

$$y = \int \sin \alpha \, ds = \int \sin \frac{at}{b} \cdot a \cdot \cos t \, dt = -\frac{a}{2} \left(\frac{b}{a-b} \cos \frac{(a-b)t}{b} + \frac{b}{a+b} \cos \frac{(a+b)t}{b} \right).$$

The clothoid. This is the curve whose natural equation is

$$R = \frac{a^2}{s}. \tag{9.2.5}$$

Thus, for the clothoid (also known as the *Cornu's spiral*), the radius of curvature is proportional to the *inverse* of the arc length. From this point of view, it is, to some extent, similar to the logarithmic spiral, where the radius of curvature was proportional to the arc length, rather than to its inverse. We include this curve here to show that even in the case of a very simple expression of the curvature in terms of the arc length (in this case the curvature is proportional to the arc length, i.e. it is a linear function), we might not be able to find a parametric representation in terms of elementary functions.

The contingency angle is easily found:

$$\alpha = \frac{s^2}{2a}, \tag{9.2.6}$$

therefore the coordinates are

$$x = \int \cos \frac{s^2}{2a} \, ds, \quad x = \int \sin \frac{s^2}{2a} \, ds. \tag{9.2.7}$$

Unfortunately, it is a well known fact from analysis that the integrals from (9.2.7) cannot be expressed in terms of elementary functions. They carry the name of *Fresnel integrals*, after the French physicists who used them first in his works on optics (the theory of diffraction, to be more specific)[1].

The catenary. For the catenary, the natural equation is

$$R = a + \frac{s^2}{a}, \tag{9.2.8}$$

where a is a non-vanishing constant. Again, instead of integrating directly, to get the contingency angle, we introduce, first, the new parameter t through the relations

$$s = a \tan t, \qquad R = \frac{a}{\cos^2 t}. \tag{9.2.9}$$

Then the contingency angle will be

$$\alpha \equiv \int \frac{1}{R(s)}\, ds = \int \frac{\cos^2 t \cdot a}{a \cdot \cos^2 t}\, dt = t.$$

The coordinates are, now, easy to find in terms on the new parameter:

$$x = \int \cos\alpha ds = \int \frac{a \cos t}{\cos^2 t}\, dt = a \int \frac{1}{\cos t}\, dt = \ln\left(\frac{1 + \sin t}{\cos t}\right),$$

and, analogously,

$$y = a\frac{1}{\cos t}.$$

It is not difficult to check that one can eliminate the parameter t from the previous two equation and one gets the usual explicit equation of the catenary, i.e.

$$y = a \cosh\frac{x}{a}. \tag{9.2.10}$$

[1] According to Gino Loria, in fact, these integrals were first considered by Euler, in 1781.

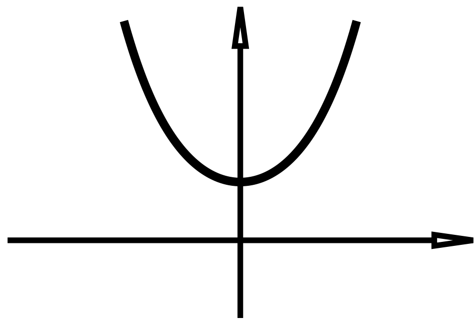


Figure 9.2: The catenary

The involute of the circle. Let us start with the natural equation

$$R^2 = 2as. \tag{9.2.11}$$

We introduce a new parameter, t, such that

$$s = \frac{at^2}{2}, \quad R = at.$$

Then the contingency angle is

$$\alpha = \int \frac{1}{R}\, ds = \int \frac{1}{at} \cdot at\, dt = t,$$

therefore

$$x = \int \cos\alpha\, ds = \int at \cos t\, dt = a(\cos t + t \sin t)$$

and

$$y = \int \sin\alpha\, ds = \int at \sin t\, dt = a(\sin t - t \cos t),$$

which are, indeed, the parametric equations of the involute of the circle.

The tractrix. Finally, we start with the natural equation

$$R^2 + a^2 = a^2 e^{-2s/a}, \tag{9.2.12}$$

where a is a non-vanishing constant. We introduce the parameter t such that

$$e^{-\frac{s}{a}} = \frac{1}{\cos t}, \quad R = a \tan t.$$

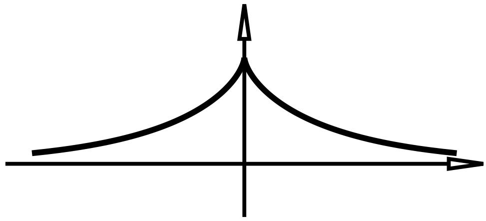


Figure 9.3: The tractrix

Then
$$\alpha = \int \frac{1}{R}\, ds = \int \frac{1}{a \tan t}(-a \tan t)\, dt = -t,$$
therefore
$$x = \int \cos\alpha\, ds = \int \cos t(-a \tan t)\, dt = a \cos t$$
and
$$y = \int \sin\alpha\, ds = \int \sin t \cdot a \cdot \tan t\, dt = a\left(\ln \frac{1+\sin t}{\cos t} - \sin t\right).$$

CHAPTER 10

General theory of surfaces

10.1 Parameterized surfaces (patches)

Definition. A regular parameterized surface (patch) in $\mathbb{R}^3$ is a smooth map $\mathbf{r} : U \to \mathbb{R}^3$, $(u, v) \to \mathbf{r}(u, v)$, where U is a domain (an open, connected subset) in $\mathbb{R}^2$, while $\mathbf{r}$ is subject to

$$\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}} \neq 0. \tag{10.1.1}$$

The condition (10.1.1) is called the *regularity condition*.

A parameterized surface is usually denoted by $(U, \mathbf{r})$, $(U, \mathbf{r} = \mathbf{r}(u, v))$ or just $\mathbf{r} = \mathbf{r}(u, v)$.

Definition. The set $\mathbf{r}(U) \subset \mathbb{R}^3$ is called the *support* of the parameterized surface $(U, \mathbf{r})$.

Remark. Usually, one and the same point of the support of a parameterized surface $(U, \mathbf{r})$ may correspond to several distinct pairs (u, v), since the map $\mathbf{r}$ is not assumed to be injective.

Definition. Two parameterized surfaces $(U, \mathbf{r})$ and $(V, \mathbf{r}_1)$ are called *equivalent* if there is a diffeomorphism $\lambda : U \to V$ such that $\mathbf{r} = \mathbf{r}_1 \circ \lambda$.

Remark. The supports of two equivalent parameterized surfaces always coincide.

Examples. 1. If $U = \mathbb{R}^2$, while $\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$, $\mathbf{a} \times \mathbf{b} \neq 0$, the the support of the parameterized surface is the plane passing through $\mathbf{r}_0$, perpendicular to the vector $\mathbf{a} \times \mathbf{b}$. This parameterized surface is called a *plane*.

2. Let $U = \{(u, v) \in \mathbb{R}^2 | \pi/2 < u < \pi/2, 0 < v < 2\pi\}$ and

$$\mathbf{r}(u, v) = (R \cos u \cos v, R \cos u \sin v, R \sin u).$$

The support of this parameterized surface is the sphere of radius R, centred at the origin of $\mathbb{R}^3$, with a meridian removed. The parameters u and v are analogues of the geographical coordinates.

3. Let $U = \mathbb{R}^2$, $\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v^3\mathbf{b}$, $\mathbf{a} \times \mathbf{b} \neq 0$. The support of this parameterized surface is identical to the support of the parameterized surface from the example 1), although the two surfaces are not equivalent, since the map $(u, v) \to (u, v^3)$ is not a diffeomorphism.

10.2 Surfaces

Definition. A subset $S \subset \mathbb{R}^3$ is called a *regular surface* if each point $a \in S$ has an open neighbourhood W in S such that there is a parameterized surface $(U, \mathbf{r})$ with $\mathbf{r}(U) = W$, while the map $\mathbf{r} : U \to W$ is a homeomorphism. The pair is called a *local parameterization* of the surface S around the point a, while the support $\mathbf{r}(U)$ is called the *domain* of the parameterization. A surface S which has a *global* parameterization (i.e. a local parameterization $(U, \mathbf{r})$ for which $\mathbf{r}(U) = S$) is called a *simple* surface.

10.2.1 Representations of surfaces

The same kind of representations we used for the case of curves are, equally, available for surfaces.

Parametrical representation. If S is a surface and $(U, \mathbf{r})$ is a local parameterization of S, then, if $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, then the equations

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \quad (u, v) \in U,$$

are called the *parametric equations* of the surface. We would like to emphasize, once more, that these are only *local* equations, they cannot be used to describe all the points of the surface, unless we are dealing with a global parameterization of a simple surface.

Explicit representation. If $f : U \to \mathbb{R}$ is a smooth function, where $U \subset \mathbb{R}^2$ is a domain, then its graph, $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$ is a simple surface. Indeed, we have the global parameterization $\mathbf{r} : U \to \mathbb{R}^3$, $\mathbf{r}(u, v) = (u, v, f(u, v))$.

Implicit representation. Let $F : V \to \mathbb{R}$, with $V \subset \mathbb{R}^3$ an open set, be a smooth function. We denote by $S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$ the 0-level set of F. If, at each point of S, the vector

$$\operatorname{grad} F = \left\{\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\}$$

is different from zero, then S is a surface. Indeed, for instance, at $(x_0, y_0, z_0) \in S$, we have $F'_z(x_0, y_0, z_0) \neq 0$, then, from the implicit function theorem, there is an open (in the topology of $\mathbb{R}^3$) neighbourhood M of (x_0, y_0, z_0) such that the set $M \cap S$ (which is an open neighbourhood of (x_0, y_0, z_0), this time in the topology of S) is the graph of a smooth function $z = f(x, y)$, therefore, according to the previous paragraph, there is a local parameterization of S around the point (x_0, y_0, z_0). We note that this parameterization is global for $M \cap S$ but, generally, it is not so for the entire S. Even if F'_z is different from zero on the entire S, it is still not sure that the function f can be defined globally. Thus, in general, unlike that case of the surfaces given explicitly, the implicitly given surfaces are *not* simple.

Examples. 1. The plane Π passing through $\mathbf{r}_0$ and having the direction given by the vectors $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{a} \times \mathbf{b} \neq 0$ is a simple surface with the global parameterization $\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$. In projection on the coordinate axes, the parametric equations of the plane will be

$$\begin{cases} x = x_0 + ua_x + vb_x \\ y = y_0 + ua_y + vb_y \\ z = z_0 + ua_z + vb_z \end{cases}, \quad (u, v) \in \mathbb{R}^2.$$

2. *Revolution surfaces*
Let C be a curve in the plane xOz, which does not intersect the z-axis, and

S – the subset of $\mathbb{R}^3$ obtained by rotating C around the z-axis. Let v be the rotation angle of the plane xOz and a' – the point of S obtained by rotating the point $a \in C$ by an angle v_0. Let $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$ be a local parameterization of the curve C around the point a, $\boldsymbol{\rho}(t) = (x(t), z(t))$. The we get the local parameterization os S around a',

$$\mathbf{r}(t, v) = (x(t)\cos(v + v_0), x(t)\sin(v + v_0), z(t)),$$

defined on the domain $U = I \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

3. *The sphere.* Let $F : \mathbb{R}^3 \to \mathbb{R}$, $F(x, y, z) = x^2 + y^2 + z^2 - R^2$. F is, obviously, a smooth function and its 0-level set

$$S_R^2 = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$$

is the sphere of radius R, with the centre at the origin. The gradient of F is

$$\operatorname{grad} F = \{2x, 2y, 2z\}$$

and, obviously, it does not vanish on the sphere S_R^2, therefore this sphere is a regular surface. Note that S_R^2 is not simple since it is compact subset of $\mathbb{R}^3$, therefore it cannot be homeomorphic to an open subset of $\mathbb{R}^2$, which is not compact.

4. *The torus.* We choose now $F : \mathbb{R}^3 \to \mathbb{R}$,

$$F(x, y, z) = (\sqrt{x^2 + y^2} - a)^2 + z^2 - b^2, \quad 0 < b < a.$$

Its 0-level set,

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 | F(x, y, z) = 0\}$$

is called the *2-dimensional torus*. If we compute the derivatives of F with respect to the coordinates, we get

$$\begin{cases} F_x' = 2\left(\sqrt{x^2 + y^2} - a\right)\frac{x}{\sqrt{x^2+y^2}} \\ F_y' = 2\left(\sqrt{x^2 + y^2} - a\right)\frac{y}{\sqrt{x^2+y^2}} \\ F_z' = 2z \end{cases},$$

therefore the gradient of F vanishes iff

$$\begin{cases} x = y = z = 0 \quad \text{or} \\ x^2 + y^2 = 0, y = 0, z = 0 \quad \text{or} \\ x = 0, x^2 + y^2 = a^2, z = 0 \quad \text{or} \\ x^2 + y^2 = a^2, z = 0 \end{cases}.$$

It is easy to check that grad F is different from zero on T^2, therefore the torus is a surface (again, it is compact, therefore it cannot be simple).

The torus can also be obtained by rotating the circle $(x-a)^2 + z^2 = b^2$ (lying in the plane xOz) around the axis Oz.

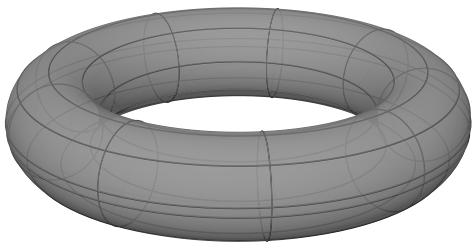


Figure 10.1: The torus

10.3 The equivalence of local parameterizations

Definition. Let S be a surface, $(U, \mathbf{r})$ – a local parameterization and $W = \mathbf{r}(U)$. Then the map $\mathbf{r}^{-1} : W \to U$ is a bijection, associating to each point of W a pair of real numbers $(u, v) \in U$. This correspondence is called a *curvilinear coordinate system* on S or a *chart* on S.

Remark. We ought to mention here that in many books the fundamental objects used to describe a surface are not the local parameterizations, but the charts. Of course, the two approaches are completely equivalent, but they came from different directions. The description of surfaces by using local parameterizations has, probably, the origin in mathematical analysis, where the surfaces are thought off either as images of functions or as graphs of functions, in any case, the central objects is *the function*. The “charts’s approach” has the origin in chartography. In fact, assigning a local chart to a surface simply means to apply that surface on a portion of a plane, in other words, what is meant is the construction of a "map" of the surface and, actually, in the advanced differential geometry, a collection of charts that cover the entire surface is called an *atlas*, exactly as happens in chartography.

Theorem 10.1. *(of the parameters' change) Let* $(U, \mathbf{r})$ *and* $(U_1, \mathbf{r}_1)$ *be two local parameterizations of a surface* S *and* $\mathbf{r}(U) = \mathbf{r}(U_1)$. *Then there is a diffeomorphism* $\lambda : U \to U_1$ *such that* $\mathbf{r} = \mathbf{r}_1 \circ \lambda$. *The diffeomorphism* λ *is called a* parameters' change.

Before proving the theorem, some remarks are in order. If a change of parameters λ does exist, then, from the relation $\mathbf{r} = \mathbf{r}_1 \circ \lambda$, we should have, of course, $\lambda = \mathbf{r}_1^{-1} \circ \mathbf{r}$. The real difficulty is to show that λ and its inverse are smooth. Although $\mathbf{r}$ is smooth, $\mathbf{r}^{-1}$ is not[1], because its domain, $\mathbf{r}_1(U_1)$, is not an open subset of the Euclidean space $\mathbb{R}^3$. We will show, however, in the following lemma, that, locally, $\mathbf{r}_1^{-1}$ is the *restriction* of a smooth function. We ought to emphasize, however, that this representation of $\mathbf{r}_1^{-1}$ is only local: in general, $\mathbf{r}_1^{-1}$ cannot be written, globally, as the restriction of a single smooth function, defined on an open set (with respect to the topology of $\mathbb{R}^3$), which contains the set $\mathbf{r}_1(U_1)$.

Lemma. *Let* $(U, \mathbf{r})$ *be a local parameterization of the surface* S, $\mathbf{r}(U) = W$ *and* $\mathbf{r}^{-1} : W \to U$ *– the inverse map. Then, for each point* $a \in W$ *there is an open set (in the topology of* $\mathbb{R}^3$*)* $B \ni a$ *and a smooth map* $G : B \to U$ *such that* $\mathbf{r}^{-1}|_{W \cap B} = G|_{W \cap B}$.

Proof of the lemma Let $\mathbf{r}(u, v) = (f_1(u, v), f_2(u, v).f_3(u, v))$ and $a = \mathbf{r}(u_0, v_0)$. Due to the regularity of $\mathbf{r}$, the Jacobi matrix

$$\begin{pmatrix} f'_{1u} & f'_{1v} \\ f'_{2u} & f'_{2v} \\ f'_{3u} & f'_{3v} \end{pmatrix}$$

has the rank two. Without restricting the generality, we may assume that

$$\begin{vmatrix} f'_{1u} & f'_{1v} \\ f'_{2u} & f'_{2v} \end{vmatrix} \neq 0.$$

Then, from the inverse function's theorem for the map

$$f : (u, v) \longrightarrow (x = f_1(u, v), y = f_2(u, v)),$$

[1] not in the classical sense, at least

there is an open neighbourhood V of the point (u_0, v_0) in U and an open neighbourhood $\widetilde{V}$ of the point $(x_0 = f_1(u_0, v_0), y_0 = f_2(u_0, v_0))$ in the plane xOy such that $f : V \to \widetilde{V}$ is a diffeomorphism. Since the map $\mathbf{r} : U \to W$ is a homeomorphism, $\mathbf{r}(V)$ is an open neighbourhood in S of the point $a = \mathbf{r}(u_0, v_0)$, therefore, in $\mathbb{R}^3$ there is an open neighbourhood B of the point a such that $\mathbf{r}(V) = B \cap S = B \cap W$. Let $p : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \to (x, y)$ be the orthogonal projection on the coordinate plane xOy. We are going to show that the map $G = (f^{-1} \circ p)|_B : B \to U$ is the map searched for. Indeed, G is smooth, as a composition of smooth maps. Moreover, to each point (x, y, z) from $B \cap W$ corresponds a single point $(u, v) = \mathbf{r}^{-1}(x, y, z)$ from V, and to each point $(x, y) \in \widetilde{V}$ – the point $(u, v) = f^{-1}(x, y)$ from V. Thus, for the points $(x, y, z) \in B \cap W$, we have

$$\mathbf{r}^{-1}(x, y, z) = f^{-1}(x, y) = f^{-1}(p(x, y, z)) = G(x, y, z).$$

$\square$

Proof of the theorem Let $(U, \mathbf{r})$ and $(U_1, \mathbf{r}_1)$ – two local parameterizations of the surface S such that $\mathbf{r}(U) = \mathbf{r}_1(U_1) = W$. We consider the map $\lambda = \mathbf{r}_1^{-1} \circ \mathbf{r} : U \to U_1$. Since $\mathbf{r}_1 : U_1 \to W$ is a homeomorphism, the same is true for $\mathbf{r}_1^{-1} : W \to U_1$ and, thus, λ is a homeomorphism, as a composition of two homeomorphisms. We have to prove only that λ and λ^{-1} are smooth. To prove the smoothness of λ este it is enough to prove that each point $(u_0, v_0) \in U$ has an open neighbourhood $V \subset U$ such that $\lambda|_V$ is smooth. We apply the previous lemma to the parameterization $(U_1, \mathbf{r}_1)$ at a point $a = \mathbf{r}_1(\lambda(u_0, v_0))$. Let $G : B \to U_1$ be the smooth map for which $\mathbf{r}_1^{-1}|_{B \cap W} = G|_{B \cap W}$ and $V = \mathbf{r}^{-1}(B \cap W)$. Then $\lambda|_V = \mathbf{r}_1^{-1} \circ \mathbf{r}|_V = (G \circ \mathbf{r})|_V$ and, therefore, $\lambda|_V$ is smooth, as a restriction of a smooth map. The smoothness of λ^{-1} is proved in the same way, replacing the parameterization $\mathbf{r}_1$ by the parameterization $\mathbf{r}$. $\square$

Locally, each surface is the support of a parameterized surface. The converse is not true, i.e. the support of an arbitrary parameterized surface is *not* a surface. Still, if we choose an arbitrary parameterized surface, by restricting the domain, we can obtain a parameterized surface whose support is a regular surface. Thus, we have:

Theorem 10.2. *Let $(U, \mathbf{r})$ be a regular parameterized surface. Then each point $(u_0, v_0) \in U$ has an open neighbourhood $V \subset U$ such that the $\mathbf{r}(V)$ is a simple surface in $\mathbb{R}^3$, for which the pair $(V, \mathbf{r}|_V)$ is a global parameterization.*

Proof The only extra condition we have to impose on V is that the map $\mathbf{r}|_V : V \to \mathbf{r}(V)$ to be a homeomorphism. Let $\mathbf{r}(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$. Without restricting the generality, we may assume that the Jacobian of the map $f : (u, v) \to (x = f_1(u, v), y = f_2(u, v))$ is different from zero at (u_0, v_0). Then, from the inverse function's theorem, there is an open neighbourhood $V \subset U$ of the point (u_0, v_0) and an open neighbourhood $\widetilde{V}$ of the point $(x_0, y_0) = f(u_0, v_0)$ such that the map $F : V \to \widetilde{V}$ is a diffeomorphism. We shall prove first the injectivity of the map $\mathbf{r}|_V : V \to \mathbf{r}(V)$. Let $(u_1, v_1), (u_2, v_2) \in V$, such that $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$. Then, in particular,

$$f_1(u_1, v_1) = f_1(u_2, v_2) \quad \text{și} \quad f_2(u_1, v_1) = f_2(u_2, v_2),$$

i.e. $f(u_1, v_1) = f(u_2, v_2)$. Or, f is diffeomorphism, so it is, in particular, an injective map, therefore $(u_1, v_1) = (u_2, v_2)$. The map $\mathbf{r} : U \to \mathbb{R}^3$ is continuous, so it is continuous also its restriction $\mathbf{r}|_V : V \to \mathbb{R}^3$. Since on $\mathbf{r}(V)$ we take the subspace topology, the map $\mathbf{r}|_V : V \to \mathbf{r}(V)$ is, also, continuous. To prove the continuity of the inverse map, we note that its is a composition of the following continuous maps: $(x, y, z) \in \mathbf{r}(V) \to (x, y) \in \widetilde{V} \to (u, v) = f^{-1}(x, y) \in V$, as seen in the proof of the lemma. $\square$

10.4 Curves on a surface

We shall say that a smooth parameterized curve $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$ lies on a surface S if its support $\boldsymbol{\rho}(I)$ is included in S. It is easy to describe the parameterized curves whose support is contained into the domain of a parameterization $(U, \mathbf{r})$ of the surface S.

Theorem 10.3. *Let $(U, \mathbf{r})$ be a parameterization of the surface S and $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$ a smooth parameterized curve whose support is included in $\mathbf{r}(U)$. Then there is a single smooth parameterized curve $(I, \tilde{\boldsymbol{\rho}})$ on U such that*

$$\boldsymbol{\rho}(t) \equiv \boldsymbol{\rho}(\tilde{\boldsymbol{\rho}}(t)). \tag{10.4.1}$$

Conversely, any smooth parameterized curve $\tilde{\boldsymbol{\rho}}$ on U defines, through the formula (10.4.1), a smooth parameterized curve on $\mathbf{r}(U)$. The regularity of $\boldsymbol{\rho}$ at t is equivalent to the regularity of $\tilde{\boldsymbol{\rho}}$ at t.

Proof Since the map $\mathbf{r} : U \to \mathbf{r}(U)$ is a homeomorphism, while $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$, from the formula (10.4.1), we can obtain $\tilde{\boldsymbol{\rho}}$, by putting

$$\tilde{\boldsymbol{\rho}} = \mathbf{r}^{-1} \circ \boldsymbol{\rho}.$$

Clearly, $\tilde{\boldsymbol{\rho}}$ is continuous, as a composition of two continuous maps. We shall check now that $\tilde{\boldsymbol{\rho}}$ is, actually, smooth. Let $t \in I$, then $\boldsymbol{\rho}(t) \in \mathbf{r}(U)$. According to the lemma from the previous paragraph there is an open neighbourhood B of the point $\boldsymbol{\rho}(t)$ in $\mathbb{R}^3$ and a smooth map $G : B \to U$ such that $\mathbf{r}^{-1}|_{B\cap\mathbf{r}(U)} = G|_{B\cap\mathbf{r}(U)}$. Therefore, the map $\tilde{\boldsymbol{\rho}}$, can be represented, in the neighbourhood of the point t, as a composition $G \circ \boldsymbol{\rho}$ of smooth maps and, thus, it is smooth. The converse affirmation can be proved even simpler, because we have

$$\boldsymbol{\rho} = \mathbf{r} \circ \tilde{\boldsymbol{\rho}}$$

and, as $\mathbf{r}$ and $\tilde{\boldsymbol{\rho}}$ are smooth, so is $\boldsymbol{\rho}$.

To verify the equivalence of the regularity conditions for $\boldsymbol{\rho}$ and $\tilde{\boldsymbol{\rho}}$, we consider the components of the path $\tilde{\boldsymbol{\rho}}$:

$$\tilde{\boldsymbol{\rho}} = (u(t), v(t)).$$

Then the equality (10.4.1) becomes

$$\boldsymbol{\rho}(t) = \mathbf{r}(u(t), v(t)).$$

Differentiating this relation, we get

$$\boldsymbol{\rho}'(t) = \mathbf{r}'_{\mathbf{u}} \cdot u'(t) + \mathbf{r}'_{\mathbf{v}} \cdot v'(t).$$

Since the vectors $\mathbf{r}'_{\mathbf{u}}$ and $\mathbf{r}'_{\mathbf{v}}$ are not colinear (because the surface, as always, is supposed to be regular), from the previous relation it follows that $\boldsymbol{\rho}'(t) = 0$ iff $u'(t) = 0$ and $v'(t) = 0$, i.e. iff $\tilde{\boldsymbol{\rho}}'(t) = 0$. □

Definition. The parameterized curve $\tilde{\boldsymbol{\rho}}(t)$ on the domain U is called the *local representation* of the parameterized curve $\boldsymbol{\rho}(t)$ in the local parameterization $(U, \mathbf{r})$, while the equations

$$\begin{cases} u = u(t) \\ v = v(t) \end{cases}$$

are called the *local equations* of $\boldsymbol{\rho}(t)$ in the considered parameterization.

Example 10.1. Let $(U, \mathbf{r} = \mathbf{r}(u, v))$ a local parameterization S and $(u_0, v_0) \in U$. We consider, in $\mathbf{r}(U) \subset S$ the paths defined by the local equations

$$\begin{cases} u = u_0 + t \\ v = v_0 \end{cases} \tag{10.4.2}$$

and

$$\begin{cases} u = u_0 \\ v = v_0 + t \end{cases} \tag{10.4.3}$$

It is easy to see that the supports of these paths lie, indeed, on S. Through each $\mathbf{r}(u_0, v_0) \in \mathbf{r}(U)$ pass exactly two such curves, one of each kind. The parameterized curves are called *coordinate lines or coordinate curves* on the surface S, in the local parameterization $(U, \mathbf{r})$.

10.5 The tangent vector space, the tangent plane and the normal to a surface

Let us denote, for any $a \in \mathbb{R}^3$, by R^3_a the space of bound vectors with the origin at a. This is, obviously, a 3-dimensional vector space, naturally isomorphic to $\mathbb{R}^3$.

Definition 10.1. A vector $\mathbf{h} \in \mathbb{R}^3_a$ is called a *tangent vector* to the surface S at the point a if there is a parameterized curve $(I, \boldsymbol{\rho}(t))$ on S and a $t_0 \in I$ such that $\boldsymbol{\rho}(t_0) = a$ and $\boldsymbol{\rho}'(t_0) = \mathbf{h}$. Thus, a tangent vector to a surface is just a tangent vector to a curve on the surface.

We shall denote by $T_a S$ the set of all the tangent vectors to the surface S at $a \in S$. The following lemma is trivial, but it plays an important role in the sequel.

Lemma. *Let $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$ be a parameterized curve on S, given by the local equations $u = u(t)$, $v = v(t)$, with respect to a local parameterization $(U, \mathbf{r})$ of S. Then we have the relation*

$$\boldsymbol{\rho}'(t) = u'(t)\mathbf{r}'_{\mathbf{u}}(u(t), v(t)) + v'(t)\mathbf{r}'_{\mathbf{v}}(u(t, v(t)). \tag{10.5.1}$$

Proof We just differentiate the relation $\boldsymbol{\rho}(t) = \mathbf{r}(u(t), v(t))$ with respect to t. $\square$

Theorem 10.4. *The set $T_a S$ is a 2-dimensional vector subspace of $\mathbb{R}^3$. If $(U, \mathbf{r})$ is a local parameterization of S, while $a = \mathbf{r}(u_0, v_0)$, then the vectors $\mathbf{r}'_{\mathbf{u}}(u_0, v_0)$ and $\mathbf{r}'_{\mathbf{v}}(u_0, v_0)$ make up a basis of this subspace, called the* natural basis *or the* coordinate basis *of the tangent space.*

Proof Let $(U, \mathbf{r})$ be a local parameterization of S, with $a = \mathbf{r}(u_0, v_0)$. if the parameterized curve $(I, \boldsymbol{\rho}(t))$ is on the surface and $\boldsymbol{\rho}(t_0) = a$, then, restricting, if necessary, the interval I, we may assume that $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$, while its local equations in this parameterization of the surface are $u = u(t), v = v(t)$. Then, from the formula (10.5.1), it follows that

$$\boldsymbol{\rho}'(t_0) = u'(t_0)\mathbf{r}_\mathbf{u}'(u_0, v_0) + v'(t_0)\mathbf{r}_\mathbf{v}'(u_0, v_0).$$

Conversely, any vector of the form

$$\mathbf{h} = \alpha\mathbf{r}_\mathbf{u}'(u_0, v_0) + \beta\mathbf{r}_\mathbf{v}'(u_0, v_0)$$

is tangent to the parameterization curve given by the local equations

$$\begin{cases} u = u_0 + \alpha t \\ v = v_0 + \beta t \end{cases},$$

which is a curve on S, passing through a for t_0, therefore $\mathbf{h} \in T_a S$. □

The vector space $T_a S$ is called the *tangent space* to S at a. As we mentioned before, $\mathbb{R}_a^3$ is naturally[2] isomorphic to $\mathbb{R}^3$. Based on this isomorphism, we may consider, when is convenient, that $T_a S$ is, in fact, a subspace of $\mathbb{R}^3$ rather than of $\mathbb{R}_a^3$. In this case, $T_a S$ is a *vectorial plane*, in the sense that it passes through the origin of $\mathbb{R}_a^3$; then the plane passing through a and having $T_a S$ as directing plane (i.e. the plane parallel to $T_a S$, passing through a), is called the tangent *plane* to S at the point a and it is denoted by $\Pi_a S$.

If $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ is a local parameterization of the surface S and $a = \mathbf{r}(u_0, v_0) = (x_0, y_0, z_0) \in S$, then, clearly, the equation of the tangent plane of S at a should be

$$\begin{vmatrix} X - x_0 & Y - y_0 & Z - z_0 \\ x_u' & y_u' & z_u' \\ x_v' & y_v' & z_v' \end{vmatrix} = 0.$$

Let, now, $(U, \mathbf{r} = \mathbf{r}(u, v))$ be a parameterization of S and $(u_0, v_0) \in U$. If we modify the arguments by $\Delta u = \alpha\Delta t$, $\Delta v = \beta\Delta t$, with $\alpha, \beta \in \mathbb{R}$ fixed, the Taylor's formula gives:

$$\mathbf{r}(u_0 + \Delta u, v_0 + \Delta v) = \mathbf{r}(u_0, v_0) + \Delta t \cdot (\alpha\mathbf{r}_\mathbf{u}'(u_0, v_0) + \beta\mathbf{r}_\mathbf{v}'(u_0, v_0)) + \Delta t \cdot \boldsymbol{\varepsilon},$$

[2]In this context, *naturally* means that there is an isomorphism which is independent of the choice of bases in the two vector spaces.

with $\lim\limits_{\Delta t\to 0} \boldsymbol{\varepsilon} = 0$. Using this formula, we shall give another characterization of the tangent plane. Let Π be a plane in $\mathbb{R}^3$, passing through $a = \mathbf{r}(u_0, v_0)$, d – the distance from the point $\Delta a = \mathbf{r}(u_0 + \Delta u, v_0 + \Delta v)$ to the plane Π, and h – the distance between the points a and Δa.

Theorem 10.5. *The plane Π is the tangent plane to the surface S at the point a iff for any modification of the arguments of the form $\Delta u = \alpha\Delta t$, $\Delta v = \beta\Delta t$, with $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$, we have*

$$\lim_{\Delta t\to 0} \frac{d}{h} = 0, \tag{10.5.2}$$

i.e. the plane and the surface have a first order contact at a.

Proof Let $\mathbf{n}$ be the versor of the normal to the plane Π, $\Delta\mathbf{r} = \mathbf{r}(u_0 + \Delta u, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$. Then $d = \Delta\mathbf{r} \cdot \mathbf{n}$, $h = \|\Delta\mathbf{r}\|$. Replacing $\Delta\mathbf{r}$ by its expression, we get:

$$\lim_{\Delta t\to 0} \frac{d}{h} = \lim_{\Delta t\to 0} \frac{\Delta t \cdot (\alpha\mathbf{r}_\mathbf{u}'(u_0, v_0) + \beta\mathbf{r}_\mathbf{v}'(u_0, v_0) + \boldsymbol{\varepsilon}) \cdot \mathbf{n}}{\|\alpha\mathbf{r}_\mathbf{u}'(u_0, v_0) + \beta\mathbf{r}_\mathbf{v}'(u_0, v_0) + \boldsymbol{\varepsilon}\|} = \pm\frac{(\alpha\mathbf{r}_\mathbf{u}'(u_0, v_0) + \beta\mathbf{r}_\mathbf{v}'(u_0, v_0)) \cdot \mathbf{n}}{\|\alpha\mathbf{r}_\mathbf{u}'(u_0, v_0) + \beta\mathbf{r}_\mathbf{v}'(u_0, v_0)\|},$$

therefore

$$\lim_{\Delta t\to 0} = 0 \iff (\alpha\mathbf{r}_\mathbf{u}'(u_0, v_0) + \beta\mathbf{r}_\mathbf{v}'(u_0, v_0)) \cdot \mathbf{n} = 0. \tag{10.5.3}$$

Necessity. If Π is the tangent plane, then the vectors $\mathbf{r}_\mathbf{u}'$ and $\mathbf{r}_\mathbf{v}'$, as directing vectors of Π, are orthogonal to $\mathbf{n}$, therefore the relation (10.5.3) is fulfilled.
Sufficiency. Let us assume, now, that (10.5.3) is fulfilled. Choosing $\alpha = 1$, $\beta = 0$, we get $\mathbf{r}_\mathbf{u}' \cdot \mathbf{n} = 0$. In the same way, for $\alpha = 0$, $\beta = 1$, one obtains $\mathbf{r}_\mathbf{v}' \times \mathbf{n} = 0$. Thus, $\mathbf{n}$ is orthogonal to the tangent plane, i.e. Π is the tangent plane. □

Definition 10.2. The straight line passing through a point of a surface, perpendicular to the tangent plane to the surface at that point, is called the *normal to the surface* at the considered point.

Thus, if $(U, \mathbf{r})$ is a parameterization of the surface around a point $a = \mathbf{r}(u_0, v_0) = (x_0, y_0, z_0) \in S$, then the a directing vector of the normal will be $\mathbf{r}_\mathbf{u}'(u_0, v_0) \times \mathbf{r}_\mathbf{v}'(u_0, v_0)$, which means that the equations of the normal at the point a will be given by:

$$\frac{X-x_0}{\begin{vmatrix} y'_u(u_0,v_0) & z'_u(u_0,v_0) \\ y'_v(u_0,v_0) & z'_v(u_0,v_0) \end{vmatrix}} = \frac{Y-y_0}{\begin{vmatrix} z'_u(u_0,v_0) & x'_u(u_0,v_0) \\ z'_v(u_0,v_0) & x'_v(u_0,v_0) \end{vmatrix}} = \frac{Z-z_0}{\begin{vmatrix} x'_u(u_0,v_0) & y'_u(u_0,v_0) \\ x'_v(u_0,v_0) & y'_v(u_0,v_0) \end{vmatrix}}. \tag{10.5.4}$$

To construct the tangent plane and the normal to a surface given in an implicit representation, the following result is very useful.

Theorem 10.6. *At the point (x_0, y_0, z_0) of the surface given through the equation*

$$F(x, y, z) = 0$$

the vector grad $F_0 = \{F'_x(x_0, y_0, z_0), F'_y(x_0, y_0, z_0), F'_z(x_0, y_0, z_0)\}$ *is perpendicular to the tangent plane to the surface at this point.*

Proof Let $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ be a local parameterization of the surface around the point $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$. Then we have the identity

$$F(x(u, v), y(u, v), z(u, v)) = 0,$$

whence, by differentiation, we get

$$\begin{cases} \underline{0} = F'_u = F'_x \cdot x'_u + F'_y \cdot y'_u + F'_z \cdot z'_u \equiv \underline{\text{grad}\, F \cdot \mathbf{r}'_{\mathbf{u}}} \\ \underline{0} = F'_v = F'_x \cdot x'_v + F'_y \cdot y'_v + F'_z \cdot z'_v \equiv \underline{\text{grad}\, F \cdot \mathbf{r}'_{\mathbf{v}}} \end{cases},$$

i.e. grad $F \perp L(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}) \equiv T_{(x_0,y_0,z_0)}S$. □

Consequence 1. *The equation of the tangent plane to the surface given by the implicit equation $F(x, y, z) = 0$ at the point (x_0, y_0, z_0) has the form*

$$(X - x_0)F'_x(x_0, y_0, z_0) + (Y - y_0)F'_y(x_0, y_0, z_0) + (Z - z_0)F'_z(x_0, y_0, z_0) = 0,$$

while the equations of the normal to the surface at the same point are

$$\frac{X - x_0}{F'_x(x_0, y_0, z_0)} = \frac{Y - y_0}{F'_y(x_0, y_0, z_0)} = \frac{Z - z_0}{F'_z(x_0, y_0, z_0}.$$

Consequence 2. *For any point a of the sphere S^2_R the tangent space $T_a S^2_R$ is orthogonal to the radius vector of the point a.*

Proof The sphere S_R^2 can be described by the equation

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - R^2 = 0,$$

whence it follows

$$\text{grad } F = 2\{x, y, z\} = 2\mathbf{a},$$

therefore the radius vector is parallel to the gradient of the function F, hence it is perpendicular to the tangent space. □

10.6 The orientation of surfaces

Definition 10.3. An *orientation* of a surface S is a choice of an orientation in each tangent space $T_a S$, i.e. a choice of the unit normal vector of $T_a S$, $\mathbf{n}(a)$. It is assumed, in this context, that the map $\mathbf{n} : S \to \mathbb{R}^3$, $a \to \mathbf{n}(a)$ is continuous. The surfaces on which is is possible to define an orientation are called *orientable*, while those on which an orientation has been already chosen – *oriented*.

Examples. a) We can define an orientation on the sphere S_R^2 using the versor of the exterior normal. It is not difficult to see that, if $\mathbf{a}$ is the radius point of the point $a \in S_R^2$, then $\mathbf{n}(a) = \frac{1}{R}\mathbf{a}$. Therefore, the map

$$\mathbf{n} : S_R^2 \to \mathbb{R}^3,$$

defining the orientation of the sphere can be represented as a composition of continuous maps:

$$S_R^2 \xrightarrow{i} \mathbb{R}^3 \xrightarrow{\frac{1}{R}} \mathbb{R}^3 : a \longrightarrow \mathbf{a} \longrightarrow \frac{1}{R}\mathbf{a}.$$

b) Let S be a simple surface, with the global parameterization $(U, \mathbf{r})$. This surface can be oriented by using the vector field

$$\mathbf{n}(u, v) = \frac{\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}}{\|\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}\|}.$$

c) Let S be a surface given by the implicit equation $F(x, y, z) = 0$. Then the surface can be oriented through the gradient vector field:

$$\mathbf{n}(x, y, z) = \frac{\text{grad } F}{\|\text{ grad } F\|}.$$

If the orientation of a surface S is given by the vector function (vector field) $\mathbf{n}(a)$, then the vector field $-\mathbf{n}(a)$ also defines an orientation on S, called the *opposite* orientation of the orientation given by $\mathbf{n}$. If the orientable surface S is connected, then each orientation of S should coincide to one of the two orientations just mentioned. Indeed, if $\mathbf{N}(a)$ is an orientation of the surface S, then we must have $\mathbf{N}(a) = \lambda\, \mathbf{n}(a)$, where λ is a continuous function on S, taking values into the finite set $\{-1, 1\}$, therefore, if S is connected, λ has to be a constant function. Thus, a connected orientable surface has only two distinct orientations. Of course, if the surface is not connected, there are more orientations, corresponding to different combinations of the two possible orientations on each connected component of the surface.

Remark. Not any surface is orientable. We consider, for instance, the support of the parameterized surface

$$\mathbf{r}(u, v) = (\cos u + v \cos \frac{u}{2} \cos u, \sin u + v \cos \frac{u}{2} \sin u, v \sin \frac{u}{2}),$$

with $u, v \in \mathbb{R}$ (the *Möbius's band*, see the next figure). It will be shown bellow that S is not orientable. (It has a single side: it is possible to move continuously the origin of the unit normal along a close path on S such that, after the "trip", the unit normal will change into its opposite.) Notice that S is not simple, as it might seem, because $\mathbf{r}$ is not a parameterization, since it is not a homeomorphism on the image.

The non-orientability of the Möbius's band. We consider two local parameterizations to describe the Möbius's band:

$$\mathbf{r} : T = \left\{(s, t) \mid -\frac{1}{2} < s < \frac{1}{2},\, 0 < t < 2\pi\right\},$$

$$\mathbf{r}(s, t) = \left(\cos t \left(1 + s \cos \frac{t}{2}\right), \sin t \left(1 + s \cos \frac{t}{2}\right), s \sin \frac{t}{2}\right)$$

and

$$\boldsymbol{\rho} : V = \left\{(u, v) \mid -\frac{1}{2} < u < \frac{1}{2},\, -\pi < v < \pi\right\},$$

$$\boldsymbol{\rho}(u, v) = \left(\cos v \left(1 + u \cos \frac{v}{2}\right), \sin v \left(1 + u \cos \frac{v}{2}\right), u \sin \frac{v}{2}\right)$$

It is not difficult to see that the domain of the diffeomorphism (parameters' change) $\Phi = \boldsymbol{\rho}^{-1} \circ \mathbf{r}$ is the set T^*, equal to T, with the segment $t = \pi$ removed. We can write Φ explicitly as $\Phi(s, t) \equiv (u, v) = (\varphi_1(s, t), \varphi_2(s, t))$, where

$$u = \varphi_1(s, t) = s, \quad \forall (s, t) \in T^*$$

and

$$v = \varphi_2(s,t) = \begin{cases} t & \text{if} \quad (s,t) \in T^*, 0 < t < \pi \\ \pi - t & \text{if} \quad (s,t) \in T^*, \pi < t < 2\pi \end{cases}.$$

The Jacobi matrix of the map Φ is, as one can readily see,

$$J(\Phi)(s,t) \equiv \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 0 < t < \pi \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \pi < t < 2\pi \end{cases}$$

The supports of $\mathbf{r}$ and $\boldsymbol{\rho}$ are, clearly, orientable, as simple surfaces. However, admitting that the Möbius's band is orientable, the two local parameterizations do not define the same orientation on it, since, as one can see from the computation we made before, the determinant of the Jacobi matrix of the parameters' change is not always positive.

On the other hand, as our surface is connected, if it is orientable, it can have only two distinct orientations, in other words, any local parameterization of S should be positively equivalent either to $\mathbf{r}$ or to $\boldsymbol{\rho}$.

Let us suppose, now, that S is orientable. This means that there is a family of local parameterizations $\mathbf{r}_1, \mathbf{r}_2, \ldots$, which are pairwise positively equivalent and their supports cover S. We may assume, without restricting the generality, that on the support of $\mathbf{r}$ all this parameterizations are positively equivalent to $\mathbf{r}$. There should

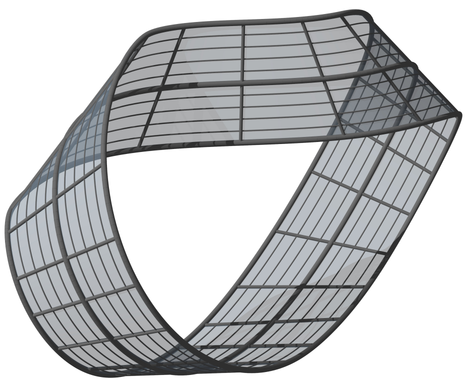


Figure 10.2: The Möbius's band

be a local parameterization in the family such that its support intersects the segment

$t = 0$. We shall suppose that this parameterization is $\mathbf{r}_1$. We may assume that the support of $\mathbf{r}_1$ is included into the support of $\boldsymbol{\rho}$ (otherwise, if necessary, we may shrink the domain of $\mathbf{r}_1$). It follows then that the Jacobi determinant of the map $\boldsymbol{\rho}^{-1} \circ \mathbf{r}_1$ should be either always positive or always negative on the domain of $\mathbf{r}_1$. On the other hand, we have, obviously, from the chain rule, that

$$J(\boldsymbol{\rho}^{-1} \circ \mathbf{r}) = J(\boldsymbol{\rho}^{-1} \circ \mathbf{r}_1) J(\mathbf{r}_1^{-1} \circ \mathbf{r}),$$

whence

$$\det J(\boldsymbol{\rho}^{-1} \circ \mathbf{r}) = \det J(\boldsymbol{\rho}^{-1} \circ \mathbf{r}_1) \det J(\mathbf{r}_1^{-1} \circ \mathbf{r}).$$

Now, in the right hand side, the last determinant is always positive, since the two parameterizations are assumed to be positively equivalent. The first determinant, from the hypothesis, is always positive or always negative. Thus, the right hand side has constant sign. On the other hand, as we saw previously, the left hand side has opposed signs on the two sides of the segment $t = 0$, whence the contradiction which shows that the Möbius's band is not orientable.

In the figure 10.6 we indicate how one can construct a Möbius band, from a strip of paper. Another example of a non-orientable surface is the so-called *Klein's bottle* (see figure **??**)

Definition. Let S be an oriented surface with the orientation $\mathbf{n}(a)$. A local parameterization $(U, \mathbf{r})$ of S is said to be *compatible* with the orientation $\mathbf{n}(a)$ if for any point $a = \mathbf{r}(u, v)$ we have

$$\mathbf{n}(a) = \frac{\mathbf{r}_\mathbf{u}' \times \mathbf{r}_\mathbf{v}'}{\|\mathbf{r}_\mathbf{u}' \times \mathbf{r}_\mathbf{v}'\|}$$

or, which is the same, if the frame $\{\mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{v}', \mathbf{n}(a)\}$ is right-handed.

10.7 Differentiable maps on a surface

Definition 10.4. Let S be a surface in $\mathbb{R}^3$. A map $f : S \to \mathbb{R}^k$ is called *differentiable* or *smooth* if for any parameterization $(U, \mathbf{r})$ of S the map $f \circ \mathbf{r} : U \to \mathbb{R}^k$ is smooth. The map $f_\mathbf{r} \equiv f \circ \mathbf{r}$ is called the *expression of f în in the curvilinear coordinates* (u, v) or the *local representation* of f with respect to the parameterization $(U, \mathbf{r})$.

Remarks. 1. On can define similarly the differentiability of maps defined on any open subset of a surface S.

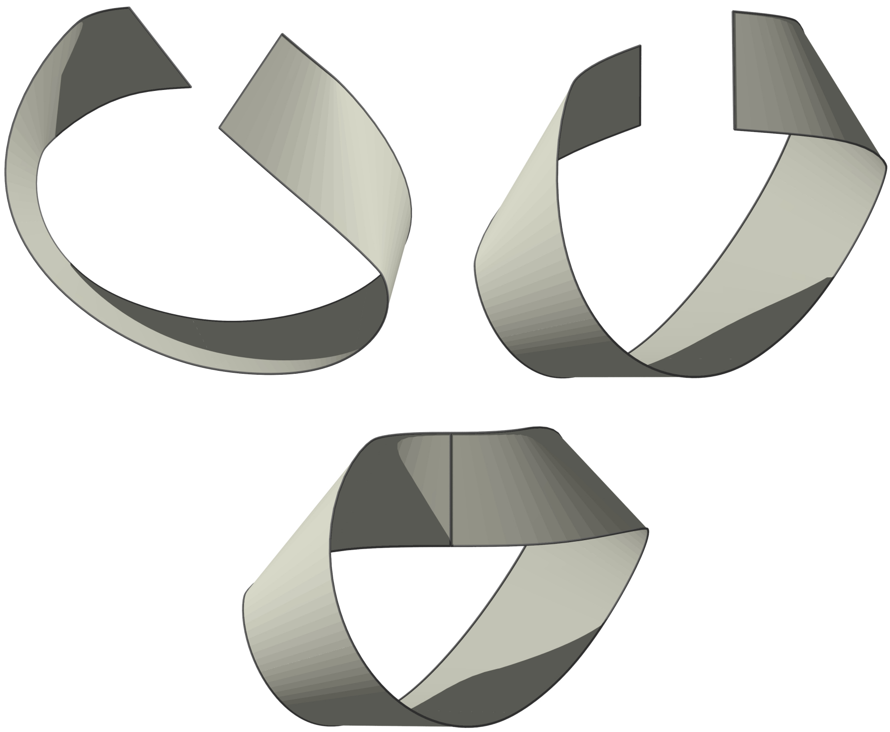


Figure 10.3: Construct your own Möbius band!

2. Any differentiable map $f : S \to \mathbb{R}^k$ is continuous, since, locally, it can be written as a composition of continuous maps: $f = f \circ (\mathbf{r} \circ \mathbf{r}^{-1}) = (f \circ \mathbf{r}) \circ \mathbf{r}^{-1} = f_{\mathbf{r}} \circ \mathbf{r}^{-1}$.

Examples. a) Any constant map $f : S \to \mathbb{R}^k : a \to A_0$, $A_0 \in \mathbb{R}^k$ is smooth, because its local representation with respect to any local parameterization of S is, equally, a constant map, hence it is differentiable.

b) If $F : \mathbb{R}^3 \to \mathbb{R}^k$ is a smooth map, then, for any surface S the map $f = F|_S : S \to \mathbb{R}^k$ is a smooth map. Indeed, for any local parameterization $(U, \mathbf{r})$ of S the local representation of f is $f_{\mathbf{r}} = F \circ \mathbf{r}$, where F and $\mathbf{r}$ are smooth maps, in the usual sense. In particular, the orthogonal projections of a surface S on the coordinate axes and planes are, all of them, smooth maps.

c) The inclusion $i : S \to \mathbb{R}^3 : a \to a$ is smooth, since for any local parameterization

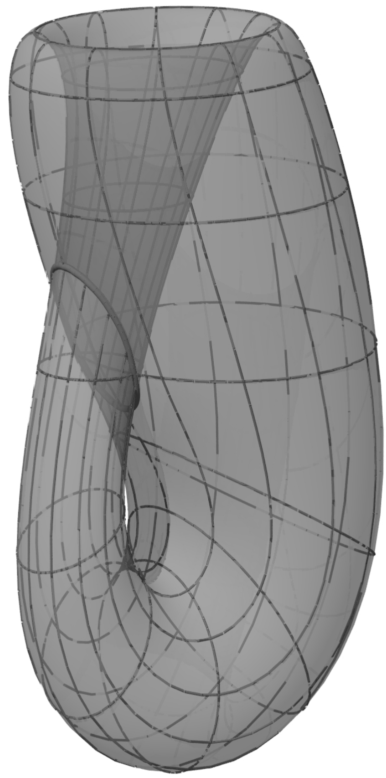


Figure 10.4: Klein's bottle

$(U, \mathbf{r})$ of S, the local representation of i is $i_{\mathbf{r}} = i \circ \mathbf{r} = \mathbf{r}$. (In fact, i is just the restriction of the identity map, $1_{\mathbb{R}^3}$ and, therefore, we can also apply the previous example).

Apparently, it is quite difficult to check whether a map defined on a surface is smooth, because we have to check the smoothness of its local representations with respect to *all* the local parameterizations of the surface, which are, of course, infinitely many. Fortunately, as the following theorem shows, it is enough to take only some of the local parameterizations, such that their domains cover the surface. In particular, if the surface is simple, it is enough to check for the global parameterization.

Theorem 10.7. *A map $f : S \to \mathbb{R}^k$ is smooth iff for any point $a \in S$ there is a local parameterization $(U, \mathbf{r})$ of the surface S with $a \in \mathbf{r}(U)$, such that the local representation $f_{\mathbf{r}} = f \circ \mathbf{r} : U \to \mathbb{R}^k$ is smooth.*

Proof The necessity is obvious, since, if F is smooth, then it local representation $f_{\mathbf{r}}$ is smooth for *any* local parameterization $(U, \mathbf{r})$ of S.

Conversely, let's suppose that that $a \in S$ is an arbitrary point of the surface and $(U, \mathbf{r})$ is a local parameterization of S around a such that the local representation $f_{\mathbf{r}} = f \circ \mathbf{r}$ is smooth. Obviously, it is enough to show that then the local representation of f in any other parameterization of S around a is, also, smooth. We choose, thus, another parameterization, $(U_1, \mathbf{r}_1)$, around a and let $W = \mathbf{r}(U) \circ \mathbf{r}_1(U_1)$. Then in $\mathbf{r}^{-1}(W) \subset U_1$, $f_{\mathbf{r}_1}$ can be represented as

$$f_{\mathbf{r}_1} = f \circ \mathbf{r}_1 = f \circ (\mathbf{r} \circ \mathbf{r}^{-1}) \circ \mathbf{r}_1 = (f \circ \mathbf{r}) \circ (\mathbf{r}^{-1} \circ \mathbf{r}_1) = f_{\mathbf{r}} \circ (\mathbf{r}^{-1} \circ \mathbf{r}_1).$$

Since both $f_{\mathbf{r}}$ (from the hypothesis) and $\mathbf{r}^{-1} \circ \mathbf{r}_1$ (from the theorem 10.1) are smooth, it follows that $f_{\mathbf{r}_1}$ is, equally smooth. □

Example 10.2. Let S be a surface and $(U, \mathbf{r})$ a local parameterization of S. As we explained earlier, the map $\mathbf{r}^{-1} : \mathbf{r}(U) \to \mathbb{R}^2$ is nor smooth in the classical sense. The reason is that its domain is not an open set of an Euclidean space, so the notion itself doesn't make sense for it. We also showed that, however, locally, $\mathbf{r}^{-1}$ is the restriction of a smooth map defined on an open set of $\mathbb{R}^3$. The notion we just defined is, actually, the natural frame of discussing this important map, which, in fact, assigns to each point of the surface (lying in $\mathbf{r}(U)$, of course), a pair of coordinates. Indeed, as a map defined on an open set of S, $\mathbf{r}^{-1}$ is smooth, as we can see easily, because the local representation of $\mathbf{r}^{-1}$ in the parameterization $(U, \mathbf{r})$ is

$$\left(\mathbf{r}^{-1}\right)_{\mathbf{r}} \equiv \mathbf{r}^{-1} \circ \mathbf{r} = 1_U.$$

The next natural step will be to define the notion of a smooth map between two surface rather then from a surface to an Euclidean space. The idea is the following. Let S_1, S_2 be two surfaces In $\mathbb{R}^3$. Then any map $F : S_1 \to S_2$ can be regarded as a map $F : S_1 \to \mathbb{R}^3$. More specifically, one can associate to F the map $i \circ F : S \to \mathbb{R}^3$, where $i : S_2 \hookrightarrow \mathbb{R}^3$ is the inclusion.

Definition 10.5. Let $S_1, S_2 \subset \mathbb{R}^3$ be two surfaces. A map $F : S_1 \to S_2$ is called *smooth* if the map $F_1 = i \circ F : S_1 \to \mathbb{R}^3$ is smooth.

Remarks. 1. It easy to see that any smooth between surfaces is continuous.

2. Let $S_1 \subset \mathbb{R}^3$ a surface and $G : \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism. Then $S_2 = G(S_1)$ is, also, a surface, while the map $G|_{S_1} : S_1 \to S_2$ is smooth.

3. Let $S_1, S_2 \subset \mathbb{R}^3$ two surfaces and $F : S_1 \to S_2$ a map. Then F is smooth iff for any $a \in S_1$, any local parameterization $(U_1, \mathbf{r}_1)$ of S_1 around a and any local parameterization $(U_2, \mathbf{r}_2)$ of S_2 around $F(a)$, the map

$$F_{\mathbf{r}_1,\mathbf{r}_2} \equiv \mathbf{r}_2^{-1} \circ F \circ \mathbf{r}_1 : U_1 \to U_2$$

is smooth (in the usual sense). $F_{\mathbf{r}_1,\mathbf{r}_2}$ is called the *local representation* of F with respect to the local parameterizations $(U_1, \mathbf{r}_1)$ and $(U_2, \mathbf{r}_2)$.

Definition 10.6. A map $F : S_1 \to S_2$ is called a *diffeomorphism* if F is bijective and both F and F^{-1} are smooth maps.

10.8 The differential of a smooth map between surfaces

The notion of a smooth map between surfaces is a natural generalization of the notion of smooth map between open sets in Euclidean spaces. The same should hold true, of course, also for the notion of differential. We shall emphasize, therefore, a property of the differential of map between Euclidean spaces that will be used to construct the generalization. Let $G : B \to \mathbb{R}^3$ be a smooth map, with $B \subset \mathbb{R}^3$ – an open set, while $G(x, y, z) = (g_1(x, y, z), g_2(x, y, z), g_2(x, y, z))$. Then, for any point $a = (x, y, z) \in B$, the differential of G at a,

$$d_a G : \mathbb{R}^3_a \to \mathbb{R}^3_{G(a)}$$

is a linear map, whose matrix is the Jacobi matrix

$$\left.\frac{D(g_1, g_2, g_3)}{D(x, y, z)}\right|_{(x_0, y_0, z_0)} = (\alpha_{ij}),\ 1 \leq i, j \leq 3,$$

where

$$\alpha_{ij} = \partial_i g_i(x_0, y_0, z_0).$$

For any vector $\mathbf{h} \in \mathbb{R}^3_a$, of components $\{h_1, h_2, h_3\}$, the vector $d_a G(\mathbf{h})$ has the components

$$\left\{ \sum_{j=1}^{3} \alpha_{1j} h_j, \sum_{j=1}^{3} \alpha_{2j} h_j, \sum_{j=1}^{3} \alpha_{3j} h_j \right\}.$$

Let us suppose now that the vector $\mathbf{h}$ is tangent to the parameterized curve $\boldsymbol{\rho}(t) = (x(t), y(t), z(t))$ at point $t = t_0$, i.e. $\mathbf{h} = \boldsymbol{\rho}'(t_0)$. We are going to show that the vector $d_aG(\mathbf{h})$ is the tangent vector to the parameterized curve $(G \circ \boldsymbol{\rho})(t)$ at $t = t_0$. To this end, we differentiate the relation

$$(G \circ \boldsymbol{\rho})(t) = (g_1(x(t), y(t), z(t)), g_2(x(t), y(t), z(t)), g_3(x(t), y(t), z(t)))$$

and we obtain:

$$\overrightarrow{(G \circ \boldsymbol{\rho})}'(t_0) = \left\{ \sum_{k=1}^{3} \frac{\partial g_1}{\partial x^k}(x_0, y_0, z_0)h_k, \sum_{k=1}^{3} \frac{\partial g_2}{\partial x^k}(x_0, y_0, z_0)h_k, \sum_{k=1}^{3} \frac{\partial g_3}{\partial x^k}(x_0, y_0, z_0)h_k \right\} =$$
$$= \left\{ \sum_{k=1}^{3} \alpha_{1k}h_k, \sum_{k=1}^{3} \alpha_{2k}h_k, \sum_{k=1}^{3} \alpha_{3k}h_k \right\},$$

where

$$\{h_1, h_2, h_3\} = \{x'(t_0), y'(t_0), z'(t_0)\} = \boldsymbol{\rho}'(t_0).$$

Thus, the differential d_aG assigns to a tangent vector to the path $\boldsymbol{\rho}(t)$ at $t = t_0$ the tangent vector to the path $G(\boldsymbol{\rho}(t))$ at $t = t_0$.

Let now $F : S_1 \to S_2$ be a smooth map between the surfaces S_1 and S_2 and $a \in S_1$. Then to any smooth path $(I, \boldsymbol{\rho})$ on S_1 corresponds a smooth path $(I, F \circ \boldsymbol{\rho})$ on S_2. If $\boldsymbol{\rho}(t)$ passes through a for $t = t_0$, then the path $F \circ \boldsymbol{\rho}(t)$ will pass through $F(a)$ for $t = t_0$.

Definition 10.7. The map $T_aS_1 \to T_{F(a)}S_2$, assigning to each tangent vector $\boldsymbol{\rho}'(t_0)$ to a parameterized curve $\boldsymbol{\rho}(t)$ on S_1, with $\boldsymbol{\rho}(t_0) = a$, the tangent vector $\overrightarrow{(F \circ \boldsymbol{\rho})}'(t_0)$ to the parameterized curve $F \circ \boldsymbol{\rho}$ at $t = t_0$ is called the *differential* of the smooth map $F : S_1 \to S_2$ at the point a and it is denoted by d_aF.

Now, there might be a little difficulty here. Namely, we could have on S_1 two different curves, which have the same tangent vector at a contact point. As the images of the two parameterized curves through F are, generally, distinct, it may happen that the images do not have the same tangent vector at the contact point. Well, as we shall see in a moment, this not actually the case. Indeed, let $a \in S_1$ and $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$, $(I_1, \boldsymbol{\rho}_1 = \boldsymbol{\rho}_1(s))$ – two parameterized curves on S_1 such that $\boldsymbol{\rho}(t_0) = \boldsymbol{\rho}_1(s_0) = a$ and $\boldsymbol{\rho}'(t_0) = \boldsymbol{\rho}'_1(s_0)$. We choose an arbitrary local parameterization $(U, \mathbf{r})$ on S_1, around a. As we are interested only on the local phenomena that happen around a, we may assume, without restricting the generality, that $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$ and $\boldsymbol{\rho}_1(I_1) \subset \mathbf{r}(U)$. Let's suppose that the local equation of the curves in the parameterization $(U, \mathbf{r})$ are

$$(\boldsymbol{\rho}) \begin{cases} u = u(t) \\ v = v(t) \end{cases},$$

and,

$$(\boldsymbol{\rho}_1)\begin{cases} u = u_1(s) \\ v = v_1(s) \end{cases},$$

respectively. Then the vectors $\boldsymbol{\rho}'(t_0)$ and $\boldsymbol{\rho_1}'(s_0)$ have, in the natural basis $\{\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}\}$ the expressions

$$\boldsymbol{\rho}'(t_0) = \{u'(t_0), v'(t_0)\}$$
$$\boldsymbol{\rho_1}'(s_0) = \{u_1'(s_0), v_1'(s_0)\}.$$

Moreover, in the chosen parameterization,

$$(F \circ \boldsymbol{\rho})(t) = F_{\mathbf{r}}(u(t), v(t))$$
$$(F \circ \boldsymbol{\rho}_1)(s) = F_{\mathbf{r}}(u_1(s), v_1(s)),$$

with $F_{\mathbf{r}} = F \circ \mathbf{r}$. Thus,

$$(\overrightarrow{F \circ \boldsymbol{\rho}})(t_0) = \frac{d}{dt}(F_{\mathbf{r}}(u(t), v(t)))(u_0, v_0) = \overrightarrow{(F_{\mathbf{r}})'_u}(u_0, v_0)u'(t_0) + \overrightarrow{(F_{\mathbf{r}})'_v}(u_0, v_0)v'(t_0)$$
$$(\overrightarrow{F \circ \boldsymbol{\rho}_1})(s_0) = \frac{d}{ds}(F_{\mathbf{r}}(u(s), v(s)))(u_0, v_0) = \overrightarrow{(F_{\mathbf{r}})'_u}(u_0, v_0)u_1'(s_0) + \overrightarrow{(F_{\mathbf{r}})'_v}(u_0, v_0)v_1'(s_0).$$

Now, since $\boldsymbol{\rho}'(t_0) = \boldsymbol{\rho_1}'(s_0)$, it follows that

$$(\overrightarrow{F \circ \boldsymbol{\rho}})'(t_0) = (\overrightarrow{F \circ \boldsymbol{\rho}_1})'(s_0),$$

i.e. the definition of the differential of F makes sense.

Theorem. *The map $dF : T_a S_1 \to T_{F(a)} S_2$ is linear.*

Proof From the computations made above, it follows immediately that if a vector $\mathbf{h} \in T_a S_1$ has in the natural basis $\{\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}\}$ of the linear space $T_a S_1$, the components $\{h_1, h_2\}$, then

$$d_a F(\mathbf{h}) = \overrightarrow{F}'_{\mathbf{r}u}(u_0, v_0)h_1 + \overrightarrow{F}'_{\mathbf{r}v}(u_0, v_0)h_2, \tag{10.8.1}$$

whence the linearity. □

Example 10.3. Let S_1, S_2 be two surfaces in $\mathbb{R}^3$, $D : \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism such that $D(S_1) = S_2$, $F = D_{S_1} : S_1 \to S_2$, $a \in S \subset \mathbb{R}^3$. Then we have

$$d_a F = d_a D|_{T_a S_1}, \tag{10.8.2}$$

where $d_a D : \mathbb{R}^3_a \to \mathbb{R}^3_{D(a)}$ is the differential of the map D at a. In particular, let S^2_R and $S^2_{\mathbf{r}}$ be the spheres of radii R and r, respectively, centred at the origin and $D : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \to \frac{r}{R}(x, y, z)$ – a homothety, which is, obviously, a diffeomorphism such that $D(S^2_R) = S^2_r$. Then, for the map $F = D|_{S^2_R} : S^2_R \to S^2_r$, we have $d_a F(\mathbf{h}) = \frac{r}{R}\mathbf{h}$.

10.9 The spherical map and the shape operator of an oriented surface

Let $S \subset \mathbb{R}^3$ be an oriented surface and S^2 – the unit sphere centred at the origin. If the orientation of S is given by the unit normal $\mathbf{n}(a)$, $a \in S$, then we can build a map $\Gamma : S \to S^2$, associating to each $a \in S$ the point on S^2 has as radius vector $\mathbf{\Gamma}(a) = \mathbf{n}(a)$. The map Γ is called the *spherical map* of the surface S. This map place a central role in the theory of surfaces. We are going to show, first of all, that Γ is smooth:

Theorem 10.8. *The spherical map $\Gamma : S \to S^2$ of an oriented surface S into the unit sphere S^2 is a smooth map between surfaces.*

Proof Let $a \in S$. We choose a local parameterization $(U, \mathbf{r})$ of the surface S around a, compatible with the orientation. Clearly, since S is orientable, such a parameterization always exists. Indeed, if we choose a parameterization $(U_1, \mathbf{r}_1)$ which is not compatible with the orientation, i.e. we have

$$\frac{\mathbf{r}'_{1u} \times \mathbf{r}'_{1v}}{\|\mathbf{r}'_{1u} \times \mathbf{r}'_{1v}\|} = -\mathbf{n}(u, v),$$

then we replace the domain U_1 by U_1^-, the symmetric of U_1 with respect to the Ov-axis, and the map $\mathbf{r}_1(u, v)$ by $\mathbf{r}_1^-(u, v) = \mathbf{r}_1(-u, v)$. It is easy to see that the pair $(U_1^-, \mathbf{r}_1^-)$ is a parameterization of the surface, compatible with the orientation.

Now we have

$$(\Gamma \circ \mathbf{r})(u, v) = \Gamma(\mathbf{r}(u, v)) = \Gamma_{\mathbf{r}}(u, v) = \mathbf{n}(u, v) = \frac{\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}}{\|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\|}.$$

Thus, the local representation of Γ is smooth, therefore Γ itself is smooth. $\square$

Examples. (i) For a plane Π the spherical map is constant.

(ii) For the sphere S^2_R the map $\Gamma : S^2_R \to S$ has the expression $\Gamma(x, y, z) = \frac{1}{R}(x, y, z)$, with $x^2 + y^2 + z^2 = R^2$.

As we saw, the tangent space to the sphere, $T_{\Gamma(a)}S^2$, is orthogonal to the radius vector $\mathbf{n}(a)$ of the point $\Gamma(a)$. On the other hand, $\mathbf{n}(a)$ is orthogonal to T_aS. Thus, if we identify $\mathbb{R}^3_a$ and $\mathbb{R}^3_{\Gamma(a)}$ to $\mathbb{R}^3$, then the subspaces, T_aS and $T_{\Gamma(a)}S^2$ coincide. Therefore, we may think of the differential $d_a\Gamma : T_aS \to T_{\Gamma(a)}S^2$ as being, in fact, a linear operator $T_aS \to T_aS$.

Definition 10.8. The linear operator $d_a\Gamma : T_aS \to T_aS$ is called the *shape operator* of the oriented surface S at the point a and it is denoted by A or A_a.

Remark. There is no general agreement regarding the definition, nor the name of the shape operator. In some books, the shape operator carries an extra minus sign. Sometimes it is called the *Weingarten mapping*, or, also the *principal, or fundamental operator*. Historically, it is true, indeed, that Weingarten was the first to write down the formulae for the differentiation of the spherical map (in other words, he was the one to find the partial derivatives of the unit normal vectors in terms of the derivatives of the radius vector). Nevertheless, the shape operator, as a linear map, was introduced in differential geometry by the Italian mathematician Cesare Burali-Forti ([**?**]), in 1912, under the name *omografia fondamentale*, i.e. fundamental homography.

Example 10.4. (i) For a plane, the shape operator vanishes.

(ii) For a sphere, the shape operator is a homothety.

Let now $(U, \mathbf{r})$ be a local parameterization of S, with the orientation. Then the local representation of the spherical map Γ of S will be given by

$$\mathbf{n}(u, v) = \frac{\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}}{\|\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}\|},$$

therefore, for the shape operator we will have:

$$A(\mathbf{h}) = \mathbf{n}'_u h_1 + \mathbf{n}'_v h_2, \tag{10.9.1}$$

where $\mathbf{h} \in T_{\mathbf{r}(u,v)}S$; (h_1, h_2) are the components of $\mathbf{h}$ with respect to the natural basis $\{\mathbf{r}'_\mathbf{u}, \mathbf{r}'_\mathbf{v}\}$. In particular, we have

$$A(\mathbf{r}'_\mathbf{u}) = \mathbf{n}'_u, \ A(\mathbf{r}'_\mathbf{v}) = \mathbf{n}'_v. \tag{10.9.2}$$

Theorem. *The shape operator A is self-adjoint, i.e. $\forall \mathbf{h}, \mathbf{p} \in T_a S$, we have*

$$A(\mathbf{h}) \cdot \mathbf{p} = \mathbf{h} \cdot A(\mathbf{p}). \tag{10.9.3}$$

Proof It is enough to make the proof for the vectors, $\mathbf{r}'_{\mathbf{u}}$ and $\mathbf{r}'_{\mathbf{v}}$. The case $\mathbf{h} = \mathbf{p}$ is trivial, therefore it is enough to check that

$$A(\mathbf{r}'_{\mathbf{u}}) \cdot \mathbf{r}'_{\mathbf{v}} = \mathbf{r}'_{\mathbf{u}} \cdot A(\mathbf{r}'_{\mathbf{v}}),$$

i.e.

$$\mathbf{n}'_u \cdot \mathbf{r}'_{\mathbf{v}} = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v. \tag{10.9.4}$$

To prove this, we start from the obvious equalities:

$$\mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n} = 0 \tag{*}$$

şi

$$\mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n} = 0 \tag{**}$$

We differentiate (*) with respect to u and (**) with respect to v and we get

$$\begin{cases} \mathbf{r}''_{uv} \cdot \mathbf{n} + \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_u = 0 \\ \mathbf{r}''_{uv} \cdot \mathbf{n} + \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v = 0 \end{cases}$$

whence, subtracting the two equalities, we get the conclusion. □

Corollary 10.1. *In each tangent space $T_a S$ there is an orthonormal basis, made up of eigenvectors of the shape operator A.*

Proof Since A is self-adjoint, the two eigenvalues λ_1, λ_2 are real. If $\lambda_1 \neq \lambda_2$, then the eigenvectors corresponding to λ_1 and λ_2 are orthogonal, hence it is enough to choose a unit vector in each eigenspace. If $\lambda_1 = \lambda_2$, then A is just a homothety and any orthonormal basis will do (since, in this case, any tangent space is an eigenvector of A). □

10.10 The first fundamental form of a surface

Let S be a surface in $\mathbb{R}^3$. Then the scalar product in $\mathbb{R}^3$ induces a scalar product in each $\mathbb{R}^3_a$ and, hence, it also induces a scalar product in each tangent space $T_a S$, $a \in S$.

Definition 10.9. The first fundamental form of a surface S is, by definition, the function φ_1, associating to each $a \in S$ the restriction of the scalar product of $\mathbb{R}^3_a$ to $T_a S$. We shall say usually, informally, that the first fundamental form is the restriction itself, but it should be kept in mind, however, what really happens. Thus, for any $a \in S$ and any $\mathbf{p}, \mathbf{q} \in T_a S$, we will have

$$\varphi_1(\mathbf{p}, \mathbf{q}) = \mathbf{p} \cdot \mathbf{q}. \tag{10.10.1}$$

Remark. In many textbooks, especially the older ones, the first fundamental form is not defined as the restriction of the scalar product to $T_a S$, but rather as the quadratic form associated to this restriction.

If $(U, \mathbf{r})$ is a local parameterization of S, then for any $(u, v) \in U$ the tangent space $\mathbb{R}^2_{(u,v)}$ of the domain U at the point (u, v) can be identified with the space $T_{\mathbf{r}(u,v)} S$, associating to the vectors $\{1, 0\}$, $\{0, 1\}$, making up a basis of $\mathbb{R}^2_{(u,v)}$, the vectors $\mathbf{r}'_{\mathbf{u}}(u, v)$ and $\mathbf{r}'_{\mathbf{v}}(u, v)$. It is easy to see that, in fact, this identification is just the linear isomorphism $dr_{(u,v)} : \mathbb{R}^2_{(u,v)} \to T_{\mathbf{r}(u,v)} S$. Using this identification, one can transport the first fundamental form φ_1 of S to the domain S (which can be seen, as a matter of fact, as being a simple surface, with the global parameterization given by the identical map of U). Thus, for any $(u, v) \in U$, in the tangent space $T_{\mathbf{r}(u,v)} U \equiv \mathbb{R}^2_{(u,v)}$ at the domain U, the scalar product of two vectors is defined by the rule

$$\widetilde{\varphi}_1(\xi, \eta) = \varphi_1(dr_{(u,v)}(\xi), dr_{(u,v)}(\eta)) = dr_{(u,v)}(\xi) \cdot dr_{(u,v)}(\eta).$$

It easy to see that, by construction, the map $dr_{(u,v)} : \mathbb{R}^2_{(u,v)} \to T_{\mathbf{r}(u,v)} S$ is isometrical with respect to the scalar products $\widetilde{\varphi}_1$ and φ_1 respectively. We introduce the notations

$$\begin{cases} E(u, v) = \mathbf{r}'_{\mathbf{u}}(u, v) \cdot \mathbf{r}'_{\mathbf{u}}(u, v) \\ F(u, v) = \mathbf{r}'_{\mathbf{u}}(u, v) \cdot \mathbf{r}'_{\mathbf{v}}(u, v) \\ G(u, v) = \mathbf{r}'_{\mathbf{v}}(u, v) \cdot \mathbf{r}'_{\mathbf{v}}(u, v) \end{cases}.$$

Then the functions E, F, G are smooth on U, while the matrix $\mathcal{G} = \left(\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right)$ is the matrix of the scalar product φ_1 on the tangent space $T_{\mathbf{r}(u,v)} S$ with respect to the basis $\{\mathbf{r}'_{\mathbf{u}}(u, v), \mathbf{r}'_{\mathbf{v}}(u, v)\}$, but it is also the matrix of the scalar product $\widetilde{\varphi}_1$ on the the tangent space $\mathbb{R}^2_{(u,v)} = T_{(u,v)} U$ with respect to the basis $\{\{1, 0\}, \{0, 1\}\}$.

Examples. 1. For the plane Π given by the global parameterization $\mathbf{r} = \mathbf{r}_0 +$

$u\mathbf{a} + v\mathbf{b}$, $\mathbf{a} \times \mathbf{b} \neq 0$, we have:

$$\begin{cases} \mathbf{r}'_{\mathbf{u}} = \mathbf{a} \\ \mathbf{r}'_{\mathbf{v}} = \mathbf{b} \end{cases} \qquad \text{hence} \qquad \begin{cases} E = \mathbf{a}^2 \\ F = \mathbf{a} \cdot \mathbf{b} \\ G = \mathbf{b}^2 \end{cases} .$$

If Π is the coordinate plane xOy, then we may set $\mathbf{r}_0 = 0$, $\mathbf{a} = \boldsymbol{\imath}$, $\mathbf{b} = \boldsymbol{\jmath}$, therefore the first fundamental form has the matrix $\mathcal{G} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

2. For the sphere S_R^2, we choose the local parameterization $(U, \mathbf{r})$, with

$$\mathbf{r}(u, v) = (R \cos u \cos v, R \cos u \sin v, R \sin u),$$

and $U = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We get immediately $E = R^2$, $F = 0$, $G = R^2 \cos^2 u$, hence the matrix of the first fundamental form is given by

$$\mathcal{G} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$$

10.10.1 First applications

The length of a segment of curve on a surface

Let S be a surface, $(U, \mathbf{r})$ – a local parameterization of S and $(I, \boldsymbol{\rho})$ – a parameterized curve with $\boldsymbol{\rho}(I) \subseteq \mathbf{r}(U)$, given by the local equations $u = u(t)$, $v = v(t)$. Then, in the natural basis, the tangent vector of $\boldsymbol{\rho}$, $\boldsymbol{\rho}'(t)$ has the components $\{u'(t), v'(t)\}$ and we can compute its length using the matrix $\mathcal{G}$. Therefore, we have, for the length of the segment of $\boldsymbol{\rho}$ between t_1 and t_2:

$$l_{t_1,t_2} = \int_{t_1}^{t_2} \|\boldsymbol{\rho}'(t)\| dt = \int_{t_1}^{t_2} \sqrt{E(t)u'^2 + 2F(t)u'v' + G(t)v'^2} dt,$$

where

$$\begin{cases} E(t) = E(u(t), v(t)) \\ F(t) = F(u(t), v(t)) \\ G(t) = G(u(t), v(t)) \end{cases} .$$

Example 10.5. We take, on the sphere S_R^2, the curve given by the local equations (in the parameterization described in the previous example) $u = 0$, $v = t$, where $t \in$

$(0, 2\pi)$ (the equator with a point removed). As we saw above, the first fundamental form of the sphere has the matrix $\mathcal{G} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & cos^2 u \end{pmatrix}$. As along the curve we have $u = 0$, it follows that $u'(t) = 0$, $v'(t) = 1$. On the other hand, $\cos^2 u = \cos^2 0 = 1$, hence, along the curve, the matrix $\mathcal{G}$ will be the identity matrix, multiplied by R^2. If we want to compute, for instance, the length of the segment of curve between $t_1 = \frac{\pi}{2}$ and $t_2 = \pi$ we will get

$$l_{\frac{\pi}{2},\pi} = \int_{\frac{\pi}{2}}^{\pi} \sqrt{R^2 \cdot 0 + 2 \cdot 0 + R^2 \cdot 1}dt = R \cdot t \Big|_{\frac{\pi}{2}}^{\pi} = \frac{\pi R}{2},$$

which was to be expected (the segment of curve is a quarter of a great circle on the sphere).

The angle of two curves on a surface

Let $(U, \mathbf{r})$ be a parameterization of a surface S, $(I, \boldsymbol{\rho} = \boldsymbol{\rho}(t))$, $(I_1, \boldsymbol{\rho}_1 = \boldsymbol{\rho}_1(s))$ – two curves on S such that $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$, $\boldsymbol{\rho}_1(I_1) \subset \mathbf{r}(U)$. We assume that the supports of the two curves intersect at $\mathbf{r}(u_0, v_0)$, i.e. there are $t_0 \in I, s_0 \in I_1$ such that:

$$\boldsymbol{\rho}(t_0) = \boldsymbol{\rho}_1(s_0) = \mathbf{r}(u_0, v_0).$$

If the local equations of the two curves are

$$(\boldsymbol{\rho}) \begin{cases} u = u_1(t) \\ v = v_1(t) \end{cases},$$

and

$$(\boldsymbol{\rho}_1) \begin{cases} u = u_2(s) \\ v = v_2(s) \end{cases},$$

respectively, then the decomposition of the tangents vector at the intersection point with respect to the natural basis will be:

$$\begin{cases} \boldsymbol{\rho}'(t_0) = \{u_1'(t_0), v_1'(t_0)\} \\ \boldsymbol{\rho}_1{}'(s_0) = \{u_2'(s_0), v_2'(s_0)\} \end{cases},$$

therefore the cosine of the angle of the curves[3] at the contact point is, as it is well known,

$$\cos\theta = \frac{\boldsymbol{\rho}'(t_0)\cdot\boldsymbol{\rho_1}'(s_0)}{\|\boldsymbol{\rho}'(t_0)\|\cdot\|\boldsymbol{\rho_1}'(s_0)\|} = \frac{Eu_1'u_2' + F(u_1'v_2' + u_2'v_1') + Gv_1'v_2'}{\sqrt{E{u_1'}^2 + 2Fu_1'v_1' + G{v_1'}^2}\cdot\sqrt{E{u_2'}^2 + 2Fu_2'v_2' + G{v_2'}^2}},$$

where

$$\begin{cases} E = E(u_0, v_0) \\ F = F(u_0, v_0) \\ G = G(u_0, v_0) \end{cases} \qquad \text{and} \qquad \begin{cases} u_1' = u_1'(t_0) \\ v_1' = v_1'(t_0) \\ u_2' = u_2'(s_0) \\ v_2' = v_2'(s_0) \end{cases}.$$

The area of a parameterized surface

Let $(U, \mathbf{r})$ be an oriented parameterized surface. There are many ways of introducing the notion of area. All of them are more or less connected to integral calculus, so we are not going to enter into any details here. Basically, as in the case of plane geometric figures, the area should be a function associating to each oriented patch a positive number, subject to some restrictions. We choose, following Stoker, the following three restrictions:

a) The area should be given by an integral of the form

$$A = \iint_U f\,du\,dv,$$

where f should depend only upon $u, v, \mathbf{r}, \mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{v}'$ (no higher derivatives of $\mathbf{r}$ should be involved!).

b) It is invariant with respect to rigid motions of the space and to parameter transformations that preserve the orientation.

c) A square of side length 1 has area 1.

It can be proved that the only formula for area that verifies these three axioms is

$$A = \iint_U \sqrt{EG - F^2}\,du\,dv \equiv \iint_U \|\mathbf{r}_\mathbf{u}' \times \mathbf{r}_\mathbf{v}'\|\,du\,dv. \tag{10.10.2}$$

[3] We are speaking, of course, about the angle of the tangent vectors.

We will give a heuristic motivation for the formula (10.10.2). It should not be taken, however, as being a "proof", because no such claim is being currently made.

The "classical" approach. Let $(U, \mathbf{r})$ be a parameterized surface and $D \subset U$ – a compact subset of U such that $\mathbf{r}(\partial U)$ is a piecewise smooth curve in $\mathbb{R}^3$. We want to define the *area* of $\mathbf{r}(D) \subset \mathbf{r}(U)$. The basic idea is that we already have a notion of area for *plane* figures, in particular for parallelograms. Thus, let $(u, v) \in D$ and $M = \mathbf{r}(u, v)$. Through M pass two coordinate lines, one from each family. Let $M_1 = \mathbf{r}(u + \Delta u, v)$, $M_2 = \mathbf{r}(u, v + \Delta v)$ be two points on these lines, shifted from M with the parameter shifts Δu and Δv, respectively, and $M' = \mathbf{r}(u+\Delta u, v+\Delta v)$. If Δu and Δv are small enough, then the projection of the curvilinear parallelogram $MM_1M'M_2$ on the tangent plane to the surface at the point M is (approximately, of course), a plane parallelogram in the tangent plane. The sides of this parallelogram are $\mathbf{r}'_{\mathbf{u}}\Delta u$ and $\mathbf{r}'_{\mathbf{v}}\Delta v$ and its area will be, then

$$\Delta\sigma = \|\mathbf{r}'_{\mathbf{u}}\Delta u \times \mathbf{r}'_{\mathbf{v}}\Delta v\| = \|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\|\Delta u \Delta v = \sqrt{EG - F^2}\Delta u \Delta v,$$

where, of course, the coefficients of the first fundamental form have been computed at the point of M. It is only natural, then, to define the area of $\mathbf{r}(D)$ as

$$A = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Sigma \Delta\sigma = \iint_D \sqrt{EG - F^2}dudv,$$

where the sum in the middle term is taken after all the small curvilinear parallelograms that cover $\mathbf{r}(U)$.

Remark. We might expect to obtain for the area of domain on a surface an interpretation similar to the one we got for the length of a segment of curve. Namely, we may discretize the domain of the parameterization and consider the images of the selected points. They will determine a polygonal surface inscribed into the surface. Then we may consider that the area of the polygons that make up this polygonal surface goes to zero and define the area of our surface as being the limit of the area of the polygonal surface. Unfortunately, as a celebrated example of H.A. Schwartz shows, it just doesn't work, because the limit is not independent on the type of polygons we consider and, in particular, for some "polygonizations" of the surface, the area may be infinite and for other ones finite. Of course, the things can be fixed, with a little care, but this is a subject that belongs to theory of integration rather than to differential geometry, therefore we shall not insist.

10.11 The matrix of the shape operator of a surface in the natural basis

Let $(U, \mathbf{r})$ be a local parameterization of the oriented surface S, compatible with the orientation. We shall denote by $\mathcal{A}$ the matrix of the shape operator A with respect to the natural basis $\{\mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{v}'\}$. Since, as we saw earlier,

$$A(\mathbf{r}_\mathbf{u}') = \mathbf{n}_u', \; A(\mathbf{r}_\mathbf{v}') = \mathbf{n}_v',$$

we have

$$(\,\mathbf{n}_u \;\; \mathbf{n}_v') = (\mathbf{r}_\mathbf{u}' \; \mathbf{r}_\mathbf{v}') \cdot \mathcal{A}. \tag{10.11.1}$$

We multiply from the left with the matrix $\begin{pmatrix} \mathbf{r}_\mathbf{u}' \\ \mathbf{r}_\mathbf{v}' \end{pmatrix}$ and we get:

$$\begin{aligned} \begin{pmatrix} \mathbf{r}_\mathbf{u}' \\ \mathbf{r}_\mathbf{v}' \end{pmatrix} \cdot (\,\mathbf{n}_u' \quad \mathbf{n}_v') &= \begin{pmatrix} \mathbf{r}_\mathbf{u}' \cdot \mathbf{n}_u' & \mathbf{r}_\mathbf{u}' \cdot \mathbf{n}_v' \\ \mathbf{r}_\mathbf{v}' \cdot \mathbf{n}_u' & \mathbf{r}_\mathbf{v}' \cdot \mathbf{n}_v' \end{pmatrix} = \\ &= \begin{pmatrix} \mathbf{r}_\mathbf{u}' \\ \mathbf{r}_\mathbf{v}' \end{pmatrix} \cdot (\mathbf{r}_\mathbf{u}' \quad \mathbf{r}_\mathbf{v}') \cdot \mathcal{A} = \begin{pmatrix} \mathbf{r}_\mathbf{u}' \cdot \mathbf{r}_\mathbf{u}' & \mathbf{r}_\mathbf{u}' \cdot \mathbf{r}_\mathbf{v}' \\ \mathbf{r}_\mathbf{v}' \cdot \mathbf{r}_\mathbf{u}' & \mathbf{r}_\mathbf{v}' \cdot \mathbf{r}_\mathbf{v}' \end{pmatrix} \cdot \mathcal{A} = \mathcal{G} \cdot \mathcal{A}. \end{aligned}$$

We introduce the functions

$$\begin{cases} L(u, v) = \mathbf{r}_\mathbf{u}' \cdot \mathbf{n}_u' \\ M(u, v) = \mathbf{r}_\mathbf{u}' \cdot \mathbf{n}_v' \\ N(u, v) = \mathbf{r}_\mathbf{v}' \cdot \mathbf{n}_v' \end{cases}, \tag{10.11.2}$$

and the matrix $\mathcal{H}$, defined by

$$\mathcal{H} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Then the last equality reads

$$\mathcal{H} = \mathcal{G} \cdot \mathcal{A}.$$

Since the scalar product on $\mathbb{R}^3$ is nondegenerate, the same is true for its restriction to any subspace and, as a consequence, the matrix $\mathcal{G}$ is invertible. If $\mathcal{G}^{-1}$ be its inverse, then for the matrix of the shape operator we get

$$\mathcal{A} = \mathcal{G}^{-1} \cdot \mathcal{H}, \tag{10.11.3}$$

where, as one can see very easily,

$$\mathcal{G}^{-1} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

If we perform the computation, we obtain:

$$\mathcal{A} = \frac{1}{EG - F^2}\begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix} \tag{10.11.4}$$

All we have to do now is to express the quantities L, M, N in terms of the derivatives of the function $\mathbf{r}$. To this end, we differentiate the relations $\mathbf{r}_{\mathbf{u}}^{'} \cdot \mathbf{n} = 0$ and $\mathbf{r}_{\mathbf{v}}^{'} \cdot \mathbf{n} = 0$ with respect to u and v and we get:

$$\begin{cases} \mathbf{r}_{\mathbf{u}^2}^{''} \cdot n + \mathbf{r}_{\mathbf{u}}^{'} \cdot \mathbf{n}_u' = 0 \\ \mathbf{r}_{\mathbf{uv}}^{''} \cdot \mathbf{n} + \mathbf{r}_{\mathbf{u}}^{'} \cdot \mathbf{n}_v' = 0 \\ \mathbf{r}_{\mathbf{uv}}^{''} \cdot \mathbf{n} + \mathbf{r}_{\mathbf{v}}^{'} \cdot \mathbf{n}_u' = 0 \\ \mathbf{r}_{\mathbf{v}^2}^{''} \cdot n + \mathbf{r}_{\mathbf{v}}^{'} \cdot \mathbf{n}_v' = 0 \end{cases},$$

whence we obtain from L, M, N the expressions:

$$\begin{cases} L = \mathbf{r}_{\mathbf{u}}^{'} \cdot \mathbf{n}_u' = -\mathbf{n} \cdot \mathbf{r}_{\mathbf{u}^2}^{''} \\ M = \mathbf{r}_{\mathbf{u}}^{'} \cdot \mathbf{n}_v' = -\mathbf{n} \cdot \mathbf{r}_{\mathbf{uv}}^{''} \\ N = \mathbf{r}_{\mathbf{v}}^{'} \cdot \mathbf{n}_v' = -\mathbf{n} \cdot \mathbf{r}_{\mathbf{v}^2}^{''} \end{cases} \tag{10.11.5}$$

or, having in mind that

$$\mathbf{n} = \frac{\mathbf{r}_{\mathbf{u}}^{'} \times \mathbf{r}_{\mathbf{v}}^{'}}{\|\mathbf{r}_{\mathbf{u}}^{'} \times \mathbf{r}_{\mathbf{v}}^{'}\|},$$

while

$$\|\mathbf{r}_{\mathbf{u}}^{'} \times \mathbf{r}_{\mathbf{v}}^{'}\| = H(= \sqrt{EG - F^2}),$$

$$\begin{cases} L = -\frac{1}{H}(\mathbf{r}_{\mathbf{u}}^{'}, \mathbf{r}_{\mathbf{v}}^{'}, \mathbf{r}_{\mathbf{u}^2}^{''}) \\ M = -\frac{1}{H}(\mathbf{r}_{\mathbf{u}}^{'}, \mathbf{r}_{\mathbf{v}}^{'}, \mathbf{r}_{\mathbf{uv}}^{''}) \\ N = -\frac{1}{H}(\mathbf{r}_{\mathbf{u}}^{'}, \mathbf{r}_{\mathbf{v}}^{'}, \mathbf{r}_{\mathbf{v}^2}^{''}) \end{cases}. \tag{10.11.6}$$

Example 10.6. For the helicoid

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = bv \end{cases}, \qquad (u, v) \in \mathbb{R}^2, \quad b > 0,$$

we can define the orientation by putting

$$\mathbf{n}(u,v) = \left\{ \frac{b\sin v}{\sqrt{b^2+u^2}}, -\frac{b\cos v}{\sqrt{b^2+u^2}}, \frac{u}{\sqrt{b^2+u^2}} \right\}.$$

On gets, after a straightforward computation:

$$\mathcal{G} = \begin{pmatrix} 1 & 0 \\ 0 & b^2+u^2 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 & \frac{b}{\sqrt{b^2+u^2}} \\ \frac{b}{\sqrt{b^2+u^2}} & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & \frac{b}{(b^2+u^2)^{3/2}} \\ \frac{b}{(b^2+u^2)^{3/2}} & 0 \end{pmatrix}.$$

10.12 The second fundamental form of an oriented surface

Definition 10.10. The second fundamental form of an oriented surface S is a map associating to each $a \in S$ the application $\varphi_2(a) : T_aS \times T_aS \to \mathbb{R}$ given by

$$\varphi_2(\xi,\eta) = -\varphi_1(A(\xi),\eta), \quad \forall \xi,\eta \in T_aS. \tag{10.12.1}$$

Remark. The minus sign in the previous definition is a consequence of our particular choice of sign in the definition of the shape operator. We found natural to choose the shape operator to be the differential of the spherical map rather then the opposite of the differential, but then in the definition of the second fundamental form we had to introduce an extra minus sign, in order to be consistent with the generally accepted definition of the second fundamental form.

Property 19. *For each $a \in S$, $\varphi_2(a)$ is a symmetrical bilinear form.*

Proof We take two arbitrary tangent vectors $\boldsymbol{\xi}, \boldsymbol{\eta} \in T_aS$ and two arbitrary real numbers $\alpha, \beta \in \mathbb{R}$. Then we have, first of all:

$$\varphi_2(\eta,\xi) = -\varphi_1(A(\eta),\xi) \underset{\text{self-adjoint}}{\overset{A}{=}} -\varphi_1(\eta, A(\xi)) \underset{\text{symmetrical}}{\overset{\varphi_1}{=}} -\varphi_1(A(\xi),\eta) = \varphi_2(\xi,\eta),$$

which means that φ_2 is symmetrical. Due to the symmetry, it is enough to prove the linearity in the first variable only. We have

$$\begin{aligned}\varphi_2(\alpha\xi_1+\beta\xi_2,\eta) &= -\varphi_1(A(\alpha\xi_1+\beta\xi_2),\eta) \underset{\text{linear}}{\overset{A}{=}} -\varphi_1(\alpha A(\xi_1)+\beta A(\xi_2),\eta) \underset{\text{bilinear}}{\overset{\varphi_1}{=}} \\ &\underset{\text{bilinear}}{\overset{\varphi_1}{=}} -\alpha\varphi_1(A(\xi_1),\eta) - \beta\varphi_1(A(\xi_2),\eta) = \alpha\varphi_2(\xi_1,\eta) + \beta\varphi_2(\xi_2,\eta),\end{aligned}$$

which shows the linearity in the first variable and concludes the proof. □

Let $(U, \mathbf{r})$ be a local parameterization of the oriented surface S, compatible with the orientation. Then the matrix $[\varphi_2]$ of the second fundamental form with respect to the canonical basis $\{\mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{v}'\}$ has the form:

$$[\varphi_2] = \begin{pmatrix} \varphi_2(\mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{u}') & \varphi_2(\mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{v}') \\ \varphi_2(\mathbf{r}_\mathbf{v}', \mathbf{r}_\mathbf{u}') & \varphi_2(\mathbf{r}_\mathbf{v}', \mathbf{r}_\mathbf{v}') \end{pmatrix}.$$

But

$$\begin{cases} \varphi_2(\mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{u}') = -\varphi_1(A(\mathbf{r}_\mathbf{u}'), \mathbf{r}_\mathbf{u}') = -\varphi_1(\mathbf{n}_u', \mathbf{r}_\mathbf{u}') = -\mathbf{n}_u' \cdot \mathbf{r}_\mathbf{u}' \\ \varphi_2(\mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{v}') = \varphi_2(\mathbf{r}_\mathbf{v}', \mathbf{r}_\mathbf{u}') = -\mathbf{n}_u' \cdot \mathbf{r}_\mathbf{v}' = -\mathbf{n}_v' \cdot \mathbf{r}_\mathbf{u}' \quad \varphi_2(\mathbf{r}_\mathbf{v}', \mathbf{r}_\mathbf{v}') = -\mathbf{n}_v' \cdot \mathbf{r}_\mathbf{v}' \end{cases},$$

and, thus, we get the matrix

$$[\varphi_2] = -\begin{pmatrix} \mathbf{n}_u' \cdot \mathbf{r}_\mathbf{u}' & \mathbf{n}_u' \cdot \mathbf{r}_\mathbf{v}' \\ \mathbf{n}_v' \cdot \mathbf{r}_\mathbf{u}' & \mathbf{n}_v' \cdot \mathbf{r}_\mathbf{v}' \end{pmatrix} = -\begin{pmatrix} L & M \\ M & N \end{pmatrix} \overset{\text{not.}}{=} \begin{pmatrix} D & D' \\ D' & D'' \end{pmatrix}.$$

It follows, this way, that the matrix of the second fundamental form in the canonical basis is just $-\mathcal{H}$. Thus, for the coefficients of the second fundamental form with respect to the natural basis, we have

$$\begin{cases} D = \mathbf{n} \cdot \mathbf{r}_{\mathbf{u}^2}'' \\ D' = \mathbf{n} \cdot \mathbf{r}_{\mathbf{uv}}'' \\ D'' = \mathbf{n} \cdot \mathbf{r}_{\mathbf{v}^2}'' \end{cases} \tag{10.12.2}$$

Remark. The reader should be careful with the notations for the coefficients of the second fundamental form. For them there is, also, used the notation e, f, g. Also, in some books, the letters L, M, N are used to denote the coefficients of the second fundamental form themselves. The notations D, D', D'' are usually credited to Gauss (in *Disquisitiones*). It should be noted, however, that for Gauss, the meaning of the symbols is a little different: they are not the coefficients of the second fundamental form as we know it, but rather these coefficients multiplied by $\sqrt{EG - F^2}$.

Example 10.7. For the sphere $S = S_R^2$, we have, as we saw before, $\mathbf{n} = \frac{1}{R}\{x, y, z\}$, therefore, as we mentioned earlier, the shape operator A is a homothety of ratio $1/R$, i.e.

$$A(\mathbf{p}) = \frac{1}{R}\mathbf{p}, \quad , \forall \mathbf{p} \in T_a S^2 R.$$

Thus,

$$\varphi_2(\mathbf{p},\mathbf{q}) = -\varphi_1\left(\frac{1}{R}\mathbf{p},\mathbf{q}\right) = -\frac{1}{R}\varphi_1(\mathbf{p},\mathbf{q}) = -\frac{1}{R}\mathbf{p}\cdot\mathbf{q}.$$

therefore, for the sphere, the first two fundamental forms are proportional. Clearly, the same is true for the plane, when the second fundamental form vanishes identically. Surprisingly as it might seem, actually, it can be shown that these are the only two surfaces with this property.

10.13 The normal curvature. The Meusnier's theorem

Let S be an oriented surface and $\mathbf{n}$ – the unit normal. We take a regular parameterized curve, $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$, lying on S.

Definition 10.11. The projection of the curvature vector $\mathbf{k}(t)$ of $\boldsymbol{\rho}$ (as a signed scalar) on $\mathbf{n}(\boldsymbol{\rho}(t))$ is called the *normal curvature* of the curve $\boldsymbol{\rho}(t)$ at t and it is denoted by $k_n(t)$.

Let $\theta(t)$ be the angle between the osculating plane of $\boldsymbol{\rho}(t)$ and $\mathbf{n}(\boldsymbol{\rho}(t))$. Then, clearly,

$$k_n(t) = k(t)\cdot\cos\theta(t), \tag{10.13.1}$$

where $k(t)$ is the curvature of the curve $\boldsymbol{\rho}(t)$.

Examples.

1. The normal curvature of any plane curve is zero (in this case the angle $\theta(t)$ is always $\frac{\pi}{2}$, therefore $\cos\theta(t)\equiv 0$).
2. If the support of a parameterized curve lies on a straight line, then its normal curvature is always zero, no matter on what surface is the curve situated, because, this time, the curvature of the curve vanishes identically.

Remark. The relation (10.13.1) has a simple geometrical interpretation (the Meusnier's theorem): the center of curvature of a curve $\boldsymbol{\rho}$, lying on a surface S, is the orthogonal projection on its osculating plane of the center of curvature of the normal section tangent to $\boldsymbol{\rho}$ at that point.

The normal curvature of a curve on a surface can be expressed easily if we know the first two fundamental forms of the surface. Indeed, we have:

Theorem 10.9. *The normal curvature of a parameterized curve $\boldsymbol{\rho}(t)$, lying on an oriented surface S, is given by the formula*

$$k_n(t) = \frac{\varphi_2(\boldsymbol{\rho}'(t),\boldsymbol{\rho}'(t))}{\varphi_1(\boldsymbol{\rho}'(t),\boldsymbol{\rho}'(t))}. \tag{10.13.2}$$

Proof As is many times the case with parameterized curves, the proof is simpler in the case of naturally parameterized curves. Since the curvature of any regular parameterized curve is invariant to a parameter change, we can replace the curve $\boldsymbol{\rho}(t)$ by a n equivalent, naturally parameterized curve, $\boldsymbol{\rho}_1(s)$, the natural parameter being the arc length. The curvature vector of the curve $\boldsymbol{\rho}_1(s)$ will be $\boldsymbol{\rho_1}''(s)$. We choose a local parameterization $(U, \mathbf{r})$ of the surface S and we assume that $\boldsymbol{\rho}_1(s)$ has in this parameterization the local equations $u = u(s),\ v = v(s)$, i.e. $\boldsymbol{\rho}_1(s) = \mathbf{r}(u(s), v(s))$. Then

$$\boldsymbol{\rho_1}''(s) = \mathbf{r}''_{\mathbf{u}^2} \cdot (u')^2 + 2\mathbf{r}''_{\mathbf{uv}} \cdot u'v' + \mathbf{r}''_{\mathbf{v}^2} \cdot (v')^2 + \mathbf{r}'_{\mathbf{u}} \cdot u'' + \mathbf{r}'_{\mathbf{v}} \cdot v''.$$

Thus, for the normal curvature of the curve $\boldsymbol{\rho}_1(s)$ we get the expression:

$$\begin{aligned} k_n(s) &= \mathbf{k}(s) \cdot \mathbf{n}(\boldsymbol{\rho}_1(s)) = \boldsymbol{\rho_1}''(s) \cdot \mathbf{n}(\boldsymbol{\rho}_1(s)) = \\ &= \mathbf{r}''_{\mathbf{u}^2} \cdot \mathbf{n} \cdot (u')^2 + 2\mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{n} \cdot u'v' + \mathbf{r}''_{\mathbf{v}^2} \cdot \mathbf{n} \cdot (v')^2 = \\ &= -L \cdot (u')^2 - 2M \cdot u'v' - N \cdot (v')^2 = \varphi_2(\boldsymbol{\rho_1}'(s), \boldsymbol{\rho_1}'(s)). \end{aligned}$$

Now, we come back to the initial parameterized curve. We have

$$\boldsymbol{\rho}'(s) = \boldsymbol{\rho_1}'(s(t)) \cdot s'(t) \qquad \text{where} \quad s'(t) \equiv \|\boldsymbol{\rho}'(t)\|.$$

Thus,

$$\boldsymbol{\rho_1}'(s(t)) = \frac{\boldsymbol{\rho}'(t)}{\|\boldsymbol{\rho}'(t)\|},$$

therefore,

$$k_n(t) = \varphi_2\left(\frac{\boldsymbol{\rho}'(t)}{\|\boldsymbol{\rho}'(t)\|}, \frac{\boldsymbol{\rho}'(t)}{\|\boldsymbol{\rho}'(t)\|}\right) = \frac{1}{\boldsymbol{\rho}'(t) \cdot \boldsymbol{\rho}'(t)} \cdot \varphi_2(\boldsymbol{\rho}'(t), \boldsymbol{\rho}'(t)) = \frac{\varphi_2(\boldsymbol{\rho}'(t), \boldsymbol{\rho}'(t))}{\varphi_1(\boldsymbol{\rho}'(t), \boldsymbol{\rho}'(t))}.$$

□

Corollary 10.2. *If two curves on an oriented surface have a common point and they have the same tangent line at this point, then the two curves have the same normal curvature at the contact point.*

Proof Let $\mathbf{p}$ and $\mathbf{q}$ be the tangent vectors to the two curves at their common point. From the hypothesis, $\mathbf{p} = \alpha\mathbf{q}$, hence, from the theorem,

$$k_n = \frac{\varphi_2(\mathbf{p}, \mathbf{p})}{\varphi_1(\mathbf{p}, \mathbf{p})} = \frac{\varphi_2(\alpha\mathbf{q}, \alpha\mathbf{q})}{\varphi_1(\alpha\mathbf{q}, \alpha\mathbf{q})} = \frac{\alpha^2\varphi_2(\mathbf{q}, \mathbf{q})}{\alpha^2\varphi_1(\mathbf{q}, \mathbf{q})} = \frac{\varphi_2(\mathbf{q}, \mathbf{q})}{\varphi_1(\mathbf{q}, \mathbf{q})}$$

□

Remark. The previous consequence can be interpreted in another way. We take a family of biregular parameterized curves on the surface, $\{\boldsymbol{\rho}^\alpha(t)\}_{\alpha\in A}$, passing through the same point and having the same tangent line at the contact point. We denote by k^α the curvature of the curve $\boldsymbol{\rho}^\alpha$ and by θ^α the angle between the normal to the surface and the osculating plane of the curve $\boldsymbol{\rho}^\alpha$. The consequence is equivalent with the affirmation that the product $k_n = k^\alpha \cos\theta^\alpha$ does not depend on the choice of the curve from the family. It makes sense, thus, to choose an arbitrary straight line in the tangent plane, passing through the tangency plane, and to speak about the normal curvature of the *surface* in the direction of this line or, in other words, we can define a map k_n on the set of all non-vanishing tangent vectors to the surface with real values, by putting

$$k_n(\mathbf{h}) = \frac{\varphi_2(\mathbf{h},\mathbf{h})}{\varphi_1(\mathbf{h},\mathbf{h})}. \tag{10.13.3}$$

The quantity $k_n(\mathbf{h})$ is called the *normal curvature of the surface in the direction of the vector* $\mathbf{h}$ (since, clearly, it depends only on the *direction* of the vector $\mathbf{h}$, but not on its length or sense.) Thus, *the normal curvature* of an oriented surface in the direction of a vector $\mathbf{h}$ is the normal curvature of an arbitrary curve, lying on the surface, passing through the origin of $\mathbf{h}$ and whose tangent line is parallel to $\mathbf{h}$.

10.14 Asymptotic directions and asymptotic lines on a surface

We saw previously that the normal curvature of a surface at a given point and in a given direction can be expressed in terms of the first two fundamental forms of the surface and that, as stressed, although it is initially defined in terms of *curves* on surface, it actually depends only on the direction of the tangent vector of the curve. It is interesting for us to identify those directions for which the normal curvature vanish.

Definition 10.12. Let S be an oriented surface and $p \in S$. A non-vanishing vector $\mathbf{h} \in T_pS$ is said to have *asymptotic direction* if the normal curvature in its direction vanishes. Alternatively, based on the previous section, we can define a vector of asymptotic direction as being one for which

$$\varphi_2(\mathbf{h},\mathbf{h}) = 0.$$

Accordingly, an *asymptotic line or curve* on a surface is a curve on the surface for which all the tangent vectors have asymptotic direction.

Theorem 10.1. *Let $p \in S$ a point of an oriented surface. Then at this point we have asymptotic directions if and only if the quadratic form associated to the second fundamental form of S at p is negatively semi-defined. If we choose a local parameterization $(U, \mathbf{r})$ of S such that $p\mathbf{r}(u_0, v_0)$ for some $(u_0, v_0) \in U$, then this condition simply means that*

$$D(u_0, v_0) \cdot D''(u_0, v_0) - D'^2(u_0, v_0) \leq 0. \tag{10.14.1}$$

Proof Let $\mathbf{h} = \{h_1, h_2\} \in T_pS$ be a non-vanishing tangent vector to the surface at the point p. Then, with respect to the local parameterization chosen, $\mathbf{h}$ has asymptotic direction if and only if

$$D(u_0, v_0)h_1^2 + 2D'(u_0, v_0)h_1h_2 + D''(u_0, v_0)h_2^2 = 0.$$

As $\mathbf{h} \neq 0$, we may assume, for instance, that $h_2 \neq 0$. Then the previous equation can be written as

$$D(u_0, v_0)\left(\frac{h_1}{h_2}\right)^2 + 2D'(u_0, v_0)\frac{h_1}{h_2} + D''(u_0, v_0) = 0.$$

This equation, obviously has *real* solutions if and only if discriminant vanishes, but this is exactly the condition 10.14.1. □

Remark. From the proof of the previous theorem it follows that when the second fundamental form is negatively defined we have two asymptotic directions, while at the points when it is degenerate we have only one asymptotic direction (or two confounded ones).

From the definition of the normal curvature of a curve on a surface, it follows immediately that

Property 20. *Any straight line lying on a surface is an asymptotic line.*

Proof Indeed, the straight lines have zero curvature, therefore their normal curvature at each point also vanishes. □

The differential equation of the asymptotic lines on a surface can be obtained directly from the definition.

Theorem 10.2. *Let S be an oriented surface and $\boldsymbol{\rho} : I \to S$ – curve on the surface. We assume that there is a local parameterization of S, $(U, \mathbf{r} = \mathbf{r}(u, v))$ such* *that $\boldsymbol{\rho}(I) \subset \mathbf{r}(U)$ and the local equations of the curve with respect to this parameterization are $u = u(t), v = v(t)$. Then $\boldsymbol{\rho}$ is an asymptotic line on S if and only if*

$$D(u(t), v(t)) \cdot u'^2(t) + 2 \cdot D'(u(t), v(t)) \cdot u'(t) \cdot v'(t) + D''(u(t), v(t)) \cdot v'^2(t) = 0. \tag{10.14.2}$$

Proof The equation (10.14.2) is just that condition for the tangent vector of the curve (which has, with respect to the natural basis of the tangent space, the components $\{u'(t), v'(t)\}$) to have asymptotic direction. □

Let us assume, now, that the curve $\boldsymbol{\rho}$ from the previous theorem is biregular, which means, in particular, that is curvature is always strictly positive. From the definition of the normal curvature it follows immediately that, if $\boldsymbol{\nu}$ is the unit principal normal vector of the curve, then the curve is anasymptotic line if and only if

$$\boldsymbol{\nu}(t) \cdot \mathbf{n}(u(t), v(t)) = 0,$$

where $\mathbf{n}$ is the unit normal vector to the surface. This actually means that the principal normal of the curve lies on the tangent plane of the surface, at each point of the curve. Thus, we obtain the following characterization of the asymptotic lines:

Theorem 10.3. *Let S be an oriented surface and $\boldsymbol{\rho}$ a biregular parameterized curve on S. Then $\boldsymbol{\rho}$ is an asymptotic line if and only if at each point its osculating plane coincide with the tangent plane to the surface at that point.*

Another immediate result concerning the asymptotic lines is the following:

Property 21. *Let S be an oriented surface and $(U, \mathbf{r} = \mathbf{r}(u, v))$ – a local parameterization of S. Then the coordinate lines $u = const$ and $v = const$ are asymptotic lines on $\mathbf{r}(U)$ if and only if $D = D'' = 0$.*

Example 10.8. Let us consider the helicoid, given by the parameterization

$$\begin{cases} x = u \cos v, \\ y = u \sin v \\ z = b \cdot v \end{cases}$$

where b is a constant. A straightforward computation leads to $D = D'' = 0$, $D'(u, v) = -b/\sqrt{b^2 + u^2}$. This means that, in this particular case, we have, at each point of the surface, two asymptotic lines and these are nothing but the coordinate lines, $u = const$ and $v = const$. We notice that the helicoid is a ruled surface (see the last chapter) and at each point, one of the coordinate lines is a straight line (or a line segment). This is, of course, the line $v = const$ (see the figure 10.14).

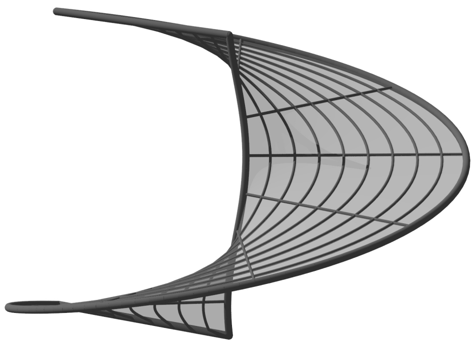


Figure 10.5: Asymptotic lines on the helicoid

10.15 The classification of points on a surface

The first fundamental form of a surface is positively defined. The second one is not. This is fortunate, as it allows us to give a classification of the points of the surface, according to the sign of the second fundamental form or, more specifically, according to the sign of its discriminant $DD'' - D'^2$.

Definition 10.13. A point $a \in S$ of an oriented surface is called

(i) *elliptic* if the second fundamental form is positively defined at a;

(ii) *parabolic* if the second fundamental form is zero, but at least one of the coefficients is different from zero;

(iii) *hyperbolic*, if the second fundamental form is negatively defined at a;

(iv) *flat*, or planar, if all the coefficients of the second fundamental form vanish at a.

It is not difficult to see that this definition do not depend on the choice of the local parameterization. Let us discuss, now separately, what happens in each particular case and, also, give some examples.

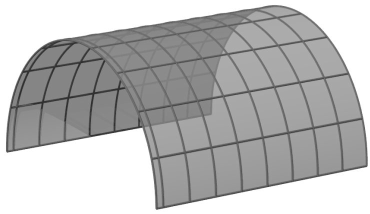


Figure 10.6: Parabolic points on a surface

Elliptic points. At an elliptic point, the normal curvature has the same signs in all directions[4]. Applying the Meusnier's theorem, this means that the centers of curvature of all normal sections lie on the same side of the surface. An example of surface that has only elliptic points is the ellipsoid, given, for instance, by the parameterization

$$\mathbf{r}(u, v) = (a \cos u \cos v, b \sin u \cos v, c \sin v). \tag{10.15.1}$$

At an elliptic point there is no real asymptotic direction, therefore no asymptotic line passes through an elliptic point.

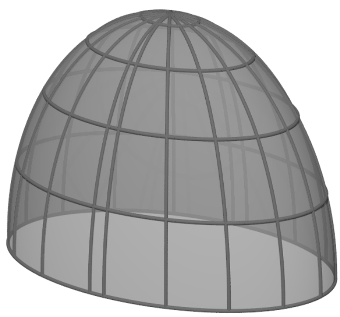


Figure 10.7: Elliptic points on a surface

Parabolic points. In this case the normal curvature does not change the sign, but there is exactly one direction where it vanishes. This is, clearly, an asymptotic direction. Thus, through a parabolic point of a surface passes only one asymptotic line. The cylinders and cones (with the apex removed) have only parabolic points.

[4]It is not necessarily positive.

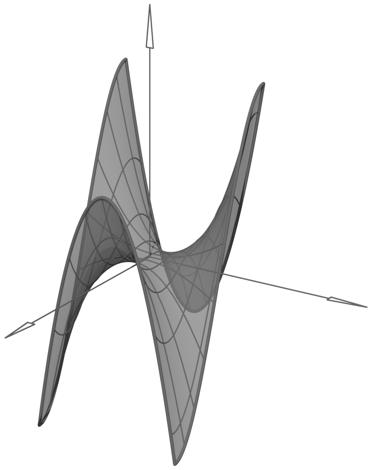


Figure 10.8: The monkey saddle

Hyperbolic points. In the case of hyperbolic points, it is possible for k_n to change sign and there are exactly two direction where it vanishes. Thus, through a hyperbolic point of a surface pass two asymptotic lines. The hyperbolic points are also called

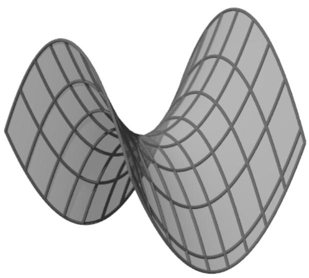


Figure 10.9: Hyperbolic points on a surface

saddle points. Such points we can find, for instance, on a hyperbolic paraboloid.

There are, of course, many surfaces on which we can meet all three kind of points (for instance on the torus).

Flat points. The shape of the surface around a flat point might be quite complicated and it is difficult to study it. In fact, in many cases, when proving a theorem in surface theory one explicitly assumes that the surface has no flat points. The surface from the figure 10.15 (the *monkey saddle* has a flat point at the origin of coordinates.

10.16 Principal directions, principal curvatures, Gauss curvature and mean curvature

Definition 10.14. The directions on the tangent plane to an oriented surface S at a point $a \in S$, $T_a S$, corresponding to the eigenvectors of the shape operators A are called the *principal directions* of the surface at the point a.

Remark. At each point, an oriented surface either has two orthogonal principal directions (if the eigenvalues of A are distinct), either all the directions are orthogonal (if the two eigenvalues coincide).

Definition 10.15. A curve (Γ) on a surface S is called a *principal line* or a *curvature line* if its tangent, at each point, has a principal direction.

Definition 10.16. A *principal curvature* of an oriented surface S at a point $a \in S$ is the normal curvature of S at a in a principal direction.

Property 22. *The principal curvatures of a surface are the eigenvalues of the shape operator, taken with opposite sign.*

Proof If $\mathbf{e}$ is an eigenvector of A, then $A(\mathbf{e}) = \lambda \cdot \mathbf{e}$, where λ is the eigenvalue corresponding to $\mathbf{e}$, therefore

$$k_n(\mathbf{e}) = \frac{\varphi_2(\mathbf{e}, \mathbf{e})}{\varphi_1(\mathbf{e}, \mathbf{e})} = \frac{-A(\mathbf{e}) \cdot \mathbf{e}}{\mathbf{e} \cdot \mathbf{e}} = \frac{-\lambda \cdot \mathbf{e} \cdot \mathbf{e}}{\mathbf{e} \cdot \mathbf{e}} = -\lambda.$$

□

Hereafter we shall denote by k_1 and k_2 the principal curvatures and we will always assume that $k_1 \geq k_2$.

Definition 10.17. An orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of the tangent space at a point of a surface is called a *basis of principal directions* of the tangent space if the vectors of the basis have principal directions.

Thus, the vectors of a basis of principal directions are subject to

$$A(\mathbf{e}_i) = -k_i \mathbf{e}_i, \quad i = \overline{1, 2}.$$

We shall fix now a point of the surface and ask the following question: find the normal curvature in the direction of a vector **e**, such that $\angle(\mathbf{e}, \mathbf{e}_1) = \theta$.

As the length of **e** is not important, we shall assume that **e** is a unit vector: $\|\mathbf{e}\| = 1$. Then $\mathbf{e} = \mathbf{e}_1 \cdot \cos\theta + \mathbf{e}_2 \cdot \sin\theta$, therefore

$$k_n(\mathbf{e}) = \frac{\varphi_2(\mathbf{e}, \mathbf{e})}{\varphi_1(\mathbf{e}, \mathbf{e})} = \frac{-A(\mathbf{e}) \cdot \mathbf{e}}{\underbrace{\mathbf{e} \cdot \mathbf{e}}_{=1}} = -A(\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta) \cdot (\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta) =$$

$$= (k_1 \cos\theta \cdot \mathbf{e}_1 + k_2 \sin\theta \cdot \mathbf{e}_2) \cdot (\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

Thus, we obtained:

Theorem 10.10. *Let S be an oriented surface. Then the normal curvature at a point of the surface, in the direction of a vector* **e**, *is given by the* Euler's formula*:*

$$k_n(\mathbf{e}) = k_1 \cos^2\theta + k_2 \sin^2\theta, \tag{10.16.1}$$

where k_1 and k_2 are the principal curvature of the surface, while $\theta = \angle(\mathbf{e}, \mathbf{e}_1)$.

An immediate consequence of the Euler's formula is:

Theorem 10.11. *The principal curvatures of a surface at a point are extreme values of the normal curvature of the surface in the direction of a vector* **e**, *when the vector* **e** *rotates around the origin of the tangent space of the surface at that point.*

Proof From the Euler's formula, we have

$$k_n(\mathbf{e}) = k_1 \cos^2\theta + k_2(1 - \cos^2\theta) = k_2 + (k_1 - k_2)\cos^2\theta.$$

It is clear now that the maximum value of the normal curvature is reached for $\theta = 0$ (we assumed that $k_1 \geq k_2$!) and, in this case, we get $k_n = k_1$, while the minimal value – for $\theta = \frac{\pi}{2}$, obtaining $k_n = k_2$. □

Definition 10.18. The quantities $K_t = k_1 \cdot k_2$ and $K_m = \frac{1}{2}(k_1 + k_2)$ are called, respectively, the *total* (Gaussian) and *mean* curvature of the surface.

The total and mean curvatures of a surface can be computed easily if the matrix of the shape operator in an arbitrary basis is known. Indeed, we have:

Proposition 10.1.

$$K_m = -\frac{1}{2}\mathcal{A} \tag{10.16.2}$$

$$K_t = \det \mathcal{A}. \tag{10.16.3}$$

Proof As it is well known from linear algebra, the determinant and the trace are invariants of any linear operator, which means that they are the same in any basis, although the matrix of the operator does change, generally, if we modify the basis. In a basis of principal directions, since the eigenvalues of the shape operator are just the opposite of the principal curvature, the matrix of the shape operator will be:

$$\mathcal{A} = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}$$

therefore

$$\begin{aligned} &\det \mathcal{A} = k_1 \cdot k_2 = K_t \\ &-\frac{1}{2}\mathcal{A} = -\frac{1}{2}(-k_1 - k_2) = \frac{1}{2}(k_1 + k_2) = K_m. \end{aligned}$$

□

Joachimstahl's theorem

We will see below how can one find the lines of curvature of a surface by integrating a differential equation. Anyway, in some special situations it is possible to find these lines using other methods. For instance, sometimes, if we know the curvature lines of a surface it is possible to find such lines on another surface. Such an instance is exemplified by the following theorem, belonging to the German mathematician Joachimstahl.

Theorem. *Let γ be a curve lying at the intersection of two regular oriented surfaces S_1 and S_2 from $\mathbb{R}^3$. Let $\mathbf{n}_i$ be the unit normals of the two surfaces ($i = \overline{1,2}$). Let us assume that S_1 and S_2 intersects under a constant angle, i.e. along the curve γ we have $\mathbf{n}_1 \cdot \mathbf{n}_2 = const.$. Then γ is a curvature line on S_1 iff it is a curvature line on S_2.*

Proof Let $\mathbf{r} = \mathbf{r}(t)$ be a local parameterization of the curve γ. Then, since $\mathbf{n}_1 \cdot \mathbf{n}_2 =$ const, we have

$$0 = \frac{d}{dt}(\mathbf{n}_1 \cdot \mathbf{n}_2) = \mathbf{n}_1' \cdot \mathbf{n}_2 + \mathbf{n}_1 \cdot \mathbf{n}_2'.$$

If γ is a principal line on S_1, then

$$\mathbf{n}_1' = -k_1 \cdot \mathbf{r}',$$

where k_1 is one of the principal curvatures of the surface S_1. On the other hand, since the curve γ lies also on S_2, we have $\mathbf{r}' \perp \mathbf{n}_2$. From here and from the previous formula we obtain that $\mathbf{n}_1' \cdot \mathbf{n}_2 = 0$, therefore

$$\mathbf{n}_1 \cdot \mathbf{n}_2' = 0.$$

Since $\mathbf{n}_2' \perp \mathbf{n}_2$ (since $\mathbf{n}_2$ has constant length), it follows from here and from the previous equation that $\mathbf{n}_2' \perp \mathbf{r}'$ or, in other words, that there is a $k_2 \in \mathbb{R}$ such that

$$\mathbf{n}_2' = -k_2\mathbf{r}',$$

i.e. γ is a line of curvature on the surface S_2, too. $\square$

Corollary 10.3. *The meridians and the parallels on a revolution surface are lines of curvature.*

Proof Let γ be the rotating curve and S the resulted revolution surface. A meridian is obtained by intersecting a plane Π_m, passing through the rotation axis and S. If $p \in \Pi_m \cap S$, then the unit normal of the surface S, $\mathbf{n}(p)$, lies on the plane Π_m, therefore the normal of the surface and the normal of the plane Π_m make up a constant angle, equal to $\frac{\pi}{2}$. As for the plane the second fundamental form vanishes identically, the same is true for the shape operator, which means that all the plane curves are lines of curvature. It follows, that, in particular, the meridian is, also, a line of curvature in the plane Π_m, which implies, using the Joachimstahl theorem, that it is a line of curvature also for the surface S.

A parallel is an intersection curve between the surface S and a plane Π_p, passing through a point of the curve γ, perpendicular to the rotation axis. It is obvious, from symmetry reasons, that along a parallel we should have $\angle(\mathbf{n}, \Pi_p) = \text{const}$ and we apply again the reasoning we used before. $\square$

10.16.1 The determination of the lines of curvature

As we saw earlier, the lines of curvature are curves on a surfaces whose tangent vectors are eigenvectors of the shape operator. Therefore, before showing how one can find the lines of curvature, we will indicate a way to find the eigenvectors of the shape operator.

Lemma. *Let* $\mathbf{r} : U \to \mathbb{R}^3$ *be a local parameterization of an oriented surface* S. *A tangent vector* $\mathbf{v} = v_1\mathbf{r}_\mathbf{u}' + v_2\mathbf{r}_\mathbf{v}'$ *has principal direction iff*

$$\begin{vmatrix} v_2^2 & -v_1v_2 & v_1^2 \\ E & F & G \\ D & D' & D'' \end{vmatrix} = 0. \tag{10.16.4}$$

Proof Since $\mathbf{v}$ has principal direction iff it is an eigenvector of the shape operator A, i.e. $A(\mathbf{v}) = \lambda \cdot \mathbf{v}$, it follows that $\mathbf{v}$ has principal direction iff $A(\mathbf{v}) \times \mathbf{v} = 0$. But, from the definition,

$$A(\mathbf{v}) = \mathcal{A} \cdot \mathbf{v} = \left(\mathcal{G}^{-1} \cdot \mathcal{H}\right) \cdot \mathbf{v} = \frac{1}{H^2} \begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$
$$= \frac{1}{H^2} \begin{pmatrix} (GL - FM)v_1 + (GM - FN)v_2 \\ (-FL + EM)v_1 + (-FM + EN)v_2 \end{pmatrix}$$

or

$$A(\mathbf{v}) = \frac{1}{H^2} \overbrace{[(GL - FM)v_1 + (GM - FN)v_2]}^{\alpha_u} \mathbf{r}_{\mathbf{u}}' +$$
$$+ \frac{1}{H^2} \underbrace{[(-FL + EM)v_1 + (-FM + EN)v_2]}_{\alpha_v} \mathbf{r}_{\mathbf{v}}'.$$

Therefore,

$$A(\mathbf{v}) \times \mathbf{v} = 0 \iff (\alpha_u \mathbf{r}_{\mathbf{u}}' + \alpha_v \mathbf{r}_{\mathbf{v}}') \times (v_1 \mathbf{r}_{\mathbf{u}}' + v_2 \mathbf{r}_{\mathbf{v}}') = 0 \iff$$
$$\iff (\alpha_u \cdot v_2 - \alpha_v \cdot v_1) \cdot (\mathbf{r}_{\mathbf{u}}' \times \mathbf{r}_{\mathbf{v}}') = 0.$$

As $\mathbf{r}_{\mathbf{u}}' \times \mathbf{r}_{\mathbf{v}}' \neq 0$, since the surface is regular, it follows that

$$A(\mathbf{v}) \times \mathbf{v} = 0 \iff \alpha_u \cdot v_2 - \alpha_v \cdot v_1 = 0$$

or, having in mind the notations we made,

$$(FL - EM)v_1^2 + (GL - EN)v_1 v_2 + (GM - FN)v_2^2 = 0.$$

Using now the coefficients of the second fundamental form instead of the components of the matrix $\mathcal{H}$, the previous relation becomes

$$(ED' - FD)v_1^2 + (ED'' - GD)v_1 v_2 + (FD'' - GD')v_2^2 = 0. \tag{10.16.5}$$

which is just another form of the relation (10.16.4) □

Corollary 10.4 (the differential equation of the lines of curvature)**.** *Let γ be a curve lying in the domain $\mathbf{r}(U)$ of a local parameterization$(\mathbf{r}, U)$ of a surface S, with the local equation $\boldsymbol{\rho}(t) = \mathbf{r}(u(t), v(t))$. Then γ is a line of curvature on S iff*

$$(ED' - FD)u'^2(t) + (ED'' - GD)u'(t)v'(t) + (FD'' - GD')v'^2(t) = 0 \tag{10.16.6}$$

or

$$(ED' - FD) + (ED'' - GD)\frac{dv}{du} + (FD'' - GD')\left(\frac{dv}{du}\right)^2 = 0. \qquad (10.16.7)$$

Proof The relation (10.16.6) express, clearly, the condition for the vector $\boldsymbol{\rho}' = u'(t)\mathbf{r}_{\mathbf{u}}' + v'(t)\mathbf{r}_{\mathbf{v}}'$ to have principal direction, while (10.16.7) follows immediately from (10.16.6), by eliminating the parameter t. □

10.16.2 The computation of the curvatures of a surface

Theorem. *Let* $\mathbf{r} : U \to \mathbb{R}^3$ *be a local parameterization of an oriented surface* S*. Then the total and the mean curvatures of* S *are given, respectively, by the formulas:*

$$K_t = \frac{DD'' - D'^2}{H^2} \qquad (10.16.1)$$

$$K_m = \frac{DG - 2D'F + D''E}{2H^2}. \qquad (10.16.2)$$

Proof As we saw earlier, the matrix of the shape operator of a surface is given by

$$\mathcal{A} = \frac{1}{H^2}\begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix} \quad \text{or} \quad \mathcal{A} = \frac{1}{H^2}\begin{pmatrix} FD' - GD & FD'' - GD' \\ FD - ED' & FD' - ED'' \end{pmatrix},$$

therefore,

$$K_t = \det \mathcal{A} = \frac{1}{H^4}\Big[F^2D'^2 - EFD'D'' - FGDD' + EGDD'' - F^2DD'' + EFD'D'' +$$

$$+FGDD' - EGD'^2\Big] = \frac{1}{H^4}\left[\underbrace{(EG - F^2)}_{H^2}\cdot(DD'' - D'^2)\right] = \frac{DD'' - D'^2}{H^2},$$

$$K_m = -\frac{1}{2}\mathrm{Tr}\mathcal{A} = -\frac{1}{2H^2}\left(2FD' - GD - ED''\right) = \frac{DG - 2D'F + D''E}{2H^2}.$$

□

Corollary. *The principal curvatures* k_1 *and* k_2 *are the roots of the equation*

$$k^2 - 2K_m \cdot k + K_t = 0, \qquad (10.16.3)$$

i.e.

$$k_1 = K_m + \sqrt{K_m^2 - K_t}, \tag{10.16.4}$$

$$k_2 = K_m - \sqrt{K_m^2 - K_t}. \tag{10.16.5}$$

Corollary. *A non-flat point of a surface is*

1. *elliptic iff* $K_t > 0$*;*
2. *parabolic iff* $K_t = 0$*;*
3. *hyperbolic iff* $K_t < 0$.

10.17 The fundamental equations of a surface

10.17.1 Introduction

We saw that, in the case of curves, the curvature and the torsion completely determine a space curve (up to a rigid motion of the space). We may ask whether a similar result holds for the case of surfaces. It is not completely obvious what are the entities that should replace the curvature and the torsion, but, clearly, the first two fundamental forms could be good candidates. Thus, we can formulate the question in the following way: If we are given a domain $U \subset \mathbb{R}^2$ and two families of symmetric bilinear forms, with the coefficients depending smoothly on the coordinates on U, such that, at each point of U, the first form is positively defined, is there any regular parameterized surface $\mathbf{r} : U \to \mathbb{R}^3$ such that the two families of bilinear forms be the two first fundamental forms of this surface? The answer is not affirmative, because, as we shall see, the coefficients of the first two fundamental forms of a surface are not independent, therefore, our initial data should satisfy some compatibility conditions (which are, in fact, the integrability conditions for a system of partial differential equation). If, however, this conditions are fulfilled, then the negative answer turns into an affirmative one. It is the aim of this section to establish the compatibility conditions and to formulate the existence and uniqueness theorem for parameterized surfaces.

10.17.2 The differentiation rules. Christoffel's coefficients

If $(U, \mathbf{r})$ is a regular parameterized surface, then, for each $(u, v) \in U$, the vectors $\mathbf{r}'_u, \mathbf{r}'_v, \mathbf{n}$ for a basis of the vector space $\mathbb{R}^3_{\mathbf{r}(u,v)}$. Therefore, in particular, the derivatives of these vectors can be expressed in terms of the vectors themselves. We already saw how to express the derivatives of the normal versor. They define, essentially, the shape operator of the surface. We shall obtain now similar formulae for the second order derivatives of the radius vector. These formulae should be of the form

$$\begin{aligned} \mathbf{r}''_{u^2} &= \Gamma^1_{11}\mathbf{r}'_u + \Gamma^2_{11}\mathbf{r}'_v + A\,\mathbf{n} \\ \mathbf{r}''_{uv} &= \Gamma^1_{12}\mathbf{r}'_u + \Gamma^2_{12}\mathbf{r}'_v + B\,\mathbf{n} \\ \mathbf{r}''_{v^2} &= \Gamma^1_{22}\mathbf{r}'_u + \Gamma^2_{22}\mathbf{r}'_v + C\,\mathbf{n} \end{aligned} \tag{10.17.1}$$

The coefficients Γ from these equations are called the *Christoffel's coefficients (of the second kind)*. The coefficients A, B, C are easily identifiable as the coefficients of the second fundamental form. To get the others, first of all, we shall express the scalar products $\mathbf{r}''_{u^2} \cdot \mathbf{r}'_u, \mathbf{r}''_{u_2} \cdot \mathbf{r}'_v$ and the analogues in terms of the coefficients of the first fundamental form and its derivatives. We have, first of all, $\mathbf{r}'^{\,2}_u = E$, whence, differentiating with respect to u, we get

$$\mathbf{r}''_{u^2} \cdot \mathbf{r}'_u = \frac{1}{2}E'_u. \tag{10.17.2}$$

Differentiating the same equality with respect to v, we get

$$\mathbf{r}''_{uv} \cdot \mathbf{r}'_u = \frac{1}{2}E'_v \tag{10.17.3}$$

Exactly in the same manner, starting from the definition of the other two coefficients of the first fundamental form, we will obtain

$$\mathbf{r}''_{uv} \cdot \mathbf{r}'_v = \frac{1}{2}G'_u \tag{10.17.4}$$

$$\mathbf{r}''_{u^2} \cdot \mathbf{r}'_v = F'_u - \frac{1}{2}E'_v \tag{10.17.5}$$

$$\mathbf{r}''_{v^2} \cdot \mathbf{r}'_u = F'_v - \frac{1}{2}G'_u \tag{10.17.6}$$

$$\mathbf{r}''_{v^2} \cdot \mathbf{r}'_v = \frac{1}{2}G'_v \tag{10.17.7}$$

Returning to our problem, we multiply the first equation of (10.17.1) successively by $\mathbf{r}'_u$ and by $\mathbf{r}'_v$. Using the formulae (10.17.2) and (10.17.5), as well as the definitions of the coefficients of the first fundamental form, we get the system

$$\begin{cases} E\Gamma^1_{11} + F\Gamma^2_{11} = \frac{1}{2}E'_u \\ F\Gamma^1_{11} + G\Gamma^2_{11} = F'_u - \frac{1}{2}E'_u \end{cases}.$$

It is very easy to solve this system and we get

$$\begin{cases} \Gamma_{11}^1 = \dfrac{GE_u' - 2FF_u' + FE_v'}{2(EG - F^2)} \\ \Gamma_{11}^2 = \dfrac{2EF_u' - EE_v' - FE_u'}{2(EG - F^2)} \end{cases} \tag{10.17.8}$$

and, exactly in the manner, starting from the other two equations from (10.17.1), we get

$$\begin{cases} \Gamma_{12}^1 = \dfrac{GE_v' - FG_u'}{2(EG - F^2)} \\ \Gamma_{12}^2 = \dfrac{EG_u' - FE_v'}{2(EG - F^2)} \end{cases} \tag{10.17.9}$$

and

$$\begin{cases} \Gamma_{22}^1 = \dfrac{2GF_v' - GG_u' - FG_v'}{2(EG - F^2)} \\ \Gamma_{22}^2 = \dfrac{EG_v' - 2FF_v' - FG_u'}{2(EG - F^2)} \end{cases}. \tag{10.17.10}$$

As for the derivatives of the normal versor, looking back at the expression of the shape operator in terms of the coefficients of the first two fundamental forms, we shall get immediately the formulae

$$\begin{cases} \mathbf{n}_u' = \dfrac{FD' - GD}{EG - F^2}\mathbf{r}_u' + \dfrac{FD - ED'}{EG - F^2}\mathbf{r}_v' \\ \mathbf{n}_v' = \dfrac{FD'' - GD'}{EG - F^2}\mathbf{r}_u' + \dfrac{FD' - ED''}{EG - F^2}\mathbf{r}_v' \end{cases} \tag{10.17.11}$$

Remark. The formulae (10.17.11) were obtained by the German mathematician Julius Weingarten, therefore they are called the *Weingartens's formulae*. Clearly, these formulae uniquely define the shape operator. This is the reason why in many books the shape operator (or its opposite) is called the *Weingarten's operator* or *Weingarten's mapping*.

Christoffel's and Weingarten's coefficients in curvature coordinates

Let us assume that in our parameterization the coordinate lines are lines of curvature. Then, as we saw earlier, we should have $F = 0$ and $D' = 0$ on the entire support of the parameterization. Then, as one can convince oneself easy, the Christoffel's coefficients become:

$$\begin{cases} \Gamma_{11}^1 = \dfrac{1}{2}\dfrac{E_u'}{E} = \dfrac{\partial}{\partial u}\ln E, \quad \Gamma_{11}^2 = -\dfrac{E_v'}{2G}, \\ \Gamma_{12}^1 = \dfrac{\partial}{\partial v}\ln E, \quad \Gamma_{12}^2 = \dfrac{\partial}{\partial u}\ln G, \\ \Gamma_{22}^1 = -\dfrac{G_u'}{E}, \quad \Gamma_{22}^2 = \dfrac{\partial}{\partial v}\ln G, \end{cases} \tag{10.17.12}$$

while the Weingarten's equation become:

$$\begin{cases} \mathbf{n}_u' = -\dfrac{D}{E}\mathbf{r}_u' \\ \mathbf{n}_v' = -\dfrac{D''}{G}\mathbf{r}_v' \end{cases} \tag{10.17.13}$$

One should not be surprised that the partial derivatives of the unit normal vector are, each of them, colinear to one of the partial derivative of the radius vector. This is, actually, nothing but the definition of the lines of curvatures.

10.17.3 The Gauss' and Codazzi-Mainardi's equations for a surface

We shall prove now that between the coefficients of the first two fundamental forms of a parameterized surface exists some relations, that we will call the *Gauss' equations* and the *Codazzi-Mainardi's equations*, respectively. We summarize them in the following theorem.

Theorem 10.12 (Gauss, Codazzi, Mainardi)**.** *In any local parameterization of a surface, the following systems of equations are fulfilled,*

$$\begin{aligned} &\frac{\partial \Gamma_{11}^2}{\partial v} - \frac{\partial \Gamma_{12}^2}{\partial u} + \Gamma_{11}^1\Gamma_{11}^2 + \Gamma_{11}^2\Gamma_{21}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{12}^2 = EK_t \\ &\frac{\partial \Gamma_{12}^1}{\partial u} - \frac{\partial \Gamma_{11}^1}{\partial v} + \Gamma_{12}^2\Gamma_{12}^1 - \Gamma_{11}^2\Gamma_{22}^1 = FK_t \\ &\frac{\partial \Gamma_{22}^1}{\partial u} - \frac{\partial \Gamma_{12}^1}{\partial v} + \Gamma_{22}^1\Gamma_{11}^1 + \Gamma_{22}^2\Gamma_{12}^1 - \Gamma_{12}^1\Gamma_{12}^1 - \Gamma_{12}^2\Gamma_{22}^1 = GK_t \\ &\frac{\partial \Gamma_{12}^2}{\partial v} - \frac{\partial \Gamma_{22}^2}{\partial u} + \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{22}^1\Gamma_{11}^2 = FK_t, \end{aligned} \tag{10.17.14}$$

called the Gauss's equations*, and the relations*

$$\begin{aligned}\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}&=D\Gamma_{12}^{1}+D'(\Gamma_{12}^{2}-\Gamma_{11}^{1})-D''\Gamma_{11}^{2}\\ \frac{\partial D'}{\partial v}-\frac{\partial D''}{\partial u}&=D\Gamma_{22}^{1}+D'(\Gamma_{22}^{2}-\Gamma_{12}^{1})-D''\Gamma_{12}^{2},\end{aligned}\tag{10.17.15}$$

called the Codazzi-Mainardi's equations. *Here E, F, G are the coefficients of the first fundamental form of the surface, while D, D', D'' are the coefficients of the second fundamental form.*

Proof To simplify a little bit the notations, let us rewrite the Weingarten's equations as

$$\begin{cases}\mathbf{n}'_u=a_{11}\mathbf{r}'_u+a_{12}\mathbf{r}'_v\\ \mathbf{n}'_v=a_{21}\mathbf{r}'_u+a_{22}\mathbf{r}'_v\end{cases}.\tag{10.17.16}$$

We have, obviously, the relation

$$\mathbf{r}'''_{u^2v}-\mathbf{r}'''_{uvu}=0.$$

But

$$\mathbf{r}''_{u^2}=\Gamma_{11}^{1}\mathbf{r}'_u+\Gamma_{11}^{2}\mathbf{b}'_v+D\,\mathbf{n},$$

hence

$$\begin{aligned}\mathbf{r}'''_{u^2v}&=\frac{\partial\Gamma_{11}^{1}}{\partial v}\mathbf{r}'_u+\Gamma_{11}^{1}\mathbf{r}''_{uv}+\frac{\partial\Gamma_{11}^{2}}{\partial v}\mathbf{r}'_v+\Gamma_{11}^{2}\mathbf{r}''_{v^2}+\frac{\partial D}{\partial v}\,\mathbf{n}+D\,\mathbf{n}'_v=\\ &=\frac{\partial\Gamma_{11}^{1}}{\partial v}\mathbf{r}'_u+\Gamma_{11}^{1}(\Gamma_{12}^{1}\mathbf{r}'_u+\Gamma_{12}^{2}\mathbf{r}'_v+D'\,\mathbf{n})+\\ &+\frac{\partial\Gamma_{11}^{2}}{\partial v}\mathbf{r}'_v+\Gamma_{11}^{2}(\Gamma_{22}^{1}\mathbf{r}'_u+\Gamma_{22}^{2}\mathbf{r}'_v+D''\,\mathbf{n})+\frac{\partial D}{\partial v}\,\mathbf{n}+D(a_{12}\mathbf{r}'_u+a_{22}\mathbf{r}'_v)=\\ &=\mathbf{r}'_u\left(\frac{\partial\Gamma_{11}^{1}}{\partial v}+\Gamma_{11}^{1}\Gamma_{12}^{1}+\Gamma_{11}^{2}\Gamma_{22}^{1}+a_{12}D\right)+\\ &+\mathbf{r}'_v\left(\frac{\partial\Gamma_{11}^{2}}{\partial v}+\Gamma_{11}^{1}\Gamma_{12}^{2}+\Gamma_{11}^{2}\Gamma_{22}^{2}+a_{22}D\right)+\\ &+\mathbf{n}\left(\frac{\partial D}{\partial v}+\Gamma_{11}^{1}D'+\Gamma_{11}^{2}D''\right).\end{aligned}$$

Analogously,

$$\mathbf{r}''_{uv}=\Gamma_{12}^{1}\mathbf{r}'_u+\Gamma_{12}^{2}\mathbf{r}'_v+D'\,\mathbf{n},$$

hence

$$\begin{aligned}
\mathbf{r}'''_{uvu} &= \frac{\partial\Gamma^1_{12}}{\partial u}\mathbf{r}'_u + \Gamma^1_{12}\mathbf{r}''_{u^2} + \frac{\partial\Gamma^2_{12}}{\partial u}\mathbf{r}'_v + \Gamma^2_{12}\mathbf{r}''_{uv} + \frac{\partial D'}{\partial u}\,\mathbf{n} + D'\,\mathbf{n}'_u = \\
&= \frac{\partial\Gamma^1_{12}}{\partial u}\mathbf{r}'_u + \Gamma^1_{12}(\Gamma^1_{11}\mathbf{r}'_u + \Gamma^2_{11}\mathbf{r}'_v + D\,\mathbf{n}) + \\
&+ \frac{\partial\Gamma^2_{12}}{\partial u}\mathbf{r}'_v + \Gamma^2_{12}(\Gamma^1_{12}\mathbf{r}'_u + \Gamma^2_{12}\mathbf{r}'_v + D'\,\mathbf{n}) + \frac{\partial D'}{\partial u}\,\mathbf{n} + D'(a_{11}\mathbf{r}'_u + a_{21}\mathbf{r}'_v) = \\
&= \mathbf{r}'_u\left(\frac{\partial\Gamma^1_{12}}{\partial u} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} + a_{11}D'\right) + \\
&+ \mathbf{r}'_v\left(\frac{\partial\Gamma^2_{12}}{\partial u} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} + a_{21}D'\right) + \\
&+ \mathbf{n}\left(\frac{\partial D'}{\partial u} + \Gamma^1_{12}D + \Gamma^2_{12}D'\right).
\end{aligned}$$

If we subtract the previous relations, we obtain

$$\begin{aligned}
0 = \mathbf{r}'''_{u^2v} - \mathbf{r}'''_{uvu} = \mathbf{r}'_u &\left(\frac{\partial\Gamma^1_{11}}{\partial v} - \frac{\partial\Gamma^1_{12}}{\partial u} + \Gamma^2_{11}\Gamma^1_{22} - \Gamma^2_{12}\Gamma^1_{12} + a_{12}D - a_{11}D'\right) + \\
+ \mathbf{r}'_v &\left(\frac{\partial\Gamma^2_{11}}{\partial v} - \frac{\partial\Gamma^2_{12}}{\partial u} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + a_{22}D - a_{21}D'\right) + \\
+ \mathbf{n} &\left(\frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} + \Gamma^1_{12}D - D'(\Gamma^2_{12} - \Gamma^1_{11}) + \Gamma^2_{12}D''\right).
\end{aligned}$$

As the vectors $\mathbf{r}'_u, \mathbf{r}'_v$ and $\mathbf{n}$ are linearly independent, a linear combination of them can vanish only if each coefficient vanishes. If we equate to zero the coefficient of $\mathbf{n}$, one sees that one gets the first of the Codazzi-Mainardi's equations. If we equate to zero the coefficient of $\mathbf{r}'_u$ follows, using the expressions of a_{21} and a_{22}, the first of the Gauss' equations, while from the coefficient of $\mathbf{r}'_v$ follow the second of the Gauss's equations. The other three equations can be obtained in the same way, using, this time, the relation

$$\mathbf{r}'''_{uv^2} = \mathbf{r}'''_{vuv}.$$

□

Corollary 10.17.1. *If the coordinate lines are lines of curvature, then the Codazzi-*

Mainardi equations can be written a lot simpler:

$$\begin{cases} \dfrac{\partial D}{\partial v} = D\dfrac{\partial \ln E}{\partial v} + \dfrac{D''}{2G}\dfrac{\partial E}{\partial v} \\ \dfrac{\partial D''}{\partial u} = \dfrac{D}{2E}\dfrac{\partial G}{\partial u} + D''\dfrac{\partial \ln G}{\partial u}. \end{cases} \tag{10.17.17}$$

10.17.4 The fundamental theorem of surface theory

The theorem we are going to prove in this section is the analogue of the existence and uniqueness theorem for space curves. It was first established by the French mathematician Ossian Bonnet, in 1860.

To shorten the formulas, in the following we shall denote the coordinates by u^1 and u^2 instead of u and v and we shall use the index notation to denote the components of the matrices of the first two fundamental forms of the surface. More precisely, for the coefficients of the first fundamental form we shall write

$$g_{11} = E, g_{12} = g_{21} = F, g_{22} = G, \tag{10.17.18}$$

while for the coefficients of the second fundamental form we shall write

$$h_{11} = D, h_{12} = h_{21} = D', h_{22} = D''. \tag{10.17.19}$$

Also, we shall denote by g the determinant of the matrix of the first fundamental form. Finally, we shall denote by $\mathbf{r_i'}$ the derivative of r with respect to the coordinate u^i and by $\mathbf{r_{ij}''}$ – the second order derivative of r with respect to the coordinates u^i and u^j, where i and j can take the values 1 and 2. Clearly, as we always assume that the surfaces are as as smooth as one expects to be (i.e., in this context, at least C^2), the order of differentiation is not relevant, therefore we shall always have $\mathbf{r_{ij}''} = \mathbf{r_{ji}''}$.

Theorem (Ossian Bonnet, 1860)**.** *Let $U \subset \mathbb{R}^2$ be an open set. On U there are given the symmetric matrix functions*

$$g_{ij} = g_{ij}(u^1, u^2),\ h_{ij} = h_{ij}(u^1, u^2),\ i, j = 1, 2 \tag{10.17.20}$$

of classes C^2 and C^1, respectively, such that for each $(u^1, u^2) \in U$ the quadratic for associated to the bilinear form whose matrix is (g_{ij}) is positively defined and, moreover, the components of the two functions verify the Gauss and Codazzi-Mainardi compatibility conditions. We choose $u_0 = (u_0^1, u_0^2) \in U$, $p_0 \in \mathbb{R}$ and the vectors

$$\mathbf{r_1'}^{(0)}, \mathbf{r_2'}^{(0)}, \mathbf{n}^{(0)} \in T_{p_0}\mathbb{R}^3, \tag{10.17.21}$$

such that $\mathbf{r}_{\mathbf{i}}'^{(0)} \cdot \mathbf{r}_{\mathbf{j}}'^{(0)} = g_{ij}(u_0)$, $\mathbf{n}^{(0)} \cdot \mathbf{r}_{\mathbf{i}}'^{(0)} = 0$, $\mathbf{n}^{(0)} \cdot \mathbf{n}^{(0)} = 1$, *while the triple* $\{\mathbf{r}_{\mathbf{1}}'^{(0)}, \mathbf{r}_{\mathbf{2}}'^{(0)}, \mathbf{n}^{(0)}\}$ *is a right-handed basis of the vector space* $T_{p_0}\mathbb{R}^3$. *Then there exists a single regular parameterized surface of class* C^3, $r : V \to \mathbb{R}^3$, *with* $V \subset U$ *– an open set, such that the following conditions are fulfilled:*

(i) $r(u_0) = p_0$ *(the surface "passes" through* p_0 *for* $u = u_0$*).*

(ii) $\dfrac{\partial r}{\partial u^i}(u_0) = \mathbf{r}_{\mathbf{i}}'^{(0)}$, $i = 1, 2$.

(iii) $\mathbf{n}(u_0) = \mathbf{n}^{(0)}$.

(iv) g_{ij} *and* h_{ij} *are the coefficients of the first two fundamental forms of the parameterized surface* r *(with respect to the orientation of* r *defined by the unit normal vector* $\mathbf{n}$*).*

Proof We consider the system of partial differential equations

$$\begin{cases} \dfrac{\partial \mathbf{r}_{\mathbf{i}}'}{\partial u^j} = \sum\limits_{k=1}^{2} \Gamma_{ij}^k \mathbf{r}_{\mathbf{k}}' + h_{ij}\mathbf{n}, \\ \dfrac{\partial \mathbf{n}}{\partial u^i} = -\sum\limits_{j=1}^{2}\sum\limits_{k=1}^{2} h_{ij} g^{jk} \mathbf{r}_{\mathbf{k}}' \end{cases} \tag{10.17.22}$$

with respect to the unknown functions $\mathbf{r}_{\mathbf{1}}', \mathbf{r}_{\mathbf{2}}', \mathbf{n}$, where the coefficients Γ_{ij}^k are computed with the formula (10.17.8)–(10.17.10). This is a linear and homogeneous system which is completely integrable, because the compatibility conditions

$$\begin{cases} \dfrac{\partial^2 \mathbf{r}_{\mathbf{i}}'}{\partial u^j \partial u^k} = \dfrac{\partial^2 \mathbf{r}_{\mathbf{i}}'}{\partial u^k \partial u^j} \\ \dfrac{\partial^2 \mathbf{n}}{\partial u^j \partial u^k} = \dfrac{\partial^2 \mathbf{n}}{\partial u^k \partial u^j} \end{cases} \tag{10.17.23}$$

are equivalent, as we saw, to the Gauss-Weingarten equations, which are satisfied, by hypothesis. Therefore, from standard results from the theory of partial differential equations of first order, it follows that there exists an open neighborhood $W \subset U$ of the point u_0 and a set of three C^2 vector functions[5] $\mathbf{r}_{\mathbf{1}}', \mathbf{r}_{\mathbf{2}}', \mathbf{n} : W \to \mathbb{R}^3$, which are solutions of the system (10.17.22), with the given initial condition at the point u_0.

[5]The order of smoothness is with one unit greater then the smallest order of smoothness of the coefficients, as the system is of first order.

Let us notice, also, that a set of initial conditions as in the theorem always exists, due, mainly, to the fact that the quadratic form associated to g_{ij} is positively defined. Indeed, we can make, for instance, the following choice:

$$\begin{cases} \mathbf{r}_1'^{(0)} = \left\{\sqrt{g_{11}(u_0)}, 0, 0\right\} \\ \mathbf{r}_2'^{(0)} = \left\{\dfrac{g_{12}(u_0)}{\sqrt{g_{11}(u_0)}}, \dfrac{\sqrt{g_{11}(u_0)g_{22}(u_0) - (g_{12}(u_0))^2}}{\sqrt{g_{11}(u_0)}}, 0\right\} \\ \mathbf{n}^{(0)} = \{0, 0, 1\} \end{cases} . \tag{10.17.24}$$

We leave to the reader to check that, indeed, these vectors verify the conditions from the hypothesis.

We consider, now, the system of partial differential equations

$$\begin{cases} \dfrac{\partial r}{\partial u^1} = \mathbf{r}_1', \\ \dfrac{\partial r}{\partial u^2} = \mathbf{r}_2' \end{cases} . \tag{10.17.25}$$

This system is, again, completely integrable, because the integrability condition

$$\frac{\partial^2 r}{\partial u^i \partial u^j} = \frac{\partial^2 r}{\partial u^j \partial u^i} \tag{10.17.26}$$

is equivalent to the condition

$$\frac{\partial \mathbf{r}_\mathbf{i}'}{\partial u^j} = \frac{\partial \mathbf{r}_\mathbf{j}'}{\partial u^i} \tag{10.17.27}$$

which is true, as one can convince oneself looking at the first equation (10.17.22), because of the symmetry of the second matrix ($h_{ji} = h_{ij}$) and because of the symmetry of the Christoffel's coefficients in the lower indices. Therefore, applying again the existence and uniqueness theorem, it follows that there exists an open neighborhood $V \subset W \subset U$ of u_0 and a single C^3 function $r : V \to \mathbb{R}^3$ such that $r(u_0) = p_0$.

We are not done yet, because we still have to show that g_{ij} and h_{ij} are the first two fundamental forms of the parameterized surface defined by r. Apparently, we also have to show that r is regular. But this follows immediately if we prove that g is the first fundamental form, because

$$g_{ij} = \frac{\partial r}{\partial u^i} \cdot \frac{\partial r}{\partial u^j}$$

and then

$$\frac{\partial r}{\partial u^i} \times \frac{\partial r}{\partial u^j} \neq 0,$$

because the square of the norm of this vector is not zero (as it is equal to the determinant of the first fundamental form, which is strictly greater then zero, since the form is positively defined). It is, actually, enough to show that, all over V, the following relations are fulfilled:

$$\begin{cases} \mathbf{r}'_\mathbf{i} \cdot \mathbf{r}'_\mathbf{j} = g_{ij}, \\ \mathbf{r}'_\mathbf{i} \cdot \mathbf{n} = 0, \\ \mathbf{n} \cdot \mathbf{n} = 1 \end{cases} . \tag{10.17.28}$$

To this end, we shall compute the derivatives with respect to the coordinates of the scalar products, taking into account the Gauss-Weingarten equations and we get the system of first order partial differential equations:

$$\begin{cases} \dfrac{\partial(\mathbf{r}'_\mathbf{i} \cdot \mathbf{r}'_\mathbf{j})}{\partial u^k} = \sum\limits_{l=1}^{2} \Gamma^l_{ik}(\mathbf{r}'_\mathbf{l} \cdot \mathbf{r}'_\mathbf{j}) + \sum\limits_{l=1}^{2} \Gamma^l_{jk}(\mathbf{r}'_\mathbf{l} \cdot \mathbf{r}'_\mathbf{i}) + h_{ik}(\mathbf{r}'_\mathbf{j} \cdot \mathbf{n}) + h_{jk}(\mathbf{r}'_\mathbf{i} \cdot \mathbf{n}) \\ \dfrac{\partial(\mathbf{r}'_\mathbf{j} \cdot \mathbf{n})}{\partial u^i} = -\sum\limits_{l,k=1}^{2} h_{il} g^{lk}(\mathbf{r}'_\mathbf{k} \cdot \mathbf{r}'_\mathbf{j}) + \sum\limits_{l=1}^{2} \Gamma^l_{ij}(\mathbf{r}'_\mathbf{l} \cdot \mathbf{n}) + h_{ij}(\mathbf{n} \cdot \mathbf{n}) \\ \dfrac{\partial(\mathbf{n} \cdot \mathbf{n})}{\partial u^i} = -2 \sum\limits_{l,k=1}^{2} h_{il} g^{lk}(\mathbf{r}'_\mathbf{k} \cdot \mathbf{n}) \end{cases} . \tag{10.17.29}$$

As probably the reader suspects already (and it is asked to check for himself), this system is, again, completely integrable, which means that it has a single solution for prescribed initial values. We "guess" that this solution is, exactly, (10.17.28) and we shall check this in the following. For the last equation, there is nothing to check: if we substitute (10.17.28), we get, simply, $0 = 0$. For the second equation, after the substitution, the left-hand side is zero, while the right-hand side becomes:

$$-\sum_{l,k=1}^{2} h_{il} g^{lk} g_{kj} + h_{ij} = -\sum_{l=1}^{2} h_{il} \delta^l_j + h_{ij} = -h_{ij} + h_{ij} = 0,$$

so we are done. Finally, the first equation becomes

$$\frac{\partial g_{ij}}{\partial u^k} = \sum_{l=1}^{2} \Gamma^l_{ik} g_{kj} + \sum_{l=1}^{2} \Gamma^l_{jk} g_{li}$$

which is very easy to prove, using the definition of the Christoffel's coefficients. Thus, the functions (10.17.28) give a solution of the system (10.17.29), which, obviously, verify the initial conditions of the theorem. As the solution is unique, for the given initial conditions we have

$$\begin{cases} \mathbf{r_i'} \cdot \mathbf{r_j'} = g_{ij}, \\ \mathbf{r_i'} \cdot \mathbf{n} = 0, \\ \mathbf{n} \cdot \mathbf{n} = 1 \\ (\mathbf{r_1'}, \mathbf{r_2'}, \mathbf{n}) > 0 \end{cases}, \tag{10.17.30}$$

which means that we proved already that

- $\mathbf{r_i'}$ are the derivatives of r;
- $\mathbf{n}$ is the unit normal of the parameterized surface given by r;
- g_{ij} are the coefficients of the first fundamental form of r.

We are left with the proof of the fact that h_{ij} are the coefficients of the second fundamental form of r. Let us denote, for the moment, by b_{ij} these coefficients. As we know, they are given by

$$b_{il} = -\mathbf{n_i'} \cdot \mathbf{r_l'} = \sum_{j,k=1}^{2} h_{ij} g^{jk} \mathbf{r_k'} \cdot \mathbf{r_l'} = \sum_{j,k=1}^{2} h_{ij} g^{jk} g_{kl} = \sum_{j=1}^{2} h_{ij} \delta_l^j = h_{il},$$

which concludes the proof of the Bonnet's theorem. □

10.18 The Gauss' egregium theorem

The theorem that we are going to prove in this section (and which is implicit in the equations of Gauss above) is one of the most important theorem in classical differential geometry. Not by accident Gauss called it, in his famous *Disquisitiones circa superficies curvas* "theorema egregium", i.e. the *remarkable theorem*. As we just saw in the previous section, the total curvature of a surface in $\mathbb{R}^3$ can be expressed in terms of the determinants of the first two fundamental forms of the surface. This would mean, in principle, that the total curvature depends both on the intrinsic data (the first fundamental form) and extrinsic data (i.e. the second fundamental form). It turns out, however, that the situation is different, i.e. we have:

Theorem 10.4 (Gauss, 1827)**.** *The total curvature of a surface of class at least C^3 depends only on the coefficients of the first fundamental form of the surface and their first order derivatives with respect to the coordinates.*

Proof There exists several proofs of this theorem (which is called, in many books, using the Latin term used by Gauss, namely *theorema egregium*. The original proof, given by Gauss in 1827, is quite complicated. The proof we are going to give here is the first different proof, belonging to the German mathematician Richard Baltzer (1867)[6], although Struik credits the formula which will be established to the Italian mathematician Brioschi. We start with the formula

$$K_t = \frac{DD'' - D'^2}{EG - F^2},$$

which we rewrite in the form

$$K_t(EG - F^2) = DD'' - D'^2, \tag{10.18.1}$$

or, having in mind the expressions of the coefficients of the second fundamental form,

$$K_t(EG - F^2)^2 = (\mathbf{r}''_{\mathbf{u}^2}, \mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}) \cdot (\mathbf{r}''_{\mathbf{v}^2}, \mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}) - (\mathbf{r}''_{\mathbf{uv}}, \mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}})^2. \tag{10.18.2}$$

The right hand side of the equation (10.18.2) can be written in a more convenient form if we notice that each term is, in fact, the product of two determinants. We shall use, now, the following formula for the product of two determinants, known from vector algebra:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) \cdot (\mathbf{d}, \mathbf{e}, \mathbf{f}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{e} & \mathbf{a}\cdot\mathbf{f} \\ \mathbf{b}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{e} & \mathbf{b}\cdot\mathbf{f} \\ \mathbf{c}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{e} & \mathbf{c}\cdot\mathbf{f} \end{vmatrix}. \tag{10.18.3}$$

[6]The name of Richard Baltzer (1818–1887) is not well known today, however, in the second half of the nineteenth century he was considered an important geometer. For the history of mathematics, there are important his "Elementhe der Mathematik", published in several editions, where was mentioned, for the first time, the Non-Euclidean geometry.

Using the formula (10.18.3), the formula (10.18.2) becomes

$$K_t(EG-F^2)^2 = (\mathbf{r}''_{\mathbf{u}^2}\cdot\mathbf{r}''_{\mathbf{v}^2} - \mathbf{r}''^{\,2}_{\mathbf{uv}})(EG-F^2)+ \\ +\begin{vmatrix} 0 & \mathbf{r}''_{\mathbf{u}^2}\cdot\mathbf{r}'_{\mathbf{u}} & \mathbf{r}''_{\mathbf{u}^2}\cdot\mathbf{r}'_{\mathbf{v}} \\ \mathbf{r}'_{\mathbf{u}}\cdot\mathbf{r}''_{\mathbf{v}^2} & E & F \\ \mathbf{r}'_{\mathbf{v}}\cdot\mathbf{r}''_{\mathbf{v}^2} & F & G \end{vmatrix} - \\ -\begin{vmatrix} 0 & \mathbf{r}''_{\mathbf{uv}}\cdot\mathbf{r}'_{\mathbf{u}} & \mathbf{r}''_{\mathbf{uv}}\cdot\mathbf{r}'_{\mathbf{v}} \\ \mathbf{r}''_{\mathbf{uv}}\cdot\mathbf{r}'_{\mathbf{u}} & E & F \\ \mathbf{r}''_{\mathbf{uv}}\cdot\mathbf{r}'_{\mathbf{v}} & F & G \end{vmatrix}. \tag{10.18.4}$$

Thus, already a part of the terms involved in the computation of the total curvature are expressed in terms of the coefficients of the first fundamental form. As one can see immediately, the remaining terms are of two kinds: either a product of a second order derivative of **r** and a first order one, either a product of two second order derivatives. The first kind of terms are easier to be taken care of. Indeed, starting from the definition of the coefficients of the first fundamental form, $E = \mathbf{r}'_{\mathbf{u}}\cdot\mathbf{r}'_{\mathbf{u}}$, $F = \mathbf{r}'_{\mathbf{u}}\cdot\mathbf{r}'_{\mathbf{v}}$, $G = \mathbf{r}'_{\mathbf{v}}\cdot\mathbf{r}'_{\mathbf{v}}$ one obtains, differentiating with respect to the coordinates, the following expressions:

$$\begin{cases} \mathbf{r}''_{\mathbf{u}^2}\cdot\mathbf{r}'_{\mathbf{u}} & = \frac{1}{2}E'_u \\ \mathbf{r}''_{\mathbf{uv}}\cdot\mathbf{r}'_{\mathbf{u}} & = \frac{1}{2}E'_v \\ \mathbf{r}''_{\mathbf{v}^2}\cdot\mathbf{r}'_{\mathbf{v}} & = \frac{1}{2}G'_v \\ \mathbf{r}''_{\mathbf{uv}}\cdot\mathbf{r}'_{\mathbf{v}} & = \frac{1}{2}G'_u \\ \mathbf{r}''_{\mathbf{u}^2}\cdot\mathbf{r}'_{\mathbf{v}} & = F'_u - \frac{1}{2}E'_v \\ \mathbf{r}''_{\mathbf{v}^2}\cdot\mathbf{r}'_{\mathbf{u}} & = F'_v - \frac{1}{2}G'_u \end{cases}. \tag{10.18.5}$$

Differentiating once more the fourth equation above with respect to u and the fifth with respect to v and subtracting them side by side, we get also the expression for the products of second order derivatives of **r**:

$$\mathbf{r}''_{\mathbf{u}^2}\cdot\mathbf{r}''_{\mathbf{v}^2} - \mathbf{r}''^{\,2}_{\mathbf{uv}} = -\frac{1}{2}G''_{u^2} + F''_{uv} - \frac{1}{2}E''_{v^2}. \tag{10.18.6}$$

Combining everything we obtained, we get the following expression (due, as we said before, to Baltzer) for the total curvature

$$K_t = \frac{1}{(EG-F^2)^2}\begin{vmatrix} -\frac{1}{2}G''_{u^2} + F''_{uv} - \frac{1}{2}E''_{v^2} & \frac{1}{2}E'_u & F'_u - \frac{1}{2}E'_v \\ F'_v - \frac{1}{2}G'_u & E & F \\ \frac{1}{2}G'_v & F & G \end{vmatrix} - \\ - \frac{1}{(EG-F^2)^2}\begin{vmatrix} 0 & \frac{1}{2}E'_v & \frac{1}{2}G'_u \\ \frac{1}{2}E'_v & E & F \\ \frac{1}{2}G'_u & F & G \end{vmatrix}, \tag{10.18.7}$$

which concludes the proof, as we got an expression of K_t which, indeed, depends only on the coefficients of the first fundamental form and their derivatives up to second order. □

Exercise 10.18.1 (Frobenius). Show that the total curvature of a surface can be written, also, in the following form, easier to remember:

$$K_t = -\frac{1}{4(EG-F^2)^2}\begin{vmatrix} E & E'_u & E'_v \\ F & F'_u & F'_v \\ G & G'_u & G'_v \end{vmatrix} + \\ + \frac{1}{2\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\left(\frac{F'_v - G'_u}{\sqrt{EG-F^2}}\right) + \frac{\partial}{\partial v}\left(\frac{F'_u - E'_v}{\sqrt{EG-F^2}}\right)\right\}. \tag{10.18.8}$$

In the particular case of an orthogonal coordinate system ($F \equiv 0$), we get the following nice “divergence” expression for the total curvature:

$$K_t = -\frac{1}{2\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left(\frac{G'_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E'_v}{\sqrt{EG}}\right)\right\}, \tag{10.18.9}$$

which will be used later for some integral formulae.

Exercise 10.18.2 (Liouville). Prove the following (slightly asymmetric) formula for the total curvature of a surface:

$$K_t = -\frac{1}{2\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\left(\frac{G'_u + \frac{F}{G}G'_v - 2F'_v}{\sqrt{EG-F^2}}\right) + \frac{\partial}{\partial v}\left(\frac{E'_v - \frac{F}{G}G'_u}{\sqrt{EG-F^2}}\right)\right\}. \tag{10.18.10}$$

10.19 Geodesics

10.19.1 Introduction

The curves we are going to study in this chapter are direct generalizations of the straight lines. More specifically, they are curves that projects on the tangent planes of the surface as straight lines. There are many different approaches to geodesics. We chose here the one we find more elementary.

10.19.2 The Darboux frame. The geodesic curvature and geodesic torsion

Let $(I, \boldsymbol{\rho})$ be a parameterized curve whose support lies on a surface S and let $M = \boldsymbol{\rho}(t_0)$ be a point of the curve, with $t_0 \in I$. As $\boldsymbol{\rho}$ is, in particular, a space curve, we can attach to it the Frenet frame at the point M, $\{M; \boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$. As we saw in the first part of the book, this frame is good enough if we want to investigate the curve $\boldsymbol{\rho}$ as an *independent* object, but it is not very helpful if one wishes to study the connections between the curve and the surface. To this end, we shall introduce another orthonormal frame, which involves both vectors related to the curve and to the surface. The first vector of the new frame will be still the unit tangent vector of the curve, $\boldsymbol{\tau}$. The second, related, this time to the surface, is the unit normal of the surface, $\mathbf{n}$. The third one, let it be denoted by $\mathbf{N}$, will be chosen in such a way that the basis $\{\boldsymbol{\tau}, \mathbf{N}, \mathbf{n}\}$ be direct, or in other words, such that $(\boldsymbol{\tau}, \mathbf{N}, \mathbf{n}) = 1$. This means, of course, that

$$\mathbf{N} = \mathbf{n} \times \boldsymbol{\tau}.$$

Clearly, $\mathbf{N}$ lies in the normal plane of the curve at M, therefore it will be called the *unit tangential normal vector* of the curve. The name *tangential* comes, of course from the fact that $\mathbf{N}$ lies, also, in the tangent plane of the surface at M.

The frame $\{M; \boldsymbol{\tau}, \mathbf{N}, \mathbf{n}\}$ is called the *Darboux frame* or the Ribaucour-Darboux frame of the surface S along the curve $\boldsymbol{\rho}$.

The next step we are going to make is to compute the derivatives of the vectors of the Darboux frame and to obtain a set of linear differential equation which is similar to the Frenet frame and which will play an important role in the following sections. To get them, the intention is exactly to use the Frenet equations. Therefore, we shall start by expressing the vectors $\mathbf{N}$ and $\mathbf{n}$ in terms of the vectors of the Frenet frame. We denote by θ the angle between the vectors $\boldsymbol{\nu}$ and $\mathbf{n}$. Then, as one can see immediately that

$$\begin{cases} \boldsymbol{\nu} = \cos(\mathbf{N}, \boldsymbol{\nu}) \cdot \mathbf{N} + \sin(\mathbf{N}, \boldsymbol{\nu})\mathbf{n} \\ \boldsymbol{\beta} = \cos(\mathbf{N}, \boldsymbol{\beta}) \cdot \mathbf{N} + \sin(\mathbf{N}, \boldsymbol{\beta})\mathbf{n} \end{cases}$$

As $(\mathbf{N}, \boldsymbol{\nu}) = \frac{\pi}{2}$ and $(\mathbf{N}, \boldsymbol{\beta}) = \pi - \theta$, we get

$$\begin{cases} \boldsymbol{\nu} = \sin\theta \mathbf{N} + \cos\theta \mathbf{n} \\ \boldsymbol{\beta} = -\cos\theta \mathbf{N} + \sin\theta \mathbf{n} \end{cases}.$$

Conversely, we get

$$\begin{cases} \mathbf{N} = \sin\theta \cdot \boldsymbol{\nu} - \cos\theta \cdot \boldsymbol{\beta} \\ \mathbf{n} = \cos\theta \cdot \boldsymbol{\nu} + \sin\theta \cdot \boldsymbol{\beta} \end{cases}.$$

Now, the derivatives of the vectors of the Darboux frame with respect to the arclength of the curve can be expressed in terms of these vectors as

$$\begin{cases} \boldsymbol{\tau} = a(s) \cdot \mathbf{N} + b(s)\mathbf{n} \\ \mathbf{N}' = c(s)\boldsymbol{\tau} + d(s) \cdot \mathbf{n} \\ \mathbf{n}' = e(s) \cdot \boldsymbol{\tau} + f(s) \cdot \mathbf{N}, \end{cases} \tag{10.19.1}$$

where a, b, c, d, e, f are smooth functions of the arclength. We remind that the derivative of a vector of the Darboux frame is perpendicular on that vector, because the frame is orthonormal. For the same reason, we deduce immediately that the six coefficients are not independent and, in fact, we have the relations: $c = -a, e = -b, f = -d$, therefore the system (10.19.1) becomes

$$\begin{cases} \boldsymbol{\tau} = a(s) \cdot \mathbf{N} + b(s)\mathbf{n} \\ \mathbf{N}' = -a(s)\boldsymbol{\tau} + d(s) \cdot \mathbf{n} \\ \mathbf{n}' = -b(s) \cdot \boldsymbol{\tau} - d(s) \cdot \mathbf{N}, \end{cases} \tag{10.19.2}$$

We shall express now the quantities a, b, d in terms of the characteristics of the curve and the angle θ. We notice, first of all, that, from the first of the Frenet's formula, we get

$$\boldsymbol{\tau}' = k\boldsymbol{\nu} = k(\sin\theta \cdot \mathbf{N} + \cos\theta \cdot \mathbf{n}),$$

hence, identifying the coefficients with those of the first equation of the system (10.19.2), we obtain

$$a = k \cdot \sin\theta; \quad b = k \cdot \cos\theta. \tag{10.19.3}$$

The quantity $k \cdot \cos\theta$ is already known to us: it is nothing but the *normal curvature* k_n of the surface that was studied previously. The function $k \cdot \sin\theta$, instead, is new. We will denote it by k_g and we will call it *geodesic* or *tangential* curvature of the curve[7]. To find the quantity d, we start from the relation

$$\mathbf{N} = \sin\theta \cdot \boldsymbol{\nu} - \cos\theta \cdot \boldsymbol{\beta}.$$

By differentiating with respect to the arclength of the curve, we get, using the last two of the Frenet's formulae:

$$\begin{aligned}\mathbf{N}' &= \theta' \cdot \sin\theta \cdot \boldsymbol{\nu} - \sin\theta(-k \cdot \boldsymbol{\tau} + \chi \cdot \boldsymbol{\beta}) + \theta' \sin\theta \cdot \boldsymbol{\beta} + \chi \cdot \cos\theta \cdot \boldsymbol{\nu} = \\ &= -k \cdot \sin\theta \cdot \boldsymbol{\tau} + (\theta' + \chi) \cdot (\cos\theta \cdot \boldsymbol{\nu} + \sin\theta \cdot \boldsymbol{\beta}) = \\ &= -k \cdot \sin\theta \cdot \boldsymbol{\tau} + (\theta' + \chi) \cdot \mathbf{n},\end{aligned}$$

and, comparing with the second equation from (10.19.2), we get

$$d = \theta' + \chi.$$

This function is denoted by χ_g and it s called the *geodesic torsion*. His geometric meaning will be made clear later. Clearly, we cannot claim, as we did with the geodesic curvature, that the geodesic torsion is the torsion of the projection of the curve on the tangent plane, as the torsion of that curve is always zero, while the geodesic torsion of the given curve is not, usually.

The geodesic curvature plays a much more important role in the differential geometry of surfaces than the geodesic torsion does, so we start by focusing on it. We notice, to begin with, that

$$k_g = \boldsymbol{\tau}' \cdot \mathbf{N} = -\boldsymbol{\tau} \cdot \mathbf{N}'. \tag{10.19.4}$$

Since, as we saw earlier, $\mathbf{N} = \mathbf{n} \times \boldsymbol{\tau}$, one obtains for k_g the expression

$$k_g = \boldsymbol{\tau}' \cdot \mathbf{N} = \boldsymbol{\tau}' \cdot (\mathbf{n} \times \boldsymbol{\tau}),$$

i.e.

$$k_g = (\boldsymbol{\tau}, \boldsymbol{\tau}', \boldsymbol{n}). \tag{10.19.5}$$

[7]Here the term "tangential" refers to the tangent plane of the surface, not to the tangent line of the curve. In fact, it can be shown that the geodesic curvature at a point of a curve lying on a surface is the signed curvature of the projection of the curve on the tangent plane to the surface at that particular point.

This formula holds for naturally parameterized curves. Let us consider, now, an arbitrary regular parameterized curve on S, given by the local equations $u = u(t), v = v(t)$. We have, therefore, $\mathbf{r} = \mathbf{r}(u(t), v(t))$. If we denote by a dot the differentiation with respect to the parameter t along the curve, we get

$$\boldsymbol{\tau} \equiv \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{1}{\dot{s}} \cdot \dot{\mathbf{r}},$$

therefore

$$\boldsymbol{\tau}' = \frac{d\boldsymbol{\tau}}{ds} = \frac{1}{\dot{s}^3}(\ddot{\mathbf{r}} \cdot \dot{s} - \dot{\mathbf{r}}\ddot{s}).$$

Thus, for the geodesic curvature one gets

$$k_g = (\boldsymbol{\tau}, \boldsymbol{\tau}', \mathbf{n}) = \left(\frac{1}{\dot{s}}\dot{\mathbf{r}}, \frac{1}{\dot{s}^3}(\ddot{\mathbf{r}} \cdot \dot{s} - \dot{\mathbf{r}}\ddot{s}), \mathbf{n}\right) = \frac{1}{\dot{s}^4}(\dot{\mathbf{r}}, \ddot{\mathbf{r}} \cdot \dot{s} - \dot{\mathbf{r}}\ddot{s}, \mathbf{n}) = \frac{1}{\dot{s}^4}(\dot{\mathbf{r}}, \dot{s} \cdot \ddot{\mathbf{r}}, \mathbf{n}),$$

i.e.

$$k_g = \frac{1}{\dot{s}^3}(\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \mathbf{n}). \tag{10.19.6}$$

The formulas we got so far for the geodesic curvature use mixture of information about the curve and about the surface, but they don't use explicitly the things we usually compute when we are given a local parameterization of a surface or, more generally, a regular parameterized surface, namely the coefficients of the first two fundamental forms. So, the next step will be to obtain a formula for the geodesic curvature in terms of the coefficients of the fundamental forms, when the curve is given by its local parametric equations with respect to a local parameterization of the surface. We shall discover, in fact, that the geodesic curvature can be expressed in terms of the coefficients of the first fundamental form and they first order partial derivatives with respect to the coordinates.

To simplify the notations, we shall use, for a while, the index notations. In other words, we shall denote the coordinates with u^1 and u^2, instead of u and v, while the first partial derivatives of the radius vector $\mathbf{r}$ with respect to the coordinates will be denoted by $\mathbf{r}'_i$ and those of second order by $\mathbf{r}''_{ij}$, $i, j = 1, 2$. Moreover, we shall use the *Einstein's summation convention*: every time an index enter twice in a monomial, once in an inferior position and once in a superior one, then one makes summation after all the allowed values of the index (in our case, 1 and 2). Thus, an expression of the form $a_i u^i$ should be read as

$$a_i u^i = a_1 u^1 + a_2 u^2.$$

Going back to our problem, we have, with the newly introduced notations:

$$\begin{cases} \dot{\mathbf{r}} & \equiv \frac{d\mathbf{r}}{dt} = \mathbf{r}'_i \dot{u}^i \\ \ddot{\mathbf{r}} & = \mathbf{r}''_{ij} \dot{u}^i \dot{u}^j + \mathbf{r}'_k \ddot{u}^k \end{cases}.$$

The decomposition of the second order partial derivatives of the radius vectors in terms of the basis $\{\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{n}\}$ is already known to us: it can be described using the Christoffel's coefficients and the second fundamental form as

$$\mathbf{r}''_{ij} = \Gamma^k_{ij} \mathbf{r}'_k + h_{ij} \cdot \mathbf{n},$$

where, as we know, h_{ij} are the coefficients of the second fundamental form of the surface. Thus, we have

$$\ddot{\mathbf{r}} = \left(\Gamma^k_{ij} \mathbf{r}'_k + h_{ij} \cdot \mathbf{n}\right) \dot{u}^i \dot{u}^j + \mathbf{r}'_k \ddot{u}^k = \left(\ddot{u}^k + \Gamma^k_{ij} \dot{u}^i \dot{u}^j\right) \mathbf{r}'_k + h_{ij} \dot{u}^i \dot{u}^j \mathbf{n}$$

or

$$\ddot{\mathbf{r}} = \left(\ddot{u}^k + \Gamma^k_{ij} \dot{u}^i \dot{u}^j\right) \mathbf{r}'_k + \varphi_2(\dot{\mathbf{r}}, \dot{\mathbf{r}}) \cdot \mathbf{n}.$$

With these in hand, we can write now

$$\begin{aligned} k_g &= \frac{1}{\dot{s}^3} (\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \mathbf{n}) = \frac{1}{\dot{s}^3} \left(\dot{u}^m \mathbf{r}'_m, \left(\ddot{u}^k + \Gamma^k_{ij} \dot{u}^i \dot{u}^j\right) \mathbf{r}'_k + \varphi_2(\dot{\mathbf{r}}, \dot{\mathbf{r}}), \mathbf{n}\right) = \\ &= \frac{1}{\dot{s}^3} \left[\dot{u}^1 \left(\ddot{u}^2 + \Gamma^2_{ij} \dot{u}^i \dot{u}^j\right) - \dot{u}^2 \left(\ddot{u}^1 + \Gamma^1_{ij} \dot{u}^i \dot{u}^j\right)\right] \left(\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{n}\right). \end{aligned}$$

But

$$\left(\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{n}\right) = \left(\mathbf{r}'_1 \times \mathbf{r}'_2\right) \cdot \mathbf{n} = \left(\mathbf{r}'_1 \times \mathbf{r}'_2\right) \cdot \frac{\mathbf{r}'_1 \times \mathbf{r}'_2}{\|\mathbf{r}'_1 \times \mathbf{r}'_2\|} = \|\mathbf{r}'_1 \times \mathbf{r}'_2\| = \sqrt{g},$$

where g is the determinant of the first fundamental form, whence

$$k_g = \frac{\sqrt{g}}{\dot{s}^3} \left[\dot{u}^1 \left(\ddot{u}^2 + \Gamma^2_{ij} \dot{u}^i \dot{u}^j\right) - \dot{u}^2 \left(\ddot{u}^1 + \Gamma^1_{ij} \dot{u}^i \dot{u}^j\right)\right]. \tag{10.19.7}$$

If, in particular, the curve is naturally parameterized, we get

$$k_g = \sqrt{g} \left[(u^1)' \left((u^2)'' + \Gamma^2_{ij} (u^i)'(u^j)'\right) - (u^2)' \left((u^1)' + \Gamma^1_{ij} (u^i)'(u^j)'\right)\right]. \tag{10.19.8}$$

In the traditional notations, the formula for the geodesic curvature of a naturally parameterized curve is

$$k_g = \sqrt{EG - F^2}\Big[\Gamma^2_{11}u'^3 + \left(2\Gamma^2_{12} - \Gamma^1_{11}\right)u'^2 v' + \left(\Gamma^2_{22} - 2\Gamma^1_{12}\right)u' v'^2 - \\ -\Gamma^1_{22}v'^3 + u'v'' - u''v'\Big]. \tag{10.19.9}$$

This equation can also be written in a more elegant form as

$$k_g = \sqrt{EG - F^2}\det\begin{pmatrix} u' & u'' + u'^2\Gamma^1_{11} + 2u'v'\Gamma^1_{12} + v'^2\Gamma^1_{22} \\ v' & v'' + u'^2\Gamma^2_{11} + 2u'v'\Gamma^2_{12} + v'^2\Gamma^2_{22}\end{pmatrix}. \tag{10.19.10}$$

If, in particular, the surface S is the plane, with the Cartesian coordinates on it, then the determinant of the first fundamental form is, of course, one, as the matrix of the fundamental form is the unit matrix, while all the Christoffel's coefficients vanish. As a consequence, in this situation the geodesic curvature of the curve (which is, in this case, a plane curve), is nothing but its *signed curvature*:

$$k_g = u'v'' - u''v' \equiv k_{\pm}.$$

Remark. It can be shown that, in fact, the geodesic curvature of a curve on a surface is nothing but the signed curvature of the projection of the curve on the tangent plane.

10.19.3 Geodesic lines

Definition 10.19. Let S be a surface. A parameterized curve $\boldsymbol{\rho} : I \to S$ is called a *geodesic line*, or simply, a *geodesic* if, at each point, its geodesic curvature vanishes.

Remark. According to the formula (10.19.12), the geodesic curvature of a curve can be written as

$$k_g = (\boldsymbol{\tau}, \boldsymbol{\tau}', \boldsymbol{n}).$$

On the other hand, from the first formula of Frenet, $\boldsymbol{\tau} \times \boldsymbol{\tau}' = k\boldsymbol{\beta}$, therefore, k_g vanishes at a point of a curve on the surface if and only if the binormal of the curve is perpendicular on the normal of the surface at that point or, in other words, *the geodesic curvature vanishes if and only if the normal of the surface is contained in the osculating plane*. Thus, the geodesic lines are exactly those lines on the surface for which the osculating plane at each of their points contain the normal of the surface at that point. In fact, in many books, this property is taken as the definition of the geodesics.

Another remark that should be made is that since, as we noticed earlier, the geodesic curvature is the signed curvature of the projection of the curve on the tangent plane, we can say that the geodesics are those curves which project on each tangent plane of the surface after a straight line. In this sense, we might say that the geodesics are the *straightest* lines on the surfaces.

The formula 10.19.9 leads to

Theorem 10.5. *The differential equation of the geodesic lines is*

$$\Gamma_{11}^{2}u'^{3}+\left(2\Gamma_{12}^{2}-\Gamma_{11}^{1}\right)u'^{2}v'+\left(\Gamma_{22}^{2}-2\Gamma_{12}^{1}\right)u'v'^{2}-\Gamma_{22}^{1}v'^{3}+u'v''-u''v'=0. \tag{10.19.11}$$

Also, from (10.19.10) we can deduce that

Theorem 10.6. *The geodesic lines verify the system of differential equations*

$$\begin{cases} u''+u'^{2}\Gamma_{11}^{1}+2u'v'\Gamma_{12}^{1}+v'^{2}\Gamma_{22}^{1}=0 \\ v''+u'^{2}\Gamma_{11}^{2}+2u'v'\Gamma_{12}^{2}+v'^{2}\Gamma_{22}^{2}=0 \end{cases}. \tag{10.19.12}$$

There is, apparently, a contradiction between the previous two theorems, as the first claim that the geodesics are solutions of a single equations, while the second – that they are a solution of a system of different equation. In fact, however, the two equations of the system (10.19.12) are not independent, because we impose the geodesics to be *naturally parameterized*, therefore, between the functions u and v there is an extra relation.

The existence (locally, at least) of geodesics passing through a point of a surface and having at that point a given tangent vector is a consequence of standard results in the theory of ordinary differential equations. Finding geodesics, is, usually, a very delicate business and can be done explicitly only in special situations.

Examples of geodesics

The geodesics of the plane. By the geometrical interpretation of the geodesics, as the curves that project on the tangent plane on straight lines, we deduce immediately that the geodesics of the plane are the straight lines and only them. On the other hand, by using cartesian coordinates, we fined immediately that the Christoffel's coefficients vanish identically, therefore the equations of geodesics become

$$\begin{cases} u'' = 0 \\ v'' = 0 \end{cases},$$

which lead to $u(s) = a_1 s + b_1$, $v(s) = a_2 s + b_2$, i.e., again, the geodesics are straight lines.

The geodesics of the sphere. The geodesics of the sphere can be found very easily from their interpretation as being those curves for which the osculating plane contains the normal to the surface. As we know, in the case of the sphere all the normals are passing through the center of the sphere, therefore the osculating planes should pass, all of them through the center, which leads immediately to the idea that the geodesics are arcs of great circles of the sphere.

Also, using a standard parameterization of the sphere (with spherical coordinates), we can find (do that!) the equations of the geodesics as:

$$\begin{cases} \ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0 \\ \ddot{\varphi} + 2\cot\theta\dot{\theta}\dot{\varphi} = 0 \end{cases}.$$

We assume that the explicit equation of the geodesic is $\theta = \theta(\varphi)$. Then

$$\dot{\theta} = \frac{d\theta}{ds} = \frac{d\theta}{d\varphi} \cdot \dot{\varphi},$$

$$\ddot{\theta} = \frac{d}{ds}\left(\frac{d\theta}{d\varphi}\right) \cdot \dot{\varphi} + \frac{d\theta}{d\varphi}\ddot{\varphi} = \frac{d\theta}{d\varphi} \cdot \ddot{\varphi} + \frac{d^2\theta}{d\varphi^2} \cdot \dot{\varphi}^2.$$

Therefore, the first equation of the system becomes:

$$\ddot{\varphi} \cdot \frac{d\theta}{d\varphi} + \frac{d^2\theta}{d\varphi^2} \cdot \dot{\varphi}^2 - \sin\theta\cos\theta \cdot \dot{\varphi}^2 = 0.$$

On the other hand, from the first equation,

$$\ddot{\varphi} = -2\cot\theta\dot{\theta} \cdot \dot{\varphi} = -2\cot\theta\frac{d\theta}{d\varphi} \cdot \dot{\varphi}^2,$$

hence:

$$\dot{\varphi}^2\left(\frac{d^2\theta}{d\varphi^2} - 2\cot\theta \cdot \frac{d\theta}{d\varphi} - \sin\theta\cos\theta\right) = 0.$$

We have two possibilities: either $\dot{\varphi} = 0$, i.e. $\varphi = const$, and, in this case, the curve is, obviously, a great circle of the sphere (a meridian), either

$$\frac{d^2\theta}{d\varphi^2} - 2\cot\theta \cdot \frac{d\theta}{d\varphi} - \sin\theta\cos\theta = 0.$$

In this case, we make the substitution $z = \cot\theta$ and, after a straightforward computation, we get

$$\frac{d^2 z}{d\varphi^2} + z = 0,$$

and this equation has the general solution

$$z = \cot\theta = A\cos\varphi + B\sin\varphi$$

or

$$A\sin\theta\cos\varphi + B\sin\theta\sin\varphi - \cos\theta = 0,$$

which is the equation of a great circle, lying in the plane passing through the origin and having as normal vector the vector $(A, B, -1)$.

10.19.4 Liouville surfaces

Definition 10.20. A surface is called a *Liouville surface* if it can be parameterized locally in such a way that the first fundamental form can be written as

$$ds^2 = (U(u) + V(v))(du^2 + dv^2), \tag{10.19.13}$$

where U and V are smooth functions of a single variable.

These surfaces have been introduced by the French mathematician Joseph Liouville in the *Note III* of the 5th edition of the book of Gaspard Monge, *Application*. In particular, the surfaces of revolution are examples of Liouville surfaces. The most important characteristic of Liouville surfaces is the fact that their geodesics an be found through quadratures. This is exactly what we are going to prove in the rest of this section.

First of all, it is an easy matter to prove that the equation of the geodesics for the metric (10.19.13) can be written as

$$(U'v' - V'u')(u'^2 + v'^2) + 2(U + V)(u'v'' - v'u'') = 0. \tag{10.19.14}$$

Here, for the functions U and V the prime denotes the derivative with respect to u and v, respectively, while for the coordinate functions u and v the prime denotes the derivative with respect to the parameter t along the geodesic. Of course, U and V are, both, functions of t, through the coordinate functions. Therefore, the equation of geodesics can be rewritten as

$$\frac{u'}{v'}\frac{dU}{dt} - \frac{v'}{u'}\frac{dV}{dt} + 2(U+V)\frac{u'v'' - v'u''}{u'^2 + v'^2} = 0. \tag{10.19.15}$$

We see that this equation actually does not involve the coordinate functions u and v, but only their derivatives. We substitute them with other two functions of t, ρ and α, defined by

$$\begin{cases} u' = \rho\cos\alpha \\ v' = \rho\sin\alpha \end{cases}. \tag{10.19.16}$$

In the sequel, the equation of the geodesics becomes

$$\sin^2\alpha\frac{dU}{dt} - \cos^2\alpha\frac{dV}{dt} + 2(U+V)\sin\alpha\cos\alpha\frac{d\alpha}{dt} = 0.$$

This equation can be also written as

$$\frac{d}{dt}\left(U\sin^2\alpha - V\cos^2\alpha\right) = 0,$$

whence

$$U\sin^2\alpha - V\cos^2\alpha = a,$$

where a is a constant. Returning to the old coordinate functions, we get

$$v'^2 U - u'^2 V = a(u'^2 + v'^2),$$

whence

$$\int\frac{du}{\sqrt{U-a}} = \pm\int\frac{dv}{\sqrt{V+a}} + b, \tag{10.19.17}$$

where b is another integration constant.

CHAPTER 11

Special classes of surfaces

11.1 Ruled surfaces

Intuitively speaking, the ruled surfaces are the surfaces generated by a moving straight line, which is subject to a further condition, usually this condition being that the moving line remains tangent to a given surface or intersects a given regular space curve. More precisely

11.1.1 General ruled surfaces

After the plane, the ruled surfaces are, no doubt, the simplest surfaces. We recall that a surface is called *ruled* if it is generated by a straight line (the *ruling*, moving in space, lying all the time on a given curve, called the *directrix*. Clearly, once the ruled surface was defined, the directrix can be replaced by another curve , obtained, for instance, by sectioning the surface by a plane or a sphere. In particular, it is, usually, convenient to choose a directrix which is, at each point, orthogonal to the ruling passing through that point (in other words, the directrix is an *orthogonal trajectory* of the rulings. As we shall see shortly, this particular choice of the ruling has as effect the diagonalization of the first fundamental form of the surface.

The simplest ruled surfaces are the *cylindrical surfaces*, for which the rulings are always parallel to a fixed direction and the *conical surfaces*, in the case of which the

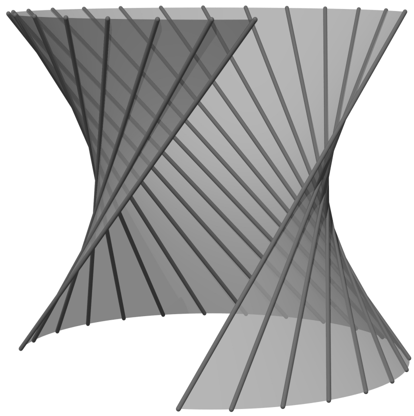


Figure 11.1: The hyperboloid with one sheet and its rulings

rulings are passing through a fixed point. The ruled surfaces have been studied in details for the first time by the French geometer Gaspard Monge, at the end of the XVIIIth century and most of the important results in the field belong to him or to some of his students. In the figure 11.1.1 we represented the hyperboloid of rotation with one sheet, together some of its rulings.

Another way of getting interesting ruled surfaces is to take as rulings the axes of the Frenet frame of a space curve. For instance, in the figure 11.1.1 we represented the surface described by the binormals of the Viviani's temple.

The parameterization of a ruled surface

We assume that the directrix of the surface has a parameterization of the form

$$\boldsymbol{\rho} = \boldsymbol{\rho}(u), \quad u \in I,$$

where I is an interval on the real axis. For each $u \in I$, we denote by $\mathbf{b}(u)$ the versor of the ruling passing through the point $\boldsymbol{\rho}(u)$. Then, if M is a point on this ruling, its coordinates will be determined by the relation

$$\mathbf{r} = \mathbf{r}(u, v) = \boldsymbol{\rho}(u) + v\mathbf{b}(u), \tag{11.1.1}$$

where v is the parameter along the ruling, i.e. if $M_0 = \boldsymbol{\rho}(\boldsymbol{u})$, then we have

$$\overrightarrow{M_0M} = v\mathbf{b}(u).$$

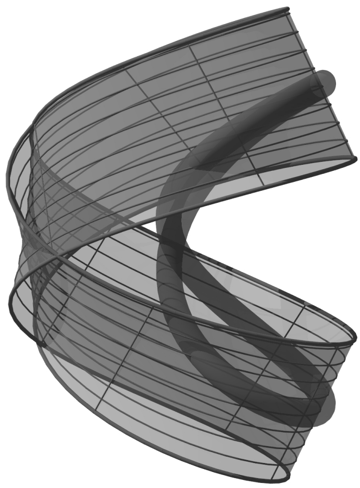


Figure 11.2: The surface of binormals of the Viviani's temple

The relation 11.1.1 provides a parameterization of the ruled surface,

$$\mathbf{r} : I \times \mathbb{R} \to S.$$

This parameterization is *not*, usually, global, as the parameterization of the directrix is not global.

With respect to this parameterization, the rulings will be coordinate lines ($u = const$), and the directrix is, equally, a coordinate line ($v = 0$). Generally, the coordinate lines $v = const$ have the property that they are "parallel" to the directrix, in the sense that all the points from such a coordinate curve lie at the same distance (equal to $|v|$) from the directrix, when we measure the distance along the ruling passing through each point.

The tangent plane and the first fundamental form of a ruled surface

To compute the coefficients of the first fundamental form of a ruled surface we need, first of all, the partial derivatives of the radius vector of a point of the surface. We have, obviously,

$$\mathbf{r}'_u = \boldsymbol{\rho}' + b\mathbf{b}'; \quad \mathbf{r}'_v = \mathbf{b}'_u. \tag{11.1.2}$$

Thus, the coefficients of the first fundamental form of the surface will be

$$\begin{aligned} E &\equiv \mathbf{r}'_u \cdot \mathbf{r}'_u = \boldsymbol{\rho}'^2 + 2v\boldsymbol{\rho}' \cdot \mathbf{b}' + v^2\mathbf{b}'^2; \\ F &\equiv \mathbf{r}'_u \cdot \mathbf{r}'_v = \boldsymbol{\rho}' \cdot \mathbf{b}; \\ G &\equiv \mathbf{r}'_v \cdot \mathbf{r}'_u = 1. \end{aligned} \tag{11.1.3}$$

It follows that the first fundamental form of a ruled surface can be written as:

$$ds^2 = \left(\boldsymbol{\rho}'^2 + 2v\boldsymbol{\rho}' \cdot \mathbf{b}' + v^2\mathbf{b}'^2\right) du^2 + 2(\boldsymbol{\rho}' \cdot \mathbf{b})\, du dv + dv^2. \tag{11.1.4}$$

To find the tangent plane at a point of a ruled surface, we notice, first of all, that the direction of the normal to the plane (and, hence, to the surface) at a given point is given by the vector $\mathbf{r}'_u \times \mathbf{r}'_v$, i.e. by the vector

$$\mathbf{N} \equiv \mathbf{r}'_u \times \mathbf{r}'_v = \boldsymbol{\rho}' \times \mathbf{b} + v\left(\mathbf{b}' \times \mathbf{b}\right). \tag{11.1.5}$$

Therefore, if $\mathbf{R}$ is the position vector of a point from the tangent plane to the surface at a point corresponding to the pair of parameters(u, v), then the equation of the tangent plane can be written under the form

$$(\mathbf{R} - \mathbf{r}) \cdot \mathbf{N} = 0,$$

i.e.

$$(\mathbf{R} - \boldsymbol{\rho} - v\mathbf{b}) \cdot \left(\boldsymbol{\rho}' \times \mathbf{b} + v\left(\mathbf{b}' \times \mathbf{b}\right)\right) = 0$$

or

$$\left(\mathbf{R}, \boldsymbol{\rho}', \mathbf{b}\right) + v\left(\mathbf{R}, \mathbf{b}', \mathbf{b}\right) - \left(\boldsymbol{\rho}, \boldsymbol{\rho}', \mathbf{b}\right) - v\left(\boldsymbol{\rho}, \mathbf{b}', \mathbf{b}\right) = 0$$

or, also,

$$\left[\mathbf{R} \times \boldsymbol{\rho}' + v\left(\mathbf{R} \times \mathbf{b}'\right) - \boldsymbol{\rho} \times \boldsymbol{\rho}' - v\left(\boldsymbol{\rho} \times \mathbf{b}'\right)\right] \cdot \mathbf{b} = 0. \tag{11.1.6}$$

A characteristic property of the ruled surfaces is described by the following proposition:

Property 23. *The tangent planes to a ruled surface in point located along the same ruling, belong to the pencil of planes determined by that ruling, or, to put it another way, the tangent plane at a point of a ruled surface contains the ruling passing through that point.*

Proof Since the ruling and the tangent plane already have a point in common (the very point of tangency), it is enough to prove that the ruling is parallel to the tangent plane or, which is the same, that it is perpendicular to the normal to the surface at the tangency point. We have, indeed,

$$\mathbf{N}\cdot\mathbf{b} = \left[\boldsymbol{\rho}'\times\mathbf{b} + v\left(\mathbf{b}'\times\mathbf{b}\right)\right]\cdot\mathbf{b} = \left(\boldsymbol{\rho}',\mathbf{b},\mathbf{b}'\right) + v\left(\mathbf{b}',\mathbf{b},\mathbf{b}'\right) = 0.$$

□

The proposition we just proved indicates how varies the tangent plane to a ruled surface along a ruling: it just rotates around the ruling. In other words, its normal vector stays all the time parallel to a fixed plane, which is perpendicular to the considered ruling.

11.1.2 The Gaussian curvature of a ruled surface

We start, of course, from the parameterization (11.1.1). Then we have

$$\mathbf{r}'_u = \boldsymbol{\rho}' + v\mathbf{b}', \quad \mathbf{r}'_v = \mathbf{b}.$$

Therefore, for the first fundamental form, we will get

$$E = \mathbf{r}'_u\cdot\mathbf{r}'_u = (\boldsymbol{\rho}' + v\mathbf{b}')\cdot(\boldsymbol{\rho}' + v\mathbf{b}') = \boldsymbol{\rho}'^2 + 2v\boldsymbol{\rho}'\cdot\mathbf{b}' + v^2\mathbf{b}'^2,$$

$$F = \mathbf{r}'_u\cdot\mathbf{r}'_v = (\boldsymbol{\rho}' + v\mathbf{b}')\cdot\mathbf{b} = \boldsymbol{\rho}'\cdot\mathbf{b},$$

while

$$G = \mathbf{r}'_v\cdot\mathbf{r}'_v = 1,$$

since $\mathbf{b}$ is a versor. Let H be the determinant of the first fundamental form. Then, for the coefficients of the second fundamental form, we will get

$$D = \frac{1}{H}\left(\mathbf{r}'_u,\mathbf{r}'_v,\mathbf{r}''_{u^2}\right) = \frac{1}{H}\left(\boldsymbol{\rho}' + v\mathbf{b}',\mathbf{b},\boldsymbol{\rho}'' + v\mathbf{b}''\right),$$

$$D' = \frac{1}{H}\left(\mathbf{r}'_u,\mathbf{r}'_v,\mathbf{r}''_{uv}\right) = \frac{1}{H}\left(\boldsymbol{\rho}' + v\mathbf{b}',\mathbf{b},\mathbf{b}'\right) = \frac{1}{H}\left(\boldsymbol{\rho}',\mathbf{b},\mathbf{b}'\right),$$

$$D'' = \frac{1}{H}\left(\mathbf{r}'_u,\mathbf{r}'_v,\mathbf{r}''_{v^2}\right) = 0.$$

Thus,

Property 24. *The total curvature of a ruled surface given by the local parameterization (11.1.1) is given by*

$$K_t = \frac{DD'' - D'^2}{EG - F^2} = -\frac{(\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}')^2}{\|(\boldsymbol{\rho}' + v\mathbf{b}') \times \mathbf{b}\|^2}. \tag{11.1.7}$$

The curvature is always negative and it vanishes if and only if

$$\left(\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}'\right) = 0. \tag{11.1.8}$$

11.1.3 Envelope of a family of surfaces

In this section, a *surface* will always be a *parameterized surface*. The theory of envelopes of surfaces is analogue to the theory of envelopes of plane curves, therefore there will be no proofs, as the proofs are completely analogue to the ones we provided for plane curves.

Definition 11.1. Let

$$\mathbf{r} = \mathbf{r}(u, v, \lambda) \tag{11.1.9}$$

be a family of surfaces. The *envelope* of the family (if there is any) is a surface which is tangent to all the surfaces from the family. The contact between the envelope and each surface is made along a curve which is called a *characteristic*. Thus, the envelope is the geometrical locus of the chracteristics of the family of surfaces.

Property 25. *The points of the envelope of the family (11.1.9) verify, beside this equation, the equation*

$$(\mathbf{r}'_u, \mathbf{r}'_v, \mathbf{r}'_\lambda) = 0. \tag{11.1.10}$$

Of course, as in the case of plane curves, these equations are also verified by the singular points of the surfaces and to get the envelope, we have to eliminate them first.

If the family of surfaces is given implicitly, i.e. through the equation

$$F(x, y, z, \lambda) = 0, \tag{11.1.11}$$

then to get the envelope, we have to add to this equation the equation

$$F'_\lambda(x, y, z, \lambda) = 0, \tag{11.1.12}$$

as we did in the case of plane curves. Something that is specific to the theory of envelopes of surfaces is the so-called *regression edge*. Let us assume that we have a family of surfaces, given, say, implicitly. Then we can form the system:

$$\begin{cases} F(x,y,z,\lambda) = 0, \\ F'_{\lambda}(x,y,z,\lambda) = 0 \\ F''_{\lambda^2}(x,y,z,\lambda) = 0 \end{cases} . \tag{11.1.13}$$

The first two equations of this system are the equations of the family of characteristic lines of the family of surfaces. If the system is compatible, then what we find solving it is a curve which, at each point, is tangent to one of the characteristics, in other words, it is the *envelope* of the characteristics. This envelope (when exists), it is called the *regression edge* of the envelope of the family of surfaces.

An example of envelope. We consider the family of spheres of constant radius, with the center of a given circle, verifying the hypothesis that the radius of the circle is greater than the radius of the spheres. Then the envelope of this family is, as one can see immediately, the torus (see the figure 11.1.3).

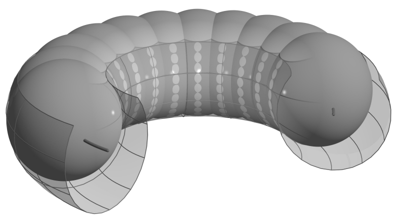


Figure 11.3: The torus as envelope of spheres

11.1.4 Developable surfaces

There is a subclass of ruled playing an important role in differential geometry, the so-called *developable surfaces*:

Definition 11.2. A ruled surface is called *developable* if the tangent plane to the surface is the same at all the points of a ruling.

We would like to find conditions for a ruled surface to be developable. We have the following result:

Property 26. *A ruled surface given by the equation*

$$\mathbf{r}(u, v) = \boldsymbol{\rho}(u) + v \cdot \mathbf{b}(u)$$

is developable if and only if

$$\left(\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}'\right) = 0. \tag{11.1.14}$$

Proof The invariance condition for the tangent plane along a ruling is equivalent to the condition of invariance of the normal line to the surface along the same ruling. As we saw previously, a normal vector to the surface is the vector

$$\mathbf{N}(u, v) = \boldsymbol{\rho}'(u) \times \mathbf{b}(u) + v \left(\mathbf{b}'(u) \times \mathbf{b}(u)\right).$$

Thus, the condition for the surface to be developable is that the direction **N** be independent on the coordinate v. This may happen, obviously,in three situations:

1. The first component of the vector **N** vanishes, i.e.

$$\boldsymbol{\rho}'(u) \times \mathbf{b}(u) = 0.$$

   In this case, the vector **N** is constant only as direction, but its norm and orientation might vary. As **b** is a versor, it cannot vanish, therefore the condition mentioned above can be fulfilled in two situations:

   (a) $\boldsymbol{\rho}' = 0$. In this case, the directrix degenerates to a point, therefore the surface is conical.

   (b) $\boldsymbol{\rho}' \parallel \mathbf{b}$. In this case, the rulings are nothing but the tangents to the directrix, i.e. the surface is generated by the tangents to a space curve.

2. The second component of the normal vector vanishes, i.e.

$$\mathbf{b}'(u) \times \mathbf{b}(u) = 0.$$

   From this relation follows, in fact, that the direction of **b** is fixed, i.e. the surface is cylindrical.

3. The two components of the normal vector are parallel, i.e. we have

$$\boldsymbol{\rho}'(u) \times \mathbf{b}(u) \parallel \mathbf{b}'(u) \times \mathbf{b}(u).$$

The three conditions we just listed are equivalent to the unique condition

$$\big(\boldsymbol{\rho}'(u) \times \mathbf{b}(u)\big) \times \big(\mathbf{b}'(u) \times \mathbf{b}(u)\big) = 0.$$

On the other hand,

$$\big(\boldsymbol{\rho}' \times \mathbf{b}\big) \times \big(\mathbf{b}' \times \mathbf{b}\big) = \big(\boldsymbol{\rho}', \mathbf{b}', \mathbf{b}\big)\, \boldsymbol{b} - \underbrace{\big(\mathbf{b}, \mathbf{b}', \mathbf{b}\big)}_{=0} \boldsymbol{\rho}' = -\big(\boldsymbol{\rho}', \mathbf{b}, \mathbf{b}'\big)\, \boldsymbol{b},$$

which concludes the proof of the proposition. □

Corollary 11.1.1. *A ruled surface is developable if and only if its Gaussian curvature vanishes identically.*

Remark. The ruled surfaces which are not developable are also called *scrolls*.

Developable surfaces as envelopes of a one-parameter family of planes. The regression edge of a developable surface

We saw previously that that there are three classes of developable surfaces, namely:

1. cylindrical surfaces;
2. conical surfaces;
3. surfaces generated by the tangents to a space curve.

We shall see, in this paragraph, that, in fact, these three classes exhaust all the developable surfaces.

In the figure 11.1.4 we give an example of developable surface with a regression edge (the tangent developable of the Viviani's temple). One can notice immediately, from the very definition of the developable ruled surfaces, that a ruled surface is developable if and only if it is the envelope of a family of planes, depending on a single parameter[1].

We consider a one-parameter family of planes

$$\mathbf{N}(\lambda) \cdot \mathbf{r} + D(\lambda) = 0. \tag{11.1.15}$$

[1] These planes are, of course, the tangent planes of the surface. Clearly, any surface is the envelope of a family of planes, namely the family of tangent planes to the surface. Nevertheless, this family depends, usually, on two parameters. In the case of developable surfaces, it depends on a single parameter, namely the parameter along the directrix of the surface.

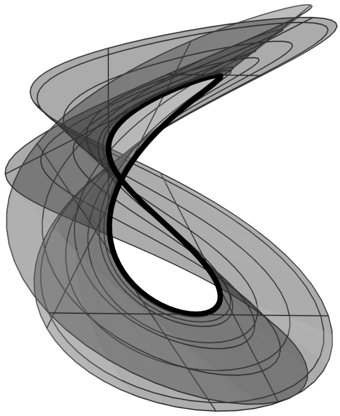


Figure 11.4: The tangent developable of Viviani's temple

We shall assume, without restricting the generality, that the normal vector $\mathbf{N}$ is a versor. By differentiating the equation (11.1.15) with respect to λ, we obtain

$$\mathbf{N}'(\lambda) \cdot \mathbf{r} + D'(\lambda) = 0. \tag{11.1.16}$$

According to the theory of envelopes of the families of surfaces, the equations (11.1.15) and (11.1.16) determine the envelope of the family of planes, if such an envelope exists[2].

The characteristic curves of the family of planes are straight lines, obtained as intersections between the planes of equations (11.1.15) and (11.1.16), respectively. Clearly, these characteristics exist only if the planes of the family are not parallel, which we admit to be true in our case.

The points of the envelope have to verify, also, the equation obtained from (11.1.16), by differentiating once more with respect to λ, i.e.

$$\mathbf{N}''(\lambda) \cdot \mathbf{r} + D''(\lambda) = 0. \tag{11.1.17}$$

We have, thus, the following system of equations:

$$\begin{cases} \mathbf{N}(\lambda) \cdot \mathbf{r} + D(\lambda) = 0 \\ \mathbf{N}'(\lambda) \cdot \mathbf{r} + D'(\lambda) = 0 \\ \mathbf{N}''(\lambda) \cdot \mathbf{r} + D''(\lambda) = 0. \end{cases} \tag{11.1.18}$$

[2] As a matter of fact, these equations describe the *discriminant set*, including, also, the singular points of the surfaces from the family. However, in this particular case, the surfaces are planes and they have no singular points whatsoever.

We can regard this system as being a system of three equations with three unknowns (the components of the radius vector $\mathbf{r}$).

Let us assume, to begin with, that the system is compatible and determined, i.e. the three planes intersect at a single point. We admit, at the first stage, that the solution of the system depends in a nontrivial manner, on the parameter λ, in other words, if

$$\mathbf{r} = \tilde{\mathbf{r}}(\lambda) \tag{11.1.19}$$

is the solution of the system, then $\tilde{\mathbf{r}}(\lambda) \neq 0$. This means, in fact, that the equation (11.1.19) describes a space curve, called the *edge of regression* of the developable surface. This is, of course, the edge of regression of the envelope of the family of planes, in the sense of the theory of envelopes of surfaces. We know from this theory that all the characteristic curves have to be tangent to the edge of regression. Therefore, we must have

$$\mathbf{N} \cdot \tilde{\mathbf{r}}' = 0 \tag{11.1.20}$$

$$\mathbf{N}' \cdot \tilde{\mathbf{r}}' = 0, \tag{11.1.21}$$

because the characteristic is obtained by the intersections of the planes of normal vectors $\mathbf{N}$ and $\mathbf{N}'$, respectively.

By differentiating the relation (11.1.20), we obtain

$$\mathbf{N}' \cdot \tilde{\mathbf{r}}' + \mathbf{N} \cdot \tilde{\mathbf{r}}'' = 0 \tag{11.1.22}$$

whence, using (11.1.21),

$$\mathbf{N} \cdot \tilde{\mathbf{r}}'' = 0. \tag{11.1.23}$$

The relations (11.1.20) and (11.1.23) tell us that the plane of the normal vector $\mathbf{N}$ contains both the vector $\tilde{\mathbf{r}}'$, and the vector $\tilde{\mathbf{r}}''$, i.e. is nothing but the *osculating plane* of the regression edge at the point associated to the value λ of the parameter..

As such, a developable surface admitting an edge of regression can be described in two different ways:

a) as the surface generated by the tangents to the edge of regression;

b) as the envelope of the family of osculating planes of the edge of regression.

In the degenerate case, when the edge of regression reduces to a point, all the characteristic curves (i.e. the generators of the surface) are passing through that point, hence the surface is a conical surface.

We shall focus now on the case when the system (11.1.18) has a vanishing determinant. In principle, two situations are possible: either the system is compatible and undetermined, either it is incompatible. We shall see that the first possibility is, in practice, excluded.

We notice, first of all, that the determinant of the linear system (11.1.18) is nothing but $(\mathbf{N}, \mathbf{N}', \mathbf{N}'')$, therefore, the condition $\Delta = 0$ is equivalent to the condition

$$\big(\mathbf{N}, \mathbf{N}', \mathbf{N}''\big) = 0. \tag{11.1.24}$$

This means, of course, that the vectors $\mathbf{N}, \mathbf{N}', \mathbf{N}''$ lie in the same plane. We denote by $\mathbf{n} = \mathbf{n}(\lambda)$ the normal versor of this plane, in other words, we have

$$\mathbf{n} \cdot \mathbf{N} = \mathbf{n} \cdot \mathbf{N}' = \mathbf{n} \cdot \mathbf{N}'' = 0. \tag{11.1.25}$$

By differentiating the first two equations from the previous system, we obtain

$$\mathbf{n}' \cdot \mathbf{N} + \mathbf{n} \cdot \mathbf{N}' = \mathbf{n}' \cdot \mathbf{N} + \mathbf{n} \cdot \mathbf{N}'' = 0,$$

whence, using (11.1.25),

$$\mathbf{n}' \cdot \mathbf{N} = \mathbf{n}' \cdot \mathbf{N}' = 0,$$

i.e. either the vector $\mathbf{n}'$ vanishes, either it is perpendicular on the vectors $\mathbf{N}$ and $\mathbf{N}'$.

Assuming that we would be in the second situation, as the vectors $\mathbf{N}$ and $\mathbf{N}'$ are not colinear, we would obtain, as one can see easily, that $\mathbf{n}'$ is parallel to $\mathbf{n}$, which is not possible since, as, $\mathbf{n}$ is a versor, we have $\mathbf{n} \cdot \mathbf{n}' = 0$.

We must have, therefore, $\mathbf{n}' = 0$, i.e. the vector $\mathbf{n}$ is constant. It follows from here that for any λ the vectors $\mathbf{N}(\lambda)$ and $\mathbf{N}'(\lambda)$ are perpendiculars on the constant vector $\mathbf{n}$, i.e. all the rulings of the family are parallel to a constant direction, and the surface is a cylindrical surface. It is easy to see that in this case the system of equations (11.1.18) is incompatible (the third plane is parallel to the straight line determined by the first two).

The considerations from this paragraph prove that, in fact, we have the following result::

Property 27. *If S is a developable ruled surface, then it is either a cylindrical surface, either a conical one, either a surface generated by the tangents to a space curve.*

Thus, the classes of surfaces mentioned in the previous paragraph do exhaust all the types of developable ruled surfaces. We mention that the plane can be considered as a limit case for any of the three classes of surfaces listed in the proposition.

11.1.5 Developable surfaces associated to the Frenet frame of a space curve

In all of this section $\mathbf{r} = \mathbf{r}(s)$ will be a smooth naturally parameterized biregular space curve. We assume, moreover, that the curve is *skew*, in other words, its torsion never vanishes.

The envelope of the family of osculating planes

The equation of osculating planes at a point of the curve is

$$(\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\beta}(s) = 0. \tag{11.1.26}$$

The parameter along the family of planes will be exactly the natural parameter s along the curve.

By differentiating the relation (11.1.26) with respect to s, we get

$$-\mathbf{r}'(s) \cdot \boldsymbol{\beta}(s) + (\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\beta}'(s) = 0$$

or, having in mind that $\mathbf{r}' \cdot \boldsymbol{\beta} = 0$ and using the third formula of Frenet,

$$(\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\nu}(s) = 0, \tag{11.1.27}$$

where we used the fact that, according to the hypotheses we made about the curve, the torsion of the curve never vanishes.

From (11.1.26) and (11.1.27) one obtains that the characteristic straight line is perpendicular both on $\boldsymbol{\beta}$ and on $\boldsymbol{\nu}$, in other words, it is nothing but the tangent to the curve .

To get the edge of regression, we differentiate once more (11.1.27) and we get

$$-\mathbf{r}'(s) \cdot \boldsymbol{\nu}(s) + (\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\nu}'(s) = 0$$

or, using the fact that $\mathbf{r}'(s) \cdot \boldsymbol{\nu}(s) = 0$,a s well as the second Frenet formula,

$$(\mathbf{R} - \mathbf{r}(s)) \cdot [\chi(s)\boldsymbol{\beta}(s) - k\boldsymbol{\nu}(s)] = 0.$$

As $\chi(s)\boldsymbol{\beta}(s) - k\boldsymbol{\nu}(s) \neq 0$ if χ and k don't vanish, we obtain that

$$\mathbf{R} = \mathbf{r}(s), \tag{11.1.28}$$

i.e. the edge of regression is nothing but the given curve.

The envelope of the family of normal planes (the polar surface)

The equation of the normal plane at an arbitrary point of a space curve is, as we know,

$$(\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\tau}(s) = 0. \tag{11.1.29}$$

We differentiate this relation once, to complete the system of equations for the characteristics and we get

$$-\mathbf{r}'(s) \cdot \boldsymbol{\tau}(s) + (\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\tau}'(s) = 0$$

or, having in mind that we have a curve parameterized by the arc length and using the first of the Frenet formulae,

$$-1 + (\mathbf{R} - \mathbf{r}(s)) \cdot k(s) \cdot \boldsymbol{\nu}(s) = 0. \tag{11.1.30}$$

It follows, then, that

$$(\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\nu}(s) = \frac{1}{k(s)}. \tag{11.1.31}$$

The equation (11.1.31) is the equation of a plane which is parallel to the rectifying plane (because it has as normal vector the versor of the principal normal). Thus, the characteristic of the envelope of the normal planes passing through the point of position vector $\mathbf{r}(s)$ is obtained by intersecting the normal plane to the curve at this point with a plane which is parallel to the rectifying plane of the curve at the same point, i.e. it is a straight line which is parallel to the binormal of the curve at the point $\mathbf{r}(s)$.

On the other hand, it is clear that this characteristic passes through the point of position vector $\mathbf{r}(s) + \frac{1}{k(s)} \cdot \boldsymbol{\nu}(s)$, which is exactly the center of curvature of the curve. Thus, the characteristics of the family of normal planes are nothing but the axes of curvature (or polar axes) of the curve. This is the reason why the envelope of the family of normal planes is called, sometimes, the *polar surface* of the curve.

To get the edge of regression of the surface, we differentiate once more the relation (11.1.30) and we get

$$-\underbrace{\mathbf{r}'(s) \cdot k(s) \cdot \boldsymbol{\nu}(s)}_{=0} + (\mathbf{R} - \mathbf{r}(s)) \cdot k'(s) \cdot \boldsymbol{\nu}(s) + (\mathbf{R} - \mathbf{r}(s)) \cdot k(s) \cdot \boldsymbol{\nu}'(s) = 0$$

or, using (11.1.31) and the second Frenet formula,

$$\frac{k'(s)}{k(s)} + (\mathbf{R} - \mathbf{r}(s)) \cdot k(s) \cdot [\chi(s) \cdot \boldsymbol{\beta}(s) - k(s)\boldsymbol{\tau}(s)] = 0$$

or, also, from (11.1.29),

$$(\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\beta}(s) = \frac{k'(s)}{k^2(s) \cdot \chi(s)}. \tag{11.1.32}$$

Comparing (11.1.29) to (11.1.32), we find out the following things:

- The straight line of intersection between the planes (11.1.29) and (11.1.32) is parallel to the osculating plane to the rectifying plane and the normal plane, i.e. is parallel to the binormal.

- This straight line passes through the point
$$\mathbf{r}(s) + \frac{k'(s)}{k^2(s) \cdot \chi(s)} \cdot \boldsymbol{\beta}(s).$$

Thus, the current point from the edge of regression is obtained by intersecting the lines

$$\mathbf{R}(\lambda) = \mathbf{r}(s) + \frac{1}{k(s)} \cdot \boldsymbol{\nu}(s) + \lambda \boldsymbol{\beta}(s) \tag{11.1.33}$$

and

$$\mathbf{R}(\mu) = \mathbf{r}(s) + \frac{k'(s)}{k^2(s)\chi(s)} \cdot \boldsymbol{\beta}(s) + \mu \boldsymbol{\nu}(s). \tag{11.1.34}$$

This point will have, thus, the position vector

$$\mathbf{R}(s) = \mathbf{r}(s) + \frac{1}{k(s)} \cdot \boldsymbol{\nu}(s) + \frac{k'(s)}{k^2(s)\chi(s)} \cdot \boldsymbol{\beta}(s). \tag{11.1.35}$$

But the right hand side of the relation (11.1.35) is nothing but the position vector of the osculating sphere of the given curve. Thus, *the edge of regression of the polar surface of a space curve is the locus of the centers of the osculating spheres of the given curve*.

The envelope of the family of rectifying planes of a space curves

The equation of the rectifying plane is

$$(\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\nu}(s) = 0. \tag{11.1.36}$$

By differentiation, we get

$$-\underbrace{\mathbf{r}'(s) \cdot \boldsymbol{\nu}(s)}_{=0} + (\mathbf{R} - \mathbf{r}(s)) \cdot \boldsymbol{\nu}'(s) = 0$$

or, using the second Frenet equation,

$$(\mathbf{R} - \mathbf{r}(s)) \cdot [\chi(s)\boldsymbol{\beta}(s) - k(s)\boldsymbol{\tau}(s)] = 0. \tag{11.1.37}$$

The points of the curve verify the equations (11.1.36) and (11.1.37), hence the curve lies on the rectifying surface and can be takes as the directrix of the surface, regarded as a ruled surface. The relations (11.1.36) and (11.1.37) indicate that the rulings (i.e. the characteristic lines of the envelope) are perpendicular both on $\boldsymbol{\nu}$, and on $\boldsymbol{\nu}'$, hence their direction is given by the vector

$$\boldsymbol{\nu} \times \boldsymbol{\nu}' = \boldsymbol{\nu} \times (\chi\boldsymbol{\beta} - k\boldsymbol{\tau}) = \chi(\boldsymbol{\nu} \times \boldsymbol{\beta}) - k(\boldsymbol{\nu} \times \boldsymbol{\tau}) = \chi\boldsymbol{\tau} + k\boldsymbol{\beta} = \boldsymbol{\delta},$$

where $\boldsymbol{\delta}$ is the Darboux vector of the curve. Thus, *the characteristics of the rectifying surface of a space curve are the Darboux axes of the curve or the instantaneous axes of rotation of the Frenet frame*.

The tangent plane to the rectifying surface at a point lying on the given curve contain both the tangent vector of the curve, $\mathbf{r}' = \boldsymbol{\tau}$, and the Darboux vector of the curve, $\boldsymbol{\delta}$. Therefore, the normal to the surface at such a point is colinear to the vector

$$\boldsymbol{\tau} \times \boldsymbol{\delta} = \boldsymbol{\tau} \times (\chi\boldsymbol{\tau} + k\boldsymbol{\beta}) = k(\boldsymbol{\tau} \times \boldsymbol{\beta}) = -k\boldsymbol{\nu}.$$

Thus, the normal to the rectifying surface of a space curve at a point of the curve coincides to the principal normal of the curve. This actually means that the *osculating plane of the curve is perpendicular on the tangent space of the surface*, i.e. *the curve is a geodesic of its rectifying surface*. This is, actually, the origin of the name of the rectifying plane and, implicitly, of the rectifying surface: the curve, when regarded as lying on its rectifying surface, is a geodesic, which is the surface analogue of a straight line. In other words, the curve is *rectified* (straightened) by its rectifying planes.

As we saw, the total of a developable surface is always zero. In many books of differential geometry it is claimed that the converse is, also, true, in other words, any surface of zero Gaussian curvature is a developable surface or a part of a developable surface. However, this is true only if we assume that the surface doesn't have any planar points, otherwise, first of all, it might not be not even a ruled surface. An example of such surface is given in the book of Klingenberg [**?**]. The surface is not, actually, described by using a parameterization or an implicit equation, but rather by prescribing the first two fundamental forms and showing that they fulfill the compatibility conditions.

11.2 Minimal surfaces

11.2.1 Definition and general properties

Definition 11.3. A *minimal* surface is, by definition, a surface whose mean curvature vanishes identically.

This is not the original definition of a minimal surface. As a matter of fact, Lagrange extended to the two-dimensional case the method of Euler for finding the extrema of a functional and he proposed the following problem:

Given a closed curve and a connected surface patch S bounded by the curve, to finde the surface such that the inclosed area shall be minimum.

We consider, thus, the area functional,

$$A = \iint_U H du dv.$$

As it is known from the variational calculus, a necessary condition for the minimum of this functional is furnished by the Lagrange equations, which, in this case, are:

$$\begin{cases} \dfrac{\partial H}{\partial x} - \dfrac{\partial}{\partial u}\left(\dfrac{\partial H}{\partial x'_u}\right) - \dfrac{\partial}{\partial v}\left(\dfrac{\partial H}{\partial x'_v}\right) = 0 \\ \dfrac{\partial H}{\partial y} - \dfrac{\partial}{\partial u}\left(\dfrac{\partial H}{\partial y'_u}\right) - \dfrac{\partial}{\partial v}\left(\dfrac{\partial H}{\partial y'_v}\right) = 0 \\ \dfrac{\partial H}{\partial z} - \dfrac{\partial}{\partial u}\left(\dfrac{\partial H}{\partial z'_u}\right) - \dfrac{\partial}{\partial v}\left(\dfrac{\partial H}{\partial z'_v}\right) = 0 \end{cases}.$$

After long, but otherwise straightforward computations, the Lagrange equations become

$$\begin{cases} \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix} K_m = 0 \\ \begin{vmatrix} x'_u & z'_u \\ x'_v & z'_v \end{vmatrix} K_m = 0 \\ \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix} K_m = 0 \end{cases},$$

and, thus, they are satisfied iff $K_m = 0$.

Let us mention, also, that, initially, Lagrange used the explicit representation $z = f(x, y)$ for the surface, in which case the area functional is

$$\iint_U \sqrt{1 + p^2 + q^2} dx dy,$$

while the stationarity condition finally becomes

$$\frac{\partial}{\partial x}\frac{p}{\sqrt{1+p^2+q^2}}+\frac{\partial}{\partial y}\frac{p}{\sqrt{1+p^2+q^2}}=0$$

or

$$(1+q^2)r-2pqs+(1+p^2)t=0.$$

It was Meusnier who showed, sixteen years later, that the minimality condition induces a relation among the coefficients of the first two fundamental forms of a parameterized surface. Eisenhart claims that Meusnier actually showed that for minimal surfaces the mean curvature vanishes. This is not quite correct: Meusnier was not aware of the true meaning of the the relation he found. In fact, the notion of mean curvature was introduced only fifty years later, by Sophie Germain.

An alternative proof of the fact that the minimal surfaces are the solution of the variational problem was found by Darboux. A particularly clear exposition of this proof can be found in the book of Hsiung.

For further convenience, we shall need the following definition:

Definition 11.4. A local parameterization $(U, \mathbf{r})$ of a surface S is called *isothermic* if, with respect to this parameterization, the coordinate lines are orthogonal (i.e. $F=0$) and $E=G$. This means, actually, that the matrix of the first fundamental form is just the unit matrix multiplied by a positive function.

On any surface which is at least C^2 there exist isothermic parameterizations. The proof of this claim is based on the theory of partial differential equations and it is not interesting for our exposition.

Property 28. *A surface is minimal iff its asymptotic directions are orthogonal.*

Proof A surface is minimal iff the coefficients of the first two fundamental forms verify the equation

$$ED''-2FD'+GD=0.$$

Let us choose, for the surface, an isothermic parameterization, with respect to which we have

$$E=G=\lambda^2(u,v),\quad F=0.$$

Then the minimality condition for the surface becomes

$$E(D''+D)=0$$

and, as $E \neq 0$, the surface being regular, we have

$$D' + D'' = 0.$$

On the other hand, the equation for the slopes of the asymptotic directions is

$$Dt^2 + 2D't + D'' = 0$$

and one can see immediately that the asymptotic directions are orthogonal iff the roots of this equation verify $t_1 t_2 = -1$, which is equivalent to $D = -D''$. $\square$

The following technical lemma will allow us to obtain a very nice characterization of minimal surfaces

Lemma 1. *Let* $\mathbf{r} = \mathbf{r}(u, v)$ *be an isothermic parameterization of a surface* S. *Then*

$$\mathbf{r}''_{\mathbf{u}^2} + \mathbf{r}''_{\mathbf{v}^2} = 2\lambda^2 K_m \, \mathbf{n}, \tag{11.2.1}$$

where $\lambda^2 = E = G$.

Proof Since $\mathbf{r}$ is isothermic, we have

$$\mathbf{r}'^{\,2}_{\mathbf{u}} = \mathbf{r}'^{\,2}_{\mathbf{v}} = \lambda^2, \quad \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{r}'_{\mathbf{v}} = 0, \tag{11.2.2}$$

therefore the mean curvature is just

$$K_m = \frac{D + D''}{\lambda^2}. \tag{11.2.3}$$

By differentiating (11.2.2), we get

$$\mathbf{r}''_{\mathbf{u}^2} \cdot \mathbf{r}'_{\mathbf{u}} = \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{r}'_{\mathbf{v}} = -\mathbf{r}'_{\mathbf{u}} \cdot \mathbf{r}''_{\mathbf{v}^2},$$

whence

$$(\mathbf{r}''_{\mathbf{u}^2} + \mathbf{r}''_{\mathbf{v}^2}) \cdot \mathbf{r}'_{\mathbf{u}} = 0,$$

and, similarly,

$$(\mathbf{r}''_{\mathbf{u}^2} + \mathbf{r}''_{\mathbf{v}^2}) \cdot \mathbf{r}'_{\mathbf{v}} = 0.$$

Thus, the vector $\mathbf{r}''_{\mathbf{u}^2} + \mathbf{r}''_{\mathbf{v}^2}$ is perpendicular both to $\mathbf{r}'_{\mathbf{u}}$ and $\mathbf{r}'_{\mathbf{v}}$, i.e. is colinear with the unit normal:

$$\mathbf{r}''_{\mathbf{u}^2} + \mathbf{r}''_{\mathbf{v}^2} = a \, \mathbf{n}. \tag{11.2.4}$$

If we multiply (11.2.4) by $\mathbf{n}$ and use the equation (11.2.2), as well as the definition of the second fundamental form, we obtain

$$a = D + D'' = 2\lambda^2 K_m, \tag{11.2.5}$$

which proves the claim. $\square$

Corollary 1. *if* $\mathbf{r} = \mathbf{r}(u, v)$ *is an isothermic parameterization of a surface* S*, then* S *is minimal iff the components of* $\mathbf{r}$ *are harmonical functions, in other words iff we have*

$$\Delta x = \Delta y = \Delta z = 0.$$

It is a classical (but nontrivial) result that on any smooth (at least C^2) surface there exists isothermic parameterization. On a minimal surface we have a very special isothermic parameterization:

Property 29. *The lines of curvature on a minimal surface form an isothermic system.*

Proof We choose as parametric lines the lines of curvature of the surface. Then, as we already know, we should have $F = D' = 0$ (both first fundamental forms are diagonal in such a kind of parameterization). Thus, the minimality condition for the surface becomes

$$ED'' + GD = 0$$

In this parameterization the Codazzi-Mainardi equations become

$$\frac{\partial D}{\partial v} - \frac{1}{2}\left(\frac{D}{E} + \frac{D''}{G}\right) = 0,$$
$$\frac{\partial D''}{\partial u} - \frac{1}{2}\left(\frac{D}{E} + \frac{D''}{G}\right) = 0,$$

which, for a minimal surface, reduce to

$$\frac{\partial D}{\partial v} = 0,$$
$$\frac{\partial D''}{\partial u} = 0,$$

i.e. D depends only on u and D'' – only on v. Since

$$\frac{E}{D} = -\frac{G}{D''},$$

we have

$$\frac{\partial^2 \log E}{\partial u \partial v} = \frac{\partial^2 \log G}{\partial u \partial v},$$

which is exactly the condition that the parametric lines are isothermic. □

Property 30. *The spherical image of a minimal surface is conformal (i.e. it preserves the angles).*

Proof Let e, f, g be the coefficients of the first fundamental form of the spherical image. We have, then,

$$e = -EK_t + DK_m, \quad f = -FK_t + D'K_m, \quad g = -GK_t + D''K_m.$$

Thus, for a minimal surface, we have

$$e = -EK_t, \quad f = -FK_t, \quad g = -GK_t,$$

i.e. the two surfaces are, indeed, conformally equivalent. □

The reciprocal of this proposition is almost true, i.e. we have

Property 31. *If the spherical map of a surface is conformal, then the surface is either minimal, either a sphere.*

Proof If the spherical map is conformal, then we have

$$-EK_t + DK_m = \mu E, \quad -FK_t + D'K_m = \mu F, \quad -GK_t + D''K_m = \mu G. \tag{11.2.6}$$

Thus,

$$DK_m = E(K_t + \mu), \quad D'K_m = F(K_t + \mu), \quad D''K_m = G(K_t + \mu). \tag{11.2.7}$$

We multiply the three relations by $D'', -2D', D$, respectively and we add the results, we get

$$2K_t K_m = K_m(K_t + \mu)$$

or

$$K_m(\mu - K_t) = 0. \tag{11.2.8}$$

On the other hand, if we multiply the relations (11.2.7) by $G, -2F, E$, respectively, we get

$$K_m^2 = 2(K_t + \mu). \tag{11.2.9}$$

If we put K_m, the (11.2.8) is satisfied, while (11.2.9) is satisfied iff $\mu = -K_t$ and we rediscover the result from the proof of the previous proposition, i.e. the spherical image of a minimal surface is conformal and the conformal factor is $\sqrt{-K_t}$. The second possibility to satisfy (11.2.8) is $K_t = \mu$. Notice that, as $\mu > 0$, we cannot have at the same time $K_t = \mu$ and $K_m = 0$, because then we would get, also, $K_t = -\mu$, i.e. $K_t = \mu = 0$.

If $K_t = \mu$, then, from (11.2.9) we get

$$K_m^2 = 4\mu = 4K_t,$$

i.e. the principal curvatures are equal at each point (all the points are umbilical). Or, as we know, the only surface with only umbilical points is the sphere (and its limit case, the plane). □

Property 32. *The spherical image of isothermic lines on a minimal surface are isothermic lines on the sphere.*

Proof This is an immediate consequence of the fact that the spherical map of a minimal surface is conformal. □

The first two minimal surfaces (apart from the plane) were discovered in the eighteenth century by the French mathematician Meusnier. These surface were the catenoid and the right helicoid. As we know, the first surface is a revolution surface, while the second is a ruled surface. For a couple of decades, nobody was able to find other examples. The surfaces found by Meusnier are very simple and they belong classes of surfaces which are relatively easy to study, but, as we shall see in the following, these classes do not contain other minimal surfaces.

11.2.2 Minimal surfaces of revolution

Property 33. *The catenoid is the only revolution surface which is, also, a minimal surface.*

Proof An arbitrary surface of revolution (around the z-axis, in our case), can be parameterized as

$$\begin{cases} x &= u\cos v \\ y &= u\sin v \\ z &= f(u) \end{cases}.$$

Therefore, the coefficients of the first two fundamental forms of the surface can be written as

$$E = 1 + f'^2, \quad F = 0, \quad G = u^2,$$
$$D = \frac{f''}{\sqrt{1+f'^2}}, \quad , D' = 0, \quad D'' = \frac{uf'}{\sqrt{1+f'^2}}.$$

Then the minimality condition for the surface reads

$$(1 + f'^2)uf' + f''u^2 = 0$$

which can be integrated and gives for f an expression of the form

$$f(u) = a \cosh^{-1} \frac{u}{a} + c,$$

where a and c are constants, which corresponds to the catenoid. □

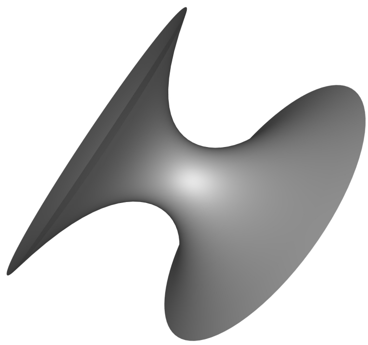


Figure 11.5: The catenoid

11.2.3 Ruled minimal surfaces

We recall that the right conoid is the surface generated by s straight line moving parallel to a plane ($z = 0$ in our case) and intersecting a line which is perpendicular to this plane (in our case – the z-axis). As we saw, the equations of a right conoid can be written as

$$\begin{cases} x &= u \cos v \\ y &= u \sin v \\ z &= f(v) \end{cases}.$$

Then the coefficients of the first two fundamental forms of the surface will be

$$E = 1, \quad F = 0, \quad G = u^2 + f'^2,$$
$$D = 0, \quad D' = -\frac{f'}{\sqrt{u^2 + f'^2}}, \quad D'' = \frac{uf''}{\sqrt{u^2 + f'^2}}.$$

We are interested to see which right conoids are minimal surfaces. Well, as we will show, there is only one.

Property 34. *The only right conoid which is a minimal surface is the right helicoid.*

Proof As we know, the condition for a surface to be minimal is that the coefficients of the first two fundamental forms verify the equation

$$ED'' - 2FD' + GD = 0,$$

which, in our case, reduces to

$$\frac{uf''}{\sqrt{u^2 + f'^2}} = 0,$$

or $f'' = 0$. But this leads to $f(v) = av + b$, where a and b are constants, i.e. the surface is a right helicoid. □

One might think that there are other ruled minimal surfaces, which are not right conoids. For such more general ruled surfaces we do not have such a nice parameterization, therefore this direct approach cannot be applied. Still, as we shall see next, there is another method to prove that, in fact, there are no other ruled minimal surfaces.

Property 35 (Catalan, 1842)**.** *The only minimal ruled surface is the right helicoid.*

Proof Let S be a ruled surface which is not a plane. Then, as we saw previously, the surface is minimal iff the asymptotic lines are orthogonal. Through each point of the surface pass exactly two asymptotic lines. On the other hand, through each point passes one straight line (the generator). To prove that the surface is a right helicoid, it is enough to prove that it has as directrix a circular cylindrical helix. Since the surface is minimal, the second asymptotic line through each point is orthogonal to the generator passing through that point. The asymptotic lines have the property that their principal normals are contained into the tangent plane. This means that for an asymptotic line from the second family, the generators that intersect it are principal normals. But they are principal normals also for any other asymptotic line that intersects them. Thus, any asymptotic line from the second family has an infinity of Bertrand mates. But then, as we know, they have to be cylindrical helices. □

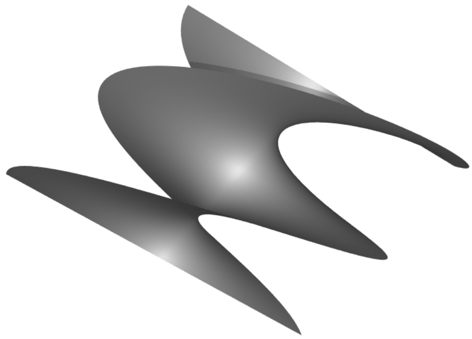


Figure 11.6: The minimal helicoid

We shall give now, following Barbosa and Colares, a more direct prove of the Catalan's theorem, using a local parameterization of the ruled surface.

Another proof of the Catalan's theorem Let $S \subset \mathbb{R}^3$ be a ruled surface. Then from the definition, S has local parameterizations of the form

$$\mathbf{r}(u, v) = \alpha(u) + v\boldsymbol{\gamma}.$$

We may choose the directrix α of the ruled surface to be an orthogonal trajectory of the family of rulings, and $\boldsymbol{\gamma}$, the directing vector of the rulings, to be a unit vector, at each point of the directrix. We shall assume, moreover, that u is the natural parameter of the directrix. A straightforward computation provides, for the coefficients of the first two fundamental forms of the surface, the expressions:

$$\begin{aligned}
E &= 1 + 2v\boldsymbol{\alpha}' \cdot \boldsymbol{\gamma}' + v^2\boldsymbol{\gamma}'^2, \\
F &= 0, \\
G &= 1, \\
D &= \frac{1}{\sqrt{1 + 2v\boldsymbol{\alpha}' \cdot \boldsymbol{\gamma}' + v^2\boldsymbol{\gamma}'^2}}(\boldsymbol{\alpha}' + v\boldsymbol{\gamma}', \boldsymbol{\gamma}, \boldsymbol{\alpha}'' + v\boldsymbol{\gamma}''), \\
D' &= \frac{1}{\sqrt{1 + 2v\boldsymbol{\alpha}' \cdot \boldsymbol{\gamma}' + v^2\boldsymbol{\gamma}'^2}}(\boldsymbol{\alpha}' + v\boldsymbol{\gamma}', \boldsymbol{\gamma}, \boldsymbol{\gamma}'), \\
D'' &= 0.
\end{aligned}$$

Thus, the minimality condition reads

$$(\boldsymbol{\alpha}' + v\boldsymbol{\gamma}', \boldsymbol{\gamma}, \boldsymbol{\alpha}'' + v\boldsymbol{\gamma}'') = 0,$$

or, which is the same,

$$(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\alpha}'' + v\left[(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' + (\boldsymbol{\gamma}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\alpha}''\right] + v^2(\boldsymbol{\gamma}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' = 0.$$

As the left hand side of the previous equation is a polynomial function in v, the equality holds for any v iff the coefficients of the polynomial function vanish simultaneously:

$$(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\alpha}'' = 0 \tag{a}$$
$$(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' + (\boldsymbol{\gamma}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\alpha}'' = 0 \tag{b}$$
$$(\boldsymbol{\gamma}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' = 0. \tag{c}$$

From the relation (a) it follows that the vector $\boldsymbol{\alpha}''$ must be contained in the plane generated by the vectors $\boldsymbol{\alpha}'$ and $\boldsymbol{\gamma}$. But, as the curve α is naturally parameterized, the vector $\boldsymbol{\alpha}'$ has unit length, which means that $\boldsymbol{\alpha}''$ is perpendicular to $\boldsymbol{\alpha}'$. As, from construction, $\boldsymbol{\gamma}$ is also perpendicular to $\boldsymbol{\alpha}'$, it follows that $\boldsymbol{\alpha}''$ is colinear to $\boldsymbol{\gamma}$. But the, the relation (b) reduces to

$$(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}) \cdot \boldsymbol{\gamma}'' = 0. \tag{b'}$$

Now, from (b') and (c) follows that the vector $\boldsymbol{\gamma}''$ should belong to the intersection of the planes determined by the vectors $\boldsymbol{\alpha}', \boldsymbol{\gamma}$ and $\boldsymbol{\gamma}', \boldsymbol{\gamma}$, respectively. Clearly, this two planes are not parallel, since they contain at least the vector $\boldsymbol{\gamma}$. Therefore, only two situations are possible: either their intersection is a straight line, either they coincide. Suppose, first, that there exists a point where $\boldsymbol{\gamma}''$ is not parallel to $\boldsymbol{\gamma}$. Then, at this point (and, by continuity, in an entire neighborhood of it), these planes have to coincide. Indeed, otherwise $\boldsymbol{\gamma}''$ cannot belong simultaneously to both planes. Moreover, in this situation we have, also $\boldsymbol{\alpha}' \parallel \boldsymbol{\gamma}'$. Indeed, both $\boldsymbol{\gamma}'$ and $\boldsymbol{\alpha}'$ are perpendicular on $\boldsymbol{\gamma}$. As the three vectors are coplanar, it follows the parallelism just mentioned. Now, as we assumed that α and γ are not just smooth, but analytic functions, the two vectors are parallel for any value of the parameter, and, also, the two planes will also coincide. The plane that contains all our vectors could, in principle, vary from point to point. In reality, instead, this not happens. Indeed, we have

$$(\boldsymbol{\gamma} \times \boldsymbol{\alpha}')' = \boldsymbol{\gamma}' \times \boldsymbol{\alpha}' + \boldsymbol{\gamma} \times \boldsymbol{\alpha}'' = 0.$$

But this means that the plane of the vectors is invariant, therefore the curve α is a plane curve and the surface is a plane.

Suppose, now, that $\boldsymbol{\gamma}''$ is always parallel to $\boldsymbol{\gamma}$. If $\boldsymbol{\alpha}'$ is always parallel to $\boldsymbol{\gamma}'$, this means, actually, that the two planes always coincide and we are, in fact, in the previous situation. Let us assume, therefore, that there exists a point (hence an entire open set) where the two vectors are not parallel. We are going to show that, in this case, α is a circular helix. As we know, to this end it is enough to prove that both the curvature and the torsion of the curve are constants.

Let $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$ the Frenet frame of the curve α. We intend to use the Frenet equation to find the curvature and the torsion, instead of using the explicit formulas we obtained when we studied the space curves. First of all, as α is naturally parameterized, we have

$$\boldsymbol{\alpha}''(u) = k(u)\boldsymbol{\nu}(u).$$

On the other hand, again, because α is naturally parameterized, we have $\boldsymbol{\alpha}''(u) \perp \boldsymbol{\alpha}'(u)$. But $\boldsymbol{\alpha}'$ is also perpendicular on $\boldsymbol{\gamma}$, and the three vectors are coplanar, which means that, as $\boldsymbol{\gamma}$ is a unit vector, we have $\boldsymbol{\gamma}'(u) = \pm\boldsymbol{\nu}$, hence, using the previous equation,

$$\pm k(u) = \boldsymbol{\alpha}''(u) \cdot \boldsymbol{\gamma}(u).$$

But, since $\boldsymbol{\alpha}' \perp \boldsymbol{\gamma}$, we have

$$0 = (\boldsymbol{\alpha}' \times \boldsymbol{\gamma})' = \boldsymbol{\alpha}' \times \boldsymbol{\gamma}' + \boldsymbol{\alpha}'' \times \boldsymbol{\gamma},$$

therefore, we have for the curvature the expression

$$\pm k(u) = -\boldsymbol{\alpha}' \times \boldsymbol{\gamma}'.$$

Thus, the derivative of the curvature will be given by

$$\pm\frac{dk}{du} = -(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}')' = -\boldsymbol{\alpha}'' \times \boldsymbol{\gamma}' - \boldsymbol{\alpha}' \times \boldsymbol{\gamma}'' = 0,$$

i.e. the curvature of α is constant. The torsion, on the other hand, can be found using the third Frenet's formula. Indeed, this formula is

$$\boldsymbol{\beta}' = -\chi\boldsymbol{\nu}.$$

As we saw, $\boldsymbol{\nu} = \pm\boldsymbol{\gamma}$. On the other hand, we have $\boldsymbol{\tau} = \boldsymbol{\alpha}'$ and, thus,

$$\boldsymbol{\beta} \equiv \boldsymbol{\tau} \times \boldsymbol{\nu} = \pm\boldsymbol{\alpha}' \times \boldsymbol{\gamma},$$

whence

$$\boldsymbol{\beta}' = \pm(\boldsymbol{\alpha}' \times \boldsymbol{\gamma})' = \pm(\boldsymbol{\alpha}'' \times \boldsymbol{\gamma} + \boldsymbol{\alpha}' \times \boldsymbol{\gamma}') = \pm\boldsymbol{\alpha}' \times \boldsymbol{\gamma}'.$$

Hence the torsion will be given by

$$\chi = -\boldsymbol{\beta} \cdot \boldsymbol{\nu} = \pm(\boldsymbol{\alpha}' \times \boldsymbol{\gamma}') \cdot \boldsymbol{\gamma}$$

and it is easy to see that the derivative of the torsion is, also, identically zero, i.e. the torsion is a constant function.

Now, from the existence and uniqueness theorem for space curves, we know that there exists a curve which has the prescribed curvature and torsion and this curve is unique, up to a motion in space. But, on the other hand, we know a curve which has constant curvature and torsion, namely the circular helix. Thus, α should be a circular helix and has a parameterization of the form

$$\alpha(u) = (A\cos au, A\sin au, bu),$$

where, since α is supposed to be naturally parameterized, the constants A, a, b are subject to $A^2a^2 + b^2 = 1$. On the other hand, as $\boldsymbol{\gamma}$ is parallel to $\boldsymbol{\alpha}''$, we have, also,

$$\boldsymbol{\gamma} = \pm\{\cos au, \sin au, 0\}.$$

If, now, we put $s = A \pm u$, $t = v$, the equation of the surface, with α and $\boldsymbol{\gamma}$ just found, becomes

$$\mathbf{r}(s,t) = (s\cos at, s\sin at, bt),$$

which means that the surface is a part of a right helicoid, as claimed. □

Corollary 11.1. *The only developable minimal surface is the plane.*

As we already mentioned, Meusnier was the one who discovered that the right helicoid is a minimal surface. It is both interested and instructive to see how Meusnier got to this result. He searched a minimal surface in an explicit representation, $z = f(x,y)$. As we saw, in this case, the minimality condition can be written as

$$(1 + f'^2_y)f''_{x^2} - 2f'_x f'y f''_{xy} + (1 + f'^2_x)f''_{y^2} = 0.$$

Meusnier was interested in solutions of the previous equation for which the level curves $f(x,y) = \text{const}$ are straight lines in the plane. Obviously, such surfaces would be ruled. He started by noticing that, for a plane curve given in the implicit form $f(x, y) = c$, the signed curvature can be written as

$$k_{\pm} = \frac{-f''_{x^2} f'^2_y + 2 f'_x f'_y f''_{xy} - f''_{y^2} f'^2_x}{\| \operatorname{grad} f \|^3}.$$

Therefore, the minimality condition becomes

$$\Delta f = k_{\pm} \| \operatorname{grad} f \|^3.$$

If the level curves are straight lines, then their signed curvature vanishes, which means that f is a harmonical function, i.e.

$$\Delta f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Now, it can be shown that the only solution of this equation for which the level curves are straight lines is

$$f(x, y) = a \operatorname{arctg} \frac{y - y_0}{z - z_0} + b,$$

with a, b, x_0, y_0 constants.

The graph of the function $z = f(x, y)$ is, then, either a a plane (if $a = 0$), either a portion of the right helicoid

$$\begin{cases} x = x_0 + u \cos v \\ y = y_0 + u \sin v \\ z = av + b. \end{cases}$$

As one can see, even the original deduction of Meusnier could make one suspect that the only ruled minimal surface is the right helicoid.

11.3 Surfaces of constant curvature

Definition 11.5. We say that a surface S has *constant curvature* if its Gaussian curvature is independent of the point, i.e. is a constant.

We already met surfaces which have zero curvature, namely the plane and the developable surfaces. We also know an example of a surface that has constant positive curvature (the sphere). We shall give, now, an example of a (parameterized) surface that has constant *negative* curvature, namely the *pseudosphere*, given by the parametric equations

$$\begin{cases} x = a \sin u \cos v \\ y = a \sin u \sin v \\ z = a \left(\ln \tan \frac{u}{2} + \cos u \right) \end{cases} . \qquad (11.3.1)$$

A straightforward computation shows that the pseudosphere has the Gaussian curvature $-1/a^2$. Loosely, we can define a surface to be *complete* if it is unextendible (like the sphere, or the plane). It turns out that any surface of constant negative curvature, if extended enough, becomes singular. In other words, there are no non-singular complete surfaces of constant negative curvature in $\mathbb{R}^3$. The pseudosphere (see the figure 11.3) is no exception. A classical theorem of Minding (1839) claims that all

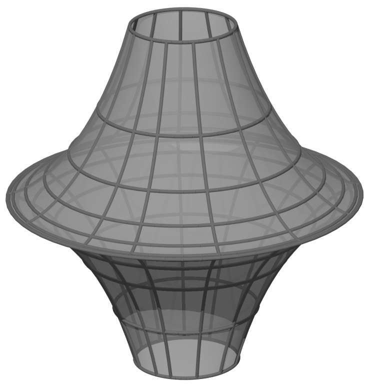


Figure 11.7: The pseudosphere

the surfaces that have the same constant curvature are locally isometric. This means that they have local parameterizations, defined on the same domain in the plane, in such a way that, for a given pair of parameters, they have the same coefficients for the first fundamental form. This result does not hold, however, if the surfaces don't have constant curvature, as the following counterexample shows.

Counterexample. Let $\mathbf{r}_1 : D_1 \equiv (0, \infty) \times (0, 2\pi) \to \mathbb{R}^3$, given by

$$\mathbf{r}_1(u_1, v_1) = (u_1 \cos v_1, u_1 \sin v_1, \ln u_1)$$

and $\mathbf{r}_2 : D_2 \equiv (0, \infty) \times (0, 2\pi) \to \mathbb{R}^3$, given by

$$\mathbf{r}_2(u_2, v_2) = (u_2 \cos v_2, u_2 \sin v_2, v_2) .$$

As one can see, the first surface is a surface of revolution, generated by the graph of the logarithmic function, while the second one is part of the right helicoid. We consider the map $\Lambda : D_1 \to D_2$, $\Lambda(u_1, v_1) = (u_2, v_2)$. As $D_1 = D_2$, clearly Λ is, in fact, the identity and it is, therefore, a diffeomorphism. An easy computation shows that the first fundamental forms for the two parameterized surfaces are, respectively:

$$E_1 = 1 + \frac{1}{u_1^2}, \quad F_1 = 0, \quad G_1 = u_1^2$$

and

$$E_2 = 1, \quad F_2 = 0, \quad G_2 = 1 + u_2^2,$$

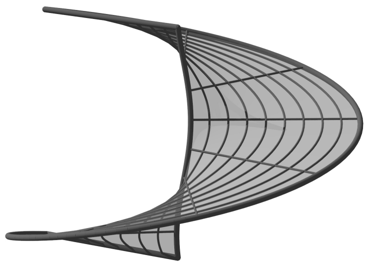
therefore they are *not* isometric (because, for instance, $E_1 \neq E_2$). However, as one can convince oneself, for their Gaussian curvatures we get

$$K_1 = -\frac{1}{(1 + u_1^2)^2}, \quad K_2 = -\frac{1}{(1 + u_2^2)^2},$$

i.e. they are equal.

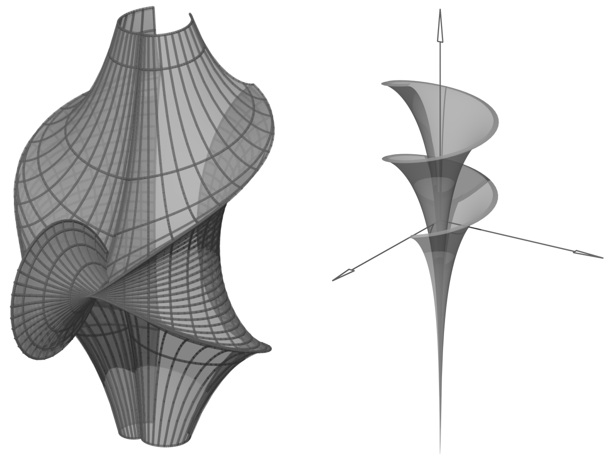
On the other hand, even if, for instance, the surfaces of the same constant negative curvature are locally isometric with the pseudosphere, they might look very different, as the surfaces from the figure 11.3, belonging to Kuen and Dini show.

(a) Surface of revolution



(b) Helicoid

Figure 11.8: Non-isometric parameterized surfaces



(a) Kuen's surface (b) Dini's surface

Figure 11.9: Surfaces of constant negative curvature

Problems

Problem 11.1. We consider, in the coordinate plane xOz, the curve given by

$$\begin{cases} x = f(u), \\ z = \varphi(u). \end{cases}$$

Find the parametrical equation of the surface obtained rotating this curve around the z-axis.

Problem 11.2. Find the equations of the *torus*, obtained by rotating the circle

$$\begin{cases} x = a + \cos u, \\ y = 0, \\ z = b \sin u, \quad b < a, \end{cases}$$

around the z-axis.

Problem 11.3. Find the equations of the *catenoid*, obtained by rotating the catenary

$$\begin{cases} x = a \cosh \dfrac{u}{a}, \\ y = 0, \\ z = u \end{cases}$$

around the z-axis.

Problem 11.4. Find the equations of the *pseudosphere*, obtained by rotating the tractrix

$$\begin{cases} x = a \sin u, \\ y = 0, \\ z = a \left(\ln \tan \frac{u}{2} + \cos u \right) \end{cases}$$

around the z-axis.

Problem 11.5. Find the equation of the tangent plane at an arbitrary point of the right helicoid

$$\mathbf{r} = \{u \cos v, u \sin v, hv\}.$$

Problem 11.6. Show that if a smooth surface Φ and a plane α have a single common point P, then the plane α is the tangent plane to the surface at the point P.

Problem 11.7. Write the equation of the tangent plane to the sphere of radius a, centred at the origin, at the point of coordinates $(0, 0, a)$ ("north pole").

Problem 11.8. Write the equations of the normal to the pseudosphere

$$\begin{cases} x = a \sin u \cos v, \\ y = a \sin u \sin v, \\ z = a \ln \tan \frac{u}{2} + a \cos u \end{cases}$$

at an arbitrary point and find the unit normal vector.

Problem 11.9. Find the unit normal vector to the right helicoid

$$\begin{cases} x = u \cos v \\ y = u \sin v \\ z = hv \end{cases} .$$

Problem 11.10. Determine the equation of the tangent plane to the surface $xy^2 + z^2 = 8$ at the point $(1, 2, 2)$ and find the unit normal vector at this point.

Problem 11.11. Show that the surfaces

$$x^2 + y^2 + z^2 = \alpha x,$$
$$x^2 + y^2 + z^2 = \beta y$$

and

$$x^2 + y^2 + z^2 = \gamma z$$

are pairwise orthogonal to each other.

Problem 11.12. Show that the normal to a surface of rotation at an arbitrary point passes through the rotation axis.

Problem 11.13. Show that the tangent plane at an arbitrary point of the surface $z = x\varphi\left(\frac{y}{x}\right)$, where φ is a smooth function of a single real variable, passes through the origin of coordinates.

Problem 11.14. Show that the tangent plane to a surface generated by the tangents to a space curve is the same at all the points of a given generator.

Problem 11.15. Write the equation of the tangent plane and the equations of the normal at the surface generated by the binormals of a curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$.

Problem 11.16. The surface Φ is generated by the binormals of a curve γ. Show that at the points of the curve γ the tangent plane to Φ coincide with the osculating plane to γ, while the normal to the surface is the principal normal to the curve γ.

Problem 11.17. A surface is generated by the principal normals of a curve γ. Establish the equation of the surface. Show that at the points of the curve γ the tangent plane to the surface coincide with the osculating plane to the curve γ.

Problem 11.18. On the normals to a surface Φ, in the same direction, there are taken segments of the same length, measured from the surface. The ends of these segments describe another surface, Φ^*, "parallel" to the surface Φ. Show that the surfaces Φ and Φ^* have, at corresponding points, common normals and parallel tangent planes.

Problem 11.19. Study the variation of the sign of the function $\mathbf{r} \cdot \mathbf{n}$ for the torus, where $\mathbf{r}$ is the radius vector of a point of the torus, while $\mathbf{n}$ is the normal to the torus at that point. Find the geometrical locus of the points from the torus at which $\mathbf{r} \cdot \mathbf{n} = 0$ (the origin of the coordinates is taken at the centre of the torus).

Problem 11.20. Find a close surface on which the function $\mathbf{r} \cdot \mathbf{n}$ is a constant.

Problem 11.21. Find the points of the torus

$$\boldsymbol{\rho} = \{(R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u\}$$

at which its normal is parallel to the vector $\mathbf{a} = \{l, m, n\}$

Problem 11.22. Write the equation of the tangent plane to the torus

$$\begin{cases} x = (3 + 2\cos u)\cos v, \\ y = x = (3 + 2\cos u)\sin v, \\ z = 2\sin u, \end{cases}$$

parallel to the plane $x + y + \sqrt{2}z + 5 = 0$.

Problem 11.23. Find the tangent plane to the surface $z = x^4 - 2xy^3$, perpendicular to the vector $\mathbf{a} = \{-2, 6, 1\}$. Find, also, the tangency point.

Problem 11.24. Find the envelope and the regression edge of the family of spheres of constant radius, equal to a, with the centers on the circle

$$\begin{cases} x^2 + y^2 = b^2, \\ z = 0. \end{cases}$$

Problem 11.25. Find the envelope of the family of spheres

$$x^2 + y^2 + (z - C)^2 = 1.$$

Problem 11.26. Find the envelope, the characteristics and the regression edge for a family of spheres of constant radius a, with centers on a given curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ (tubular surface).

Problem 11.27. Find the envelopes and the regression edge for a family of spheres passing through the origin of the coordinates and with the centers on the curve

$$\begin{cases} x = t^3, \\ y = t^2, \\ z = t. \end{cases}$$

Problem 11.28. Find the envelope of a family of planes forming with the coordinate solid angle $x > 0, y > 0, z > 0$ a triangular pyramid of constant volume V.

Problem 11.29. Find the envelope and the regression edge for the family of normal planes to the helix

$$\begin{cases} x = a\cos t, \\ y = a\sin t, \\ z = bt. \end{cases}$$

Problem 11.30. Find the envelopes, the characteristics and the regression edge for the family of normal planes to a space curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ (*the polar surface of the curve*).

Problem 11.31. Find the envelope, the characteristics and the regression edge for the family of rectifying planes of a space curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$.

Problem 11.32. Find the envelope, the characteristics and the regression edge of the family of osculating planes of a space curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$.

Problem 11.33. Find the envelope, the characteristics and the regression edge of the family of planes

$$(\mathbf{r}, \mathbf{n}(\alpha)) + D(\alpha) = 0,$$

where $\|\mathbf{n}\| = 1$, $\|\mathbf{n}'(\alpha)\| \neq 0$, $\|\mathbf{n}''(0)\| \neq 0$.

Problem 11.34. Find the envelope and the characteristics of the family of circular cylinders of constant radius r:

a) $(x - C)^2 + y^2 = r^2$;

b) $x^2 + (y - C)^2 = r^2$;

c) $(x - C)^2 + (y - C)^2 = r^2$.

Problem 11.35. Write the equation of the tangent plane at an arbitrary point of the surface generated by the tangents to the space curve $\mathbf{r} = \mathbf{r}(s)$. Study its position when the tangency point moves along a generators of the surface.

Problem 11.36. Show that the osculating plane to the regression edge of a developable surface coincides with the tangent plane to the surface at that point.

Problem 11.37. Show that the intersection curve between a developable surface and the osculating plane at its regression edge has, for arbitrary point of this edge, the curvature equal to $\frac{3}{4}$ from the curvature of the regression edge at that point.

Problem 11.38. A *Catalan surface* is a skew ruled surface whose generators are parallel to a given plane, the *directing plane*. Show that a ruled surface

$$\mathbf{r} = \mathbf{r}_1(u) + v\mathbf{b}(u)$$

is a Catalan surface iff

$$(\mathbf{b}, \mathbf{b}', \mathbf{b}'') = 0, \quad \mathbf{b}'' \neq 0.$$

Problem 11.39. Find the first fundamental form of the second degree surfaces

1. $x^2 + y^2 + z^2 = r^2$ (the sphere);
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (the ellipsoid);
3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (the hyperboloid with one sheet);
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ (the hyperboloid with two sheets);
5. $\frac{x^2}{p} + \frac{y^2}{q} = 2z$ (the elliptical paraboloid);
6. $\frac{x^2}{p} - \frac{y^2}{q} = 2z$ (the hyperbolical paraboloid);
7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ (the cone);
8. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (the elliptical cylinder);
9. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (the hyperbolical cylinder);
10. $y^2 = 2px$ (the parabolical cylinder).

Problem 11.40. Specify which of the following quadratic forms can play the role of the first fundamental form for a surface:

1. $ds^2 = du^2 + 4dudv + dv^2$;
2. $ds^2 = du^2 + 4dudv + 4dv^2$;
3. $ds^2 = du^2 - 4dudv + 6dv^2$;
4. $ds^2 = du^2 + 4dudv - 2dv^2$.

Problem 11.41. Find the transformation formulas for the coefficients of the first fundamental form and of the quantity

$$H = \sqrt{EG - F^2}$$

for a parameters' change.

Problem 11.42. Show that any revolution surface has a parameterization for which the first fundamental form can be written under the form

$$ds^2 = du^2 + G(u)dv^2.$$

Problem 11.43. A coordinate net on a surface is called a *Tchebysheff net* if the segments of coordinate lines from a family intercepted between two coordinate lines from the other family have constant length. Show that a coordinate net on a surface is a Tchebysheff net iff $E'_v = 0, G'_u = 0$.

Problem 11.44. Find a coordinate transformation such that the first fundamental form of the pseudosphere,

$$ds^2 = a^2 \operatorname{ctg}^2 u du^2 + a^2 \sin^2 u dv^2$$

becomes

$$ds^2 = d\overline{u}^2 + G(\overline{u})d\overline{v}^2.$$

Problem 11.45. Find the equation of a curve which intersect the meridians of a revolution surface under a constant angle α (the *loxodrom*).

Problem 11.46. Find the loxodroms of the sphere.

Problem 11.47. Find the orthogonal trajectories of the rectilinear generators of a conical surface.

Problem 11.48. Find the differential equation of the curve which intersects the rectilinear generators of a tangent developable under a constant angle α.

Problem 11.49. Find the orthogonal trajectories of the family of curves

$$u + v = \text{const},$$

lying on the sphere

$$\begin{cases} x = R \cos u \cos v, \\ y = R \cos u \sin v, \\ z = R \sin u. \end{cases}$$

Problem 11.50. On the circular cone

$$\begin{cases} x = u \cos v, \\ y = u \sin v, \\ z = u, \end{cases}$$

one considers the family of curves

$$v = u^2 + \alpha,$$

where α is a parameter. Find the orthogonal trajectories of this family.

Problem 11.51. Find the equations of the curves from the sphere

$$\begin{cases} x = R \cos u \sin v, \\ y = R \sin u \sin v, \\ z = R \cos v \end{cases}$$

which bissect the angle between the meridians and the parallels.

Problem 11.52. Find the intersection angle between the curves

$$u + v = 0, \quad u - v = 0$$

from the right helicoid

$$\begin{cases} x = u \cos v, \\ y = u \sin v, \\ z = av. \end{cases}$$

Problem 11.53. Find the perimeter and the interior angles of the curvilinear triangle

$$u = \pm \frac{1}{2} a v^2, \quad v = 1,$$

from the surface with the first fundamental form

$$ds^2 = du^2 + (u^2 + a^2) dv^2.$$

Problem 11.54. On the surface with the first fundamental form

$$ds^2 = du^2 + \sinh^2 u dv^2$$

find the length of the arc of the curve $u = v$ between the points $M_1(u_1, v_1)$ and $M_2(u_2, v_2)$.

Problem 11.55. Find the angle between the curves

$$v = u + 1 \quad \text{şi} \quad v = 3 - u$$

from the surface

$$\begin{cases} x = u \cos c, \\ y = u \sin v, \\ z = u^2. \end{cases}$$

Problem 11.56. On a sphere there is given a right angle triangle, whose sides are arcs of great circles of the sphere. Find:

1. the relation between the sides of the triangle;
2. the area of the triangle.

Problem 11.57. Find the area of the quadrilater from the right helicoid

$$\begin{cases} x = u \cos v, \\ y = u \sin v, \\ z = av. \end{cases}$$

bounded by the curves

$$u = 0, \quad u = a, \quad v = 0, \quad v = 1.$$

Problem 11.58. Find the second fundamental form of the following surfaces of rotation:

1. $x = f(u) \cos v, \quad , y = f(u) \sin v, \quad z = \varphi(u)$ – the surface of rotation of rotation axis Oz;
2. $x = R \cos u \cos v, \quad y = R \cos u \sin v, \quad z = R \sin u$ – the sphere;
3. $x = a \cos u \cos v, \quad y = a \cos u \sin v, \quad z = c \sin u$ – the ellipsoid of rotation;
4. $x = a \cosh u \cos v, \quad y = a \cosh u \sin v, \quad z = c \sinh u$ – the hyperboloid of rotation with one sheet;
5. $x = a \sinh u \cos v, \quad y = a \sinh u \sin v, \quad z = c \cosh u$ – the hyperboloid of rotation with two sheets;

6. $x = u\cos v,\quad y = u\sin v,\quad z = u^2$ – the paraboloid of rotation;
7. $x = R\cos v,\quad y = R\sin v,\quad z = u$ – the circular cylinder;
8. $x = u\cos v,\quad y = u\sin v,\quad z = ku$ – the circular cone;
9. $x = (a + b\cos u)\cos v,\quad (a + b\cos u)\sin v,\quad z = b\sin u$ – the torus;
10. $x = a\cosh\frac{u}{a}\cos v,\quad y = a\cosh\frac{u}{a}\sin v,\quad z = u$ – the catenoid;
11. $x = a\sin u\cos v,\quad y = a\sin u\sin v,\quad z = a\left(\ln\operatorname{tg}\frac{u}{2} + \cos u\right)$ – the pseudo-sphere.

Problem 11.59. Find the principal curvatures and the principal directions of the right helicoid

$$x = u\cos v,\quad y = u\sin v,\quad z = av.$$

Problem 11.60. Find the principal curvatures at the vertices of the hyperboloid with two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Problem 11.61. Compute the principal curvatures of the surface

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z$$

at the point $M(0,0,0)$.

Problem 11.62. Show that at each point of the surface

$$x = u\cos v,\quad y = u\sin v,\quad z = \lambda u$$

one of the normal principal sections is a straight line.

Problem 11.63. Find the curvatures of the normal sections of the surface $y = \frac{1}{2}x^2$:

1. at an arbitrary point and in an arbitrary direction;
2. at the points of the lines obtained by sectioning the surface with planes $z = k$ and in directions tangents to this curves;
3. at the point $M(2,2,4)$, in the direction of the tangent to the curve

$$y = \frac{1}{2}x^2,\quad z = x^2.$$

Problem 11.64. On the surface

$$x = u^2 + v^2, \quad y = u^2 - v^2, \quad z = uv$$

we take the point $P(u = 1, v = 1)$.

1. Compute the principal curvatures of the surface at P.
2. Find the equations of the tangents PT_1, PT_2 to the principal normal sections at the indicated point.
3. Find the curvature of the normal section passing through the tangent to the curve $v = u^2$.

Problem 11.65. We are given the surface

$$z = 2x^2 + \frac{9}{2}y^2.$$

1. Find the equation of the Dupin's indicatrix at the origin of coordinates.
2. Compute, at the origin of coordinate, the curvature radius of the normal section curve whose angle with the x-axis is $45°$.

Problem 11.66. In the tangent plane at the point M to a surface on draws n straight lines passing through the origin and intercepting between them angle equal to $\frac{\pi}{n}$. Show that

$$K_m = \frac{1}{n}\left(\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n}\right),$$

where $\frac{1}{r_i}$ are the normal curvatures of the curves from the surface which are tangent to the given straight lines, while K_m is the mean curvature of the surface at the point M.

Problem 11.67. Through a point M of an ellipsoid of rotation one draws all the possible curves, lying on the ellipsoid. Find the geometrical locus of the curvature centers of these curves at M.

Problem 11.68. Find the surfaces for which the first fundamental form is a perfect square.

Problem 11.69. Through a point M of a surface one considers all the possible plane sections. Find the equation of the surface which contains the centers of the osculating circles of these sections.

Problem 11.70. A parabolical cylinder intersects with a plane which is perpendicular to its generators after a parabola C.Let M be the vertex of this parabola and MT – the tangent to the parabola at the vertex M. On what curve lie the foci of the parabolas obtained sectioning the cylinder with planes passing through the straight line MT?

Problem 11.71. Starting from the fact that an ellipse can be projected onto a circle, find the curvature radius of an ellipse at one of its vertices, using Meusnier's theorem.