Theoretical Summaries

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Contents

1 Vectors

1.1 Theoretical Summary

1.1.1 The Vector Space of Free Vectors

Definiţia 1.1. An *oriented segment* or a *bound vector* $\overline{AB}$ is an ordered pair of points (A, B). The point A is called the *origin* or *point of application* of the vector, while the point B is called the *head* or *extremity* of the vector. The distance between the points A and B is called the *magnitude* of the vector $\overline{AB}$ and is denoted by $\|\overline{AB}\|$. Alternatively, the bound vector $\overline{AB}$ can be regarded as the segment $[AB]$ with an arbitrarily chosen orientation on it. A bound vector is represented by means of an arrow starting at the point A and having its tip at the point B. In the case where the points A and B coincide, the vector $\overline{AB} \equiv \overline{AA}$ is called the *zero vector* with its origin at A and is also denoted by $\overline{0}_A$ or, if the point A is understood, simply by $\overline{0}$.

Definiţia 1.2. We say that two bound vectors $\overline{AB}$ and $\overline{CD}$ have the *same direction* if the lines supporting the two oriented segments, AB and CD, are parallel (or coincide).

If the bound vectors $\overline{AB}$ and $\overline{CD}$ have the same direction, and their supporting lines do not coincide, we say that they also have the *same sense* if the lines AC and BD intersect within the interior of the trapezoid $ABDC$. Otherwise, we say that the two vectors have opposite senses.

Definiţia 1.3. We say that two bound vectors $\overline{AB}$ and $\overline{CD}$ are *equipollent* and write $\overline{AB} \sim \overline{CD}$ if they have the same magnitude, the same direction, and the same sense.

The equipollence relation is an equivalence relation on the set of all bound vectors.

Definiţia 1.4. A *free vector* is defined as an equivalence class with respect to the equipollence relation. The free vector determined by the oriented segment $\overline{AB}$ is denoted by $\overrightarrow{AB}$.

Thus,

$$\overrightarrow{AB} = \left\{ \overline{CD} \mid \overline{CD} - \text{oriented segment, i.e. } \overline{CD} \sim \overline{AB} \right\}$$

If we do not wish to single out a representative, the free vectors are denoted by lowercase letters, usually from the beginning of the alphabet: $\mathbf{a}, \mathbf{b}, \ldots$.

Addition of Free Vectors

Let $\mathbf{a}$ and $\mathbf{b}$ be any two free vectors. We choose an arbitrary point O in space and construct a point A such that $\overrightarrow{OA} = \mathbf{a}$ and a point B such that $\overrightarrow{AB} = \mathbf{b}$.

Definiţia 1.5 (the triangle rule)**.** The *sum* of the vectors $\mathbf{a}$ and $\mathbf{b}$ is defined and denoted by $\mathbf{a} + \mathbf{b}$, namely the vector $\overrightarrow{OB}$.

The set $\mathcal{V}$ of free vectors in space forms an abelian group with respect to vector addition. The neutral element is the zero vector (the equivalence class of bound vectors of zero magnitude), and the opposite of a free vector $[\overline{AB}]$ is the free vector $[\overline{BA}]$.

Scalar Multiplication of Vectors

Definiţia 1.6. If $\mathbf{a}$ is a free vector and $\lambda \in \mathbb{R}$ is a real number, we define the *product of the vector* $\mathbf{a}$ *with the scalar* λ as a vector, denoted by $\lambda\mathbf{a}$, characterised as follows:

(i) The magnitude of $\lambda\mathbf{a}$ is given by

$$\|\lambda\mathbf{a}\| := |\lambda| \cdot \|\mathbf{a}\|,$$

the product on the right being a product of real numbers;

(ii) The direction of $\lambda\mathbf{a}$ coincides with the direction of $\mathbf{a}$;

(iii) The sense of $\lambda\mathbf{a}$ coincides with the sense of $\mathbf{a}$ if $\lambda > 0$ or is the opposite of the sense of $\mathbf{a}$ if $\lambda < 0$.

With respect to the operations of addition and scalar multiplication, the set of free vectors is a real vector space. This space has dimension 3 if the oriented segments from which we start are in space, or dimension 2 if we are working with vectors situated in a plane. All concepts related to vector spaces (linear combinations, linear dependence, bases, etc.) apply to this space.

1.1.2 Affine Coordinate Systems

Definiţia 1.7. Consider an oriented basis (the order of the vectors is fixed!) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the space of free vectors and an arbitrary point O in space. The system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called a *frame* or an *affine coordinate system* in $\mathbb{R}^3$. The coordinates of a point M are, by definition, the components of the vector $\overrightarrow{OM}$ (the *position vector of M with respect to the origin*), relative to the chosen basis. We write $M = M(x_1, x_2, x_3)$ or $M = M(x, y, z)$.

Similarly, coordinates in the plane are defined, but the basis consists of only two vectors.

Definiţia 1.8. An oriented basis in the plane $\{\mathbf{e}_1, \mathbf{e}_2\}$ is called *direct* or *right-handed* if, when we rotate the first vector to superimpose it onto the second, along the shortest path, the rotation is performed in the positive trigonometric direction (i.e. counterclockwise). Otherwise, the basis is called *inverse* or *left-handed*.

An oriented basis in space $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called *direct* or *right-handed* if, as seen from the tip of the third vector, the rotation of the first vector to superimpose it onto the second, along the shortest path, is performed in the positive trigonometric direction (i.e. counterclockwise). Otherwise, the basis is called *inverse* or *left-handed*.

A coordinate system is right-handed or left-handed according to the oriented basis that defines it.

Definiţia 1.9. A coordinate system is called *orthogonal* if the vectors of the basis are pairwise perpendicular. If, in addition, the vectors of the basis are of unit length, the system (or frame) is called *orthonormal*.

In this compendium, unless stated otherwise, all bases (in the plane and in space) are assumed to be orthonormal and right-handed. Such a basis in the plane is denoted by $\{\mathbf{i}, \mathbf{j}\}$, and in space by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

The lines passing through the origin and parallel to the coordinate vectors are called *coordinate axes*: Ox – parallel to $\mathbf{i}$, Oy – parallel to $\mathbf{j}$, Oz – parallel to $\mathbf{k}$.

1.1.3 The Scalar Product of Two Free Vectors

If $\mathbf{a}$ and $\mathbf{b}$ are two free vectors, the *scalar product* of the two vectors is a real number, denoted by $\mathbf{a} \cdot \mathbf{b}$, given by

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos\alpha, \tag{1.1.1}$$

where α is the angle between the two vectors.

Properties

(a) For any free vector $\mathbf{a}$, we have $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathbf{a}^2}$.

(b) The scalar product is commutative: for any two vectors **a** and **b** we have $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.

(c) The scalar product is linear in each of its factors. If **a**, **b**, and **c** are three free vectors, and λ and μ are two real numbers, then

$$(\lambda\mathbf{a} + \mu\mathbf{b}) \cdot \mathbf{c} = \lambda \cdot (\mathbf{a} \cdot \mathbf{c}) + \mu \cdot (\mathbf{b} \cdot \mathbf{c}).$$

(d) Two vectors are perpendicular if and only if their scalar product is zero (with the convention that the zero vector is considered perpendicular to any vector).

(e) The cosine of the angle between two free vectors **a** and **b** is given by

$$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}.$$

The Expression of the Scalar Product in Coordinates

If we have chosen an orthonormal coordinate system with respect to which the vectors **a** and **b** are given by $\mathbf{a} = \mathbf{a}(a_1, a_2, a_3)$, respectively $\mathbf{b} = \mathbf{b}(b_1, b_2, b_3)$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Consequently:

- The length of a vector **a** is given by

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

- The cosine of the angle between two free vectors **a** and **b** is given by

$$\cos\alpha = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

- Two vectors **a** and **b** are perpendicular if and only if

$$a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

1.1.4 The Cross Product of Two Vectors

The *cross product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a *vector*, denoted $\mathbf{a} \times \mathbf{b}$, defined as follows:

1. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, then $\mathbf{a} \times \mathbf{b} = 0$.
2. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are not collinear, then the cross product $\mathbf{a} \times \mathbf{b}$ is a vector such that:
   (a) $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \sin\alpha$, where α is the angle between the two vectors.
   (b) $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
   (c) The sense of the vector $\mathbf{a} \times \mathbf{b}$ is chosen such that the ordered triplet of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ is right-handed.

Properties

Among the properties of the cross product we mention:

- Two free vectors are collinear if and only if their cross product is equal to zero.
- The cross product is *anticommutative*: if $\mathbf{a}$ and $\mathbf{b}$, then
$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$
- The cross product is linear in each factor: if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors, and λ and μ are two real numbers, then
$$(\lambda\mathbf{a} + \mu\mathbf{b}) \times \mathbf{c} = \lambda\,(\mathbf{a} \times \mathbf{c}) + \mu\,(\mathbf{b} \times \mathbf{c})\,.$$
- The area of the parallelogram constructed on two vectors is equal to the magnitude of the cross product of the two vectors, and the area of the triangle constructed on the two vectors is equal to half of this magnitude.
- Although it makes sense to speak of cross products involving three factors (in contrast to the scalar product), in general the cross product is *not associative*: if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors, then
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})\,.$$

The Expression of the Cross Product in Coordinates

If we have two vectors $\mathbf{a}(a_1, a_2, a_3)$ and $\mathbf{b}(b_1, b_2, b_3)$, given by their components relative to an orthonormal basis, then their cross product can be written

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \cdot \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \cdot \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \mathbf{k}. \tag{1.1.2}$$

1.1.5 The Mixed Product (Scalar Triple Product) of Three Vectors

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three vectors. The *mixed product* of these three vectors is defined as the scalar

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \tag{1.1.3}$$

Properties

1. The magnitude of the mixed product of three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is equal to the volume of the parallelepiped constructed on the three vectors:
$$V = |(\mathbf{a}, \mathbf{b}, \mathbf{c})|\,.$$
2. The volume of the tetrahedron constructed on three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is given by
$$V = \frac{1}{6}\,|(\mathbf{a}, \mathbf{b}, \mathbf{c})|\,.$$
3. The mixed product is linear in each argument (i.e. it is trilinear).
4. Three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar if and only if their mixed product is zero.
5. Three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form a right-handed system if and only if
$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) > 0.$$

The Expression of the Mixed Product in Coordinates

If the vectors $\mathbf{a}(a_1, a_2, a_3)$, $\mathbf{b}(b_1, b_2, b_3)$ and $\mathbf{c}(c_1, c_2, c_3)$ are given by their components with respect to an orthonormal basis, then

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{1.1.4}$$

1.1.6 Other Vector Products

The Double Cross Product

Consider three arbitrary vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. The *double cross product* of these three vectors (in this order!) is one of the vectors $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. For these we have the expressions:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a} \tag{1.1.5}$$

and

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}. \tag{1.1.6}$$

Products of Four Vectors

There are two products of four vectors that appear in applications. Consider four arbitrary vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$.

1. The *cross product of the four vectors* is defined as the triple cross product $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$. It is calculated by the formula

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{a},\mathbf{b},\mathbf{d})\,\mathbf{c} - (\mathbf{a},\mathbf{b},\mathbf{c})\,\mathbf{d} = (\mathbf{a},\mathbf{c},\mathbf{d})\,\mathbf{b} - (\mathbf{b},\mathbf{c},\mathbf{d})\,\mathbf{a}. \qquad (1.1.7)$$

2. The *scalar product of the four vectors* is defined as the scalar $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$. It is calculated by the formula

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d}) = \begin{vmatrix} (\mathbf{a}\cdot\mathbf{c}) & (\mathbf{a}\cdot\mathbf{d}) \\ (\mathbf{b}\cdot\mathbf{c}) & (\mathbf{b}\cdot\mathbf{d}) \end{vmatrix}. \qquad (1.1.8)$$

2 The Line in the Plane

2.1 Theoretical Summary

Let Δ be a line in the plane π, fixed in this chapter. A nonzero vector $\mathbf{a}$, collinear with the line Δ, is called a *direction vector* of the line. Every line has an infinite number of direction vectors since any nonzero vector collinear with a direction vector is itself a direction vector. A direction vector of unit length is called a *unit direction vector* of the line. Any line has exactly two unit direction vectors, which are opposite vectors. Thus, if $\mathbf{a}$ is any direction vector of the line Δ, the two unit direction vectors of the line are given by

$$\mathbf{v}_{\pm} = \pm \frac{\mathbf{a}}{\|\mathbf{a}\|}.$$

Now, consider a line Δ in the plane, with direction vector $\mathbf{a}$, and let M_0 be any point on the line, which has position vector $\mathbf{r}_0$ relative to a fixed point O in the plane. Then, the position vector $\mathbf{r}$ of any point M on the line can be written as

$$\mathbf{r} = \mathbf{r}_0 + t \cdot \mathbf{a}, \tag{2.1.1}$$

where t is a real number. Equation (2.1.1) is called the *vector equation* of the line Δ.

If, relative to an affine coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2\}$ with origin at O, the point M_0 has coordinates (x_0, y_0), and the vector $\mathbf{a}$ has components (l, m), then the vector equation (2.1.1) can be written as a system of two scalar equations:

$$\begin{cases} x = x_0 + l \cdot t, \\ y = y_0 + m \cdot t, \end{cases} \qquad t \in \mathbb{R}, \tag{2.1.2}$$

where (x, y) are the coordinates of the current point M on the line in the chosen coordinate system. Equations (2.1.2) are called the *parametric equations* of the line Δ.

If both l and m are nonzero, we can eliminate the parameter t between the two equations in system (2.1.2) and obtain the so-called *canonical equation* of the line Δ:

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}. \tag{2.1.3}$$

Equation (2.1.3) can also be written when either l or m is zero, with the convention that if the denominator in one of the fractions is zero, this indicates that the numerator must be identically zero. For example, the equation

$$\frac{x-x_0}{0}=\frac{y-y_0}{m}$$

should be replaced with the equation

$$x-x_0=0.$$

We will use this convention throughout this collection.

Let $M_0(x_0,y_0)$ and $M_1(x_1,y_1)$ be two distinct points on the line Δ. Then, the vector $\overrightarrow{M_0M_1}$ is a nonzero vector collinear with the line, so it is a direction vector. Since the components of the vector $\overrightarrow{M_0M_1}$ are (x_1-x_0, y_1-y_0), the canonical equation of the line Δ can be written as:

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}. \tag{2.1.4}$$

This is the *equation of the line passing through two points*.

Now, consider a line Δ in the plane, passing through the point $M_0(x_0,y_0)$ and having direction vector $\mathbf{a}(l,m)$ such that $l\neq 0$ (the line is not "vertical"). Then, the canonical equation of the line, (2.1.3), can be rewritten in the form

$$y-y_0=\frac{m}{l}(x-x_0).$$

The number $k\equiv\dfrac{m}{l}$ is called the *slope* of the line. If the coordinate basis is orthonormal, then k represents the angle that the line makes with the positive direction of the Ox axis and is called the *gradient* of the line. In what follows, the coordinate basis will always be orthonormal. The equation

$$y-y_0=k(x-x_0) \tag{2.1.5}$$

is the equation of the line Δ which *passes through M_0 and has slope k*.

It is easy to see that equation (2.1.5) can be rewritten in the form

$$y=kx+b, \tag{2.1.6}$$

where $b = y_0 - kx_0$. This form of the line equation is called the *explicit equation*. Another commonly used name is the *slope-intercept equation*, since b is the ordinate of the point where the line intersects the Oy axis (*the intercept* of the line on the Oy axis).

Returning to the vector equation of the line (2.1.1), we can rewrite this equation in the form

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{a}. \tag{*}$$

Let $\mathbf{n}$ be a nonzero vector perpendicular to the direction vector of the line (a *normal vector* to the line). There is an infinite number of such vectors, all collinear with each other. If we take the scalar product of both sides of equation (*) with $\mathbf{n}$, we obtain that the position vectors of all points on the line satisfy the equation

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0. \tag{2.1.7}$$

It is easy to see that the converse is also true (any point whose position vector satisfies equation (2.1.7) belongs to the line).

By using vector components, equation (2.1.7) can be written in the form

$$n_1(x - x_0) + n_2(y - y_0) = 0$$

or

$$n_1 x + n_2 y - n_1 x_0 - n_2 y_0 = 0.$$

Thus, the equation of a line in the plane can be written in the form

$$Ax + By + C = 0, \tag{2.1.8}$$

where the coefficients A and B are not both zero and represent the components of a normal vector to the line. This equation is called the *general equation* of the line. It can be used for lines of any direction, including vertical lines.

Observație. Since the vector $\mathbf{n}(A, B)$ is a normal vector to the line given by equation (2.1.8), it is clear that the vector $\mathbf{a}(-B, A)$ is a direction vector of the line, as it is a (nonzero!) vector perpendicular to the normal vector. Thus, from the general equation of the line, we can immediately identify both a normal vector and a direction vector.

If the line does not pass through the origin and is not parallel to either of the coordinate axes, then the general equation of the line can be rewritten in the form

$$\frac{x}{a} + \frac{y}{b} - 1 = 0. \tag{2.1.9}$$

This equation is called the *intercept form* of the line equation because the real numbers a and b represent the x-intercept and y-intercept of the line, respectively, i.e., they are the *intercepts* of the line on the two coordinate axes.

Sometimes, it is useful to write the equation of a line passing through a given point as the intersection of two lines, without explicitly determining the coordinates of the intersection point. This is done using the *pencil of lines* equation. Thus, if

$$A_1x + B_1y + C_1 = 0 \tag{2.1.10}$$

and

$$A_2x + B_2y + C_2 = 0 \tag{2.1.11}$$

are two concurrent lines, meaning that

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \neq 0, \tag{2.1.12}$$

then any line passing through their intersection point has an equation of the form

$$\lambda\,(A_1x + B_1y + C_1) + \mu\,(A_2x + B_2y + C_2) = 0, \tag{2.1.13}$$

where λ and μ are two real parameters that cannot both be zero.

The set of all lines in the plane whose equations can be written in the form (2.1.13) for some values of the parameters forms the *pencil of lines determined by lines* (2.1.10) *and* (2.1.11).

Observaţie. Sometimes, the equation of the pencil of lines is written in the form

$$A_1x + B_1y + C_1 + \alpha\,(A_2x + B_2y + C_2) = 0, \tag{2.1.14}$$

where α can take any real value, including 0. However, it is worth noting that equation (2.1.14) does not describe line (2.1.11), which would correspond to $\alpha = \infty$.

The *angle* between two lines is the angle between their direction vectors (or, equivalently, the angle between their normal vectors). Since changing the direction of one of the two tangent vectors replaces the angle with its supplement, this means that we actually have two angles, one acute and one obtuse.

If the two lines are given by their general equations (2.1.10) and (2.1.11), then their normal vectors are $\mathbf{n}_1(A_1, B_1)$ and $\mathbf{n}_2(A_2, B_2)$, so the cosines of the two angles between the lines are given by

$$\cos\theta = \pm\frac{A_1A_2 + B_1B_2}{\sqrt{A_1^2 + B_1^2}\sqrt{A_2^2 + B_2^2}}. \tag{2.1.15}$$

If the lines are given by their explicit equations, $y = k_1x + b_1$ and $y = k_2x + b_2$, then we can express the tangents of the two angles between the lines:

$$\tan\theta = \pm\frac{k_1 - k_2}{1 + k_1k_2}. \tag{2.1.16}$$

To determine the *acute* angle between two lines, it is sufficient to replace the right-hand sides of formulas (2.1.15) and (2.1.16) with their absolute values.

These formulas allow us to establish the conditions for parallelism and perpendicularity of two lines.

If the lines are given by their general equations, then:

- The lines are perpendicular if and only if

$$A_1A_2 + B_1B_2 = 0; \tag{2.1.17}$$

- The lines are parallel if and only if

$$A_1B_2 - A_2B_1 = 0. \tag{2.1.18}$$

If the lines are given by their explicit equations, then:

- The lines are perpendicular if and only if

$$1 + k_1k_2 = 0; \tag{2.1.19}$$

- The lines are parallel if and only if

$$k_1 = k_2. \tag{2.1.20}$$

If, in the general equation of the line, we replace the normal vector with one of the two unit normal vectors such that the constant term becomes negative, we obtain the so-called *normal (Hesse) form* of the line equation:

$$\cos\alpha \cdot x + \sin\alpha \cdot y - p = 0, \tag{2.1.21}$$

where p is the distance from the origin to the line (the length of the perpendicular dropped from the origin to the line), and α is the angle the line makes with the Ox axis. If the line passes through the origin, either of the two unit normal vectors to the line can be chosen.

If we start from the general equation of the line (2.1.8), the normal equation will be

$$\frac{A}{\pm\sqrt{A^2+B^2}}x + \frac{B}{\pm\sqrt{A^2+B^2}}y + \frac{C}{\pm\sqrt{A^2+B^2}} = 0, \tag{2.1.22}$$

where the sign in the denominator is chosen so that the constant term is negative. If the constant term is zero, either sign can be chosen.

The *distance* from a point $M_0(x_0, y_0)$ to a line given by the general equation (2.1.8) is given by

$$d(M_0, \Delta) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}. \tag{2.1.23}$$

3 The Line and the Plane in Space

3.1 Theoretical Summary

3.1.1 The Plane in Space

We fix an affine coordinate system in space, with origin at point O.

Let Π be a plane in space and $M_0(x_0, y_0, z_0)$ a point in the plane. If $\mathbf{a}_1(l_1, m_1, n_1)$ and $\mathbf{a}_2(l_2, m_2, n_2)$ are two non-collinear vectors parallel to the plane Π, then the position vector $\mathbf{r}$ of an arbitrary point $M(x, y, z)$ in the plane can be written as

$$\mathbf{r} = \mathbf{r}_0 + u\mathbf{a}_1 + v\mathbf{a}_2, \tag{3.1.1}$$

where $\mathbf{r}_0$ is the position vector of point M_0, and u, v are real numbers. Equation (3.1.1) is called the *vector equation of the plane* Π.

If we project equation (3.1.1) onto the coordinate axes, we obtain the *parametric equations of the plane* Π:

$$\begin{cases} x = x_0 + l_1 \cdot u + l_2 \cdot v, \\ y = y_0 + m_1 \cdot u + m_2 \cdot v, \qquad u, v \in \mathbb{R}. \\ z = z_0 + n_1 \cdot u + n_2 \cdot v, \end{cases} \tag{3.1.2}$$

The vector equation (3.1.1) actually expresses the condition that the vectors $\mathbf{r} - \mathbf{r}_0$, $\mathbf{a}_1$ and $\mathbf{a}_2$ be linearly independent, therefore, in this case, coplanar. But the condition for the three vectors to be coplanar, as we already know, is equivalent to the condition that their mixed product is zero:

$$(\mathbf{r} - \mathbf{r}_0, \mathbf{a}_1, \mathbf{a}_2) = 0. \tag{3.1.3}$$

We call this equation the *non-parametric vector equation of the plane passing through a given point and parallel to two (non-collinear) given vectors*.

Equation (3.1.3) can be rewritten using the coordinate expression of the mixed product of three vectors, yielding the *scalar equation of the plane passing through a given point and parallel to two (non-collinear) given vectors*:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0. \tag{3.1.4}$$

Three non-collinear points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, $M_3(x_3, y_3, z_3)$ determine a plane Π. Two non-collinear vectors that are parallel to this plane are, for example,

$$\overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1) \quad \text{and} \quad \overrightarrow{M_1M_3}(x_3 - x_1, y_3 - y_1, z_3 - z_1).$$

Thus, the plane Π determined by the *three non-collinear points* is the plane passing through M_1 and parallel to $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_1M_3}$, therefore its equation is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0. \tag{3.1.5}$$

Equation (3.1.5) can also be written in a more symmetrical and easier-to-remember form, though less practical:

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \tag{3.1.6}$$

The most commonly used form for representing a plane is the *general equation*,

$$Ax + By + Cz + D = 0, \tag{3.1.7}$$

where A, B, C, D are real numbers such that A, B, C do not all vanish simultaneously.

The vector $\mathbf{n}(A, B, C)$ is a *normal* vector to the plane.

If all four coefficients in the general equation of the plane are non-zero (which is equivalent to the plane not being parallel to any coordinate axis and not passing through the origin), then the equation can be rewritten as

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0. \tag{3.1.8}$$

Equation (3.1.8) is called the *intercept form of the plane*, because the numbers a, b, and c are the x-, y-, and z-coordinates, respectively, of the points where the plane intersects the Ox, Oy, and Oz axes, i.e., the *intercepts* of the plane on the three coordinate axes.

A particular form of the general equation of the plane is obtained if we replace the normal vector to the plane with one of its unit vectors. Then we can choose one of the two unit normal vectors such that the general equation of the plane becomes

$$\cos\alpha \cdot x + \cos\beta \cdot y + \cos\gamma \cdot z - p = 0, \tag{3.1.9}$$

where p is the distance from the origin to the plane (the length of the perpendicular dropped from the origin to the plane), and α, β, and γ are the angles made by the unit normal vector with the axes Ox, Oy, and Oz, respectively.

Equation (3.1.9) is called the *normal form* or the *Hesse form* of the general equation of the plane. If the plane passes through the origin, then p is zero and either of the two unit vectors may be used.

If the plane is given by the general equation (3.1.7), then its normal form can be written as

$$\frac{Ax + By + Cz + D}{\pm\sqrt{A^2 + B^2 + C^2}} = 0, \tag{3.1.10}$$

where the sign in the denominator is chosen so that the constant term,

$$\frac{D}{\pm\sqrt{A^2 + B^2 + C^2}},$$

is negative.

A plane divides space into two open half-spaces:

negative half-space (the one containing the origin) and

positive half-space (the one opposite the negative half-space).

If the plane passes through the origin, then an arbitrary unit normal vector is chosen. The positive half-space is the one containing the endpoint of the normal unit vector when the vector is attached to a point on the plane.

The *signed distance* of a point $M_0(x_0, y_0, z_0)$ relative to a plane Π given in the normal form (3.1.9) is the real number

$$\delta(M_0, \Pi) = \cos\alpha \cdot x_0 + \cos\beta \cdot y_0 + \cos\gamma \cdot z_0 - p. \tag{3.1.11}$$

If the plane is given by an arbitrary general equation of the form (3.1.7), then the signed distance of the point to the plane is given by

$$\delta(M_0, \Pi) = \frac{Ax_0 + By_0 + C_0 z + D}{\pm\sqrt{A^2 + B^2 + C^2}}, \tag{3.1.12}$$

with the same rule for choosing the sign of the square root as in the case of the normal equation of the plane.

The signed distance of a point from a plane is the signed length of the perpendicular dropped from the point to the plane, the length being considered negative when the point lies in the negative half-space defined by the plane and positive when it lies in the positive half-space.

The *distance* $d(M_0, \Pi)$ from a point M_0 to a plane Π is the length of the perpendicular dropped from the point to the plane, that is,

$$d(M_0, \Pi) = |\delta(M_0, \Pi)|. \tag{3.1.13}$$

Thus, if the plane is given in the normal form, then

$$d(M_0, \Pi) = |\cos\alpha \cdot x_0 + \cos\beta \cdot y_0 + \cos\gamma \cdot z_0 - p|\,, \tag{3.1.14}$$

while if the plane is given by an arbitrary general equation, then

$$d(M_0, \Pi) = \frac{|Ax_0 + By_0 + C_0 z + D|}{\sqrt{A^2 + B^2 + C^2}}. \tag{3.1.15}$$

The *angle* between two planes is the planar angle associated with the dihedral angle formed by the planes. It is equal to the angle formed by the normal vectors of the two planes. As in the case of the line in the plane, there are actually four angles between the two planes, two by two equal (opposite at the vertex). Also, as in the case of the line in the plane, the formula we provide gives the cosine of one of these angles.

If the two planes, say Π_1 and Π_2, are given by their general equations

$$A_1 x + B_1 y + C_1 z + D_1 = 0, \tag{3.1.16}$$

and

$$A_2 x + B_2 y + C_2 z + D_2 = 0, \tag{3.1.17}$$

then their normal vectors are $\mathbf{n}_1\,(A_1, B_1, C_1)$ and $\mathbf{n}_2\,(A_2, B_2, C_2)$, respectively, and the cosine of the angle α formed by the two planes (actually, by the two normal vectors) is given by

$$\cos\alpha = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}. \tag{3.1.18}$$

The angle α is acute if $\cos\alpha > 0$ and obtuse if $\cos\alpha < 0$. If we wish to obtain the acute angle, we use the formula

$$\cos\alpha = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}. \tag{3.1.19}$$

The planes Π_1 and Π_2 are *perpendicular* if their normal vectors are perpendicular, so the condition for perpendicularity is

$$A_1A_2 + B_1B_2 + C_1C_2 = 0, \tag{3.1.20}$$

while they are *parallel* when their normal vectors are parallel, so the condition for parallelism is written as

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}. \tag{3.1.21}$$

The bisector planes of two concurrent planes given by their general equations (3.1.16) and (3.1.17) are the planes that form equal angles with the two given planes. They also represent the *locus of points in space equidistant from the two planes*, therefore they are given by the equation

$$\frac{|A_1x + B_1y + C_1z + D_1|}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{|A_2x + B_2y + C_2z + D_2|}{\sqrt{A_2^2 + B_2^2 + C_2^2}} \tag{3.1.22}$$

or

$$\frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \pm\frac{A_2x + B_2y + C_2z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}}, \tag{3.1.23}$$

where each choice of sign corresponds to one bisector plane.

3.1.2 The Line in Space

As in the case of the line in the plane, we shall call a *direction vector* of a line Δ any nonzero vector $\mathbf{a}$ which is collinear with the line. Once again, we have an infinite number of direction vectors, any two of them being collinear.

If we choose an origin O in space and a point M_0 on the line Δ, then the position vector $\mathbf{r}$ of an arbitrary point M on the line can be written as

$$\mathbf{r} = \mathbf{r}_0 + t \cdot \mathbf{a}, \tag{3.1.24}$$

where t is a real number, $\mathbf{r}_0$ is the position vector of point M_0, and $\mathbf{a}$ is the direction vector of the line. Equation (3.1.24) is called the *vector equation of the line passing through point M_0 and having direction vector* $\mathbf{a}$.

If we choose a coordinate system with origin at O, relative to which point M has coordinates (x, y, z), point M_0 has coordinates (x_0, y_0, z_0), and the vector $\mathbf{a}$ has components (l, m, n), then equation (3.1.24) can be replaced by the system of scalar equations

$$\begin{cases} x = x_0 + l \cdot t, \\ y = y_0 + m \cdot t, \\ z = z_0 + n \cdot t, \end{cases} \qquad t \in \mathbb{R}. \tag{3.1.25}$$

Equations (3.1.25) are called the *parametric equations of the line Δ, which passes through point $M_0(x_0, y_0, z_0)$ and is parallel to vector* **a** *(or has direction vector* **a***).*

If we eliminate the parameter t from the parametric equations (3.1.25), we obtain the *canonical equations of the line passing through point $M_0(x_0, y_0, z_0)$ and having direction vector* **a**:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}. \tag{3.1.26}$$

Here too, as in the case of the canonical equation of a line in the plane, we use the convention that if one of the denominators is zero, then the corresponding numerator must also be taken as zero. In other words, for example, the system of equations

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{0}$$

must be replaced with the system

$$\frac{x - x_0}{l} = \frac{y - y_0}{m}, \quad z - z_0 = 0,$$

while the system

$$\frac{x - x_0}{l} = \frac{y - y_0}{0} = \frac{z - z_0}{0}$$

is replaced with the system

$$y - y_0 = 0, \quad z - z_0 = 0.$$

We can easily write the equations of a line passing through two distinct points in space, $M_0(x_0, y_0, z_0)$ and $M_1(x_1, y_1, z_1)$. A direction vector of this line is

$$\mathbf{a} = \overrightarrow{M_0M_1}(x_1 - x_0, y_1 - y_0, z_1 - z_0),$$

thus the canonical equations of this line are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}. \tag{3.1.27}$$

Finally, another way to represent a line is as the intersection of two planes:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases} \tag{3.1.28}$$

where the rank of the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$$

must be equal to 2, a condition equivalent to requiring that the normal vectors to the two planes are not parallel, hence the planes intersect.

If a line Δ is given by the intersection of two planes (3.1.28), then a direction vector of the line can be obtained by computing the cross product of the two normal vectors of the planes:

$$\mathbf{a} = \mathbf{n}_1(A_1, B_1, C_1) \times \mathbf{n}_2(A_2, B_2, C_2),$$

that is,

$$\mathbf{a} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \mathbf{i} + \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \mathbf{k}.$$

The *distance* from a point $M_1(x_1, y_1, z_1)$ to a line Δ passing through point $M_0(x_0, y_0, z_0)$ and having direction vector $\mathbf{a}$ is given by the formula

$$d(M_1, \Delta) = \frac{\|(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{a}\|}{\|\mathbf{a}\|}, \tag{3.1.29}$$

where $\mathbf{r}_0$ and $\mathbf{r}_1$ are the position vectors of points M_0 and M_1, respectively.

The *angle* between two lines Δ_1 and Δ_2 is the angle formed by their direction vectors $\mathbf{a}_1(l_1, m_1, n_1)$ and $\mathbf{a}_2(l_2, m_2, n_2)$. As in the case of two lines in the plane, we have one acute angle and one obtuse angle. The cosines of these two angles are thus given by the formula

$$\cos\alpha = \pm\frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}. \tag{3.1.30}$$

Of course, the positive cosine corresponds to the acute angle, while the negative cosine corresponds to the obtuse angle.

If we wish to determine the cosine of the acute angle, we use the formula

$$\cos\alpha = \frac{|l_1 l_2 + m_1 m_2 + n_1 n_2|}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}. \tag{3.1.31}$$

The two lines are *perpendicular* if and only if their direction vectors are perpendicular, hence the condition for perpendicularity is

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0. \tag{3.1.32}$$

On the other hand, the lines are *parallel* if and only if their direction vectors are collinear, that is,

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}, \tag{3.1.33}$$

with the same convention regarding zero components as in the case of the canonical equations of the line.

3.1.3 The Line and the Plane

Relative Position of Two Planes

Two planes, Π_1 and Π_2, given by their general equations

$$A_1 x + B_1 y + C_1 z + D_1 = 0$$

and

$$A_2 x + B_2 y + C_2 z + D_2 = 0$$

- intersect along a line if
$$\operatorname{rg}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2;$$
- are parallel if
$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2};$$
- coincide if
$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}.$$

Relative Position of Three Planes

We consider three planes, given by their general equations:

$$\begin{cases} (\Pi_1)\ A_1 x + B_1 y + C_1 z + D_1 = 0, \\ (\Pi_2)\ A_2 x + B_2 y + C_2 z + D_2 = 0, \\ (\Pi_3)\ A_3 x + B_3 y + C_3 z + D_3 = 0. \end{cases} \tag{3.1.34}$$

Let Δ be the determinant of system (3.1.34):

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix},$$

m – the coefficient matrix,

$$m = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

and M – the augmented matrix,

$$M = \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{pmatrix}.$$

Then:

(a) If $\Delta \neq 0$, then the planes intersect at a single point.

(b) If $\Delta = 0$, $\operatorname{rg} m = 2$, $\operatorname{rg} M = 3$, and the normal vectors to the three planes are pairwise non-collinear, then the planes intersect in pairs along lines, and the three resulting lines are parallel.

(c) If $\operatorname{rg} m = 2$, $\operatorname{rg} M = 3$, but two of the three normal vectors to the planes are collinear[1], then two of the planes (the ones with collinear normal vectors) are parallel, and the third intersects the other two along lines.

(d) If $\operatorname{rg} m = 2$, $\operatorname{rg} M = 2$, and the normal vectors are pairwise non-collinear, then the planes are pairwise distinct and pass through the same line.

(e) If $\operatorname{rg} m = 2$, $\operatorname{rg} M = 2$, and two of the three normal vectors are collinear, then two of the planes coincide (those with collinear normal vectors), and the third intersects them along a line.

(f) If $\operatorname{rg} m = 1$, $\operatorname{rg} M = 3$, then the planes are distinct and parallel.

(g) If $\operatorname{rg} m = 1$, $\operatorname{rg} M = 2$, then two of the planes coincide, and the third is parallel to them.

(h) If $\operatorname{rg} m = 1$, $\operatorname{rg} M = 1$, then all three planes coincide.

Bundles of Planes and Pencils of Planes

Given two distinct intersecting planes

$$A_1x + B_1y + C_1z + D_1 = 0, \tag{3.1.35}$$
$$A_2x + B_2y + C_2z + D_2 = 0, \tag{3.1.36}$$

the *bundle of planes* determined by these two planes is the set of all planes in space that pass through the line of intersection of the two planes. The equation of an arbitrary plane from the bundle can be written as

$$\alpha(A_1x + B_1y + C_1z + D_1) + \beta(A_2x + B_2y + C_2z + D_2) = 0, \tag{3.1.37}$$

where α and β are two real numbers that cannot both vanish simultaneously. The line of intersection of the given planes is called the *axis of the bundle*.

[1] All three cannot be collinear since $\operatorname{rg} m = 2$!

Let

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases} \tag{3.1.38}$$

be the equations of three planes passing through the point $S(x_0, y_0, z_0)$ such that the condition

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0 \tag{3.1.39}$$

is satisfied.

The *pencil of planes* or *star of planes* centred at S_0 is the set of all planes that pass through this point. The equation of an arbitrary plane in the pencil can be written as

$$\alpha(A_1x+B_1y+C_1z+D_1)+\beta(A_2x+B_2y+C_2z+D_2)+\gamma(A_3x+B_3y+C_3z+D_3) = 0, \tag{3.1.40}$$

where α, β, and γ are three real numbers not all simultaneously zero.

Relative Position of a Line with Respect to a Plane

Given a plane Π

$$Ax + By + Cz + D = 0 \tag{3.1.41}$$

and a line Δ, given by its parametric equations

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \\ z = z_0 + nt, \end{cases} \tag{3.1.42}$$

then:

- if

  $$Al + Bm + Cn \neq 0$$

  (that is, if the direction vector of the line is not perpendicular to the normal vector of the plane), then the line and the plane have a common point (the line *intersects* the plane);

- if

  $$Al + Bm + Cn = 0, \quad Ax_0 + By_0 + Cz_0 + D \neq 0$$

  (that is, the direction vector of the line is perpendicular to the normal vector of the plane, and there exists a point on the line that does not lie on the plane), then the line is *parallel* to the plane;

- if
$$Al + Bm + Cn = 0, \quad Ax_0 + By_0 + Cz_0 + D = 0$$
(that is, the direction vector of the line is perpendicular to the normal vector of the plane and there exists a point on the line that lies on the plane), then the line is *contained in* the plane.

Equation of the Plane Determined by Two Intersecting Lines

The intersecting lines

$$(\Delta_1): \frac{x-x_0}{l_1} = \frac{y-y_0}{m_1} = \frac{z-z_0}{n_1} \tag{3.1.43}$$

and

$$(\Delta_2): \frac{x-x_0}{l_2} = \frac{y-y_0}{m_2} = \frac{z-z_0}{n_2}, \tag{3.1.44}$$

which pass through point $M_0(x_0, y_0, z_0)$, determine a plane with equation

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0. \tag{3.1.45}$$

Equation of the Plane Determined by a Line and a Point

The line

$$(\Delta): \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \tag{3.1.46}$$

and the point $M_2(x_2, y_2, z_2)$ which does not lie on the line, determine a plane with equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \end{vmatrix} = 0. \tag{3.1.47}$$

Equation of the Plane Determined by Two Parallel Lines

The parallel (and distinct!) lines

$$(\Delta_1): \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \tag{3.1.48}$$

and

$$(\Delta_2): \frac{x-x_2}{l} = \frac{y-y_2}{m} = \frac{z-z_2}{n}, \tag{3.1.49}$$

which pass through the points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$, determine a plane with equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0. \tag{3.1.50}$$

Projection of a Point onto a Plane

The projection of a point $M_0(x_0, y_0, z_0)$ onto a plane with equation

$$Ax + By + Cz + D = 0$$

is the foot of the perpendicular dropped from the point onto the plane. Its coordinates are obtained by solving the system of equations

$$\begin{cases} Ax + By + Cz + D = 0, \\ \dfrac{x - x_0}{A} = \dfrac{y - y_0}{B} = \dfrac{z - z_0}{C}. \end{cases}$$

Projection of a Point onto a Line in Space

The projection of a point $M_1(x_1, y_1, z_1)$ onto a line

$$\Delta : \frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$$

is the point itself, if it lies on the line, or the foot of the perpendicular dropped from the point to the line, if the point does not lie on the line. In the latter case, the coordinates of the projection are obtained by solving the system

$$\begin{cases} \dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}, \\ (x - x_1) \cdot l + (y - y_1) \cdot m + (z - z_1) \cdot n = 0. \end{cases} \tag{3.1.51}$$

Projection of a Line onto a Plane

The projection of the line

$$(\Delta) : \frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} \tag{3.1.52}$$

onto the plane

$$(\Pi) : Ax + By + Cz + D = 0 \tag{3.1.53}$$

is their point of intersection, if the line is perpendicular to the plane. If the line Δ is not perpendicular to the plane Π, then its projection onto the plane is the intersection between plane Π and the plane perpendicular to Π which passes through line Δ. The equations of the projection are

$$\begin{cases} \begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ l & m & n \\ A & B & C \end{vmatrix} = 0, \\ Ax + By + Cz + D = 0. \end{cases} \tag{3.1.54}$$

Relative Position of Two Lines in Space

We consider two lines in space:

$$(\Delta_1)\ \frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \tag{3.1.55}$$

and

$$(\Delta_2)\ \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}, \tag{3.1.56}$$

which pass through points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ respectively, and have direction vectors $\mathbf{a}_1(l_1, m_1, n_1)$ and $\mathbf{a}_2(l_2, m_2, n_2)$.

Then:

- the lines Δ_1 and Δ_2 are parallel and distinct if vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are collinear, but vectors $\mathbf{a}_1$ and $\overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ are not collinear;
- the lines Δ_1 and Δ_2 coincide if vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ are collinear.

Now suppose that vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are not collinear (i.e., the corresponding lines are not parallel). Then:

- the lines are intersecting if vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ are coplanar, i.e., if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0; \tag{3.1.57}$$

- the lines are skew (non-coplanar) if the three vectors are non-coplanar, i.e., if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \neq 0. \tag{3.1.58}$$

Common Perpendicular of Two Skew Lines

The common perpendicular of two skew lines

$$(\Delta_1)\ \frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \tag{3.1.59}$$

and

$$(\Delta_2)\ \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}, \tag{3.1.60}$$

which pass through points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$, and have direction vectors $\mathbf{a}_1(l_1, m_1, n_1)$ and $\mathbf{a}_2(l_2, m_2, n_2)$, such that

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \neq 0,$$

is the unique line which intersects both lines and is perpendicular to them.

The equations of the common perpendicular are obtained by intersecting a plane that passes through the first line and the common perpendicular with a plane that passes through the second line and the common perpendicular. Thus, the equations of the common perpendicular are:

$$\begin{cases} \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = 0, \\ \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix} = 0, \end{cases} \tag{3.1.61}$$

where $(\beta_1, \beta_2, \beta_3)$ are the components of the direction vector of the common perpendicular, $\mathbf{v} = \mathbf{a}_1 \times \mathbf{a}_2$.

Distance Between Two Skew Lines (Length of the Common Perpendicular)

The *distance between two skew lines* Δ_1 and Δ_2 is the distance between the points where the common perpendicular of the two lines intersects them. Thus, it is the *length of the segment on the common perpendicular between the points of intersection with the two lines*, or, with a slight abuse of language, the *length of the common perpendicular* of the given lines.

This distance is calculated as the distance between any point on the first line and the plane passing through the second line and parallel to the first line. Thus, we have

$$d(\Delta_1, \Delta_2) = \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_2 & m_2 & n_2 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}} \right|. \tag{3.1.62}$$

The notations are the same as in the previous paragraph.

Angle Between a Line and a Plane

If a line is perpendicular to a plane, then the angle between the line and the plane is equal to $\frac{\pi}{2}$. Otherwise, the *angle between a line and a plane is the angle between the line and its projection onto the plane, i.e., the angle between the direction vector of the line and the direction vector of its projection onto the plane.*

If the line has direction vector $\mathbf{a}(l, m, n)$, and the plane has general equation

$$Ax + By + Cz + D = 0,$$

then the angle between the line and the plane is the angle φ given by

$$\sin\varphi = \frac{|Al + Bm + Cn|}{\sqrt{A^2 + B^2 + C^2}\sqrt{l^2 + m^2 + n^2}}. \tag{3.1.63}$$

The line is *parallel* to the plane if the direction vector of the line is perpendicular to the normal vector of the plane, that is, if

$$Al + Bm + Cn = 0, \tag{3.1.64}$$

and it is *perpendicular* to the plane if the two vectors mentioned are parallel, that is, if

$$\frac{A}{l} = \frac{B}{m} = \frac{C}{n}. \tag{3.1.65}$$

4

Conics in Reduced Equations

4.1 Theoretical Summary

4.1.1 Ellipse

Canonical Equation

An *ellipse* is the geometric locus of points in the plane for which the sum of the distances to two fixed points in the plane, F_1 and F_2, called *foci*, located at a distance $2c$ from each other, is a constant length equal to $2a$. If we choose the coordinate system such that the Ox axis passes through the two foci, and the Oy axis is the perpendicular bisector of the segment determined by the foci, then the equation of the ellipse can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{4.1.1}$$

where $b = \sqrt{a^2 - c^2}$. a is called the *major semi-axis* of the ellipse, and b – the *minor semi-axis*. The length $2c$ is called the *focal distance*.

Eccentricity and Focal Radii

The *eccentricity* of the ellipse is the number

$$\varepsilon = \frac{c}{a} = \sqrt{1 - \left(\frac{b^2}{a^2}\right)}, \tag{4.1.2}$$

which measures the deviation of the shape of the ellipse from that of a circle.

The *focal radii* of a point $M(x, y)$ on the ellipse are the distances from that point to the two foci of the ellipse. They are expressed, in terms of the major semi-axis and the eccentricity of the ellipse, by the formulas

$$\begin{cases} r_1 = a + \varepsilon x, \\ r_2 = a - \varepsilon x. \end{cases} \tag{4.1.3}$$

Tangent and Normal at a Point on the Ellipse

A *tangent* to an ellipse is a line which has a double contact with the ellipse (the line and the ellipse have two coinciding points in common). The *normal* to the ellipse at a point is the line passing through that point and perpendicular to the tangent at that point.

If $M(x_0, y_0)$ is a point on the ellipse, then the equation of the tangent at M_0 is

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1. \tag{4.1.4}$$

The equation of the normal is

$$a^2 y_0 x - b^2 x_0 y - (a^2 - b^2) x_0 y_0 = 0. \tag{4.1.5}$$

Tangents to the Ellipse Parallel to a Given Direction

If the given direction is not vertical and it is identified by a slope k, then there are two tangents with this slope, given by the equations

$$y = kx \pm \sqrt{a^2k^2 + b^2}. \tag{4.1.6}$$

There are two vertical tangents, given by the equations

$$x = \pm a. \tag{4.1.7}$$

Tangents to the Ellipse Passing Through a Point Outside the Ellipse

Let $M_1(x_1, y_1)$ be a point outside the ellipse. If $x_1 \neq \pm a$ (i.e., the point does not lie on one of the two vertical tangents), then through M_1 pass two tangents to the ellipse, which have slopes given by

$$k_{1,2} = \frac{-x_1 y_1 \pm \sqrt{b^2 x_1^2 + a^2 y_1^2 - a^2 b^2}}{x_1^2 - a^2}. \tag{4.1.8}$$

If M_1 has coordinates of the form $(\pm a, y_1)$, then we have a vertical tangent (either $x = a$ or $x = -a$, depending on the point) and a non-vertical tangent with slope

$$k = \pm \frac{y_1^2 - b^2}{2ay_1} \tag{4.1.9}$$

(y_1 cannot be zero, as the point M_1 is outside the ellipse). The sign corresponds to the chosen point M_1.

4.1.2 Hyperbola

Canonical Equation

A *hyperbola* is the geometric locus of points in the plane for which the absolute value of the difference of the distances to two fixed points in the plane, F_1 and F_2, called *foci*, located at a distance $2c$ from each other, is a constant length equal to $2a$. If we choose the coordinate system such that the Ox axis passes through the two foci, and the Oy axis is the perpendicular bisector of the segment determined by the foci, then the equation of the hyperbola can be written as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{4.1.10}$$

where $b = \sqrt{c^2 - a^2}$. a and b are *semi-axes* of the hyperbola. The length $2c$ is called the *focal distance*.

Eccentricity and Focal Radii

The *eccentricity* of the hyperbola is the number

$$\varepsilon = \frac{c}{a} = \sqrt{1 + \left(\frac{b^2}{a^2}\right)}. \tag{4.1.11}$$

Unlike in the case of the ellipse, the eccentricity of a hyperbola is always greater than 1.

The *focal radii* of a point $M(x, y)$ on the hyperbola are the distances from that point to the two foci of the hyperbola. They are expressed, in terms of the first semi-axis of the hyperbola and its eccentricity, by the formulas

$$\begin{cases} r_1 = a + \varepsilon x, \\ r_2 = a - \varepsilon x. \end{cases} \tag{4.1.12}$$

Asymptotes of the Hyperbola

Unlike the ellipse, the hyperbola is an unbounded curve. It consists of two branches. The curve has two asymptotes. Each is an asymptote for both branches— for one at $+\infty$, for the other at $-\infty$. The equations of the asymptotes are

$$y = \pm\frac{b}{a}x. \tag{4.1.13}$$

Tangent and Normal at a Point on the Hyperbola

A *tangent* to a hyperbola is a line which has a double contact with the hyperbola (the line and the hyperbola have two coinciding points in common). The *normal* to the hyperbola at a point is the line passing through that point and perpendicular to the tangent at that point.

If $M(x_0, y_0)$ is a point on the hyperbola, then the equation of the tangent at M_0 is

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1. \tag{4.1.14}$$

The equation of the normal is

$$a^2y_0x + b^2x_0y - (a^2 + b^2)x_0y_0 = 0. \tag{4.1.15}$$

Tangents to the Hyperbola Parallel to a Given Direction

If the given direction is not vertical and it is identified by a slope k, then there are two tangents with this slope, given by the equations

$$y = kx \pm \sqrt{a^2k^2 - b^2}. \tag{4.1.16}$$

There are two vertical tangents, given by the equations

$$x = \pm a. \tag{4.1.17}$$

Unlike the ellipse, the hyperbola does not have tangents of just any slope. The slope k of a tangent to the hyperbola must satisfy the condition

$$k^2 > \frac{b^2}{a^2}.$$

Tangents to the Hyperbola Passing Through a Point Outside It

Let $M_1(x_1, y_1)$ be a point in the plane outside the hyperbola, so that its coordinates satisfy the inequality

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1. \tag{4.1.18}$$

Then from M_1 two tangents can be drawn to the hyperbola. If x_1 is not equal to $\pm a$, then both tangents are non-vertical, and their slopes are given by

$$k_{1,2} = \frac{x_1 y_1 \pm \sqrt{a^2 y_1^2 - b^2 x_1^2 + a^2 b^2}}{x_1^2 - a^2}. \tag{4.1.19}$$

If M_1 has coordinates of the form $(\pm a, y_1)$, then we have a vertical tangent (either $x = a$ or $x = -a$, depending on the point) and a non-vertical tangent with slope

$$k = \pm \frac{y_1^2 + b^2}{2ay_1} \tag{4.1.20}$$

(y_1 cannot be zero, since the point M_1 is outside the hyperbola). The sign corresponds to the chosen point M_1.

4.1.3 Parabola

Definition and Canonical Equation

A *parabola* is the geometric locus of points in the plane that are equidistant from a fixed line Δ, called the *directrix*, and a fixed point F, called the *focus*. The distance between the fixed point and the fixed line is denoted by p and is called the *parameter of the parabola*.

If we choose as the Ox axis the perpendicular to the directrix passing through the focus, oriented from the directrix toward the focus, and as the Oy axis the perpendicular bisector of the segment on the Ox axis between the directrix and the focus, oriented so as to obtain a right-handed coordinate system, then the *canonical* equation of the parabola is written as

$$y^2 = 2px. \tag{4.1.21}$$

The Ox axis is the only axis of symmetry of the curve. The parabola intersects the axis of symmetry at a single point (the *vertex of the parabola*), which, in the chosen coordinate system, coincides with the origin.

Tangent at a Point on the Parabola

If $M_0(x_0, y_0)$ is a point on the parabola given by the canonical equation (4.1.21), then the equation of the tangent to the parabola at the point M_0 is obtained by *doubling*, that is, it has the form

$$yy_0 = p(x + x_0). \tag{4.1.22}$$

Tangent to the Parabola of a Given Slope

For any slope k, non-zero, there is a unique tangent to the parabola with slope k. The equation of this tangent is

$$y = kx + \frac{p}{2k}. \tag{4.1.23}$$

Observaţie. There is no tangent to the parabola with slope $k = 0$ (i.e., horizontal or, equivalently, parallel to the axis of symmetry of the parabola).

Tangents to the Parabola from a Point Outside the Parabola

Consider a point $M_1(x_1, y_1)$ in the plane of the parabola (4.1.21), outside the parabola, such that its coordinates satisfy the inequality

$$y_1^2 > 2px_1. \tag{4.1.24}$$

From M_1 two distinct tangents can always be drawn to the parabola. We consider two cases:

1. If the point M_1 lies on the Oy axis (i.e., has coordinates of the form $(0, y_1)$), then one of the tangents is vertical (namely, the Oy axis itself), and the other tangent has the equation

$$y - y_1 = \frac{p}{2y_1}. \tag{4.1.25}$$

2. If the point does not lie on the Oy axis, then there are two oblique tangents, with equations of the form

$$y - y_1 = k(x - x_1), \tag{4.1.26}$$

where the slope k of the tangent is one of the two (distinct!) roots of the equation

$$2x_1k^2 - 2y_1k + p = 0. \tag{4.1.27}$$

5

Quadrics in Reduced Equations

5.1 Theoretical Summary

Quadrics are surfaces in space whose affine coordinates satisfy a second-degree equation of the form

$$\begin{aligned}&a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_{33}z^2 + \\ &+ a_{14}x + a_{24}y + a_{34}z + a_{44} = 0.\end{aligned} \tag{5.1.1}$$

There are only a few classes of such surfaces. Through a coordinate transformation, the equation of any one of them can be brought into a so-called *canonical form*, which does not contain second-degree cross terms and, with a few exceptions, does not contain first-degree terms either. We will study exclusively these quadrics using the canonical equation.

5.1.1 The Ellipsoid

An *ellipsoid* is the set of points in space whose coordinates (x, y, z) satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{5.1.2}$$

where a, b and c are strictly positive real numbers called the *semi-axes* of the ellipsoid.

The coordinate planes are *planes of symmetry* of the ellipsoid, while the coordinate axes are *axes of symmetry*.

The points where the ellipsoid intersects the axes of symmetry (six in total) are called the *vertices* of the ellipsoid. They have coordinates $(\pm a, 0, 0)$, $(0, \pm b, 0)$, and $(0, 0, \pm c)$ respectively.

The ellipsoid is a bounded figure. It is contained within the parallelepiped

$$[-a, a] \times [-b, b] \times [-c, c].$$

The Surface of Revolution Ellipsoid

If in equation (5.1.2) two of the semi-axes corresponding to two coordinate axes are equal, the ellipsoid is called a *surface of revolution ellipsoid*, since it is in fact a surface of revolution about the third axis. For instance, a surface of revolution ellipsoid around the Oz axis has the equation

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1. \tag{5.1.3}$$

Evidently, if all three semi-axes are equal, we obtain a sphere of radius a centred at the origin.

The Tangent Plane to the Ellipsoid

If (x_0, y_0, z_0) is a point on the ellipsoid given by equation (5.1.2), then the equation of the tangent plane to the ellipsoid at this point is obtained by doubling, that is, it is written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1. \tag{5.1.4}$$

5.1.2 Second-Degree Cone

A *second-degree cone* or *elliptic cone* is the set of points in space whose coordinates relative to an orthonormal frame satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \tag{5.1.5}$$

where a, b, c are strictly positive real numbers.

The second-degree cone is a particular case of a *conical surface*, having its vertex at the origin, while its generators intersect the ellipse

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \\ z = c. \end{cases} \tag{5.1.6}$$

Hereafter, we will use the term *generators* of the cone through a point on it to refer to the line passing through the origin and through the point (which, evidently, lies entirely within the cone).

Like the ellipsoid, the second-degree cone has the coordinate planes as its *planes of symmetry*, the coordinate axes as *axes of symmetry*, and the origin as its *centre of symmetry*. All three axes intersect the surface at a single point (the origin), so the cone has a single *vertex*, which is the apex of the cone viewed as a conical surface.

Intersections with Coordinate Planes

The intersection of the cone with a plane parallel to the xOy plane, with equation $z = h$, reduces to the origin if $h = 0$ (i.e., the plane coincides with xOy), or, if $h \neq 0$, is the ellipse

$$\begin{cases} \dfrac{x^2}{a^2h^2/c^2} + \dfrac{y^2}{b^2h^2/c^2} = 1 \\ z = h \end{cases},$$

with semi-axes $a|h|/c$ and $b|h|/c$.

The intersection with planes parallel to the xOz plane, of equation $y = h$, reduces to the pair of lines

$$\begin{cases} \dfrac{x}{a} \pm \dfrac{z}{c} = 0, \\ y = 0, \end{cases}$$

if $h = 0$, or to the hyperbola

$$\begin{cases} \dfrac{z^2}{c^2h^2/b^2} - \dfrac{x^2}{a^2h^2/b^2} = 1 \\ y = h \end{cases},$$

with semi-axes $c|h|/b$ and $a|h|/b$, if $h \neq 0$.

The intersection with planes parallel to the yOz plane, of equation $x = h$, reduces to the pair of lines

$$\begin{cases} \dfrac{y}{b} \pm \dfrac{z}{c} = 0, \\ x = 0, \end{cases}$$

if $h = 0$, or to the hyperbola

$$\begin{cases} \dfrac{y^2}{b^2h^2/a^2} - \dfrac{z^2}{c^2h^2/a^2} = 1 \\ x = h \end{cases},$$

with semi-axes $b|h|/a$ and $c|h|/a$, if $h \neq 0$.

The Tangent Plane to the Cone

The tangent plane to the cone of equation (5.1.5) at a point on it, different from the vertex, with coordinates (x_0, y_0, z_0), is obtained, as with the ellipsoid, by doubling, i.e., it is

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = 0. \tag{5.1.7}$$

The tangent plane at a point on a cone is the same at all points of the generator that passes through that point.

The Surface of Revolution Cone

If in equation (5.1.5) we set $a = b$, the resulting conical surface, given by the equation

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 0, \tag{5.1.8}$$

is called a *surface of revolution cone*. This surface is both a conical surface and a surface of revolution around the Oz axis.

5.1.3 One-Sheeted Hyperboloid

A *one-sheeted hyperboloid* is the set of points in space whose coordinates, relative to an orthonormal coordinate system, satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \tag{5.1.9}$$

where a, b, c are three strictly positive real numbers, called the *semi-axes* of the hyperboloid.

The points where the Ox and Oy axes intersect the surface, that is, the points with coordinates $(\pm a, 0, 0)$ and $(0, \pm b, 0)$, are called the *vertices* of the hyperboloid.

The one-sheeted hyperboloid has the coordinate planes as planes of symmetry, the coordinate axes as axes of symmetry, and the origin as the centre of symmetry.

Intersections of the Hyperboloid with Planes Parallel to the Coordinate Planes

The intersection between the hyperboloid and a plane parallel to the xOy plane, with equation $z = h$, is an ellipse

$$\begin{cases} z = h, \\ \dfrac{x^2}{\left(a\sqrt{\dfrac{h^2}{c^2}+1}\right)^2} + \dfrac{y^2}{\left(b\sqrt{\dfrac{h^2}{c^2}+1}\right)^2} = 1, \end{cases}$$

with axes parallel to the Ox and Oy axes and semi-axes $a\sqrt{\frac{h^2}{c^2}+1}$ and $b\sqrt{\frac{h^2}{c^2}+1}$. For $h=0$, we obtain the ellipse with minimal semi-axes, which is called the *waist ellipse*.

The intersection between the hyperboloid and a plane parallel to the yOz plane, with equation $x=h$, is given by the system

$$\begin{cases} x = h, \\ \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{h^2}{a^2}. \end{cases}$$

There are three possible cases:

(a) If $|h| < |a|$, then we have a hyperbola with equations

$$\begin{cases} x = h, \\ \dfrac{y^2}{\left(b\sqrt{1-\frac{h^2}{a^2}}\right)^2} - \dfrac{z^2}{\left(c\sqrt{1-\frac{h^2}{a^2}}\right)^2} = 1. \end{cases}$$

(b) If $|h| = |a|$, we obtain a pair of intersecting lines

$$\begin{cases} x = h, \\ \dfrac{y}{b} \pm \dfrac{z}{c} = 0. \end{cases}$$

(c) If $|h| > |a|$, then the intersection is the hyperbola

$$\begin{cases} x = h, \\ \dfrac{z^2}{\left(c\sqrt{\frac{h^2}{a^2}-1}\right)^2} - \dfrac{y^2}{\left(b\sqrt{\frac{h^2}{a^2}-1}\right)^2} = 1. \end{cases}$$

Intersections between planes parallel to the xOz plane and the hyperboloid are treated completely analogously to the intersections with planes parallel to the yOz plane.

Tangent Plane to the One-Sheeted Hyperboloid

If (x_0, y_0, z_0) is a point on the one-sheeted hyperboloid, then the equation of the tangent plane at that point is written by doubling, i.e.,

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = 1. \tag{5.1.10}$$

Straight Line Generators of the One-Sheeted Hyperboloid

On the one-sheeted hyperboloid lie two families of lines called *straight line generators* of the hyperboloid. The equations of the generators in the first family are

$$\begin{cases} \lambda\left(\frac{x}{a}+\frac{z}{c}\right)=\mu\left(1+\frac{y}{b}\right), \\ \mu\left(\frac{x}{a}-\frac{z}{c}\right)=\lambda\left(1-\frac{y}{b}\right), \end{cases} \tag{5.1.11}$$

where λ and μ are two real numbers not both zero, while the equations of the generators in the second family are

$$\begin{cases} \alpha\left(\frac{x}{a}+\frac{z}{c}\right)=\beta\left(1-\frac{y}{b}\right), \\ \beta\left(\frac{x}{a}-\frac{z}{c}\right)=\alpha\left(1+\frac{y}{b}\right), \end{cases} \tag{5.1.12}$$

where α and β are again real parameters not both zero.

The straight line generators have several remarkable properties, including:

- Through every point of the hyperboloid pass exactly two generators, one from each family,
- Any two generators from the same family are non-coplanar,
- Each generator from one family intersects all generators from the other family,
- All generators (from both families) intersect the waist ellipse of the hyperboloid,
- The two generators passing through a point on the hyperboloid lie in the tangent plane at that point.

Surface of Revolution One-Sheeted Hyperboloid

If the first two semi-axes of the hyperboloid are equal: $a = b$, then we obtain the so-called *surface of revolution one-sheeted hyperboloid*, with equation

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{c^2}=1. \tag{5.1.13}$$

This is a surface of revolution about the Oz axis. The intersections between the surface of revolution hyperboloid and planes parallel to the xOy plane are circles.

5.1.4 Two-Sheeted Hyperboloid

The *two-sheeted hyperboloid* is the set of points in space whose coordinates relative to an orthonormal frame satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \tag{5.1.14}$$

where a, b, c are strictly positive real constants, called the *semi-axes* of the hyperboloid.

The two-sheeted hyperboloid consists of two disjoint subsets, called *sheets*, corresponding to the cases $z > 0$ and $z < 0$.

The Oz axis intersects the surface in two points, called the *vertices* of the two-sheeted hyperboloid.

The two-sheeted hyperboloid has the coordinate planes as planes of symmetry, the coordinate axes as axes of symmetry, and the origin as the centre of symmetry.

Intersections of the Two-Sheeted Hyperboloid with Planes Parallel to the Coordinate Planes

The intersection of the surface with a plane parallel to the xOy plane, with equation $z = h$, is described by the system

$$\begin{cases} z = h, \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{h^2}{c^2} - 1. \end{cases}$$

This may be:

(a) the empty set, if $|h| < |c|$;

(b) a point, if $h = \pm c$;

(c) an ellipse

$$\begin{cases} z = h, \\ \dfrac{x^2}{\left(a\sqrt{\frac{h^2}{c^2} - 1}\right)^2} + \dfrac{y^2}{\left(b\sqrt{\frac{h^2}{c^2} - 1}\right)^2} = 1, \end{cases}$$

with semi-axes $a\sqrt{\frac{h^2}{c^2} - 1}$ and $b\sqrt{\frac{h^2}{c^2} - 1}$, if $|h| > |c|$.

The intersection of the two-sheeted hyperboloid with a plane parallel to the yOz plane, with equation $x = h$, is given by

$$\begin{cases} x = h, \\ \dfrac{z^2}{c^2} - \dfrac{y^2}{b^2} = \dfrac{h^2}{a^2} + 1. \end{cases}$$

This intersection, regardless of the value of h, is a hyperbola with equation

$$\begin{cases} x = h, \\ \dfrac{z^2}{\left(c\sqrt{\frac{h^2}{a^2} + 1}\right)^2} - \dfrac{y^2}{\left(b\sqrt{\frac{h^2}{a^2} + 1}\right)^2} = 1, \end{cases}$$

with semi-axes $c\sqrt{\frac{h^2}{a^2} + 1}$ and $b\sqrt{\frac{h^2}{a^2} + 1}$.

The intersections with planes parallel to the zOx plane are similar to those obtained with planes parallel to yOz.

Surface of Revolution Two-Sheeted Hyperboloid

If the first two semi-axes of the hyperboloid are equal, we obtain the *surface of revolution two-sheeted hyperboloid*, with equation

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = -1.$$

This is a surface of revolution about the Oz axis.

Tangent Plane to the Two-Sheeted Hyperboloid

The equation of the tangent plane to the two-sheeted hyperboloid at a point (x_0, y_0, z_0) is obtained by doubling:

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = -1,$$

5.1.5 Elliptic Paraboloid

An *elliptic paraboloid* is the set of points in space whose coordinates relative to an orthonormal frame satisfy the equation

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z, \tag{5.1.15}$$

where p and q are strictly positive real numbers, called the *parameters of the paraboloid*.

The elliptic paraboloid has two symmetry planes, the xOz and yOz planes, and one axis of symmetry, the Oz axis. It has no centre of symmetry.

The Oz axis intersects the paraboloid at a single point, the origin, which is called the *vertex* of the elliptic paraboloid.

Intersections with Planes Parallel to the Coordinate Planes

The intersection of the elliptic paraboloid with a plane parallel to the xOy plane, with equation $z = h$, is

$$\begin{cases} z = h, \\ \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2h. \end{cases}$$

This intersection may be:

(a) the empty set, if $h < 0$;

(b) a single point (the origin), if $h = 0$;

(c) an ellipse

$$\begin{cases} z = h, \\ \dfrac{x^2}{2ph} + \dfrac{y^2}{2qh} = 1, \end{cases}$$

with semi-axes $\sqrt{2ph}$ and $\sqrt{2qh}$, if $h > 0$.

The intersection with a plane parallel to yOz, of equation $x = h$, is

$$\begin{cases} x = h, \\ y^2 = 2qz - \dfrac{qh^2}{p}. \end{cases}$$

This is a parabola for any h. Intersections with planes parallel to zOx are also parabolas, obtained similarly.

Surface of Revolution Elliptic Paraboloid

If the two parameters of the paraboloid are equal: $p = q$, we obtain the *surface of revolution elliptic paraboloid*, with equation

$$x^2 + y^2 = 2pz. \tag{5.1.16}$$

For this paraboloid (which is indeed a surface of revolution about the Oz axis), the intersections with planes parallel to xOy are circles (possibly degenerated to a point or imaginary).

Tangent Plane to the Elliptic Paraboloid

The tangent plane to the elliptic paraboloid at a point (x_0, y_0, z_0) is obtained from the paraboloid's equation by doubling:

$$\frac{xx_0}{p} + \frac{yy_0}{q} = p(z + z_0). \tag{5.1.17}$$

5.1.6 Hyperbolic Paraboloid

A *hyperbolic paraboloid* is the set of points in space whose coordinates relative to an orthonormal frame satisfy the equation

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z. \tag{5.1.18}$$

Here p and q are strictly positive real numbers called the *parameters of the hyperbolic paraboloid.*

The hyperbolic paraboloid has two symmetry planes (yOz and zOx) and one axis of symmetry (Oz). The surface has no centre of symmetry.

The Oz axis intersects the surface at the origin (the *vertex* of the hyperbolic paraboloid).

Intersections with Planes Parallel to the Coordinate Planes

The intersection with a plane parallel to xOy, with equation $z = h$, is:

$$\begin{cases} z = h, \\ \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2h. \end{cases}$$

This is:

(a) a hyperbola

$$\begin{cases} z = h, \\ \dfrac{y^2}{-2qh} - \dfrac{x^2}{-2ph} = 1, \end{cases}$$

with semi-axes $\sqrt{-2qh}$ and $\sqrt{-2ph}$, if $h < 0$;

(b) a pair of lines

$$\begin{cases} z = 0, \\ \dfrac{x}{\sqrt{p}} \pm \dfrac{y}{\sqrt{q}} = 0, \end{cases}$$

if $h = 0$;

(c) a hyperbola

$$\begin{cases} z = h, \\ \dfrac{x^2}{2ph} - \dfrac{y^2}{2qh} = 1, \end{cases}$$

with semi-axes $\sqrt{2ph}$ and $\sqrt{2qh}$, if $h > 0$.

The intersection with a plane parallel to yOz, of equation $x = h$, is

$$\begin{cases} x = h, \\ y^2 = -2qz + \dfrac{qh^2}{p}, \end{cases}$$

a parabola for any value of h. Similarly, the intersections with planes parallel to zOx are also parabolas.

Tangent Plane to the Hyperbolic Paraboloid

The tangent plane to the hyperbolic paraboloid at a point (x_0, y_0, z_0) is obtained by doubling:

$$\frac{xx_0}{p} - \frac{yy_0}{q} = z + z_0. \tag{5.1.19}$$

Straight Line Generators of the Hyperbolic Paraboloid

As in the case of the one-sheeted hyperboloid, the hyperbolic paraboloid has two families of lines, called *straight line generators*.

Equations of the first family:

$$\begin{cases} \lambda \left(\dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} \right) = 2\mu z, \\ \mu \left(\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} \right) = \lambda, \end{cases} \tag{5.1.20}$$

with real parameters λ and μ not both zero.

Equations of the second family:

$$\begin{cases} \alpha \left(\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} \right) = 2\beta z, \\ \beta \left(\dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} \right) = \alpha, \end{cases} \tag{5.1.21}$$

where α and β are also real parameters, not both zero.

These generators enjoy the following properties:

- Through every point on the surface pass exactly two straight line generators, one from each family.
- The generators through a point lie in the tangent plane at that point.
- Any two generators from the same family are non-coplanar.
- Each generator from one family intersects all generators from the other family.

5.1.7 Elliptic Cylinder

An *elliptic cylinder* is the set of points in space whose coordinates relative to an orthonormal system satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{5.1.22}$$

where a and b are real numbers called the *semi-axes* of the elliptic cylinder.

The *planes of symmetry* of the elliptic cylinder are: the yOz plane, the xOz plane, and any plane parallel to the xOy plane. As a result, it has an infinite number of *axes of symmetry*: the Oz axis and any line parallel to the other two axes that intersects Oz, as well as an infinite number of *centres of symmetry*: all points on the Oz axis.

Intersections of the Elliptic Cylinder with Planes Parallel to the Coordinate Planes

The intersection between the elliptic cylinder and a plane parallel to the xOy plane, with equation $z = h$, is, for any value of h, an ellipse with semi-axes a and b, given by

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \\ z = h. \end{cases}$$

The intersection between the cylinder and a plane parallel to the yOz plane, with equation $x = h$, is given by

$$\begin{cases} \dfrac{y^2}{b^2} = 1 - \dfrac{h^2}{a^2}, \\ x = h. \end{cases}$$

This intersection is:

- a pair of lines parallel to the Oz axis:

$$\begin{cases} y = \pm b\sqrt{1 - \dfrac{h^2}{a^2}}, \\ x = h, \end{cases}$$

if $|h| < |a|$;

- a single line parallel to Oz:

$$\begin{cases} y = 0, \\ x = h, \end{cases}$$

if $|h| = |a|$;

- the empty set, if $|h| > |a|$.

Intersections of the elliptic cylinder with planes parallel to the xOz plane are similar to those with planes parallel to the yOz plane.

Tangent Plane at a Point on the Elliptic Cylinder

The equation of the tangent plane to the elliptic cylinder at a point (x_0, y_0, z_0) is obtained by doubling:

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1. \tag{5.1.23}$$

Surface of Revolution Elliptic Cylinder

If the two semi-axes of the elliptic cylinder are equal: $a = b$, then the cylinder is called a *cylinder of revolution* or *circular cylinder*. Its equation is

$$x^2 + y^2 = a^2. \tag{5.1.24}$$

5.1.8 Hyperbolic Cylinder

A *hyperbolic cylinder* is the set of points in space whose coordinates relative to an orthonormal system satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{5.1.25}$$

where a and b are real numbers called the *semi-axes* of the hyperbolic cylinder.

The symmetries of the hyperbolic cylinder are the same as those of the elliptic cylinder.

Intersections of the Hyperbolic Cylinder with Planes Parallel to the Coordinate Planes

The intersection between the hyperbolic cylinder and a plane parallel to the xOy plane, with equation $z = h$, is, for any h, a hyperbola with semi-axes a and b, given by

$$\begin{cases} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, \\ z = h. \end{cases}$$

The intersection with a plane parallel to the yOz plane, with equation $x = h$, is given by

$$\begin{cases} \dfrac{y^2}{b^2} = \dfrac{h^2}{a^2} - 1, \\ x = h. \end{cases}$$

This intersection is:

- the empty set, if $|h| < a$;
- a line parallel to the Oz axis:
$$\begin{cases} y = 0, \\ z = h, \end{cases}$$
if $|h| = a$;
- a pair of lines parallel to the Oz axis:
$$\begin{cases} y = \pm b\sqrt{\dfrac{h^2}{a^2} - 1}, \\ x = h, \end{cases}$$
if $|h| > a$.

The intersection of the hyperbolic cylinder with a plane parallel to the zOx plane, with equation $y = h$, is, for any real h, a pair of lines parallel to the Oz axis:

$$\begin{cases} x = \pm a\sqrt{\dfrac{h^2}{b^2} + 1}, \\ y = h. \end{cases}$$

Tangent Plane at a Point on the Hyperbolic Cylinder

As for all quadrics, the tangent plane to the hyperbolic cylinder at a point (x_0, y_0, z_0) is given by doubling:

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1. \tag{5.1.26}$$

5.1.9 Parabolic Cylinder

A *parabolic cylinder* is the set of points in space whose coordinates relative to an orthonormal coordinate system satisfy the equation

$$y^2 = 2px, \tag{5.1.27}$$

where p is a strictly positive real number, called the *parameter* of the cylinder.

The parabolic cylinder has one plane of symmetry, the xOz plane, and infinitely many axes of symmetry: all lines parallel to the Ox axis that intersect the Oz axis. The cylinder has no centre of symmetry.

Intersections of the Parabolic Cylinder with Planes Parallel to the Coordinate Planes

The intersection of the parabolic cylinder with a plane parallel to the xOy plane, with equation $z = h$, is, for any value of h, a parabola with parameter p, given by

$$\begin{cases} y^2 = 2px, \\ z = h. \end{cases}$$

The intersection with a plane parallel to the yOz plane, with equation $x = h$, is described by the system

$$\begin{cases} y^2 = 2ph, \\ x = h. \end{cases}$$

This intersection is:

- the empty set, if $h < 0$;
- the Oz axis, if $h = 0$;
- a pair of lines parallel to the Oz axis:

  $$\begin{cases} y = \pm\sqrt{2ph}, \\ x = h, \end{cases}$$

  if $h > 0$.

Finally, the intersection of the parabolic cylinder with a plane parallel to the zOx plane, with equation $y = h$, is, for any real value of h, a line parallel to the Oz axis, given by

$$\begin{cases} x = \dfrac{h^2}{2p}, \\ y = h. \end{cases}$$

6 Surface Generation

6.1 Theoretical Summary

6.1.1 Cylindrical Surfaces

A *cylindrical surface* is a surface generated by a family of lines, called *generators*, which move in space parallel to a fixed line, called the *directing line*, and satisfy an additional condition. This additional condition is usually that the generators intersect a given curve, called the *directrix*, but other conditions can also be imposed, such as requiring the generators to be tangent to a given surface.

Assume that the directing line is given as the intersection of two planes, i.e., by a system of equations of the form

$$(\Delta)\ \begin{cases} P_1(x, y, z) = 0, \\ P_2(x, y, z) = 0, \end{cases} \tag{6.1.1}$$

while the directrix is given as the intersection of two surfaces, i.e., by another system of equations

$$(C)\ \begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{6.1.2}$$

Since the generators are parallel to the directing line, their equations can be written in the form

$$(G_{\lambda,\mu})\ \begin{cases} P_1(x, y, z) = \lambda, \\ P_2(x, y, z) = \mu, \end{cases} \tag{6.1.3}$$

where λ and μ are two arbitrary real parameters.

The condition that the generators intersect the directrix leads to the system

$$\begin{cases} P_1(x, y, z) = \lambda, \\ P_2(x, y, z) = \mu, \\ F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{6.1.4}$$

This is a system of four equations in the unknowns x, y, and z. The requirement that the system be consistent implies the existence of a dependency between the parameters λ and μ,

$$\varphi(\lambda, \mu) = 0. \tag{6.1.5}$$

Then the equation of the cylindrical surface is written as

$$\varphi\left(P_1(x, y, z), P_2(x, y, z)\right) = 0. \tag{6.1.6}$$

6.1.2 Conical Surfaces

A *conical surface* is a surface generated by a family of moving lines, called *generators*, which all pass through a fixed point called the *vertex of the conical surface*, and which satisfy an additional condition. As in the case of cylindrical surfaces, the condition is usually that the generators intersect a given curve, called the *directrix* of the conical surface, but, again, other conditions may be imposed, such as requiring the generators to be tangent to a given surface.

Assume that the vertex of the cone is given as the intersection of three planes, i.e., as the unique solution to a consistent and determined linear system:

$$(V) \begin{cases} P_1(x, y, z) = 0, \\ P_2(x, y, z) = 0, \\ P_3(x, y, z) = 0, \end{cases} \tag{6.1.7}$$

and that the directrix is given as the intersection of two surfaces:

$$(C) \begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{6.1.8}$$

A generator of the conical surface (i.e., a line through the vertex) can be written as

$$(G_{\lambda,\mu}) \begin{cases} P_1(x, y, z) = \lambda P_3(x, y, z), \\ P_2(x, y, z) = \mu P_3(x, y, z), \end{cases} \tag{6.1.9}$$

where λ and μ are two arbitrary real parameters.

The condition that the generators intersect the directrix leads to the system

$$\begin{cases} P_1(x, y, z) = \lambda P_3(x, y, z), \\ P_2(x, y, z) = \mu P_3(x, y, z), \\ F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{6.1.10}$$

The system is consistent only if there exists a relation between the two parameters:

$$\varphi(\lambda, \mu) = 0. \tag{6.1.11}$$

Then the equation of the conical surface is

$$\varphi\left(\frac{P_1(x, y, z)}{P_3(x, y, z)}, \frac{P_2(x, y, z)}{P_3(x, y, z)}\right). \tag{6.1.12}$$

6.1.3 Conoid Surfaces (Conoids with a Director Plane)

A *conoid surface* or a *conoid with director plane* is a surface generated by a family of lines, called *generators*, which intersect a given line, called the *directing line*, remain parallel to a given plane, called the *director plane*, and satisfy an additional condition, usually that of intersecting a given curve, called the *directrix*.

If the directing line is given as the intersection of two planes:

$$(\Delta) \ \begin{cases} P_1(x, y, z) = 0, \\ P_2(x, y, z) = 0, \end{cases} \tag{6.1.13}$$

and the director plane has the equation

$$(\Pi) \ P(x, y, z) = 0, \tag{6.1.14}$$

then the equations of the generators can be written as

$$(G_{\lambda,\mu}) \ \begin{cases} P_1(x, y, z) = \lambda P_2(x, y, z), \\ P(x, y, z) = \mu, \end{cases} \tag{6.1.15}$$

where λ and μ are real parameters.

Assume now that the directrix is given by

$$(C) \ \begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{6.1.16}$$

Then the condition that the generators intersect the directrix is given by the system

$$\begin{cases} P_1(x, y, z) = \lambda P_2(x, y, z), \\ P(x, y, z) = \mu, \\ F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{6.1.17}$$

System (6.1.17) is consistent if and only if there exists a compatibility relation between the parameters:

$$\varphi(\lambda, \mu) = 0. \tag{6.1.18}$$

Then the equation of the conoid surface is

$$\varphi\left(\frac{P_1(x, y, z)}{P_2(x, y, z)}, P(x, y, z)\right) = 0. \tag{6.1.19}$$

6.1.4 Surfaces of Revolution

A *surface of revolution* (also called a *revolution surface*) is a surface generated by a curve (called the *directrix*) that rotates in space around a line, called the *axis of revolution*.

The method for deducing the equation of a surface of revolution is similar to that used for deducing the equations of other generated surfaces. Specifically, consider a bi-parametric family of circles lying in planes perpendicular to the axis of revolution and with centres on the axis (*generating circles*). The algorithm proceeds analogously to the others:

- Require that the generating circles intersect the directrix.
- Deduce the compatibility condition.
- Obtain the equation of the surface of revolution by substituting the parameters from the generating circles into the compatibility condition.

Assume the axis of revolution is given by its canonical equations,

$$(\Delta)\ \frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}. \tag{6.1.20}$$

Then the equations of the generating circles will be

$$(G_{\lambda,\mu})\ \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda^2, \\ lx + my + nz = \mu. \end{cases} \tag{6.1.21}$$

If the directrix is given by

$$(C)\ \begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0, \end{cases} \tag{6.1.22}$$

then the condition that the generating circles intersect the directrix is given by the system

$$\begin{cases} (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=\lambda^2, \\ lx+my+nz=\mu, \\ F(x,y,z)=0, \\ G(x,y,z)=0. \end{cases} \tag{6.1.23}$$

This system is consistent if and only if there exists a relation between the two parameters:

$$\varphi(\lambda,\mu)=0. \tag{6.1.24}$$

Then the equation of the surface of revolution is written as

$$\varphi\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2},\, lx+my+nz\right)=0. \tag{6.1.25}$$