The differential geometry of curves and surfaces

Paul A. Blaga

Universitatea “Babeş-Bolyai”

May 17, 2025

Course Content

The course will consist of the following parts:

1. Curves in space
2. Plane curves
3. Elements of the general theory of surfaces

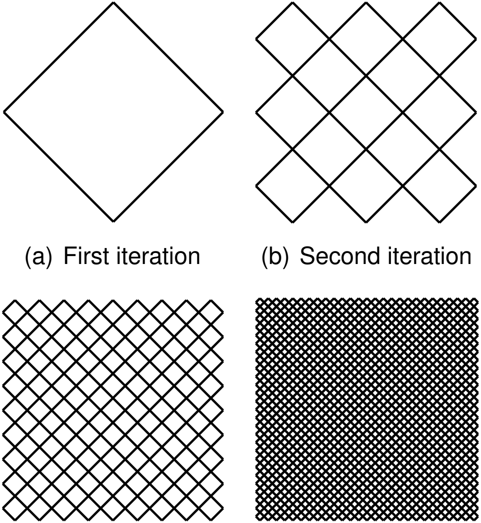
Recommended Software

- For symbolic (and numerical) computation: Matlab, Maple, Mathematica, Maxima
- For programming: Python, C++, Java
- For plotting: Matlab, Maple, Mathematica, Maxima, Geogebra, Gnuplot

- Intuitively, a curve = deformation of a straight line (1-dimensional objects)
- Graphs of functions are examples of curves (given by $y = f(x)$)
- In general, conics are curves, but they are not graphs. They are given *implicitly*, by $f(x, y) = 0$.
- Curves may be described parametrically:

$$\begin{cases} x = f(t), \\ y = g(t) \end{cases} .$$

- In general, we require the involved functions to be at least C^1, otherwise we may encounter pathological cases (Peano curve).



(a) First iteration (b) Second iteration

(c) Third iteration (d) Fourth iteration

Figure: Peano curve (first four iterations)

Let I be an interval on the real axis $\mathbb{R}$. We will not always assume that the interval is open. Sometimes, it is actually important that the interval be closed. In particular, it may be unbounded and may even coincide with the whole real axis.

Definition

A *parametrised curve* (or *path*) of class C^k ($k > 0$) in Euclidean space $\mathbb{R}^3$ is a C^k mapping

$$\mathbf{r} : I \to \mathbb{R}^3 : t \to (x(t), y(t), z(t)). \tag{1}$$

A parametrised curve is usually denoted as $(I, \mathbf{r})$, $(I, \mathbf{r} = \mathbf{r}(t))$, or, when the interval is understood, simply $\mathbf{r} = \mathbf{r}(t)$.

We note that a path is of class C^k if the (real-valued) functions x, y, z are C^k. If the interval is not open, we will assume, first of all, that the functions we work with are of class C^k in the interior of the interval and that all their derivatives up to order k have finite one-sided limits at the ends of the interval, if those endpoints belong to the interval.

A path is called *compact, semi-open*, or *open* if its domain I is, respectively, compact, semi-open, or open.

If the interval I is bounded below, above, or on both sides, then the image of any endpoint of I is called an *endpoint* of the path. If, in particular, the curve is compact and the two endpoints coincide, the path is called *closed*. An alternative term used for a closed curve is *loop*.

Sometimes it may be necessary to consider paths which are of class C^k at all points of the interval I, except for a finite number of points. The following definition is more precise.

Definition

We say that a parametrised curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is C^k *piecewise* if there exists a finite subdivision $(a = a_0, a_1, \ldots, a_n = b)$ of the interval $[a, b]$ such that the restriction of $\mathbf{r}$ to each compact subinterval $[a_{i-1}, a_i]$ is of class C^k, where $i \in \{1, \ldots, n\}$.

Remark

It is not difficult to show that a parametrised curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is C^k piecewise if and only if the following conditions are simultaneously satisfied:

(i) The set
$$S = \left\{ t \in [a, b] \,|\, f^{(k)} \text{ does not exist} \right\}$$
is finite.

(ii) $f^{(k)}$ is continuous on $[a, b] \setminus S$.

(iii) $f^{(k)}$ has finite left and right limits at each point in S.

Parametrised Curves

From now on, we will always assume that the smoothness order k is sufficiently high and we will no longer mention it (with a few exceptions). Instead, we will use the general term *smooth parametrised curve*, meaning of class at least C^1, and whenever derivatives of order k appear – at least C^k.

The image $\mathbf{r}(I) \subset \mathbb{R}^3$ of the interval I under the mapping (1) is called the *support* of the path $(I, \mathbf{r})$.

If $\mathbf{r}(t_0) = a$, we say that the parametrised curve *passes through* the point a for $t = t_0$. Sometimes, for brevity, we refer to this point as the *point* t_0 of the parametrised curve.

1. Let $\mathbf{r}_0 \in \mathbb{R}^3$ be an arbitrary point and $\mathbf{a} \in \mathbb{R}^3$ a non-zero vector, and $I = \mathbb{R}$. The parametrised curve $\mathbb{R} \to \mathbb{R}^3$, $t \to \mathbf{r}_0 + t\mathbf{a}$ is called a *line*. Its support is the straight line passing through $\mathbf{r}_0$ (for $t = 0$) and having direction given by the vector $\mathbf{a}$.
2. $I = \mathbb{R}$, $\mathbf{r}(t) = \mathbf{r}_0 + t^3\mathbf{a}$. The support of this path is the same line.
3. $I = \mathbb{R}$, $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, $a, b \in \mathbb{R}$. The support of this parametrised curve is called a *circular cylindrical helix* (see figure 14).
4. $I = [0, 2\pi]$, $\mathbf{r}(t) = (\cos t, \sin t, 0)$. The support of the path is the unit circle, lying in the *xOy* plane, centred at the origin.
5. $I = [0, 2\pi]$, $\mathbf{r}(t) = (\cos 2t, \sin 2t, 0)$. The support of the curve is the same as in the previous example.
6. $I = \mathbb{R}$, $\mathbf{r}(t) = (t^2, t^3, 0)$. This curve has a point of return at $t = 0$.

Parametrised Curves

Definition

A parametrised curve (1) is said to be *regular at* $t = t_0$ if $\mathbf{r}'(t_0) \neq 0$ and *regular* if it is regular for each $t \in I$.

As we shall see later, the notion of regularity of a function at a point is related to the existence of a well-defined tangent to the curve at that point.

The curves from the previous example are regular, except for those at items 2 and 6, which are not regular for $t = 0$.

Parametrised Curves

Remark

The fact that the *same* support can correspond to both a regular and a non-regular curve suggests that the absence of regularity at a point does not necessarily mean that the corresponding point on the support has any particular geometric feature. It simply means that regularity *guarantees* the absence of such features. Indeed, if we re-examine curves 2 and 6 from the previous example, we immediately notice that, although both are non-regular at $t = 0$, only the second curve exhibits a geometric singularity (a point of return), while the first one's support is a straight line, with no special points.

Each path corresponds to a subset of $\mathbb{R}^3$, its support. However, as examples 1 and 2 show, different parametrised curves may have the same support. A parametrised curve can be thought of as a subset of $\mathbb{R}^3$ together with a parametrisation.

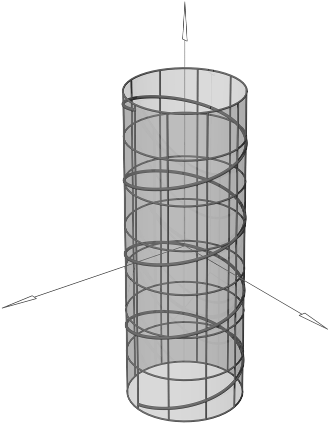


Figure: Circular helix

Parametrised Curves

The support of a parametrised curve corresponds to our intuitive image of the curve as a one-dimensional geometric object. As we shall see later, the support of a parametrised curve may have self-intersections or return points which, for many reasons, are undesirable in applications. Regularity conditions eliminate return points, but not self-intersections. To eliminate the latter, additional conditions must be imposed.

As we have seen, different parametrised curves can have the same support. Ultimately, the support as a point set is what we are interested in. Therefore, it is necessary to identify the relation between parametrised curves that define the same support. For reasons that will be clarified later, at least for the moment, we are only interested in regular curves. Therefore, for example, the transition from one parametric representation to another must not affect the regularity of the curve.

Definition

Let $(I, \mathbf{r} = \mathbf{r}(t))$, $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ be two parametrised curves. A diffeomorphism $\lambda : I \to J : t \to s = \lambda(t)$ such that $\mathbf{r} = \boldsymbol{\rho} \circ \lambda$, i.e. $\mathbf{r}(t) \equiv \boldsymbol{\rho}(\lambda(t))$, is called a *change of parameter* or *reparametrisation*. Two parametrised curves for which there exists a change of parameter are called *equivalent*, while the points t and $s = \lambda(t)$ are called *corresponding*.

Parametrised Curves

Remarks

1. The relation defined above is an equivalence relation on the set of all parametrised curves.
2. Reparametrisation has a simple kinematic interpretation. If we interpret the parametric equations of a path as the equations of motion of a particle, then the support of the curve is the trajectory of the particle, while the vector $\mathbf{r}'(t)$ is the velocity of the particle. The effect of a reparametrisation is to change (in magnitude) the speed at which the trajectory is traversed. Also, if $\lambda'(t) < 0$, then the trajectory is traversed in the opposite direction after reparametrisation. It is worth noting that the two velocity vectors of two equivalent parametrised curves at corresponding points have the same direction. They may have different magnitudes and opposite directions.

Parametrised Curves

Example

The parametrised curves from examples 1 and 2 are not equivalent, although, as we mentioned, they have the same support.

Sometimes, the equivalence classes determined by the relation defined above between parametrised curves are called *curves*. We will not use this approach here, as we want curves to be more general objects than the supports of parametrised curves. In particular, as we will see shortly, curves usually cannot be represented globally by the same set of parametric equations. It is enough to think about the unit circle, centred at the origin. One of the most commonly used parametric representations of the circle is

$$\begin{cases} x = \cos\theta, \\ y = \sin\theta \end{cases}.$$

Parametrised Curves

If we let the parameter vary only within the interval $(0, 2\pi)$, then one point of the circle is not represented. Of course, we can extend the interval, but then the same point corresponds to multiple values of the parameter, which again is not acceptable.
Among all the parametrised curves equivalent to a given parametrised curve, there is one that has special theoretical value and which simplifies many proofs in the theory of curves, although, in most cases, it is very difficult to find it analytically, and thus its practical value is limited.

Definition

We say that a parametrised curve is *naturally parametrised* if $\|\mathbf{r}'(s)\| = 1$ for all $s \in I$. Usually, the natural parameter is denoted by s.

Parametrised Curves

Remark

It is immediately apparent that any smooth naturally parametrised curve $(I, \mathbf{r} = \mathbf{r}(s))$ is *regular*, because clearly $\mathbf{r}'(s)$ cannot vanish anywhere, since its norm does not vanish.

It is not at all obvious that for any smooth (and regular!) parametrised curve, there exists another equivalent one which is naturally parametrised. To construct such a curve, we first need another notion. The *arc length* of a path $(I, \mathbf{r} = \mathbf{r}(t))$ between the points t_1 and t_2 is the real number[1]

$$l_{t_1,t_2} = \left| \int_{t_1}^{t_2} \|\mathbf{r}'(t)\| dt \right|.$$

[1] Although the integrand is positive, we have not assumed $t_1 < t_2$, hence the integral may be negative, and the modulus is needed if we want to obtain a positive quantity.

Parametrised Curves

Remark

There is a strong motivation for defining arc length in this way. Let us assume, to fix ideas, that $t_1 < t_2$. We choose an arbitrary subdivision $t_1 = a_0 < a_1 < \cdots < a_n = t_2$ of the segment $[t_1, t_2]$ and examine the polygonal line $\gamma_n = \mathbf{r}(a_0)\mathbf{r}(a_1)\cdots\mathbf{r}(a_n)$. The length of this polygonal line is the sum of the lengths of its segments. It can be shown that if the parametrised curve $(I, \mathbf{r})$ is "good enough" (for example, at least once continuously differentiable), then the limit of the length of the polygonal line γ_n, as the norm of the subdivision tends to zero, exists and equals the arc length. It should also be noted that the definition of arc length makes sense for piecewise smooth curves as well, since in this case, the tangent vector has only a finite number of discontinuity points, and hence its norm is integrable.

We will prove that the arc lengths of two equivalent parametrised curves between corresponding points are equal; therefore, the arc length is, in a sense, a characteristic of the support[2].
Indeed, let $\mathbf{r}(t) = \rho(\lambda(t))$, then $\mathbf{r}'(t) = \lambda'(t)\boldsymbol{\rho}'(\lambda(t))$. Hence,

$$\left|\int_{t_1}^{t_2} \|\mathbf{r}'(t)\| dt\right| = \left|\int_{t_1}^{t_2} \|\boldsymbol{\rho}'(\lambda(t))\| \cdot |\lambda'| dt\right| = $$
$$= \left|\int_{t_1}^{t_2} \|\boldsymbol{\rho}'(\lambda)\| \underbrace{\lambda'(t) dt}_{d\lambda}\right| = \left|\int_{\lambda_1}^{\lambda_2} \|\boldsymbol{\rho}'(\lambda)\| d\lambda\right|.$$

[2]We say “in a sense” because we may represent the same point set as the support of another parametrised curve which is not equivalent to the original. The new path may very well have a different arc length between the same points of the support.

Parametrised Curves

For naturally parametrised curves, $(I, \mathbf{r} = \mathbf{r}(s))$,

$$l_{s_1, s_2} = |s_2 - s_1|.$$

In particular, if $0 \in I$ (which we may always assume, since translation is a diffeomorphism), then $l_{0,s} = |s|$, i.e., abstracting from the sign, the natural parameter is the arc length.

Proposition

For every regular parametrised curve, there exists an equivalent naturally parametrised curve.

Parametrised Curves

Proof.

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a regular parametrised curve, $t_0 \in I$, and

$$\lambda : I \to \mathbb{R}, \quad t \to \int_{t_0}^{t} \|\mathbf{r}'(\tau)\| d\tau.$$

The function λ is smooth and strictly increasing, since $\lambda'(t) = \|\mathbf{r}'(t)\| > 0$. Therefore, its image will be an open interval J, and the function $\lambda : I \to J$ will be a diffeomorphism. The parametrised curve $(J, \rho(s) = \mathbf{r}(\lambda^{-1}(s))$ is equivalent to $(I, \mathbf{r})$ and is naturally parametrised, because $\boldsymbol{\rho}'(s) = \mathbf{r}'(\lambda^{-1}(s))(\lambda^{-1})'(s)$, while

$$(\lambda^{-1})'(s) = \frac{1}{\lambda'(\lambda^{-1}(s))} = \frac{1}{\|\mathbf{r}'(\lambda^{-1}(s))\|}$$

□

Proof.

Therefore,

$$\|\rho'(s)\| = \|\mathbf{r}'(\lambda^{-1}(s))\| \cdot |(\lambda^{-1})'(s)| = 1.$$

$\square$

Remark

In the proof of the previous proposition, we essentially used the fact that all points of the curve are regular. On an interval where the curve has singular points, there does not exist a naturally parametrised curve equivalent to it.

Parametrised Curves

Example

For the circular helix

$$\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt, \end{cases}$$

we obtain, via a change of parameter,

$$s(t) = \int_0^t \|\mathbf{r}'(\tau)\| d\tau = \int_0^t \|\{-a\sin\tau, a\cos\tau, b\}\| d\tau = \sqrt{a^2+b^2},$$

therefore,

$$t = \frac{s}{\sqrt{a^2+b^2}}.$$

Example

Thus, the natural parametrisation of the helix is given by the equations

$$\begin{cases} x = a\cos\frac{s}{\sqrt{a^2+b^2}} \\ y = a\sin\frac{s}{\sqrt{a^2+b^2}} \\ z = b\frac{s}{\sqrt{a^2+b^2}}. \end{cases}$$

Parametrised Curves

Remark

It is worth noting that, in general, the natural parameter along a curve cannot be expressed in closed form (i.e., using only elementary functions) in terms of the parameter along the curve. This is in fact impossible even for very simple curves, such as the ellipse

$$\begin{cases} x = a\cos t \\ y = b\sin t, \end{cases}$$

with $a \neq b$, for which the arc length can only be expressed using elliptic functions (from which the name of these functions actually derives!). Therefore, although the natural parameter is very important for theoretical considerations and proofs, as we will see, we will hardly use it in concrete examples.

Curbe regulate

Definition

O submulţime $M \subset \mathbb{R}^3$ se numeşte *curbă regulară* (sau *o subvarietate 1-dimensională a lui* $\mathbb{R}^3$) dacă, pentru fiecare punct $a \in M$, există o curbă parametrizată regulară $(I, \mathbf{r})$, al cărei suport, $\mathbf{r}(I)$, este o vecinătate deschisă în M a punctului a (adică este o mulţime de forma $M \cap U$, unde U este o vecinătate deschisă a lui a în $\mathbb{R}^3$), în timp ce aplicaţia $\mathbf{r} : I \to \mathbf{r}(I)$ este un omeomorfism, în raport cu topologia de subspaţiu a lui $\mathbf{r}(I)$. O curbă parametrizată cu aceste proprietăţi se numeşte *parametrizare locală* a curbei M în jurul punctului a. Dacă pentru o curbă M există o parametrizare locală $(I, \mathbf{r})$ care este *globală*, adică pentru care $\mathbf{r}(I) = M$, curba se numeşte *simplă*.

Curbe regulate

Exemple

1. Orice dreaptă din $\mathbb{R}^3$ este o curbă simplă, deoarece are o parametrizare globală, dată de o funcţie de forma $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, $\mathbf{r}(t) = \mathbf{a} + \mathbf{b} \cdot t$, unde $\mathbf{a}$ şi $\mathbf{b}$ sunt vectori constanţi, $\mathbf{b} \neq 0$.
2. Elicea circulară este o curbă regulară simplă, cu parametrizarea globală $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, dată de $\mathbf{r}(t) = (a\cos t, b\sin t, bt)$.
3. Un cerc în $\mathbb{R}^3$ este o curbă regulară, dar nu este simplă, deoarece nici un interval deschis nu poate fi omeomorf cu cercul, care este o submulţime compactă a lui $\mathbb{R}^3$.

Astfel, o curbă regulară este, pur şi simplu, o submulţime a lui $\mathbb{R}^3$ obţinută “lipind în mod neted” suporturi de curbe parametrizate. Dacă examinăm cu atenţie definiţia curbei, remarcăm că nu orice curbă parametrizată poate fi utilizată ca parametrizare locală a unei curbe. Pentru o curbă parametrizată $(I, \mathbf{r})$ arbitrară, aplicaţia $\mathbf{r} : I \to \mathbb{R}^3$ nu este injectivă şi, astfel, nu poate fi o parametrizare locală. Menţionăm, de asemenea, că, chiar dacă funcţia este injectivă, $\mathbf{r} : I \to \mathbf{r}(I)$ poate să nu fie un omeomorfism (chiar dacă aplicaţia este continuă şi bijectivă, inversa ei ar putea să nu fie continuă).

Curbe regulate

Dacă, de exemplu, considerăm curba parametrizată $(I, \mathbf{r})$, cu $I = \mathbb{R}$ şi $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$,

$$\mathbf{r}(t) = (\cos t, \sin t, 0),$$

atunci suportul acestei curbe parametrizate este cercul unitate în planul de coordonate xOy, cu centrul în origine. Nu trebuie, totuşi, să tragem concluzia că cercul este o curbă simplă, deoarece $\mathbf{r}$ *nu* este un omeomorfism pe imagine (de fapt, aplicaţia nu este nici măcar injectivă, deoarece este periodică).

Curbe regulate

Să presupunem, acum, că $(I, \mathbf{r} = \mathbf{r}(t))$ şi $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(\tau))$ sunt două parametrizări locale ale unei curbe regulare M, în jurul aceluiaşi punct $a \in M$. Atunci, după cum ne putem aştepta, cele două curbe parametrizate sunt echivalente, dacă restrângem intervalele de definiţie astfel încât drumurile să aibă acelaşi suport. Mai precis, are loc următoarea teoremă:

Theorem

Fie $M \subset \mathbb{R}^3$ o curbă regulară şi $(I, \mathbf{r} = \mathbf{r}(t))$, $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(\tau))$ – două parametrizări locale ale lui M astfel încât $W \equiv \mathbf{r}(I) \cap \boldsymbol{\rho}(J) \neq \emptyset$. Atunci $(\mathbf{r}^{-1}(W), \mathbf{r}|_{\mathbf{r}^{-1}(W)})$ şi $(\boldsymbol{\rho}^{-1}(W), \boldsymbol{\rho}|_{\boldsymbol{\rho}^{-1}(W)})$ sunt curbe parametrizate echivalente.

Regular Curves

It follows from the definition that any regular curve is, locally, the support of a parametrised curve. Globally, this observation is only true if the curve is simple. Moreover, in general, the support of a parametrised curve is not a regular curve. Let us consider, for example, the lemniscate of $(\mathbb{R}, \mathbf{r}(t) = (x(t), y(t), z(t)))$, where

$$\begin{cases} x(t) = \frac{t(1+t^2)}{1+t^4} \\ y(t) = \frac{t(1-t^2)}{1+t^4} \\ z = 0 \end{cases} .$$

$\mathbf{r}$ is continuous, even bijective, but its inverse is not continuous. In fact, the support has a self-intersection, since $\lim_{t\to-\infty} \mathbf{r} = \lim_{t\to\infty} \mathbf{r} = \mathbf{r}(0)$ (see the following figure). Nevertheless, we can always restrict the domain of a regular parametrised curve such that the support of the restriction is a regular curve.

Regular Curves

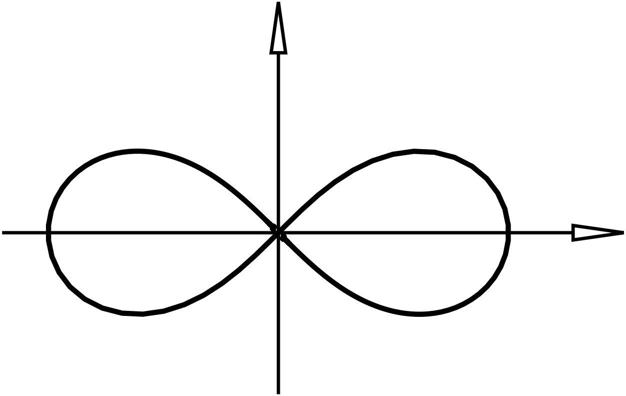


Figure: Bernoulli's lemniscate

Theorem

Let $(I, \mathbf{r} = \mathbf{r}(t))$ *be a regular parametrised curve. Then each point* $t_0 \in I$ *has a neighbourhood* $W \subset I$ *such that* $\mathbf{r}(W)$ *is a regular curve.*

Remark

The previous theorem plays a conceptually very important role. Essentially, it tells us that any *local* property of a regular parametrised curve is also valid for regular curves, provided that the property is invariant under a change of parameter, *without* assuming that the parametrised curve, as a mapping, is a homeomorphism onto its image. Of course, we must take all necessary precautions when investigating *global* properties of regular curves.

A regular curve $M \subset \mathbb{R}^3$ is called *plane* if it is contained in a plane π. We will usually assume that the plane π coincides with the coordinate plane xOy and, therefore, we will use only the coordinates x and y to describe such curves.

Parametric Representation. We choose a local parametrisation $(I, \mathbf{r}(t) = (x(t), y(t))$ of the curve. Then the support $\mathbf{r}(I)$ of this local parametrisation will be an open subset of the curve. For a global parametrisation of a simple curve, $\mathbf{r}(I)$ is the entire curve. Thus, each point a of the curve has an open neighbourhood which is the support of the parametrised curve

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} . \tag{2}$$

Equations (2) are called the *parametric equations* of the curve in the neighbourhood of point a.

Explicit Representation. Let $f : I \to \mathbb{R}$ be a smooth function defined on an open interval of the real axis. Then its graph

$$C = \{(x, f(x)) \mid x \in I\}, \tag{3}$$

is a simple curve, which has the global parametrisation

$$\begin{cases} x = t \\ y = f(t) \end{cases} . \tag{4}$$

Equation

$$y = f(x) \tag{5}$$

is called the *explicit equation* of the curve (3).

Implicit Representation. Let $F : D \to \mathbb{R}$ be a smooth function defined on a domain $D \subset \mathbb{R}^2$, and let

$$C = \{(x, y) \in D \mid F(x, y) = 0\} \tag{6}$$

be the level set 0 of the function F. In general, C is not a regular curve. All we can say about this set is that it is a closed subset of the plane. However, if at the point $(x_0, y_0) \in C$, the vector $\operatorname{grad} F = \{\partial_x F, \partial_y F\}$ does not vanish, for example $\partial_y F(x_0, y_0) \neq 0$, then, by the implicit function theorem, there exist:

- an open neighbourhood U of the point (x_0, y_0) in $\mathbb{R}^2$;
- a smooth function $y = f(x)$ defined on an open neighbourhood $I \subset \mathbb{R}$ of the point x_0,

such that

$$C \cap U = \{(x, f(x) | x \in I\}.$$

If $\operatorname{grad} F \neq 0$ at all points of C, then C is a regular curve (although, in general, not a simple one).

Exemple

1. $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x^2 + y^2 - 1$. Let
$$(x_0, y_0) \in C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 = 0\}.$$
Then we have
$$\operatorname{grad} F(x_0, y_0) = \{2x_0, 2y_0\}.$$
Clearly, since $x_0^2 + y_0^2 = 1$, the vector $\operatorname{grad} F$ cannot vanish on C and, therefore, C is a curve (the unit circle centred at the origin).

Exemple

2. $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x^2 - y^2$. C is not a curve in this case (the gradient vanishes at the origin). In fact, the set C has a self-intersection at the origin (C is nothing but the union of the two bisectors of the coordinate axes, see figure **??**). It may not be very clear why we have problems in the neighbourhood of the origin for this "curve". The truth is that there is no neighbourhood of the origin (on C) that is homeomorphic to an open interval on the real axis. A neighbourhood of the origin on C is the intersection between a neighbourhood of the origin in the plane and the set C. Now, if we shrink the neighbourhood of the origin in the plane, its intersection with C will have a cross shape. If we remove the origin from the cross, there remain four connected components. On the other hand, suppose there exists a homeomorphism f from the cross to an open interval on the real axis. If we remove from

Remark

The condition that the gradient of F be non-zero is only a *sufficient* condition for the equation $F(x, y) = 0$ to represent a curve. If the gradient of F vanishes at a point, we cannot claim that the equation describes a curve in the neighbourhood of that point, but neither can we claim the opposite. Let us consider, as a trivial example, the equation

$$F(x, y) \equiv (x - y)^2 = 0.$$

Remark

Then we have

$$\operatorname{grad} F(x, y) = 2\{x - y, -(x - y)\}$$

and if we denote

$$C = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\},$$

then $\operatorname{grad} F = 0$ at all points of C. Yet, clearly, C *is* a curve (it is easy to see that it is the first bisector of the coordinate axes, i.e., a straight line).

Parametric Representation. As in the case of plane curves, through a local parametrisation

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \tag{7}$$

we can represent either the entire curve, or just a neighbourhood of one of its points.

Explicit Representation. If $f, g : I \to \mathbb{R}$ are two smooth functions defined on an open interval of the real axis, then the set

$$C = \{(x, f(x), g(x)) \in \mathbb{R}^3 \mid x \in I\} \tag{8}$$

is a smooth curve with global parametrisation given by

$$\begin{cases} x = t \\ y = f(t) \\ z = g(t) \end{cases} . \tag{9}$$

The equations

$$\begin{cases} y = f(x) \\ z = g(x) \end{cases} \tag{10}$$

are called the *explicit equations* of the curve.

Implicit Representation. Let $F, G : D \to \mathbb{R}$, defined on a domain $D \subset \mathbb{R}^3$. We consider the set

$$C = \{(x, y, z) \in D \mid F(x, y, z) = 0,\ G(x, y, z) = 0\},$$

in other words, the set of solutions of the system

$$\begin{cases} F(x, y, z) = 0, \\ G(x, y, z) = 0. \end{cases} \tag{11}$$

In general, the set C is not a regular curve. However, if at a point $a = (x_0, y_0, z_0) \in C$ the rank of the Jacobian matrix

$$\begin{pmatrix} \partial_x F & \partial_y F & \partial_z F \\ \partial_x G & \partial_y G & \partial_z G \end{pmatrix} \tag{12}$$

is equal to two, then there exists an open neighbourhood $U \subset D$ of the point (x_0, y_0, z_0) such that $C \cap U$ — the set of solutions to system (11) in U — is a curve.

Indeed, let us suppose, for example, that

$$\det \begin{pmatrix} \partial_y F(a) & \partial_z F(a) \\ \partial_y G(a) & \partial_z G(a) \end{pmatrix} \neq 0.$$

Then, from the implicit function theorem, it follows that there exists an open neighbourhood $U \subset D$ such that the set $C \cap U$ can be written as

$$C \cap U = \{(x, f(x), g(x)) | x \in W\},$$

where W is an open neighbourhood in $\mathbb{R}$ of the point x_0, while $y = f(x)$, $z = g(x)$ are smooth functions defined on W.

Clearly, $C \cap U$ is a simple curve, and the pair $(W, \mathbf{r}(t) = (t, f(t), g(t))$ is a global parametrisation of it.
If the rank of matrix (12) is two everywhere, then C is a curve (though, in general, not a simple one).

Example (Viviani's Window)

An important example of a spatial curve given by implicit equations is the so-called *Viviani's window*[a]. This curve is obtained as the intersection of the sphere with centre at the origin and radius $2a$ and the circular cylinder of radius a with axis parallel to the Oz axis, located at a distance a from that axis. In other words, the equations of Viviani's window are

$$\begin{cases} x^2 + y^2 + z^2 = 4a^2, \\ (x-a)^2 + y^2 = a^2. \end{cases}$$

[a]Vincenzo Viviani (1622–1703) was an Italian mathematician and architect who was in contact with Galileo Galilei in the latter's final years and liked to describe himself as "Galileo's last student".

It is instructive to do some calculations for the case of Viviani's window. As we shall see, it is not globally a curve. We will need to remove a point in order to obtain, in fact, a regular curve. The shape of Viviani's window is easy to understand. It is similar to a Bernoulli lemniscate placed on the surface of a sphere.
Let us therefore consider

$$\begin{cases} F(x,y,z) = x^2 + y^2 + z^2 - 4a^2, \\ G(x,y,z) = (x-a)^2 + y^2 - a^2. \end{cases}$$

Then the equations of the curve become

$$\begin{cases} F(x,y,z) = 0, \\ G(x,y,z) = 0. \end{cases}$$

We now have

$$\begin{pmatrix} F'_x & F'_y & F'_z \\ G'_x & G'_y & G'_z \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ 2(x-a) & 2y & 0 \end{pmatrix} = 2\begin{pmatrix} x & y & z \\ x-a & y & 0 \end{pmatrix}.$$

To obtain a singular point, the following system of equations must be satisfied:

$$\begin{cases} y = 0 \\ yz = 0 \\ (x-a)z = 0 \end{cases}.$$

Clearly, the only solution of the system that also satisfies the curve's equations is $x = 2a, y = z = 0$. Thus, Viviani's window (see figure 4) is a regular curve everywhere, except at the point with these coordinates.

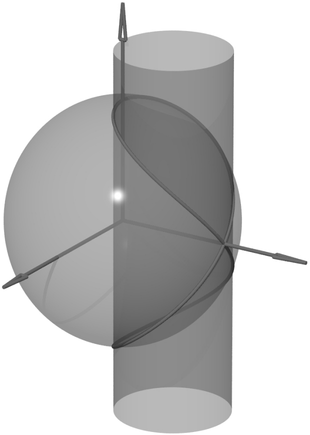


Figure: Viviani's window

It is not difficult to prove that Viviani's window is, in fact, the support of the parametrised curve

$$\mathbf{r}(t) = (a(1+\cos t), a\sin t, 2a\sin\frac{t}{2}),$$

with $t \in (-2\pi, 2\pi)$. When we compute $\mathbf{r}'(t)$, we obtain

$$\mathbf{r}'(t) = \{-a\sin t, a\cos t, a\cos\frac{t}{2}\},$$

which shows that this parametrised curve is regular. In particular, the existence of this regular parametric representation of Viviani's window shows that the point with coordinates $x = 2a, y = z = 0$ is, in fact, a point of self-intersection, not a singular point.

Tangent and Normal Plane

Definition

For a parametrised curve $\mathbf{r} = \mathbf{r}(t)$, the vector $\mathbf{r}'(t_0)$ is called the *tangent vector* or the *velocity vector* of the curve at point t_0. If the point t_0 is regular, then the line passing through $\mathbf{r}(t_0)$ and having the direction given by the vector $\mathbf{r}'(t_0)$ is called the *tangent to the curve at the point* $\mathbf{r}(t_0)$ (or at the point t_0).

The vector equation of the tangent is therefore:

$$\mathbf{R}(\tau) = \mathbf{r}(t_0) + \tau\mathbf{r}'(t_0). \tag{13}$$

Tangent and Normal Plane

Example

The circular cylindrical helix has the parametrisation

$$\mathbf{r}(t) = (a\cos t, a\sin t, bt),$$

therefore, for a point t_0,

$$\mathbf{r}'(t_0) = \{-a\sin t_0, a\cos t_0, b\}.$$

Thus, the equation of the tangent to the helix is

$$\begin{aligned}\mathbf{R}(\tau) =&(a\cos t_0 - \tau a\sin t_0, a\sin t_0 + \tau a\cos t_0, bt_0 + \tau b) = \\ =&(a(\cos t_0 - \tau\sin t_0), a(\sin t_0 + \tau\cos t_0), b(t_0 + \tau)).\end{aligned}$$

Property

Tangent vectors to two equivalent parametrised curves at corresponding points are collinear, while the tangents coincide.

Proof.

Let $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ be the two equivalent parametrised curves and $\lambda : I \to J$ the reparametrisation, i.e. $\mathbf{r} = \boldsymbol{\rho}(\lambda(t))$. Then, according to the chain rule,

$$\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t),$$

where $\lambda'(t) \neq 0$. □

Tangent and Normal Plane

Remarks

1. Clearly, $\mathbf{r}'$ and $\boldsymbol{\rho}'$ have the same orientation when $\lambda' > 0$ (the reparametrisation does not change the direction in which the support of the parametrised curve is traversed), and opposite orientations when $\lambda' < 0$.
2. Since the reparametrisation generally alters the tangent vector, we cannot define the tangent vector at a point of a regular curve using a local parametrisation. However, as we have seen, only the orientation and length of the tangent vector may vary, not the direction. Thus, it makes sense to speak of the tangent at a point of a regular curve defined by *any* local parametrisation of the curve around the chosen point.

Tangent and Normal Plane

We can use a more “geometric” way to define the tangent to a parametrised curve. Let $\mathbf{r}(t_0 + \Delta t)$ be a point on the curve close to the point $\mathbf{r}(t_0)$. Then, from Taylor's formula, we have

$$\mathbf{r}(t_0 + \Delta t) = \mathbf{r}(t_0) + \Delta t \cdot \mathbf{r}'(t_0) + \Delta t \cdot \epsilon, \tag{14}$$

where $\lim\limits_{\Delta t \to 0} \epsilon = 0$. We consider an arbitrary line π, which passes through $\mathbf{r}(t_0)$ and has direction given by the unit vector $\mathbf{m}$. Let

$$d(\Delta t) \stackrel{\text{def}}{=} d((\mathbf{r}(t_0 + \Delta t), \pi).$$

Tangent and Normal Plane

Theorem

The line π is the tangent to the parametrised curve $\mathbf{r} = \mathbf{r}(t)$ at point t_0 if and only if

$$\lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|} = 0. \tag{15}$$

Proof.

From Taylor's formula (14), we have

$$\Delta \mathbf{r} \equiv \mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0) = \Delta t \cdot \mathbf{r}'(t_0) + \Delta t \cdot \epsilon.$$

The distance $d(\Delta t)$ is equal to

$$\|\Delta \mathbf{r} \times \mathbf{m}\| = |\Delta t| \|\mathbf{r}'(t_0) \times \mathbf{m} + \epsilon \times \mathbf{m}\|.$$

□

Proof.

Thus,

$$\lim_{\Delta t\to 0}\frac{d(\Delta t)}{|\Delta t|}=\lim_{\Delta t\to 0}\|\mathbf{r}'(t_0)\times\mathbf{m}+\underbrace{\epsilon\times\mathbf{m}}_{\to 0}\|=\|\mathbf{r}'(t_0)\times\mathbf{m}\|.$$

Now, if the line π is the tangent at t_0, then the vectors $\mathbf{r}'(t_0)$ and $\mathbf{m}$ are collinear, hence $\mathbf{r}'(t_0)\times\mathbf{m}=0$.

Conversely, if condition (15) is satisfied, then $\|\mathbf{r}'(t_0)\times\mathbf{m}\|=0$, thus either $\mathbf{r}'(t_0)=0$ (which cannot happen, since the parametrised curve is regular), or the vectors $\mathbf{r}'(t_0)$ and $\mathbf{m}$ are collinear, i.e. π is the tangent at t_0. □

Remark

Condition (15) is expressed by the requirement that the tangent and the curve have a *first-order contact* (or a tangential contact). Another way to interpret this formula is that the tangent is the limiting position of a line determined by the chosen point and a nearby point on the curve, as the nearby point approaches the given one indefinitely.

From now on, unless otherwise specified, all parametrised curves will be considered regular.

Tangent and Normal Plane

Definition

Let $\mathbf{r} = \mathbf{r}(t)$ be a parametrised curve and $t_0 \in I$. The *normal plane* at the point $\mathbf{r}(t_0)$ of the curve $\mathbf{r} = \mathbf{r}(t)$ is, by definition, the plane passing through $\mathbf{r}(t_0)$ and perpendicular to the tangent to the curve at the point $\mathbf{r}(t_0)$.

If $\mathbf{r} = \mathbf{r}(t)$ is a *planar* parametrised curve (i.e. its support lies in a plane, which we shall assume to be the coordinate plane xOy), then the *normal* to the curve at the point $\mathbf{r}(t_0)$ is, by definition, the line passing through $\mathbf{r}(t_0)$ and perpendicular to the tangent to the curve at the point $\mathbf{r}(t_0)$.

Remark

Since it makes sense to define the tangent at a point of a regular curve using any local parametrisation of the curve around that point, the same holds true for the normal plane (the normal line, in the case of planar curves).

Tangent and Normal Plane

The vector equation of the normal plane (or of the normal line) follows immediately from the definition:

$$(\mathbf{R} - \mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = 0. \tag{16}$$

Parametric Representation If we start from the vector equation (13) of the tangent and project it onto the coordinate axes, we obtain the parametric equations of the tangent, namely, for spatial curves,

$$\begin{cases} X(\tau) = x(t_0) + \tau x'(t_0), \\ Y(\tau) = y(t_0) + \tau y'(t_0), \\ Z(\tau) = z(t_0) + \tau z'(t_0), \end{cases} \tag{17}$$

and for planar curves,

$$\begin{cases} X(\tau) = x(t_0) + \tau x'(t_0), \\ Y(\tau) = y(t_0) + \tau y'(t_0). \end{cases} \tag{18}$$

If we eliminate the parameter τ, we obtain the canonical equations:

$$\frac{X-x}{x'}=\frac{Y-y}{y'}=\frac{Z-z}{z'}, \tag{19}$$

for spatial curves, and

$$\frac{X-x}{x'}=\frac{Y-y}{y'}, \tag{20}$$

for planar curves.

As for the equation of the normal plane (or the normal line), we obtain it from equation (16), by expressing it as

$$\{X-x, Y-y, Z-z\}\cdot\{x', y', z'\}=0,$$

for spatial curves and

$$\{X-x, Y-y\}\cdot\{x', y'\}=0,$$

for planar curves.

Expanding the scalar products, we obtain:

$$(X - x)x' + (Y - y)y' + (Z - z)z' = 0, \tag{21}$$

for the equation of the normal plane to a spatial curve, and for the normal to a planar curve,

$$(X - x)x' + (Y - y)y' = 0. \tag{22}$$

Explicit Representation If we have a spatial curve given by the equations

$$\begin{cases} y = f(x) \\ z = g(x) \end{cases},$$

then we can construct a (global) parametrisation

$$\begin{cases} x = t \\ y = f(t) \\ z = g(t). \end{cases}$$

For the derivatives, we immediately obtain the expressions

$$\begin{cases} x' = 1 \\ y' = f' \\ z' = g', \end{cases}$$

which, after substitution into equations (19), lead us, for the tangent, to the equations

$$X - x = \frac{Y - f(x)}{f'(x)} = \frac{Z - g(x)}{g'(x)}, \tag{23}$$

while for the normal plane, after substituting the derivatives into equation (21), we obtain

$$X - x + (Y - f(x))f'(x) + (Z - g(x))g'(x) = 0. \tag{24}$$

For a planar curve given explicitly,

$$y = f(x),$$

we have the parametric representation

$$\begin{cases} x = t \\ y = f(t), \end{cases}$$

and thus the equation of the tangent is

$$X - x = \frac{Y - f(x)}{f'(x)} \tag{25}$$

or, in a more familiar form,

$$Y - f(x) = f'(x)(X - x), \tag{26}$$

For the normal to a planar curve, we have

$$X - x + (Y - f(x))f'(x) = 0 \tag{27}$$

or

$$Y - f(x) = -\frac{1}{f'(x)}(X - x). \tag{28}$$

Tangent and Normal Plane

Particular Representations

Implicit Representation We consider a curve given by the implicit equations

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0. \end{cases} \tag{29}$$

Let us suppose that at a point (x_0, y_0, z_0)

$$\det \begin{pmatrix} F'_y & F'_z \\ G'_y & G'_z \end{pmatrix} \neq 0$$

Then, as we saw above, in a neighbourhood of this point, the curve can be represented as

$$\begin{cases} y = f(x) \\ z = g(x). \end{cases} \tag{30}$$

Thus, the system (29) can be written as

$$\begin{cases} F(x, f(x), g(x)) = 0 \\ G(x, f(x), g(x)) = 0. \end{cases}$$

Calculating the total derivatives with respect to x of F and G, we obtain the system

$$\begin{cases} F'_x + f'(x)F'_y + g'(x)F'_z = 0 \\ G'_x + f'(x)G'_y + g'(x)G'_z = 0, \end{cases}$$

therefore

$$\begin{cases} f'F'_y + g'F'_z = -F'_x \\ f'G'_y + g'G'_z = -G'_x. \end{cases}$$

From this system, we can obtain f' and g' using Cramer's rule:

$$\Delta = \begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix} \overset{\text{not}}{=} \frac{D(F,G)}{D(y,z)} \overset{\text{hyp}}{\neq} 0,$$

$$\Delta_{f'} = \begin{vmatrix} -F'_x & F'_z \\ -G'_x & G'_z \end{vmatrix} = \begin{vmatrix} F'_z & F'_x \\ G'_z & G'_x \end{vmatrix} \overset{\text{not}}{=} \frac{D(F,G)}{D(z,x)}$$

$$\Delta_{g'} = \begin{vmatrix} F'_y & -F'_x \\ G'_y & -G'_x \end{vmatrix} = \begin{vmatrix} F'_x & F'_y \\ G'_x & G'_y \end{vmatrix} \overset{\text{not}}{=} \frac{D(F,G)}{D(x,y)}.$$

Thus,

$$\begin{cases} f' = \dfrac{\frac{D(F,G)}{D(z,x)}}{\frac{D(F,G)}{D(y,z)}} \\ g' = \dfrac{\frac{D(F,G)}{D(x,y)}}{\frac{D(F,G)}{D(y,z)}} \end{cases} \tag{31}$$

As we saw above, for the curve (30) the equations of the tangent are

$$X - x_0 = \frac{Y - f(x_0)}{f'(x_0)} = \frac{Z - g(x_0)}{g'(x_0)}.$$

Using (31), this relation becomes

$$X - x_0 = \frac{Y - f(x_0)}{\frac{\frac{D(F,G)}{D(z,x)}}{\frac{D(F,G)}{D(y,z)}}} = \frac{Z - g(x_0)}{\frac{\frac{D(F,G)}{D(x,y)}}{\frac{D(F,G)}{D(y,z)}}},$$

from which, taking into account that $f(x_0) = y_0$ and $g(x_0) = z_0$, we get

$$\frac{X - x_0}{\frac{D(F,G)}{D(y,z)}} = \frac{Y - y_0}{\frac{D(F,G)}{D(z,x)}} = \frac{Z - z_0}{\frac{D(F,G)}{D(x,y)}}.$$

For a planar curve

$$F(x, y) = 0,$$

if, at a point (x_0, y_0) the condition $F'_y \neq 0$ is satisfied, then, from the implicit function theorem, at least locally, we can write $y = f(x)$, hence the equation of the curve can be written as

$$F(x, f(x)) = 0.$$

Differentiating this relation with respect to x, we obtain

$$F'_x + f' F'_y = 0 \Longrightarrow f' = -\frac{F'_x}{F'_y}.$$

Thus, from the equation of the tangent:

$$Y - y_0 = f'(x_0)(X - x_0),$$

we deduce that

$$Y - y_0 = -\frac{F'_x}{F'_y}(X - x_0)$$

or

$$(X - x_0)F'_x + (Y - y_0)F'_y = 0,$$

while for the normal we obtain the equation

$$(X - x_0)F'_y - (Y - y_0)F'_x = 0.$$

Osculating Plane

Definition

A parametrised curve $\mathbf{r} = \mathbf{r}(t)$ is called *biregular* (or in *general position*) at the point t_0 if the vectors $\mathbf{r}'(t_0)$ and $\mathbf{r}''(t_0)$ are not collinear, that is,

$$\mathbf{r}'(t_0) \times \mathbf{r}''(t_0) \neq 0.$$

The parametrised curve is called *biregular* if it is biregular at every point of its domain.

Remark

It is not difficult to verify that the notion of a biregular point is independent of the parametrisation: if a point is biregular for a given parametrised curve, then its corresponding point under any change of parameter is also a biregular point.

Osculating Plane

Definition

Let $(I, \mathbf{r})$ be a parametrised curve and $t_0 \in I$ a biregular point. The *osculating plane* of the curve at $\mathbf{r}(t_0)$ is the plane that passes through $\mathbf{r}(t_0)$ and is parallel to the vectors $\mathbf{r}'(t_0)$ and $\mathbf{r}''(t_0)$, i.e., the equation of the plane is

$$(R - \mathbf{r}(t_0), \mathbf{r}'(t_0), \mathbf{r}''(t_0)) = 0, \tag{32}$$

or, by expanding the mixed product,

$$\begin{vmatrix} X - x_0 & Y - y_0 & Z - z_0 \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0. \tag{33}$$

Osculating Plane

Theorem

The osculating planes of two equivalent parametrised curves at corresponding biregular points coincide.

Proof.

Let $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \boldsymbol{\rho} = \rho(s))$ be two equivalent parametrised curves and let $\lambda : I \to J$ be the change of parameter. Then

$$\begin{aligned}
\mathbf{r}(t) &= \rho(\lambda(t)),\\
\mathbf{r}'(t) &= \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t),\\
\mathbf{r}''(t) &= \boldsymbol{\rho}''(\lambda(t)) \cdot (\lambda'(t))^2 + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t).
\end{aligned}$$

Since $\lambda'(t) \neq 0$, from these relations it follows that the vector systems $\{\mathbf{r}'(t), \mathbf{r}''(t)\}$ and $\{\boldsymbol{\rho}'(\lambda(t)), \boldsymbol{\rho}''(\lambda(t))\}$ are equivalent, i.e., they generate the same vector subspace of $\mathbb{R}^3$. $\square$

Proof.

Thus, the osculating planes of the two curves at the corresponding (biregular) points t_0 and $\lambda(t_0)$ have the same direction subspace, therefore they are parallel. On the other hand, they have a common point (since $\mathbf{r}(t_0) = \rho(\lambda(t_0))$, hence they must coincide. □

Remark

From the previous theorem it follows that the notion of osculating plane also makes sense for regular curves.

Just as in the case of the tangent, there is a more geometric way to define the osculating plane, which is at the same time more general, as it can also be applied to the case of points that are not biregular.

Osculating Plane

Let $\mathbf{r}(t_0)$ and $\mathbf{r}(t_0 + \Delta t)$ be two neighbouring points on a parametrised curve, with $\mathbf{r}(t_0)$ biregular. We consider a plane α, with normal unit vector $\mathbf{e}$, which passes through $\mathbf{r}(t_0)$, and we denote by $d(\Delta t) = d(\mathbf{r}(t_0 + \Delta t), \alpha)$.

Theorem

α is the osculating plane to the parametrised curve $\mathbf{r} = \mathbf{r}(t)$ at the biregular point $\mathbf{r}(t_0)$ if and only if

$$\lim_{\Delta t \to 0} \frac{d(\Delta t)}{|\Delta t|^2} = 0,$$

i.e., the plane and the curve have second-order contact.

Osculating Plane

Remarks

a) The previous theorem justifies the name ”osculating plane”. In fact, the name (coined by Johann Bernoulli) comes from the Latin verb *osculare*, which means *to kiss*, and highlights the fact that, among all the planes passing through a given point of a curve, the osculating plane has the highest order of contact (“the closest”).

b) If we define the osculating plane via contact, we can also define the notion of osculating plane for points that are not biregular, but in this case, any plane that passes through the tangent is osculating in the sense that it has second-order contact with the curve.

c) To say that the osculating plane at a biregular point is the only plane that has second-order contact with the curve is the same as saying that the osculating plane is the limiting position of a plane determined by the considered point and two neighbouring points, as these points indefinitely approach the given one.

d) We can also define the osculating plane at a biregular point of a parametrised curve as the limiting position of a plane that passes through the tangent at the given point and a neighbouring point on the curve, as the neighbouring point indefinitely approaches the given one. A natural question we might ask is: what happens in the case of planar parametrised curves? The answer is given by the following proposition, whose proof is left to the reader:

Property

If a biregular parametrised curve is planar, i.e., its support lies in a plane π, then the osculating plane at every point of this curve coincides with the curve's plane. Conversely, if a biregular parametrised curve has the same osculating plane at every one of its points, then the curve is planar and its support lies in the osculating plane.

Curvature of a Curve

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a regular parametrised curve. Let $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$ be a naturally parametrised curve equivalent to it. Then $\|\boldsymbol{\rho}'(s)\| = 1$, while the vector $\boldsymbol{\rho}''(s)$ is perpendicular to $\boldsymbol{\rho}'(s)$.
It can be shown that $\boldsymbol{\rho}''(s)$ does not depend on the choice of the naturally parametrised curve equivalent to the given curve $\mathbf{r} = \mathbf{r}(t)$. Indeed, if $(J_1, \boldsymbol{\rho}_1 = \boldsymbol{\rho}_1(\tilde{s}))$ is another naturally parametrised curve equivalent to the given one, with change of parameter $\tilde{s} = \lambda(s)$, then from the condition

$$\|\boldsymbol{\rho}'(s)\| = \|\boldsymbol{\rho}_1'(\lambda(s))\| = 1,$$

we obtain $|\lambda'(s)| = 1$ for all $s \in J$. Hence, $\lambda' = \pm 1$ and therefore $\tilde{s} = \pm s + s_0$, where s_0 is a constant. It follows that

$$\boldsymbol{\rho}''(s) = \boldsymbol{\rho}_1''(\tilde{s}) \underbrace{(\lambda'(s))^2}_{=1} + \boldsymbol{\rho}'_1 \cdot \underbrace{\lambda''(s)}_{=0} = \boldsymbol{\rho}''_1(\tilde{s}).$$

Curvature of a Curve

Definition

The vector $\mathbf{k} = \rho''(s(t))$ is called the *curvature vector* of the parametrised curve $\mathbf{r} = \mathbf{r}(t)$ at the point t, and its norm $k(t) = \|\rho''(s(t))\|$ is called the *curvature* of the parametrised curve at the point t.

We now express the curvature vector $\mathbf{k}(t)$ in terms of $\mathbf{r}(t)$ and its derivatives. We choose the arc length as natural parameter. Then we have

$$\begin{aligned}\mathbf{r}(t) &= \rho(s(t)) \Rightarrow \\ \mathbf{r}'(t) &= \rho'(s(t)) \cdot s'(t) \\ \mathbf{r}''(t) &= \rho''(s(t)) \cdot (s'(t))^2 + \rho'(s(t)) \cdot s''(t),\end{aligned}$$

where

$$s'(t) = \|\mathbf{r}'(t)\|, \; s''(t) = \|\mathbf{r}'(t)\|' = \frac{d}{dt}\left(\sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)}\right) = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}.$$

Curvature of a Curve

Thus, we have

$$\mathbf{k}(t) = \rho''(s(t)) = \frac{\mathbf{r}''}{\|\mathbf{r}'\|^2} - \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|^4} \cdot \mathbf{r}'. \tag{34}$$

Now, since the vectors ρ' and ρ'' are perpendicular, and ρ' has unit length, we have

$$k(t) = \|\mathbf{k}(t)\| = \|\rho''\| = \|\rho' \times \rho''\|.$$

Substituting $\rho' = \dfrac{\mathbf{r}'}{\|\rho'\|}$ and ρ'' using formula (34), we obtain

$$k(t) = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}. \tag{35}$$

Remarks

1. From formula (35) it follows that a parametrised curve $\mathbf{r} = \mathbf{r}(t)$ is biregular at a point t_0 if and only if $k(t_0) \neq 0$.
2. Since for 2 parametrised curves the naturally parametrised curves equivalent to them are equivalent among themselves, it follows that the notion of curvature also makes sense for regular curves.

Exemple

1. For the straight line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}$, the curvature vector (and hence also the curvature) vanishes identically.

Exemple

2. For the circle $S^1_R = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = R^2,\ z = 0\}$, we choose the parametrisation

$$\begin{cases} x = R\cos t \\ y = R\sin t \qquad\qquad 0 < t < 2\pi. \\ z = 0 \end{cases}$$

Then

$$\mathbf{r}'(t) = \{-R\sin t, R\cos t, 0\}, \quad \mathbf{r}''(t) = \{-R\cos t, R\sin t, 0\}$$

and hence $\|\mathbf{r}'(t)\| = R$, $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$.

Curvature of a Curve

Therefore, for the curvature vector we obtain

$$\mathbf{k}(t) = \left\{-\frac{1}{R}\cos t, -\frac{1}{R}\sin t, 0\right\} = -\frac{1}{R}\{x, y, z\}, \quad k(t) = \frac{1}{R}.$$

Remarks

1. The calculations we have done above explain why the reciprocal of the curvature of a curve is called the *radius of curvature* of the curve.
2. We have seen that the curvature of a straight line is identically zero. The converse of this statement is also true, at least in a certain sense, as shown by the following proposition.

Property

If the curvature of a regular parametrised curve is identically zero, then the support of the curve lies on a straight line.

Curvature of a Curve

Proof.

We assume, from the very beginning, that we are dealing with a naturally parametrised curve $(I, \rho = \rho(s))$. From the hypothesis, $\rho''(s) = 0$, hence $\rho'(s) = \mathbf{a} = \text{const}$, $\rho(s) = \rho_0 + s\mathbf{a}$, i.e., the support $\rho(I)$ lies on a straight line. Since two equivalent parametrised curves have the same support, the result remains valid also for curves that are not naturally parametrised. $\square$

Remark

The fact that a parametrised curve has zero curvature means only that the *support* of the curve lies on a straight line, but it does not necessarily mean that the parametrised curve is an affine map from $\mathbb{R}$ to $\mathbb{R}^3$ or a restriction of such a map, nor that the curve is equivalent to such a particular parametrised curve.

Curvature of a Curve

We can consider, as before, the parametrised curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$, $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}t^3$, where $\mathbf{a}$ is a non-zero constant vector in $\mathbb{R}^3$. Then we have, immediately, that $\mathbf{r}'(t) = 2\mathbf{a}t^2$ and $\mathbf{r}''(t) = 3\mathbf{a}t$, i.e., the velocity and the acceleration of the curve are parallel, i.e., the curve has zero curvature, but, as we have already seen, this parametrised curve is not equivalent to a curve with an affine parametrisation.

Let us consider a naturally parametrised curve $(I, \mathbf{r} = \mathbf{r}(s))$. We denote by $\Delta\varphi(s)$ the measure of the angle formed by the unit vectors $\mathbf{r}'(s)$ and $\mathbf{r}'(s + \Delta s)$. Then

$$\|\mathbf{r}'(s+\Delta s) - \mathbf{r}'(s)\| = 2\left|\sin\frac{\Delta\varphi(s)}{2}\right|.$$

Therefore,

$$k(s) = \|\mathbf{r}''(s)\| = \left\|\lim_{\Delta s\to 0}\frac{\mathbf{r}'(s+\Delta s) - \mathbf{r}'(s)}{\Delta s}\right\| = \lim_{\Delta s\to 0}\frac{2\left|\sin\frac{\Delta\varphi(s)}{2}\right|}{|\Delta s|} =$$

$$= \lim_{\Delta s\to 0}\left|\frac{\Delta\varphi(s)}{\Delta s}\right|\cdot\frac{\left|\sin\frac{\Delta\varphi(s)}{2}\right|}{\left|\frac{\Delta\varphi(s)}{2}\right|} = \lim_{\Delta s\to 0}\left|\frac{\Delta\varphi(s)}{\Delta s}\right| = \left|\frac{d\varphi}{ds}\right|.$$

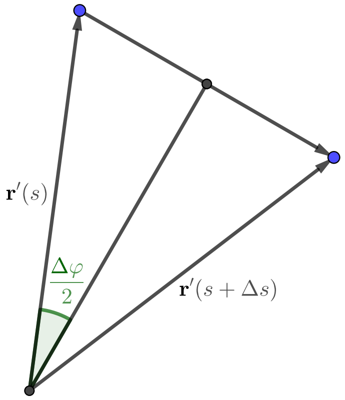


Figure: Geometric meaning of curvature

Thus, taking into account that $\Delta\varphi(s)$ is the measure of the angle between the tangents to the curve at s and $s + \Delta s$, the last formula gives us:

Property

The curvature of a parametrised curve is the modulus of the rate of rotation of the tangent to the curve, when the point of tangency moves along the curve with unit speed.

Frenet Frame of a Parametrised Curve

In every point of a biregular parametrised curve $(I, \mathbf{r} = \mathbf{r}(t))$ we can construct an affine frame of the space $\mathbb{R}^3$. The idea is that, if we want to investigate the local properties of a parametrised curve around a given point on the curve, then it is much easier to do so if we do not use the standard coordinate system of $\mathbb{R}^3$, but rather a coordinate system with the origin at a given point of the curve, while the coordinate axes have certain relations with the local properties of the curve. Such a coordinate system was constructed, independently, in the mid-19th century, by the French mathematicians Frenet and Serret.

Frenet Frame of a Parametrised Curve

Definition

The *Frenet frame* (or *moving frame*) of a biregular parametrised curve $(I, \mathbf{r} = \mathbf{r}(t))$ at the point $t_0 \in I$ is an orthonormal frame of the space $\mathbb{R}^3$, with the origin at the point $\mathbf{r}(t_0)$, the unit coordinate vectors being the vectors $\{\boldsymbol{\tau}(t_0), \boldsymbol{\nu}(t_0), \boldsymbol{\beta}(t_0)\}$, where:

- $\boldsymbol{\tau}(t_0)$ is the unit tangent vector to the curve at t_0, that is,

$$\boldsymbol{\tau}(t_0) = \frac{\mathbf{r}'(t_0)}{\|\mathbf{r}'(t_0)\|};$$

- $\boldsymbol{\nu}(t_0) = \mathbf{k}(t_0)/k(t_0)$ is the unit vector of the curvature vector:

$$\boldsymbol{\nu}(t_0) = \frac{\mathbf{k}(t_0)}{k(t_0)}$$

  and is called the *unit principal normal* at the point t_0;

Definition

- $\boldsymbol{\beta}(t_0) = \boldsymbol{\tau}(t_0) \times \boldsymbol{\nu}(t_0)$ is called the *unit binormal* at the point t_0.

- The direction axis $\boldsymbol{\tau}(t_0)$ is, evidently, the *tangent* to the curve at t_0.
- The direction axis $\boldsymbol{\nu}(t_0)$ is called the *principal normal*. In fact, this line lies in the normal plane (as it is perpendicular to the tangent), but it is also contained in the osculating plane. Thus, *the principal normal is the normal contained in the osculating plane*.
- The direction axis $\boldsymbol{\beta}(t_0)$ is called the *binormal*. *The binormal is the normal perpendicular to the osculating plane*.
- The plane determined by the vectors $\{\boldsymbol{\tau}(t_0), \boldsymbol{\nu}(t_0)\}$ is the osculating plane (*O*.*P*. in the figure).
- The plane determined by the vectors $\{\boldsymbol{\nu}(t_0), \boldsymbol{\beta}(t_0)\}$ is the normal plane (*N*.*P*. in the figure).

Frenet Frame of a Parametrised Curve

- The plane determined by the vectors $\{\boldsymbol{\tau}(t_0), \boldsymbol{\beta}(t_0)\}$ is called the *rectifying plane*, for reasons that will become clear later (*R.P.* in the figure).

For a naturally parametrised curve $(J, \boldsymbol{\rho} = \boldsymbol{\rho}(s))$, the expressions for the vectors of the Frenet trihedron are quite simple:

$$\begin{cases} \boldsymbol{\tau}(s) & = \boldsymbol{\rho}'(s) \\ \boldsymbol{\nu}(s) & = \dfrac{\boldsymbol{\rho}''(s)}{\|\boldsymbol{\rho}''(s)\|} \\ \boldsymbol{\beta}(s) & \equiv \boldsymbol{\tau}(s) \times \boldsymbol{\nu}(s) = \dfrac{\boldsymbol{\rho}'(s) \times \boldsymbol{\rho}''(s)}{\|\boldsymbol{\rho}''(s)\|} \end{cases}. \tag{36}$$

Frenet Frame of a Parametrised Curve

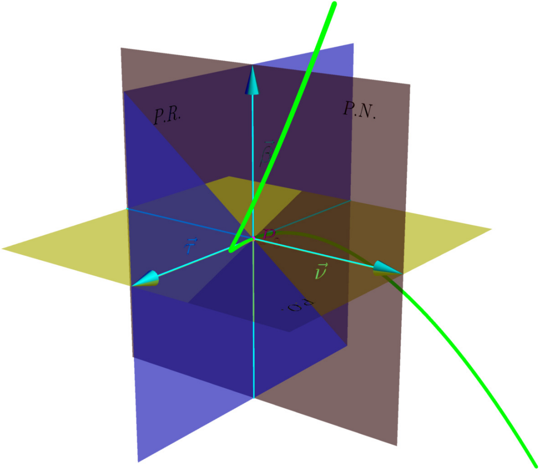


Figure: Frenet trihedron of the curve $\mathbf{r}(t) = (t, t^2, t^3)$ at $t = 0$.

Frenet Frame of a Parametrised Curve

For a general biregular parametrised curve $(I, \mathbf{r} = \mathbf{r}(t))$ the situation is somewhat more complicated. Thus, obviously, from the definition, at any point $t \in I$ we have

$$\boldsymbol{\tau}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}. \tag{37}$$

Therefore, since

$$\mathbf{k}(t) = \frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^2} - \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^4} \cdot \mathbf{r}'(t)$$

and

$$k(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3},$$

we obtain

$$\boldsymbol{\nu}(t) \equiv \frac{\mathbf{k}(t)}{k(t)} = \frac{\|\mathbf{r}'(t)\|}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \cdot \mathbf{r}''(t) - \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\| \cdot \|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \cdot \mathbf{r}'(t), \tag{38}$$

while

$$\beta(t) \equiv \boldsymbol{\tau}(t) \times \boldsymbol{\nu}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}. \tag{39}$$

Remark

The above calculations show that, in practice, for a general parametrised curve $(I, \mathbf{r} = \mathbf{r}(t))$ it is simpler to calculate $\boldsymbol{\tau}$ and β directly, and then to compute $\boldsymbol{\nu}$ using the formula

$$\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}.$$

A notion defined for a parametrised curve also makes sense for a regular curve if and only if it is invariant under a change of parameter, in other words, if it does not change when we replace the parametrised curve with another equivalent parametrised curve. The Frenet frame is “almost” invariant, that is, we have:

Theorem

Let $(I, \mathbf{r} = \mathbf{r}(t))$ *and* $\rho = \rho(u)$ *be two equivalent parametrised curves, with change of parameter* $\lambda : I \to J$*,* $u = \lambda(t)$*. Then, at the corresponding points* t *and* $u = \lambda(t)$*, their Frenet frames coincide if* $\lambda'(t) > 0$*. If* $\lambda'(t) < 0$*, then the origins and the principal normal unit vectors coincide, while the other two pairs of unit vectors have the same directions but opposite orientations.*

Proof.

Since $\mathbf{r}(t) = \boldsymbol{\rho}(\lambda(t))$, the origins of the Frenet frames always coincide. As we have seen above, the curvature vectors of two equivalent parametrised curves coincide, and thus the same is true for the principal normal unit vectors. From $\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t)$ it follows that the Frenet frames coincide for $\lambda'(t) > 0$. If $\lambda'(t) < 0$, then the tangent vectors $\boldsymbol{\rho}'(u)$ and $\mathbf{r}'(t)$ have opposite orientations, and the same is true for their unit vectors. From $\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$ it follows that, in this case, the binormal unit vectors also have opposite orientations, although their directions coincide. □

Remark

From the above, it follows that we can speak about the Frenet frame of an *oriented* curve, i.e., a curve for which we have chosen a family of local parametrisations that are positively equivalent to each other. The choice of such a family (an *orientation*) actually means the choice of a direction of traversal.

Frenet Formulas. Torsion

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a biregular parametrised curve. Then the vectors $\boldsymbol{\tau}(t)$, $\boldsymbol{\nu}(t)$, $\boldsymbol{\beta}(t)$ are in fact smooth vector functions with respect to the parameter t. We want to find their derivatives with respect to t, more precisely, the decompositions of these derivatives with respect to the vectors $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$. From the definition, we have

$$\boldsymbol{\tau} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} = \frac{\mathbf{r}'}{\sqrt{\mathbf{r}'^2}}.$$

Therefore,

$$\boldsymbol{\tau}' = \frac{\mathbf{r}'' \cdot \|\mathbf{r}'\| - \mathbf{r}' \cdot \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|}}{\|\mathbf{r}'\|^2} = \frac{\mathbf{r}'' \cdot \|\mathbf{r}'\|^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')}{\|\mathbf{r}'\|^3} =$$

$$= \|\mathbf{r}'\| \cdot \underbrace{\frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}}_{k} \left[\underbrace{\frac{\|\mathbf{r}'\|}{\|\mathbf{r}' \times \mathbf{r}''\|} \cdot \mathbf{r}'' - \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\| \cdot \|\mathbf{r}' \times \mathbf{r}''\|} \cdot \mathbf{r}'}_{\boldsymbol{\nu}} \right].$$

Frenet Formulas. Torsion

Thus,

$$\boldsymbol{\tau}' = \|\mathbf{r}'\| k \cdot \boldsymbol{\nu}. \tag{40}$$

Furthermore,

$$\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu} \ \Rightarrow \ \boldsymbol{\beta}' = \boldsymbol{\tau}' \times + \boldsymbol{\tau} \times \boldsymbol{\nu}' = k(\underbrace{\boldsymbol{\nu} \times \boldsymbol{\nu}}_{=0}) + \boldsymbol{\tau} \times \boldsymbol{\nu}' \ \Rightarrow \boldsymbol{\beta}' = \boldsymbol{\tau} \times \boldsymbol{\nu}',$$

hence $\boldsymbol{\beta}' \perp \boldsymbol{\tau}$. On the other hand,

$$\boldsymbol{\beta} \cdot \boldsymbol{\beta} = 1 \ \Rightarrow \ \boldsymbol{\beta}' \cdot \boldsymbol{\beta} = 0 \ \Rightarrow \ \boldsymbol{\beta}' \perp \boldsymbol{\beta}.$$

Thus, the vector $\boldsymbol{\beta}'$ is collinear with the vector $\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}$ and we can write

$$\boldsymbol{\beta}' = -\|\mathbf{r}'\| \cdot \chi \boldsymbol{\nu},$$

where χ is a proportionality factor, whose significance will be clarified later.

Frenet Formulas. Torsion

We now differentiate the equality

$$\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}.$$

We have

$$\boldsymbol{\nu}' = \boldsymbol{\beta}' \times \boldsymbol{\tau} + \beta \times \boldsymbol{\tau}' = -\|\mathbf{r}'\| \cdot \chi(\boldsymbol{\nu} \times \boldsymbol{\tau}) + \|\mathbf{r}'\| \cdot k(\boldsymbol{\beta} \times \boldsymbol{\nu}),$$

hence,

$$\boldsymbol{\nu}' = \|\mathbf{r}'\|(-k\boldsymbol{\tau} + \chi\boldsymbol{\beta}). \tag{41}$$

Thus, we obtain the equations

$$\begin{cases} \boldsymbol{\tau}'(t) = \|\mathbf{r}'\| k(t)\boldsymbol{\nu}(t) \\ \boldsymbol{\nu}'(t) = \|\mathbf{r}'\|(-k(t)\boldsymbol{\tau}(t) + \chi(t)\boldsymbol{\beta}(t)) \\ \boldsymbol{\beta}'(t) = -\|\mathbf{r}'\|\chi(t)\boldsymbol{\nu}(t). \end{cases} \tag{42}$$

These equations are called the *Frenet formulas* for the parametrised curve $\mathbf{r} = \mathbf{r}(t)$.

If we are dealing with a naturally parametrised curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$, then the Frenet equations are slightly simpler:

$$\begin{cases} \boldsymbol{\tau}'(t) = k(t)\boldsymbol{\nu}(t) \\ \boldsymbol{\nu}'(t) = -k(t)\boldsymbol{\tau}(t) + \chi(t)\boldsymbol{\beta}(t) \\ \boldsymbol{\beta}'(t) = -\chi(t)\boldsymbol{\nu}(t). \end{cases} \tag{42'}$$

Definition

The quantity $\chi(t)$ is called the *torsion* (or *second curvature*) of the biregular parametrised curve $\mathbf{r} = \mathbf{r}(t)$ at the point t.

Frenet Formulas. Torsion

We will first compute the torsion for a naturally parametrised curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$. For such a curve, the unit vectors of the Frenet trihedron are given by the expressions:

$$\begin{cases} \boldsymbol{\tau} = \boldsymbol{\rho}' \\ \boldsymbol{\nu} = \frac{1}{k}\boldsymbol{\rho}'' \\ \boldsymbol{\beta} = \frac{1}{k}\boldsymbol{\rho}' \times \boldsymbol{\rho}''. \end{cases}$$

From the third Frenet formula, we have:

$$\boldsymbol{\beta}' \cdot \boldsymbol{\nu} = -\chi(s) \cdot (\underbrace{\boldsymbol{\nu} \times \boldsymbol{\nu}}_{=1}) = -\chi(t).$$

But, on the other hand, from the definition,

$$\boldsymbol{\beta}' = \left(\frac{1}{k}\right)' + \frac{1}{k}\underbrace{\boldsymbol{\rho}'' \times \boldsymbol{\rho}''}_{=0} + \frac{1}{k}\boldsymbol{\rho}' \times \boldsymbol{\rho}'''.$$

Frenet Formulas. Torsion

Thus,

$$\chi = -\beta' \cdot \nu = -\underbrace{\left(\frac{1}{k}\right)' (\rho' \times \rho'') \cdot \frac{1}{k}\rho''}_{=0} - \frac{1}{k}(\rho' \times \rho''') \cdot \frac{1}{k}\rho'',$$

hence,

$$\chi = \frac{1}{k^2}\left(\rho', \rho'', \rho'''\right). \tag{43}$$

The following theorem gives us a way to compute the torsion of any biregular parametrised curve:

Theorem

If $(I, \mathbf{r} = \mathbf{r}(t))$ and $(J, \rho = \rho(u))$ are two positively equivalent biregular parametrised curves, with parameter change $\lambda : I \to J$, $\lambda' > 0$, then they have the same torsion at the corresponding points t and $u = \lambda(t)$.

Frenet Formulas. Torsion

Proof.

Let $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$ and $\{\boldsymbol{\tau_1}, \boldsymbol{\nu_1}, \boldsymbol{\beta_1}\}$ be the Frenet frames of the two parametrised curves at the corresponding points t and $u = \lambda(t)$, and χ and χ_1 their torsions at these points. Then

$$\boldsymbol{\beta_1}(\lambda(t)) = \boldsymbol{\beta}(t)$$
$$\boldsymbol{\nu_1}(\lambda(t)) = \boldsymbol{\nu}(t),$$

$$\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t) \ \Rightarrow\ \mathbf{r}'(t) = \frac{d}{du}(\boldsymbol{\beta_1}(u))\lambda'(t).$$

From the last Frenet equation for the curve $\mathbf{r}$, we obtain

$$\boldsymbol{\beta}'(t) \cdot \boldsymbol{\nu}(t) = -\|\mathbf{r}'(t)\| \cdot \chi(t).$$

□

Proof.

Therefore,

$$\chi(t) = -\frac{1}{\|\mathbf{r}'(t)\|} \cdot \boldsymbol{\beta}'(t) \cdot \boldsymbol{\nu}(t) =$$
$$= -\frac{1}{\|\boldsymbol{\rho}'(\lambda(t))\| \cdot \lambda'(t)} \cdot \boldsymbol{\beta}_{\mathbf{1}}{}'(\lambda(t)) \cdot \lambda'(t) \cdot \boldsymbol{\nu}_{\mathbf{1}}(\lambda(t)) =$$
$$= -\frac{1}{\|\boldsymbol{\rho}'(\lambda(t))\|} \cdot (-\|\boldsymbol{\rho}'(\lambda(t))\| \cdot \chi_1(\lambda(t))) = \chi_1(\lambda(t)),$$

where we have once again used the last Frenet equation, but this time for the curve $\boldsymbol{\rho}$, as well as the fact that the vector $\boldsymbol{\nu}_{\mathbf{1}}(\lambda(t))$ has unit length. □

Frenet Formulas. Torsion

Let now $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ be a naturally parametrised curve, positively equivalent to the parametrised curve $\mathbf{r} = \mathbf{r}(t)$, where $s = \lambda(t)$ is the change of parameter. Then $\mathbf{r}$ and its derivatives up to third order can be expressed in terms of $\boldsymbol{\rho}$ and its derivatives as follows:

$$\begin{aligned}
\mathbf{r}(t) &= \boldsymbol{\rho}(\lambda(t)) \\
\mathbf{r}'(t) &= \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t) \\
\mathbf{r}''(t) &= \boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'^2(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t) \\
\mathbf{r}'''(t) &= \boldsymbol{\rho}'''(\lambda(t)) \cdot \lambda'^3(t) + 3\boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'(t) \cdot \lambda''(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'''(t).
\end{aligned}$$

Frenet Formulas. Torsion

Thus, the mixed product of the first three derivatives of **r** becomes

$$\left(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)\right) = \Big(\boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'(t), \boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'^2(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda''(t),$$

$$\boldsymbol{\rho}'''(\lambda(t)) \cdot \lambda'^3(t) + 3\boldsymbol{\rho}''(\lambda(t)) \cdot \lambda'(t) \cdot \lambda''(t) + \boldsymbol{\rho}'(\lambda(t)) \cdot \lambda'''(t)\Big) =$$

$$= \lambda'^6(t) \left(\boldsymbol{\rho}'(\lambda(t)), \boldsymbol{\rho}''(\lambda(t)), \boldsymbol{\rho}'''(\lambda(t))\right).$$

All other mixed products in the second member vanish, since two of the factors are collinear. Thus

$$\left(\boldsymbol{\rho}'(\lambda(t)), \boldsymbol{\rho}''(\lambda(t)), \boldsymbol{\rho}'''(\lambda(t))\right) = \frac{1}{\lambda'^6(t)} \left(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)\right).$$

Frenet Formulas. Torsion

But, since ρ is naturally parametrised and the two curves are positively equivalent, we have $\lambda' = \|\mathbf{r}'\|$, hence the previous formula becomes

$$(\rho', \rho'', \rho''') = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{\|\mathbf{r}'\|^6}.$$

Moreover (see (35)), the curvature can be expressed in terms of the derivatives of **r** by the formula

$$k = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3},$$

therefore, from the expression of the torsion (43) and the previous relation, we obtain

$$\chi(t) = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{\|\mathbf{r}' \times \mathbf{r}''\|^2}. \tag{44}$$

Frenet Formulas. Torsion

Geometric Meaning of Torsion

Torsion is, in a certain sense, analogous to curvature (this is why, in older books, it is called the *second curvature*). What we mean is that torsion too can be interpreted as the rate of rotation of a line, this time referring to the binormal. In other words, we have

Property

If $(I, \mathbf{r} = \mathbf{r}(s))$ *is a naturally parametrised curve and* $\Delta\alpha$ *is the angle between the osculating planes of the curve at* $\mathbf{r}(s)$ *and* $\mathbf{r}(s + \Delta s)$ *(in other words, the angle between the binormals of the curve at those points), then we have*

$$\chi(s) = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s}.$$

We note that, this time, unlike the case of curvature, torsion is the *algebraic value* of the limit, not the absolute value. However, we must mention that the curvature of a space curve is *defined* to be positive, since no geometric meaning for the sign of curvature could be found. As we shall see below, for planar curves we can define a *signed curvature* whose absolute value equals the curvature and which will help us obtain additional information about the curve.
As we have already said, torsion is analogous to curvature. Thus, curvature measures the deviation of the curve from a straight line. On the other hand, torsion measures the deviation of the curve from a planar curve.

More precisely, we have

Theorem

The support of a biregular parametrised curve lies in a plane if and only if the torsion of the curve is identically zero.

Proof.

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a biregular parametrised curve such that $\mathbf{r}(I) \subset \pi$, where π is a plane. Then, obviously, the vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are parallel to this plane which, as we know, is the osculating plane of the curve. Therefore, $\beta(t) = \text{const}$, so we have

$$0 = \beta'(t) = -\underbrace{\|\mathbf{r}'\|}_{\neq 0} \cdot \chi(t) \cdot \underbrace{\boldsymbol{\nu}(t)}_{\neq 0} \Rightarrow \chi(t) \equiv 0.$$

$\square$

Proof.

Conversely, if $\chi(t) \equiv 0$, then the binormal unit vector $\beta(t)$ is always equal to a constant vector $\boldsymbol{\beta_0}$. But $\beta(t) = \mathbf{r}'(t) \times \mathbf{r}''(t)$, therefore the vector $\mathbf{r}'(t)$ is always perpendicular to the constant vector $\boldsymbol{\beta_0}$. Thus, we have the chain of implications

$$\mathbf{r}'(t) \cdot \boldsymbol{\beta_0} = (\mathbf{r} \cdot \boldsymbol{\beta_0})' = 0 \ \Rightarrow \ \mathbf{r} \cdot \boldsymbol{\beta_0} = const = \mathbf{r_0} \cdot \boldsymbol{\beta_0} \ \Rightarrow \ (\mathbf{r} - \mathbf{r_0}) \times \boldsymbol{\beta_0} = 0,$$

meaning that the support $\mathbf{r}(I)$ of the curve lies in a plane perpendicular to the constant vector $\boldsymbol{\beta_0}$ and passing through the point $\mathbf{r}_0$. □

We have seen that the curvature of a parametrised curve vanishes identically if and only if the support of the curve lies on a straight line. On the other hand, we know that the curvature of a circle is constant and equals the inverse of the radius of the circle. We might expect the converse to also be true — in other words, if the curvature of a parametrised curve is constant, then the support of the curve lies on a circle. Unfortunately, this statement is not true. It suffices to think of the circular cylindrical helix which has constant curvature (and also constant torsion). We do, however, have the following weaker result:

Property

If $(I, \mathbf{r} = \mathbf{r}(s))$ *is a naturally parametrised curve, with curvature* k *equal to a strictly positive constant* k_0*, while its torsion is identically zero, then the support of the curve lies on a circle of radius* $1/k_0$*.*

Proof.

Since the torsion is identically zero, the curve is planar. We consider the parametrised curve $(I, \mathbf{r}_1 = \mathbf{r}_1(s))$, where

$$\mathbf{r}_1 = \mathbf{r} + \frac{1}{k_0}\boldsymbol{\nu}. \tag{*}$$

Differentiating with respect to s and using the second Frenet formula for $\mathbf{r}$, we obtain

$$\mathbf{r}_1^{'} = \mathbf{r}' + \frac{1}{k_0}\boldsymbol{\nu}' = \boldsymbol{\tau} + \frac{1}{k_0}(-k_0\boldsymbol{\tau}) = \boldsymbol{\tau} - \boldsymbol{\tau} = 0.$$

Thus, the curve $\mathbf{r}_1$ reduces to a point, for example $\mathbf{r}_1(s) \equiv \mathbf{c} = \text{const}$. □

Proof.

But from (*) we obtain

$$\|\mathbf{r} - c\| = \left\| \frac{1}{k_0}\boldsymbol{\nu} \cdot \boldsymbol{\nu} \right\| = \frac{1}{k_0},$$

which means that every point of the support of the curve **r** lies at a (constant) distance $1/k_0$ from the fixed point **c**, that is, the support $\mathbf{r}(I)$ lies on a circle of radius $1/k_0$, centred at **c**. □

Another interesting situation is when the support of the curve lies not in a plane, but on a sphere. In this case, we have

Property

If a naturally parametrised curve $(I, \mathbf{r} = \mathbf{r}(s))$ *has its support on a sphere centred at the origin and of radius a, then the curvature of the curve satisfies the inequality*

$$k \geq \frac{1}{a}.$$

Proof.

The distance from a point of the curve to the origin is equal to $\|\mathbf{r}\|$, i.e., we have $\mathbf{r}^2 = a^2$. Differentiating this equality, we obtain $\mathbf{r} \cdot \mathbf{r}' = 0$ or $\mathbf{r} \cdot \boldsymbol{\tau} = 0$. □

Proof.

If we differentiate once more, we obtain

$$\mathbf{r}' \cdot \boldsymbol{\tau} + \mathbf{r} \cdot \boldsymbol{\tau}' = 0,$$

or

$$1 + \mathbf{r} \cdot \boldsymbol{\tau}' = 0 \quad \Longleftrightarrow \quad 1 + k\mathbf{r} \cdot \boldsymbol{\nu} = 0 \quad \Longrightarrow k\mathbf{r} \cdot \boldsymbol{\nu} = -1.$$

From the properties of the scalar product, we have

$$|\mathbf{r} \cdot \boldsymbol{\nu}| \leq \|\mathbf{r}\| \cdot \|\boldsymbol{\nu}\| = \|\mathbf{r}\| = a,$$

therefore,

$$k = |k| = \frac{1}{|\mathbf{r} \cdot \boldsymbol{\nu}|} \geq \frac{1}{\|\mathbf{r}\| \cdot \|\boldsymbol{\nu}\|} = \frac{1}{a}.$$

□

In Figure 7 we provide an example of a curve lying on a sphere. It is the so-called *spherical helix*, as it is both a general helix (see the following section) and a spherical curve.

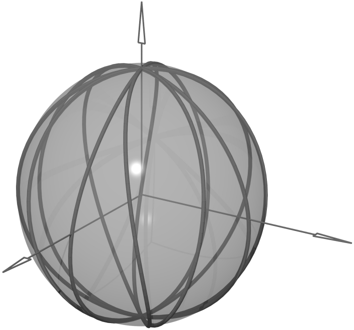


Figure: A spherical helix

Definition

A parametrised curve $(I, \mathbf{r})$ is called a *general helix* if its tangents make a constant angle with a fixed direction in space.

The following theorem was formulated in 1802 by the French mathematician Paul Lancret, but the first known proof is due to another famous French mathematician, A. de Saint Venant (1845).

Theorem (Lancret, 1802)

A space curve with curvature $k > 0$ is a general helix if and only if the ratio between its torsion and curvature is constant.

Proof.

Let us first suppose that we are dealing with a curve parametrised by arc length. To prove the first implication, suppose that **r** is a general helix and let **c** be the unit vector of the fixed direction:

$$\boldsymbol{\tau} \cdot \mathbf{c} = \cos \alpha_0 = \mathit{const}.$$

Differentiating the above relation, we obtain

$$\boldsymbol{\tau}' \cdot \mathbf{c} = 0,$$

hence

$$k \cdot \mathbf{c} \cdot \boldsymbol{\nu} = 0.$$

□

Proof.

Since by hypothesis $k > 0$, it follows that

$$\mathbf{c} \cdot \boldsymbol{\nu} = 0,$$

that is, at each point of the curve, $\mathbf{c} \perp \boldsymbol{\nu}$. This means that $\mathbf{c}$ lies in the rectifying plane and therefore,

$$\boldsymbol{\beta} \cdot \mathbf{c} = \sin \alpha_0.$$

Differentiating the relation $\boldsymbol{\nu} \cdot \mathbf{c} = 0$, we obtain, noting that $\mathbf{c}$ is constant and using the second Frenet formula:

$$(-k\boldsymbol{\tau} + \chi\boldsymbol{\beta}) \cdot \mathbf{c} = 0.$$

□

Proof.

We are thus led to

$$-k \cdot \cos \alpha_0 + \chi \cdot \sin \alpha_0 = 0,$$

that is,

$$\frac{\chi}{k} = \cot \alpha_0 = const.$$

Conversely, suppose that

$$\frac{\chi(s)}{k(s)} = c_0 = const,$$

or

$$c_0 \cdot k - \chi = 0.$$

Proof.

On the other hand, from the first and third Frenet formulas, we obtain

$$(c_0 \cdot k - \chi)\boldsymbol{\nu} = c_0\boldsymbol{\tau}' + \boldsymbol{\beta}' = 0.$$

Integrating once, we obtain

$$c_0\boldsymbol{\tau} + \boldsymbol{\beta} = \mathbf{c}^*,$$

where $\mathbf{c}^* \neq 0$ is a constant vector. □

Proof.

Define

$$\mathbf{c} := \frac{\mathbf{c}^*}{\|\mathbf{c}^*\|} = \frac{c_0\boldsymbol{\tau} + \boldsymbol{\beta}}{\|c_0\boldsymbol{\tau} + \boldsymbol{\beta}\|} = \frac{c_0\boldsymbol{\tau} + \boldsymbol{\beta}}{\sqrt{1 + c_0^2}},$$

while

$$\mathbf{c} \cdot \boldsymbol{\tau} = \frac{c_0}{\sqrt{1 + c_0^2}} = const \leq 1.$$

Therefore, the vectors **c** and $\boldsymbol{\tau}$ make a constant angle, and the curve is a general helix. □

We conclude this paragraph by noting, as a historical curiosity, that although this theorem is attributed to Lancret in most books, in reality he (like Saint-Venant) proved only one implication: on a general helix, the ratio between curvature and torsion is constant. The converse implication was stated and proved only later, by Joseph Bertrand (see, for example, Eisenhart's book [**?**]).

Local Behaviour of a Parametrised Curve Around a Biregular Point

Let $(I, \mathbf{r})$ be a naturally parametrised curve. We shall assume that $0 \in I$, as an interior point, and that $M_0 \equiv \mathbf{r}(0)$ is a biregular point of the curve. We shall use the following notation: $\boldsymbol{\tau}_0 = \boldsymbol{\tau}(0)$, $\boldsymbol{\nu}_0 = \boldsymbol{\nu}(0)$, $\boldsymbol{\beta}_0 = \boldsymbol{\beta}(0)$.
We expand $\mathbf{r}$ in a Taylor series around the origin. If we stop at third order in s, we obtain

$$\mathbf{r}(s) = \mathbf{r}(0) + s\mathbf{r}'(0) + \frac{1}{2}s^2\mathbf{r}''(0) + \frac{1}{6}s^3\mathbf{r}'''(0) + o(s^3). \tag{45}$$

We want to express the derivatives of $\mathbf{r}$ in terms of the Frenet vectors $\boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0$. Since $\mathbf{r}$ is naturally parametrised, we have

$$\mathbf{r}'(0) = \boldsymbol{\tau}_0. \tag{46}$$

On the other hand, from the definition of the curvature vector we have

$$\mathbf{r}''(0) = \mathbf{k}(0) = k(0) \cdot \boldsymbol{\nu}_0. \tag{47}$$

Local Behaviour of a Parametrised Curve Around a Biregular Point

Furthermore, if we differentiate the relation

$$\mathbf{r}''(s) = \mathbf{k}(s) = k(s) \cdot \boldsymbol{\nu} \tag{48}$$

we obtain

$$\begin{aligned}\mathbf{r}'''(s) &= k'(s)\boldsymbol{\nu} + k(s)\boldsymbol{\nu}' = k'(s)\boldsymbol{\nu} + k(s)(-k(s)\boldsymbol{\tau} + \chi(s)\boldsymbol{\beta}) = \\ &= -k^2(s)\boldsymbol{\tau} + k'(s)\boldsymbol{\nu} + k(s)\chi(s)\boldsymbol{\beta},\end{aligned} \tag{49}$$

where we have used the second Frenet equation.
Substituting $s = 0$ into (49), we obtain

$$\mathbf{r}'''(0) = -k^2(0)\boldsymbol{\tau}_0 + k'(0)\boldsymbol{\nu}_0 + k(0)\chi(0)\boldsymbol{\beta}_0. \tag{50}$$

Local Behaviour of a Parametrised Curve Around a Biregular Point

Thus, relation (45) becomes

$$\begin{aligned}\mathbf{r}(s)-\mathbf{r}(0) &= s\boldsymbol{\tau}_0+\frac{1}{2}s^2k(0)\boldsymbol{\nu}_0+\\ &+\frac{1}{6}s^3\left(-k^2(0)\boldsymbol{\tau}_0+k'(0)\boldsymbol{\nu}_0+k(0)\chi(0)\boldsymbol{\beta}_0\right)+o(s^3)\end{aligned} \tag{51}$$

or

$$\begin{aligned}\mathbf{r}(s)-\mathbf{r}(0) &= \left(s-\frac{1}{6}k^2(0)s^3+o(s^3)\right)\boldsymbol{\tau}_0+\\ &+\left(\frac{1}{2}k(0)s^2+\frac{1}{6}k'(0)s^3+o(s^3)\right)\boldsymbol{\nu}_0+\\ &+\left(\frac{1}{6}k(0)\chi(0)s^3+o(s^3)\right)\boldsymbol{\beta}_0.\end{aligned} \tag{52}$$

Local Behaviour of a Parametrised Curve Around a Biregular Point

We now consider a coordinate frame with origin at M_0 and axes given by the Frenet frame at this point. The position vector of a point on the curve relative to this frame is simply the vector $\mathbf{r}(s) - \mathbf{r}(0) \equiv \overrightarrow{M_0 M}$, where $M = \mathbf{r}(s)$. Therefore, projecting (52) onto the axes, we obtain

$$\begin{cases} x(s) & = s - \frac{1}{6}k^2(0)s^3 + o(s^3) \\ y(s) & = \frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3) \\ z(s) & = \frac{1}{6}k(0)\chi(0)s^3 + o(s^3) \end{cases} . \tag{53}$$

Local Behaviour of a Parametrised Curve Around a Biregular Point

It is not difficult to observe that locally, sufficiently close to 0, we have the following relations between the coordinates, which actually give us the equations of the projections of the curve onto the coordinate planes of the Frenet frame:

$$\begin{cases} y = \frac{1}{2}k(0)x^2, \\ z = \frac{1}{6}k(0)\chi(0)x^3, \\ z^2 = \frac{2}{9}\frac{\chi^2(0)}{k(0)}y^3. \end{cases} \tag{54}$$

Thus, the projection of the curve onto the *xOy* plane (the osculating plane) is a parabola, the projection onto the *xOz* plane (the rectifying plane) is a cubic curve, while the projection onto the *yOz* plane (the normal plane) is a semicubical parabola.

Local Behaviour of a Parametrised Curve Around a Biregular Point

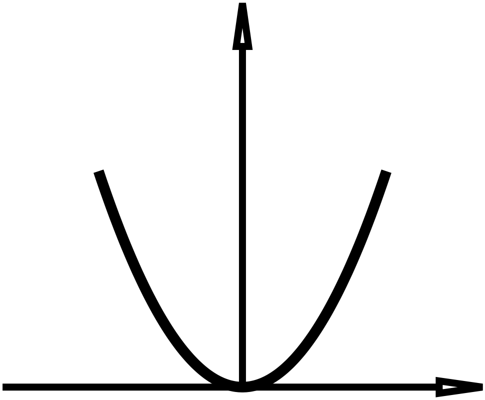


Figure: Projection onto the osculating plane

Local Behaviour of a Parametrised Curve Around a Biregular Point

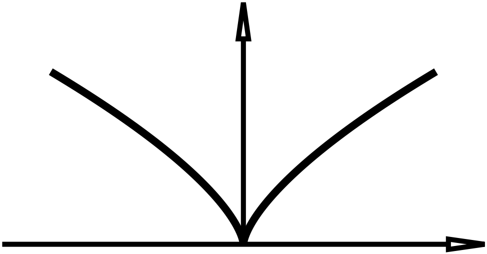


Figure: Projection onto the normal plane

Local Behaviour of a Parametrised Curve Around a Biregular Point

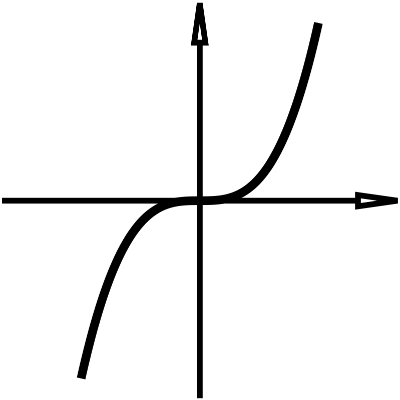


Figure: Projection onto the rectifying plane

The Contact between a Space Curve and a Plane

We consider a natural parametrised curve $(I, \mathbf{r})$. We assume, as before, that 0 is an interior point of the interval I and that the point $M_0 = \mathbf{r}(0)$ on the curve is regular. We consider a plane curve Π passing through the point M_0. We choose as a coordinate frame the Frenet frame of the curve $\mathbf{r}$ at the point M_0, $\mathcal{R}_0 = \{M_0; \boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0\}$. Relative to this coordinate frame, the Cartesian equation of the plane Π will have the form

$$F(x, y, z) \equiv ax + by + cz = 0. \tag{55}$$

Now, if $x = x(s), y = y(s), z = z(s)$ are the local equations of the curve relative to the Frenet frame (see (53)), the condition for intersection between the plane and the curve is $F(x(s), y(s), z(s)) = 0$, that is

The Contact between a Space Curve and a Plane

$$a\left(s-\frac{1}{6}k^2(0)s^3+o(s^3)\right)+b\left(\frac{1}{2}k(0)s^2+\frac{1}{6}k'(0)s^3+o(s^3)\right)+ \\ +c\left(\frac{1}{6}k(0)\chi(0)s^3+o(s^3)\right)=0 \tag{56}$$

or

$$as+\frac{1}{2}bk(0)s^2+\frac{1}{6}\left(-ak^2(0)+bk'(0)+ck(0)\chi(0)\right)s^3+o(s^3)=0. \tag{57}$$

We now have several possibilities:

a) If $a \neq 0$ (i.e., the plane does not contain the tangent), then the plane has a zero-order contact with the curve (intersection, they have only one common point or the *curve pierces the plane*).

The Contact between a Space Curve and a Plane

b) If $a = 0$, then the intersection equation has $s = 0$ as a double root, which means that the plane has a first-order contact with the curve (tangential contact). It is immediately observed that, in this case, the plane Π passes through the tangent to the curve at the point M_0 (which, in our chosen coordinate system, is the line with equations $y = 0, z = 0$, i.e., the Ox axis).

c) If we want at least a second-order contact (osculating contact), then the coefficient of s^2 in the intersection equation must also vanish. This can only happen if $b = 0$ (as the point M_0 is regular, the curvature is non-zero at this point). Thus, we have osculating contact if we impose $a = 0$ and $b = 0$. But this choice leads us to the equation $z = 0$ for the plane Π, meaning the plane is already completely determined (the osculating plane).

The Contact between a Space Curve and a Plane

d) Since the plane Π is completely determined by the condition of having second-order contact with the curve, we cannot have more special cases such as a higher-order contact (superosculating). Nevertheless, if we examine the intersection equation, we can conclude that the osculating plane has superosculating contact with the curve at the planar points of the curve, i.e., at the points where the torsion of the curve vanishes. At all other points, the contact is only second-order (osculating).

The Contact between a Space Curve and a Sphere. The Osculating Sphere

As in the previous paragraph, we consider a natural parametrised curve $(I, \mathbf{r})$, and we assume that 0 is an interior point of the interval I and that $\mathbf{r}(0)$ is a regular point on the curve. We choose as a coordinate frame the Frenet frame of the curve $\mathbf{r}$ at the point $M_0 = \mathbf{r}(0)$, $\mathcal{R}_0 = \{M_0; \boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0\}$. Then an arbitrary sphere passing through the point M_0 will have, in this coordinate system, the equation

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0, \tag{58}$$

where $\Omega(a, b, c)$ is the centre of the sphere. Then, the condition for intersection will be again,

$$F(x(s), y(s), z(s)) = 0,$$

where $x = x(s), y = y(s), z = z(s)$ are the local equations of the curve relative to the Frenet frame at the point M_0.

The Contact between a Space Curve and a Sphere. The Osculating Sphere

Thus, in our case, we have

$$F(x(s), y(s), z(s)) = \left(s - \frac{1}{6}k^2(0)s^3 + o(s^3)\right)^2 + \\ + \left(\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)\right)^2 + \\ + \left(\frac{1}{6}k(0)\chi(0)s^3 + o(s^3)\right)^2 - 2a\left(s - \frac{1}{6}k^2(0)s^3 + o(s^3)\right) - \\ - 2b\left(\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)\right) - 2c\left(\frac{1}{6}k(0)\chi(0)s^3 + o(s^3)\right) = 0. \tag{59}$$

or

$$-2as + (1 - bk(0))s^2 + \frac{1}{3}(ak^2(0) - bk'(0) - ck(0)\chi(0))s^3 + o(s^3). \tag{60}$$

The Contact between a Space Curve and a Sphere. The Osculating Sphere

The following discussion is also similar to the case of the contact between a curve and a plane. However, here we have more cases to consider.

a) If $a \neq 0$, then $s = 0$ is a simple root of the intersection equation. Thus, in this case, the sphere and the curve have a zero-order contact (intersecting contact).

b) The sphere and the curve have a first-order contact (tangential contact) if and only if the intersection equation has a double root at the origin. This clearly happens if and only if $a = 0$. This means the first coordinate of the centre of the sphere is 0, i.e., the centre lies in the normal plane to the curve at M_0. It is easy to observe that, in this case, the tangent to the curve at M_0 lies in the tangent plane to the sphere at the same point, which justifies, again, the name *tangential contact*.

The Contact between a Space Curve and a Sphere. The Osculating Sphere

c) For osculating contact (second-order), the coefficient of s^2 in the intersection equation must also vanish and we obtain

$$b = \frac{1}{k(0)} = R(0), \tag{61}$$

where $R(0)$ is the radius of curvature of the curve at the point M_0. Thus, in this case, the centre of the osculating sphere lies on the line of intersection between the planes $x = 0$ (the normal plane) and $y = R(0)$. This line, which is easily seen to be parallel to the binormal of the curve at M_0, is called the *axis of curvature* or *polar axis* of the curve. It intersects the osculating plane at M_0 of the curve in the centre of curvature of the curve at M_0. Thus, any sphere whose centre lies on the axis of curvature of a curve has osculating contact with it.

The Contact between a Space Curve and a Sphere. The Osculating Sphere

d) We say that the sphere has a *superosculating* contact with the curve if they have contact of at least third order, which means that in the intersection equation, the coefficients of powers up to three of s must all vanish simultaneously.

1) Suppose first that the torsion of the curve at the point M_0 does not vanish, i.e., $\chi(0) \neq 0$. In this case, it is immediately seen that a superosculating contact exists between the sphere and the curve if and only if

$$\begin{cases} a = 0, \\ b = \frac{1}{k(0)}, \\ c = \frac{k'(0)}{k^2(0)\chi(0)}, \end{cases} \tag{62}$$

that is, in this case the sphere is uniquely determined and we call it the *osculating sphere*[3].

[3] In some books, all spheres that have osculating contact with the curve are called osculating spheres, and the one with superosculating contact is called the *superosculating sphere*.

The Contact between a Space Curve and a Sphere. The Osculating Sphere

2) If $\chi(0) = 0$, while $k'(0) \neq 0$, then, as can be easily seen, the coefficient of s^3 never vanishes, hence there is no sphere that has a superosculating contact with the curve. Nevertheless, as we saw in the previous paragraph, in this case the osculating plane has a superosculating contact with the curve and we can think of the osculating plane as playing the role of an osculating sphere of infinite radius.

3) If both $\chi(0)$ and $k'(0)$ vanish, then the coefficient of s^3 in the intersection equation vanishes for any value of c, in other words, in this case, any sphere which has osculating contact with the curve also has superosculating contact. Thus, in this case, we have an *infinity* of osculating spheres.

Definition

A *displacement* of $\mathbb{R}^3$ is a mapping $D : \mathbb{R}^3 \to \mathbb{R}^3$, $D(x) = \mathcal{A} \cdot x + \mathbf{b}$, where $\mathcal{A} \in M_{3\times 3}(\mathbb{R})$ is an orthogonal matrix, $\mathcal{A}^t \cdot \mathcal{A} = I_3$, with determinant equal to one: $\det \mathcal{A} = 1$, while $\mathbf{b} \in \mathbb{R}^3$ is a constant vector. The linear mapping $A : \mathbb{R}^3 \to \mathbb{R}^3$, $A(x) = \mathcal{A} \cdot x$ is called the *homogeneous part* of the displacement.

Remarks

(i) A displacement of $\mathbb{R}^3$ is nothing other than a rotation followed by a translation.

Remarks

(ii) If we do not require that $\det \mathcal{A} = 1$, we also obtain an isometry of $\mathbb{R}^3$. However, in this case (if $\det \mathcal{A} = -1$), a transformation $D(x) = \mathcal{A} \cdot x + b$ is no longer reduced to a rotation and translation, we must also add a reflection with respect to a plane. In many books, the term "motion" only implies that the matrix $\mathcal{A}$ is orthogonal, while what we call a "displacement" is referred to as a "proper displacement".

Definition

Let $D : \mathbb{R}^3 \to \mathbb{R}^3$ be a displacement, with homogeneous part $A : \mathbb{R}^3 \to \mathbb{R}^3$. The *image* of the parametrised curve $(I, \mathbf{r} = \mathbf{r}(t))$ under D is, by definition, the parametrised curve $(I, \mathbf{r}_1 = (D \circ \mathbf{r})(t))$.

Remark

Since A is a non-degenerate linear mapping, the image of a regular parametrised curve is also a regular parametrised curve.

Theorem

Let D be a displacement of $\mathbb{R}^3$, with homogeneous part A, and $(I, \mathbf{r} = \mathbf{r}(t))$ a biregular parametrised curve, $(I, \mathbf{r}_1 = D \circ \mathbf{r})$ its image under D, and $\{\mathbf{r}(t); \boldsymbol{\tau}(t), \boldsymbol{\nu}(t), \boldsymbol{\beta}(t)\}$ the Frenet frame of the curve $\mathbf{r}$ at t. Then the frame $\{\mathbf{r}_1(t); A(\boldsymbol{\tau}(t)), A(\boldsymbol{\nu}(t)), A(\boldsymbol{\beta}(t))\}$ is the Frenet frame of $\mathbf{r}_1$ at t.

Proof.

Let $\mathbf{r}(t) = (x(t), y(t), z(t))$, $D(x, y, z) = (x_1, y_1, z_1)$, where

$$\begin{cases} x_1 = \alpha_{11}x + \alpha_{12}y + \alpha_{13}z + b_1 \\ y_1 = \alpha_{21}x + \alpha_{22}y + \alpha_{23}z + b_2 \\ z_1 = \alpha_{31}x + \alpha_{32}y + \alpha_{33}z + b_3 \end{cases} \tag{63}$$

Then

$$\mathbf{r}_1(t) = (x_1(t), y_1(t), z_1(t)),$$

so

$$\mathbf{r}'_1(t) = A(\mathbf{r}'(t)), \quad \mathbf{r}''_1(t) = A(\mathbf{r}''(t)). \tag{64}$$

□

Proof.

Since A is a linear isometry that preserves the orientation of $\mathbb{R}^3$, we have

$$\|\mathbf{r}'_1\| = \|A(\mathbf{r}')\| = \|\mathbf{r}'\|,\ \ \mathbf{r}'_1 \cdot \mathbf{r}''_1 = \mathbf{r}' \cdot \mathbf{r}'',$$
$$\mathbf{r}'_1 \times \mathbf{r}''_1 = A(\mathbf{r}' \times \mathbf{r}''),\ \ \|\mathbf{r}'_1 \times \mathbf{r}''_1\| = \|\mathbf{r}' \times \mathbf{r}''\|.$$

Then:

$$\tau_1 = \frac{\mathbf{r}'_1}{\|\mathbf{r}'\|} = \frac{A(\mathbf{r}')}{\|A(\mathbf{r}')\|} = \frac{A(\mathbf{r}')}{\|\mathbf{r}'\|} = A\left(\frac{\mathbf{r}'}{\|\mathbf{r}'\|}\right) = A(\tau)$$

$\square$

Proof.

$$\begin{aligned}\boldsymbol{\nu}_1 &= \frac{\|\mathbf{r}_1'\|}{\|\mathbf{r}_1'\times\mathbf{r}_1''\|}\mathbf{r}_1'' - \frac{\mathbf{r}_1'\cdot\mathbf{r}_1''}{\|\mathbf{r}_1'\|\cdot\|\mathbf{r}_1'\times\mathbf{r}_1''\|}\mathbf{r}_1' = \frac{\|\mathbf{r}'\|}{\mathbf{r}'\times\mathbf{r}''}\cdot A(\mathbf{r}'') - \\ &- \frac{\mathbf{r}'\cdot\mathbf{r}''}{\|\mathbf{r}'\|\cdot\|\mathbf{r}'\times\mathbf{r}''\|}\cdot A(\mathbf{r}') = A(\boldsymbol{\nu}) \\ \boldsymbol{\beta}_1 &= \frac{\mathbf{r}_1'\times\mathbf{r}_1''}{\|\mathbf{r}_1'\times\mathbf{r}_1''\|} = A(\boldsymbol{\beta}).\end{aligned}$$

$\square$

Corollary

The parametrised curves $(I, \mathbf{r})$ *and* $(I, \mathbf{r}_1 = D \circ \mathbf{r})$ *have the same curvature and torsion.*

Proof.

We have

$$k_1 = \frac{\|\mathbf{r}_1' \times \mathbf{r}_1''\|}{\|\mathbf{r}_1'\|} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|} = k.$$

For the torsion, the situation is slightly more complex. From the theorem, we have

$$\beta_1(t) = A(\beta(t)) \quad \text{and} \quad \nu_1(t) = A(\nu(t)).$$

Proof.

Since A is a linear operator, a similar result holds for the derivatives of the two Frenet vectors, i.e., we have

$$\beta_1'(t) = A(\beta'(t)) \quad \text{and} \quad \nu_1'(t) = A(\nu'(t)).$$

Using the last Frenet equations for the two curves, we obtain the equalities

$$-\chi_1(t)\nu_1(t) = A(-\chi(t)\nu(t)) = -\chi(t)A(\nu(t)) = -\chi(t)\nu_1(t),$$

and thus the two torsions are equal, as stated. $\square$

Theorem

Let $(I, \mathbf{r} = \mathbf{r}(t))$ and $(I, \mathbf{r}_1 = \mathbf{r}_1(t))$ be two biregular parametrised curves. If $k(t) = k_1(t)$, $\chi(t) = \chi_1(t)$ and $\|\mathbf{r}'(t)\| = \|\mathbf{r}_1'(t)\|$ $\forall t \in I$, then there exists a unique displacement D of $\mathbb{R}^3$ such that $\mathbf{r}_1 = D \circ \mathbf{r}$.

Proof.

Let $t_0 \in I$ be an arbitrary point and D a displacement of $\mathbb{R}^3$ that maps the Frenet frame $\{\mathbf{r}(t_0); \boldsymbol{\tau}_0, \boldsymbol{\nu}_0, \boldsymbol{\beta}_0\}$ of $\mathbf{r}$ at t_0 onto the Frenet frame $\{\mathbf{r}_1(t_0); \boldsymbol{\tau}_{10}, \boldsymbol{\nu}_{10}, \boldsymbol{\beta}_{10}\}$ of the curve $\mathbf{r}_1$ at the same point. Clearly, there exists a unique displacement with this property. Let $(I, \mathbf{r}_2(t) = D \circ \mathbf{r}(t))$ be the image of the curve $\mathbf{r}$ through D, and k_2, χ_2 the curvature and torsion of the parametrised curve $\mathbf{r}_2$. □

Proof.

Then

$$k_2(t) \equiv k(t) \equiv k_1(t)$$

$$\chi_2(t) \equiv \chi(t) \equiv \chi_1(t)$$

and, moreover,

$$\|\mathbf{r}_2'(t)\| \equiv \|\mathbf{r}_1'(t)\|.$$

Hence, the vector functions $\boldsymbol{\tau}_1(t)$, $\boldsymbol{\nu}_1(t)$, $\boldsymbol{\beta}_1(t)$ and $\boldsymbol{\tau}_2(t)$, $\boldsymbol{\nu}_2(t)$, $\boldsymbol{\beta}_2(t)$ which give the Frenet frame are solutions of the same Frenet system. $\square$

Proof.

$$\begin{cases} \boldsymbol{\tau}' = \|\mathbf{r}_1'\| k_1 \boldsymbol{\nu} \\ \boldsymbol{\nu}' = -\|\mathbf{r}_1'\| k_1 \boldsymbol{\tau} + \|\mathbf{r}_1'\| \chi_1 \boldsymbol{\beta} \\ \boldsymbol{\beta}' = -\|\mathbf{r}_1'\| \chi_1 \boldsymbol{\nu}. \end{cases}$$

Since the solutions coincide at $t = t_0$, by uniqueness of the Cauchy problem, they must coincide globally. In particular, we have

$$\boldsymbol{\tau}_1(t) \equiv \boldsymbol{\tau}_2(t) \quad \text{or} \quad \frac{\mathbf{r}_1'(t)}{\|\mathbf{r}_1'(t)\|} = \frac{\mathbf{r}_2'(t)}{\|\mathbf{r}_2'(t)\|},$$

so

$$\mathbf{r}_1'(t) - \mathbf{r}_2'(t) = 0 \ \Rightarrow \ \mathbf{r}_1(t) - \mathbf{r}_2(t) = \text{const}.$$

Proof.

But for $t = t_0$, $\mathbf{r}_1(t_0) - \mathbf{r}_2(t_0) = 0$, hence the two functions coincide for all t, therefore $\mathbf{r}_1(t) \equiv \mathbf{r}_2(t) = D \circ \mathbf{r}(t)$.

As for the uniqueness of D, we note that for any other point $t_1 \in I$, since $\mathbf{r}_1 \equiv \mathbf{r}_2$, D maps the Frenet frame of the curve $\mathbf{r}$ at t_1 to the Frenet frame of the curve $\mathbf{r}_1$ at t_1. □

Remark

For naturally parametrised curves, the condition $\|\mathbf{r}'(t)\| = \|\mathbf{r}_1'(t)\|$ is always satisfied.

Theorem

Let $f(s)$ and $g(s)$ be two smooth functions defined on an interval I, such that $f(s) > 0$, $\forall t \in I$. Then there exists a naturally parametrised curve $(I, \mathbf{r} = \mathbf{r}(s))$ for which $f(s) = k(s)$ $\forall s \in I$ and $g(s) = \chi(s)$ $\forall s \in I$. This curve is uniquely defined, up to a displacement of $\mathbb{R}^3$.

Proof.

Let $\{\mathbf{r}_0; \mathbf{T_0}, \mathbf{N_0}, \mathbf{B_0}\}$ be a right-handed orthonormal frame in $\mathbb{R}^3$. We consider the system of linear differential equations

$$\begin{cases} \mathbf{T}'(s) = f(s)\mathbf{N}(s) \\ \mathbf{N}'(s) = -f(s)\mathbf{T}(s) + g(s)\mathbf{B}(s) \\ \mathbf{B}'(s) = -g(s)\mathbf{N}(s) \end{cases} \tag{65}$$

Proof.

with respect to the vector functions $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$.
If we denote

$$X(s) = (\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)), \tag{66}$$

then the system (65) can be written as

$$X'(s) = A(s) \cdot X(s), \tag{67}$$

with

$$A(s) = \begin{pmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{pmatrix}.$$

□

Proof.

In the theory of ordinary differential equations, it is shown that system (67) has a unique solution that satisfies

$$X(s_0) = (\mathbf{T_0}, \mathbf{N_0}, \mathbf{B_0}),$$

where $s_0 \in I$, while the columns of the matrix $X(s_0)$ are the vectors $\mathbf{T_0}, \mathbf{N_0}, \mathbf{B_0}$ of the initial orthonormal basis.
We shall first prove that for any $s \in I$ the vectors in $(\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s))$ form an orthonormal basis. It is sufficient to show that, for any $s \in I$, $X(s)$ is orthogonal, i.e., $X^t(s) \cdot X(s) = I_3$. $\square$

Proof.

We have

$$\frac{d}{dt}\left(X^t \cdot X\right) = \frac{d}{dt}\left(X^t(s)\right) \cdot X + X^t \cdot \frac{d}{dt}\left(X(s)\right) = \\ = X^t(A^t X + AX) = X^t(A^t + A)X.$$

but since A is antisymmetric, $A^t + A = 0$, therefore

$$\frac{d}{dt}\left(X^t \cdot X\right) = 0 \Rightarrow X^t \cdot X = const.$$

On the other hand, from the initial condition, $\left(X^t \cdot X\right)(s_0) = I_3$, so $X^t(s) \cdot X(s) = I_3$ for all $s \in I$. $\square$

Proof.

Now let us define

$$\mathbf{r} = \mathbf{r}_0 + \int_{s_0}^{s} \mathbf{T}(s)ds, \tag{sol}$$

where $\mathbf{r}_0$ is the origin of the initial frame, while $\mathbf{T}(s)$ is the first column of $X(s)$. We will show that $(I, \mathbf{r}(s))$ is the desired curve. Clearly, we have:

$$\mathbf{r}'(s) = \mathbf{T}(s),$$
$$\|\mathbf{r}'(s)\| = \|\mathbf{T}(s)\| = 1,$$
$$\mathbf{r}''(s) = \mathbf{T}'(s) = f(s)\mathbf{N}(s).$$

□

Proof.

We immediately notice that $\mathbf{r}'(s) \times \mathbf{r}''(s) \neq 0$, hence $\mathbf{r}(s)$ is a biregular, naturally parametrised curve. On the other hand,

$$\mathbf{r}'''(s) = (f(s)\mathbf{N})' = f'\mathbf{N} + f\mathbf{N}' = f'\mathbf{N} + f(-f\mathbf{T} + g\mathbf{B}) = -f^2\mathbf{T} + f'\mathbf{N} + fg\mathbf{B},$$

so

$$(\mathbf{r}', \mathbf{r}'', \mathbf{r}''') = (\mathbf{T}, f\mathbf{N}, -f^2\mathbf{T}+f'\mathbf{N}+fg\mathbf{B}) = (\mathbf{T}, f\mathbf{N}, fg\mathbf{B}) = f^2 g\,\underbrace{(\mathbf{T}, \mathbf{N}, \mathbf{B})}_{=1} = f^2 g.$$

□

Proof.

All that remains is to compute the curvature and torsion:

$$k(s) = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = |f(s)| = f(s),$$

$$\chi(s) = \frac{f^2(s)g(s)}{f^2(s)} = g(s),$$

so the curve $\mathbf{r}$ satisfies the conditions of the theorem.

The uniqueness of $\mathbf{r}$, up to a displacement, follows from the previous theorem. □

In this section, unless otherwise stated, all curves are *parametrised curves*. Let

$$\mathbf{r} = \mathbf{r}(t, \lambda) \tag{68}$$

be a family of parametrised curves depending smoothly on the parameter λ.

Definition

The *envelope* of the family (68) is a parametrised curve (J, Γ) which, at each of its points, is tangent to a curve from the family.

Theorem

The points of the envelope of the family $\mathbf{r}(t, \lambda)$ *satisfy*

$$\mathbf{r} = \mathbf{r}(t, \lambda) \tag{69}$$

$$\mathbf{r}'_{\boldsymbol{\lambda}} \times \mathbf{r}'_{\boldsymbol{t}} = 0. \tag{70}$$

Proof.

If Γ is the envelope of the family (γ_λ), and *P* is a point on Γ, then *P* is a point of tangency between Γ and a curve from the family, corresponding to a certain value of the parameter λ. Thus, the equation of Γ will be of the form

$$\mathbf{r}_1 = \mathbf{r}_1(\lambda).$$

On the other hand, *P* lies on one of the curves γ_λ and therefore satisfies

$$\mathbf{r}_1 = \mathbf{r}(t(\lambda), \lambda).$$

The condition of tangency between Γ and γ_λ is written

$$\mathbf{r}'_{1\lambda} \parallel \mathbf{r}'_t$$

Proof.

or

$$\mathbf{r}'_{1\lambda} \times \mathbf{r}'_t = 0$$

or, again, since $\mathbf{r}'_{1\lambda} = \mathbf{r}'_t \cdot t'_\lambda + \mathbf{r}'_\lambda$,

$$(\mathbf{r}'_t \cdot t'_\lambda + \mathbf{r}'_\lambda) \times \mathbf{r}'_t = 0,$$

and the theorem is proved, since $\mathbf{r}'_t \times \mathbf{r}'_t = 0$. □

Remarks

1. The set of points described by equations (69) and (70) is called the *discriminant set* of the family γ_λ. It contains not only the support of the envelope, but also the singular points of the curves in the family, for which $\mathbf{r}'_t = 0$, and thus no tangent exists.

Remarks

2. The equation $\mathbf{r}'_\lambda \times \mathbf{r}'_t = 0$ can also be written as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'_\lambda & y'_\lambda & 0 \\ x'_t & y'_t & 0 \end{vmatrix} = 0 \Leftrightarrow x'_\lambda y'_t - x'_t y'_\lambda = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{x'_\lambda}{x'_t} = \frac{y'_\lambda}{y'_t}. \tag{71}$$

Example

Let us consider the family of curves

$$\mathbf{r}(t,\lambda)=(\lambda+a\cos t,\lambda+a\sin t),\qquad \lambda,t\in\mathbb{R},a>0.$$

Clearly, we are dealing with a family of circles of radius a, with centres on the first bisector of the coordinate axes. Then, we have

$$\begin{aligned}\mathbf{r}'_\lambda &= \{1,1\},\\ \mathbf{r}'_t &= \{-a\sin t, a\cos t\},\end{aligned}$$

therefore, the points of the envelope (and only them, since circles have no singular points) satisfy

Example

$$\begin{cases} x = \lambda + a\cos t \\ y = \lambda + a\sin t \\ x'_\lambda \cdot y'_t = x'_t \cdot y'_\lambda \end{cases}$$

or

$$\begin{cases} x = \lambda + a\cos t \\ y = \lambda + a\sin t \\ a\cos t = -a\sin t. \end{cases}$$

Eliminating t, we obtain the parametric equations of the envelope:

$$\begin{cases} x(\lambda) = \lambda \pm \frac{a}{\sqrt{2}} \\ y(\lambda) = \lambda \mp \frac{a}{\sqrt{2}}. \end{cases}$$

Property

The points of the envelope of a family of plane curves given by the implicit equation

$$F(x, y, \lambda) = 0 \tag{72}$$

satisfy the system of equations

$$\begin{cases} F(x, y, \lambda) = 0 \\ F'_\lambda(x, y, \lambda) = 0 \end{cases} \tag{73}$$

Proof.

Locally, around each point of a curve from the family, we can parametrise the curve, that is, represent it as

$$\begin{cases} x = x(t, \lambda) \\ y = y(t, \lambda) \end{cases} .$$

Substituting into the equation of the family, we obtain

$$F(x(t, \lambda), y(t, \lambda), \lambda) = 0,$$

from which, differentiating with respect to t and λ, we obtain the system:

□

Proof.

$$\begin{cases} F'_x x'_t + F'_y y'_t = 0 \\ F'_x x'_\lambda + F'_y y'_\lambda + F'_\lambda = 0 \end{cases}.$$

But, from (71),

$$x'_\lambda = K x'_t, \;\; y'_\lambda = K y'_t,$$

with $K =$ const., therefore the second equation above becomes

$$K(\underbrace{F'_x x'_t + F'_y y'_t}_{=0}) + F'_\lambda = 0$$

or

$$F'_\lambda = 0.$$

□

Example

Let us again consider the family of circles from the previous paragraph, this time given by the implicit equation

$$F(x, y, \lambda) \equiv (x-\lambda)^2 + (y-\lambda)^2 - a^2 = 0.$$

Then, the second equation of the discriminant set will be

$$F'_\lambda(x, y, \lambda) = -2(x + y - 2\lambda) = 0,$$

from which we obtain

$$\lambda = \frac{x+y}{2},$$

Example

which, after substituting into the equation of the family, gives us

$$(x - y)^2 = 2a^2,$$

that is, we again obtain the same equations of the envelope, namely

$$y = x \pm a\sqrt{2}.$$

Property

Let us suppose we are given a family of curves depending smoothly on two parameters, λ and μ

$$F(x, y, \lambda, \mu) = 0, \tag{74}$$

where the parameters λ and μ are linked by a relation

$$\varphi(\lambda, \mu) = 0, \tag{75}$$

then the points of the envelope satisfy the system

$$\begin{cases} F(x, y, \lambda, \mu) = 0 \\ \varphi(\lambda, \mu) = 0 \\ \dfrac{D(F, \varphi)}{D(\lambda, \mu)} = 0 \end{cases} . \tag{76}$$

Proof.

From the equation

$$\varphi(\lambda, \mu) = 0,$$

we may assume, for example, that

$$\mu = \mu(\lambda),$$

therefore, substituting into F and φ,

$$\begin{cases} F(x, y, \lambda, \mu(\lambda)) = 0, \\ \varphi(\lambda, \mu(\lambda)) = 0. \end{cases}$$

□

Proof.

Differentiating these two equations with respect to λ, we obtain:

$$\begin{cases} F'_\lambda + F'_\mu \mu'_\lambda = 0 \\ \varphi'_\lambda + \varphi'_\mu \mu'_\lambda = 0. \end{cases}$$

Eliminating the derivative μ'_λ between the two equations, we obtain the third equation in (76), as required. $\square$

Definition

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a parametrised plane curve. The *evolute* of $\mathbf{r}$ is, by definition, the envelope of the family of normals to the curve.

We have the following result:

Property

The parametric equations of the evolute of the curve $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t))$ *are*

$$\begin{cases} X = x - \dfrac{y'(x'^2 + y'^2)}{x'y'' - x''y'} \\ Y = y + \dfrac{x'(x'^2 + y'^2)}{x'y'' - x''y'} \end{cases} \tag{77}$$

Proof.

As is known, the equation of the normal to a plane curve is

$$F(X, Y, t) = (X - x(t)) \cdot x'(t) + (Y - y(t)) \cdot y'(t) = 0.$$

$\square$

Proof.

The relations satisfied by the points of the envelope of the family of normals (and *only* by them, since in this case the curves of the family are straight lines, and thus have no singular points) are (see (73)):

$$\begin{cases} F(X, Y, t) = 0 \\ F'_t(X, Y, t) = 0 \end{cases},$$

that is

$$\begin{cases} x'(t)X + y'(t)Y = x(t) \cdot x'(t) + y(t) \cdot y'(t) \\ x''(t)X + y''(t)Y = x'^2(t) + x''(t) \cdot x(t) + y'^2(t) + y(t) \cdot y''(t) \end{cases}.$$

Equations (77) now follow immediately by solving this linear system in X and Y. □

Example

For the ellipse

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$$

we obtain, after calculation,

$$\begin{cases} X = \frac{a^2-b^2}{a}\cos^3 t \\ Y = \frac{b^2-a^2}{b}\sin^3 t \end{cases}$$

or, after eliminating the parameter t,

$$a^{\frac{2}{3}}X^{\frac{2}{3}} + b^{\frac{2}{3}}Y^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

Example

The curve described by this equation is called the *stretched astroid* (see the figure below).

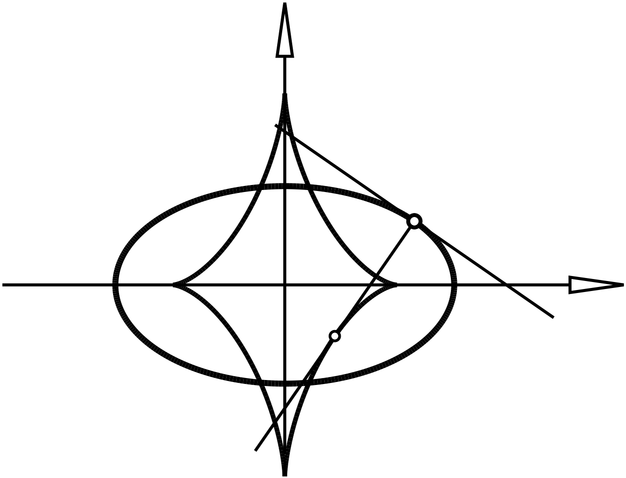


Figure: Evolute of an ellipse

As we have seen, in the case of an arbitrary space curve, the curvature is always a positive scalar. Of course, this concept of curvature can be equally applied to plane curves. However, it turns out that in this particular case, we can obtain more information about the curve if we use a slightly different notion of curvature, allowing the curvature to have a sign. To define the curvature of a plane curve, we shall use a technical trick that will allow us to construct the definition in a coordinate-independent manner.

Definition

A *complex structure* on $\mathbb{R}^2$ is the map $J : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$J(x, y) = (-y, x).$$

Remark

Applying J simply means rotating the vector $\{x, y\}$ by $\frac{\pi}{2}$ or multiplying the complex number $x + iy$ by the imaginary unit i (from which the name originates).

A few obvious properties of the complex structure are contained in the following proposition:

Property

a) $J\mathbf{v} \cdot J\mathbf{w} = \mathbf{v} \cdot \mathbf{w}$.
b) $(J\mathbf{v}) \cdot \mathbf{v} = 0$.
c) $J(J\mathbf{v}) = -\mathbf{v}$ (i.e., $J^2 = -id$).

All these properties follow immediately from the geometric interpretation of the complex structure.

Anticipating slightly, let us say a few words about the type of curvature we will define. Recall that the curvature of an arbitrary parametrised space curve $\mathbf{r} = \mathbf{r}(t)$ can be computed using the formula

$$k(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Now, if $\mathbf{r}$ is a *plane* curve, with support located in the xOy coordinate plane, then the vectors $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are also in this plane. Therefore, their cross product is a vector directed along the z-axis and, thus, the norm of the vector is simply the absolute value of its z-component. The idea behind the definition of signed curvature is to replace the absolute value with the component itself. For this purpose, the following characterisation of the cross product of two vectors in the xOy plane will prove very useful.

Property

Let $\mathbf{u}(x_1, y_1), \mathbf{v}(x_2, y_2) \in \mathbb{R}^2$*. Then*

$$\mathbf{u} \times \mathbf{v} = [\mathbf{v} \cdot J\mathbf{u}] \cdot \mathbf{k}.$$

Proof.

As is known, the cross product of the vectors **u** and **v** (viewed as vectors in $\mathbb{R}^3$, with the last component zero) can be computed using the formula

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix} = (x_1 y_2 - x_2 y_1) \cdot \mathbf{k}.$$

□

Proof.

On the other hand,

$$\mathbf{v} \cdot J\mathbf{u} = \{x_2, y_2\} \cdot \{-y_1, x_1\} = -x_2 y_1 + x_1 y_2,$$

from which the stated equality follows. □

We are now ready to define the curvature of a plane curve.

Definition

Let $\mathbf{r} = \mathbf{r}(t)$ be a parametrised plane curve. The *signed curvature* of $\mathbf{r}$ is, by definition, the quantity

$$k_{\pm} = \frac{\mathbf{r}'' \cdot J\mathbf{r}'}{\|\mathbf{r}'\|^3}. \tag{78}$$

Remark

According to proposition 12, the signed curvature is the projection of the curvature vector onto the z-axis. Since the curvature vector is parallel to the z-axis, we have

$$|k_{\pm}| = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = k.$$

Another immediate, but important, result from the computational point of view is the following:

Property

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a parametrised plane curve. If $\mathbf{r}(t) = (x(t), y(t))$, then the signed curvature of $\mathbf{r}$ can be expressed as

$$k_{\pm}(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\left(x'^2(t) + y'^2(t)\right)^{3/2}}.$$

Corollary

If $y = f(x)$ is the explicit equation of a plane curve, then its signed curvature is given by

$$k_{\pm}(x) = \frac{f''(x)}{(1 + f'^2)^{3/2}}.$$

Remark

The previous consequence shows that for an explicitly given plane curve (i.e., the graph of a function of a single variable), the sign of the signed curvature is, in fact, the sign of the second derivative of the function f, that is, as is well known from analysis, the sign of the signed curvature indicates the convexity or concavity of the function.

Just as with the torsion of a space curve, the signed curvature of a plane curve is "almost" invariant under a reparametrisation, i.e., we have

Theorem

Let $(I, \mathbf{r} = \mathbf{r}(t))$ *be a parametrised plane curve,* $(J, \rho = \rho(u))$ *– a parametrised curve equivalent to it, with reparametrisation* $\lambda : I \to J$*,* $u = \lambda(t)$*. Then*

$$k_{\pm}[\rho](u) = \operatorname{sgn}(\lambda') \cdot k_{\pm}[\mathbf{r}](t).$$

Proof.

We have

$$\begin{aligned}
\mathbf{r}(t) &= \rho(\lambda(t)) \\
\mathbf{r}'(t) &= \rho'(\lambda(t)) \cdot \lambda'(t) \\
\mathbf{r}''(t) &= \rho''(\lambda(t)) \cdot \lambda'^2(t) + \rho'(\lambda(t)) \cdot \lambda''(t)
\end{aligned}$$

□

Proof.

$$\begin{aligned}\mathbf{r}'' \cdot J\boldsymbol{\rho}' &= (\boldsymbol{\rho}''\lambda'^2 + \boldsymbol{\rho}'\lambda'') \cdot J(\lambda'\boldsymbol{\rho}') = \\ &= \lambda'^3\boldsymbol{\rho}'' \cdot J\boldsymbol{\rho}' + \lambda'\lambda'' \underbrace{\boldsymbol{\rho}' \cdot J\boldsymbol{\rho}'} = 0 = \lambda'^3\boldsymbol{\rho}'' \cdot J\boldsymbol{\rho}'\end{aligned}$$

$$\|\mathbf{r}'\|^3 = |\lambda'^3|\|\boldsymbol{\rho}'\|^3$$

therefore

$$k_\pm[\mathbf{r}](t) = \frac{\mathbf{r}'' \cdot J\mathbf{r}'}{\|\mathbf{r}'\|^3} = \frac{\lambda'^3}{|\lambda'^3|} \cdot \frac{\boldsymbol{\rho}''(u) \cdot J\boldsymbol{\rho}'(u)}{\|\boldsymbol{\rho}'(u)\|^3} = \operatorname{sgn}(\lambda') \cdot k_\pm[\boldsymbol{\rho}](u),$$

from which

$$k_\pm[\boldsymbol{\rho}](u) = \operatorname{sgn}(\lambda') \cdot k_\pm[\mathbf{r}](t).$$

Remark

The previous theorem shows that the signed curvature is invariant under any *positive* reparametrisation, and thus it makes sense to define it also for regular *oriented* plane curves.

The curvature vector of a naturally parametrised plane curve can easily be expressed in terms of the signed curvature:

Lemma

Let $(I, \mathbf{r} = \mathbf{r}(s))$ *be a naturally parametrised plane curve. Then*

$$\mathbf{r}''(s) = k_{\pm}(s) \cdot J\mathbf{r}'(s).$$

Proof.

We have $\mathbf{r}'^2(s) = 1$ (the curve is naturally parametrised), hence $\mathbf{r}' \cdot \mathbf{r}'' = 0$, from which it follows that $\mathbf{r}'' \perp \mathbf{r}'$, or, equivalently, $\mathbf{r}'' \parallel J\mathbf{r}'$. On the other hand, from the definition of signed curvature, $k_\pm(s) = \mathbf{r}''(s) \cdot J\mathbf{r}'(s)$. If we write $\mathbf{r}''(s) = \alpha(s) \cdot J\mathbf{r}'(s)$, then we must have $\mathbf{r}''(s) \cdot J\mathbf{r}'(s) = \alpha(s) \cdot [J\mathbf{r}'(s)]^2 = \alpha(s)$, hence $\alpha(s) = k_\pm(s)$. $\square$

For the signed curvature of a plane curve, we have a geometric interpretation similar to that of the curvature of a space curve, only now the sign is also taken into account. First, we need the following definition.

Definition

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a parametrised plane curve. The *angle of rotation* of $\mathbf{r}$ is the function $\theta[\mathbf{r}] : I \to \mathbb{R}$, defined by:

$$\boldsymbol{\tau}(t) = \{\cos\theta[\mathbf{r}](t), \sin\theta[\mathbf{r}](t)\} = \exp(i\theta[\mathbf{r}](t)), \tag{79}$$

where $\boldsymbol{\tau}(t)$ is the unit tangent vector, i.e., $\theta[\mathbf{r}]$ is the angle between the unit tangent and the positive direction of the *Ox*-axis.

Remark

This definition seems very innocent and natural. After all, it should be clear to anyone that θ is exactly the angle formed by the unit tangent vector with the positive direction of the x-axis. In reality, however, for an arbitrary plane curve, it is not at all evident that a *continuous* angle function can be found, let alone a smooth one. Such functions *do* exist, and any two such functions differ by an integer multiple of 2π.

The following lemma provides the connection between the signed curvature and the variation of the rotation angle. The content of this lemma is quite similar to the geometric interpretation of the curvature of a space curve. In fact, when a plane curve is viewed as a particular case of a space curve, the variation of the rotation angle equals (in absolute value) the variation of the angle of contingency, so, in reality, the geometric interpretation of the *absolute* curvature of a plane curve (viewed as a space curve) is a particular case of this lemma.

Lemma

If $(I, \mathbf{r} = \mathbf{r}(t))$ is a parametrised plane curve, θ is its rotation angle, and $k_\pm$ is its signed curvature, then:

$$\frac{d\theta}{dt} = \|\mathbf{r}'(t)\| k_\pm(t).$$

Proof.

From the definition of the unit tangent vector, we have $\boldsymbol{\tau}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, hence

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} + \mathbf{r}'(t)\frac{d}{dt}\left(\frac{1}{\|\mathbf{r}'(t)\|}\right).$$

On the other hand, if we use the expression of $\boldsymbol{\tau}(t)$ as a function of the angle θ, given by (79), we obtain for $\frac{d\boldsymbol{\tau}}{dt}$ the formula

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{d\theta}{dt}\{-\sin\theta(t), \cos\theta(t)\} = \frac{d\theta}{dt}J\boldsymbol{\tau}(t).$$

$\square$

Proof.

Combining the two relations, we obtain the equality

$$\frac{\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} + \mathbf{r}'(t)\frac{d}{dt}\left(\frac{1}{\|\mathbf{r}'(t)\|}\right) = \frac{d\theta}{dt}J\boldsymbol{\tau}(t) \equiv \frac{d\theta}{dt}\cdot\frac{J\mathbf{r}''(t)}{\|J\mathbf{r}'(t)\|}.$$

Taking the scalar product with $J\mathbf{r}'(t)$ on both sides and noting that $J\mathbf{r}'(t)\cdot\mathbf{r}'(t) = 0$ and $J\mathbf{r}'(t)\cdot J\mathbf{r}'(t) = \|\mathbf{r}'(t)\|^2$, we get:

$$\frac{\mathbf{r}''(t)\cdot J\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{d\theta}{dt}\cdot\|\mathbf{r}'(t)\|,$$

from which, using the definition of signed curvature, it follows that

$$\frac{d\theta}{dt}\cdot\|\mathbf{r}'(t)\| = k_\pm(t)\cdot\|\mathbf{r}'(t)\|^2$$

Proof.

or, after simplification,

$$\frac{d\theta}{dt} = k_{\pm}(t) \cdot \|\mathbf{r}'(t)\|,$$

which is what had to be proved. □

Corollary

For a naturally parametrised curve, $(I, \mathbf{r} = \mathbf{r}(s))$, we have

$$k_{\pm}(s) = \frac{d\theta}{ds}.$$

Remark

From the previous formula, we obtain

$$k \equiv \|k_{\pm}\| = \left|\frac{d\theta}{ds}\right|,$$

which is exactly the formula for the curvature of an arbitrary space curve which, therefore, remains valid, as expected, for the particular case of plane curves.

Definition

A point $\Omega \in \mathbb{R}^2$ is called the *centre of curvature* at $\mathbf{r}_0 = \mathbf{r}(t_0)$ of a parametrised plane curve $\mathbf{r} : I \to \mathbb{R}^2$ if there exists a circle (γ), centred at Ω, which is tangent to the curve at $\mathbf{r}_0 = \mathbf{r}(t_0)$, with $t_0 \in I$, such that the signed curvatures of $\mathbf{r}$ and γ at $\mathbf{r}_0$ coincide. It follows that the position of Ω for an arbitrary $t \in I$ is given by:

$$\Omega(t) = \mathbf{r}(t) + \frac{1}{k_{\pm}(t)} \frac{J\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Remark

The notion of centre of curvature is invariant under a reparametrisation: if $(J, \rho = \rho(u))$ is equivalent to $\mathbf{r}$, with reparametrisation $\lambda : I \to J$, then $\mathbf{r}'(t) = \boldsymbol{\rho}'(\lambda(t))\lambda'(t)$ and $k_{\pm}[\mathbf{r}](t) = \operatorname{sgn}(\lambda')k_{\pm}[\rho](\lambda(t))$. Clearly, issues may only arise when $\lambda' < 0$, but in this case $J\mathbf{r}'$ changes orientation and $k_{\pm}$ changes sign, so overall the situation remains unchanged.

We have defined the evolute of a plane curve as the envelope of the family of normals to the curve. The following theorem provides an alternative approach.

Property

The evolute of a plane curve is the locus of the centres of curvature of the curve.

Proof.

The centre of curvature of a curve for an arbitrary parameter value is

$$\Omega(t) = \mathbf{r}(t) + \frac{\|\mathbf{r}'(t)\|^2}{\mathbf{r}''(t)\cdot J\mathbf{r}'(t)}\cdot J\mathbf{r}'(t) = (x(t), y(t)) + \frac{x'^2 + y'^2}{x'y'' - x''y'}\{-y', x'\}.$$

$\square$

Proof.

Thus, if $\Omega(t) = (X(t), Y(t))$, projecting the previous equation onto the coordinate axes, we obtain the parametric equations of the locus described by the centres of curvature:

$$\begin{cases} X(t) = x(t) - \dfrac{y'(x'^2 + y'^2)}{x'y'' - x''y'}, \\ Y(t) = y(t) + \dfrac{x'(x'^2 + y'^2)}{x'y'' - x''y'}, \end{cases}$$

which are precisely the equations of the evolute. □

From the previous observation we immediately deduce:

Corollary

The definition of the evolute also makes sense for regular curves (in other words, two equivalent parametrised curves have the same evolute).

Example

It can be shown that the evolute of the astroid

$$\begin{cases} x(t) = a\cos^3 t, \\ y = a\sin^3 t. \end{cases}$$

is also an astroid (see the following figure).

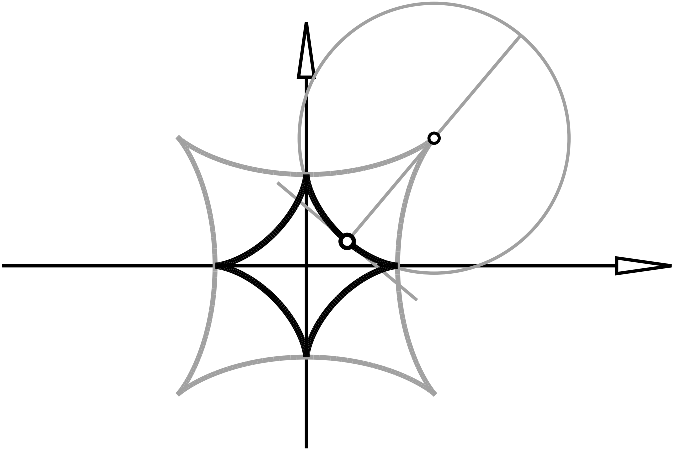


Figure: Evolute of an astroid

Example

The evolute of a cycloid

$$\begin{cases} x(t) = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$$

is also a cycloid (see the following figure).

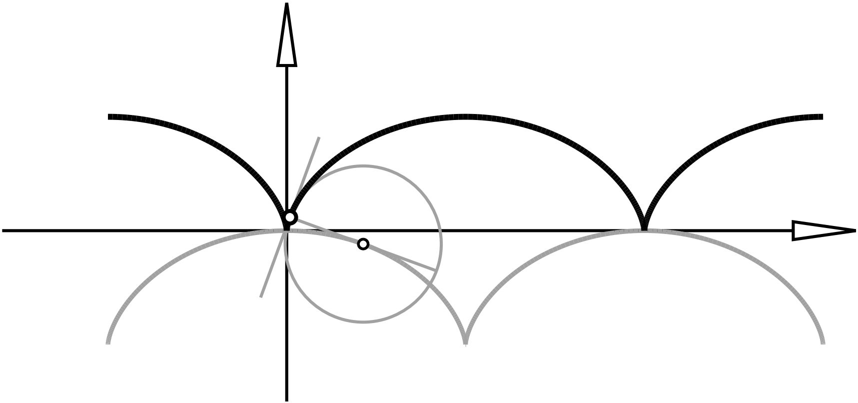


Figure: Evolute of a Cycloid

Another interesting plane curve associated to a given plane curve is the so-called *involute*, which is, as we shall soon see, in a certain sense, the inverse of the evolute.

Definition

Let $(I, \mathbf{r} = \mathbf{r}(s))$ be a naturally parametrised curve and $c \in I$. The *involute* of $\mathbf{r}$ with origin at $\mathbf{r}(c)$ (or, more briefly, at c) is the parametrised curve $(I, \rho[\mathbf{r}, c] = \rho[\mathbf{r}, c](s))$, where

$$\rho[\mathbf{r}, c](s) = \mathbf{r}(s) + (c - s)\mathbf{r}'(s).$$

Remark

In general, s is *not* a natural parameter along the curve $\boldsymbol{\rho}$.

If $(I, \mathbf{r} = \mathbf{r}(t))$ is an arbitrary parametrised curve, we can replace the parameter t by the arc length $s = \int\limits_0^t \|\mathbf{r}'(\tau)\| d\tau$ and define the involute of $\mathbf{r}$ as the involute of the equivalent naturally parametrised curve, the natural parameter being the arc length. It is easy to see that the following proposition holds:

Property

Let $(I, \mathbf{r} = \mathbf{r}(t))$ *be a parametrised curve. The involute of* $\mathbf{r}$ *with origin at* $c \in I$ *is given by*

$$\rho(t) = \mathbf{r}(t) + (c - s(t))\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

where $s = s(t)$ *is the arc length of* $\mathbf{r}$*.*

Example

Let $\mathbf{r}(t) = (a\cos t, a\sin t)$ be a circle. Then

$$\begin{cases} \mathbf{r}'(t) = \{-a\sin t, a\cos t\} \\ x'^2 + y'^2 = a^2 \\ s(t) = \int\limits_0^t a dt = at, \end{cases}$$

so the equation of the involute is

$$\rho(t) = (a\cos t, a\sin t) + \frac{(c-at)}{a}\{-a\sin t, a\cos t\}$$

Example

or, projected onto axes,

$$\begin{cases} X(t) = a\cos t - (c - at)\sin t \\ Y(t) = a\sin t + (c - at)\cos t \end{cases}.$$

We have depicted, in the figure on the next page, an involute of the circle of radius 1.5, with origin at the point of parameter 0.

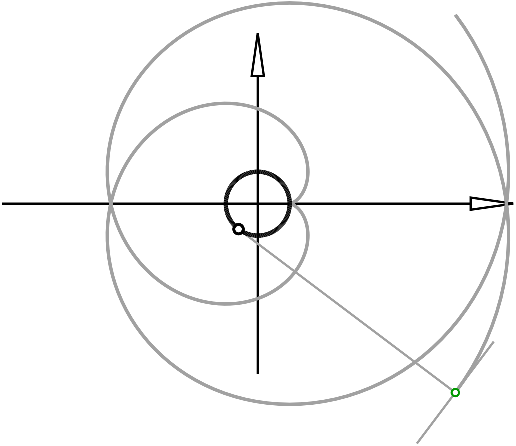


Figure: An involute of a circle

The following lemma establishes a connection between the signed curvature of a parametrised curve and that of its involute, and will serve as a tool to establish a relation between evolute and involute.

Lemma

Let $(I, \mathbf{r} = \mathbf{r}(s))$ *be a naturally parametrised curve and* ρ *its involute with origin at* $c \in I$*. Then the signed curvature of* ρ *is given by*

$$k_{\pm}[\rho](s) = \frac{\operatorname{sgn}(k_{\pm}[\mathbf{r}](s))}{|c - s|}.$$

Proof.

We have

$$\begin{aligned}
\rho'(s) &= \mathbf{r}'(s) + (c-s)\mathbf{r}''(s) - \mathbf{r}'(s) = (c-s)\mathbf{r}''(s) = \\
&= (c-s)k_{\pm}[\mathbf{r}](s) \cdot J\mathbf{r}'(s) \\
\rho''(s) &= -k_{\pm}[\mathbf{r}](s) \cdot J\mathbf{r}'(s) + (c-s)(k_{\pm}[\mathbf{r}](s))' \cdot J\mathbf{r}'(s) + \\
&+ (c-s)k_{\pm}[\mathbf{r}](s) \cdot J\mathbf{r}''(s) \\
&= [-k_{\pm}[\mathbf{r}](s) + (c-s)(k_{\pm}[\mathbf{r}](s))'] \cdot J\mathbf{r}'(s) - \\
&- (c-s)(k_{\pm}[\mathbf{r}](s))^2 \cdot \mathbf{r}'(s),
\end{aligned}$$

from which

$$J\rho' = -(c-s)k_{\pm}[\mathbf{r}](s) \cdot \mathbf{r}'(s),$$

□

Proof.

while

$$\rho''(s) \cdot J\rho'(s) = (c - s)^2 \cdot (k_{\pm}[\mathbf{r}](s))^3.$$

The conclusion now follows from the definition of signed curvature. □

The following theorem gives us a connection between the involute and the evolute. In many textbooks, this connection is actually taken as the definition of the involute.

Theorem

Let $(I, \mathbf{r} = \mathbf{r}(s))$ be a naturally parametrised curve and ρ – its involute with origin at $c \in I$. Then the evolute of ρ is $\mathbf{r}$.

Proof.

The evolute of ρ is given, as is known, by the equation

$$\rho_1(s) = \rho(s) + \frac{1}{k_\pm[\rho](s)} \cdot \frac{J\rho'(s)}{\|\rho'(s)\|}.$$

Using the previous lemma to express the signed curvature of ρ as a function of the signed curvature of $\mathbf{r}$, we obtain

$$\rho_1(s) = \mathbf{r}(s)+(c-s)\mathbf{r}'(s)+\frac{|c-s|}{\operatorname{sgn}(k_\pm[\mathbf{r}](s))}\cdot\frac{(c-s)k_\pm[\mathbf{r}](s)\cdot J^2\mathbf{r}'(s)}{\|(c-s)k_\pm[\mathbf{r}](s)\cdot J\mathbf{r}'(s)\|} = \mathbf{r}(s).$$

□

Definition

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a parametrised curve. The *osculating circle* of $\mathbf{r}$ at a point $t \in I$ is the circle with centre at the centre of curvature $\Omega(t)$, with radius equal to the radius of curvature $\frac{1}{k(t)}$ of the curve at that point.

Just as the osculating plane at a point of a space curve can be regarded as the limiting position of the plane determined by three neighbouring points, as they approach the given point, the osculating circle is the limiting position of a circle determined by three neighbouring points, as these approach the given point. More precisely, we have:

Theorem

Let $(I, \mathbf{r} = \mathbf{r}(t))$ be a parametrised plane curve and $t_1 < t_2 < t_3 \in I$. Let $C(t_1, t_2, t_3)$ be the circle passing through $\mathbf{r}(t_1), \mathbf{r}(t_2), \mathbf{r}(t_3)$. Suppose that for some value $t \in I$ of the parameter we have $k_\pm(t) \neq 0$. Then the osculating circle of $\mathbf{r}$ at the point $\mathbf{r}(t)$ is the circle

$$C = \lim_{\substack{t_1 \to t \\ t_2 \to t \\ t_3 \to t}} C(t_1, t_2, t_3).$$

Proof.

Let $A(t_1, t_2, t_3)$ be the centre of the circle $C(t_1, t_2, t_3)$ and $f : I \to \mathbb{R}$ the function defined by $f(t) = \|\mathbf{r}(t) - A\|^2$. $\square$

Proof.

Then clearly f is smooth and we have:

$$\begin{cases} f'(t) = 2\mathbf{r}' \cdot (\mathbf{r}(t) - A) \\ f''(t) = 2\mathbf{r}''(t) \cdot (\mathbf{r}(t) - A) + 2\|\mathbf{r}'(t)\|^2 \end{cases}.$$

Since f is differentiable and $f(t_1) = f(t_2) = f(t_3)$, by the mean value theorem it follows that there exist two points $u_1, u_2 \in I$, with $t_1 < u_1 < t_2 < u_2 < t_3$ such that

$$f'(u_1) = f'(u_2) = 0.$$

□

Proof.

On the other hand, if we apply the mean value theorem once more, this time to the derivative f', which is also differentiable, it follows that there exists $v \in (u_1, u_2)$ such that

$$f''(v) = 0.$$

Now, as $t_1, t_2, t_3 \to t$, we also have $u_1, u_2, v \to t$, so at the limit we must obtain:

$$\begin{cases} \mathbf{r}'(t) \cdot (\mathbf{r}(t) - A(t)) = 0 \\ \mathbf{r}''(t) \cdot (\mathbf{r}(t) - A(t)) = -\|\mathbf{r}'(t)\|^2 \end{cases} . \tag{*}$$

$\square$

Proof.

Here

$$A(t) = \lim_{\substack{t_1 \to t \\ t_2 \to t \\ t_3 \to t}} A(t_1, t_2, t_3).$$

From (*) and the definition of signed curvature, it follows that

$$\mathbf{r}(t) - A(t) = \frac{-1}{k_{\pm}(t)} \cdot \frac{J\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

therefore C is the osculating circle of the curve $\mathbf{r}$ at the point $\mathbf{r}(t)$. □

The existence and uniqueness theorem for parametrised plane curves is similar to the corresponding theorem for space curves and can be proven in the same way, so we shall omit the proof here.

Theorem

Let $f : I \to \mathbb{R}$ be a continuous function. Then there exists a regular, naturally parametrised plane curve $(I, \mathbf{r} = \mathbf{r}(s))$ such that $\forall s \in I$, $k_\pm(s) = f(s)$. The curve $\mathbf{r}$ is unique up to a rigid motion of $\mathbb{R}^2$.

To give an example, we shall find the curve $\mathbf{r}$ for the particular case when the function f is a constant α, for any real value of the parameter s.

Starting from the geometric interpretation of the signed curvature, we obtain

$$\alpha = k_{\pm}(s) = \frac{d\theta}{ds},$$

therefore θ (the rotation angle) will be an affine function of s:

$$\theta = \alpha s + \theta_0,$$

where θ_0 is a constant. On the other hand, from the definition of the rotation angle, we obtain:

$$\boldsymbol{\tau}(s) \equiv \left\{\frac{dx}{ds}, \frac{dy}{ds}\right\} = \{\cos\theta(s), \sin\theta(s)\} = \{\cos(\alpha s + \theta_0), \sin(\alpha s + \theta_0)\}$$

from which results the system of differential equations:

$$\begin{cases} \frac{dx}{ds} = \cos(\alpha s + \theta_0) \\ \frac{dy}{ds} = \sin(\alpha s + \theta_0) \end{cases}.$$

Since the equations are separated, the system can be easily integrated and we obtain the solution:

$$\begin{cases} x = \frac{1}{\alpha}[\sin\alpha s\cos\theta_0 + \cos\alpha s\sin\theta_0] + x_0, \\ y = \frac{1}{\alpha}[-\cos\alpha s\cos\theta_0 + \sin\alpha s\sin\theta_0] + y_0 \end{cases}, \tag{*}$$

where x_0 and y_0 are two integration constants. The solution (*) can be written in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha}\sin\alpha s\cos\theta_0 + \frac{1}{\alpha}\cos\alpha s\sin\theta_0 \\ -\frac{1}{\alpha}\cos\alpha s\cos\theta_0 + \frac{1}{\alpha}\sin\alpha s\sin\theta_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

or, alternatively,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{2} - \theta_0\right) & \sin\left(\frac{\pi}{2} - \theta_0\right) \\ -\sin\left(\frac{\pi}{2} - \theta_0\right) & \cos\left(\frac{\pi}{2} - \theta_0\right) \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha}\cos\alpha s \\ \frac{1}{\alpha}\sin\alpha s \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

which shows that any plane curve with constant positive signed curvature α can be obtained from the curve

$$\begin{cases} x = \frac{1}{\alpha}\cos\alpha s \\ y = \frac{1}{\alpha}\sin\alpha s \end{cases}$$

by applying a rotation followed by a translation, i.e. a *rigid motion*. But this is a circle of radius $1/\alpha$ centred at the origin. The conclusion is that, abstracting from a rigid motion of the plane, *the only plane curve with constant positive curvature α is the circle of radius $1/\alpha$.*

Parametrised Surfaces (Sheets)

Definition

A *parametrised surface* (sheet) in $\mathbb{R}^3$ is a smooth map $\mathbf{r} : U \to \mathbb{R}^3$, $(u, v) \to \mathbf{r}(u, v)$, where U is a domain (an open and connected subset) of $\mathbb{R}^2$, while $\mathbf{r}$ satisfies the condition

$$\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}} \neq 0. \tag{80}$$

Condition (80) is called the *regularity condition*.

A parametrised surface is usually denoted by $(U, \mathbf{r})$, $(U, \mathbf{r} = \mathbf{r}(u, v))$, or simply $\mathbf{r} = \mathbf{r}(u, v)$, if the domain of definition is implied.

Definition

The set $\mathbf{r}(U) \subset \mathbb{R}^3$ is called the *support* of the parametrised surface $(U, \mathbf{r})$.

Parametrised Surfaces (Sheets)

Remark

Generally, the same point in the support of a parametrised surface $(U, \mathbf{r})$ may correspond to several distinct pairs (u, v), since the function $\mathbf{r}$ is not assumed to be injective.

Definition

Two parametrised surfaces $(U, \mathbf{r})$ and $(V, \mathbf{r}_1)$ are called *equivalent* if there exists a diffeomorphism $\lambda : U \to V$ such that $\mathbf{r} = \mathbf{r}_1 \circ \lambda$.

Remark

The supports of two equivalent parametrised surfaces always coincide.

Parametrised Surfaces (Sheets)

Exemple

1. If $U = \mathbb{R}^2$, while $\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$, $\mathbf{a} \times \mathbf{b} \neq 0$, then the support of the parametrised surface is the plane passing through the position vector point $\mathbf{r}_0$ and orthogonal to the vector $\mathbf{a} \times \mathbf{b}$. This parametrised surface is called a *plane*.
2. Let $U = \{(u, v) \in \mathbb{R}^2 | \pi/2 < u < \pi/2, 0 < v < 2\pi\}$ and

$$\mathbf{r}(u, v) = (R \cos u \cos v, R \cos u \sin v, R \sin u).$$

   The support of this parametrised surface is the sphere of radius R, centred at the origin of $\mathbb{R}^3$, except for one meridian. The parameters u and v are analogous to geographic coordinates.

Parametrised Surfaces (Sheets)

Exemple

3. Let $U = \mathbb{R}^2$, $\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v^3\mathbf{b}$, $\mathbf{a} \times \mathbf{b} \neq 0$. The support of this parametrised surface is the same as the support of the surface in example 1), but the two surfaces are not equivalent, since the map $(u, v) \to (u, v^3)$ is not a diffeomorphism.

Definition

A subset $S \subset \mathbb{R}^3$ is called a *regular surface* if every point $a \in S$ has an open neighbourhood W in S such that there exists a parametrised surface $(U, \mathbf{r})$ with $\mathbf{r}(U) = W$, while the map $\mathbf{r} : U \to W$ is a homeomorphism. The pair is called a *local parametrisation* of the surface S around the point a, while the support $\mathbf{r}(U)$ is called the *domain* of the parametrisation. A surface S that admits a *global* parametrisation (i.e., a local parametrisation $(U, \mathbf{r})$ for which $\mathbf{r}(U) = S$) is called a *simple* surface.

The ways in which we described curves are equally available for surfaces.

Parametric representation. If S is a surface, and $(U, \mathbf{r})$ is a local parametrisation of S, then, if $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, the equations

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}, \quad (u,v) \in U,$$

are called the *parametric equations* of the surface. Let us emphasise once again that these are *local* equations; they cannot be used to describe all points on the surface unless we are dealing with a global parametrisation of a simple surface.

Explicit representation. If $f : U \to \mathbb{R}$ is a smooth map, where $U \subset \mathbb{R}^2$ is a domain, then its graph, $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$, is a simple surface. Indeed, we have the global parametrisation $\mathbf{r} : U \to \mathbb{R}^3$, $\mathbf{r}(u, v) = (u, v, f(u, v))$.

Implicit representation. Let $F : V \to \mathbb{R}$ be a smooth function, with $V \subset \mathbb{R}^3$ an open subset. We denote by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$$

the level set 0 of F. If, at each point of S, the vector

$$\operatorname{grad} F = \left\{ \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\}$$

is non-zero, then S is a surface.

Indeed, if, for example, at $(x_0, y_0, z_0) \in S$, we have $F'_z(x_0, y_0, z_0) \neq 0$, then, by the implicit function theorem, it follows that there exists an open neighbourhood (in the topology of $\mathbb{R}^3$) M of (x_0, y_0, z_0) such that the set $M \cap S$ (which is an open neighbourhood of (x_0, y_0, z_0), this time in the topology of S) is the graph of a smooth function $z = f(x, y)$. Therefore, as follows from the previous paragraph, there exists a local parametrisation of S around the point (x_0, y_0, z_0). We remark that this parametrisation is global for $M \cap S$ but, in general, not for the entire S. Even if F'_z is non-zero over the entire set S, it is still not certain that the function f can be defined globally. Thus, unlike explicitly given surfaces, implicitly defined ones are generally not simple.

Exemple

1. The plane Π that passes through the point with position vector $\mathbf{r}_0$ and has direction given by the vectors $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{a} \times \mathbf{b} \neq 0$, is a simple surface, with the global parametrisation $\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$. In projection onto the coordinate axes, the parametric equations of the plane are

$$\begin{cases} x = x_0 + ua_x + vb_x \\ y = y_0 + ua_y + vb_y \\ z = z_0 + ua_z + vb_z \end{cases}, \quad (u, v) \in \mathbb{R}^2.$$

Exemple

2. *Surfaces of revolution*
Let C be a curve in the xOz plane, which does not intersect the Oz axis, and let S be the subset of $\mathbb{R}^3$ obtained by rotating C about the Oz axis. Let v be the angle of rotation in the xOz plane, and let a' be the point of S obtained by rotating the point $a \in C$ through an angle v_0. Let $(I, \rho = \rho(t))$ be a local parametrisation of the curve C around the point a, $\rho(t) = (x(t), z(t))$. Then we obtain a local parametrisation of S around a',

$$\mathbf{r}(t, v) = (x(t)\cos(v + v_0), x(t)\sin(v + v_0), z(t)),$$

defined on the domain $U = I \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Exemple

3. *The Sphere.* Let $F : \mathbb{R}^3 \to \mathbb{R}$, $F(x, y, z) = x^2 + y^2 + z^2 - R^2$. F is clearly a smooth function, and its level set 0

$$S_R^2 = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$$

is the sphere of radius R, centred at the origin. The gradient of F is

$$\operatorname{grad} F = \{2x, 2y, 2z\}$$

and clearly does not vanish on the sphere S_R^2, hence this sphere is a regular surface. It is worth noting that S_R^2 is not simple, since it is a compact subset of $\mathbb{R}^3$, and therefore cannot be homeomorphic to an open subset of $\mathbb{R}^2$, which is not compact.

Exemple

4. *The Torus.* Now let us choose $F : \mathbb{R}^3 \to \mathbb{R}$,

$$F(x, y, z) = (\sqrt{x^2 + y^2} - a)^2 + z^2 - b^2, \;\; 0 < b < a.$$

Its level set 0,

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 | F(x, y, z) = 0\}$$

is called the *two-dimensional torus*. By computing the partial derivatives of F with respect to the coordinates, we obtain

$$\begin{cases} F'_x = 2\left(\sqrt{x^2 + y^2} - a\right) \frac{x}{\sqrt{x^2+y^2}} \\ F'_y = 2\left(\sqrt{x^2 + y^2} - a\right) \frac{y}{\sqrt{x^2+y^2}} \end{cases},$$

therefore, the gradient of F vanishes if and only if

$$\begin{cases} x = y = z = 0 \quad \text{or} \\ x^2 + y^2 = 0, y = 0, z = 0 \quad \text{or} \\ x = 0, x^2 + y^2 = a^2, z = 0 \quad \text{or} \\ x^2 + y^2 = a^2, z = 0 \end{cases} .$$

It is easy to verify that $\operatorname{grad} F$ is non-zero on T^2, therefore the torus is a surface (again, it is compact, hence it cannot be simple).
The torus can also be obtained by rotating the circle $(x-a)^2 + z^2 = b^2$ (lying in the plane xOz) around the Oz axis.

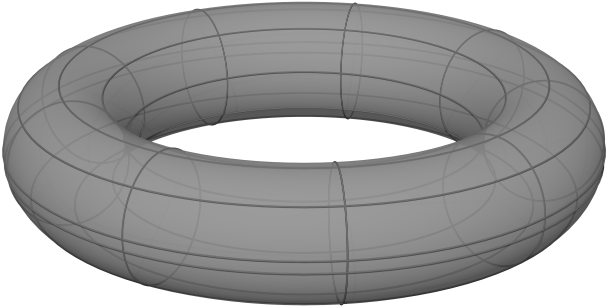


Figure: The Torus

Definition

Let S be a surface, $(U, \mathbf{r})$ – a local parameterisation of it and $W = \mathbf{r}(U)$. Then the map $\mathbf{r}^{-1} : W \to U$ is a bijection, which associates to each point in W a pair of real numbers $(u, v) \in U$. This correspondence is called a *curvilinear coordinate system* on S or a *chart* on S.

Theorem

(on change of parameters) Let $(U, \mathbf{r})$ and $(U_1, \mathbf{r}_1)$ be two local parameterisations of a surface S and $\mathbf{r}(U) = \mathbf{r}(U_1)$. Then there exists a diffeomorphism $\lambda : U \to U_1$ such that $\mathbf{r} = \mathbf{r}_1 \circ \lambda$. The diffeomorphism λ is called a change of parameters or reparameterisation.

Before proving the theorem, we need to make a few observations. If there exists a change of parameters λ, then, from the relation $\mathbf{r} = \mathbf{r}_1 \circ \lambda$, it follows, of course, that $\lambda = \mathbf{r}_1^{-1} \circ \mathbf{r}$. The real difficulty is to prove that both λ and its inverse are smooth. Although $\mathbf{r}$ is smooth, $\mathbf{r}^{-1}$ is not[4], since its domain, $\mathbf{r}_1(U_1)$, is not an open subset of Euclidean space $\mathbb{R}^3$. We will nevertheless prove, in the following lemma, that locally $\mathbf{r}_1^{-1}$ is the *restriction* of a smooth function. We emphasise that this representation of $\mathbf{r}_1^{-1}$ is only local: in general, $\mathbf{r}_1^{-1}$ cannot be written, on its entire domain, as the restriction of a *single* map defined on an open set (in the topology of $\mathbb{R}^3$) containing the set $\mathbf{r}_1(U_1)$.

[4] at least not in the classical sense

Lemma

Let $(U, \mathbf{r})$ be a local parameterisation of the surface S, $\mathbf{r}(U) = W$ and $\mathbf{r}^{-1} : W \to U$ – the inverse map. Then, for each point $a \in W$, there exists an open set (in the topology of $\mathbb{R}^3$) $B \ni a$ and a smooth map $G : B \to U$ such that $\mathbf{r}^{-1}|_{W \cap B} = G|_{W \cap B}$.

Proof of the lemma.

Let $\mathbf{r}(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$ and $a = \mathbf{r}(u_0, v_0)$. Due to the regularity of $\mathbf{r}$, the Jacobian matrix

$$\begin{pmatrix} f'_{1u} & f'_{1v} \\ f'_{2u} & f'_{2v} \\ f'_{3u} & f'_{3v} \end{pmatrix}$$

□

Proof.

has rank two. Without loss of generality, we can assume that

$$\begin{vmatrix} f'_{1u} & f'_{1v} \\ f'_{2u} & f'_{2v} \end{vmatrix} \neq 0.$$

Then, from the inverse function theorem for the map

$$f : (u, v) \longrightarrow (x = f_1(u, v), y = f_2(u, v)),$$

it follows that there exists an open neighbourhood V of the point (u_0, v_0) in U and an open neighbourhood $\widetilde{V}$ of the point $(x_0 = f_1(u_0, v_0), y_0 = f_2(u_0, v_0))$ in the xOy plane such that $f : V \to \widetilde{V}$ is a diffeomorphism. Since the map $\mathbf{r} : U \to W$ is a homeomorphism, $\mathbf{r}(V)$ is an open neighbourhood in S of the point $a = \mathbf{r}(u_0, v_0)$, therefore, in $\mathbb{R}^3$, there exists an open neighbourhood B of the point a

Proof.

Let $p : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \to (x, y)$ be the orthogonal projection onto the coordinate plane xOy. We will show that the map $G = (f^{-1} \circ p)|_B : B \to U$ is the one we are looking for. Indeed, G is smooth, being a composition of smooth maps. Moreover, each point (x, y, z) in $B \cap W$ corresponds to a unique point $(u, v) = \mathbf{r}^{-1}(x, y, z)$ in V, and each point $(x, y) \in \widetilde{V}$ – to the point $(u, v) = f^{-1}(x, y)$ in V. Thus, for the points $(x, y, z) \in B \cap W$, we have

$$\mathbf{r}^{-1}(x, y, z) = f^{-1}(x, y) = f^{-1}(p(x, y, z)) = G(x, y, z).$$

□

Proof of the theorem.

Let $(U, \mathbf{r})$ and $(U_1, \mathbf{r}_1)$ be two local parameterisations of the surface S such that $\mathbf{r}(U) = \mathbf{r}_1(U_1) = W$. Consider the map $\lambda = \mathbf{r}_1^{-1} \circ \mathbf{r} : U \to U_1$. Since $\mathbf{r}_1 : U_1 \to W$ is a homeomorphism, the same is true for $\mathbf{r}_1^{-1} : W \to U_1$, and thus λ is a homeomorphism, as the composition of two homeomorphisms. We only need to prove that λ and λ^{-1} are smooth. To prove the smoothness of λ it is sufficient to prove that each point $(u_0, v_0) \in U$ has an open neighbourhood $V \subset U$ such that $\lambda|_V$ is smooth. We apply the previous lemma to the parameterisation $(U_1, \mathbf{r}_1)$ at the point $a = \mathbf{r}_1(\lambda(u_0, v_0))$. Let $G : B \to U_1$ be the smooth map such that $\mathbf{r}_1^{-1}|_{B\cap W} = G|_{B\cap W}$ and $V = \mathbf{r}^{-1}(B \cap W)$. Then $\lambda|_V = \mathbf{r}_1^{-1} \circ \mathbf{r}|_V = (G \circ \mathbf{r})|_V$ and, therefore, $\lambda|_V$ is smooth, being the restriction of a smooth map. The smoothness of λ^{-1} is proved in the same way, replacing the parameterisation $\mathbf{r}_1$ with the parameterisation $\mathbf{r}$. □

Locally, each surface is the support of a parametrised surface. The converse statement is not true, that is, the support of an arbitrary parametrised surface is *not* a surface. However, if we take an arbitrary parametrised surface and restrict its domain, we can obtain a parametrised surface whose support is a regular surface. Thus, we have:

Theorem

Let $(U, \mathbf{r})$ be a regular parametrised surface. Then each point $(u_0, v_0) \in U$ has an open neighbourhood $V \subset U$ such that $\mathbf{r}(V)$ is a simple surface in $\mathbb{R}^3$, for which the pair $(V, \mathbf{r}|_V)$ is a global parameterisation.

Proof.

The only additional condition we must impose on V is that the map $\mathbf{r}|_V : V \to \mathbf{r}(V)$ should be a homeomorphism. Let $\mathbf{r}(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$. Without loss of generality, we can assume that the Jacobian of the map $f : (u, v) \to (x = f_1(u, v), y = f_2(u, v))$ is non-zero at (u_0, v_0). Then, from the inverse function theorem it follows that there exists an open neighbourhood $V \subset U$ of the point (u_0, v_0) and an open neighbourhood $\widetilde{V}$ of the point $(x_0, y_0) = f(u_0, v_0)$ such that the map $F : V \to \widetilde{V}$ is a diffeomorphism. We will first prove the injectivity of the map $\mathbf{r}|_V : V \to \mathbf{r}(V)$. $\square$

Proof.

Let $(u_1, v_1), (u_2, v_2) \in V$ such that $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$. Then, in particular,

$$f_1(u_1, v_1) = f_1(u_2, v_2) \quad \text{and} \quad f_2(u_1, v_1) = f_2(u_2, v_2),$$

that is, $f(u_1, v_1) = f(u_2, v_2)$. Now, f is a diffeomorphism, hence it is, in particular, injective, therefore $(u_1, v_1) = (u_2, v_2)$. The map $\mathbf{r} : U \to \mathbb{R}^3$ is continuous, hence so is its restriction $\mathbf{r}|_V : V \to \mathbb{R}^3$. Since on $\mathbf{r}(V)$ we use the subspace topology, the map $\mathbf{r}|_V : V \to \mathbf{r}(V)$ is also continuous. To prove the continuity of the inverse map, we note that it is the composition of the following continuous maps: $(x, y, z) \in \mathbf{r}(V) \to (x, y) \in \widetilde{V} \to (u, v) = f^{-1}(x, y) \in V$, as seen in the proof of the lemma. $\square$

Curves on a Surface

We say that a smooth parametrised curve $(I, \rho = \rho(t))$ lies on a surface S if its support $\rho(I)$ is contained in S. In particular, we can easily describe parametrised curves whose support is contained in the domain of a parameterisation $(U, \mathbf{r})$ of the surface S.

Theorem

Let $(U, \mathbf{r})$ be a parameterisation of the surface S and $(I, \rho = \rho(t))$ a smooth parametrised curve whose support is contained in $\mathbf{r}(U)$. Then there exists a unique smooth parametrised curve $(I, \tilde{\rho})$ on U such that

$$\rho(t) \equiv \rho(\tilde{\rho}(t)). \tag{81}$$

Conversely, any smooth parametrised curve $\tilde{\rho}$ on U defines, via formula (81), a smooth parametrised curve on $\mathbf{r}(U)$. The regularity of ρ in t is equivalent to the regularity of $\tilde{\rho}$ in t.

Proof.

Since the map $\mathbf{r} : U \to \mathbf{r}(U)$ is a homeomorphism, and $\rho(I) \subset \mathbf{r}(U)$, from formula (81) we can obtain $\tilde{\rho}$ by setting

$$\tilde{\rho} = \mathbf{r}^{-1} \circ \rho.$$

Clearly, $\tilde{\rho}$ is continuous, being the composition of two continuous maps. We now verify that $\tilde{\rho}$ is in fact smooth. If $t \in I$, then $\rho(t) \in \mathbf{r}(U)$. According to the lemma from the previous paragraph, there exists an open neighbourhood B of the point $\rho(t)$ in $\mathbb{R}^3$ and a smooth map $G : B \to U$ such that $\mathbf{r}^{-1}|_{B \cap \mathbf{r}(U)} = G|_{B \cap \mathbf{r}(U)}$. Therefore, the map $\tilde{\rho}$ can be locally represented near the point t as a composition of smooth maps $G \circ \rho$ and hence is smooth. $\square$

Curves on a Surface

Proof.

The converse statement can be proved even more simply, since we have

$$\boldsymbol{\rho} = \mathbf{r} \circ \tilde{\boldsymbol{\rho}}$$

and since $\mathbf{r}$ and $\tilde{\boldsymbol{\rho}}$ are smooth, so is $\boldsymbol{\rho}$.

To verify the equivalence of the regularity conditions for $\boldsymbol{\rho}$ and $\tilde{\boldsymbol{\rho}}$, we consider the components of the path $\tilde{\boldsymbol{\rho}}$:

$$\tilde{\boldsymbol{\rho}} = (u(t), v(t)).$$

Then equality (81) becomes

$$\boldsymbol{\rho}(t) = \mathbf{r}(u(t), v(t)).$$

□

Proof.

Differentiating this relation, we obtain

$$\rho'(t) = \mathbf{r}'_{\mathbf{u}} \cdot u'(t) + \mathbf{r}'_{\mathbf{v}} \cdot v'(t).$$

Since the vectors $\mathbf{r}'_{\mathbf{u}}$ and $\mathbf{r}'_{\mathbf{v}}$ are not collinear (as the surface is assumed, as usual, to be regular), it follows from the previous relation that $\rho'(t) = 0$ if and only if $u'(t) = 0$ and $v'(t) = 0$, that is, if and only if $\tilde{\rho}^{'}(t) = 0$. $\square$

Curves on a Surface

Definition

The parametrised curve $\tilde{\rho}(t)$ on the domain U is called the *local representation* of the parametrised curve $\rho(t)$ in the local parameterisation $(U, \mathbf{r})$, and the equations

$$\begin{cases} u = u(t) \\ v = v(t) \end{cases}$$

are called the *local equations* of $\rho(t)$ in the given parameterisation.

Example

Let $(U, \mathbf{r} = \mathbf{r}(u, v))$ be a local parameterisation of S and $(u_0, v_0) \in U$. We consider, in $\mathbf{r}(U) \subset S$, the paths defined by the local equations

$$\begin{cases} u = u_0 + t \end{cases} \tag{82}$$

Example

and

$$\begin{cases} u = u_0 \\ v = v_0 + t \end{cases} \tag{83}$$

It is easy to see that the supports of these two paths indeed lie on S (more precisely, in $\mathbf{r}(U)$). Through each $\mathbf{r}(u_0, v_0) \in \mathbf{r}(U)$ pass exactly two such curves, one of each type. These curves are called *coordinate lines or coordinate curves* on the surface S, in the local parameterisation $(U, \mathbf{r})$.

Tangent Vector Space, Tangent Plane and Normal to a Surface

Let us denote, for any $a \in \mathbb{R}^3$, by R_a^3 the space of vectors anchored at a. This is clearly a three-dimensional vector space, naturally isomorphic to $\mathbb{R}^3$.

Definition

A vector $\mathbf{h} \in \mathbb{R}_a^3$ is called a *tangent vector* to the surface S at the point a if there exists a parametrised curve $(I, \rho(t))$ on S and a $t_0 \in I$ such that $\rho(t_0) = a$ and $\rho'(t_0) = \mathbf{h}$. Thus, a tangent vector to a surface is simply a tangent vector to a curve on the surface.

We denote by T_aS the set of tangent vectors to the surface S at $a \in S$. The following lemma is trivial, but will play an essential role in what follows.

Tangent Vector Space, Tangent Plane and Normal to a Surface

Lemma

Let $\rho = \rho(t)$ be a parametrised curve on S, given by the local equations $u = u(t)$, $v = v(t)$ with respect to a local parameterisation $(U, \mathbf{r})$ of S. Then we have the relation

$$\rho'(t) = u'(t)\mathbf{r}'_{\mathbf{u}}(u(t), v(t)) + v'(t)\mathbf{r}'_{\mathbf{v}}(u(t), v(t)). \tag{84}$$

Proof.

We simply differentiate the relation $\rho(t) = \mathbf{r}(u(t), v(t))$ with respect to t. □

Tangent Vector Space, Tangent Plane and Normal to a Surface

Theorem

The set T_aS is a two-dimensional subspace of $\mathbb{R}^3_a$. If $(U, \mathbf{r})$ is a local parameterisation of S, and $a = \mathbf{r}(u_0, v_0)$, then the vectors $\mathbf{r}'_{\mathbf{u}}(u_0, v_0)$ and $\mathbf{r}'_{\mathbf{v}}(u_0, v_0)$ form a basis of this subspace, called the natural basis or coordinate basis of the tangent space.

Proof.

Let $(U, \mathbf{r})$ be a local parameterisation of S, with $a = \mathbf{r}(u_0, v_0)$. If the parametrised curve $(I, \rho(t))$ is on the surface and $\rho(t_0) = a$, then, possibly after shrinking the interval I, we may assume that $\rho(I) \subset \mathbf{r}(U)$, and its local equations, in this parameterisation of the surface, are $u = u(t), v = v(t)$. Then, from formula (84) it follows that

$$\rho'(t_0) = u'(t_0)\mathbf{r}'_{\mathbf{u}}(u_0, v_0) + v'(t_0)\mathbf{r}'_{\mathbf{v}}(u_0, v_0).$$

Proof.

Conversely, any vector of the form

$$\mathbf{h} = \alpha \mathbf{r}'_{\mathbf{u}}(u_0, v_0) + \beta \mathbf{r}'_{\mathbf{v}}(u_0, v_0)$$

is tangent to the parametrised curve given by the local equations

$$\begin{cases} u = u_0 + \alpha t \\ v = v_0 + \beta t \end{cases},$$

which is a curve on S passing through a at $t = t_0$, hence $\mathbf{h} \in T_a S$. □

Tangent Vector Space, Tangent Plane and Normal to a Surface

The vector space T_aS is called the *tangent space* to S at a. As mentioned above, $\mathbb{R}^3_a$ is naturally[5] isomorphic to $\mathbb{R}^3$. Based on this isomorphism, we may consider, when convenient, that T_aS is in fact a subspace of $\mathbb{R}^3$ rather than of $\mathbb{R}^3_a$. In this case, T_aS is a *vector plane*, in the sense that it passes through the origin of $\mathbb{R}^3$; then the plane passing through a and having T_aS as direction plane (i.e., the plane passing through a and parallel to T_aS) is called the *tangent plane* to S at the point a and is denoted by Π_aS.
If $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ is a local parameterisation of the surface S, and $a = \mathbf{r}(u_0, v_0) = (x_0, y_0, z_0) \in S$, then clearly the equation of the tangent plane to S at a must be

$$\begin{vmatrix} X - x_0 & Y - y_0 & Z - z_0 \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{vmatrix} = 0.$$

Tangent Vector Space, Tangent Plane and Normal to a Surface

Let now $(U, \mathbf{r} = \mathbf{r}(u, v))$ be a parameterisation of S and $(u_0, v_0) \in U$. If we vary the arguments with $\Delta u = \alpha \Delta t$, $\Delta v = \beta \Delta t$, with $\alpha, \beta \in \mathbb{R}$ fixed, then from Taylor's formula we get:

$$\mathbf{r}(u_0 + \Delta u, v_0 + \Delta v) = \mathbf{r}(u_0, v_0) + \Delta t \cdot (\alpha \mathbf{r}_{\mathbf{u}}'(u_0, v_0) + \beta \mathbf{r}_{\mathbf{v}}'(u_0, v_0)) + \Delta t \cdot \varepsilon,$$

with $\lim\limits_{\Delta t \to 0} \varepsilon = 0$. Using this formula, we will provide another characterisation of the tangent plane. Let Π be a plane in $\mathbb{R}^3$ passing through $a = \mathbf{r}(u_0, v_0)$, d – the distance from the point $\Delta a = \mathbf{r}(u_0 + \Delta u, v_0 + \Delta v)$ to the plane Π, and h – the distance between the points a and Δa.

Tangent Vector Space, Tangent Plane and Normal to a Surface

Theorem

The plane Π *is the tangent plane to the surface S at the point a if and only if for any variation of the parameters of the form* $\Delta u = \alpha \Delta t$, $\Delta v = \beta \Delta t$, *with* $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$, *we have*

$$\lim_{\Delta t \to 0} \frac{d}{h} = 0, \tag{85}$$

that is, the plane and the surface have first-order contact at a.

Proof.

Let $\mathbf{n}$ be the unit normal vector to the plane Π,

$$\Delta \mathbf{r} = \mathbf{r}(u_0 + \Delta u, v_0 + \Delta v) - \mathbf{r}(u_0, v_0).$$

Tangent Vector Space, Tangent Plane and Normal to a Surface

Proof.

Then $d = \Delta\mathbf{r} \cdot \mathbf{n}$, $h = \|\Delta\mathbf{r}\|$. Substituting $\Delta\mathbf{r}$ with its expression, we obtain:

$$\lim_{\Delta t\to 0}\frac{d}{h} = \lim_{\Delta t\to 0}\frac{\Delta t\cdot(\alpha\mathbf{r}'_{\mathbf{u}}(u_0,v_0)+\beta\mathbf{r}'_{\mathbf{v}}(u_0,v_0)+\varepsilon)\cdot\,\mathbf{n}}{|\Delta t|\cdot\|\alpha\mathbf{r}'_{\mathbf{u}}(u_0,v_0)+\beta\mathbf{r}'_{\mathbf{v}}(u_0,v_0)+\varepsilon\|} =$$
$$= \pm\frac{(\alpha\mathbf{r}'_{\mathbf{u}}(u_0,v_0)+\beta\mathbf{r}'_{\mathbf{v}}(u_0,v_0))\cdot\,\mathbf{n}}{\|\alpha\mathbf{r}'_{\mathbf{u}}(u_0,v_0)+\beta\mathbf{r}'_{\mathbf{v}}(u_0,v_0)\|},$$

hence

$$\lim_{\Delta t\to 0}\frac{d}{h} = 0 \iff (\alpha\mathbf{r}'_{\mathbf{u}}(u_0,v_0)+\beta\mathbf{r}'_{\mathbf{v}}(u_0,v_0))\cdot\,\mathbf{n} = 0. \tag{86}$$

□

Proof.

Necessity. If Π is the tangent plane, then the vectors $\mathbf{r}'_{\mathbf{u}}$ and $\mathbf{r}'_{\mathbf{v}}$, as direction vectors of the plane Π, are perpendicular to $\mathbf{n}$, therefore relation (86) holds.
Sufficiency. Now suppose that (86) holds. Choosing $\alpha = 1$, $\beta = 0$, we get $\mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n} = 0$. Similarly, for $\alpha = 0$, $\beta = 1$, we obtain $\mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n} = 0$. Thus, $\mathbf{n}$ is orthogonal to the tangent plane, that is, Π is indeed the tangent plane. □

Tangent Vector Space, Tangent Plane and Normal to a Surface

Definition

The line that passes through a point of the surface and is perpendicular to the tangent plane to the surface at that point is called the *normal to the surface* at the given point.

Thus, if $(U, \mathbf{r})$ is a parameterisation of the surface around the point $a = \mathbf{r}(u_0, v_0) = (x_0, y_0, z_0) \in S$, then a direction vector of the normal to the surface will be $\mathbf{r}'_{\mathbf{u}}(u_0, v_0) \times \mathbf{r}'_{\mathbf{v}}(u_0, v_0)$, which means that the equations of the normal at the point a will be:

$$\frac{X - x_0}{\begin{vmatrix} y'_u(u_0, v_0) & z'_u(u_0, v_0) \\ y'_v(u_0, v_0) & z'_v(u_0, v_0) \end{vmatrix}} = \frac{Y - y_0}{\begin{vmatrix} z'_u(u_0, v_0) & x'_u(u_0, v_0) \\ z'_v(u_0, v_0) & x'_v(u_0, v_0) \end{vmatrix}} = \frac{Z - z_0}{\begin{vmatrix} x'_u(u_0, v_0) & y'_u(u_0, v_0 \\ x'_v(u_0, v_0) & y'_v(u_0, v_0 \end{vmatrix}} \tag{87}$$

Tangent Vector Space, Tangent Plane and Normal to a Surface

To determine the tangent plane and the normal to a surface given by an implicit representation, the following result is very useful.

Theorem

At the point (x_0, y_0, z_0) of the surface given by the equation

$$F(x, y, z) = 0$$

the vector $\operatorname{grad} F_0 = \{F'_x(x_0, y_0, z_0), F'_y(x_0, y_0, z_0), F'_z(x_0, y_0, z_0)\}$ *is perpendicular to the tangent plane to the surface at this point.*

Tangent Vector Space, Tangent Plane and Normal to a Surface

Proof.

Let $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ be a local parameterisation of the surface around the point $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$. Then we have the identity

$$F(x(u,v), y(u,v), z(u,v)) = 0,$$

from which, by differentiation, we obtain

$$\begin{cases} \underline{0} = F'_u = F'_x \cdot x'_u + F'_y \cdot y'_u + F'_z \cdot z'_u \equiv \underline{\operatorname{grad} F \cdot \mathbf{r}'_{\mathbf{u}}} \\ \underline{0} = F'_v = F'_x \cdot x'_v + F'_y \cdot y'_v + F'_z \cdot z'_v \equiv \underline{\operatorname{grad} F \cdot \mathbf{r}'_{\mathbf{v}}} \end{cases},$$

that is, $\operatorname{grad} F \perp L(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}) \equiv T_{(x_0,y_0,z_0)}S$. □

Tangent Vector Space, Tangent Plane and Normal to a Surface

Corollary

The equation of the tangent plane to the surface given by the implicit equation $F(x, y, z) = 0$ at the point (x_0, y_0, z_0) is of the form

$$(X - x_0)F'_x(x_0, y_0, z_0) + (Y - y_0)F'_y(x_0, y_0, z_0) + (Z - z_0)F'_z(x_0, y_0, z_0) = 0,$$

while the equations of the normal to the surface at the same point are

$$\frac{X - x_0}{F'_x(x_0, y_0, z_0)} = \frac{Y - y_0}{F'_y(x_0, y_0, z_0)} = \frac{Z - z_0}{F'_z(x_0, y_0, z_0)}.$$

Tangent Vector Space, Tangent Plane and Normal to a Surface

Corollary

For any point a on the sphere S_R^2, the tangent space $T_aS_R^2$ is perpendicular to the radius vector at point a.

Proof.

The sphere S_R^2 can be described by the equation

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - R^2 = 0,$$

from which it follows that

$$\operatorname{grad} F = 2\{x, y, z\} = 2\mathbf{a},$$

therefore the radius vector is parallel to the gradient of the function F, hence it is perpendicular to the tangent space. □

The orientation of surfaces

Definition

An *orientation* of a surface S is a choice of an orientation in each tangent space T_aS, i.e. a choice of the unit normal vector of T_aS, $\mathbf{n}(a)$. It is assumed, in this context, that the map $\mathbf{n} : S \to \mathbb{R}^3$, $a \to \mathbf{n}(a)$ is continuous. The surfaces on which is is possible to define an orientation are called *orientable*, while those on which an orientation has been already chosen – *oriented*.

Exemple

a) We can define an orientation on the sphere S_R^2 using the versor of the exterior normal. It is not difficult to see that, if $\mathbf{a}$ is the radius vector of the point $a \in S_R^2$, then $\mathbf{n}(a) = \frac{1}{R}\mathbf{a}$. Therefore, the map

$$\mathbf{n} : S_R^2 \to \mathbb{R}^3,$$

defining the orientation of the sphere can be represented as a composition of continuous maps:

$$S_R^2 \xrightarrow{i} \mathbb{R}^3 \xrightarrow{\frac{1}{R}} \mathbb{R}^3 : a \longrightarrow \mathbf{a} \longrightarrow \frac{1}{R}\mathbf{a}.$$

The orientation of surfaces

Exemple

b) Let S be a simple surface, with the global parameterization $(U, \mathbf{r})$. This surface can be oriented by using the vector field

$$\mathbf{n}(u, v) = \frac{\mathbf{r}_\mathbf{u}' \times \mathbf{r}_\mathbf{v}'}{\|\mathbf{r}_\mathbf{u}' \times \mathbf{r}_\mathbf{v}'\|}.$$

c) Let S be a surface given by the implicit equation $F(x, y, z) = 0$. Then the surface can be oriented through the gradient vector field:

$$\mathbf{n}(x, y, z) = \frac{\operatorname{grad} F}{\|\operatorname{grad} F\|}.$$

If the orientation of a surface S is given by the vector function (vector field) $\mathbf{n}(a)$, then the vector field $-\mathbf{n}(a)$ also defines an orientation on S, called the *opposite* orientation of the orientation given by $\mathbf{n}$. If the orientable surface S is connected, then each orientation of S should coincide to one of the two orientations just mentioned. Indeed, if $\mathbf{N}(a)$ is an orientation of the surface S, then we must have $\mathbf{N}(a) = \lambda\,\mathbf{n}(a)$, where λ is a continuous function on S, taking values into the finite set $\{-1, 1\}$, therefore, if S is connected, λ has to be a constant function. Thus, a connected orientable surface has only two distinct orientations. Of course, if the surface is not connected, there are more orientations, corresponding to different combinations of the two possible orientations on each connected component of the surface.

The orientation of surfaces

Remark

Not any surface is orientable. We consider, for instance, the support of the parameterized surface

$$\mathbf{r}(u, v) = (\cos u + v\cos\frac{u}{2}\cos u, \sin u + v\cos\frac{u}{2}\sin u, v\sin\frac{u}{2}),$$

with $u, v \in \mathbb{R}$ (the *Möbius's band*, see the next figure). It can be shown that S is not orientable. (It has a single side: it is possible to move continuously the origin of the unit normal along a close path on S such that, after the "trip", the unit normal will change into its opposite.) Notice that S is not simple, as it might seem, because $\mathbf{r}$ is not a parameterization, since it is not a homeomorphism on the image.

The orientation of surfaces

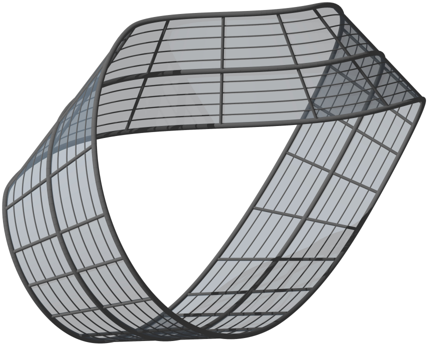


Figure: The Möbius's band

The Differential of a Smooth Map Between Surfaces

The notion of a smooth map between surfaces is a natural generalisation of the notion of a smooth map between open subsets of Euclidean spaces. The same should happen with the notion of differential. We will therefore highlight a property of the differential of a smooth map between Euclidean spaces, which will be used to construct the generalisation we are looking for.
Let $G : B \to \mathbb{R}^3$ be a smooth map, with $B \subset \mathbb{R}^3$ an open set, and

$$G(x, y, z) = (g_1(x, y, z), g_2(x, y, z), g_3(x, y, z)).$$

Then, for any point $a = (x, y, z) \in B$, the differential of G at a,

$$d_a G : \mathbb{R}^3_a \to \mathbb{R}^3_{G(a)}$$

is a linear map, whose matrix is the Jacobian matrix

$$\left.\frac{D(g_1, g_2, g_3)}{D(x, y, z)}\right|_{(x_0, y_0, z_0)} = (\alpha_{ij}),\ 1 \le i, j \le 3,$$

The Differential of a Smooth Map Between Surfaces

where

$$\alpha_{ij} = \partial_i g_i(x_0, y_0, z_0).$$

For any vector $\mathbf{h} \in \mathbb{R}^3_a$, from $\{h_1, h_2, h_3\}$, the vector $d_a G(\mathbf{h})$ has components

$$\left\{ \sum_{j=1}^{3} \alpha_{1j} h_j, \sum_{j=1}^{3} \alpha_{2j} h_j, \sum_{j=1}^{3} \alpha_{3j} h_j \right\}.$$

Let us now suppose that the vector $\mathbf{h}$ is tangent to the parametrised curve $\rho(t) = (x(t), y(t), z(t))$ at the point $t = t_0$, that is $\mathbf{h} = \rho'(t_0)$. We shall prove that the vector $d_a G(\mathbf{h})$ is the tangent vector to the parametrised curve $(G \circ \rho)(t)$ at $t = t_0$.

The Differential of a Smooth Map Between Surfaces

To this end, we differentiate the relation

$$(G\circ\rho)(t)=(g_1(x(t),y(t),z(t)),g_2(x(t),y(t),z(t)),g_3(x(t),y(t),z(t)))$$

and obtain:

$$\begin{aligned}(\overrightarrow{G\circ\rho})'(t_0)=&\left\{\sum_{k=1}^{3}\frac{\partial g_1}{\partial x^k}(x_0,y_0,z_0)h_k,\sum_{k=1}^{3}\frac{\partial g_2}{\partial x^k}(x_0,y_0,z_0)h_k,\right.\\&\left.\sum_{k=1}^{3}\frac{\partial g_3}{\partial x^k}(x_0,y_0,z_0)h_k\right\}=\\&=\left\{\sum_{k=1}^{3}\alpha_{1k}h_k,\sum_{k=1}^{3}\alpha_{2k}h_k,\sum_{k=1}^{3}\alpha_{3k}h_k\right\},\end{aligned}$$

where

$$\{h_1,h_2,h_3\}=\{x'(t_0),y'(t_0),z'(t_0)\}=\rho'(t_0).$$

The Differential of a Smooth Map Between Surfaces

Thus, the differential d_aG associates to the tangent vector to the path $\rho(t)$ at $t = t_0$ the tangent vector to the path $G(\rho(t))$ at $t = t_0$.
Let now $F : S_1 \to S_2$ be a smooth map between the surfaces S_1 and S_2, and $a \in S_1$. Then, to every smooth path (I, ρ) on S_1 corresponds a smooth path $(I, F \circ \rho)$ on S_2. If $\rho(t)$ passes through a at $t = t_0$, then the path $F \circ \rho(t)$ will pass through $F(a)$ at $t = t_0$.

Definition

The map $T_aS_1 \to T_{F(a)}S_2$, which assigns to each tangent vector $\rho'(t_0)$ to a parametrised curve $\rho(t)$ on S_1, with $\rho(t_0) = a$, the tangent vector $(\overrightarrow{F \circ \rho})'(t_0)$ to the parametrised curve $F \circ \rho$ at $t = t_0$, is called the *differential* of the smooth map $F : S_1 \to S_2$ at the point a, and is denoted by d_aF.

There might be a slight difficulty here. On S_1, we can have two different curves that share the same tangent vector at a contact point. Since the images of two parametrised curves through F are, in general, distinct, it might happen that these images do not share the same tangent vector at the point of contact. However, as we shall immediately see, this is not the case. Indeed, let $a \in S_1$ and $(I, \rho = \rho(t))$, $(I_1, \rho_1 = \rho_1(s))$ be two parametrised curves on S_1 such that $\rho(t_0) = \rho_1(s_0) = a$ and $\rho'(t_0) = \rho'_1(s_0)$. We choose an arbitrary local parametrisation $(U, \mathbf{r})$ on S_1, around a.

The Differential of a Smooth Map Between Surfaces

Since we are only interested in the local phenomena occurring around a, we may assume, without loss of generality, that $\rho(I) \subset \mathbf{r}(U)$ and $\rho_1(I_1) \subset \mathbf{r}(U)$. Let us suppose that the local equations of the curves in the parametrisation $(U, \mathbf{r})$ are

$$(\rho) \begin{cases} u = u(t) \\ v = v(t) \end{cases} ,$$

and respectively

$$(\rho_1) \begin{cases} u = u_1(s) \\ v = v_1(s) \end{cases} .$$

Then, the vectors $\boldsymbol{\rho}'(t_0)$ and $\boldsymbol{\rho_1}'(s_0)$ have, in the natural basis $\{\mathbf{r}'_\mathbf{u}, \mathbf{r}'_\mathbf{v}\}$, the expressions

$$\boldsymbol{\rho}'(t_0) = \{u'(t_0), v'(t_0)\}$$
$$\boldsymbol{\rho_1}'(s_0) = \{u_1'(s_0), v_1'(s_0)\}.$$

The Differential of a Smooth Map Between Surfaces

Furthermore, in the chosen parametrisation,

$$(F \circ \rho)(t) = F_{\mathbf{r}}(u(t), v(t))$$
$$(F \circ \rho_1)(s) = F_{\mathbf{r}}(u_1(s), v_1(s)),$$

with $F_{\mathbf{r}} = F \circ \mathbf{r}$. Thus,

$$\begin{aligned}(\overrightarrow{F \circ \rho})(t_0) &= \frac{d}{dt}(F_{\mathbf{r}}(u(t), v(t)))(u_0, v_0) = \\ &= \overrightarrow{(F_{\mathbf{r}})'_u}(u_0, v_0)u'(t_0) + \overrightarrow{(F_{\mathbf{r}})'_v}(u_0, v_0)v'(t_0) \\ (\overrightarrow{F \circ \rho_1})(s_0) &= \frac{d}{ds}(F_{\mathbf{r}}(u(s), v(s)))(u_0, v_0) = \\ &= \overrightarrow{(F_{\mathbf{r}})'_u}(u_0, v_0)u_1'(s_0) + \overrightarrow{(F_{\mathbf{r}})'_v}(u_0, v_0)v_1'(s_0).\end{aligned}$$

Now, since $\boldsymbol{\rho}'(t_0) = \boldsymbol{\rho_1}'(s_0)$, it follows that

$$(\overrightarrow{F \circ \rho})'(t_0) = (\overrightarrow{F \circ \rho_1})'(s_0),$$

which means that the definition of F is well-defined.

The Differential of a Smooth Map Between Surfaces

Theorem

The map $dF : T_aS_1 \to T_{F(a)}S_2$ is linear.

Proof.

From the above computation, it follows immediately that if a vector $\mathbf{h} \in T_aS_1$ has, in the natural basis $\{\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}\}$ of the vector space T_aS_1, the components $\{h_1, h_2\}$, then

$$d_aF(\mathbf{h}) = \overrightarrow{F'_{\mathbf{r}}}_u(u_0, v_0)h_1 + \overrightarrow{F'_{\mathbf{r}}}_v(u_0, v_0)h_2, \tag{88}$$

from which linearity follows. $\square$

Example

Let S_1, S_2 be two surfaces in $\mathbb{R}^3$, $D : \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism such that $D(S_1) = S_2$, $F = D_{S_1} : S_1 \to S_2$, $a \in S \subset \mathbb{R}^3$. Then we have

$$d_a F = d_a D|_{T_a S_1}, \tag{89}$$

where $d_a D : \mathbb{R}^3_a \to \mathbb{R}^3_{D(a)}$ is the differential of the map D at a. In particular, let S^2_R and S^2_r be the spheres of radii R and r, respectively, centred at the origin, and $D : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \to \frac{r}{R}(x, y, z)$ a homothety, which is clearly a diffeomorphism such that $D(S^2_R) = S^2_r$. Then, for the map $F = D|_{S^2_R} : S^2_R \to S^2_r$, we have $d_a F(\mathbf{h}) = \frac{r}{R}\mathbf{h}$.

The Gauss Map and the Shape Operator of a Surface

Let $S \subset \mathbb{R}^3$ be an oriented surface and S^2 the unit sphere centred at the origin. If the orientation of S is given by the unit normal vector $\mathbf{n}(a)$, $a \in S$, then we can construct a map $\Gamma : S \to S^2$, which assigns to each $a \in S$ the point on S^2 whose position vector is $\boldsymbol{\Gamma}(a) = \mathbf{n}(a)$. The map Γ is called the *Gauss map* of the surface S. This map plays a central role in the theory of surfaces. We shall first prove that Γ is smooth:

Theorem

The Gauss map $\Gamma : S \to S^2$ of an oriented surface S to the unit sphere S^2 is a smooth map between surfaces.

Proof.

Let $a \in S$. We choose a local parametrisation $(U, \mathbf{r})$ of the surface S around a, compatible with the orientation. Clearly, since S is orientable, such a parametrisation always exists. Indeed, if we choose a parametrisation $(U_1, \mathbf{r}_1)$ that is not compatible with the orientation, that is, we have

$$\frac{\mathbf{r}'_{\mathbf{1}u} \times \mathbf{r}'_{\mathbf{1}v}}{\|\mathbf{r}'_{\mathbf{1}u} \times \mathbf{r}'_{\mathbf{1}v}\|} = -\mathbf{n}(u, v),$$

then we replace the domain U_1 with U_1^-, the symmetric of U_1 with respect to the Ov axis, and the map $\mathbf{r}_1(u, v)$ with $\mathbf{r}_1^-(u, v) = \mathbf{r}_1(-u, v)$. It is easy to see that the pair $(U_1^-, \mathbf{r}_1^-)$ is a parametrisation of the surface, compatible with the orientation. □

The Gauss Map and the Shape Operator of a Surface

Proof.

Now we have

$$(\Gamma \circ \mathbf{r})(u, v) = \Gamma(\mathbf{r}(u, v)) = \Gamma_{\mathbf{r}}(u, v) = \mathbf{n}(u, v) = \frac{\mathbf{r}_{\mathbf{u}}' \times \mathbf{r}_{\mathbf{v}}'}{\|\mathbf{r}_{\mathbf{u}}' \times \mathbf{r}_{\mathbf{v}}'\|}.$$

Thus, the local representation of Γ is smooth, hence Γ itself is smooth. □

Exemple

1. For a plane Π, the Gauss map is constant.
2. For the sphere S_R^2, the Gauss map $\Gamma : S_R^2 \to S$ has the expression $\Gamma(x, y, z) = \frac{1}{R}(x, y, z)$, with $x^2 + y^2 + z^2 = R^2$.

The Gauss Map and the Shape Operator of a Surface

As we have seen, the tangent space to the sphere, $T_{\Gamma(a)}S^2$, is perpendicular to the position vector $\mathbf{n}(a)$ of the point $\Gamma(a)$. On the other hand, $\mathbf{n}(a)$ is perpendicular to T_aS. Thus, if we identify $\mathbb{R}^3_a$ and $\mathbb{R}^3_{\Gamma(a)}$ with $\mathbb{R}^3$, then the subspaces T_aS and $T_{\Gamma(a)}S^2$ coincide. Therefore, we can think of the differential $d_a\Gamma : T_aS \to T_{\Gamma(a)}S^2$ as, in fact, a linear operator $T_aS \to T_aS$.

Definition

The linear operator $d_a\Gamma : T_aS \to T_aS$ is called the *shape operator* of the oriented surface S at the point a, and is denoted by A or A_a.

Remark

There is no general consensus regarding either the definition of the shape operator or its name. In some books, a minus sign is included in the definition. Sometimes it is called the *Weingarten operator*, or also the *principal or fundamental operator*. Historically, it is true that Julius Weingarten was the first to write the derivative formulas for the Gauss map (in other words, he found the partial derivatives of the surface normal unit vector in terms of the partial derivatives of the position vector). Nevertheless, the shape operator, as a linear map, was introduced in differential geometry by the Italian geometer Cesare Burali-Forti ([**?**]), in 1912, under the name *omografia fondamentale*, that is, fundamental homography.

Aplicaţia sferică şi operatorul de formă al unei suprafeţe

Example

1. (i) Pentru un plan, operatorul de formă se anulează.
2. (ii) Pentru sferă, operatorul de formă este o omotetie.

Fie acum $(U, \mathbf{r})$ o parametrizare locală a lui S, compatibilă cu orientarea. Atunci, reprezentarea locală a aplicaţiei sferice Γ a lui S va fi dată de

$$\mathbf{n}(u, v) = \frac{\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}}{\|\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}\|},$$

de aceea, pentru operatorul de formă vom avea:

$$A(\mathbf{h}) = \mathbf{n}'_u h_1 + \mathbf{n}'_v h_2, \tag{90}$$

unde $\mathbf{h} \in T_{\mathbf{r}(u,v)}S$; (h_1, h_2) sunt componentele lui $\mathbf{h}$ faţă de baza naturală $\{\mathbf{r}'_\mathbf{u}, \mathbf{r}'_\mathbf{v}\}$.

The First Fundamental Form of a Surface

Let S be a surface in $\mathbb{R}^3$. Then the scalar product in $\mathbb{R}^3$ induces a scalar product in each $\mathbb{R}^3_a$ and, therefore, also induces a scalar product in each tangent space T_aS, $a \in S$.

Definition

The first fundamental form of a surface S is, by definition, the function φ_1, which assigns to each $a \in S$ the restriction of the scalar product on $\mathbb{R}^3_a$ to T_aS. We shall usually, by abuse of language, say that the first fundamental form is the restriction itself, but we must understand what is actually happening. Thus, for any $a \in S$ and any $\mathbf{p}, \mathbf{q} \in T_aS$, we have

$$\varphi_1(\mathbf{p}, \mathbf{q}) = \mathbf{p} \cdot \mathbf{q}. \tag{91}$$

The First Fundamental Form of a Surface

Remark

In many textbooks, especially older ones, the first fundamental form is not defined as the restriction of the scalar product to T_aS, but rather as the quadratic form associated with this restriction.

If $(U, \mathbf{r})$ is a local parametrisation of S, then for any $(u, v) \in U$, the tangent space $\mathbb{R}^2_{(u,v)}$ to the domain U at the point (u, v) can be identified with the space $T_{\mathbf{r}(u,v)}S$ by associating the vectors $\{1, 0\}, \ \{0, 1\}$, which form a basis of $\mathbb{R}^2_{(u,v)}$, with the vectors $\mathbf{r}'_{\mathbf{u}}(u, v)$ and $\mathbf{r}'_{\mathbf{v}}(u, v)$. It is easy to see that, in fact, this identification is precisely the linear isomorphism $dr_{(u,v)} : \mathbb{R}^2_{(u,v)} \to T_{\mathbf{r}(u,v)}S$.

The First Fundamental Form of a Surface

Using this identification, we can transfer the first fundamental form φ_1 of S to the domain U (which can be regarded, in fact, as a simple surface, with the global parametrisation given by the inclusion of U in $\mathbb{R}^3$). Thus, for any $(u, v) \in U$, in the tangent space $T_{\mathbf{r}(u,v)}U \equiv \mathbb{R}^2_{(u,v)}$ to the domain U, the scalar product of two vectors is defined by the rule

$$\widetilde{\varphi}_1(\xi, \eta) = \varphi_1(dr_{(u,v)}(\xi), dr_{(u,v)}(\eta)) = dr_{(u,v)}(\xi) \cdot dr_{(u,v)}(\eta).$$

It is easy to see that, by construction, the map $dr_{(u,v)} : \mathbb{R}^2_{(u,v)} \to T_{\mathbf{r}(u,v)}S$ is an isometry with respect to the scalar products $\widetilde{\varphi}_1$ and φ_1, respectively. We introduce the notations

$$\begin{cases} E(u, v) = \mathbf{r}'_{\mathbf{u}}(u, v) \cdot \mathbf{r}'_{\mathbf{u}}(u, v) \\ F(u, v) = \mathbf{r}'_{\mathbf{u}}(u, v) \cdot \mathbf{r}'_{\mathbf{v}}(u, v) \\ G(u, v) = \mathbf{r}'_{\mathbf{v}}(u, v) \cdot \mathbf{r}'_{\mathbf{v}}(u, v) \end{cases}.$$

The First Fundamental Form of a Surface

Then the functions E, F, G are smooth on U, while the matrix $\mathcal{G} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is the matrix of the scalar product φ_1 on the tangent space $T_{\mathbf{r}(u,v)}S$ with respect to the basis $\{\mathbf{r}'_\mathbf{u}(u,v), \mathbf{r}'_\mathbf{v}(u,v)\}$, but it is also the matrix of the scalar product $\widetilde{\varphi}_1$ on the tangent space $\mathbb{R}^2_{(u,v)} = T_{(u,v)}U$ with respect to the basis $\{\{1,0\},\{0,1\}\}$.

Exemple

1. For the plane Π, given by the global parametrisation $\mathbf{r} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$, with $\mathbf{a} \times \mathbf{b} \neq 0$, we have:

$$\begin{cases} \mathbf{r}'_\mathbf{u} = \mathbf{a} \\ \mathbf{r}'_\mathbf{v} = \mathbf{b} \end{cases} \qquad \text{thus} \qquad \begin{cases} E = \mathbf{a}^2 \\ F = \mathbf{a} \cdot \mathbf{b} \\ G = \mathbf{b}^2 \end{cases} .$$

If Π is the xOy coordinate plane, then we can set $\mathbf{r}_0 = 0$, $\mathbf{a} = \imath$, $\mathbf{b} = \jmath$, hence, the first fundamental form has the matrix $\mathcal{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The First Fundamental Form of a Surface

Exemple

2. For the sphere S_R^2, we choose the local parametrisation $(U, \mathbf{r})$, with
$$\mathbf{r}(u, v) = (R\cos u \cos v, R\cos u \sin v, R\sin u),$$
and $U = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We immediately obtain that $E = R^2, F = 0, G = R^2\cos^2 u$, so the matrix of the first fundamental form is given by
$$\mathcal{G} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$$

Length of a curve segment on a surface. Let S be a surface, $(U, \mathbf{r})$ a local parametrisation of S and (I, ρ) a parametrised curve with $\rho(I) \subseteq \mathbf{r}(U)$, given by the local equations $u = u(t), v = v(t)$. Then, in the natural basis, the tangent vector of the curve ρ, $\rho'(t)$ has components $\{u'(t), v'(t)\}$ and we can compute its length using the matrix $\mathcal{G}$. Thus, the length of the segment of the curve ρ between t_1 and t_2 is:

$$l_{t_1,t_2} = \int_{t_1}^{t_2} \|\boldsymbol{\rho}'(t)\| dt = \int_{t_1}^{t_2} \sqrt{E(t)u'^2 + 2F(t)u'v' + G(t)v'^2} dt,$$

where

$$\begin{cases} E(t) = E(u(t), v(t)) \\ F(t) = F(u(t), v(t)) \\ G(t) = G(u(t), v(t)) \end{cases}.$$

Example

We choose, on the sphere S^2_R, the curve given by the local equations $u = 0, v = t$ (in the parametrisation described in the previous example), where $t \in (0, 2\pi)$ (the equator minus one point). As we saw above, the first fundamental form of the sphere has the matrix $\mathcal{G} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}$. Along the curve we have $u = 0$, hence $u'(t) = 0$, $v'(t) = 1$. On the other hand, $\cos^2 u = \cos^2 0 = 1$, so along the curve, the matrix $\mathcal{G}$ is the identity matrix multiplied by R^2. If we want to calculate, for example, the length of the curve segment between $t_1 = \frac{\pi}{2}$ and $t_2 = \pi$, we obtain

$$l_{\frac{\pi}{2},\pi} = \int\limits_{\frac{\pi}{2}}^{\pi} \sqrt{R^2 \cdot 0 + 2 \cdot 0 + R^2 \cdot 1} dt = R \cdot t \Big|_{\frac{\pi}{2}}^{\pi} = \frac{\pi R}{2},$$

The angle between two curves on a surface. Let $(U, \mathbf{r})$ be a parametrisation of the surface S, and $(I, \rho = \rho(t))$, $(I_1, \rho_1 = \rho_1(s))$ be two curves on S such that $\rho(I) \subset \mathbf{r}(U)$, $\rho_1(I_1) \subset \mathbf{r}(U)$. Assume that the supports of the two curves intersect at $\mathbf{r}(u_0, v_0)$, i.e., there exist $t_0 \in I$, $s_0 \in I_1$ such that:

$$\rho(t_0) = \rho_1(s_0) = \mathbf{r}(u_0, v_0).$$

If the local equations of the two curves are

$$(\rho) \begin{cases} u = u_1(t) \\ v = v_1(t) \end{cases},$$

and respectively

$$(\rho_1) \begin{cases} u = u_2(s) \\ v = v_2(s) \end{cases},$$

then the decompositions of the tangent vectors at the point of

$$\begin{cases} \boldsymbol{\rho}'(t_0) = \left\{u_1'(t_0), v_1'(t_0)\right\} \\ \boldsymbol{\rho_1}'(s_0) = \left\{u_2'(s_0), v_2'(s_0)\right\} \end{cases},$$

therefore, the cosine of the angle between the curves[6] at the point of contact is, as is known,

$$\cos\theta = \frac{\boldsymbol{\rho}'(t_0)\cdot\boldsymbol{\rho_1}'(s_0)}{\|\boldsymbol{\rho}'(t_0)\|\cdot\|\boldsymbol{\rho_1}'(s_0)\|} =$$
$$= \frac{Eu_1'u_2' + F(u_1'v_2' + u_2'v_1') + Gv_1'v_2'}{\sqrt{Eu_1'^2 + 2Fu_1'v_1' + Gv_1'^2}\cdot\sqrt{Eu_2'^2 + 2Fu_2'v_2' + Gv_2'^2}},$$

where

$$\begin{cases} E = E(u_0, v_0) \\ F = F(u_0, v_0) \\ G = G(u_0, v_0) \end{cases} \quad \text{and} \quad \begin{cases} u_1' = u_1'(t_0) \\ v_1' = v_1'(t_0) \\ u_2' = u_2'(s_0) \\ \ldots \end{cases}.$$

Area of a parametrised surface. Let $(U, \mathbf{r})$ be a parametrised surface. There are many ways to introduce the notion of area. All are more or less related to integral calculus, so we shall not enter into any detailed discussion here. Essentially, as in the case of plane geometric figures, area should be a function that associates to each oriented parametrised surface a positive number, subject to certain constraints. Following Stoker, we choose the following three constraints:

a) The area must be given by an integral of the form

$$A = \iint_U f du dv,$$

where f must depend only on $u, v, \mathbf{r}, \mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}$ (higher-order derivatives of $\mathbf{r}$ must not appear!).

b) It is invariant under translations of the plane and under orientation-preserving changes of parameters.

c) A unit square has area 1.

It can be shown that the only formula for area that satisfies the three axioms above is

$$A = \iint\limits_U \sqrt{EG - F^2} dudv \equiv \iint\limits_U \|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\| dudv. \tag{92}$$

We shall give a heuristic motivation for formula (92). This should not be taken as a “proof”; we do not claim that it is one.

The “classical” approach. Let $(U, \mathbf{r})$ be a parametrised surface and $D \subset U$ a compact subset of U such that $\mathbf{r}(\partial D)$ is a piecewise smooth curve in $\mathbb{R}^3$. We want to define the *area* of $\mathbf{r}(D) \subset \mathbf{r}(U)$. The basic idea is that we already have a notion of area for *plane* figures, particularly for parallelograms. So let $(u, v) \in D$ and $M = \mathbf{r}(u, v)$. Through M pass two coordinate lines, one from each family. Let $M_1 = \mathbf{r}(u + \Delta u, v)$, $M_2 = \mathbf{r}(u, v + \Delta v)$ be two points on these lines, corresponding to parameter increments of M by Δu and Δv, respectively, and $M' = \mathbf{r}(u + \Delta u, v + \Delta v)$.

If Δu and Δv are sufficiently small, then the projection of the curvilinear polygon $MM_1M'M_2$ onto the tangent plane to the surface at the point M is (approximately, of course) a planar parallelogram in the tangent plane. The sides of this parallelogram are $\mathbf{r}'_\mathbf{u}\Delta u$ and $\mathbf{r}'_\mathbf{v}\Delta v$, and its area will therefore be

$$\Delta\sigma = \|\mathbf{r}'_\mathbf{u}\Delta u \times \mathbf{r}'_\mathbf{v}\Delta v\| = \|\mathbf{r}'_\mathbf{u} \times \mathbf{r}'_\mathbf{v}\|\Delta u\Delta v = \sqrt{EG - F^2}\Delta u\Delta v,$$

where, of course, the coefficients of the first fundamental form are evaluated at the point M. It is therefore natural to define the area of $\mathbf{r}(D)$ as

$$A = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Sigma\Delta\sigma = \iint\limits_D \sqrt{EG - F^2}dudv,$$

where the sum in the middle term is taken over all the small curvilinear parallelograms covering $\mathbf{r}(D)$.

Remark

We might expect to obtain for the area of a domain on a surface an interpretation similar to that obtained for the length of a curve segment. Namely, we could discretise the domain and consider the images of the selected points. These will determine a polyhedral surface inscribed in our surface. Then we could consider that the area of the polygons composing this polyhedral surface tends to zero, and define the area of the surface as the limit approached by the area of the polyhedral surface. Unfortunately, as shown by a famous example due to H.A. Schwarz, this approach does not work, because the limit is not independent of the type of polyhedral surfaces used to approximate the surface and, in particular, for certain “polygonalisations” of the surface, the area can be infinite, while for others it is finite.

The Matrix of the Shape Operator in the Natural Basis

Let $(U, \mathbf{r})$ be a local parametrisation of the oriented surface S, compatible with the orientation. We denote by $\mathcal{A}$ the matrix of the shape operator A with respect to the natural basis $\{\mathbf{r}_\mathbf{u}', \mathbf{r}_\mathbf{v}'\}$. Since, as we have seen earlier,

$$A(\mathbf{r}_\mathbf{u}') = \mathbf{n}_u', \; A(\mathbf{r}_\mathbf{v}') = \mathbf{n}_v',$$

we have

$$(\, \mathbf{n}_u \;\; \mathbf{n}_v') = (\mathbf{r}_\mathbf{u}' \; \mathbf{r}_\mathbf{v}') \cdot \mathcal{A}. \tag{93}$$

We multiply on the left by the matrix $\begin{pmatrix} \mathbf{r}_\mathbf{u}' \\ \mathbf{r}_\mathbf{v}' \end{pmatrix}$ and obtain:

$$\begin{pmatrix} \mathbf{r}_\mathbf{u}' \\ \mathbf{r}_\mathbf{v}' \end{pmatrix} \cdot (\, \mathbf{n}_u' \quad \mathbf{n}_v') = \begin{pmatrix} \mathbf{r}_\mathbf{u}' \cdot \mathbf{n}_u' & \mathbf{r}_\mathbf{u}' \cdot \mathbf{n}_v' \\ \mathbf{r}_\mathbf{v}' \cdot \mathbf{n}_u' & \mathbf{r}_\mathbf{v}' \cdot \mathbf{n}_v' \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbf{r}_\mathbf{u}' \\ \mathbf{r}_\mathbf{v}' \end{pmatrix} \cdot (\mathbf{r}_\mathbf{u}' \quad \mathbf{r}_\mathbf{v}') \cdot \mathcal{A} = \begin{pmatrix} \mathbf{r}_\mathbf{u}' \cdot \mathbf{r}_\mathbf{u}' & \mathbf{r}_\mathbf{u}' \cdot \mathbf{r}_\mathbf{v}' \\ \mathbf{r}_\mathbf{v}' \cdot \mathbf{r}_\mathbf{u}' & \mathbf{r}_\mathbf{v}' \cdot \mathbf{r}_\mathbf{v}' \end{pmatrix} \cdot \mathcal{A} = \mathcal{G} \cdot \mathcal{A}.$$

The Matrix of the Shape Operator in the Natural Basis

We introduce the functions

$$\begin{cases} L(u,v) = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_u \\ M(u,v) = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v \\ N(u,v) = \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_v \end{cases}, \tag{94}$$

and the matrix $\mathcal{H}$, defined by

$$\mathcal{H} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Then the last equation becomes

$$\mathcal{H} = \mathcal{G} \cdot \mathcal{A}.$$

The Matrix of the Shape Operator in the Natural Basis

Since the scalar product in $\mathbb{R}^3$ is non-degenerate, this remains true for its restriction to any subspace and, as a consequence, the matrix $\mathcal{G}$ is invertible. If $\mathcal{G}^{-1}$ is its inverse, then for the matrix of the shape operator we obtain

$$\mathcal{A} = \mathcal{G}^{-1} \cdot \mathcal{H}, \tag{95}$$

where, as can be easily seen,

$$\mathcal{G}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

If we carry out the computations, we obtain:

$$\mathcal{A} = \frac{1}{EG - F^2} \begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix} \tag{96}$$

The Matrix of the Shape Operator in the Natural Basis

All we have to do is express the quantities L, M, N in terms of the derivatives of the function **r**. To this end, we differentiate the relations $\mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n} = 0$ and $\mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n} = 0$ with respect to u and v respectively, and obtain:

$$\begin{cases} \mathbf{r}''_{\mathbf{u^2}} \cdot n + \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_u = 0 \\ \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{n} + \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v = 0 \\ \mathbf{r}''_{\mathbf{uv}} \cdot \mathbf{n} + \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_u = 0 \\ \mathbf{r}''_{\mathbf{v^2}} \cdot n + \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_v = 0 \end{cases},$$

from which we obtain the expressions for L, M, N:

$$\begin{cases} L = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_u = -\mathbf{n} \cdot \mathbf{r}''_{\mathbf{u^2}} \\ M = \mathbf{r}'_{\mathbf{u}} \cdot \mathbf{n}'_v = -\mathbf{n} \cdot \mathbf{r}''_{\mathbf{uv}} \\ N = \mathbf{r}'_{\mathbf{v}} \cdot \mathbf{n}'_v = -\mathbf{n} \cdot \mathbf{r}''_{\mathbf{v^2}} \end{cases} \tag{97}$$

The Matrix of the Shape Operator in the Natural Basis

or, taking into account that

$$\mathbf{n} = \frac{\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}}{\|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\|},$$

while

$$\|\mathbf{r}'_{\mathbf{u}} \times \mathbf{r}'_{\mathbf{v}}\| = H(= \sqrt{EG - F^2}),$$

$$\begin{cases} L = -\frac{1}{H}(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}, \mathbf{r}''_{\mathbf{u^2}}) \\ M = -\frac{1}{H}(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}, \mathbf{r}''_{\mathbf{uv}}) \\ N = -\frac{1}{H}(\mathbf{r}'_{\mathbf{u}}, \mathbf{r}'_{\mathbf{v}}, \mathbf{r}''_{\mathbf{v^2}}) \end{cases}. \tag{98}$$

The Matrix of the Shape Operator in the Natural Basis

Example

For the helicoid

$$\begin{cases} x = u\cos v \\ y = u \sin v \\ z = bv \end{cases}, \qquad (u,v)\in\mathbb{R}^2, \quad b>0,$$

we can define the orientation by setting

$$\mathbf{n}(u,v) = \left\{ \frac{b\sin v}{\sqrt{b^2+u^2}}, -\frac{b\cos v}{\sqrt{b^2+u^2}}, \frac{u}{\sqrt{b^2+u^2}} \right\}.$$

We obtain, after a straightforward calculation:

$$\mathcal{G} = \begin{pmatrix} 1 & 0 \\ 0 & b^2+u^2 \end{pmatrix}, \mathcal{H} = \begin{pmatrix} 0 & \frac{b}{\sqrt{b^2+u^2}} \\ \frac{b}{\sqrt{b^2+u^2}} & 0 \end{pmatrix}, \mathcal{A} = \begin{pmatrix} 0 & \frac{b}{(b^2+u^2)^{3/2}} \\ \frac{b}{(b^2+u^2)^{3/2}} & 0 \end{pmatrix}$$

The Second Fundamental Form of an Oriented Surface

Definition

The second fundamental form of an oriented surface S is the map which assigns to each $a \in S$ the application $\varphi_2(a) : T_aS \times T_aS \to \mathbb{R}$ given by

$$\varphi_2(\xi, \eta) = -\varphi_1(A(\xi), \eta), \quad \forall \xi, \eta \in T_aS. \tag{99}$$

Remark

The minus sign in the previous definition is a consequence of the sign convention we chose in defining the shape operator. It seemed natural to define the shape operator as the differential of the Gauss map, rather than as the opposite of the differential, but then, in the definition of the second fundamental form, we must introduce an extra minus sign in order to recover the generally accepted definition.

The Second Fundamental Form of an Oriented Surface

Property

For each $a \in S$, $\varphi_2(a)$ is a symmetric bilinear form.

Proof.

Let us take two arbitrary tangent vectors $\boldsymbol{\xi}, \boldsymbol{\eta} \in T_aS$ and two real numbers $\alpha, \beta \in \mathbb{R}$. Then we have, first of all:

$$\varphi_2(\eta, \xi) = -\varphi_1(A(\eta), \xi) \underset{\text{self-adjoint}}{\overset{A}{=}} -\varphi_1(\eta, A(\xi)) \underset{\text{symmetric}}{\overset{\varphi_1}{=}} -\varphi_1(A(\xi), \eta) =$$
$$= \varphi_2(\xi, \eta),$$

which means that φ_2 is symmetric. By symmetry, it is enough to prove linearity only in the first variable. □

The Second Fundamental Form of an Oriented Surface

Proof.

We have

$$\begin{aligned}\varphi_2(\alpha\xi_1+\beta\xi_2,\eta) &= -\varphi_1(A(\alpha\xi_1+\beta\xi_2),\eta) = \\ &\underset{\text{linear}}{\overset{A}{=}} -\varphi_1(\alpha A(\xi_1)+\beta A(\xi_2),\eta) \underset{\text{bilinear}}{\overset{\varphi_1}{=}} \\ &= -\alpha\varphi_1(A(\xi_1),\eta)-\beta\varphi_1(A(\xi_2),\eta) = \\ &= \alpha\varphi_2(\xi_1,\eta)+\beta\varphi_2(\xi_2,\eta),\end{aligned}$$

which proves linearity in the first variable and completes the proof. □

The Second Fundamental Form of an Oriented Surface

Let $(U, \mathbf{r})$ be a local parametrisation of the oriented surface S, compatible with the orientation. Then the matrix $[\varphi_2]$ of the second fundamental form with respect to the canonical basis $\{\mathbf{r}_{\mathbf{u}}', \mathbf{r}_{\mathbf{v}}'\}$ has the form:

$$[\varphi_2] = \begin{pmatrix} \varphi_2(\mathbf{r}_{\mathbf{u}}', \mathbf{r}_{\mathbf{u}}') & \varphi_2(\mathbf{r}_{\mathbf{u}}', \mathbf{r}_{\mathbf{v}}') \\ \varphi_2(\mathbf{r}_{\mathbf{v}}', \mathbf{r}_{\mathbf{u}}') & \varphi_2(\mathbf{r}_{\mathbf{v}}', \mathbf{r}_{\mathbf{v}}') \end{pmatrix}.$$

But

$$\begin{cases} \varphi_2(\mathbf{r}_{\mathbf{u}}', \mathbf{r}_{\mathbf{u}}') = -\varphi_1(A(\mathbf{r}_{\mathbf{u}}'), \mathbf{r}_{\mathbf{u}}') = -\varphi_1(\mathbf{n}_u', \mathbf{r}_{\mathbf{u}}') = -\mathbf{n}_u' \cdot \mathbf{r}_{\mathbf{u}}' \\ \varphi_2(\mathbf{r}_{\mathbf{u}}', \mathbf{r}_{\mathbf{v}}') = \varphi_2(\mathbf{r}_{\mathbf{v}}', \mathbf{r}_{\mathbf{u}}') = -\mathbf{n}_u' \cdot \mathbf{r}_{\mathbf{v}}' = -\mathbf{n}_v' \cdot \mathbf{r}_{\mathbf{u}}' \\ \varphi_2(\mathbf{r}_{\mathbf{v}}', \mathbf{r}_{\mathbf{v}}') = -\mathbf{n}_v' \cdot \mathbf{r}_{\mathbf{v}}' \end{cases},$$

and thus, we obtain the matrix

$$[\varphi_2] = -\begin{pmatrix} \mathbf{n}_u' \cdot \mathbf{r}_{\mathbf{u}}' & \mathbf{n}_u' \cdot \mathbf{r}_{\mathbf{v}}' \\ \mathbf{n}_v' \cdot \mathbf{r}_{\mathbf{u}}' & \mathbf{n}_v' \cdot \mathbf{r}_{\mathbf{v}}' \end{pmatrix} = -\begin{pmatrix} L & M \\ M & N \end{pmatrix} \overset{\text{not.}}{=} \begin{pmatrix} D & D' \\ D' & D'' \end{pmatrix}.$$

The Second Fundamental Form of an Oriented Surface

It follows, therefore, that the matrix of the second fundamental form in the canonical basis is nothing but $-\mathcal{H}$. Thus, for the coefficients of the second fundamental form with respect to the natural basis we have

$$\begin{cases} D = \mathbf{n} \cdot \mathbf{r}''_{\mathbf{u^2}} \\ D' = \mathbf{n} \cdot \mathbf{r}''_{\mathbf{uv}} \\ D'' = \mathbf{n} \cdot \mathbf{r}''_{\mathbf{v^2}} \end{cases} \tag{100}$$

Remark

The reader should pay attention to the notations used for the coefficients of the second fundamental form. The notation e, f, g is also used. Moreover, in some books, the letters L, M, N are used to denote the coefficients of the second fundamental form themselves.

The Second Fundamental Form of an Oriented Surface

Remark

The notations D, D', D'' are usually attributed to Gauss (in *Disquisitiones*). However, it must be mentioned that for Gauss the meaning of these symbols is slightly different: they do not denote the coefficients of the second fundamental form, but these coefficients multiplied by $\sqrt{EG - F^2}$.

The Second Fundamental Form of an Oriented Surface

Example

For the sphere $S = S^2_R$ we have, as previously seen, $\mathbf{n} = \frac{1}{R}\{x, y, z\}$, therefore, as we already know, the shape operator A is a homothety of ratio $1/R$, i.e.,

$$A(\mathbf{p}) = \frac{1}{R}\mathbf{p}, \quad \forall \mathbf{p} \in T_a S^2 R.$$

Thus,

$$\varphi_2(\mathbf{p}, \mathbf{q}) = -\varphi_1\left(\frac{1}{R}\mathbf{p}, \mathbf{q}\right) = -\frac{1}{R}\varphi_1(\mathbf{p}, \mathbf{q}) = -\frac{1}{R}\mathbf{p} \cdot \mathbf{q}.$$

Hence, for the sphere, the first and second fundamental forms are proportional. Clearly, the same holds for the plane, where the second fundamental form is identically zero. It can be shown that only these two surfaces possess this property.

Normal Curvature. Meusnier's Theorem

Let S be an oriented surface and $\mathbf{n}$ the unit normal vector. We consider a regular parametrised curve $\rho = \rho(t)$ lying on S.

Definition

The projection of the curvature vector $\mathbf{k}(t)$ of the curve ρ (considered as a signed scalar) onto $\mathbf{n}(\rho(t))$ is called the *normal curvature* of the curve $\rho(t)$ at t, relative to the surface, and is denoted by $k_n(t)$.

Let $\theta(t)$ be the angle between the osculating plane of $\rho(t)$ and $\mathbf{n}(\rho(t))$. Then, clearly,

$$k_n(t) = k(t) \cdot \cos\theta(t), \tag{101}$$

where $k(t)$ is the curvature of the curve $\rho(t)$.

Remark

The angle θ is, in fact, the angle between the surface normal vector and the principal normal vector of the curve.

Exemple

1. The normal curvature of a plane curve (with respect to its plane) is zero (In this case, the angle $\theta(t)$ is always $\dfrac{\pi}{2}$, hence $\cos\theta(t) \equiv 0$).
2. If the support of a parametrised curve lies on a straight line, then its normal curvature is zero, regardless of which surface it lies on, since in this case the curvature of the curve itself is identically zero.

Normal Curvature. Meusnier's Theorem

Remark

Relation (101) has a simple geometric interpretation (Meusnier's theorem): *the centre of curvature of a curve ρ lying on a surface S is the orthogonal projection onto the osculating plane of the centre of curvature of the normal section tangent to ρ at that point.*

The normal curvature of a curve on a surface can easily be expressed if the two fundamental forms of the surface are known. Indeed, we have:

Theorem

The normal curvature of a regular parametrised curve $\rho(t)$ lying on an oriented surface S is given by the formula

$$k_n(t) = \frac{\varphi_2(\boldsymbol{\rho}'(t), \boldsymbol{\rho}'(t))}{\varphi_1(\boldsymbol{\rho}'(t), \boldsymbol{\rho}'(t))}. \tag{102}$$

Proof.

As is often the case with parametrised curves, the proof is simpler for curves parametrised by arc length. Since the curvature of any regular parametrised curve is invariant under reparametrisation, we can replace the curve $\rho(t)$ with a naturally parametrised curve equivalent to it, $\rho_1(s)$, where s is arc length. The curvature vector of the curve $\rho_1(s)$ will be $\boldsymbol{\rho_1}''(s)$. We choose a local parametrisation $(U, \mathbf{r})$ of the surface S, compatible with the orientation, and assume that $\rho_1(s)$ has, in this parametrisation, the local equations $u = u(s),\ v = v(s)$, i.e., $\rho_1(s) = \mathbf{r}(u(s), v(s))$. Then

$$\boldsymbol{\rho_1}''(s) = \mathbf{r}''_{\mathbf{u^2}} \cdot (u')^2 + 2\mathbf{r}''_{\mathbf{uv}} \cdot u'v' + \mathbf{r}''_{\mathbf{v^2}} \cdot (v')^2 + \mathbf{r}'_{\mathbf{u}} \cdot u'' + \mathbf{r}'_{\mathbf{v}} \cdot v''.$$

□

Proof.

Thus, for the normal curvature of the curve $\rho_1(s)$ we obtain:

$$\begin{aligned} k_n(s) &= \mathbf{k}(s)\cdot\,\mathbf{n}(\rho_1(s)) = \boldsymbol{\rho_1}''(s)\cdot\,\mathbf{n}(\rho_1(s)) = \\ &= \mathbf{r}''_{\mathbf{u^2}}\cdot\,\mathbf{n}\cdot(u')^2 + 2\mathbf{r}''_{\mathbf{uv}}\cdot\,\mathbf{n}\cdot u'v' + \mathbf{r}''_{\mathbf{v^2}}\cdot\,\mathbf{n}\cdot(v')^2 = \\ &= -L\cdot(u')^2 - 2M\cdot u'v' - N\cdot(v')^2 = \varphi_2(\boldsymbol{\rho_1}'(s), \boldsymbol{\rho_1}'(s)). \end{aligned}$$

We now return to the initially parametrised curve. We have

$$\boldsymbol{\rho}'(s) = \boldsymbol{\rho_1}'(s(t))\cdot s'(t) \qquad \text{where} \quad s'(t) \equiv \|\boldsymbol{\rho}'(t)\|.$$

Hence,

$$\boldsymbol{\rho_1}'(s(t)) = \frac{\boldsymbol{\rho}'(t)}{\|\boldsymbol{\rho}'(t)\|}.$$

□

Normal Curvature. Meusnier's Theorem

Proof.

therefore,

$$k_n(t) = \varphi_2\left(\frac{\rho'(t)}{\|\rho'(t)\|}, \frac{\rho'(t)}{\|\rho'(t)\|}\right) =$$
$$= \frac{1}{\rho'(t)\cdot\rho'(t)} \cdot \varphi_2(\rho'(t), \rho'(t)) = \frac{\varphi_2(\rho'(t), \rho'(t))}{\varphi_1(\rho'(t), \rho'(t))}.$$

□

Corollary

If two curves on an oriented surface share a common point and the same tangent at that point, then the two curves have the same normal curvature at the point of contact.

Proof.

Let **p** and **q** be the tangent vectors to the two curves at their common point. From the hypothesis, $\mathbf{p} = \alpha\mathbf{q}$, thus, by the theorem,

$$k_n = \frac{\varphi_2(\mathbf{p}, \mathbf{p})}{\varphi_1(\mathbf{p}, \mathbf{p})} = \frac{\varphi_2(\alpha\mathbf{q}, \alpha\mathbf{q})}{\varphi_1(\alpha\mathbf{q}, \alpha\mathbf{q})} = \frac{\alpha^2\varphi_2(\mathbf{q}, \mathbf{q})}{\alpha^2\varphi_1(\mathbf{q}, \mathbf{q})} = \frac{\varphi_2(\mathbf{q}, \mathbf{q})}{\varphi_1(\mathbf{q}, \mathbf{q})}$$

□

Remark

The previous consequence can be interpreted differently. Consider a family of biregular parametrised curves on the surface, $\{\boldsymbol{\rho}^\alpha(t)\}_{\alpha\in A}$, all passing through the same point and having the same tangent at the point of contact. We denote by k^α the curvature of the curve $\boldsymbol{\rho}^\alpha$ and by θ^α the angle between the surface normal vector and the osculating plane of the curve $\boldsymbol{\rho}^\alpha$.

Remark

The consequence is equivalent to the statement that the product $k_n = k^\alpha \cos\theta^\alpha$ does not depend on the choice of curve from the family. It thus makes sense to choose any line in the tangent plane passing through the origin of the tangent space (i.e., through the point of tangency) and speak about the *normal curvature of the surface* in the direction of this line, or, equivalently, we can define an application k_n on the set of all non-zero tangent vectors to the surface with real values, by setting

$$k_n(\mathbf{h}) = \frac{\varphi_2(\mathbf{h}, \mathbf{h})}{\varphi_1(\mathbf{h}, \mathbf{h})}. \tag{103}$$

Remark

The quantity $k_n(\mathbf{h})$ is called the *normal curvature of the surface in the direction of the vector* $\mathbf{h}$ (since it clearly depends only on the *direction* of the vector $\mathbf{h}$, not on its length or sense). Thus, the *normal curvature* of an oriented surface in the direction of a non-zero vector $\mathbf{h}$ is the normal curvature of some parametrised curve passing through the origin of $\mathbf{h}$ whose tangent vector is collinear with $\mathbf{h}$.

Asymptotic Directions and Asymptotic Lines on a Surface

We have seen earlier that the normal curvature on a surface at a given point and in a given direction can be expressed in terms of the first and second fundamental forms of the surface at that point (evaluated on a tangent vector to the surface at that point, having the same direction as the given direction), and that, although it was initially defined using *curves* on the surface, it in fact depends only on the direction of the tangent vectors to these curves at the contact point. It is of interest to identify those directions for which the normal curvature vanishes.

Asymptotic Directions and Asymptotic Lines on a Surface

Definition

Let S be an oriented surface and $p \in S$. We say that a non-zero vector $\mathbf{h} \in T_pS$ has an *asymptotic direction* if the normal curvature in its direction vanishes. Alternatively, based on the previous section, we can say that a vector has an asymptotic direction if

$$\varphi_2(\mathbf{h}, \mathbf{h}) = 0.$$

Accordingly, we say that a curve on the surface is an *asymptotic line* or *asymptotic curve* if all its tangent vectors have an asymptotic direction.

Asymptotic Directions and Asymptotic Lines on a Surface

Asymptotic directions do not exist at every point of a surface. The following result provides a necessary and sufficient condition for the existence of such directions.

Theorem

Let $p \in S$ be a point on an oriented surface. Then at this point there exist asymptotic directions if and only if the quadratic form associated with the second fundamental form of S at p is not positive definite. If we choose a local parametrisation $(U, \mathbf{r})$ of S around p such that $p = \mathbf{r}(u_0, v_0)$ for a pair $(u_0, v_0) \in U$, then this condition simply means that

$$D(u_0, v_0) \cdot D''(u_0, v_0) - D'^2(u_0, v_0) \leq 0. \tag{104}$$

Asymptotic Directions and Asymptotic Lines on a Surface

Proof.

Let $\mathbf{h} = \{h_1, h_2\} \in T_pS$ be a non-zero vector tangent to the surface at the point p. Then, in the chosen local parametrisation, $\mathbf{h}$ has an asymptotic direction if and only if

$$D(u_0, v_0)h_1^2 + 2D'(u_0, v_0)h_1h_2 + D''(u_0, v_0)h_2^2 = 0.$$

Since $\mathbf{h} \neq 0$, we may assume, for example, that $h_2 \neq 0$. Then the above equation can be rewritten as

$$D(u_0, v_0)\left(\frac{h_1}{h_2}\right)^2 + 2D'(u_0, v_0)\frac{h_1}{h_2} + D''(u_0, v_0) = 0.$$

Evidently, this equation (with unknown h_1/h_2) has *real* solutions if and only if its discriminant is positive, which is precisely condition 104. $\square$

Asymptotic Directions and Asymptotic Lines on a Surface

Remark

From the proof of the previous theorem it follows that when the second fundamental form is negative definite, we have two asymptotic directions, while at points where it is degenerate we have only one (or, more precisely, two coincident ones).

From the definition of the normal curvature of a curve on a surface it immediately follows that

Property

Any straight line lying on a surface is an asymptotic line of the surface.

Proof.

Indeed, straight lines have zero curvature, which means that their normal curvature, relative to any surface on which they lie, also

Asymptotic Directions and Asymptotic Lines on a Surface

The differential equation of the asymptotic lines on a surface can be obtained directly from the definition.

Theorem

Let S be an oriented surface and $\rho : I \to S$ a curve on the surface. Suppose there exists a local parametrisation of S, $(U, \mathbf{r} = \mathbf{r}(u, v))$, such that $\rho(I) \subset \mathbf{r}(U)$, and the local equations of the curve in this parametrisation are $u = u(t), v = v(t)$. Then ρ is an asymptotic line on S if and only if

$$\begin{aligned} &D(u(t), v(t)) \cdot u'^2(t) + 2 \cdot D'(u(t), v(t)) \cdot u'(t) \cdot v'(t) + \\ &+ D''(u(t), v(t)) \cdot v'^2(t) = 0. \end{aligned} \tag{105}$$

Proof.

Equation (105) is the condition that the tangent vector to the curve

Asymptotic Directions and Asymptotic Lines on a Surface

Suppose now that the curve ρ in the previous theorem is biregular, which means, in particular, that its curvature is strictly positive at all points. From the definition of normal curvature it immediately follows that, if ν is the principal normal vector of the curve, then the curve is an asymptotic line if and only if

$$\nu(t) \cdot \mathbf{n}(u(t), v(t)) = 0,$$

where **n** is the unit normal to the surface. This in fact means that the principal normal of the curve lies in the tangent plane to the surface at each point of the curve. Thus we obtain the following characterisation of asymptotic lines:

Theorem

Let S be an oriented surface and ρ a biregular parametrised curve on S. Then ρ is an asymptotic line if and only if, at every point, the osculating plane coincides with the tangent plane to the surface at that

Asymptotic Directions and Asymptotic Lines on a Surface

Another immediate result concerning asymptotic lines is the following:

Property

Let S be an oriented surface and $(U, \mathbf{r} = \mathbf{r}(u, v))$ a local parametrisation of S. Then the coordinate lines $u = const$ and $v = const$ are asymptotic lines on $\mathbf{r}(U)$ if and only if $D = D'' = 0$.

Asymptotic Directions and Asymptotic Lines on a Surface

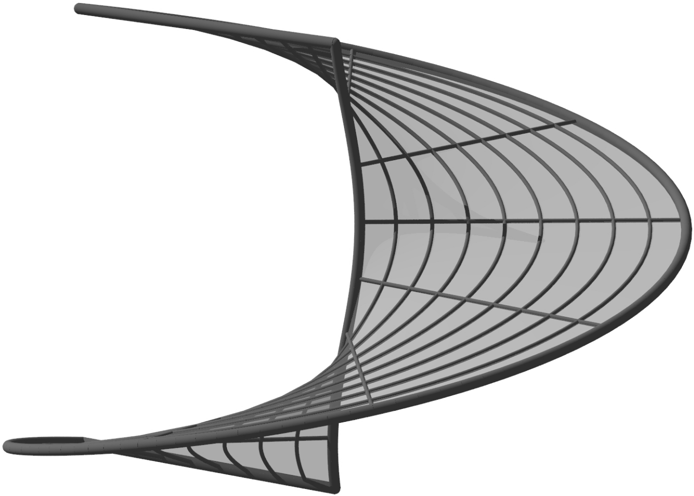
Example

Let us consider the helicoid, given by the (global!) parametrisation

$$\begin{cases} x = u\cos v, \\ y = u\sin v \\ z = b\cdot v \end{cases}$$

where b is a constant. A straightforward calculation leads to $D = D'' = 0$, $D'(u,v) = -b/\sqrt{b^2+u^2}$. This means, in this particular case, we have two asymptotic lines, and they are precisely the coordinate lines, $u = const$ and $v = const$. We note that the helicoid is a ruled surface, and that in every point, one of the coordinate lines is a straight line (or a straight line segment). This is, of course, the line $v = const$ (see the next figure).

Asymptotic Directions and Asymptotic Lines on a Surface



Classification of Points on a Surface

The first fundamental form of a surface is positive definite at all points of the surface. The second fundamental form, however, may be positive definite, negative definite, or degenerate at different points of the surface. We can therefore make a classification of the points on the surface based on the sign of the discriminant $DD'' - D'^2$ of the second fundamental form, which tells us whether the form is definite (positive or negative) or degenerate at a given point.

Definition

A point $a \in S$ on an oriented surface is called

(i) *elliptic*, if the second fundamental form is positive definite at a;

(ii) *parabolic*, if the discriminant of the second fundamental form is zero at a, but at least one of the coefficients is non-zero;

Classification of Points on a Surface

Definition

(iii) *hyperbolic*, if the second fundamental form is negative definite at a;

(iv) *flat* or *planar*, if all the coefficients of the second fundamental form vanish at a.

It is not difficult to see that this definition does not depend on the choice of local parametrisation (compatible with the orientation of the surface).

We will now discuss separately what happens in each case and we will also provide some examples.

Elliptic points. At an elliptic point, the normal curvature has the same sign in all directions[7]. If we apply Meusnier's theorem, this means that the centres of curvature of all normal sections lie on the same side of the surface. An example of a surface that has only elliptic points is the ellipsoid given, for example, by the parametrisation

$$\mathbf{r}(u, v) = (a \cos u \cos v, b \sin u \cos v, c \sin v). \tag{106}$$

At an elliptic point of a surface, there are no asymptotic directions, and therefore no asymptotic line passes through an elliptic point.

[7] It is not necessarily positive.

Classification of Points on a Surface

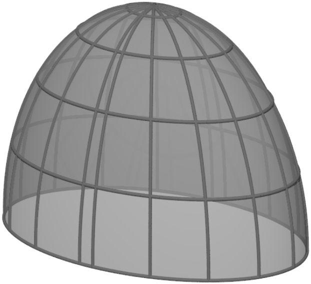


Figure: Elliptic points on a surface

Parabolic points. In this case, the normal curvature does not change sign, but there exists exactly one direction in which it vanishes. This is clearly an asymptotic direction. Thus, through a parabolic point of a surface, there passes a single asymptotic line.

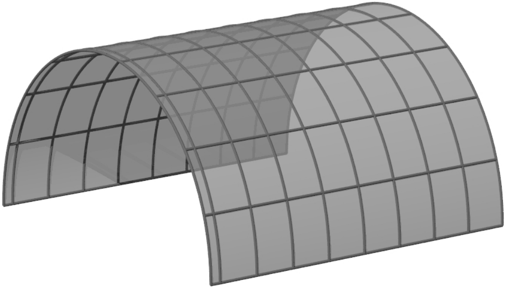


Figure: Parabolic points on a surface

Cylinders and cones (with their vertices removed) have only parabolic points.

Hyperbolic points. In the case of hyperbolic points, k_n can change its sign and there exist exactly two directions in which it vanishes. Thus, through a hyperbolic point of the surface pass two distinct asymptotic

Classification of Points on a Surface

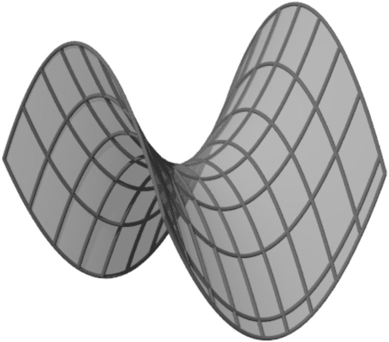
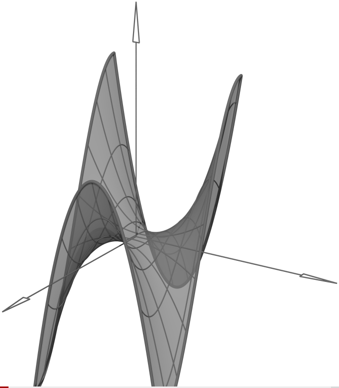


Figure: Hyperbolic points on a surface

Hyperbolic points of a surface are also called *saddle points*. Such points are found, for example, on a hyperbolic paraboloid.
There are, of course, surfaces on which all three types of points can be

Classification of Points on a Surface

Planar points. The shape of a surface near a planar point can be quite complicated and difficult to study. In fact, in many cases, in the proof of a theorem in surface theory, it is explicitly assumed that the surface has no planar points. The surface in the figure below (*the monkey saddle*) has a planar point at the origin of the coordinates.



Principal Directions, Principal Curvatures, Gaussian Curvature and Mean Curvature

Definition

The directions in the tangent plane to an oriented surface S at a point $a \in S$, T_aS, which correspond to the eigenvalues of the shape operator A, are called the *principal directions* of the surface at point a.

Remark

At each point, an oriented surface either has two orthogonal eigen-directions (if the eigenvalues of A are distinct), or all directions are principal (if the two eigenvalues coincide).

Definition

A curve (Γ) on an oriented surface S is called a *principal line* or a *line of curvature* if its tangent at each point has a principal direction.

Principal Directions, Principal Curvatures, Gaussian Curvature and Mean Curvature

Definition

The *principal curvature* of an oriented surface S at a point $a \in S$ is the normal curvature of S at a in a principal direction.

Property

The principal curvatures of an oriented surface are the eigenvalues of the shape operator, taken with opposite sign.

Proof.

If $\mathbf{e}$ is an eigenvector of A, then $A(\mathbf{e}) = \lambda \cdot \mathbf{e}$, where λ is the corresponding eigenvalue. Hence,

$$k_n(\mathbf{e}) = \frac{\varphi_2(\mathbf{e}, \mathbf{e})}{\varphi_1(\mathbf{e}, \mathbf{e})} = \frac{-A(\mathbf{e}) \cdot \mathbf{e}}{\mathbf{e} \cdot \mathbf{e}} = \frac{-\lambda \cdot \mathbf{e} \cdot \mathbf{e}}{\mathbf{e} \cdot \mathbf{e}} = -\lambda.$$

From now on, we denote the principal curvatures by k_1 and k_2 and we always assume $k_1 \geq k_2$.

Definition

An orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of the tangent space at a point on a surface is called a *basis of principal directions* of the tangent space if the basis vectors have principal directions.

Thus, the vectors of a principal direction basis satisfy

$$A(\mathbf{e}_i) = -k_i \mathbf{e}_i, \quad i = \overline{1,2}.$$

Now, fix a point on the surface and consider the following problem: to determine the normal curvature in the direction of a vector $\mathbf{e}$ such that $\angle(\mathbf{e}, \mathbf{e}_1) = \theta$.

Principal Directions, Principal Curvatures, Gaussian Curvature and Mean Curvature

Since the length of **e** is not important, we assume that **e** is a unit vector: $\|\mathbf{e}\| = 1$. Then $\mathbf{e} = \mathbf{e}_1 \cdot \cos\theta + \mathbf{e}_2 \cdot \sin\theta$, hence

$$\begin{aligned} k_n(\mathbf{e}) &= \frac{\varphi_2(\mathbf{e},\mathbf{e})}{\varphi_1(\mathbf{e},\mathbf{e})} = \frac{-A(\mathbf{e})\cdot\mathbf{e}}{\underbrace{\mathbf{e}\cdot\mathbf{e}}_{=1}} = \\ &= -A(\mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta)\cdot(\mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta) = \\ &= (k_1\cos\theta\cdot\mathbf{e}_1 + k_2\sin\theta\cdot\mathbf{e}_2)\cdot(\mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta) = \\ &= k_1\cos^2\theta + k_2\sin^2\theta. \end{aligned}$$

Thus, we obtain:

Theorem

Let S be an oriented surface. Then the normal curvature at a point of the surface, in the direction of a vector $\mathbf{e}$, is given by Euler's formula:

$$k_n(\mathbf{e}) = k_1 \cos^2\theta + k_2 \sin^2\theta, \tag{107}$$

where k_1 and k_2 are the principal curvatures of the surface, while $\theta = \angle(\mathbf{e}, \mathbf{e}_1)$.

Principal Directions, Principal Curvatures, Gaussian Curvature and Mean Curvature

An immediate consequence of Euler's theorem is:

Theorem

The principal curvatures of a surface at a point are the extreme values of the normal curvature of the surface in the direction of a vector **e**, *as the vector* **e** *rotates around the origin of the tangent space to the surface at that point.*

Proof.

From Euler's formula, we get

$$k_n(\mathbf{e}) = k_1 \cos^2\theta + k_2(1 - \cos^2\theta) = k_2 + (k_1 - k_2)\cos^2\theta.$$

It is clear that the maximum value of the normal curvature is reached when $\theta = 0$ (we assumed $k_1 \geq k_2$!), and in this case we get $k_n = k_1$, while the minimum value ... for $\theta = \frac{\pi}{2}$, yielding $k_n = k_2$.

Principal Directions, Principal Curvatures, Gaussian Curvature and Mean Curvature

Definition

The quantities $K_t = k_1 \cdot k_2$ and $K_m = \frac{1}{2}(k_1 + k_2)$ are called the *total (or Gaussian) curvature* and the *mean curvature* of the surface, respectively.

The total and mean curvature of a surface can be easily computed if the matrix of the shape operator is known in some basis. Indeed, we have:

Property

$$K_m = -\frac{1}{2} \operatorname{Tr} \mathcal{A} \tag{108}$$

$$K_t = \det \mathcal{A}. \tag{109}$$

Principal Directions, Principal Curvatures, Gaussian Curvature and Mean Curvature

Proof.

It is well known from linear algebra that the determinant and the trace are invariants of any linear operator, which means they remain the same in any basis, even though the matrix of the operator generally changes with a change of basis. In a basis of principal directions, the matrix of the shape operator is:

$$\mathcal{A} = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}$$

Hence,

$$\det \mathcal{A} = k_1 \cdot k_2 = K_t$$
$$-\frac{1}{2}\operatorname{Tr}\mathcal{A} = -\frac{1}{2}(-k_1 - k_2) = \frac{1}{2}(k_1 + k_2) = K_m.$$

We will now see how the lines of curvature of a surface can be found by integrating a differential equation. However, in certain special situations it is possible to determine these lines by other methods. One such example is illustrated by the following theorem, due to the German mathematician Joachimsthal.

Theorem

Let γ be a curve located at the intersection of two regular oriented surfaces S_1 and S_2 in $\mathbb{R}^3$. Let $\mathbf{n}_i$ be the unit normal vectors to the two surfaces ($i = 1, 2$). Suppose that S_1 and S_2 intersect at a constant angle, that is, along the curve γ we have $\mathbf{n}_1 \cdot \mathbf{n}_2 = const.$. Then γ is a line of curvature on S_1 if and only if it is also a line of curvature on S_2.

Proof.

Let $\mathbf{r} = \mathbf{r}(t)$ be a local parametrisation of the curve γ. Then, since $\mathbf{n}_1 \cdot \mathbf{n}_2 = \text{const}$, we have

$$0 = \frac{d}{dt}(\mathbf{n}_1 \cdot \mathbf{n}_2) = \mathbf{n}_1' \cdot \mathbf{n}_2 + \mathbf{n}_1 \cdot \mathbf{n}_2'.$$

If γ is a principal line on S_1, then

$$\mathbf{n}_1' = -k_1 \cdot \mathbf{r}',$$

where k_1 is one of the principal curvatures of the surface S_1. On the other hand, since the curve γ is also on S_2, we have $\mathbf{r}' \perp \mathbf{n}_2$. □

Proof.

From this and the previous formula, we obtain that $\mathbf{n}_1' \cdot \mathbf{n}_2 = 0$, hence

$$\mathbf{n}_1 \cdot \mathbf{n}_2' = 0.$$

Since $\mathbf{n}_2' \perp \mathbf{n}_2$ (because $\mathbf{n}_2$ has constant length), it follows from this and the previous equation that $\mathbf{n}_2' \perp \mathbf{r}'$ or, in other words, that there exists $k_2 \in \mathbb{R}$ such that

$$\mathbf{n}_2' = -k_2 \mathbf{r}',$$

i.e., γ is also a line of curvature on the surface S_2. □

Corollary

Meridians and parallels on a surface of revolution are lines of