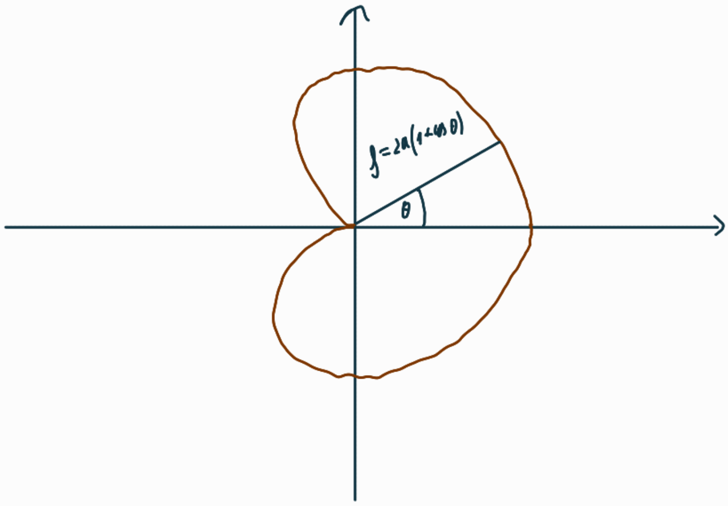
1) a cardioid is the plane curve whose implicit equation is $x^2+y^2=2a(x+\sqrt{x^2+y^2})$, $a>0$
calculate the arclength of the cardioid!

- a parameterization of cardiods can be found using polar coordinates



$x=\rho\cos\theta$

$y=\rho\sin\theta$

$x^2+y^2=2a(x+\sqrt{x^2+y^2}) \iff$

$\iff \rho^2=2a(\rho\cos\theta+\rho) \iff$

$\iff \rho=2a(\cos\theta+1) \longleftarrow$ the polar equation of the cardioid

$$\gamma:\begin{cases} x=2a(\cos\theta+1)\cos\theta \\ y=2a(\cos\theta+1)\sin\theta \end{cases} \qquad \theta\in[0,2\pi]$$

$$l(\gamma)=\int_0^{2\pi}\|\gamma'(\theta)\|\,d\theta$$

$x=2a(\cos^2\theta+\cos\theta)$

$y=2a(\cos\theta\sin\theta+\sin\theta)$

$$\gamma'(\theta)=(2a(2(-\sin\theta)\cos\theta-\sin\theta),\ 2a(\cos^2\theta\cdot\sin^2\theta+\cos\theta))=$$

$$=(-2a(2\sin\theta\cos\theta+\sin\theta),\ 2a(\cos(2\theta)+\cos\theta))$$

$$\|\gamma'(\theta)\|^2=4a^2(\sin(2\theta)+\sin\theta)^2+4a^2(\cos2\theta+\cos\theta)^2=$$

$$=4a^2(\sin^2 2\theta+2\sin2\theta\sin\theta+\sin^2\theta+\cos^2 2\theta+2\cos2\theta\cos\theta+\cos^2\theta)=$$

$$=4a^2(2+2\sin2\theta\sin\theta+2\cos2\theta\cos\theta)=$$

$$=8a^2(1+\underbrace{\sin2\theta\sin\theta+\cos2\theta\cos\theta}_{\cos(2\theta-\theta)=\cos\theta})$$

$$=8a^2(\underbrace{1+\cos\theta}_{2\cos^2\frac{\theta}{2}})=16a^2\cos^2\frac{\theta}{2} \Rightarrow \|\gamma'(\theta)\|=4a\left|\cos\frac{\theta}{2}\right|$$

$$\Rightarrow \ell(\gamma) = \int_0^{2\pi} 4a \left|\cos\frac{\theta}{2}\right| d\theta = 4a\left(\int_0^{2\pi}\right.$$

$$= 8a^2(1+ \underbrace{\sin 2\theta \sin\theta + \cos 2\theta \cos\theta}_{\cos(2\theta-\theta)=\cos\theta}) = 8a^2\underbrace{(1+\cos\theta)}_{2\cos^2\frac{\theta}{2}} =$$

$$= 16a^2\cos^2\frac{\theta}{2} \Rightarrow \|\gamma'(\theta)\| = 4a\left|\cos\frac{\theta}{2}\right|$$

$$\ell(\gamma) = \int_0^{2\pi} 4a\cos\frac{\theta}{2} = 8a\sin\frac{\theta}{2}\Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} 4a\left|\cos\frac{\theta}{2}\right| d\theta = 4a\left(\int_0^{\pi}\cos\frac{\theta}{2}d\theta - \int_{\pi}^{2\pi}\cos\frac{\theta}{2}d\theta\right)$$

$$= 4a\left(2\sin\frac{\theta}{2}\Big|_0^{\pi} - 2\sin\frac{\theta}{2}\Big|_{\pi}^{2\pi}\right) = 4a(2+2) = 16a$$

2) let $a>0$ and let $\gamma:[0,2\pi]\longrightarrow\mathbb{R}^2$ be the P.P. defined by $\gamma(t) = (a(t-\sin t), a(1-\cos t))$, calculate $I = \int_\gamma y^2 ds$

$$I = \int_0^{2\pi} a^2(1-\cos t)^2 \cdot \|\gamma'(t)\| dt = \int_0^{2\pi} a^2(1-\cos t)^2 \cdot 2a\sin\frac{t}{2} dt =$$

$$\|\gamma'(t)\| = 2a\sin\frac{t}{2}$$

$$= 2a^3\int_0^{2\pi}\underbrace{(1-\cos t)^2}_{2\sin^2\frac{t}{2}}\cdot\sin\frac{t}{2}dt = 2a^3\int_0^{2\pi} 4\sin^5\frac{t}{2}dt$$

$$= 8a^3\int_0^{2\pi}\sin^5\frac{t}{2}dt = 8a^3\int_0^{\pi}\sin^5 x\, 2dx = 16a^3\int_0^{\pi}\sin^5 x\, dx =$$

$$\frac{t}{2} = x \Rightarrow t = 2x \Rightarrow dt = 2dx$$

$$= 16a^3\int_0^{\pi}\sin x\,(1-\cos^2 x)^2 dx =$$

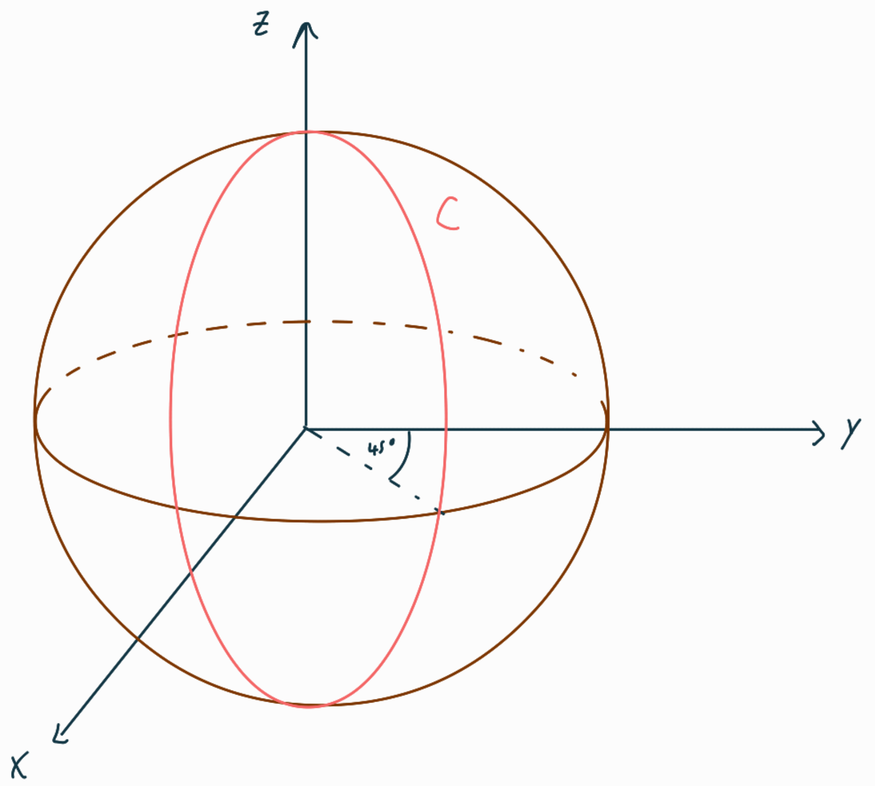
$$u = \cos x \qquad du = -\sin x\, dx$$

$$= 16a^3 \int_1^{-1} (1-u^2)^2 \cdot du = 16a^3 \int_{-1}^{1} 1 - 2u^2 + u^4 \, du = \frac{256a^3}{15}$$

3) calculate $I = \int_\Gamma \frac{z}{x^2+y^2+1} \, ds$ if Γ is the simple curve who's image is the arc of the circle $x^2+y^2+z^2=1$, $x=y$, lying in the first octant

$$I = \int_\gamma \frac{z}{x^2+y^2+1} \, ds$$

where $\gamma \in \Gamma$ is a parameterization of the arc of the circle in the statement



- a parameterization of C can be obtained using spherical coordinates

$$\rho = 1 \qquad \begin{cases} x = \sin\varphi \cos\theta \\ y = \sin\varphi \sin\theta \\ z = \cos\varphi \end{cases} \qquad \cos\theta = \sin\theta \implies \theta = \frac{\pi}{4}$$

$$\begin{cases} x = \frac{\sqrt{2}}{2} \sin\varphi \\ y = \frac{\sqrt{2}}{2} \sin\varphi \\ z = \cos\varphi \end{cases} \qquad \varphi \in [0, \tfrac{\pi}{2}] \text{ because } 1^{st} \text{ octant}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos\varphi}{\frac{1}{2}\sin^2\varphi + \frac{1}{2}\sin^2\varphi + 1} \cdot \|\gamma'(\varphi)\| \, d\varphi =$$

$$\gamma'(\varphi) = \left(\frac{\sqrt{2}}{2}\cos\varphi, \frac{\sqrt{2}}{2}\cos\varphi, -\sin\varphi\right)$$

$$\|\gamma'(\varphi)\| = \sqrt{\frac{1}{2}\cos^2\varphi + \frac{1}{2}\cos^2\varphi + \sin^2\varphi} = 1$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos\varphi}{\sin^2\varphi + 1} \, d\varphi = \int_0^1 \frac{1}{u^2+1} \, du = \arctan\Big|_0^1 = \frac{\pi}{4}$$

$u = \sin\varphi$

$du = \cos\varphi \, d\varphi$

4) calculate $I = \int_\gamma (x-y)\,dx - (x+y)\,dy$ if $\gamma : [0; \frac{\pi}{4}] \to$

$\gamma(t) = (e^t \cdot \cos t, e^t \sin t)$

$I = \int_\gamma \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = (x-y)\vec{i} - (x+y)\vec{j}$

$$I = \int_0^{\frac{\pi}{4}} (e^t\cos t - e^t \sin t)(e^t\cos t)'\,dt - \int_0^{\frac{\pi}{4}} (e^t\cos t + e^t\sin t)(e^t\sin t)'\,dt =$$

$$= \int_0^{\frac{\pi}{4}} e^t(\cos t - \sin t)(e^t\cos t - e^t\sin t)\,dt - \int_0^{\frac{\pi}{4}} e^t(\cos t + \sin t)(e^t\sin t + e^t\cos t)\,dt =$$

$$= \int_0^{\frac{\pi}{4}} e^{2t}(\cos t - \sin t)^2\,dt - \int_0^{\frac{\pi}{4}} e^{2t}(\cos t + \sin t)^2\,dt =$$

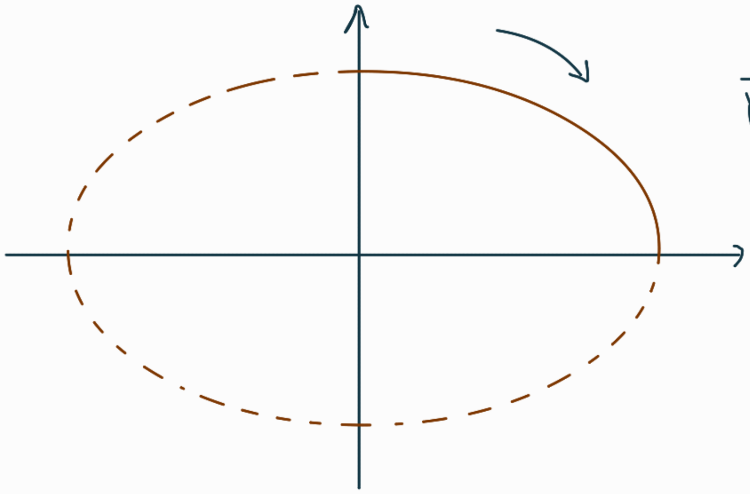
$$= \int_0^{\frac{\pi}{4}} e^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t - \cos^2 t - 2\sin t\cos t - \sin^2 t)\,dt =$$

$$= -\int_0^{\frac{\pi}{4}} 2e^{2t}\sin 2t\,dt = \int_0^{\frac{\pi}{2}} e^u \sin u\,du = \frac{e^{\frac{\pi}{2}}+1}{2}$$

$u = 2t \Rightarrow du = 2dt$

5) let $a, b > 0$, calculate $\int_\Gamma \frac{1}{x+a}\,dy$ if Γ is the simple curve whose image is the arc of the elipse $\mathcal{E} = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ traced clockwise

$I = \int_\gamma \frac{1}{x+a}\,dy$ $\quad\gamma \in \Gamma$ is a parameterization of $\mathcal{E}$



$$\bar{\gamma}: \begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \qquad t \in [0, \tfrac{\pi}{2}]$$

$$I = \int_{\gamma} \frac{1}{x+a}\,dy = -\int_{\bar{\gamma}} \frac{1}{x+a}\,dy = -\int_0^{\frac{\pi}{2}} \frac{1}{a\cos t + a}(b\sin t)'\,dt = -\frac{b}{a}\int_0^{\frac{\pi}{2}} \frac{\cos t}{\cos t + 1}$$

$$\operatorname{tg}\frac{t}{2} = x \qquad \Rightarrow t = 2\arctan x \qquad \Rightarrow dt = \frac{2}{x^2+1}\,dx$$

$$\sin t = \frac{1-x^2}{1+x^2}$$

$$\cos t = \frac{2x}{1+x^2}$$

$$= -\frac{b}{a}\int_0^1 \frac{\frac{1-x^2}{1+x^2}}{\frac{1-x^2}{1+x^2}+1}\,\frac{2}{x^2+1}\,dx = -\frac{b}{a}\int_0^1 \frac{1-x^2}{1+x^2}\cdot\frac{x^2+1}{2}\cdot\frac{2}{x^2+1}\,dx = etc\ldots$$