Exercise 1

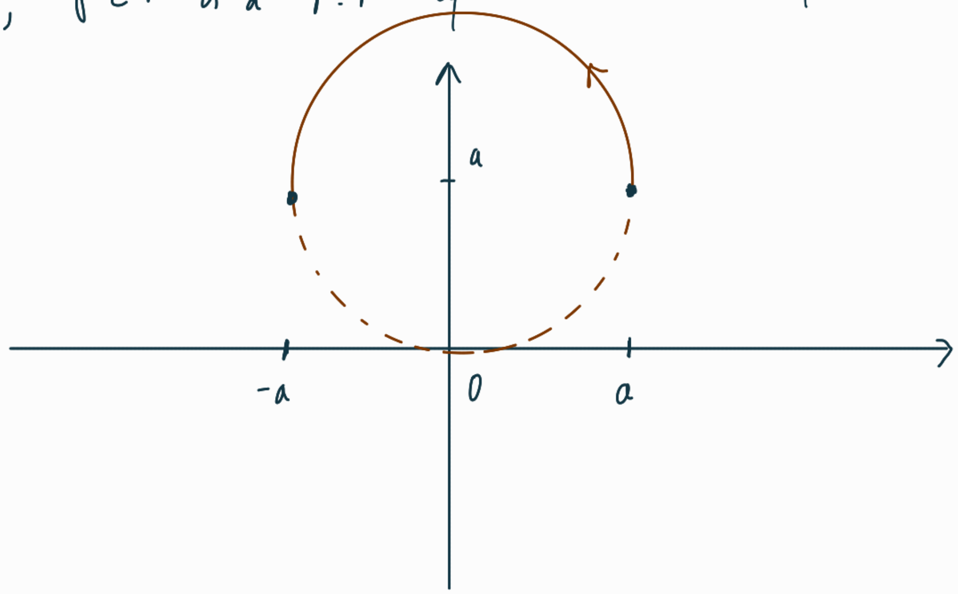
Given $a>0$ Consider the points $A(-a,a)$, $B(a,a)$

Calculate $I=\int_\Gamma [3a(x^2+y^2)-y^3]\,dx+3xy\,(2a-y)\,dy$

If Γ is the simple curve whose image is the semicircle of diameter AB, not containing the origin, traced counterclockwise.

sol1: $I=\int_\Gamma \vec{F}\cdot d\vec{r}$ $\quad \vec{F}(x,y)=[\overbrace{3a\cdot(x^2+y^2)-y^3}^{P(x,y)}]\,\vec{i}+\overbrace{3xy\,(2a-y)}^{Q(x,y)}\,\vec{j}$

$I=\int_\gamma \vec{F}\cdot d\vec{r}$, $\gamma\in\Gamma$ is a P.P. of the semicircle from the statement



$$\left.\begin{array}{l}(x-x_0)^2+(y-y_0)^2=r^2 \longrightarrow x^2+(y-a)^2=a^2\\[1em] x^2+y^2=r^2 \longrightarrow \begin{array}{l}x=r\cos t\\ y=r\sin t\end{array}\end{array}\right\}\Rightarrow \gamma:\begin{cases}x=a\cos t\\ y=a\sin t+a\end{cases} \quad t\in[0,\pi]$$

$$I=\int_0^\pi [3a(a^2\cos^2 t+a^2(1+\sin t)^2)-a^3(1+\sin t)^3]\,(a\cos t)'\,dt$$
$$+\int_0^\pi 3\cdot a\cos t\cdot a(1+\sin t)(2a-a-a\sin t)\cdot(a+a\sin t)'\,dt$$

sol 2: we check if $\vec{F}$ is conservative or not

? $\vec{F}$ is conservative $\overset{\text{Poincaré}}{\Leftrightarrow}$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow 6ay - 3y^2 \overset{\text{true}}{=} 3y(2a-y) \Rightarrow$

$\Rightarrow \vec{F}$ is a conservative V.F. $\Rightarrow$ by Leibniz - Newton $\Rightarrow$

$\Rightarrow I = U \Big|_B^A = U(-a, a) - U(a, a) =$

where U is a scalar potential for $\vec{F}$

U is a scalar potential for $\vec{F} \Leftrightarrow \nabla U = F \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \frac{\partial U}{\partial x} = P \\ \frac{\partial U}{\partial y} = Q \end{cases}$$

- $\frac{\partial U}{\partial x}(x,y) = P(x,y) = 3ax^2 + 3ay^2 - y^3 \Leftrightarrow$

$\Rightarrow U(x,y) = \int (3ax^2 + 3ay^2 - y^3)\,dx = 3a\frac{x^3}{3} + 3ay^2x - y^3x + \varphi(y) \Rightarrow$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial U}{\partial y}(x,y) = 6ayx - 3y^2x + \varphi'(y) \\ \bullet \text{ but } \frac{\partial U}{\partial y}(x,y) \text{ must be } = \text{ to } Q(x,y) = 6axy - 3xy^2 \end{array}\right\} \Rightarrow \varphi'(y) = 0$$

$\Rightarrow \varphi(y) = c \quad \Rightarrow U = ax^3 + 3ay^2x - y^3x + c$

$\Rightarrow I = U(-a,a) - U(a,a) = (-a^4 - 3a^4 + a^4 + c) - (a^4 + 3a^4 - a^4 + c) =$

$\boxed{= -6a^4}$

2) Calculate $I = \int_{(2,0)}^{(1,1)} (y^3+6xy^2+4x^3)\,dx + (3xy^2+6xy^2+2y)\,dy$

$I = \int_{(0,2)}^{(1,1)} \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \underbrace{(y^3+6xy^2+4x^3)}_{P(x,y)}\vec{i} + \underbrace{(3xy^2+6xy^2+2y)}_{Q(x,y)}\vec{j}$

we must check if the work of F does not depend on the path of integration $\Leftrightarrow$

$\Leftrightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Poincaré

$$\left.\begin{aligned} &\frac{\partial P}{\partial y}(x,y) = 3y^2 + 12xy \\ &\frac{\partial Q}{\partial x}(x,y) = 3y^2 + 12xy \end{aligned}\right\} \Rightarrow F \text{ does not depend on the path}$$

By Leibniz-Newton: $I = U(1,1) - U(0,2)$, where U is a scalar potential of $\vec{F}$

U is a scalar potential for $\vec{F} \Leftrightarrow \dfrac{\partial U}{\partial x} = P$ and $\dfrac{\partial U}{\partial y} = Q$

- $\dfrac{\partial U}{\partial x}(x,y) = P(x,y) = y^3 + 6xy^2 + 4x^3$

$U(x,y) = \int y^3+6xy^2+4x^3\,dx = xy^3 + 3x^2y^2 + x^4 + \varphi(y) \Rightarrow \dfrac{\partial U}{\partial y}(x,y) = 3xy^2 + 6x^2y + \varphi'(y)$

but $\dfrac{\partial U}{\partial y}(x,y) = Q(x,y) = 3xy^2 + 6xy + 2y$

$\Rightarrow \varphi'(y) = 2y \Rightarrow \varphi(y) = y^2 + C$

$\Rightarrow U(1,1) - U(0,2) = (6+C) - (16+C) = -10$

3) $I = \int_{(-1,-1,-1)}^{(1,0,1)} (3x^2y+z^3)\,dx + (3y^2z+x^3)\,dy + (3xz^2+y^3)\,dz$, $I = ?$

$I = \int_{(-1,-1,-1)}^{(1,0,1)} \vec{F}\cdot d\vec{r}$ where

$\vec{F}(x,y,z) = \underbrace{(3x^2y+z^3)}_{P(x,y,z)}\vec{i} + \underbrace{(3y^2z+x^3)}_{Q(x,y,z)}\vec{j} + \underbrace{(3xz^2+y^3)}_{R(x,y,z)}\vec{k}$

we check if the work of F is independent on the path of integration $\Leftrightarrow$

Poincaré

$(\Leftrightarrow)\quad \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$ and $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$ $(\Leftrightarrow)$

$(\Leftrightarrow)\quad 3x^2 = 3x^2$ and $3y^2 = 3y^2$ and $3z^2 = 3z^2$ $\Rightarrow$

$\Rightarrow I = U(1,0,1) - U(-1,-1,-1)$ where U is a scalar potential (S.P) for $\vec{F}$

U is a S.P. for $\vec{F}$ $(\Leftrightarrow)$ $\dfrac{\partial U}{\partial x} = P$ and $\dfrac{\partial U}{\partial y} = Q$ and $\dfrac{\partial U}{\partial z} = R$

$$\frac{\partial U}{\partial x}(x,y,z) = P(x,y,z) = 3x^2 y + z^3 \Rightarrow U(x,y,z) = \int 3x^2 y + z^3\, dx = x^3 y + z^3 x + \varphi(y,z)$$

$$\left.\begin{aligned} &\frac{\partial U}{\partial y}(x,y,z) = x^3 + \frac{\partial \varphi}{\partial y}(y,z) \\ &\text{but } \frac{\partial U}{\partial y}(x,y,z) = Q(x,y,z) = 3y^2 z + x^3 \end{aligned}\right\} \Rightarrow \frac{\partial \varphi}{\partial y}(y,z) = 3y^2 z \Rightarrow$$

$$\Rightarrow \varphi(y,z) = \int 3y^2 z\, dz = y^3 z + \psi(z) \Rightarrow U(x,y,z) = x^3 y + z^3 y + y^3 z + \psi(z)$$

$$\frac{\partial U}{\partial z}(x,y,z) = y$$