sem 3:

1) a) $\lim\limits_{(x,y)\to(0,0)} xy \sin \frac{1}{x^2+y^2}$

$$0 \le \left|xy \sin\frac{1}{x^2+y^2} - 0\right| = |x||y| \underbrace{\left|\sin\frac{1}{x^2+y^2}\right|}_{\in[0,1]} \le |x|\cdot|y|$$

thus

0

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lecture 3 ex: $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim\limits_{(x,y,z)\to(0,0,0)} \left|\frac{xyz}{x^2+y^2+z^2} - 0\right| = \lim \frac{|x|\cdot|y|\cdot|z|}{x^2+y^2+z^2}$

solution 1:

$0 \le \sqrt{xy} \le \sqrt{\frac{x^2+y^2}{2}} \Rightarrow |x|\cdot|y| \le \frac{x^2+y^2}{2} \le \frac{x^2+y^2+z^2}{2} \quad /\cdot \frac{1}{x^2+y^2+z^2}$

$\Rightarrow \frac{|x|\cdot|y|}{x^2+y^2+z^2} \le \frac{1}{2} \quad /\cdot |z| \Rightarrow 0 \le \frac{|x|\cdot|y|\cdot|z|}{x^2+y^2+z^2} \le \frac{|z|}{2}$

thus

0

solution 2: $0 \le \sqrt[3]{|x|\cdot|y|\cdot|z|} \le \sqrt{\frac{x^2+y^2+z^2}{3}} \Rightarrow |x|\cdot|y|\cdot|z| \le \left(\frac{x^2+y^2+z^2}{3}\right)^{\frac{3}{2}} \quad /\cdot \frac{1}{x^2+y^2+z^2}$

$\Rightarrow 0 \le \frac{|x|\cdot|y|\cdot|z|}{x^2+y^2+z^2} \le \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{27}}$

lecture 3 ex: $f(x,y) = \frac{xy}{x^2+y^2}$ $\quad \exists ? \lim\limits_{(x,y)\to(0,0)}$

$$\left.\begin{aligned} &\lim_{k\to\infty}\left(\frac{1}{k}, \frac{1}{k}\right) = (0,0) \quad \lim_{k\to\infty} f\left(\frac{1}{k}, \frac{1}{k}\right) = \frac{1}{2} \\ &\lim_{k\to\infty}\left(\frac{1}{k}, 0\right) = (0,0) \quad \lim_{k\to\infty} f\left(\frac{1}{k}, 0\right) = 0 \end{aligned}\right\} \Rightarrow \not\exists$$

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sem 3 1) h) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy}$?

$f(x,y) = \frac{x^3+y^3}{xy}$

$$\left.\begin{aligned} &\lim_{k\to\infty}\left(\frac{1}{k}, \frac{1}{k}\right) = (0,0) \quad \lim_{k\to\infty} f\left(\frac{1}{k}, \frac{1}{k}\right) = \lim_{k\to\infty} \frac{\frac{1}{k^3}+\frac{1}{k^3}}{\frac{1}{k}\cdot\frac{1}{k}} = \lim_{k\to\infty} \frac{\frac{2}{k^3}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{2}{k} = 0 \\ &\lim_{k\to\infty}\left(\frac{1}{k^2}, \frac{1}{k}\right) = (0,0) \quad \lim_{k\to\infty} f\left(\frac{1}{k^2}, \frac{1}{k}\right) = \lim_{k\to\infty} \frac{\frac{1}{k^6}+\frac{1}{k^3}}{\frac{1}{k^3}} = \lim_{k\to\infty} \frac{1}{k^3} + 1 = 1 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \not\exists$

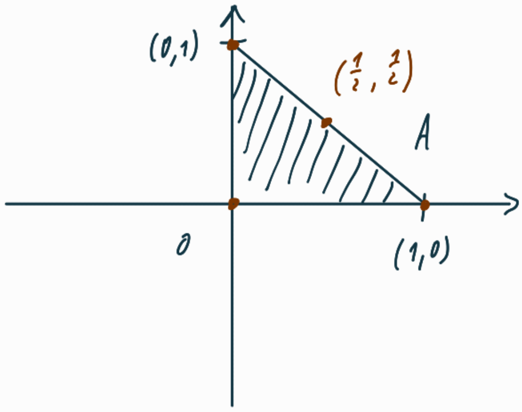
g) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ $\qquad f(x,y) = \frac{x^2-y^2}{x^2+y^2}$

$\lim\limits_{k\to\infty}\left(\frac{1}{k}, \frac{1}{k}\right) = (0,0) \qquad \lim\limits_{k\to\infty} f\left(\frac{1}{k}, \frac{1}{k}\right) = 0$

$\lim\limits_{k\to\infty}\left(\frac{1}{k}, 0\right) = (0,0) \qquad \lim\limits_{k\to\infty} f\left(\frac{1}{k}, 0\right) = 1$

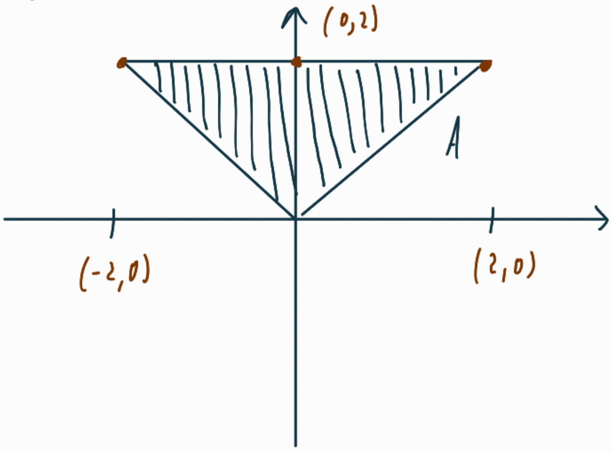
$\left.\begin{matrix} \\ \end{matrix}\right\} \Rightarrow \nexists$

sem 3 $\quad$ 3) a) $A = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x+y \leq 1\}$ compact?



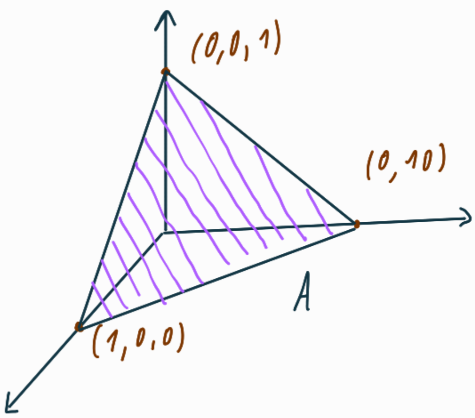
$\Rightarrow$ bounded

b) $A = \{(x,y) \in \mathbb{R}^2 \mid |x| \leq y \leq 2\}$



$\Rightarrow$ bounded

c) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z = 1\}$



sem 4: 1) $f:\mathbb{R}\times(0,\infty)\to\mathbb{R}$, $f(x,y)=\ln(2\sqrt{x^2+y^2}-x)-\ln y$

prove: $x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=0 \quad \forall (x,y)\in\mathbb{R}\times(0,\infty)$

sol: $$\frac{\partial f}{\partial x}(x,y)=\frac{1}{2\sqrt{x^2+y^2}-x}\cdot\left(\frac{2\cdot 2x}{2\cdot\sqrt{x^2+y^2}}-1\right)=\frac{\frac{2x-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}-x}=\frac{2x-\sqrt{x^2+y^2}}{(2\sqrt{x^2+y^2}-x)\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{\frac{2y}{\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}-x}-\frac{1}{y}=\frac{2y^2-(2\sqrt{x^2+y^2}-x)\sqrt{x^2+y^2}}{y(2\sqrt{x^2+y^2}-x)\sqrt{x^2+y^2}}$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{2x^2-x\sqrt{x^2+y^2}}{(2\sqrt{x^2+y^2}-x)\sqrt{x^2+y^2}}+\frac{2y^2-(2\sqrt{x^2+y^2}-x)\sqrt{x^2+y^2}}{(2\sqrt{x^2+y^2}-x)\sqrt{x^2+y^2}}=0$$

2) $f:(0,\infty)\times\mathbb{R}^2\to\mathbb{R}$

$f(x,y,z)=(xy+z^2)\cos\left(\frac{yz}{x^2}\right)$ satisfies

$$x\frac{\partial f}{\partial x}(x,y,z)+y\frac{\partial f}{\partial y}(x,y,z)+z\frac{\partial f}{\partial z}(x,y,z)=2f(x,y,z) \quad \forall (x,y,z)\in(0,\infty)\times\mathbb{R}$$

$$\frac{\partial f}{\partial x}(x,y,z)=y\cdot\cos\left(\frac{yz}{x^2}\right)-(xy+z^2)\cdot\sin\left(\frac{yz}{x^2}\right)\cdot\frac{yz\cdot(-2)}{x^3}=y\cos\frac{yz}{x^2}+\frac{2yz}{x^3}(xy+z^2)\sin\frac{yz}{x^2}$$

$$\frac{\partial f}{\partial y}(x,y,z)=x\cos\frac{yz}{x^2}-(xy+z^2)\cdot\sin\left(\frac{yz}{x^2}\right)\cdot\frac{z}{x^2}$$

$$\frac{\partial f}{\partial z}(x,y,z)=2z\cos\frac{yz}{x^2}-(xy+z^2)\sin\left(\frac{yz}{x^2}\right)\frac{y}{x^2}$$

$$x\frac{\partial f}{\partial x}(x,y,z)+y\frac{\partial f}{\partial y}(x,y,z)+z\frac{\partial f}{\partial z}(x,y,z)=$$

$$=xy\cos\frac{yz}{x^2}+\cancel{\frac{2yz}{x^2}\sin\left(\frac{yz}{x^2}\right)\cdot(xy+z^2)}+xy\cos\left(\frac{yz}{x^2}\right)-\cancel{\frac{yz}{x^2}(xy+z^2)\sin\left(\frac{yz}{x^2}\right)}+2z^2\cos\left(\frac{yz}{x^2}\right)-\cancel{\frac{yz}{x^2}(xy+z^2)\sin\left(\frac{yz}{x^2}\right)}=$$

$$=2(xy-z^2)\cos\frac{yz}{x^2}=2f(x,y,z)$$

6) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x\sqrt{x^2+y^2}$, determine $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\nabla f(3,4)$ and $df(3,4)$

$$\frac{\partial f}{\partial x}(x,y)=\sqrt{x^2+y^2}+\frac{x\cdot 2x}{2\sqrt{x^2+y^2}}=\sqrt{x^2+y^2}+\frac{x^2}{\sqrt{x^2+y^2}}=\frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{2yx}{2\sqrt{x^2+y^2}}=\frac{xy}{\sqrt{x^2+y^2}}$$

$$\nabla f(3,4) = \left(\frac{34}{5}, \frac{12}{5}\right)$$

$$df(x,y) \in L(\mathbb{R}^2, \mathbb{R})$$

$$\boxed{df(x,y)(h_1,h_2) = h_1 \frac{\partial f}{\partial x} + h_2 \frac{\partial f}{\partial y}} \Rightarrow df(3,4)(h_1,h_2) = \frac{34}{5} h_1 + \frac{12}{5} h_2$$

sem 5: 1) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \quad f(x,y) = (x^2 - y, xy + x^2, e^{x^2-y^2})$

$df(1,1) = ?$

$$J(f)(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \\ \frac{\partial f_3}{\partial x}(x,y) & \frac{\partial f_3}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ y+2x & x \\ 2xe^{x^2-y^2} & -2ye^{x^2-y^2} \end{pmatrix}$$

$$[df(1,1)] = J(f)(1,1) = \begin{pmatrix} 2 & -1 \\ 3 & 1 \\ 2 & -2 \end{pmatrix}$$

$$df(1,1)(h_1,h_2) = J(f)(1,1)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

2) surface $\Sigma: xyz = 1$, $M(a,b,c)$ s.t. $M \in \Sigma$

T_Σ in M, $T_\Sigma \cap O_x = A \quad T_\Sigma \cap O_y = B \quad T_\Sigma \cap O_z = C$

V_{OABC} does not depend on M

$$f(x,y,z) = xyz - 1$$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\right) = (yz, xz, xy)$$

$$\Rightarrow \nabla f(a,b,c) = (bc, ac, ab)$$

$$\Rightarrow \exists T_\Sigma: bcx + acy + abz + k = 0 \text{ s.t. } M \in T_\Sigma \Rightarrow$$

$$\Rightarrow bca + acb + abc + k = 0 \Rightarrow 1+1+1+k = 0 \Rightarrow k = -3 \Rightarrow$$

$$\Rightarrow T_\Sigma: bcx + acy + abz - 3 = 0$$

$$A = O_x \cap T_\Sigma \Rightarrow bcx = 3 \Rightarrow x = \frac{3}{bc} \Rightarrow OA\left(\frac{3}{bc}, 0, 0\right)$$

$$B\left(0, \frac{3}{ac}, 0\right)$$

$$C\left(0, 0, \frac{3}{ab}\right)$$

$$V_{OABC} = \frac{|OA| \cdot |OB| \cdot |OC|}{6} = \frac{\left|\frac{3}{bc}\right| \cdot \left|\frac{3}{ac}\right| \cdot \left|\frac{3}{ab}\right|}{6} = \frac{27}{6|a^2b^2c^2|} = \frac{9}{2 \cdot 1} = \frac{9}{2} \Rightarrow \text{true}$$

3) is it diff : $f(x,y) = \begin{cases} (x,y) \neq (0,0), & \frac{x^3+y^3}{x^2+y^2} \\ (x,y) = (0,0), & 0 \end{cases}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(x^3+y^3)'(x^2+y^2) - (x^3+y^3)(x^2+y^2)'}{(x^2+y^2)^2} = \frac{3x^2(x^2+y^2) - 2x(x^3+y^3)}{(x^2+y^2)^2} \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{3y^2(x^2+y^2) - 2y(x^3+y^3)}{(x^2+y^2)} \quad \forall (x,y) \in \mathbb{R}^2 \in \{(0,0)\}$$

$\Rightarrow$ f is part. diff. on $\mathbb{R}^2 \setminus \{0,0\}$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both cont. on $\mathbb{R}^2 \setminus \{0,0\}$ $\Bigg\} \Rightarrow f$ is diff. on $\mathbb{R}^2 \setminus \{(0,0)\}$

$\mathbb{R}^2 \setminus \{0,0\}$ is an open set

f diff at $(0,0)$?

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x \to 0} \frac{x - 0}{x - 0} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y \to 0} \frac{y - 0}{y - 0} = 1$$

$\Bigg\} \Rightarrow$ they exist $\Rightarrow$

$\Rightarrow$ we calcute $\displaystyle\lim_{\substack{h_1 \to 0 \\ h_2 \to 0}} \frac{f(0+h_1, 0+h_2) - f(0,0) - h_1 \frac{\partial f}{\partial x}(0,0) - h_2 \frac{\partial f}{\partial y}(0,0)}{\sqrt{h_1^2 + h_2^2}} =$

$$= \lim_{\substack{h_1 \to 0 \\ h_2 \to 0}} \frac{\frac{h_1^3 + h_2^3}{h_1^2 + h_2^2} - h_1 - h_2}{\sqrt{h_1^2 + h_2^2}} = \lim_{\substack{h_1 \to 0 \\ h_2 \to 0}} \frac{\frac{h_1^3 + h_2^3 - h_1^3 - h_1 h_2^2 - h_1^2 h_2 - h_2^3}{h_1^2 + h_2^2}}{\sqrt{h_1^2 + h_2^2}} = \lim_{\substack{h_1 \to 0 \\ h_2 \to 0}} \frac{-h_1 h_2^2 - h_1^2 h_2}{(h_1^2 + h^2)^{\frac{3}{2}}} = 0 \Rightarrow$$

$\Rightarrow$ f is diff. at $(0,0)$ $\Rightarrow$ f is diff on $\mathbb{R}^2$

4) $f:\mathbb{R}^2 \to \mathbb{R} \quad f(x,y) = \sqrt[3]{x^3-y^3}$, is f diff on $\mathbb{R}^2$

$A = \{(x,x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{3}\cdot(x^3-y^3)^{-\frac{2}{3}}\cdot 3x^2 = \frac{x^2}{(x^3-y^3)^{\frac{2}{3}}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{-y^2}{(x^3-y^3)^{\frac{3}{2}}}$$

$$\left.\begin{array}{l} f \text{ is part. diff. on } \mathbb{R}^2\setminus A \\ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ cont. on } \mathbb{R}^2\setminus A \\ \mathbb{R}^2\setminus A \text{ is an open set} \end{array}\right\} \Rightarrow f \text{ diff on } \mathbb{R}^2\setminus A$$

now we study diff of f on A

$$\frac{\partial f}{\partial x}(a,a) = \lim_{x\to a}\frac{f(x,a)-f(a,a)}{x-a} = \lim_{x\to a}\frac{\sqrt[3]{x^3-a^3}}{x-a} = \lim_{x\to a}\frac{\sqrt[3]{x^2+ax+a^2}}{\sqrt[3]{(x-a)^2}} = \frac{\sqrt{a^2+a\cdot a+a^2}}{+0} = \infty \quad \forall a \in \mathbb{R}^*$$

$\Rightarrow$ if $a \in \mathbb{R}^* \Rightarrow f$ is not diff at (a,a)

we study diff. at $(0,0)$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}(0,0) = \lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0}\frac{\sqrt[3]{x^3}}{x} = 1 \\ \frac{\partial f}{\partial y}(0,0) = \lim\limits_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = \lim\limits_{y\to 0}\frac{\sqrt[3]{-y^3}}{y} = -1 \end{array}\right\} \Rightarrow \text{they exist} \Rightarrow$$

$$\Rightarrow \text{calculate} \lim_{\substack{h_1\to 0\\ h_2\to}}\frac{f(0+h_1,0+h_2)-f(0,0)-h_1\frac{\partial f}{\partial x}(0,0)-h_2\frac{\partial f}{\partial y}(0,0)}{\sqrt{h_1^2+h_2^2}} =$$

$$= \lim_{\substack{h_1\to 0\\ h_2\to 0}}\frac{\sqrt[3]{h_1^3-h_2^3}-h_1+h_2}{\sqrt{h_1^2+h_2^2}} \quad (1)$$

we conside $g(x,y) = \dfrac{\sqrt[3]{x^3-y^3}-x+y}{\sqrt{x^2+y^2}}$

$$\lim_{k\to\infty}\left(\frac{1}{k},\frac{1}{k}\right) = (0,0) \qquad \lim_{k\to\infty} f\left(\frac{1}{k},\frac{1}{k}\right) = 0$$

$\lim_{k\to\infty}\left(\frac{1}{k},-\frac{1}{k}\right)=(0,0)$ $\quad\lim_{k\to\infty} f\left(\frac{1}{k},\frac{-1}{k}\right)\neq 0$ thus (1) does not exist $\Rightarrow$

$\Rightarrow$ f is not diff. at $(0,0)$ $\Rightarrow$ f is not diff. on A

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sem 6: 1) $f=f(u,v):\mathbb{R}^2\to\mathbb{R}$, $F:\mathbb{R}^3\to\mathbb{R}$, $F(x,y,z)=f(x^2-y+2yz,\ z^3e^{xy})$

express the part. derivative of F in terms of first order part. derivatives of f

$F=f\circ g$ $\quad g(x,y,z)=(\underbrace{x^2-y+2yz}_{u(x,y,z)},\ \underbrace{z^3e^{xy}}_{v(x,y,z)})$

$$\frac{\partial F}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial F}{\partial x}(x,y,z)=\frac{\partial f}{\partial u}(g(x,y,z))\cdot\frac{\partial u}{\partial x}(x,y,z)+\frac{\partial f}{\partial v}(g(x,y,z))\cdot\frac{\partial v}{\partial x}(x,y,z)=$$
$$=2x\cdot\frac{\partial f}{\partial u}(x^2-y+2yz,\ z^3e^{xy})+yz^3e^{xy}\frac{\partial f}{\partial v}(x^2-y+2yz,\ z^3e^{xy})$$

$$\frac{\partial F}{\partial y}(x,y,z)=\frac{\partial f}{\partial u}(g(x,y,z))\cdot\frac{\partial u}{\partial y}(x,y,z)+\frac{\partial f}{\partial v}(g(x,y,z))\cdot\frac{\partial v}{\partial y}(x,y,z)=$$
$$=(-1+2z)\frac{\partial f}{\partial u}(x^2-y+2yz,\ z^3e^{xy})+xz^3e^{xy}\frac{\partial f}{\partial v}(x^2-y+2yz,\ z^3e^{xy})$$

$$\frac{\partial F}{\partial z}(x,y,z)=\frac{\partial f}{\partial u}(g(x,y,z))\cdot\frac{\partial u}{\partial z}(x,y,z)+\frac{\partial f}{\partial v}(g(x,y,z))\cdot\frac{\partial v}{\partial z}(x,y,z)=$$
$$=2y\frac{\partial f}{\partial u}(x^2-y+2yz,\ z^3e^{xy})+3z^2e^{xy}\frac{\partial f}{\partial v}(x^2-y+2yz,\ z^3e^{xy})$$

2) Let $f:\mathbb{R}^3\to\mathbb{R}^2$ be diff on $\mathbb{R}^3$ and let $F:\mathbb{R}^2\to\mathbb{R}^2$

$F(x,y)=f(\cos x+\sin y,\ \sin x+\cos y,\ e^{x-y})$

a) prove: if f is con. diff. on $\mathbb{R}^3$ then F is con. diff on $\mathbb{R}^3$

b) determine $dF\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ if $J(f)(1,1,1)=\begin{pmatrix}1&3&4\\2&-1&3\end{pmatrix}$

solution: $F = f \circ g$ where $g: \mathbb{R}^2 \to \mathbb{R}^3$ $g(x,y) = (\underbrace{\cos x + \sin y}_{u(x,y)}, \underbrace{\sin x + \cos y}_{v(x,y)}, \underbrace{e^{x-y}}_{w(x,y)}) = (\ldots)$

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial f}{\partial u}(g(x,y)) \cdot \frac{\partial u}{\partial x}(x,y) + \frac{\partial f}{\partial v}(g(x,y)) \cdot \frac{\partial v}{\partial x}(x,y) + \frac{\partial f}{\partial w}(g(x,y)) \cdot \frac{\partial w}{\partial x}(x,y) =$$

$$\left.\begin{array}{l} = -\sin x \frac{\partial f}{\partial u}(\ldots) + \cos x \frac{\partial f}{\partial v}(\ldots) + e^{x-y} \frac{\partial f}{\partial w}(\ldots) \\ f \text{ is con. diff. on } \mathbb{R}^3 \Rightarrow \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w} \text{ are con. diff. on } \mathbb{R}^3 \end{array}\right\} \Rightarrow$$

$\Rightarrow \frac{\partial F}{\partial x}$ is con. diff on $\mathbb{R}^2$ (1)

$$\frac{\partial F}{\partial y}(x,y) = \frac{\partial f}{\partial u}(g(x,y)) \cdot \frac{\partial u}{\partial y}(x,y) + \frac{\partial f}{\partial v}(g(x,y)) \cdot \frac{\partial v}{\partial y}(x,y) + \frac{\partial f}{\partial w}(g(x,y)) \cdot \frac{\partial w}{\partial y}(x,y) =$$

$$\left.\begin{array}{l} = \cos y \frac{\partial f}{\partial u}(\ldots) - \sin y \frac{\partial f}{\partial v}(\ldots) - e^{x-y} \frac{\partial f}{\partial w}(\ldots) \\ f \text{ is con. diff on } \mathbb{R}^3 \Rightarrow \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w} \text{ are con. diff. on } \mathbb{R}^3 \end{array}\right\} \Rightarrow$$

$\Rightarrow \frac{\partial F}{\partial y}$ is con. diff on $\mathbb{R}^2$ (2)

(1),(2) $\Rightarrow$ F is con. diff on $\mathbb{R}^2$

b) $dF(\frac{\pi}{2}, \frac{\pi}{2}) \in L(\mathbb{R}^2, \mathbb{R}^2)$ $\quad [dF(\frac{\pi}{2}, \frac{\pi}{2})] = J(F(\frac{\pi}{2}, \frac{\pi}{2})) =$

$= J(f)(g(\frac{\pi}{2}, \frac{\pi}{2})) \cdot J(g)(\frac{\pi}{2}, \frac{\pi}{2}) = J(f)(1, 1, 1) \cdot J(g)(\frac{\pi}{2}, \frac{\pi}{2}) =$

$$J(g)(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial w}{\partial x}(x,y) & \frac{\partial w}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} -\sin x & \cos y \\ \cos x & -\sin y \\ e^{x-y} & -e^{x-y} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix}$$

$$\Rightarrow dF(\tfrac{\pi}{2}, \tfrac{\pi}{2})(h_1, h_2) = \begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 3h_1 - 7h_2 \\ h_1 - 2h_2 \end{pmatrix}$$

$dF(\frac{\pi}{2}, \frac{\pi}{2})(h_1, h_2) = (3h_1 - 7h_2, h_1 - 2h_2)$

3) with the aid of polar coordinates determine all diff. functions

$f:(0,\infty)\times(0,\infty)\to\mathbb{R}$, satisfying $x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=\sqrt{x^2+y^2}$ $\forall (x,y)\in(0,\infty)\times(0,\infty)$ (1)

sol: assume (1) is true

consider the function F defined by: $F(\rho,\theta)=f(\rho\cos\theta,\rho\sin\theta)$

$F:(0,\infty)\times(0,\frac{\pi}{2})\to\mathbb{R}$, $F=f\circ g$ where $g(\rho,\theta)=(\underbrace{\rho\cos\theta}_{x(\rho,\theta)},\underbrace{\rho\sin\theta}_{y(\rho,\theta)})=(\dots)$

$$\frac{\partial F}{\partial\rho}(\rho,\theta)=\frac{\partial f}{\partial x}(g(\rho,\theta))\cdot\frac{\partial x}{\partial\rho}(\rho,\theta)+\frac{\partial f}{\partial y}(g(\rho,\theta))\cdot\frac{\partial y}{\partial\rho}(\rho,\theta)=$$

$$=\cos\theta\frac{\partial f}{\partial x}(\dots)+\sin\theta\frac{\partial f}{\partial y}(\dots)\ \Big/\cdot\rho$$

$$\Rightarrow \rho\frac{\partial F}{\partial\rho}(\rho,\theta)=\rho\cos\theta\frac{\partial f}{\partial x}(\rho\cos\theta,\rho\sin\theta)+\rho\sin\theta\frac{\partial f}{\partial y}(\rho\cos\theta,\rho\sin\theta)\overset{(1)}{=}$$

$$=\sqrt{(\rho\cos\theta)^2+(\rho\sin\theta)^2}=\sqrt{\rho^2}=\rho \quad\Rightarrow\quad \rho\frac{\partial F}{\partial\rho}(\rho,\theta)=\rho \Rightarrow \frac{\partial F}{\partial\rho}(\rho,\theta)=1 \Rightarrow$$

$$\Rightarrow F(\rho,\theta)=\int 1\,d\rho=\rho+\varphi(\theta)$$

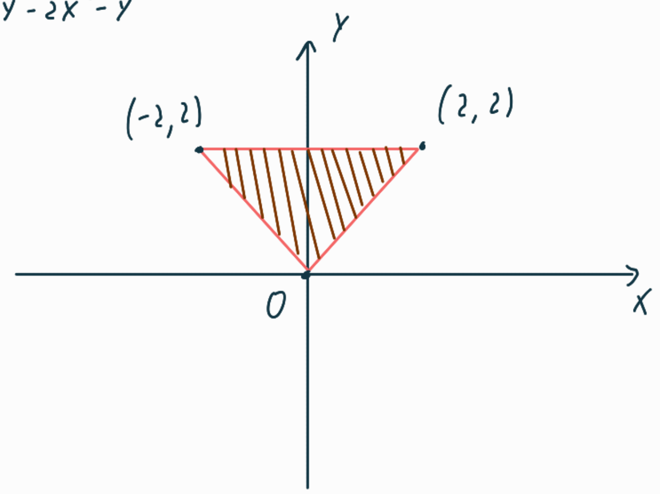
$$F=f\circ g\ /\circ g^{-1}\ \Rightarrow f=F\circ g^{-1}$$

$$g:\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}\quad\Rightarrow g^{-1}:\begin{cases}\rho=\sqrt{y^2+x^2}\\ \theta=\operatorname{arctg}\frac{y}{x}\end{cases}$$

$f(x,y)=\sqrt{x^2+y^2}+h(\operatorname{arctg}\frac{y}{x})$ where $h:(0,\frac{\pi}{2})\to\mathbb{R}$ is an arbitrary diff. function

4) $A=\{(x,y)\in\mathbb{R}^2 \mid |x|\le y\le 2\}$ and $f:A\to\mathbb{R}$

$f(x,y)=4x+3y-2x^2-y^2$



$m=\min(f(A))=?$ $\quad M=\max(f(A))=?$

$m_1 = \min(bd(A))$ $\quad M_1 = \max(bd(A))$ $\quad$ where $C = \{(x,y) \in \text{int}A \mid \nabla f(x,y) = (0,0)\}$

$m_2 = \min(C)$ $\quad M_2 = \max(C)$

$B_1 = \{(x,x) \in \mathbb{R}^2 \mid x \in [0,2]\}$

$B_2 = \{(x,-x) \in \mathbb{R}^2 \mid x \in [-2,0)\}$ $\qquad bdA = B_1 \cup B_2 \cup B_3$

$B_3 = \{(x,2) \in \mathbb{R}^2 \mid x \in (-2,2)\}$

for B_1: $f(x,x) = 4x + 3x - 2x^2 - x^2 = -3x^2 + 7x$

$g(x) = -3x^2 + 7x$ $\qquad g'(x) = -6x + 7$ $\qquad g'(x) = 0 \Rightarrow x = \frac{7}{6}$

x	0		$\frac{7}{6}$		2
$g'(x)$		$+ + + + + +$	0	$- - - - - -$	
$g(x)$	0	$\nearrow$		$\searrow$	2

$g\left(\frac{7}{6}\right) = \frac{49}{12}$

$g(0) = 0$

$g(2) = 2$

for B_2: $f(x,-x) = 4x - 3x - 2x^2 - x^2 = -3x^2 + x$

$h(x) = -3x^2 + x$ $\qquad h'(x) = -6x + 1$ $\qquad h'(x) = 0 \Rightarrow x = \frac{1}{6}$

x	-2		0	$\frac{1}{6}$
$h'(x)$		$+ + + + + + + + + +$		0
$h(x)$	-14	$\nearrow$	0	

for B_3 $f(x,2) = 4x + 6 - 2x^2 - 4 = -2x^2 + 4x + 2$

$i(x) = -2x^2 + 4x + 2$ $\qquad i'(x) = -4x + 4$ $\qquad i'(x) = 0 \Rightarrow x = 1$

x	-2		0	1		2
$i'(x)$		$+ + + + + +$		0	$- -$	
$i(x)$	-14	$\nearrow$		4	$\searrow$	-2

$\Rightarrow m_1 = -14$

$M_1 = \frac{49}{12}$

$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = (4-4x, 3-2y)$

$\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} 4-4x=0 \\ 3-2y=0 \end{cases} \Rightarrow x=1 \quad y=\frac{3}{2}$

$f(1, \frac{3}{2}) = \frac{17}{4}$

$m_2 = M_2 = \frac{17}{4}$

$\Rightarrow m = -14$

$M = \frac{17}{4}$

sem 7 2) $f(x,y,z) = x+y+z$, $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1$ and $2x+y+2z=1\}$

$m = \min(f(C)) = ?$ $M = \max(f(C))$

$\left.\begin{array}{l} f \text{ con.} \\ C \text{ compact} \end{array}\right\} \overset{\text{W.t.}}{\Rightarrow}$ f is bounded and reaches its bounds

$\Rightarrow \exists$ 2 points $(a,b,c) \in C$ and $(a',b',c') \in C$ s.t.

$\left.\begin{array}{l} f(a,b,c) = \min(f(C)) \\ f(a',b',c') = \max(f(C)) \end{array}\right\} \Rightarrow$ by the Lagrange multipliers rule

$\exists \lambda_0, \mu_0, \lambda_0', \mu_0' \in \mathbb{R}$ s.t. (a,b,c,λ_0,μ_0) and $(a',b',c',\lambda_0',\mu_0')$ are crit. points of the Lagrange function

let: $F_1(x,y,z) = x^2+y^2+z^2-1$

$F_2(x,y,z) = 2x+y+2z-1$

$L(x,y,z,\lambda,\mu) = f(x,y,z) + \lambda F_1(x,y,z) + \mu F_2(x,y,z) =$

$= x+y+z+\lambda(x^2+y^2+z^2-1) + \mu(2x+y+2z-1)$

the crit. points of L are the solutions of the system:

$$\begin{cases} \frac{\partial L}{\partial x}(x,y,z,\lambda,\mu) = 0 \iff 1 + 2x\lambda + 2\mu = 0 & (1) \\ \frac{\partial L}{\partial y}(x,y,z,\lambda,\mu) = 0 \iff 1 + 2y\lambda + \mu = 0 & (2) \\ \frac{\partial L}{\partial z}(x,y,z,\lambda,\mu) = 0 \iff 1 + 2z\lambda + 2\mu = 0 & (3) \\ \frac{\partial L}{\partial \lambda}(x,y,z,\lambda,\mu) = 0 \iff x^2+y^2+z^2 - 1 = 0 & (4) \\ \frac{\partial L}{\partial \mu}(x,y,z,\lambda,\mu) = 0 \iff 2x + y + 2z = 0 & (5) \end{cases}$$

$(1)-(3) \Rightarrow 2\lambda(x-z) = 0 \Rightarrow \lambda = 0 \Rightarrow \mu = -1$ and $\mu = \frac{-1}{2}$ contradiction

$\Rightarrow x - z = 0 \Rightarrow x = z$

$\Rightarrow 2x + y + 2x = 1 \Rightarrow y = -4x+1 \Rightarrow$

$\Rightarrow x^2 + 16x^2 - 8x + 1 + x^2 = 1 \Rightarrow$

$\Rightarrow 18x^2 - 8x = 0 \Rightarrow$ $x = 0 \Rightarrow y = 1,\ z = 0$
OR
$x = \frac{4}{9} \Rightarrow y = \frac{-7}{9},\ z = \frac{4}{9}$

$$\left.\begin{aligned} f(0,1,0) &= 1 \\ f\left(\tfrac{4}{9}, \tfrac{-7}{9}, \tfrac{4}{9}\right) &= \tfrac{1}{9} \end{aligned}\right\} \Rightarrow \begin{aligned} (0,1,0) &= \max(F(C)) \\ \left(\tfrac{4}{9}, \tfrac{-7}{9}, \tfrac{4}{9}\right) &= \min(F(C)) \end{aligned}$$

3) consider $f(x,y,z) = x^2+y^2+z^2 - 2x + 2\sqrt{2}\cdot y + 2z$ and $B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$

determine $\min f(B)$, $\max f(B)$

solution:

$$\left.\begin{aligned} &B \text{ is compact} \\ &f \text{ con. on } B \end{aligned}\right\} \overset{\text{Wt.}}{\Longrightarrow} f \text{ is bounded and reaches its bounds} \Rightarrow$$

$\exists\ (a,b,c) \in B$ and $(a',b',c') \in B$ s.t. $f(a,b,c) = \min(f(B))$
$f(a',b',c') = \max(f(B))$

if $(a,b,c) \in \operatorname{int} B$ or $(a',b',c') \in \operatorname{int} \overset{\text{Fermat}}{\Longrightarrow}$

$\Rightarrow \nabla f(a,b,c) = (0,0,0)$ or $\nabla f(a',b',c') = (0,0,0)$

$\nabla f(x,y,z) = (0,0,0) \Rightarrow$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 0 & \Rightarrow 2x - 2 = 0 \\ \frac{\partial f}{\partial y}(x,y,z) = 0 & \Rightarrow 2y + 2\sqrt{2} = 0 \\ \frac{\partial f}{\partial z}(x,y,z) = 0 & \Rightarrow 2z + 2 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -\sqrt{2} \\ z = -1 \end{cases} \Rightarrow$$

$\Rightarrow$ the only crit. point of f is $(1, -\sqrt{2}, -1) \notin B \Rightarrow \notin \text{int} \Rightarrow$ can't use Fermat

$\Rightarrow (a,b,c)$ and (a',b',c') are constrained extrema $\Rightarrow$ Lagrangemultipliers rule

$\Rightarrow \exists\, \lambda_0, \lambda_0' \in \mathbb{R}$ s.t. (a,b,c,λ_0) and (a',b',c',λ_0') are crit. points of L

let $F(x,y,z) = x^2+y^2+z^2-1$

$L(x,y,z,\lambda) = f(x,y,z) + \lambda F(x,y,z) = x^2+y^2+z^2-2x+2\sqrt{2}y+2z+\lambda(x^2+y^2+z^2-1)$

the crit. points of L are the solutions of $\nabla L(x,y,z,\lambda) = (0,0,0,0)$

$$\begin{cases} \frac{\partial L}{\partial x}(x,y,z,\lambda) = 0 \Leftrightarrow 2x - 2 + 2x\lambda = 0 & \Rightarrow x = \frac{1}{\lambda+1} \\ \frac{\partial L}{\partial y}(x,y,z,\lambda) = 0 \Leftrightarrow 2y + 2\sqrt{2} + 2y\lambda = 0 & \Rightarrow y = \frac{-\sqrt{2}}{\lambda+1} \\ \frac{\partial L}{\partial z}(x,y,z,\lambda) = 0 \Leftrightarrow 2z + 2 + 2z\lambda = 0 & \Rightarrow z = \frac{-1}{\lambda+1} \\ \frac{\partial L}{\partial \lambda}(x,y,z,\lambda) = 0 \Leftrightarrow x^2+y^2+z^2-1 = 0 & \Rightarrow \frac{4}{(\lambda+1)^2} = 1 \Rightarrow \end{cases}$$

$\Rightarrow \lambda = 1 \Rightarrow x = \frac{1}{2}\quad y = \frac{-\sqrt{2}}{2}\quad z = \frac{-1}{2}$

or

$\lambda = -3 \Rightarrow x = -\frac{1}{2}\quad y = \frac{\sqrt{2}}{2}\quad z = \frac{1}{2}$

$f\left(\frac{1}{2}, \frac{-\sqrt{2}}{2}, -\frac{1}{2}\right) = -3$

$f\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right) = 5$

$\Rightarrow \min f(B) = -3$

$\max f(B) = 5$

sem 8: 1) a) $f(x,y,z) = 2x^2 - xy + 2xz - y + y^3 + z^2$ find crit. points of f and their natures

$\nabla f(x,y,z) = (0,0,0) \Rightarrow$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 0 \iff 4x - y + 2z = 0 \Rightarrow y = -2z \\ \frac{\partial f}{\partial y}(x,y,z) = 0 \iff -x - 1 + 3y^2 = 0 \Rightarrow z - 1 + 12z^2 = 0 \Rightarrow z_1 = -\frac{1}{3} \quad z_2 = \frac{1}{4} \\ \frac{\partial f}{\partial z}(x,y,z) = 0 \iff 2x + 2z = 0 \Rightarrow x = -z \end{cases}$$

the crit. points of f are $\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ and $\left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)$

$$H(f)(x,y,z) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 6y & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$H(f)\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 4 \quad \Delta_2 = 15 \quad \Delta_3 = 14 \Rightarrow$

$\Rightarrow \left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ is a local minimum of f

$$H(f)\left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 4 \quad \Delta_2 = -13 \quad \Delta_3 = -14$

$\Rightarrow \left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)$ is a saddle point of f

1) c) $f(x,y,z) = x^2 + y^2 + z^2 - 2xyz$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 0 \iff 2x - 2yz = 0 \Rightarrow x = yz \\ \frac{\partial f}{\partial y}(x,y,z) = 0 \iff 2y - 2xz = 0 \Rightarrow y = xz = y^2 z \Rightarrow y = yz^2 = 0 \Rightarrow y \cdot (1 - z^2) = 0 \Rightarrow z = \pm 1 \\ \frac{\partial f}{\partial x}(x,y,z) = 0 \iff 2z - 2xy = 0 \end{cases}$$

or

$y = 0 \Rightarrow x = 0, z = 0$

$z = 1 \Rightarrow x = y \Rightarrow 2 - 2x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow y = \pm 1$

$z = -1 \Rightarrow x = -y \Rightarrow -2 + 2x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow y = \pm 1$

$\Rightarrow$ the crit. points of f are $(1,1,1), (-1,-1,1), (1,1,-1), (-1,-1,-1),$
$(0,0,0)$

$$H(f)(x,y,z) = \begin{pmatrix} \frac{\partial f}{\partial^2 x}(x,y,z) & \frac{\partial f}{\partial y \partial x}(x,y,z) & \frac{\partial f}{\partial z \partial x}(x,y,z) \\ \frac{\partial f}{\partial x \partial y}(x,y,z) & \frac{\partial f}{\partial^2 y}(x,y,z) & \frac{\partial f}{\partial z \partial y}(x,y,z) \\ \frac{\partial f}{\partial x \partial z}(x,y,z) & \frac{\partial f}{\partial y \partial z}(x,y,z) & \frac{\partial f}{\partial^2 z}(x,y,z) \end{pmatrix} = \begin{pmatrix} 2 & -2z & -2y \\ -2z & 2 & -2x \\ -2y & -2x & 2 \end{pmatrix}$$

$$H(f)(0,0,0) = \begin{array}{c} \begin{matrix} \Delta_1 & \Delta_2 & \Delta_3 \end{matrix} \\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array} \qquad \Delta_1 = 2 \qquad \Delta_2 = 4 \qquad \Delta_3 = 8 \Rightarrow \text{local minimizer}$$

$$H(f)(1,1,1) = \begin{pmatrix} \overset{h_1h_1}{2} & \overset{h_1h_2}{-2} & \overset{h_1h_3}{-2} \\ \overset{h_2h_1}{-2} & \overset{h_2h_2}{2} & \overset{h_2h_3}{-2} \\ \overset{h_3h_1}{-2} & \overset{h_3h_2}{-2} & \overset{h_3h_3}{2} \end{pmatrix} \qquad \Delta_1 = 2 \qquad \underline{\Delta_2 = 0} \Rightarrow \text{we can't apply Sylvester}$$

$$d^2 f(1,1,1)(h_1,h_2,h_3) = 2h_1^2 + 2h_2^2 + 2h_3^2 - 4h_1h_2 - 4h_1h_3 - 4h_2h_3 = E$$

$$\left.\begin{array}{l} \text{if } h_1 = 1 \text{ and } h_2 = h_3 = 0 \Rightarrow E = 2 > 0 \\ \text{if } h_1 = h_2 = h_3 \Rightarrow \quad E = -6 < 0 \end{array}\right\} \Rightarrow d^2 f(1,1,1) \text{ is an indefinite}$$

quadratic form $\Rightarrow$

$\Rightarrow (1,1,1)$ is saddle point

same thing for the other points

2) a) $$\int_1^6 \int_2^3 \frac{1}{(x+y)^2}\, dx\, dy = \int_{x=1}^{x=6} \left(\int_{y=2}^{y=3} \frac{1}{(x+y)^2}\, dy \right) dx =$$

$$= \int_{x=1}^{x=6} \left(\int_{y=2}^{y=3} \frac{u(y)'}{u(y)^2}\, dy \right) dx = \int_{x=1}^{x=6} \left(\frac{-1}{x+y} \Bigg|_{y=2}^{y=3} \right) dx =$$

$$= \int_1^6 \left(\frac{-1}{x+3} - \frac{-1}{x+2} \right) dx = \ln(x+2) \Big|_1^6 - \ln(x+3) \Big|_1^6 = \ln \frac{8 \cdot 4}{9 \cdot 3} = \ln \frac{32}{27}$$

$$\boxed{\int u(x)^r \cdot u'(x)\, dx = \frac{u(x)^{r+1}}{r+1} + c \quad \forall r \in \mathbb{R} \setminus \{-1\}}$$

b) $\displaystyle\int_0^1\int_0^1 \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}}\,dx\,dy = \int_{y=0}^{y=1}\left(\frac{1}{2}\int_{x=0}^{x=1}\frac{2x}{(1+x^2+y^2)^{\frac{3}{2}}}\,dx\right)dy =$

$$= \frac{1}{2}\int_{y=0}^{y=1}\left(\int_{x=0}^{x=1} u'(x)\cdot u(x)^{-\frac{3}{2}}\,dx\right)dy = \frac{1}{2}\int_{y=0}^{y=1}\left(\frac{1}{\sqrt{1+x^2+y^2}}\cdot(-2)\Big|_{x=0}^{x=1}\right)dy =$$

$$= -\int_{y=0}^{y=1}\left(\frac{1}{\sqrt{y^2+2}}-\frac{1}{\sqrt{y^2+1}}\right)dy = -\left(\ln\left|y+\sqrt{y^2+2}\right|\Big|_0^1 - \ln\left|y+\sqrt{y^2+1}\right|\Big|_0^1\right)=$$

$$= -\left(\ln(1+\sqrt{3}) - \ln\sqrt{2} - \ln(1+\sqrt{2}) + \ln 1\right) = -\left(\ln\frac{1+\sqrt{3}}{\sqrt{2}\cdot(1+\sqrt{2})}\right) = -\ln\frac{1+\sqrt{3}}{2+\sqrt{2}} = \ln\frac{2+\sqrt{2}}{1+\sqrt{3}}$$

c) $\displaystyle I = \int_1^2\int_1^2\int_1^2 \frac{1}{(x+y+z)^3}\,dx\,dy\,dz = \int_{x=1}^{x=2}\int_{y=1}^{y=2}\left(\frac{1}{(x+y+z)^3}\,dz\right)dx\,dy =$

$$= \int_{x=1}^{x=2}\int_{y=1}^{y=2}\left(\frac{1}{(-2)(x+y+z)^2}\Big|_1^2\right)dy\,dx = -\frac{1}{2}\int_{x=1}^{x=2}\left(\int_{y=1}^{y=2}\frac{1}{(x+y+2)^2}-\frac{1}{(x+y+1)^2}\,dy\right)dx$$

$$= -\frac{1}{2}\int_{x=1}^{x=2}\left(\frac{1}{x+y+1}\Big|_1^2 - \frac{1}{x+y+2}\Big|_1^2\right)dx = -\frac{1}{2}\int_1^2\left(\frac{1}{x+3}-\frac{1}{x+2}-\frac{1}{x+4}+\frac{1}{x+3}\right)dx =$$

$$= -\frac{1}{2}\left(2\ln(x+3) - \ln(x+4) - \ln(x+2)\right)\Big|_1^2 = \dots = \frac{1}{2}\ln\frac{128}{125}$$

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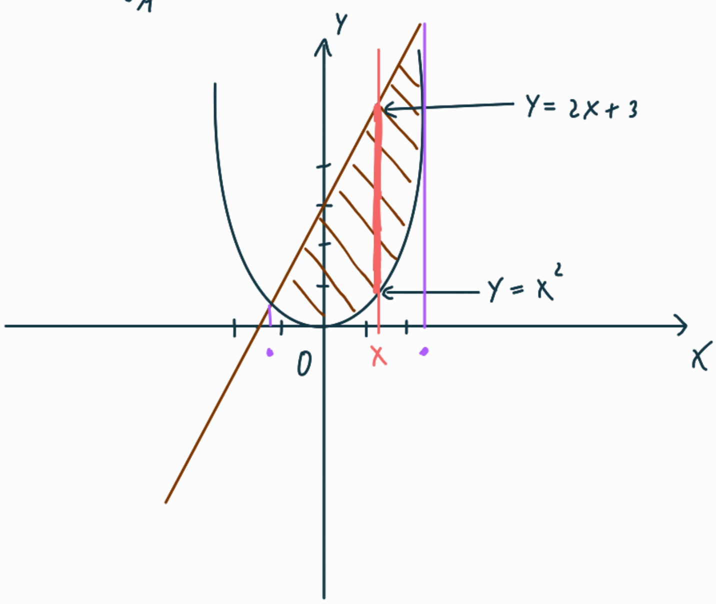
sem 9: 1) A is bounded by $y=x^2$ and $y=2x+3$

$I=\iint_A (x+2y)\,dx\,dy$



$$\begin{cases} y=x^2 \\ y=2x+3 \end{cases} \Rightarrow x^2 = 2x+3 \Rightarrow$$

$\Rightarrow x^2-2x-3=0$

$x_1=-1$

$x_2=3$

$$I = \int_{x=-1}^{x=3}\left(\int_{y=x^2}^{y=2x+3}(x+2y)\,dy\right)dx = \int_{x=-1}^{x=3}\left(xy+y^2\right)\Big|_{y=x^2}^{y=2x+3}dx =$$

$$= \int_{x=-1}^{x=3} 2x^2 + 3x + 4x^2 + 12x + 9 - x^3 - x^4 \, dx$$

$$= \int_{x=-1}^{x=3} \left(-x^4 - x^3 + 6x^2 + 15x + 9\right) dx = \dots$$

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test Mate-info 28.06

1) $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2$

$\nabla f(x,y) = (0,0)$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 & \Leftrightarrow 3x^2 + 3y^2 - 6x = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 & \Leftrightarrow 6xy - 6y = 0 \Rightarrow 6y(x-1) = 0 \end{cases} \Rightarrow y = 0 \Rightarrow x = 0 \text{ or } x = 2$$

OR

$x = 1 \Rightarrow y = \pm 1$

the crit points of f are $(0,0)$, $(2,0)$, $(1,1)$, $(1,-1)$

$$H(f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y \partial x}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 6x-6 & 6y \\ 6y & 6x-6 \end{pmatrix}$$

$\Delta_1 = 6x - 6$

$\Delta_2 = 36x^2 - 72x + 36 - 36y^2 =$
$= 36(x^2 - 2x + 1 - y^2)$

$\Rightarrow \Delta_1(0,0) = -6 \quad \Delta_2(0,0) = 36 \Rightarrow$ local max.

$\Delta_1(2,0) = 6 \quad \Delta_2(2,0) = 36 \Rightarrow$ local min.

$\Delta_1(1,1) = 0 \Rightarrow d^2 f$

$\Delta_1(1,-1) = 0 \Rightarrow d^2 f$

$$H(f)(1,1) = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix} = 12 h_1 h_2 = E$$

if $h_1 = h_2 = 1 \Rightarrow E = 12 > 0$
if $h_1 = 1 \; h_2 = -1 \Rightarrow E = -12 < 0$ $\Big\} \Rightarrow$

$\Rightarrow$ indefinite quadratic form $\Rightarrow$

$\Rightarrow (1,1)$ saddle point

$$H(f)(1,-1) = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} = -12 h_1 h_2 = E$$

indefinite quadratic form $\Rightarrow$

$\Rightarrow (1,-1)$ saddlepoint

2) $f(x,y,z) = \begin{cases} \dfrac{x^3-y^3-z^3}{x^2+y^2+z^2} & , (x,y,z) \neq (0,0,0) \\ 0 & , (x,y,z) = (0,0,0) \end{cases}$ is f diff. on $\mathbb{R}^3$?

$$\left.\begin{aligned} \frac{\partial f}{\partial x}(x,y,z) &= \frac{3x^2(x^2+y^2+z^2) - 2x(x^3-y^3-z^3)}{(x^2+y^2+z^2)^2} \\ \frac{\partial f}{\partial y}(x,y,z) &= \frac{-3y^2(x^2+y^2+z^2) - 2y(x^3-y^3-z^3)}{(x^2+y^2+z^2)^2} \\ \frac{\partial f}{\partial z}(x,y,z) &= \frac{-3z^2(x^2+y^2+z^2) - 2z(x^3-y^3-z^3)}{(x^2+y^2+z^2)^2} \end{aligned}\right\} \text{part. diff. on } \mathbb{R}^3 \setminus \{(0,0,0)\}$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}$ con. on $\mathbb{R}^3 \setminus \{0,0,0\}$

$\mathbb{R}^2 \setminus \{(0,0,0)\}$ open set

$\Rightarrow f$ is diff. on $\mathbb{R}^3 \setminus \{(0,0,0)\}$

$$\left.\begin{aligned} \frac{\partial f}{\partial x}(0,0,0) &= \lim_{x \to 0} \frac{f(x,0,0) - f(0,0,0)}{x-0} = 1 \\ \frac{\partial f}{\partial y}(0,0,0) &= \lim_{y \to 0} \frac{f(0,y,0) - f(0,0,0)}{y-0} = -1 \\ \frac{\partial f}{\partial z}(0,0,0) &= \lim_{z \to 0} \frac{f(0,0,z) - f(0,0,0)}{z-0} = -1 \end{aligned}\right\} \text{they exist}$$

$$\lim_{\substack{h_1 \to 0 \\ h_2 \to 0 \\ h_3 \to 0}} \frac{f(0+h_1, 0+h_2, 0+h_3) - f(0,0,0) - h_1 \frac{\partial f}{\partial x}(0,0,0) - h_2 \frac{\partial f}{\partial y}(0,0,0) - h_3 \frac{\partial f}{\partial z}(0,0,0)}{\sqrt{h_1^2+h_2^2+h_3^2}} =$$

$$= \lim_{\substack{h_1 \to 0 \\ h_2 \to 0 \\ h_3 \to 0}} \frac{\frac{h_1^3 - h_2^3 - h_3^3}{h_1^2+h_2^2+h_3^2} - h_1 + h_2 + h_3}{\sqrt{h_1^2+h_2^2+h_3^2}} = \lim_{\substack{h_1 \to 0 \\ h_2 \to 0 \\ h_3 \to 0}} \frac{\frac{h_1^3 - h_2^3 - h_3^2 - h_1^3 - h_1h_2^2 - h_1h_3^2 + h_2^3 + h_1^2h_2 + h_3^2h_2 + h_3^3 + h_1^2h_3 + h_2^2h_3}{h_1^2+h_2^2+h_3^2}}{\sqrt{h_1^2+h_2^2+h_3^2}}$$

$$= \lim_{\substack{h_1 \to 0 \\ h_2 \to 0 \\ h_3 \to 0}} \frac{-h_1h_2^2 - h_1h_3^2 + h_2h_1^2 + h_2h_3^2 + h_3h_1^2 + h_3h_2^2}{(h_1^2+h_2^2+h_3^2)^{\frac{3}{2}}} = \lim_{\substack{h_1 \to 0 \\ h_2 \to 0 \\ h_3 \to 0}} \frac{h_1^2(h_2+h_3) + h_2^2(h_3-h_1) + h_3^2(h_2-h_1)}{(h_1^2+h_2^2+h_3^2)^{\frac{3}{2}}}$$

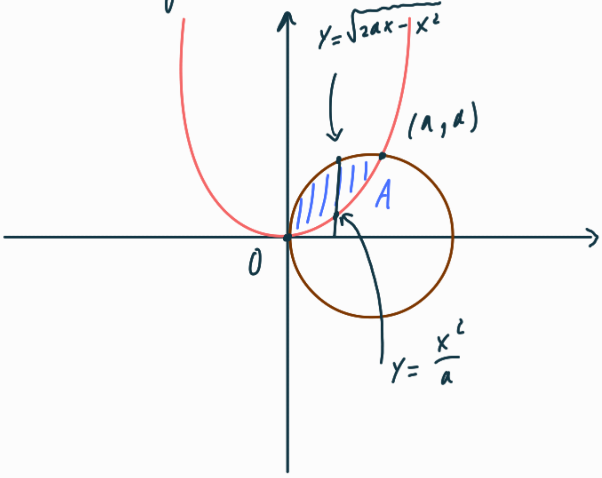
$\lim_{k\to\infty}\left(\frac{1}{k}, \frac{1}{k}, \frac{1}{k}\right) = (0,0,0)$ => we can calculate the limit by using $\frac{1}{k}, \frac{1}{k}, \frac{1}{k}$

$$\lim_{k\to\infty} \frac{\frac{1}{k^2}\left(\frac{1}{k}+\frac{1}{k}\right) + \frac{1}{k^2}\left(\frac{1}{k}-\frac{1}{k}\right) + \frac{1}{k}\left(\frac{1}{k}-\frac{1}{k}\right)}{\left(\frac{1}{k^2}+\frac{1}{k^2}+\frac{1}{k^2}\right)^{\frac{3}{2}}} = \lim_{k\to\infty} \frac{\frac{2}{k^3}}{\frac{3}{k^3}} \neq 0 \quad \Rightarrow$$

$\Rightarrow f$ is not diff. at $(0,0,0)$

3) $C: x^2+y^2 = 2ax \quad , \quad a>0$

$P: x^2 = ay$



$$\begin{cases} x^2+y^2 = 2ax \\ x^2 = ay \quad \Rightarrow y = \frac{x^2}{a} \end{cases}$$

$\Rightarrow x^2 + \frac{x^4}{a^2} = 2ax \quad / : x \quad \overset{x\neq 0}{\Rightarrow}$

$\Rightarrow x + \frac{x^3}{a^2} = 2a \Rightarrow$

$\Rightarrow \frac{x}{a} + \frac{x^3}{a^3} = 2$

$\Rightarrow x = a$

if $x=0 \Rightarrow y = \frac{0}{a} = 0$

$$\iint_A dx\,dy = \int_{x=0}^{x=a} \int_{y=\frac{x^2}{a}}^{y=\sqrt{2ax-x^2}} dy\,dx = \int_{x=0}^{x=a} \left(\sqrt{2ax-x^2} - \frac{x^2}{a}\right) dx =$$

$$= \int_0^a \sqrt{-a^2+2ax-x^2+a^2}\,dx - \frac{x^3}{3a}\Big|_0^a = \int_0^a \sqrt{a^2-(x-a)^2}\,dx - \frac{a^2}{3} =$$

$$= \int_{t=-a}^{t=0} \sqrt{a^2-t^2}\,dt - \frac{a^2}{3} = \int_{u=-\frac{\pi}{2}}^{u=0} \sqrt{a^2-a^2\sin^2 u}\; a\cos u\,du - \frac{a^2}{3} = \int_{-\frac{\pi}{2}}^{0} |a\cos u|\, a\cos u\,du - \frac{a^2}{3} =$$

$t = x-a$

$dt = dx$

$t = a\sin u \Rightarrow u = \arcsin\frac{t}{a}$

$dt = a\cos u\,du$

$$= I = \int_{-\frac{\pi}{2}}^{0} a^2 \cos^2 u \, du - \frac{a^2}{3} = a^2 \int_{-\frac{\pi}{2}}^{0} (\sin u)' \cos u \, du - \frac{a^2}{3} = a^2 \left(\sin u \cos u \Big|_{-\frac{\pi}{2}}^{0} - \int_{-\frac{\pi}{2}}^{0} -\sin^2 u \, du \right) - \frac{a^2}{3} =$$

$$= a^2 \left(\int_{-\frac{\pi}{2}}^{0} 1 - \cos^2 u \, du \right) - \frac{a^2}{3} \Rightarrow I = a^2 \left(\int_{-\frac{\pi}{2}}^{0} 1 \, du - I \right) - \frac{a^2}{3}$$

$$\Rightarrow I = a^2 \times \Big|_{-\frac{\pi}{2}}^{0} - a^2 I - \frac{a^2}{3} \Rightarrow I + a^2 I = a^2 \frac{\pi}{2} - \frac{a^2}{3} \Rightarrow I = \frac{a^2 \frac{\pi}{2} - \frac{a^2}{3}}{a^2 + 1}$$

2) $f(x,y,z) = \begin{cases} \dfrac{x^3 - y^3 - z^3}{x^2+y^2+z^2} & , (x,y,z) \neq (0,0,0) \\ 0 & , (x,y,z) = (0,0,0) \end{cases}$ is f con. at $(0,0,0)$

f is con. at $(0,0,0)$ if $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z) = f(0,0,0)$ $\Rightarrow$

$$\Rightarrow \lim_{(x,y,z)\to(0,0,0)} \left| \frac{x^3 - y^3 - z^3}{x^2+y^2+z^2} - f(0,0,0) \right| = 0 \Rightarrow \lim_{(x,y,z)\to(0,0,0)} \left| \frac{x^3 - y^3 - z^3}{x^2+y^2+z^2} \right| = 0$$

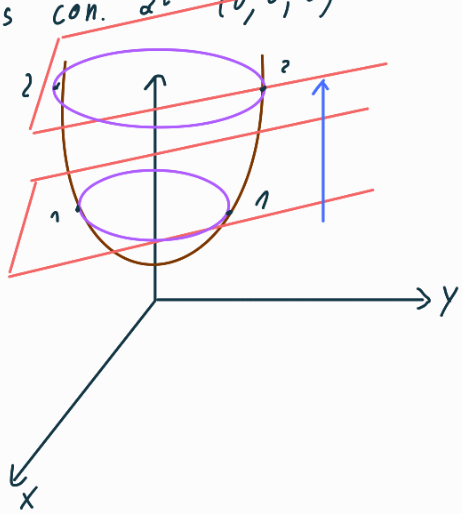
$$\boxed{0 \leq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 \cdot x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}}}$$

$$0 \leq \left| \frac{x^3 - y^3 - z^3}{x^2+y^2+z^2} \right| \leq \left| \frac{x^3 + y^3 + z^3}{x^2+y^2+z^2} \right| \leq \frac{|x^3| + |y^3| + |z^3|}{x^2+y^2+z^2} =$$

$$= \underbrace{\frac{x^2}{x^2+y^2+z^2}}_{\leq 1} |x| + \underbrace{\frac{y^2}{x^2+y^2+z^2}}_{\leq 1} |y| + \underbrace{\frac{z^2}{x^2+y^2+z^2}}_{\leq 1} |z| \leq |x| + |y| + |z| = 0 \Rightarrow$$

$$\Rightarrow \lim_{(x,y,z)\to(0,0,0)} \left| \frac{x^3 - y^3 - z^3}{x^2+y^2+z^2} \right| = 0 \Rightarrow f \text{ is con. at } (0,0,0)$$

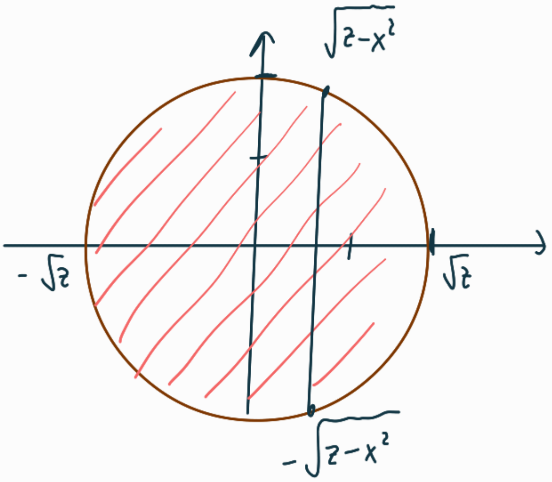
4) $P: z = x^2 + y^2$, $\pi_1: z = 1$, $\pi_2: z_2 = 0$



$\pi_1 \cap P: x^2 + y^2 = 1 \Rightarrow x = \sqrt{1 - y^2}$

$\pi_2 \cap P: x^2 + y^2 = 4 \Rightarrow x = \sqrt{4 - y^2}$

$$\iiint_V dx\,dy\,dz = \int_{z=1}^{z=4}\int_{x=-\sqrt{z}}^{x=\sqrt{z}}\int_{y=-\sqrt{z-x^2}}^{y=\sqrt{z-x^2}} dy\,dx\,dz = \frac{15\pi}{2}$$



$$\int_\gamma \|\gamma'(t)\|\,dt = \int_0^{2\pi} \|\gamma'(t)\|\,dt = l$$

5) $\gamma: [0, 2\pi] \longrightarrow \mathbb{R}^3$

$\gamma(t) = (\cos^3 t,\ \sin^3 t,\ \cos 2t)$

$\gamma'(t) = (-3\sin t\cos^2 t,\ 3\cos t\sin^2 t,\ -2\sin 2t) = (-3\sin t\cos^2 t,\ 3\cos t\sin^2 t,\ -4\sin t\cos t)$

$\|\gamma'(t)\|^2 = 9\sin^2 t\cos^4 t + 9\cos^2 t\sin^4 t + 16\sin^2 t\cos^2 t = \sin^2\cos^2 t\,(9\cos^2 t + 9\sin^2 t + 16) =$

$= 25\cos^2 t\sin^2 t \Rightarrow \|\gamma'(t)\| = |5\sin t\cos t|$

$l = \int_0^{2\pi} |5\cos t\sin t|\,dt = 5\cdot 4\int_0^{\frac{\pi}{2}} \sin t\cos t\,dt = 20\int_0^{\frac{\pi}{2}} \sin t\cdot(\sin t)'\,dt =$

$= 20\left(\frac{\sin^2 t}{2}\Big|_0^{\frac{\pi}{2}}\right) = 20\cdot\frac{1}{2} = 10$

4) again $P: z = x^2 + y^2$, $\pi_1: z = 1$, $\pi_2: z = 4$

$$\begin{cases} x = \rho\cos\theta & \theta\in[0, 2\pi] \\ y = \rho\sin\theta & \rho\in[0, \sqrt{z}] \\ z = z & z\in[1, 4] \end{cases}$$

$$\iiint_V dx\,dy\,dz = \iiint_V \rho\,d\rho\,d\theta\,dz = \int_{z=1}^{z=4}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{z}} \rho\,d\rho\,d\theta\,dz =$$

$$= \int_{z=1}^{z=4}\int_{\theta=0}^{\theta=2\pi}\left(\frac{\rho^2}{2}\Big|_{\rho=0}^{\rho=\sqrt{z}}\right) d\theta\,dz = \int_{z=1}^{z=4}\int_{\theta=0}^{\theta=2\pi} \frac{z}{2}\,d\theta\,dz = \int_{z=1}^{z=4} \frac{z}{2}\left(\theta\Big|_{\theta=0}^{\theta=2\pi}\right) =$$

$$= \int_1^4 \frac{z}{2}\,2\pi = \pi\left(\frac{z^2}{2}\Big|_1^4\right) = \frac{16-1}{\pi} = \frac{15}{\pi}$$

other test

4) $\gamma : [0, \frac{\pi}{4}] \to \mathbb{R}^3$

$\gamma(t) = (\cos t, \sin t, 2t)$

$\vec{F}(x,y,z) = yz\vec{i} + xz\vec{j} + (x+y)\vec{k} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

$\int_\gamma \vec{F} \cdot d\vec{r} = \int_0^{\frac{\pi}{4}} (2t\sin t)\cdot(\cos t)' dt + \int_0^{\frac{\pi}{4}} (2t\cos t)(\sin t)' dt + \int_0^{\frac{\pi}{4}} (\sin t + \cos t)(2t)' dt =$

$= \int_0^{\frac{\pi}{4}} -2t\sin^2 t + 2t\cos^2 t + 2\sin t + 2\cos t \, dt = 2\int_0^{\frac{\pi}{4}} (\cos^2 t - \sin^2 t)t + \sin t + \cos t \, dt =$

$= 2\int_0^{\frac{\pi}{4}} t\cos 2t + \sin t + \cos t \, dt = \int_0^{\frac{\pi}{4}} t\cdot(\sin 2t)' dt - 2\cos t \Big|_0^{\frac{\pi}{4}} + 2\sin t \Big|_0^{\frac{\pi}{4}} =$

$= t\sin 2t \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin 2t \, dt - (\sqrt{2} - 2) + \sqrt{2} - 0 =$

$= \frac{\pi}{4} - \frac{1}{2}\int 2\cdot \sin 2t \, dt + 2 = \frac{\pi}{4} + 2 - \frac{1}{2}\left(-\cos 2t \Big|_0^{\frac{\pi}{4}}\right) = \frac{\pi}{4} + 2 - \frac{1}{2} = \frac{\pi + 8 - 2}{4} = \frac{\pi + 6}{4}$

$$\boxed{\begin{gathered} u(x)^r = u'(x)\cdot \frac{u(x)^{r+1}}{r+1} \\ \int \cos(u(x))\cdot u'(x)\, dx = \sin(u(x)) \end{gathered}}$$

pentru integraliști :>

$\vec{F}$ conservative $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \end{cases} \Leftrightarrow \begin{cases} z = z \\ x \neq 1 \end{cases} \Rightarrow \text{not conservative}$$

$$\boxed{\text{prin party: } \int g(x)' h(x)\, dx = g(x)h(x) - \int g(x) h'(x)\, dx}$$

pentru levi integraliști

nature : $\nabla f(x_1, x_2, \dots x_n) = 0_n$ crit points are $(a_1, a_2, \dots a_n) = a$

$\Rightarrow$ if $H(f)(a_1, a_2, \dots a_n)$

$$\begin{array}{l} \\ \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{array} \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a)\, h_1h_1 & + \frac{\partial^2 f}{\partial x_2 \partial x_1}(a)\, h_1h_2 & \dots + & \frac{\partial^2 f}{\partial x_n \partial x_1}(a)\, h_1h_n \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(a)\, h_2h_1 & + \frac{\partial^2 f}{\partial x_2^2}(a)\, h_2h_2 & \dots + & \frac{\partial^2 f}{\partial x_n \partial x_1}(a)\, h_2h_n \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(a)\, h_nh_1 & + \frac{\partial^2 f}{\partial x_2 \partial x_n}(a)\, h_nh_2 & \dots + & \frac{\partial^2 f}{\partial x_n^2}(a)\, h_nh_n \end{pmatrix}$$

for $d^2 f$

- if $\forall i \ \Delta_i > 0 \Rightarrow$ local min
- if $\forall i \ (-1)^i \Delta_i > 0 \Rightarrow$ local max

! $\Delta_i \neq 0$ otherwise $\Rightarrow d^2 f$

- otherwise saddle point

$h = (h_1, h_2, \dots h_n)$, $d^2 f(a)(h) = \phi(h)$, $d^2 f(a) \in L(\mathbb{R}^n, \mathbb{R})$

- ϕ is said to be:
  - positive definite if $\phi(h) > 0$, $\forall h \in \mathbb{R}^n$
  - positive semidefinite if $\phi(h) \geq 0$, $\forall h \in \mathbb{R}^n$
  - negative definite if $\phi(h) < 0$, $\forall h \in \mathbb{R}^n$
  - negative semidefinite if $\phi(h) \leq 0$, $\forall h \in \mathbb{R}^n$

  $\Big\}\ h \neq 0_n$
  - indefinite if $\exists h, h' \in \mathbb{R}^n$ s.t. $\phi(h) < 0 < \phi(h')$

- p. def. : $h_1^2 + h_2^2 > 0 \ \forall h \in \mathbb{R}^2 \setminus \{(0,0)\}$
- n. def : $-h_1^2 - h_2^2 < 0 \ \forall h \in \mathbb{R}^2 \setminus \{(0,0)\}$
- p. sdef. : $h_1^2 \geq 0 \ \forall h \in \mathbb{R}^2 \setminus \{(0,0)\}$! $\phi(0,1) = 0$
- n. sdef. : $-h_1^2 \leq 0 \ \forall h \in \mathbb{R}^2 \setminus \{(0,0)\}$
- indef : $h_1^2 - h_2^2 \ \forall h \in \mathbb{R}^2$

p. def $\Rightarrow$ max	indef $\Rightarrow$ saddle
n. def $\Rightarrow$ min	othewise womp

Fermat: $A \subset \mathbb{R}^n$ resrtiction, $f: A \to \mathbb{R}$! 1 restriction max.

$(a_1, a_2, \dots a_n) = a$ crit. point $(\nabla f(a) = 0_n)$

- if $a \in A \Rightarrow a$ can local min./max.

otherwise L.M.R.

- Lagrange multiplayer rule:

/gamemode lagrange

$$\left.\begin{array}{l} F_1(x_1, \dots x_n) \\ \vdots \\ F_m(x_1, \dots x_n) \end{array}\right\} \text{restriction function}$$

$\exists \lambda_1, \dots \lambda_n \in \mathbb{R}$ s.t. $(x_1', \dots x_n', \lambda_1', \dots \lambda_n')$ is crit. point of $L(x_1, \dots x_n, \lambda_1, \dots \lambda_n)$

$$L(x_1, \dots x_n, \lambda_1, \dots \lambda_n) = f(x_1, \dots x_n) + \lambda_1 F(x_1, \dots x_n) + \dots + \lambda_m F_m(x_1, \dots x_n)$$

$\Rightarrow \underbrace{(x_1', \dots x_n')}_{\text{2 values}}$ is min./max of f

$$J(f)(x, y, z) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_m} \end{pmatrix}$$

$$[df(a)(h_1, \dots h_n) = J(f)(a) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \quad df(a) \in L(\mathbb{R}^n, \mathbb{R}^m)$$

$f: \mathbb{R}^n \to \mathbb{R}^m$

$$F = f \circ g \Rightarrow J(F)(a) = J(f)(g(a)) \cdot J(g)(a)$$

for continuity at $a \in \mathbb{R}^n$: $\lim\limits_{(x_1, \dots x_n) \to a} \frac{f(x) - f(a)}{|x - a|} = f(a) \quad (f: \mathbb{R}^n \to \mathbb{R})$

for differentiability at $a \in \mathbb{R}^n$

$\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n}$ con diff. on $\mathbb{R}^n \setminus \{a\}$

$\mathbb{R}^n \setminus \{a\}$ open

$$\exists \frac{\partial f}{\partial x_i}(a) = \lim_{x_i \to a_i} \frac{f(a_1, \dots x_i, \dots a_n) - f(a)}{x_i - a_1}$$

$\Rightarrow$ if $\lim\limits_{\substack{h_1 \to a_1 \\ \vdots \\ h_n \to a_n}} \dfrac{f(a_1+h_1, \dots a_n+h_n) - f(a) - h_1 \frac{\partial f}{\partial x_1}(a) - \dots - h_n \frac{\partial f}{\partial x_n}(a)}{\sqrt{h_1^2 + \dots + h_n^2}} = 0 \Rightarrow f$ diff. on $\mathbb{R}^n$

otherwise not

$\sin^2 t = \dfrac{1-\cos 2x}{2}$ $\quad\quad \cos^2 t = \dfrac{1+\cos 2x}{2}$

$\sin 2t = 2 \sin t \cos t$ $\quad\quad \cos 2t = \cos^2 t - \sin^2 t$ $\quad\quad tg^2 x + 1 = \dfrac{1}{\cos^2 x}$

$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$ $\quad\quad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$ $\quad dx\,dy = \rho\, d\rho\, d\theta$ $\quad\quad \begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$ $\quad dx\,dy\,dz = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$

$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \\ z = z \end{cases}$ $\quad dx\,dy\,dz = \rho\, d\rho\, d\theta\, dz$ $\quad\quad$ jeometer noledge

az a szép az a szép akinek a szeme kék, akinek a szeme kék

Jordan measure: A/V

A: $\iint_A dx\,dy$

V: $\iiint_V dx\,dy\,dz$

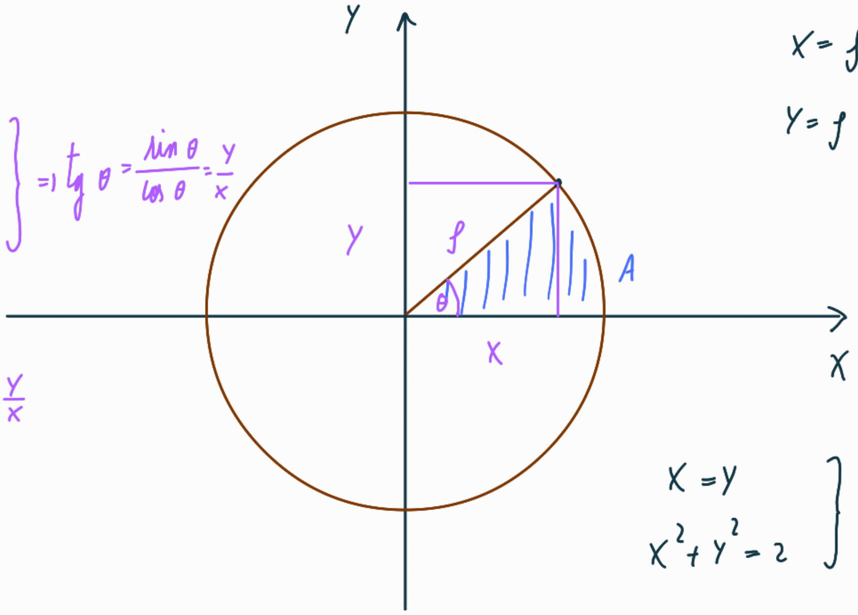
center of mass: dont forget about axis of rotation

$(x, y) \Rightarrow \dfrac{\iint_A x\,dx\,dy}{\iint_A dx\,dy} = x \quad \dfrac{\iint_A y\,dx\,dy}{\iint_A dx\,dy} = y$

$(x, y, z) \Rightarrow \dfrac{\iiint_V x\,dx\,dy\,dz}{\iiint_V dx\,dy\,dz} = x$ etc.

arclengths of γ : $l(\gamma) = \int_\gamma \|\gamma'(t)\| \, dt$

4) $I = \iint_A \frac{x}{y^2+1} \, dx \, dy \qquad A = \{(x,y) \in \mathbb{R}^2 \mid x \geq y \geq 0 , \ x^2+y^2 \leq 2\}$



$\sin\theta = \frac{y}{\rho}$, $\cos\theta = \frac{x}{\rho}$ $\Rightarrow \operatorname{tg}\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$

$\Rightarrow \theta = \operatorname{arctg}\frac{y}{x}$

$x = \rho\cos\theta \qquad \rho \in [0, \sqrt{2}]$

$y = \rho\sin\theta \qquad \theta \in [0, \frac{\pi}{4}]$

$\left.\begin{array}{l} x = y \\ x^2+y^2 = 2 \end{array}\right\} \Rightarrow 2x^2 = 2 \Rightarrow x = 1$

$\Rightarrow \operatorname{arctg}\frac{y}{x} = \frac{\pi}{4}$

$$\iint_A \frac{x}{y^2+1}\,dx\,dy = \iint_A \frac{\rho\cos\theta}{\rho^2\sin^2\theta+1}\,\rho\,d\rho\,d\theta = \int_0^{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{\rho^2\cos\theta}{\rho^2\sin^2\theta+1}\,d\theta\,d\rho =$$

$$= \int_0^{\sqrt{2}} \rho \int \frac{(\rho\sin\theta)'}{(\rho\sin\theta)^2+1^2}\,d\theta\,d\rho = \int_0^{\sqrt{2}} \rho\left(\operatorname{arctg}\rho\sin\theta\Big|_0^{\frac{\pi}{4}}\right)d\rho =$$

$$= \int_0^{\sqrt{2}} \rho\left(\operatorname{arctg}\frac{\rho\sqrt{2}}{2}\right)d\rho = \int_0^{\sqrt{2}}\left(\frac{\rho^2}{2}\right)'\operatorname{arctg}\frac{\rho\sqrt{2}}{2}\,d\rho = \frac{\rho^2}{2}\operatorname{arctg}\frac{\rho\sqrt{2}}{2}\Big|_0^{\sqrt{2}} - \frac{1}{2}\int_0^{\sqrt{2}}\frac{\sqrt{2}\rho^2}{\rho^2+2}\,d\rho =$$

$$= \frac{\pi}{4} - \frac{1}{2}\int_0^{\sqrt{2}} \frac{\sqrt{2}\rho^2+2\sqrt{2}}{\rho^2+2} - \frac{2\sqrt{2}}{\rho^2+2}\,d\rho = \frac{\pi}{4} - \frac{1}{2}\int_0^{\sqrt{2}}\frac{\sqrt{2}(\rho^2+2)}{\rho^2+2}\,d\rho + \sqrt{2}\int_0^{\sqrt{2}}\frac{1}{\rho^2+2^2}\,d\rho =$$

$$= \frac{\pi}{4} - \frac{\sqrt{2}}{2}(\sqrt{2}) + \sqrt{2}\,\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\rho}{\sqrt{2}}\Big|_0^{\sqrt{2}} = \frac{\pi}{4} - 1 + 1\cdot\frac{\pi}{4} = \frac{\pi}{2} - 1 = \frac{\pi-2}{2}$$