SEMINARS 6 and 7

1) Solve the following systems of linear equations:

$$\text{a)} \begin{cases} x_1 + x_2 + 2x_3 = -1 \\ 2x_1 - x_2 + 2x_3 = -4 \\ 4x_1 + x_2 + 4x_3 = -2 \end{cases} (\text{în } \mathbb{R}^3); \quad \text{b)} \begin{cases} 3x_1 + 4x_2 + x_3 + 2x_4 = 3 \\ 6x_1 + 8x_2 + 2x_3 + 5x_4 = 7 \\ 9x_1 + 12x_2 + 3x_3 + 10x_4 = 13 \end{cases} (\text{în } \mathbb{R}^4);$$

$$\text{c)} \begin{cases} x_1 + x_2 - 3x_3 = -1 \\ 2x_1 + x_2 - 2x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \\ x_1 + 2x_2 - 3x_3 = 1 \end{cases} \quad (\text{în } \mathbb{R}^3).$$

2) Discuss on the real parameter α the consistency of the following systems, then solve them:

$$a) \begin{cases} 5x_1 - 3x_2 + 2x_3 + 4x_4 = 3 \\ 4x_1 - 2x_2 + 3x_3 + 7x_4 = 1 \\ 8x_1 - 6x_2 - x_3 - 5x_4 = 9 \\ 7x_1 - 3x_2 + 7x_3 + 17x_4 = \alpha \end{cases}, \quad b) \begin{cases} 2x_1 - x_2 + 3x_3 + 4x_4 = 5 \\ 4x_1 - 2x_2 + 5x_3 + 6x_4 = 7 \\ 6x_1 - 3x_2 + 7x_3 + 8x_4 = 9 \\ \alpha x_1 - 4x_2 + 9x_3 + 10x_4 = 11 \end{cases};$$

$$c) \begin{cases} \alpha x_1 + x_2 + x_3 = 1 \\ x_1 + \alpha x_2 + x_3 = 1 \\ x_1 + x_2 + \alpha x_3 = 1 \end{cases}.$$

3) Using elementary operations, determine the ranks of the following matrices:

$$\text{a)} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & -3 \end{pmatrix}; \text{b)} \begin{pmatrix} 1 & 2 & 1 & -2 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 2 & -3 \end{pmatrix}; \text{c)} \begin{pmatrix} 3 & 0 & 3 & 0 & 3 \\ 0 & 2 & 0 & 2 & 0 \\ 3 & 2 & 0 & 3 & 2 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix};$$

$$\text{d)} \begin{pmatrix} 2 & \alpha & -2 & 2 \\ 4 & -1 & 2\alpha & 5 \\ 2 & 10 & -12 & 1 \end{pmatrix} \quad (\alpha \in \mathbb{C}).$$

4) Are these matrices invertible? If yes, find their inverses:

$$\text{a)} \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 4 & 1 & 4 \end{pmatrix}; \text{b)} \begin{pmatrix} 3 & 4 & 2 \\ 6 & 8 & 5 \\ 9 & 12 & 10 \end{pmatrix}; \text{c)} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Definition. A square matrix resulted from the identilastty matrix after performing only one elementary operation is called **elementary matrix**.

5) Show that any elementary matrix has an inverse and that the inverse of any elementary matrix is also an elementary matrix.

6) Let $m, n \in \mathbb{N}^*$. Show that any elementary operation on a matrix $A = (a_{ij}) \in M_{m,n}(K)$ is the result of the multiplication of A with an elementary matrix. More precisely, any elementary operation on the rows (columns) of A results by multiplying A on the left

(right) side with the elementary matrix resulted by performing the same elementary operation on I_m (I_n, respectively).

7) (HOMEWORK) Let $n \in \mathbb{N}^*$. For any elementary matrix $E \in M_n(K)$ and any matrix $A \in M_n(K)$ we have

$$\det(EA) = \det E \cdot \det A = \det(AE).$$

8) Show that any invertible matrix is a product of elementary matrices.

9) Let $n \in \mathbb{N}^*$. For any matrices $A, B \in M_n(K)$ we have $\det(AB) = \det A \cdot \det B$.