5.1

(1) Is $\lambda=2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$?

$\lambda=2$ is an eigenvalue iff $A\underline{x}=2\underline{x}$ has a nontrivial sol.

$(A-2I)\underline{x}=0$

$$A-2I=\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}-\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

$\downarrow$

cols lin. dep

$\downarrow$

nontrivial sol

$\lambda=2$ ✓

(3) is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$?

is $A\underline{x}$ a multiple of $\underline{x}$?

$$\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}=\begin{bmatrix} 1 \\ 29 \end{bmatrix}\neq\lambda\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is NOT an eigenvalue

(4) is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$?

is $A\underline{x}$ a multiple of $\underline{x}$?

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \boxed{\lambda = 2}$$

5 is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$

is $A\underline{x}$ a multiple of $\underline{x}$?

$$\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \rightarrow \boxed{\lambda = 0}$$

7 Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$?

$$A - 4I = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$$

$$(A - 4I)\underline{x} = 0$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$(A - 4I)\underline{x} = 0$ has a nontrivial sol, so $\lambda = 4$ ✓

$$\begin{cases} x_1 + x_3 = 0 \\ -x_2 - x_3 = 0 \\ x_3 \text{ free} \end{cases}$$

if $x_3 = 1$
$x_2 = -1$ and $x_1 = -1$

$\underline{x} = (-1, -1, 1)$

(11) $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$ $\lambda = 10$

$$A - 10I = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix}$$

$(A - 10I)\underline{x} = 0$

$$\left[\begin{array}{cc|c} -6 & -2 & 0 \\ -3 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x_1 = (-1/3)x_2 \\ x_2 \text{ free} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/3\, x_2 \\ x_2 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

a basis for eigenspace corresponding to $\lambda = 10$ is $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ $\left(\text{or also } \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right)$

(13) $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ $\lambda = 1, 2, 3$

- for $\lambda = 1$:

$$A - 1I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$(A-1I)\underline{x}=0$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} x_1 = 0 \\ x_3 = 0 \\ x_2 \text{ free} \end{cases}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

a basis for $\lambda = 1$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

- for $\lambda = 2$:

$$A-1I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

$$(A-1I)\underline{x}=0$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} x_1 = -1/2\, x_3 \\ x_2 = x_3 \\ x_3 \text{ free} \end{cases}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2\, x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$$

a basis vector for $\lambda = 2$ is $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

• for $\lambda=3$:

$$A-1I=\begin{bmatrix}4&0&1\\-2&1&0\\-2&0&1\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}=\begin{bmatrix}1&0&1\\-2&-2&0\\-2&0&-2\end{bmatrix}$$

$(A-1I)\underline{x}=0$

$$\left[\begin{array}{ccc|c}1&0&1&0\\-2&-2&0&0\\-2&0&-2&0\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]\quad\begin{cases}x_1=-x_3\\x_2=x_3\\x_3\text{ free}\end{cases}$$

$$\underline{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_3\\x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

a basis vector for $\lambda=2$ is $\begin{bmatrix}-1\\1\\1\end{bmatrix}$

15 $A=\begin{bmatrix}8&3&-4\\-1&4&4\\2&6&-1\end{bmatrix}$ $\qquad\lambda=7$

$$A-7I=\begin{bmatrix}8&3&-4\\-1&4&4\\2&6&-1\end{bmatrix}-\begin{bmatrix}7&0&0\\0&7&0\\0&0&7\end{bmatrix}=\begin{bmatrix}1&3&-4\\-1&-3&4\\2&6&-8\end{bmatrix}$$

$(A-7I)\underline{x}=0$:

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ -1 & -3 & 4 & 0 \\ 2 & 6 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x_1 = -3x_2 + 4x_3 \\ x_2, x_3 \text{ free} \end{cases}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_2 + 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

a basis for given eigenspace is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\}$

33) if λ is an eigenvalue of A, then there is a nonzero vector $\underline{x}$ such that $A\underline{x} = \lambda \underline{x}$.
Since A is invertible, $A^{-1}A\underline{x} = A^{-1}(\lambda x)$, and so $\underline{x} = \lambda(A^{-1}\underline{x})$.
Since $\underline{x} \neq \underline{0}$ (and A is invertible), λ cannot be 0.
Then $\lambda^{-1}\underline{x} = A^{-1}\underline{x}$, which shows that λ^{-1} is an eigenvalue of A^{-1}.

$\boxed{5.2}$ 1) $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$$

The characteristic polynomial is:

$$\det(A - \lambda I) = (2-\lambda)^2 - 7^2 = 4 - 4\lambda + \lambda^2 - 49 = \underline{\lambda^2 - 4\lambda - 45}$$

$$\lambda^2-4\lambda-45=(\lambda-9)(\lambda+5)=0$$

$$\downarrow \qquad \downarrow$$

$$\underline{\underline{\lambda_1=9}} \qquad \underline{\underline{\lambda_2=-5}}$$

③ $A=\begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$

$$A-\lambda I=\begin{bmatrix} 3-\lambda & -2 \\ 1 & -1-\lambda \end{bmatrix}$$

the characteristic polynomial:

$$\det(A-\lambda I)=(3-\lambda)(-1-\lambda)-(-2)(1)=-3-2\lambda+\lambda^2+2$$
$$=\lambda^2-2\lambda-1$$

$$\lambda_1=1+\sqrt{2} \quad \lambda_2=1-\sqrt{2}$$

⑤ $A=\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

$$A-\lambda I=\begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}$$

$$\det(A-\lambda I)=(2-\lambda)(4-\lambda)-1(-1)=8-6\lambda+\lambda^2+1=$$
$$=\lambda^2-6\lambda+9=(\lambda-3)^2$$

$\lambda=3$ with multiplicity 2

⑦ $A=\begin{bmatrix} 5 & 3 \\ \end{bmatrix}$

$$\begin{bmatrix} -4 & 4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (5-\lambda)(4-\lambda) - 3(-4) = 20 - 9\lambda + \lambda^2 + 12$$
$$= \lambda^2 - 9\lambda + 32 \quad = \nexists$$

no real eigenvalues

(9) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 3-\lambda & -1 \\ 0 & 6 & 0-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)\det\begin{bmatrix} 3-\lambda & -1 \\ 6 & -\lambda \end{bmatrix} - 2\cdot\det\begin{bmatrix} 0 & -1 \\ 6 & -\lambda \end{bmatrix}$$

$$= (1-\lambda)\big((3-\lambda)(-\lambda) + 6\big) - 2(6)$$
$$= (1-\lambda)(\lambda^2 - 3\lambda + 6) - 12$$
$$= \lambda^3 + 4\lambda^2 - 9\lambda - 6$$

(11) $A = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 6-\lambda & 0 & 0 \\ 5 & 4-\lambda & 3 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4-\lambda & 3 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (6-\lambda) \cdot \det \begin{bmatrix} 4-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} =$$

$$= (6-\lambda)(4-\lambda)(2-\lambda) = (\lambda^2 - 10\lambda + 24)(2-\lambda)$$
$$= 2\lambda^2 - 20\lambda + 48 - \lambda^3 + 10\lambda^2 - 24\lambda$$
$$= \underline{-\lambda^3 + 12\lambda^2 - 44\lambda + 48}$$

(15) $A = \begin{bmatrix} 7 & -5 & 3 & 0 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

$$\det(A-2I) = \det \begin{bmatrix} 7-\lambda & -5 & 3 & 0 \\ 0 & 3-\lambda & 7 & -5 \\ 0 & 0 & 5-\lambda & -3 \\ 0 & 0 & 0 & 7-\lambda \end{bmatrix} = (7-\lambda)^2(3-\lambda)(5-\lambda)$$

$\lambda_1 = 7$ with multiplicity 2
$\lambda_2 = 3$
$\lambda_3 = 5$

$\boxed{5.2}$

(1) $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$$

characteristic polynomial:

$$\det(A-\lambda I) = (2-\lambda)^2 - 7^2 = \lambda^2 - 4\lambda - 45 = (\lambda-9)(\lambda+5)$$

$\lambda_1 = 9 \qquad \lambda_2 = -5$

(3) $A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 3-\lambda & -2 \\ 1 & -1-\lambda \end{bmatrix}$

polynomial:

$$\det(A-\lambda I) = (3-\lambda)(-1-\lambda) + 2 = -3 - 2\lambda + \lambda^2 + 2 =$$
$$= \lambda^2 - 2\lambda - 1$$

$\lambda_1 = 1 - \sqrt{2} \qquad \lambda_2 = 1 + \sqrt{2}$

(5) $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}$

polynom:

$$\det(A-\lambda I) = (2-\lambda)(4-\lambda) + 1 = \lambda^2 - 6\lambda + 9 = (x-3)^2$$

$\lambda = 3$ with mult. 2

(7) $A = \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{bmatrix}$

$$\det(A-\lambda I) = (5-\lambda)(4-\lambda) + 12 = \lambda^2 - 9\lambda + 32$$

$\lambda^2 - 9\lambda + 32 = 0 \qquad \lambda \notin R \qquad$ no real eigenvalues

(9) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 3-\lambda & -1 \\ 0 & 6 & -\lambda \end{bmatrix} = (-1)^2(1-\lambda)\begin{bmatrix} 3-\lambda & -1 \\ 6 & -\lambda \end{bmatrix} + (-1)^3 \cdot 2 \begin{bmatrix} 0 & -1 \\ 6 & -\lambda \end{bmatrix}$$

$$= (1-\lambda)(-3\lambda+\lambda^2+6) - 2\cdot 6 = -3\lambda+\lambda^2+6+3\lambda^2-\lambda^3-6\lambda-12$$

$$= \underline{-\lambda^3+4\lambda^2-9\lambda-6}$$

(11) $A = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

$$\det(A-\lambda I) = \det\begin{bmatrix} 6-\lambda & 0 & 0 \\ 5 & 4-\lambda & 3 \\ 1 & 0 & 2-\lambda \end{bmatrix} = (-1)^2(6-\lambda)\det\begin{bmatrix} 4-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} =$$

$$= (6-\lambda)\ (4-\lambda)(2-\lambda) = -\lambda^3+9\lambda^2-26\lambda+24$$

(15) $A = \begin{bmatrix} 7 & -5 & 3 & 0 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

hint: use the fact that the determinant of a triangular matrix is the product of the diagonal entries

$$\det(A-\lambda I) = \det\begin{bmatrix} 7-\lambda & -5 & 3 & 0 \\ 0 & 3-\lambda & 7 & -5 \\ 0 & 0 & 5-\lambda & -3 \\ 0 & 0 & 0 & 7-\lambda \end{bmatrix} = (7-\lambda)^2(3-\lambda)(5-\lambda)$$

$\lambda_1 = 7$ with mult. 2

$\lambda_2 = 3$
$\lambda_3 = 5$

(16) $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 0 & 0 & 0 \\ 8 & -4-\lambda & 0 & 0 \\ 0 & 7 & 1-\lambda & 0 \\ 1 & -5 & 2 & 1-\lambda \end{bmatrix} = (5-\lambda)(-4-\lambda)(1-\lambda)^2$$

$\lambda_1 = 5 \qquad \lambda_2 = -4 \qquad \lambda_3 = 1$ with mult. 2

(18) find h for a given matrix such that the eigenspace for $\lambda = 6$ is two-dimensional

$A = \begin{bmatrix} 6 & 3 & 9 & -5 \\ 0 & 9 & h & 2 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

$(A - 6I)x = 0 \qquad \left[\begin{array}{cccc|c} 0 & 3 & 9 & -5 & 0 \\ 0 & 3 & h & 2 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 3 & 9 & 0 & 0 \\ 0 & 0 & h-9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

for a 2D eigenspace, the system needs 2 free var.
This happens iff $h = 9$

(20) show that A and A^T have the same polynomial

$$\det(A^T-\lambda I) = \det(A^T-\lambda I^T)$$
$$= \det(A-\lambda I)^T \quad \text{(transpose prop.)}$$
$$= \det(A-\lambda I)$$

$\boxed{5.3}$

(1) let $A=PDP^{-1}$, compute A^4

$$P=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix} \qquad D=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}$$

$$A^4=PD^4P^{-1}$$

$$P^{-1}=\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix} \qquad D^4=\begin{bmatrix}81 & 0\\ 0 & 1\end{bmatrix}$$

$$A^4=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}\begin{bmatrix}81 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix} = \begin{bmatrix}162 & 5\\ 81 & 3\end{bmatrix}\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix}481 & -800\\ 240 & -399\end{bmatrix}}}$$

(3) compute A^k (k = an arbitrary positive integer)

$$\begin{bmatrix}a & 0\\ 3(a-b) & b\end{bmatrix}=\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}$$

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^k & 0 \\ 3a^k & b^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} a^k & 0 \\ 3a^k - 3b^k & b^k \end{bmatrix}$$

$$= \begin{bmatrix} a^k & 0 \\ 3(a^k - b^k) & b^k \end{bmatrix}$$

(5) $$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/3 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$$

By Diagonalization theorem, eigenvectors from the cols of the left factor, and they correspond respectively to the eigenvalues on the diagonal of the middle factor

$\lambda_1 = 5 : \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\qquad$ $\lambda_2 = 1 : \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

(7) diagonalize: $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

since A is triangular: $\lambda_1 = 1 \quad \lambda_2 = -1$

- for $\lambda_1 = 1$:

$(A - 1I) = \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}$ $\qquad$ $(A - 1I)\underline{x} = 0 : \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 6 & -2 & 0 \end{array}\right]$

$$\begin{cases} x_1 = 1/3\, x_2 \\ x_2 \text{ free} \end{cases}$$ general sol: $x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

basis for a vector eigenspace: $\underline{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

- for $\lambda_2 = -1$:

$(A + 1I) = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$ $\quad (A+1I)\underline{x} = 0: \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right]$

$$\begin{cases} x_1 = 0 \\ x_2 \text{ free} \end{cases}$$ general sol: $x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

basis for a vector eigenspace: $\underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

8 diagonalize $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

since the matrix is triangular: $\lambda = 5$ (with mult 2)

$(A - 5I) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad (A-5I)\underline{x} = 0: \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$$\begin{cases} x_2 = 0 \\ x_1 \text{ free} \end{cases}$$ general sol: $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

since we cannot generate an eigenvector basis for $\mathbb{R}^2$, A is not diagonalizable

10 diagonalize: $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

1) find eigenvalues by computing char. polynom.

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - 12$$

$$= \lambda^2 - 3\lambda - 10$$

$\lambda_1 = 5 \qquad \lambda_2 = -2$

• for $\lambda_1 = 5$:

$A - 5I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$ $\qquad (A-5I)\underline{x} = 0$: $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$\sim$

$\begin{cases} x_1 = x_2 \\ x_2 \text{ free} \end{cases}$ $\qquad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

general sol: $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

basis for a vector eigenspace: $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

• for $\lambda_2 = -2$:

$(A+2I) = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$ $\qquad (A+2I)\underline{x} = 0$: $\left[\begin{array}{cc|c} 4 & 3 & 0 \\ 4 & 3 & 0 \end{array}\right]$

$\sim$

$\begin{cases} x_1 = -3/4\, x_2 \\ x_2 \text{ free} \end{cases}$ $\qquad \left[\begin{array}{cc|c} 4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

general sol: $x_2 \begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$

basis for a vector eigenspace: $\underline{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$\Rightarrow$ from $\underline{v_1}$ and $\underline{v_2}$ construct $P = [\underline{v}_1 \ \underline{v}_2]$

$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$ then set $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$, where the

eigenvalues in D correspond to $\underline{v_1}$ and $\underline{v_2}$

(11) $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$ $\quad \lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

- for $\lambda_3 = 3$:

$A - 3I = \begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$

$$[A - 3I \ \ 0] = \left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

general sol: $x_3 \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}$

basis vector for the eigenspace: $\underline{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

- for $\lambda_2 = 2$

$$A - 2I = \begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

$$[A-2I \quad 0] = \left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 2/3\, x_3 \\ x_2 = x_3 \\ x_3 \text{ free} \end{cases}$$

general sol: $x_3 \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix}$

basis vector for the eigenspace: $\underline{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

- for $\lambda_3 = 1$

$$A - 1I = \begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix}$$

$$[A-1I \quad 0] = \left[\begin{array}{ccc|c} -2 & 4 & -2 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

general sol: $x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

basis vector for the eigenspace: $\underline{v_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12 $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \lambda_2 = 4$

- for $\lambda_1 = 1$

$A - I = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$[A-I \ \ 0] = \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

general sol: $x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

- for $\lambda_2 = 4$

$$A-4I = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$[A-4I \;\; 0] = \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

general sol: $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

basis for the eigenspace: $\{\underline{v_2}, \underline{v_3}\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(17) $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

since A is triangular, eigenvalues

$\lambda_1 = 4$ (with mult. 2)

$\lambda_2 = 5$

- for $\lambda_1 = 4$

$$A-4I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A - 4I \quad 0] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

general sol: $x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

basis for eigenspace: $\underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

since $\lambda = 5$ must have only 1D eigenspace, we can find at most 2 independent eigenvectors for A, so A is not diagonalizable