$\boxed{4.1}$

1. let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$

a) $u, v \in V$ $\quad\quad$ $u + v \overset{?}{\in} V$

u and v are both positive
u+v then must be positive
thus, $u + v \in V$ ✓

b) example: $\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad u \in V$

$c = -1$ $\quad\quad$ $c \cdot \underline{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\quad c \cdot \underline{u} \notin V$

2. $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$

a) $\underline{u} \in W$ $\quad\quad$ $c \cdot \underline{u} \overset{?}{\in} W$

$c \cdot \underline{u} = c \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ $\quad\quad$ $(cx)(cy) \overset{?}{\geq} 0$

$c^2(xy) \geq 0$

yes, because $xy \geq 0$
and c^2 is always ≥ 0

b) $\underline{u} = \begin{bmatrix} -1 \\ -10 \end{bmatrix}$ $\quad\quad$ $\underline{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\underline{u} + \underline{v} = \begin{bmatrix} 2 \\ -9 \end{bmatrix} \notin W$

③ $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$

find an example to show that H is not a subspace of $\mathbb{R}^2$

$\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad u \in H \quad (1^2 + 1^2 \leq 1)$

$c = 5 \qquad c \cdot \underline{u} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \qquad c \cdot \underline{u} \notin H$

since H is not closed under scalar multiplication, H is not a subspace of $\mathbb{R}^2$

⑤ $p(\underline{t}) = at^2 \qquad a \in \mathbb{R}$

$\{ at^2 \mid a \in \mathbb{R} \}$

is this set a subspace of P_n?

check 3 criteria:

1) Closure under addition:

let $p(t) = a_1 t^2$ and $q(t) = a_2 t^2$

$p(t) + q(t) = (a_1 + a_2) t^2$

since $a_1, a_2 \in \mathbb{R}$, $a_1 + a_2 \in \mathbb{R}$, $(a_1 + a_2) t^2 \in \mathbb{R}$

2) Closure under scalar multiplication:

$$c \cdot p(t) = c\left(at^2\right) = (ca)t^2$$

since $c, a \in \mathbb{R}$, $ca \in \mathbb{R}$, $cat^2 \in \mathbb{R}$

3) zero vector

$$P_n = 0 + 0t + \ldots + 0t^n$$

$p(t) = 0t^2$ when $a = 0$

$p(t)$ is a subspace of P_n

6. $p(t) = a + t^2 \qquad a \in \mathbb{R}$

$\{a + t^2 \mid a \in \mathbb{R}\}$

is this set a subspace of P_n?

check 3 criteria:

1) Closure under addition:

let $p(t) = a_1 + t^2$ and $q(t) = a_2 + t^2$

$p(t) + q(t) = a_1 + t^2 + a_2 + t^2 = (a_1 + a_2) + 2t^2$

$p(t)+q(t) = a_1+t^2+a_2+t^2 = (a_1+a_2)+2t^2$

since $a_1, a_2 \in \mathbb{R}$, $a_1+a_2 \in \mathbb{R}$, $(a_1+a_2)+2t^2 \in \mathbb{R}$

2) Closure under scalar multiplication:

$c \cdot p(t) = c(a+t^2) = ca + ct^2$

since $c, a \in \mathbb{R}$, $ca \in \mathbb{R}$, $ca+ct^2 \in \mathbb{R}$

3) zero vector

$P_n = 0 + 0t + \ldots + 0t^n$

$p(t) = 0 + t^2$ when $a = 0$

the set of all polynomials doesn't contain $0+t^2$

$p(t)$ is NOT a subspace of P_n

9) H is set of vectors $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$

find vector v in $\mathbb{R}^3$ such that $H = \text{Span}\{v\}$

why does this show that H is a subspace of $\mathbb{R}^3$?

$H = \text{Span}\{v\}$ where $v = \begin{bmatrix} 1 \\ 3 \\ \vdots \end{bmatrix}$

$\begin{bmatrix} 2 \end{bmatrix}$

thus H is a subspace of $\mathbb{R}^3$

(10) H is set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$

show that H is a subspace of $\mathbb{R}^3$

$H = \text{Span}\{v\}$ $\quad \underline{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

H is a subspace of $\mathbb{R}^3$

(12) W is a set of all vectors of a form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$

show W is a subspace of $\mathbb{R}^4$

$W = \text{Span}\{\underline{u}, \underline{v}\}$ $\quad \underline{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$

W is a subset of $\mathbb{R}^4$

(13) $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ $\quad v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ $\quad w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

a) $w \notin \{v_1, v_2, v_3\}$

size $\{v_1, v_2, v_3\} = 3$

b) size Span $\{v_1, v_2, v_3\}$ = infinitely many

c) w is in the subspace spanned by $\{v_1, v_2, v_3\}$ iff equation $x_1 \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3} = w$ has a solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \underset{R_3 : R_3 + R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right] \underset{R_3 : R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so the equation has a solution.

w is in a subspace ✓

(14)
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{array}\right] \underset{R_3 : R_3 + R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right] \underset{R_3 : R_3 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

the equation has no solution

w is not in the subspace

Q(20) The set of all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ is denoted $C[a, b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a, b]$

a) what facts about continuous functions should be proved in order to demonstrate that $C[a, b]$ is indeed a subspace as claimed?

1) the constant function $f(t) = 0$ is continuous
2) the sum of 2 continuous fun. is also continuous
3) a constant multiple of a continuous fun. is continuous

b) Show that $\{f \text{ in } C[a,b] : f(a) = f(b)\}$ is a subspace of $C[a,b]$

1) let $H = \{f \text{ in } C[a,b] : f(a) = f(b)\}$
let $g(t) = 0$ for all t in $[a,b]$
then $g(a) = g(b) = 0$, so g is in H

2) let g and h be in H
then $g(a) = g(b)$ $h(a) = h(b)$, and $(g+h)(a) = g(a) + h(a)$
$= g(b) + h(b) = (g+h)(b)$, so $g+h$ is in H

3) let $g \in H$
then $g(a) = g(b)$ and $(cg)(a) = cg(a) = cg(b) =$
$= (cg)(b)$, so $c \cdot g \in H$

Thus H is a subspace of $C[a,b]$

$\boxed{4.2}$

2 is $w = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ in $Nul(A)$?

$$A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \qquad A w = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

w is in $\mathrm{Nul}(A)$ ✓

(4) $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

$$[A|0] = \left[\begin{array}{cccc|c} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{array}\right] \underset{R_1 : R_1 - 3R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 7x_3 - 6x_4 \\ x_2 = -4x_3 + 2x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

spanning set of $\mathrm{Nul}(A)$ is $\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(6) $$[A|0] = \left[\begin{array}{ccccc|c} 1 & 5 & -4 & -3 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{R_1 : R_1 - 5R_2}{\sim}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & -8 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = -6x_3 + 8x_4 - x_5 \\ x_2 = 2x_3 - x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \\ x_5 \text{ free} \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

spanning set for Null(A) is $\left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(7) is it a vector space?

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a+b+c = 2 \right\}$$

zero vector not in W. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $0+0+0 \neq 2$

!

$W = \{ \begin{bmatrix} r \\ \dots \end{bmatrix} \dots \}$

$$\left\{ \begin{bmatrix} \vdots \\ s \\ t \end{bmatrix} : 5r-1 = s+2t \right\}$$

zero vec.: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad 5(0)-1 = 0+2(0)$

$\quad -1 \neq 0$

(9)

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{l} a-2b=4c \\ 2a = c+3d \end{array} \right\}$$

zero vec.: $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$: $\begin{array}{l} 0-2(0)=4(0) \rightarrow 0=0 \checkmark \\ 2(0) = 0+3(0) \rightarrow 0=0 \checkmark \end{array}$

$W = \mathrm{nul}(A) \qquad A = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix}$

(11) $$\left\{ \begin{bmatrix} b-2d \\ 5+d \\ b+3d \\ d \end{bmatrix} : b, d \text{ real} \right\}$$

zero vector is not in W

(15) find A such that the given set is $\mathrm{Col}(A)$

$$\left\{ \begin{bmatrix} 2s+3t \\ \vdots \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r,s,t \text{ real} \right\}$$

$$r\begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

23

Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ $\qquad \underline{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

is $\underline{w}$ in col(A) $\qquad 6\underline{w}$

$$\begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

24 $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ $\qquad \underline{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$$[A \mid \underline{w}] = \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

consistent, so $\underline{w}$ is in Col(A)

$$A\underline{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{w}$ is in Nul(A)

$\boxed{4.3}$

(1) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ → echelon form
→ 3 pivots
⇒ lin. indep. , span $\mathbb{R}^3$

basis for $\mathbb{R}^3$

(2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ since the zero vector is a part of the given set the set is NOT lin. indep. and thus cannot be a basis for $\mathbb{R}^3$

only 2 pivots, do not span $\mathbb{R}^3$

(3) $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9/2 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$ only 2 pivots, not a basis for $\mathbb{R}^3$

not lin. indep.
not span $\mathbb{R}^3$

(4) $\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3 pivots

basis for $\mathbb{R}^3$

(6) $\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ cannot span $\mathbb{R}^3$
(cannot have a pivot in each row)

lin. indep.

(7) $\begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

cannot span $\mathbb{R}^3$
not a basis for $\mathbb{R}^3$

lin. indep.

(8) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & -2 \\ 3 & -1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$

cannot have a pivot in each col → cannot be lin. indep → cannot be a basis for $\mathbb{R}^3$

pivot in each row → vectors spans $\mathbb{R}^3$

(9) $Ax = 0$

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 3 & -2 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 3x_3 - 2x_4 \\ x_2 = 5x_3 - 4x_4 \\ x_3, x_4 \text{ free} \end{cases} \qquad x = x_3 \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{basis } \operatorname{Nul}(A) = \left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(12) basis for a set of vectors in $\mathbb{R}^2$ on the line $y=5x$

$5x-y=0 \qquad A=[5\ -1]$

basis of Nul(A) $\qquad Ax=0$

$[5\ -1 | 0] \qquad \begin{cases} y=5x \\ x \text{ free} \end{cases} \qquad x=\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

basis Nul(A) $= \left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$

(13) $A=\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

B is REF(A)

$\text{Col}(A) = \left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$

Nul(A)? $\qquad Ax=0$

$\begin{cases} x_1 = -6x_3 - 5x_4 \\ x_2 = (-5/2)x_3 - (3/2)x_4 \\ x_3, x_4 \text{ free} \end{cases} \qquad \text{Nul}(A) = \left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\text{Row}(A) = \{ [1\ 0\ 6\ 5], [0\ 2\ 5\ 3] \}$

pivot rows B

(14) $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$Col(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$$

$Nul(A) = Ax = 0$

$$\begin{cases} x_1 = -2x_2 - 4x_4 \\ x_3 = 7/5\, x_4 \\ x_5 = 0 \\ x_2, x_4 \text{ free} \end{cases}$$

$$Nul(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$Row(A) = \{ [1\ 2\ 0\ 4\ 5], [0\ 0\ 5\ -7\ 8], [0\ 0\ 0\ 0\ -9] \}$$

$\boxed{4.4}$ (1) $B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$ $\quad [x]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$$\underline{x} = 5 \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3 \cdot \begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

(3) $B = \left\{ \begin{bmatrix} 1 \\ -8 \\ \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ \end{bmatrix} \right\}$ $\quad [x]_B = \begin{bmatrix} 2 \\ -3 \\ \end{bmatrix}$

$$\left(\begin{bmatrix} 6 \end{bmatrix}, \begin{bmatrix} 7 \end{bmatrix}, \begin{bmatrix} -4 \end{bmatrix}\right) \qquad \begin{bmatrix} 0 \end{bmatrix}$$

$$\underline{x} = 2\begin{bmatrix} 1 \\ -8 \\ 6 \end{bmatrix} + (-3)\begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -9 \end{bmatrix}$$

(5) $\underline{b_1} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \underline{b_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$[\underline{b_1}\ \underline{b_2} \mid \underline{x}] = \left[\begin{array}{cc|c} 1 & 2 & -2 \\ -3 & -5 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -5 \end{array}\right]$$

$$[\underline{x}]_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

(7) $\underline{b_1} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} \quad \underline{b_2} = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} \quad \underline{b_3} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$

$$[b_1\, b_2\, b_3 \mid x] = \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad [\underline{x}]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

(9) $B = \left\{\begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix}\right\}$

change from B to standard basis in $\mathbb{R}^2$

$$P_B = [\underline{b_1}\ \underline{b_2}] = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}$$

(13) $B = \{1+t^2,\ t+t^2,\ 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$

find a coordinate vector $\underline{p}(t) = 1+4t+7t^2$ relative to B

$$c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2) = 1+4t+7t^2$$

$$\begin{cases} c_1 + c_3 = 1 \\ c_2 + 2c_3 = 4 \\ c_1 + c_2 + c_3 = 7 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$[\underline{p}]_B = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

(21) $\underline{v_1} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \underline{v_2} = \begin{bmatrix} 2 \\ -8 \end{bmatrix} \quad \underline{v_3} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

$\underline{v_1}, \underline{v_2}, \underline{v_3}$ span $\mathbb{R}^2$ but don't form a basis

find 2 ways of expressing $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a lin. combination of $\underline{v_1}, \underline{v_2}, \underline{v_3}$

$$x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -8 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 5 \end{array}\right] \quad \{ x_1 = 5+5x$

$$\begin{bmatrix} 1 & 2 & -3 & | & 1 \\ -3 & -8 & 7 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & | & 5 \\ 0 & 1 & 1 & | & -2 \end{bmatrix} \qquad \begin{cases} x_1 = 5 + 5x_3 \\ x_2 = -2 - x_3 \\ x_3 \text{ free} \end{cases}$$

$x_3 = 0$ or $x_3 = 1$ to produce $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{4.5}$

① $\left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s,t \in \mathbb{R} \right\}$

This subspace is $H = \text{Span}\{\underline{v}_1, \underline{v}_2\}$ where $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

Since $\underline{v}_1$ and $\underline{v}_2$ are not multiples of each other, $\{\underline{v}_1, \underline{v}_2\}$ is lin. indep. and thus a basis for H.

Hence the dimension of H is 2.

③ $\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

the subspace is $H = \text{Span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ where $\underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ \end{bmatrix}$ and $\underline{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ \end{bmatrix}$

$$\begin{bmatrix} 1 \end{bmatrix} \quad \begin{bmatrix} 2 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

$\underline{v}_1 \neq 0$, $\underline{v}_2$ is not a multiple of $\underline{v}_1$, since $\underline{v}_3$ 1st entry is a non-zero, $\underline{v}_3$ is not a lin. combination of $\underline{v}_1$ and $\underline{v}_2$

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ lin. indep. and form a basis for H

(5) $$\left\{ \begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

This subspace is $H = \text{Span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ where

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

$\underline{v}_1 = (-2)\,\underline{v}_3$ $\qquad$ $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are lin. dep.

$H = \text{Span}\{\underline{v}_1, \underline{v}_2\}$ $\qquad$ ($\underline{v}_3$ can be removed by the Spanning Set Theorem)

Since $\underline{v}_1$ and $\underline{v}_2$ are not multiples of each other

$\{\underline{v}_1, \underline{v}_2\}$ is lin indep. and form a basis for H

$\dim(H) = 2$

(7)

$\{(a,b,c): a-3b+c=0,\ b-2c=0,\ 2b-c=0)$

$$\begin{cases} a-3b+c=0 \\ b-2c=0 \\ 2b-c=0 \end{cases} \qquad H = \text{Nul}(A)$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -1 \end{bmatrix} \qquad [A|0] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ only trivial sol

$H = \text{Nul}\, A = \{0\}$

(9) find the dim of the subspace spanned by the given vectors

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(\text{Col}\, A) = 2$

||

dim. of the subspace spanned by vectors

(10)

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -1 & 2 & 3 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 pivots

$\dim(\text{Col}\, A) = 3$ $\qquad$ $\text{Row}\, A = 3$

$\dim(\mathrm{Col}\,A) = 3$ Row $A = 3$

(11) $\begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 3 pivot

Col $A = 3$ Row $A = 3$

$A\underline{x} = 0$: 2 free var (x_3 and x_4)

Nul $A = 2$

(12) $\begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 3 pivots

Col $A = 3$ Row $A = 3$

$A\underline{x} = 0$: 3 free var $\rightarrow$ Nul $A = 3$

(33) If a 4×7 matrix A has rank 4, find:

↳ 4 lin indep. rows or cols

$A\underline{x} = 0$: 3 free var : Nul$(A) = 3$

also: Nul(A) = # cols of A − rank A
= 7 − 4 = $\underline{\underline{3}}$

rank(A^T) = rank(A) (because row space A = col space A^T)

rank$(A^T) = 4$

(35) suppose a 5×9 matrix A has 4 pivot cols.

Is $\text{Col } A = \mathbb{R}^5$?

- since has 5 rows and 4 pivot cols, its rank $= 4$
- Col A is a subspace of $\mathbb{R}^5$ with dimension 4

Is $\text{Nul}(A) = \mathbb{R}^4$?

$\text{Nul}(A) = 9 - 4 = 5$ (5 free var in $A\underline{x} = 0$)

- since $\text{Nul}(A) = 5$, the null space of A has dimension 5 and therefore cannot be equal to $\mathbb{R}^4$

(37) If nullity $= 4$ and A is 5×6 matrix, what are the dimension of col and row spaces of A?

$\text{Rank } A = 6 - 4 = 2$ (2 pivots)

so the dimension of row and col spaces are 2

(38) If nullity of 7×6 matrix A is 5, what are the dim. of the col and row spaces of A?

$\text{Rank } A = 6 - 5 = 1$ (1 free var)

so, the dim. of the col and row spaces is 1

(39) A is 7×5 matrix.
largest Rank A?

at most 5 pivots → Rank A = at most 5

A is 5×7 matrix.

largest Rank A ?

– at most 5 pivots → Rank A = at most 5