$\boxed{H2}$ ii.c)

$$\dot{y} = (y+t)^2 - 1$$
up to $t = 1$
$$y(0) = 2/3$$
$h = 0.5$ and $h = 0.25$
solution $y(t) = 2/(3-2t) - t$

case $h = 0.5$:

$$t(0) = 0 \qquad y(0) = 2/3$$

$$k1 = h \cdot f(t(0), y(0)) = h \cdot f(0, 2/3) = 0.5 \cdot \left[(2/3+0)^2 - 1\right] = -5/18$$

$$k2 = h \cdot f(t(0) + 2/3 h,\ y(0) + 2/3 k_1) = h \cdot f(1/3, 13/27) =$$
$$= 0.5 \cdot \left[(13/27 + 1/3)^2 - 1\right] = 0.5 \cdot -245/729 = -245/1458$$

$$y(0.5) = y(0) + 1/4 (k_1 + 3k_2) = 2/3 + 1/4 \left(-5/18 + 3 \cdot -245/1458\right)$$
$$= \underline{229/486 \approx 0.471} \quad (3sf)$$

approx

$$y(0.5) = 2/(3 - 2 \cdot 0.5) - 0.5 = 0.5$$

actual sol.

absolute error: $|0.5 - 229/486| = 7/243 \approx 0.029$ (2sf)

relative error: $\dfrac{|0.5 - 229/486|}{|0.5|} = 14/243 \approx 0.058$ (2sf)

$$= \underline{5.8 \ \%}$$

$$t(1) = 0.5 \qquad y(0.5) = 229/486$$

$$k1 = h \cdot f(t(1), y(0.5)) = h \cdot f(0.5, 229/486) = 0.5\left[(229/486 + 0.5)^2 - 1\right]$$

$= -3353/118098 \approx -0.0284 \quad (3sf)$

$k2 = h \cdot f(t(1) + 2/3 h,\ y(0.5) + 2/3 k_1) = h \cdot f(0.5 + 2/3 \cdot 0.5,\ 229/486 + 2/3 \cdot -3353/118098)$

$= h \cdot f(5/6,\ 160235/354294) = 0.5 \cdot (0.652765) = 0.3263825$

$y(1) = y(0.5) + 1/4 (k_1 + 3k_2) = 229/486 + 1/4 (-3353/118098 + 3 \cdot (0.3263825))$

approx

$= 229/486 + 1/4 (0.950756)$

$= 229/486 + 0.237689$

$= \underline{\underline{0.708882}}$

$y(1) = 2/(3 - 2 \cdot 1) - 1 = 1$

actual sol.

absolute error $|1 - 0.708882| \approx 0.29 \quad (2sf)$

relative error $\dfrac{|1 - 0.708882|}{|1|} \approx 0.29 \quad (2sf)$

$= 29\%$

case $h = 0.25$:

$t(0) = 0 \qquad y(0) = 2/3$

$k1 = h \cdot f(t(0), y(0)) = h \cdot f(0, 2/3) = 0.25 [(2/3)^2 - 1] = -5/36$

$k2 = h \cdot f(t(0) + 2/3 h,\ y(0) + 2/3 k_1) = h \cdot f(1/6,\ 31/54) =$

$= 0.25[(31/54 + 1/6)^2 - 1] = 0.25 \cdot (-329/729) = -329/2916$

$$y(0.25) = y(0) + 1/4(k_1 + 3k_2) = 2/3 + 1/4\left(-5/36 + 3\left(-329/2916\right)\right)$$
$$= \underline{0.547325}$$

$$y(0.25) = 2/(3 - 2 \cdot 0.25) - 0.25 = 0.55$$

← actual sol.

absolute error: $|0.55 - 0.547325| = 0.0027$ (2sf)

relative error: $\dfrac{|0.55 - 0.547325|}{|0.55|} = 0.0049$ (2sf)
$= 0.49\,\%$

$t(1) = 0.25 \qquad y(0.25) = 0.547325$

$$k1 = h \cdot f(0.25, 0.547325) = 0.25 \cdot (-0.364273) = -0.09106825$$
$$k2 = h \cdot f(0.25 + 2/3 \cdot 0.25,\ 0.547325 + 2/3 \cdot (-0.09106825)) =$$
$$= 0.25 \cdot f(5/12, 0.486613) = 0.25 \cdot [(0.486613 + 5/12)^2 - 1]$$
$$= -0.0460215$$

$$y(0.5) = y(0.25) + 1/4(k_1 + 3k_2) =$$
$$= 0.547325 + 1/4[-0.09106825 + 3 \cdot (-0.0460215)]$$
$$= 0.490042$$
$$\approx \underline{0.49} \quad \text{(2sf)}$$

$t(2) = 0.5 \qquad y(0.5) = 0.490042$

$$k1 = h \cdot f(0.5, 0.490042) = 0.25 \cdot [(0.490042 + 0.5)^2 - 1]$$
$$= -0.00495421$$
$$k2 = h \cdot f(0.5 + 2/3 \cdot 0.25,\ 0.490042 + 2/3(-0.00495421))$$
$$= h \cdot f(2/3, 0.486739) = 0.25 \cdot [(0.486739 + 2/3)^2 - 1]$$

$= h \cdot f(3/4, 0.486751) = 0.25 \cdot [(0.486751 + 3/4)^2 - 1]$
$= 0.0825862$

$$y(0.75) = 0.490042 + 1/4\,(-0.00495421 + 3 \cdot 0.0825862)$$
$$= \underline{\underline{0.550743}}$$

$t(3) = 0.75 \qquad y(0.75) = 0.550743$

$$k1 = h \cdot f(0.75, 0.550743) = 0.25 \cdot [(0.550743 + 0.75)^2 - 1]$$
$$= 0.172983$$
$$k2 = h \cdot f(0.75 + 2/3 \cdot 0.25 \;,\; 0.550743 + 2/3 \cdot 0.172983)$$
$$= h \cdot f(11/12 \;,\; 0.666065) =$$
$$= 0.25 \cdot [(0.666065 + 11/12)^2 - 1] = 0.37626$$

$$y(1) = 0.550743 + 1/4\,(0.172983 + 3 \cdot 0.37626)$$
$$= \underline{\underline{0.876184}}$$ ← approx.

absolute error: $|1 - 0.876184| = 0.12$ (2sf)

relative error: $\dfrac{|1 - 0.876184|}{|1|} = 0.12$ (2sf)

$= \underline{\underline{12\%}}$

→ of course the accuracy is better with smaller step size.
→ the relative error dropped from 29% to 12% when using step size 0.25 compared to 0.5