1. $f(x) = x^3 - 2x^2 - 1$ $\quad [2,3]$ $\quad P_0 = 3$, $P_1 = 2$

$f(P_0) = f(3) = 8$

$f(P_1) = f(2) = -1$

$f(P_2) = f(2.1111) = -0.5049$

$$P_2 = P_1 - \frac{P_1 - P_0}{f(P_1) - f(P_0)} \cdot f(P_1) = 2 - \frac{2-3}{-1-8} \cdot (-1) = \underline{2.1111}$$

$$P_3 = P_2 - \frac{P_2 - P_1}{f(P_2) - f(P_1)} \cdot f(P_2) = 2.1111 - \frac{2.1111 - 2}{-0.5049 - (-1)} \cdot (-0.5049) = \underline{\underline{2.2244}}$$

$f(P_3) = f(2.2244) = \underline{\underline{0.1103}}$

error: $|P_3 - P_2| = 0.1133 = \underline{\underline{0.11}}$

The best bracket is $[P_2, P_3]$ as f changes sign in the interval

2. $\dot{y} = 1 - t/y$ $\quad w_0 = y(1) = 5$ $\quad t \in [1,2]$ $\quad h = 1/3$

$w_1 = y(1\,1/3) = 5.59341564$

$w_2 = y(1\,2/3) = w_1 + h/2\,(3f(t_1, w_1) - f(t_0, w_0))$

$f(t_0, w_0) = f(1, 5) = 0.8000$

$f(t_1, w_1) = f(1\,1/3, 5.5934) = 0.7616$

$w_2 = 5.5934 + 1/6\,(3 \cdot 0.7616 - 0.8000) = \underline{5.8409}$

$f(t_2, w_2) = f(1\,2/3, 5.8409) = 0.7147$

$w_3 = y(2) = w_2 + h/2\,(3f(t_2, w_2) - f(t_1, w_1))$

$= 5.8409 + 1/6\,(3 \cdot 0.7147 - 0.7616) = 6.0713$

absolute error: $\underline{0.68}$

relative error: 10.02 %

(3) $a_0 = f(x_0) = 0.64$

$a_1 = f[x_0, x_1] = -1.2$

$a_2 = f[x_0, x_1, x_2] = 2.9253$

$a_3 = f[x_0, x_1, x_2, x_3] = -1.4401$

$$f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{0.24-0.64}{2^{2/3}-2^{1/3}} = -1.2000$$

$$f[x_1,x_2] = \frac{1.69-0.24}{2-2^{2/3}} = -2.1750$$

$$f[x_2,x_3] = \frac{0.17-1.69}{3-2} = -1.5200$$

$$f[x_0,x_1,x_2] = \frac{-2.1750-(-1.2)}{2-2^{1/3}} = 2.9253$$

$$f[x_1,x_2,x_3] = \frac{-1.52-(-2.175)}{3-2.6667} = 1.9652$$

$$f[x_0,x_1,x_2,x_3] = \frac{1.9652-2.9253}{3-2.3333} = -1.4401$$

$$p(x) = 0.64 + (x-2.3333)\big(-1.2+(x-2.6667)\big(2.9253+(x-2)\cdot(-1.4401)\big)\big)$$

$p(2.5) = 0.3787 \approx 0.38$

error: $n+1 = 4$ nodes at $x = 2.5$

$$|f(x)-p(x)| \leq \frac{\prod_{i=0}^{n}|x-x_i|}{(n+1)!} f''''(\xi)$$

$$|f(x)-p(x)| \leq \frac{|(2.5-2.3333)(2.5-2.6667)(2.5-2)(2.5-3)|}{4!} \cdot 10 = \underline{0.00029}$$

(4) for $x=1$ use 3-point forward:

$$f'(1) = \big(-3f(1)+4f(1.25)-f(1.5)\big)/2(0.25)$$

$= (-3 \cdot (-0.23925) + 4 \cdot 0.48679 - 0.9516) / 0.5$

$= \underline{3.42662}$

for $x = 1.25$ use 3-point centered:

$f'(1.25) = (f(1.5) - f(1)) / 2(0.25) = (0.9516 - (-0.23925)) / 0.5$

$= \underline{2.38170}$

estimate: $\int_1^2 \sqrt{1 + f'(x)^2}\, dx$ using Trapezoid:

for $x = 1$ $= \sqrt{1 + f'(1)^2} = \sqrt{1 + 3.42662^2} = 3.56956$

for $x = 1.25 = \sqrt{1 + f'(1.25)^2} = \sqrt{1 + 2.38170^2} = 2.58312$

$T_4(f; 1, 2) = h\left(\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2} f(x_4)\right)$

$= 0.25\left(\frac{1}{2} \cdot 3.56956 + 2.58312 + 1.08423 + 1.15489 + \frac{1}{2} \cdot 1.84445\right)$

$= 1.8823$

(5) $m = 3$

$a_0 = \frac{1}{3} \sum_{i=-3}^{2} f(x_i) \cancel{\cos(0)} = \frac{1}{3}(0.017 + 0.620 + 0.761 + 0.477 + 0.169 + 0.038)$

1

$\underline{a_0 = 0.6940}$

$a_1 = \frac{1}{3} \sum_{i=-3}^{2} f(x_i) \cos(x_i) = \underline{0.1987}$

$a_2 = \frac{1}{3} \sum_{i=-3}^{2} f(x_i) \cos(2x_i) = \frac{1}{3}(0.017 - 0.31 - 0.3805 + 0.477 - 0.0845 - 0.019)$

$\underline{a_2 = -0.1}$

$b_1 = \frac{1}{3} \sum_{i=-3}^{2} f(x_i) \sin(x_i) = \frac{1}{3}(-0.5369 - 0.6590 + 0.1464 + 0.0329) = \underline{-0.3389}$

$b_2 = \frac{1}{3} \sum_{i=-3}^{2} f(x_i) \sin(2x_i) = \frac{1}{3}(0.5369 - 0.6590 + 0.1464 - 0.0329) = \underline{-0.0029}$

$S_1 = 0.6940/2 + 0.1987 \cos(x) - 0.3386 \sin(x)$

$S_2 = 0.6940/2 + 0.1987 \cos(x) - 0.3386 \sin(x) - 0.1000 \cos(2x) - 0.0029 \sin(2x)$

$S_2(\pi/2) = 0.1084 \approx 0.108$ 3dp

⑥ $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 6 & 2 \\ 1 & 2 & 7 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ $\quad x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2 steps of conjugate gradient:

$r^{(0)} = b - \underbrace{\cancel{Ax^{(0)}}}_{\bar{0}} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \qquad v^{(1)} = r^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

$t^{(1)} = \langle r^{(0)}, r^{(0)} \rangle / \langle v^{(1)}, A, v^{(1)} \rangle = 17 / \langle \begin{bmatrix} 7 \\ 7 \\ 29 \end{bmatrix}, v^{(1)} \rangle = 17/123 = \underline{0.1382}$

$x^{(1)} = x^{(0)} + t^{(1)} v^{(1)} = \bar{0} + 0.1382 \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.1382 \\ 0.0000 \\ 0.5528 \end{bmatrix}$

$r^{(1)} = r^{(0)} - t^{(1)} A v^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - 0.1382 \begin{bmatrix} 3 & -1 & 1 \\ -1 & 6 & 2 \\ 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.0326 \\ -0.9674 \\ -0.0078 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - 0.1382 \begin{bmatrix} 7 \\ 7 \\ 29 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 0.9674 \\ 0.9674 \\ 4.0078 \end{bmatrix} \,//$

$s^{(2)} = \langle r^{(1)}, r^{(1)} \rangle / \langle r^{(0)}, r^{(0)} \rangle = 0.9370/17 = 0.0551$

$v^{(2)} = r^{(1)} + s^{(2)} v^{(1)} = \begin{bmatrix} 0.0326 \\ -0.9674 \\ -0.0078 \end{bmatrix} + 0.0551 \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.0877 \\ -0.9674 \\ 0.2126 \end{bmatrix}$

$t^{(2)} = \langle r^{(1)}, r^{(1)} \rangle / \langle v^{(2)}, A, v^{(2)} \rangle = 0.9370 / \langle \begin{bmatrix} 1.4431 \\ -5.4669 \\ -0.3589 \end{bmatrix}, v^{(2)} \rangle = 0.9370/5.3389 = 0.1755$

$x^{(2)} = x^{(1)} + t^{(2)} v^{(2)} = \begin{bmatrix} 0.1382 \\ 0.0000 \\ 0.5528 \end{bmatrix} + 0.1755 \begin{bmatrix} 0.0877 \\ -0.9674 \\ 0.2126 \end{bmatrix} = \begin{bmatrix} 0.1536 \\ -0.1700 \\ 0.5901 \end{bmatrix}$

$r^{(2)} = r^{(1)} - t^{(2)} A v^{(2)} = \begin{bmatrix} 0.0326 \\ -0.9674 \\ -0.0078 \end{bmatrix} - 0.1755 \begin{bmatrix} 3 & -1 & 1 \\ -1 & 6 & 2 \\ 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} 0.0877 \\ -0.9674 \\ 0.2126 \end{bmatrix} = \begin{bmatrix} -0.2207 \\ -0.0080 \\ 0.0552 \end{bmatrix}$ $r^{(2)}$

$= \begin{bmatrix} 0.0326 \\ -0.9674 \\ -0.0078 \end{bmatrix} - 0.1755 \begin{bmatrix} 1.4431 \\ -5.4669 \\ -0.3589 \end{bmatrix} =$

norm $r^{(2)} = \underline{0.23}$