- adding vectors in $\mathbb{R}^n \rightarrow$ another vector in $\mathbb{R}^n$
- scaling vector in $\mathbb{R}^n \rightarrow$ another vector in $\mathbb{R}^n$

Definition of a vector space V:

- a nonempty set of objects (vectors) with the following rules:
  1) $\underline{u}, \underline{v} \in V \rightarrow \underline{u}+\underline{v}$ is another vector in V
  2) $\underline{u} \in V$, $c \in \mathbb{R} \rightarrow c \cdot \underline{u}$ is another vector in V
  3) $\underline{u}+\underline{v} = \underline{v}+\underline{u}$
  4) $\underline{u}+(\underline{v}+\underline{w}) = (\underline{u}+\underline{v})+\underline{w}$

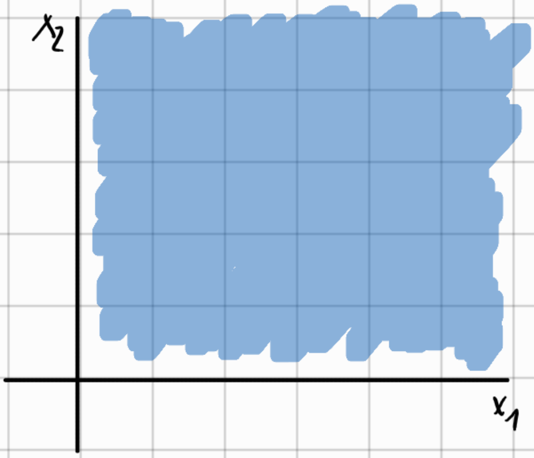
  5+6) $\exists\ \underline{0} \in V$: $\underline{u}+\underline{0}=\underline{u}$ and $\underline{u}+(-\underline{u})=\underline{0}$

  7) $1 \cdot \underline{u} = \underline{u}$

  8-10) see the book

(EX) Is $\mathbb{R}^2_+$ a vector space?

$\hookrightarrow \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0\}$



① Closed under addition?

Let $\underline{u}, \underline{v} \in \mathbb{R}^2_+$, then $\underline{u}+\underline{v} = \begin{bmatrix} \overbrace{u_1+v_1}^{\geq 0} \\ \underbrace{u_2+v_2}_{\geq 0} \end{bmatrix}$

so $\underline{u}+\underline{v} \in \mathbb{R}^2_+$ ✓

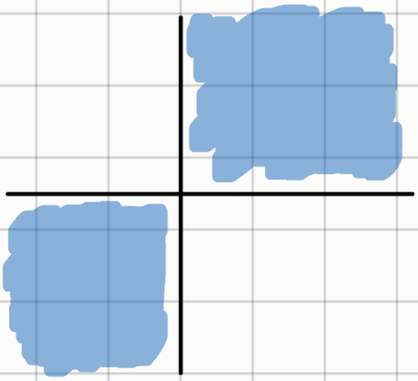
② Closed under scalar multiplication?

No, e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2_+$, but $(-3) \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \end{bmatrix} \notin \mathbb{R}^2_+$ ✗

So $\mathbb{R}^2_+$ is NOT a vector space

... is NOT a vector space

(EX) Is $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geq 0\}$ a vector space?



No, eg. $\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$\underline{u} + \underline{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(EX) Is P_n a vector space?

↳ set of polynomials of degree at most n

(ie. all polynomials of the form $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$)

(1) let $\underline{p}, \underline{q} \in \mathbb{P}_n$

$\underline{p}(x) = a_0 + a_1 x + \ldots + a_n x^n$

$\underline{q}(x) = b_0 + b_1 x + \ldots + b_n x^n$

$(\underline{p} + \underline{q})(x) = \underline{p}(x) + \underline{q}(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$

so, $\underline{p} + \underline{q} \in \mathbb{P}_n$ ✓

(2) let $\underline{p} \in \mathbb{P}_n$ an $c \in \mathbb{R}$

$(c \cdot \underline{p})(x) = c \cdot \underline{p}(x) = c(a_0 + a_1 x + \ldots + a_n x^n)$

$= (c \cdot a_0) + (c \cdot a_1)x + \ldots + (c \cdot a_n)x^n$

so $c \cdot \underline{p} \in \mathbb{P}_n$ ✓

(3)-(10) DIY

But what is $\underline{0}$? $\underline{0}(x) = 0$ for every x

---

(EX) set of polynomials in the form
$p(x) = 8 + a_1 x + a_2 x^2 + \dots + a_n x^n$

(1) ✗ ⟹ so NOT a vector space

(2) ✗

Similarly: $\mathbb{P}$ is also a vector space
↳ set of all polynomials

Let V be a vector space and let $\emptyset \neq W \subseteq V$
When is W also a vector space?
(1) $\underline{w_1}, \underline{w_2} \in W \Rightarrow \underline{w_1} + \underline{w_2} \in W$ (closed under addition)
(2) $\underline{w} \in W$, $c \in \mathbb{R} \Rightarrow c \cdot \underline{w} \in W$ (closed under scalar mult)
(3) $\underline{0} \in W$ (if $w \neq \emptyset$, then (3) follows from (2))

(All the other axioms are fullfiled because $W \subseteq V$ and V is a vector space)

W is called a subspace of V.

(EX) Vector space V. $W \neq \emptyset$ Is W a vector space?
$\emptyset \neq W \subseteq V$

① ✓
② ✓
③ ✗ $\rightarrow$ no, W is NOT a vector space

(EX) $\mathbb{R}^3$ is a vector space.

$$W = \left\{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\} \qquad W \subseteq \mathbb{R}^3$$

① ✗
② ✗
③ ✗ $\rightarrow$ no W is NOT a vector space

(EX)

$$W = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} \qquad W \subseteq \mathbb{R}^3$$

① if $\underline{w_1}, \underline{w_2} \in W$, then $\underline{w_1} + \underline{w_2} = \begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix}$

② ✓

③ ✓

$$= \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \end{bmatrix} \in W \checkmark$$

So, W is a subspace of $\mathbb{R}^3$

(EX) $W = \text{Span}\left(\left\{\begin{bmatrix} 2 \end{bmatrix}\right\}\right)$ Subspace of $\mathbb{R}^3$?

$$\left\{\begin{bmatrix} 4 \\ 7 \end{bmatrix}\right\}$$

(W is a line in $\mathbb{R}^3$)

⓪ ✓ $W \subseteq \mathbb{R}^3$

① ✓

② ✓

③ ✓

Yes, W is a subspace of $\mathbb{R}^3$

Theorem: Let V be a vector space.
If $\underline{v_1}, \dots, \underline{v_p} \in V$, then $W = \text{Span}\{\underline{v_1}, \dots, \underline{v_p}\}$ is a subspace of V.

Proof:

⓪ we need to show $W \subseteq V$
Use that V is a vector space.
So, $w_1 \cdot \underline{v_1} \in V, \dots, w_p \cdot \underline{v_p} \in V$ (because V is closed under scalar multiplication)
And also $w_1 \underline{v_1} + \dots + w_p v_p \in V$ (because V is closed under addition)

① If $\underline{u}, \underline{v} \in W$, then $\underline{u} + \underline{v} = c_1 \underline{v_1} + \dots + c_p \underline{v_p} + d_1 \underline{v_1} + \dots + d_p \underline{v_p} = (c_1 + d_1)\underline{v_1} + \dots + (c_p + d_p)\underline{v_p}$

So, $\underline{u} + \underline{v} \in W$

② If $\underline{u} \in W$, $c \in \mathbb{R}$, then $c \cdot \underline{u} = c \cdot (c_1 \underline{v_1} + \dots + c_p v_p)$
$= (c \cdot c_1)\underline{v_1} + \dots + (c \cdot c_p) \cdot \underline{v_p}$

So, $c \cdot \underline{u} \in W$

③ $\underline{0} = 0 \cdot \underline{v_1} + \ldots + 0 \cdot \underline{v_p}$ So, $\underline{0} \in W$

Theorem: Let A be an $m \times n$ matrix. The set of all solutions of $A\underline{x} = 0$ is a subspace of $\mathbb{R}^n$

$= \text{Nul}(A)$

$= W$

Proof:

⓪ A is an $m \times n$, so $\underline{x}$ needs to have n elements, so, $W \subseteq \mathbb{R}^n$

① If $\underline{u}, \underline{v} \in W$, then $A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{0} = \underline{0}$

② If $\underline{u} \in W$, $c \in \mathbb{R}$, then $A(c \cdot \underline{u}) = c(A\underline{u}) =$
$= c \cdot \underline{0} = \underline{0}$

③ $A\underline{0} = \underline{0}$ So, $\underline{0} \in W$

Three important subspaces associated with an $m \times n$ matrix A.

① $\text{Nul}(A) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{0}\}$ is a subspace of $\mathbb{R}^n$

$$[A | \underline{0}] = \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -6 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underline{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$Nul(A) = Span\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}\right\}$$

Note:

- # vectors in the spanning set = # free var.
- the vectors in the spanning set are always lin. independent.
- $A \sim B \Rightarrow Nul(A) = Nul(B)$

② $Col(A) = Span\{\underline{a_1}, \dots, \underline{a_n}\}$ is a subspace of $\mathbb{R}^m$

Be careful $A = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} = B$

$$Col(A) = Span\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\} = Span\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

$$\neq$$

$$Span\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} = Col(B)$$

Hence, $A \sim B \not\Rightarrow Col(A) = Col(B)$

③ $Row(A) = Span\{r_1, \dots, r_m\}$ is a subspace of $\mathbb{R}^n$

$= Col(A^T)$

Note: $\boxed{A \sim B \Rightarrow Row(A) = Row(B)}$

idea: $A \sim B$

$\Rightarrow$ rows of B are lin. combinations of the rows of A

$\Rightarrow$ any lin. comb. of the rows of B are also lin. comb. of the rows of A

$\Rightarrow \mathrm{Row}(B) \subseteq \mathrm{Row}(A)$

$A \sim B \Rightarrow B \sim A \Rightarrow \ldots \Rightarrow \mathrm{Row}(A) \subseteq \mathrm{Row}(B)$