Sequences

$\{a_n\}$ = list of numbers $a_1, a_{2}, \dots, a_n$ in a given order

term

index

a sequence can be seen as a function : $f\ \mathbb{N} \to \mathbb{R}$:
$n \to a_n$

$\sqrt{n}$: $1, \sqrt{2}, \sqrt{3}, \dots$

$\frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \dots$

$(-1)^n$: $-1, 1, -1, \dots$

$\frac{(-1)^n \cdot x^{2n}}{(2n)!}$: $1, \frac{-x^2}{2}, \frac{x^4}{4!}, \frac{-x^6}{6!}, \frac{x^8}{8!}, \dots$

$n=0$

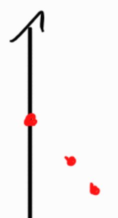
$a_1 = 1$
$a_2 = 1$
$a_n = a_{n-1} + a_{n-2}$ Fibonacci sequence

$a_3 = a_2 + a_1 = 2$
$a_4 = a_3 + a_2 = 3$

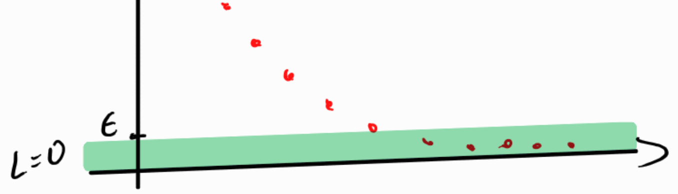
↳ Does the sequence converge ?

$a_n \to L$ if $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ such that
if $n > N$, $|a_n - L| < \epsilon$

$a_n = \frac{1}{n}$ $\frac{1}{n} \to 0$



converges



* if a_n does not converge, it diverges

* if $a_n \to \infty$, it diverges to infinity

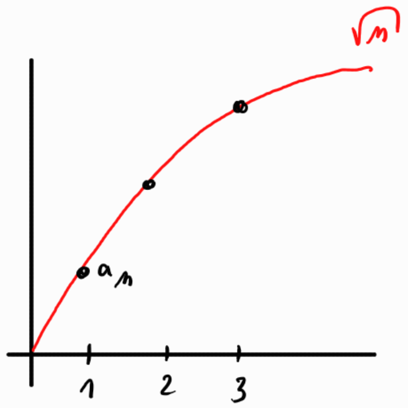
$\hookrightarrow \forall M > 0 \quad \exists N$ such that if $n > N$, $a_n > M$

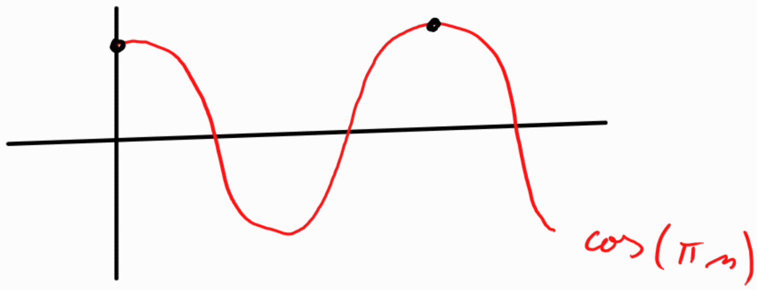
* if a sequence can be seen as a REAL function

if $f(x)$ $(x \in \mathbb{R})$ is defined, and $a_n = f(n)$

then $\lim\limits_{x \to n} f(x) = L \implies a_n \longrightarrow L$

oposite is not true





* for $a_n \to A$, $b_n \to B$

then $(a_n + b_n) \longrightarrow (A+B)$

$(a_n - b_n) \longrightarrow (A-B)$

$a_n \cdot b_n \longrightarrow A \cdot B$

$k \cdot a_n \longrightarrow k \cdot A$

* for $a_n \to L$, $f(x)$ continuous at L

then $f(a_n) \longrightarrow f(L)$

EX converges?

$$\left(1+\frac{1}{n}\right)^n \qquad \ln\left[\left(1+\frac{1}{n}\right)^n\right] = n \cdot \ln\left(1+\frac{1}{n}\right)$$

$$\lim_{x\to\infty}\left(x\cdot \ln\left(1+\frac{1}{x}\right)\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{H}{=}$$

$$\boxed{\lim_{x\to\pm\infty}\left(\frac{1}{x^p}\right)=0}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

$\hookrightarrow$ so $\left(1+\frac{1}{n}\right)^n \to e^1 = e$

* squeeze theorem for sequences

$a_n \le b_n \le c_n$ and $a_n \to L$, $c_n \to L$ then $b_n \to L$

$$-\frac{1}{n} \le \frac{\cos(n)}{n} \le \frac{1}{n} \quad \text{, so } \frac{\cos(n)}{n} \to 0$$

$-\frac{1}{n} \to 0$, $\frac{1}{n} \to 0$

* a sequence $\{a_n\}$ is

BOUNDED ABOVE: if $\exists M$ such that $\forall n \quad a_n \le M$

BELOW: if $\exists M$ such that $\forall n \quad a_n \ge M$

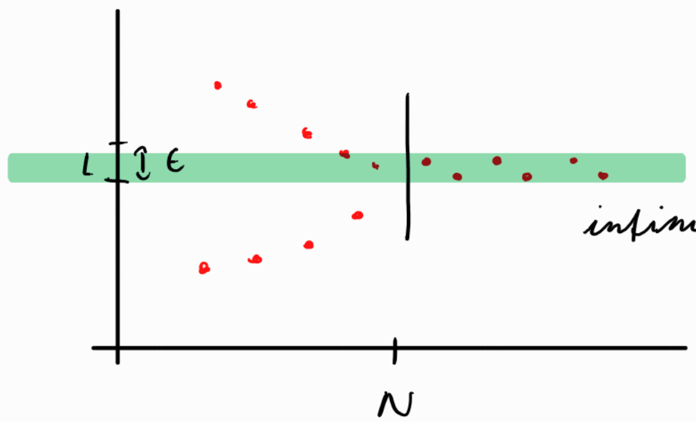
BOUNDED: bounded above and below

* increasing : if $\forall n$ $a_{n+1} > a_n$
decreasing : if $\forall n$ $a_{n+1} < a_n$ } monotonous
alternating : if $\forall n$ $sgn(a_n) = -sgn(a_{n+1})$
$(a_n \neq 0)$

positive / negative

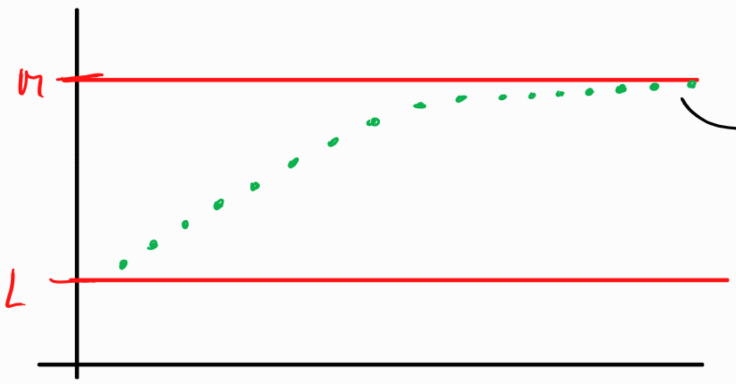
• every converging sequence is bounded:

$a_n \to L$



infinite num. of terms
but within $[L-\epsilon ; L+\epsilon]$

• a monotonous bounded sequence always converges



converging

INFINITE SERIES

series = a formal sum of infinitely many terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + a_{n+1} + \dots$$

↳ a series is a sequence of partial sums

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

$n \to \infty$

the series $\sum_{n=1}^{\infty} a_n$ converges if the sequence s_n converges

EX: <u>Geometric Series</u> $a_n = a \cdot r^{n-1}$, $r = \dfrac{a_{n+1}}{a_n}$

$$s_1 = a$$
$$s_2 = a + ar$$
$$s_3 = a + ar + ar^2$$
$$\vdots$$
$$s_n = a + ar + \dots + ar^{n-1}$$
$$s_n = a \cdot \frac{1 - r^n}{1 - n}$$

if $|r| < 1$, $r^n \to 0$ and $s_n \to \dfrac{a}{1-n}$

$r > 1$, $r^n \to +\infty$ and $s_n \to +\infty$

($-\infty$ if $a < 0$)

$r < -1$, r^n diverges

$n = 1$ diverges

EX: Harmonic Series $\quad a_n = \frac{1}{n}$

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Which of the following is true?

A If the sequence a_n converges, then the series Σa_n also converges

If the sequence a_n diverges, then the series Σa_n also diverges

If the sequence $a_n \to L > 0$, then the series Σa_n diverges

If the series Σa_n converges, then the sequence a_n also converges

E If $a_n \to 0$, then Σa_n converges

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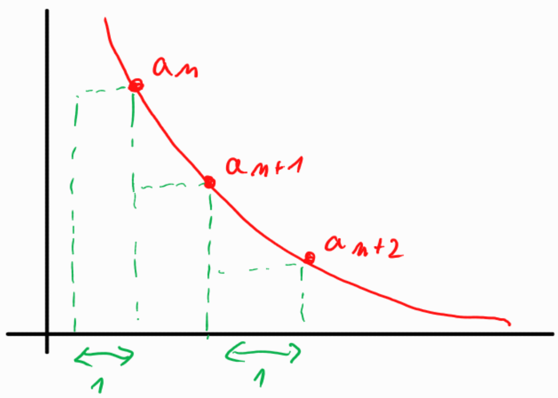
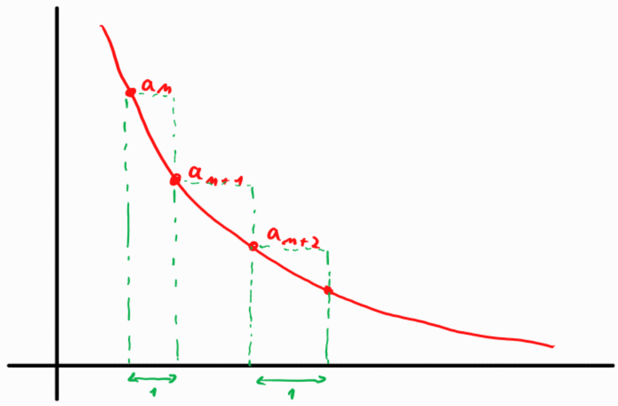
1) POSITIVE SERIES $(a_n \geq 0)$

1) Integral test $\quad$ if $a_n = f(n)$, f is non-increasing on $[N, \infty)$

then $\sum a_n \quad \int_N^{\infty} f(x)\,dx$ both diverge or both converge





$$\sum_{n \geq 1}^{n} \frac{1}{n} \longrightarrow \int_1^{\infty} \frac{dx}{x} = \ln|x| \Big|_1^{\infty} = \infty$$

$$\sum_{n>1} \frac{1}{n^2} \longrightarrow \int_1 \frac{dx}{x^2} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^2} = \lim_{R\to\infty} -\frac{1}{x}\Big|_1^R = 1$$

so $\sum_{n>1}^{n} \frac{1}{n^2}$ converges

$$\sum \frac{1}{\sqrt{n}} \longrightarrow \int_1^\infty \frac{dx}{\sqrt{n}} = \lim_{R\to\infty} \int_1^R \frac{dx}{\sqrt{x}} = \lim_{R\to\infty} 2\sqrt{x}\Big|_1^R = \infty$$

↳ diverges to ∞

2) Comparison test

if $\{a_n\}, \{b_n\}$ positive sequences

and $0 \le a_n \le b_n$, $k > 0$

- if $\sum a_n \to \infty$, then $\sum b_n \to \infty$
- if $\sum b_n$ converges, then $\sum a_n$ converges

$$\sum_{n=1}^\infty \frac{3n+1}{n^3} \le \sum_{n=1}^\infty \frac{3n+n}{n^3} = \sum_{n=1}^\infty \frac{4n}{n^3} = 4\sum_{n=1}^\infty \frac{1}{n^2}$$

↳ converges

converges

Ex

$$\sum_{n=2}^\infty \frac{1}{\sqrt{n}-1} \qquad \sqrt{n}-1 < \sqrt{n} \longleftrightarrow \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$$

compare to $\sum \frac{1}{\sqrt{n}}$

$\sum \frac{1}{n}$

if $\sum \frac{1}{\sqrt{n}}$ diverges,

$\sum \frac{1}{\sqrt{n}-1}$ also diverges to ∞

power < 1 try diverging	power > 1 try converging

3) <u>Limit comparison test</u>

for $\{a_n\}, \{b_n\}$ positive

if $\lim_{n \to \infty} \frac{a_n}{b_n} = L$ (can be ∞)

- if $L < +\infty$ and $\sum b_n$ converges, then $\sum a_n$ converges
- if $L > 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

Proof: if $L < +\infty$, $\exists N$ such that $\frac{a_n}{b_n} < L + 1$

$$a_n < (L+1) b_n$$

if $L > 0$, $\exists N$ such that $\frac{a_n}{b_n} > \frac{L}{2}$

$$b_n \leq \frac{2}{L} a_n$$

EX

$$\sum_{n=4} \frac{n+5}{n^3 - 2n + 3}$$

$$a_n = \frac{n+5}{n^3 - 2n + 3} \qquad b_n = \frac{1}{n^2}$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n+5}{n^3 - 2n + 3} \cdot n^2 = 1$$

$0 < 1 < \infty \Rightarrow \sum \frac{n+5}{n^3 - 2n + 3}$ and $\sum \frac{1}{n^2}$ behave the same

$\Rightarrow$ both converge!

$$\sum \frac{\ln(n)}{n^2}$$

- compare to $\sum \frac{1}{n^2}$ $\lim\limits_{n\to\infty} \frac{\ln(n)}{\cancel{n^2}} \cancel{n^2} = +\infty$ not working

- compare to $\sum \frac{1}{n}$ $\lim\limits_{n\to\infty} \frac{\ln(n)}{n^{\cancel{2}}} \cancel{n} = 0$ not working

$1 < \ln(n) < n$

* compare to $\sum \frac{1}{\sqrt{n}\, n}$ (converges) $\lim\limits_{n\to\infty} \frac{\ln(n)}{n^2} \sqrt{n}\, n =$

$$= \lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} \overset{H}{=} 0$$

$\Rightarrow \sum \frac{\ln(n)}{n^2}$ converges

Ratio test (comparing to geometric series)

for positive $\{a_n\}$

if $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$

if $\rho < 1$; $\sum a_n$ converges

if $\rho > 1$; $\sum a_n$ diverges

if $\rho = 1$, no info, try different test

$$\lim_{n\to\infty} (a_n)^{1/n} = \sigma$$

$\sigma < 1$: $\sum a_n$ converges

$\sigma > 1$: $\sum a_n$ diverges

$\sigma = 1$: no info

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

if $\rho < 1$, choose $\rho < r < 1$

$\exists N \quad n \geq N, \ \frac{a_{n+1}}{a_n} < 1 \Rightarrow a_{n+1} < r \cdot a_n$

$a_{n+k} < r^k \cdot a_n$

↳ converging geo. series

if $\rho > 1$, choose $r > \rho > 1$

$\exists N \quad n \geq N \quad \frac{a_{n+1}}{a_n} > r \Rightarrow a_{n+1} > r \cdot a_n$

$a_{n+k} > r^k \cdot a_n$

↳ diverging geo. series

Ex $\sum_{n=1} \frac{1}{n!}$

↳ ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} =$

$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$0 < 1$, so $\sum \frac{1}{n!}$ converges

$$\sum_{n=1} \frac{2^n}{n!}$$

↳ ratio test: $\lim\limits_{n \to \infty} \dfrac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim\limits_{n \to \infty} \dfrac{2}{n+1} = 0$

$0 < 1$, so $\sum \frac{2^n}{n!}$ converges