A **basis** of a vector space V is a set $\{\underline{v}_1, \ldots, \underline{v}_p\}$ of vectors V such that:

→ $\{\underline{v}_1, \ldots, \underline{v}_p\}$ is lin. indep.

→ it spans the entire vector space: $\text{Span}\{\underline{v}_1, \ldots, \underline{v}_p\} = V$

EX A basis for $\mathbb{R}^3$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ ← standard basis

But also $\left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right\}$ why?

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 6 & 7 & 5 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* pivot in every col → lin indep.
* pivot in every row → spans $\mathbb{R}^3$

EX The standard basis for $\mathbb{P}_n$ is: $\{1, t, t^2, \ldots, t^n\}$

But for $\mathbb{P}_2$ also: $\{1+t^2,\ t-3t^2,\ 1+t-3t^2\}$ (see later)

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2 views of basis B:

* it is **maximal**: adding any vector $\underline{v} \in V$ to B, makes B lin dep.
* it is **minimal**: removing any vector $\underline{v}$ from B, causes B no longer span V

dim(V) = # of vectors in a basis of V

$\dim(\mathbb{R}^3) = 3$ $\dim(\mathbb{P}_n) = n+1$

EX $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right\}$ basis for $\mathbb{R}^3$

$\underline{x} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$ How to find the weights?

$$\left[\begin{array}{ccc|c} 3 & -4 & -2 & 5 \\ 0 & 1 & 1 & 2 \\ -6 & 7 & 5 & 4 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

So, $[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -6 \\ 8 \end{bmatrix}$

Let $\mathcal{B} = \{ b_1, \dots, b_n \}$ be a basis for V:
For every $\underline{x} \in V$, we can find weights $\underbrace{c_1, \dots, c_n}$
such that $\underline{x} = c_1 \underline{b_1} + c_2 \underline{b_2} + \dots + c_n \underline{b_n}$

coordinates of $\underline{x}$ with the basis $\mathcal{B}$

Notation: $[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \end{bmatrix}$

$$\begin{bmatrix} \vdots \\ c_n \end{bmatrix}$$

Note: this set of coordinates is unique

(Ex) $B = \{1+t^2,\ t-3t^2,\ 1+t-3t^2\}$ is a basis for $\mathbb{P}_2$

$p(t) = 6+3t-t^2$ Find $[p]_B$

$$6+3t-t^2 = c_1(1+t^2) + c_2(t-3t^2) + c_3(1+t-3t^2)$$

$$\left.\begin{aligned} c_1 \quad\quad\quad + c_3 &= 6 \\ c_2 + c_3 &= 3 \\ c_1 - 3c_2 - 3c_3 &= -1 \end{aligned}\right\} \text{SLE}$$

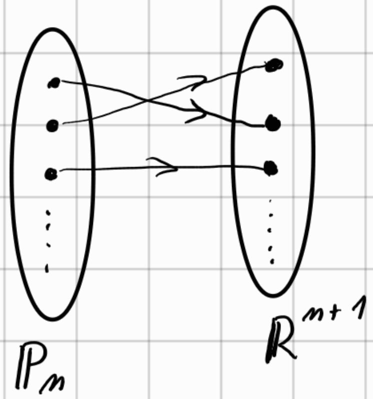
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 1 & -3 & -3 & -1 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

So, $[p]_B = \begin{bmatrix} 8 \\ 5 \\ -2 \end{bmatrix}$

check: $p(t) = 8(1+t^2) + 5(t-3t^2) - 2(1+t-3t^2)$ ✓

Theorem: Let $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ be a basis for V. Then, the lin. transformation $T: V \to \mathbb{R}^n$ defined by $T(\underline{x}) = [\underline{x}]_B$ is a bijection

Hence, vectors can be abstract objects, but using coordinates, we can translate them to real numbers.



one-to-one

(EX) Show that $\{1+t^2, t-3t^2, 1+t-3t^2\}$ form a basis for $\mathbb{P}_2$. The coordinate mapping of the standard basis $\{1, t, t^2\}$ produces the coordinate vectors:

$$\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix}, \begin{bmatrix}1\\1\\-3\end{bmatrix}}_{\text{form a basis in } \mathbb{R}^3 \text{ ?}}$$

YES

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & -3 & -3\end{bmatrix} \sim \dots \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

So, $\{1+t^2, t-3t^2, 1+t-3t^2\}$ forms a basis for $\mathbb{P}_2$

What about spaces and dimensions for Nul(A), Col(A) and Row(A) ?

$\text{Nul}(A) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{0}\}$

$\underline{x} = x_2\begin{bmatrix}3 \\ \dots\end{bmatrix} + x_3\begin{bmatrix}-2 \\ \dots\end{bmatrix}$ $\qquad$ $\text{Nul}(A) = \text{Span}\left\{\begin{bmatrix}3 \\ \dots\end{bmatrix}, \begin{bmatrix}-2 \\ \dots\end{bmatrix}\right\}$

$$\begin{bmatrix} \vdots \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \left\{ \begin{bmatrix} \vdots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis for $Nul(A)$: $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad \dim(Nul(A)) = 2$

In general:

- → $\dim(Nul(A)) = \#$ free var.
- → the vectors from parametric vector form, form a basis for $Nul(A)$

The Spanning set theorem:

let $\{\underline{v}_1, \ldots, \underline{v}_p\} \in V$ and let $H = Span\{\underline{v}_1, \ldots, \underline{v}_p\}$

* if $\underline{v}_k$ is a lin. combination of the other vectors, then $\{\underline{v}_1, \ldots, \underline{v}_p\} \setminus \{\underline{v}_k\}$ still spans H

* if $H \neq \{\underline{0}\}$, then some subset of $\{\underline{v}_1, \ldots, \underline{v}_p\}$ spans H

$Col(A) = Span\{\underline{a}_1, \ldots, \underline{a}_n\}$

* 1^{st} strategy (based on Spaning Set Theorem): keep deleting cols if they are lin. dep. on others. The cols that you end up with form a basis for $Col(A)$

Easy when A is in REF.

$B = \begin{bmatrix} 1 & 3 & 0 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{b}_1 \quad \underline{b}_2 \quad \underline{b}_3 \quad \underline{b}_4$

$$\left.\begin{array}{l} \underline{b}_4 = -1\,\underline{b}_1 - 2\,\underline{b}_3 \\ \underline{b}_2 = 3\underline{b}_1 \end{array}\right\} \text{remove } \underline{b}_2, \underline{b}_4$$

So, $\underline{b}_1$ and $\underline{b}_3$ form a basis and $\dim(\mathrm{Col}(B)) = 2$

pivot cols (true in general)

Linear dependance relation between cols do not change under row operations

If $A \sim B$, then $A\underline{x} = \underline{0}$ and $B\underline{x} = \underline{0}$ have the same sol. set

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 2 & 6 & -1 & 0 \\ 1 & 3 & 0 & -1 \end{bmatrix} \sim \dots \sim B$$

So we also have

$\underline{a}_2 = 3\underline{a}_1$

$\underline{a}_4 = -1\underline{a}_1 - 2\underline{a}_3$

So, $\{\underline{a}_1, \underline{a}_3\}$ forms a basis and $\dim(\mathrm{Col}(A)) = 2$

- 2^{nd} strategy: the pivot cols of A form a basis for $\mathrm{Col}(A)$

↳ be careful ! of A ! NOT REF !

Hence, $\dim(\mathrm{Col}(A)) = \#$ pivot cols

$\text{Row}(A) = \text{Span}\{\underline{r_1}, \dots, \underline{r_m}\}$ and $A \sim B \Rightarrow \text{Row}(A) = \text{Row}(B)$

If B is RREF, then the nonzero rows of B are lin. indep. and thus form a basis for both $\text{Row}(B)$ and $\text{Row}(A)$

$$B = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\dim(\text{Row}(A)) = \#$ pivot rows

$\text{Rank}(A) = \dim(\text{Col}(A)) = \#$ pivot cols $= \#$ pivot rows
$= \dim(\text{Row}(A))$

$\#$ pivot col $= n - \#$ nonpivot col
$= n - \#$ free var
$= \boxed{n - \dim(\text{Nul}(A))}$

Recall a lin. transformation $T: \underline{x} \to A\underline{x}$

$A: m \times n$
domain: $\mathbb{R}^n$
codomain: $\mathbb{R}^m$

$*$ Range $= \{T(\underline{x}) : \underline{x} \in \mathbb{R}^n\} = \{A\underline{x} : \underline{x} \in \mathbb{R}^n\} = \text{Col}(A)$

Suppose, T is surjective $\Leftrightarrow$ Range $=$ codomain
$\Leftrightarrow \text{Col}(A) = \mathbb{R}^m$

$\iff \dim(\text{Col}(A)) = m \iff \text{Rank}(A) = m$

* Kernel $= \{\underline{x} \in \mathbb{R}^n : T(\underline{x}) = \underline{0}\} = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{0}\} = \text{Nul}(A)$

So, T is injective $\iff T(\underline{x}) = \underline{0}$ has only the trivial sol. $\iff$ Kernel $= \{\underline{0}\} \iff \text{Nul}(A) = \{\underline{0}\}$
$\iff \dim(\text{Nul}(A)) = 0 \iff \text{Rank}(A) = n$
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