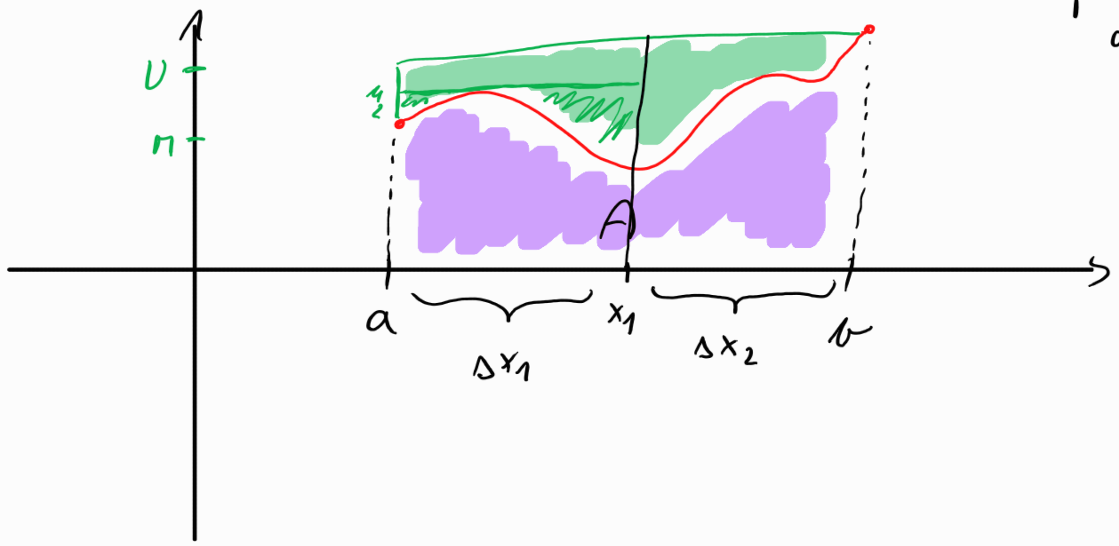
1) Definite integrals = area under the graph

2) Indefinite integrals = antiderivatives

3) Fundamental theorem of Calculus

Definite Integrals

$f(x)$ continuous on $[a, b]$



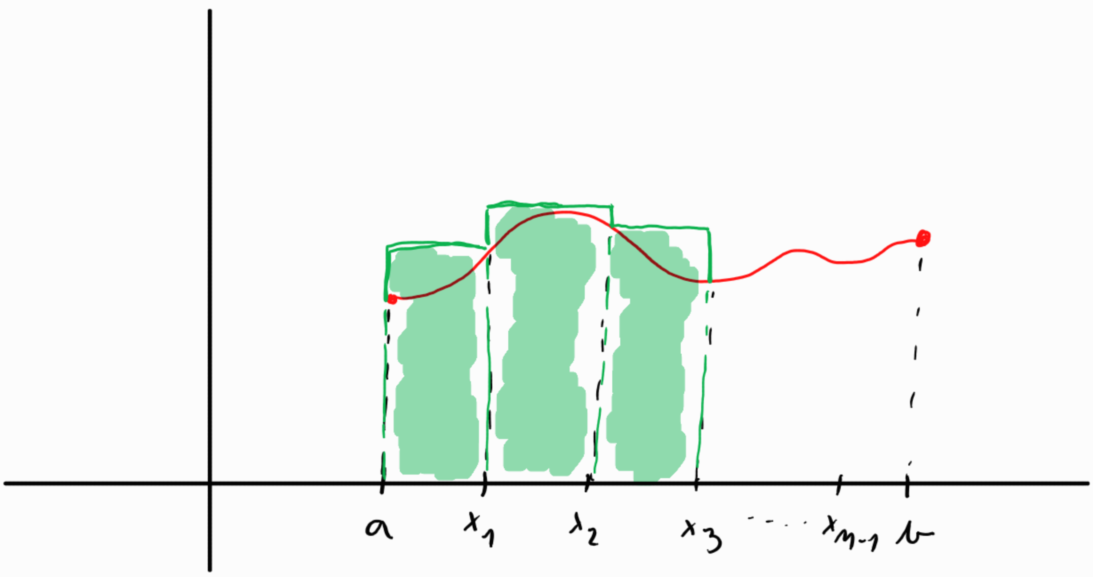
→ upper and lower boundaries of A?

upper boundary $(b-a)\cdot U \geq A$

$\Delta x_1 \cdot M_1 + \Delta x_2 \cdot M_2 \geq A$

↳ $a = x_0 < x_1 < x_2 < \dots < x_n$

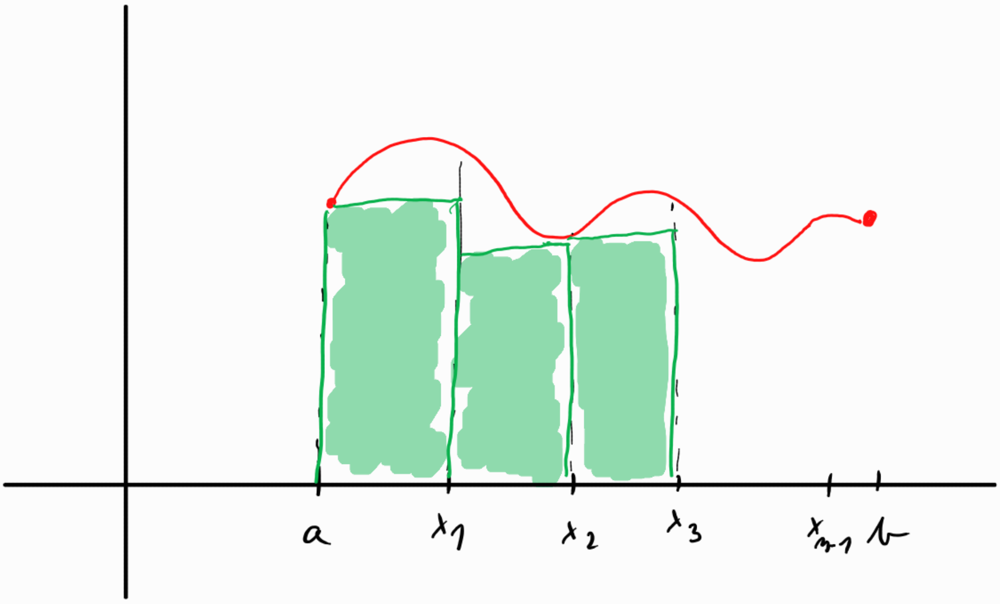
↳ "partition P" of $[a,b]$



$U(f,P) \geq A$

→ upper Riemann sum:

$$U(f,P) = \sum_{k=1}^{n} \Delta x_k \cdot U_k$$



$L(f,P) \leq A$

$$L(f,P) = \sum_{k=1}^{n} \Delta x_k \cdot e_k$$

+ if there is a unique value A, such that for all partitions P, $L(f,P)' \leq A \leq U(f,P)$ then, f is integrable and

$$A = \int_a^b f(x)\,dx$$

↳ $$\lim_{\substack{\|P\| \to 0 \\ n \to \infty}} U(f,P) = \lim_{\substack{\|P\| \to 0 \\ n \to \infty}} L(f,P) = \int_a^b f(x)\,dx$$

$$\|P\| = \max(\Delta x_n)$$

egration
imits

$$\int^{b} f(x)\, dx$$

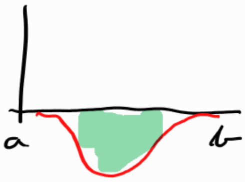
→ differential $(\Delta x \to 0)$

a

integrand

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

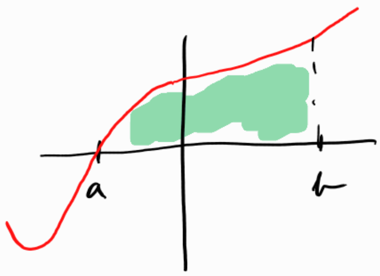
$$\int_a^b f(x)\,dx < 0 \implies$$



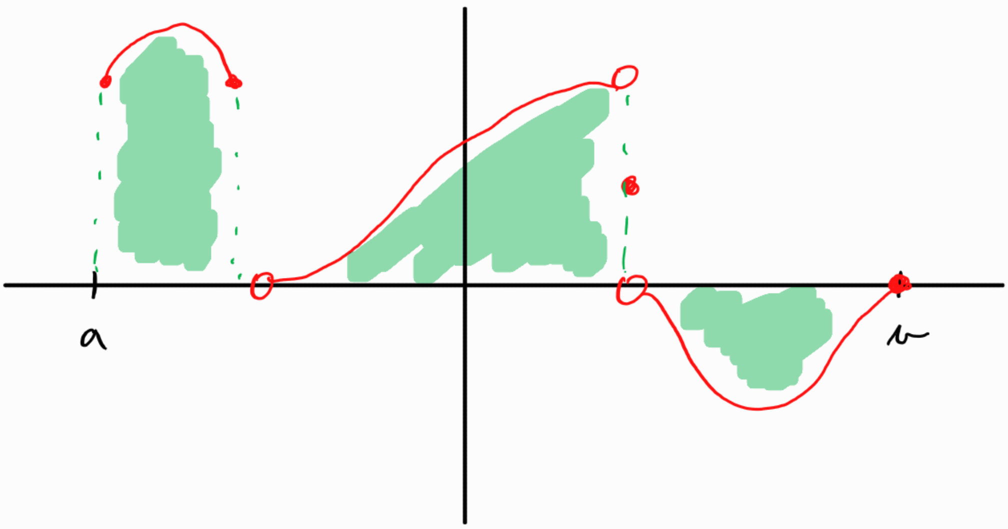
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$f(x) = 3x+2 \qquad x \geq 0$$

$$f(x) = 2\cos(x) \qquad x < 0$$



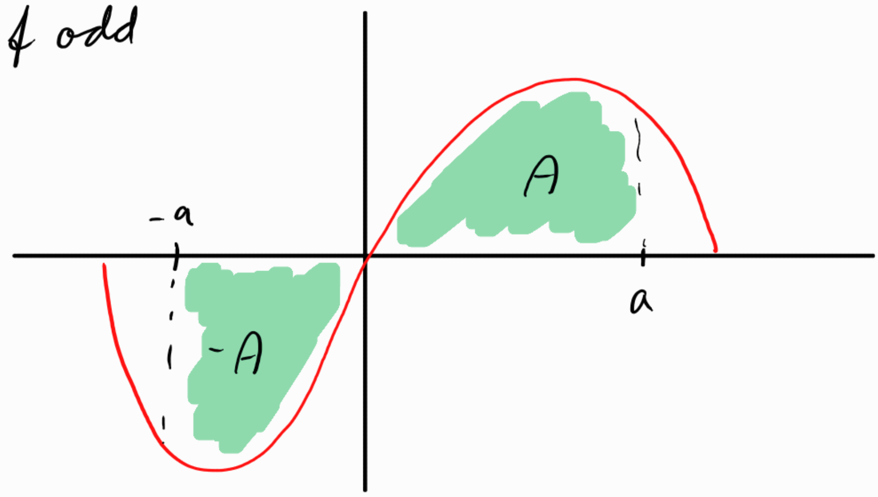
$$\int_a^b f(x)\,dx = \int_a^0 2\cos(x)\,dx + \int_0^b (3x+2)\,dx$$



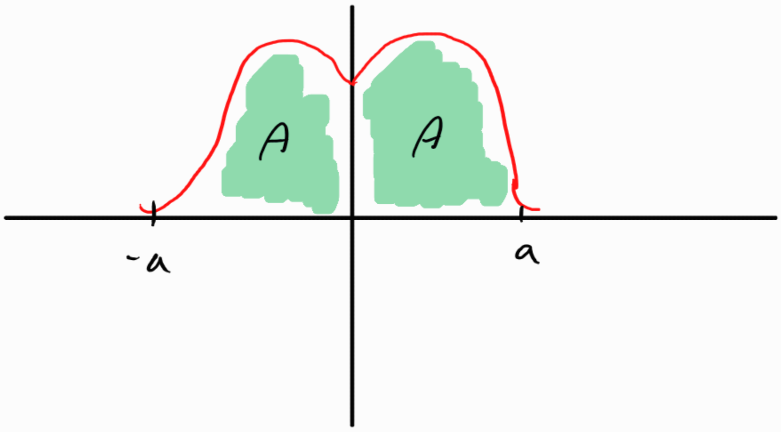
piecewise continuous f (functions continuous on the whole domain except a finite num. of points)

- $\int_{-a}^{a} f(x)\,dx = 0$ for $f(x)$ odd

f odd

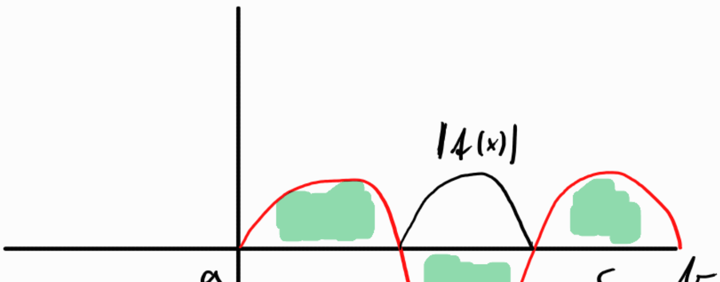


- $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ for $f(x)$ even



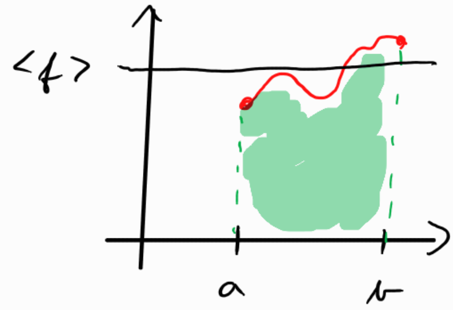
$$\int_{b}^{a} |f(x)|\,dx \geq \left| \int_{b}^{a} f(x)\,dx \right|$$

$A + |B| + C$ $\qquad$ $|A+B+C|$



• *mean of a function*:

$$\langle f \rangle = \bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx$$



$$\langle f \rangle (b-a) = \int_a^b f(x)\,dx$$

$$\exists c \in [a,b] : \quad f(c) = \langle f \rangle$$

$$\int_{-1}^{1} \frac{dx}{|x|}$$ *integral* <u>*not defined*</u> *at* $x=0$

$$\int_0^1 \frac{dx}{|x|} \rightarrow \lim_{a \to 0} \int_a^1 \frac{dx}{|x|}$$

$f(x) = 1$ for $x \in Q$

$f(x) = 0$ for $x \notin Q$

$$\int_0^1 f(x) = \text{DOES NOT EXIST}$$

INDEFINITE INTEGRALS

$\int_a^b f(x)\,dx$ → *definite integral* → NUMBER

$\int g(x)\,dx$ → *indefinite integral* → FUNCTION $F(x)$

$$\frac{d}{dx} F(x) = f(x)$$

$F(x) + C \quad \frac{d}{dx}\left(F(x)\right) = f(x)$

integration constant

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int x\,dx = \frac{x^2}{2} + C$$

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

* Fundamental Theorem of Calculus

function f continuous on an interval I, $a \in I$

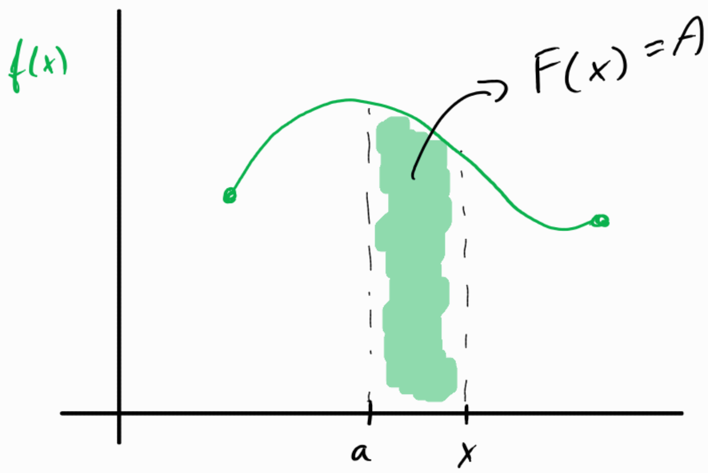
- $F(x) = \int_a^x f(t)\,dt$

  then $F(x)$ is differentiable on I

  and $\frac{d}{dx}\left(F(x)\right) = f(x)$

- if for a $g(x)$, $g'(x) = f(x)$

  then $\int_a^b f(t)\,dt = g(b) - g(a)$

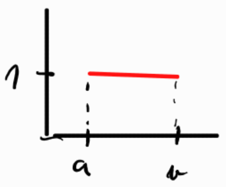


- $\int_2^0 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{2^2}{2} - 0 = 2$

- $\int dx = \int 1 \cdot dx = x + C$

$\Downarrow$

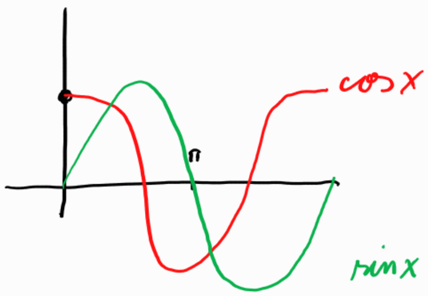
$f(x)$ (which function has derivative $= 1$)



$\int_a^b dx = 1 \cdot (b-a)$

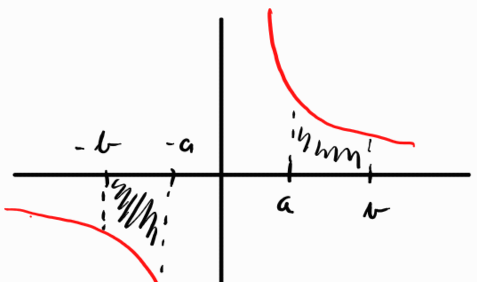
- $\int x^r dx \quad (r \neq -1) \quad = \frac{x^{r+1}}{r+1} + C$

- $\int \cos x\,dx = \sin x + C$



- $\int e^x dx = e^x + C$

- $\int \frac{dx}{x} = \int \frac{1}{x} dx = \ln|x| + C$



$\int_a^b \frac{dx}{x} = \int_{-a}^{-b} \frac{dx}{x}$

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot(-1)$$
$x<0$
$$= \frac{1}{x}$$

- not every integral can be solved!

  eg. $\int e^{-x^2}dx$

1) SUBSTITUTION

$$\underbrace{\int \frac{d}{dx}\left(f(g(x))\right)dx}_{f(g(x))+C} = \int f'(g(x))\,g'(x)\,dx \rightarrow \text{chain rule}$$

$$f(g(x)) + C = \int f'(u)\,du$$
$$f(u) + C$$

$g(x) = u$
$g(x)dx = du$

EX.
$$\int \sin(ax)\,dx = \int \sin u \,\frac{du}{a} = \frac{1}{a}\int \sin u \, du$$
$u = ax$
$du = a\cdot dx$
$$= \frac{1}{a}\left(-\cos u + C\right) \quad \left(= \frac{1}{a}(-\cos u) + C\right)$$
$$= -\frac{1}{a}\cos(ax) + C$$

$$\int_0^{\pi/2} \sin(2x)\,dx = \frac{1}{2}\int_0^{\pi}\sin(u)\,du = \frac{1}{2}\left[-\cos(u)\right]_0^{\pi}$$
$u = 2x$
$$= \frac{1}{2}\left(1-(-1)\right) = 1$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\cdot\int\frac{a\,dx}{ax+b} = \frac{1}{a}\int\frac{du}{u} = \frac{1}{2}\ln|u|$$
$u = ax+b$
$du = a\cdot dx$
$$= \frac{1}{2}\ln|ax+b|$$

$$\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$u = \cos x$

$du = -\sin x \cdot dx$

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln(x^2+1) + C$$

$u = x^2+1$

$du = 2x\,dx$

---

2) INTEGRATION BY PARTS

product rule: $\int \frac{d}{dx}(U \cdot V)\,dx = \int U'V\,dx + \int UV'\,dx$

$$U \cdot V = \int V\,dU + \int U\,dV$$

$$\int U\,dV = U \cdot V - \int V\,dU$$

EX. $\int \underbrace{x}_{U}\underbrace{e^x dx}_{dV} = x \cdot e^x - \int e^x dx = xe^x - e^x + C$

$U = x \qquad dU = dx$

$dV = e^x dx \qquad V = e^x$

$\int dx$

$\int \underbrace{\ln(x)}_{U}\underbrace{dx}_{dV} = x \cdot \ln(x) - \int \cancel{x}\frac{dx}{\cancel{x}} = x \cdot \ln(x) - x + C$

$U = \ln(x) \qquad dU = \frac{dx}{x}$

$dV = dx \qquad V = x$

$\int x\,\underbrace{\sin x\,dx} = -\cos x \cdot x - \int \cos x\,dx = -x \cdot \cos x + \sin x + C$

$$U = x \qquad dU = dx$$

$$dV = \sin x \, dx \qquad V = -\cos x$$

---

3) RATIONAL FUNCTIONS

$$R(x) = \frac{P(x)}{Q(x)}$$

1) If degree of $P(x) \geq Q(x)$, do division

$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{P_2(x)}{Q(x)} \quad \text{with degree } P_2 < \text{degree } Q$$

easy to integrate

$$\frac{x^2+2}{x^2+2x+4} = \frac{(x^2+2x+4)+(-2x-2)}{x^2+2x+4} = 1 - 2\,\frac{x+1}{x^2+2x+4}$$

2) $Q(x) = (x-x_1)(x-x_2)\dots(x-x_n)$ factorise $Q(x)$

3) $$\frac{P_2(x)}{Q(x)} = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \frac{A_3}{x-x_3} + \dots + \frac{A_n}{x-x_n}$$

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A_1}{x-2} + \frac{A_2}{x+2} =$$

$$= \frac{A_1(x+2) + A_2(x-2)}{(x+2)(x-2)}$$

$$\Rightarrow \underbrace{(A_1+A_2)}_{0}x + 2\underbrace{(A_1-A_2)}_{\frac{1}{2}} = 1$$

$$\Rightarrow A_1 = -A_2$$

$$2(A_1+A_1)=4A_1=1 \Rightarrow A_1=\frac{1}{4}, \quad A_2=-\frac{1}{4}$$

$$\frac{1}{x^2-4}=\frac{1}{4}\cdot\frac{1}{x-2}-\frac{1}{4}\cdot\frac{1}{x+2}$$

* $Q(x)=(x-x_1)^2$

$$\frac{P_2(x)}{Q(x)}=\frac{A_1}{x-x_1}+\frac{A_2}{(x-x_1)^2}$$

* $Q(x)=(x-x_1)(x^2+bx+c)$

$$\frac{P_2(x)}{Q(x)}=\frac{A_1}{x-x_1}+\frac{Bx+C}{x^2+bx+c}$$

EX.

$$\int\frac{1}{x^2-4}=\int\left(\frac{1}{4(x-2)}-\frac{1}{4(x+2)}\right)dx=\frac{1}{4}\int\frac{dx}{x-2}-\frac{1}{4}\int\frac{dx}{x+2}=$$

$$=\frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+C$$

$$=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C$$

$$\int\frac{dx}{x^2-5x+6}=\int\frac{dx}{(x-2)(x-3)}=\frac{A_1}{x-2}+\frac{A_2}{x-3}=\frac{A_1(x-3)+A_2(x-2)}{(x-2)(x-3)}=$$

$$\Rightarrow A_1(x-3)+A_2(x-2)=1$$

$$A_1x-3A_1+A_2x-2A_2=1$$

$$x\underbrace{(A_1+A_2)}_{0}-3A_1-2A_2=1$$

$$\begin{cases} A_1+A_2=0 \quad \Rightarrow A_1=-A_2 \end{cases}$$

$$-3A_1 - 2A_2 = 1$$

$$3A_2 - 2A_2 = 1$$

$$A_2 = 1 \qquad A_1 = -1$$

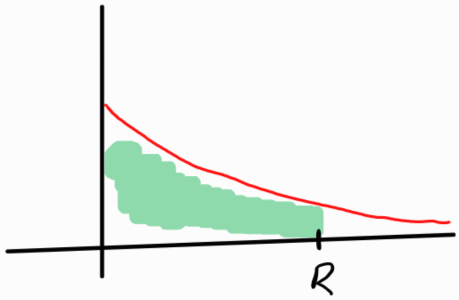
$$-\frac{1}{x-2} + \frac{1}{x-3} \qquad \int\left(-\frac{1}{x-2}\right)dx + \int\left(\frac{1}{x-3}\right)dx =$$

$$= -\ln|x-2| + \ln|x-3| + C$$

$$= -\ln\left|\frac{x-3}{x-2}\right| + C$$

IMPROPER INTEGRAL

$$\int_a^{\infty} f(x)\,dx \qquad \lim_{x \to a} f(x) = \pm\infty \qquad \int_a^b f(x)\,dx$$



* improper integrals might converge / diverge to ∞

* how to calculate?

$$\int_0^{\infty} e^{-x}\,dx = \lim_{R\to\infty} \int_0^R e^{-x}\,dx = \lim_{R\to\infty} \left[-e^{-x}\right]\Big|_0^R =$$

$$= \lim_{R\to\infty} \left(-e^{-R} + 1\right) = 1$$

converging (limit exists)

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_1^R \frac{dx}{x^2} = \lim_{R\to\infty} \left(-\frac{1}{x}\right)\Big|_1^R =$$

$$= \lim_{R\to\infty} \left(\frac{-1}{R} + 1\right) = \underline{\underline{1}}$$

$$\int_1^{+\infty} \frac{dx}{\sqrt{x}} = \lim_{R\to\infty} \int_1^R \frac{dx}{\sqrt{x}} = \lim_{R\to\infty} \left(2\sqrt{x} \Big|_1^R \right) = \lim_{R\to\infty} \left(2\sqrt{R} - 2\right) = \underline{\underline{\infty}}$$

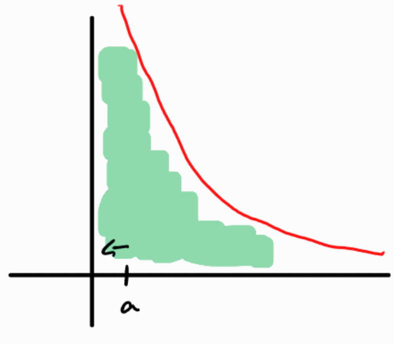
<u>diverging</u> to ∞

$$\int_0^{\infty} \sin x \, dx = \lim_{R\to\infty} \int_0^R \sin x \, dx = \lim_{R\to\infty} (-\cos x)\Big|_0^R =$$

$$= \lim_{R\to\infty} \left(-\cos R + 1\right) = DNE$$

<u>diverging</u>

$$\int_0^1 \frac{1}{x^2} dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x^2} dx =$$



$$= \lim_{a\to 0^+} \left(-\frac{1}{x}\right)\Big|_a^1 = \lim_{a\to 0} \left(\frac{1}{a} - 1\right) = +\infty$$

<u>diverging</u> to $+\infty$

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \left(2\sqrt{x}\right)\Big|_a^1 = \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right)$$

$$= \underline{\underline{2}}$$

<u>converging</u>

---

* if $f(x) \leq g(x)$ (on the domain of interest)

- if $\int_A^B f(x)\,dx$ diverges, then $\int_A^B g(x)\,dx$ diverges

- if $\int_A^B f(x)\,dx$ converges, then $\int_A^B g(x)\,dx$ converges

$$\int_0^1 \frac{dx}{x\sqrt{1-x}}$$

observe: $\frac{1}{x} \leq \frac{1}{x\sqrt{1-x}}$ on $(0,1)$

and $\int_0^1 \frac{dx}{x} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{x} =$

$= \lim_{a \to 0^+} \left(\ln(x) \Big|_a^1 \right) = \lim_{a \to 0^+} \left(-\ln(a) \right) = \underline{\underline{+\infty}}$

so $\int_0^1 \frac{dx}{x\sqrt{1-x}}$ diverges!