(1)

$$\begin{array}{cccc} a_1 & a_2 & a_3 & b \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 2 - 5x_3 \\ x_2 = 3 - 4x_3 \\ x_3 \text{ free} \end{cases}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2-5x_3 \\ 3-4x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ -4x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

$c_1 = 2$
$c_2 = 3$
$c_3 = 0$

$c_1 = -3$
$c_2 = -1$
$c_3 = 1$

(2) let $T: \mathbb{R}^3 \to \mathbb{R}^5$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

according to excercise:

$$\begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad T(\vec{u} + \vec{v}) \neq T(\vec{u}) + T(\vec{v})$$

$$\begin{bmatrix} \vdots \\ 2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} \vdots \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ satisfy

where:

$$\vec{u} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{u} + \underline{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

property not satisfied, so this is NOT a linear transformation

(3)

$$A = \begin{bmatrix} p & 0 & 0 \\ 1 & p & 2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\det(A) = p \cdot (-1)^2 \cdot \begin{bmatrix} p & 2 \\ -1 & 2 \end{bmatrix} = p(2p+2)$$

$$p = 0 \qquad p = -1$$

$$\boxed{p = 0 \text{ or } p = -1}$$

(4)

$$A = \begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}$$

every row sum $= 3$
$\det(A) = 6$

$$A - \lambda I = \begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & a & b \\ c & -\lambda & d \\ e & f & -\lambda \end{bmatrix}$$

$a + b = 3$
$c + d = 3$
$e + f = 3$

if every row adds to the same number x, then x is the eigenvalue

$trace(A) = 0 = \lambda_1 + \lambda_2 + \lambda_3$ $\quad$ $\lambda_1 = 3$

$$\begin{cases} 0 = \lambda_1 + \lambda_2 + \lambda_3 \\ 6 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \\ 3 = \lambda_1 \end{cases} \qquad \begin{cases} -3 = \lambda_2 + \lambda_3 \\ 2 = \lambda_2 \cdot \lambda_3 \end{cases} \rightarrow \lambda_2 = \frac{2}{\lambda_3}$$

$$-3 = \frac{2}{\lambda_3} + \lambda_3 \qquad -3\lambda_3 = 2 + \lambda_3^2$$

$$0 = \lambda_3^2 + 3\lambda_3 + 2 \quad \begin{cases} \lambda_{3.1} = -2 \\ \lambda_{3.2} = -1 \end{cases}$$

$$\boxed{\lambda_1 = 3 \; ; \; \lambda_2 = -2 \; ; \; \lambda_3 = -1}$$

(5) $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$

$Nul(A) \Rightarrow A\underset{\sim}{x} = 0$

$$A\underline{x} = 0 : \left[\begin{array}{cccc|c} 1 & -4 & 9 & -7 & 0 \\ -1 & 2 & -4 & 1 & 0 \\ 5 & -6 & 10 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -4 & 9 & -7 & 0 \\ 0 & -2 & 5 & -6 & 0 \\ 0 & 14 & -35 & 42 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -4 & 9 & -7 & 0 \\ 0 & -2 & 5 & -6 & 0 \end{array}\right. \quad \sim \left[\begin{array}{cccc|c} 1 & -4 & 9 & -7 & 0 \\ 0 & -2 & 5 & -6 & 0 \end{array}\right.$$

$$\left[\begin{array}{cccc|c} 0 & -2 & 5 & -6 & 0 \\ 0 & 2 & -5 & 6 & 0 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 0 & -2 & 5 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 5 & 0 \\ 0 & -2 & 5 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 5 & 0 \\ 0 & 1 & -5/2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = x_3 - 5x_4 \\ x_2 = 5/2\, x_3 - 3x_4 \\ x_3, x_4 \text{ free} \end{cases}$$

$$\underline{x} = \begin{bmatrix} x_3 - 5x_4 \\ 5/2\, x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathrm{Nul}(A) = \left\{ \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathrm{Nul}(A) \perp \mathrm{Row}(A)$$

$$\underline{v_1} = \begin{bmatrix} 1 \\ -4 \\ 9 \\ 7 \end{bmatrix} \qquad \underline{v_2} = \begin{bmatrix} -1 \\ 2 \\ -4 \\ 1 \end{bmatrix}$$

answer

$$\begin{bmatrix} 1 & -4 & 9 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = 2 - 20 + 18 + 0 = \begin{bmatrix} 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 9 & 7 \end{bmatrix} \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} = -5 + 12 + 0 + 7 = 0$$

(6) $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ → lin indep.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -8 \end{bmatrix}$$

$\text{Row}(A) = \{ [1\ -1\ 0], [2\ 1\ 1] \}$

a) $\underline{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2 \\ -3 \\ -9 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} = 1$ → not orthogonal

b) $\underline{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -2 \\ -3 \\ -9 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = -2 - 3 - 9$ → not orthogonal

c) $\underline{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} -1 \\ -1 \\ -8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} = -1+1+0 = 0$

$\downarrow$

orthogonal

d) $\underline{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} -1 \\ -1 \\ -8 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = -2-1-8 \neq 0$

$\downarrow$

not orthogonal

$$\underline{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

7b

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix} \qquad \lambda_1 = -2$$

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -14 \\ 0 \end{bmatrix} \qquad \lambda_2 = 7$$

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix} \qquad \lambda_3 = 7$$

$$\boxed{D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}}$$

7c

$\underline{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\underline{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow$ lin indep.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow$ lin indep.

$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow$ free var. $\rightarrow$ lin dep.

7d