Find the particular solution of the differential eq. given the initial condition:

$f'(x) = 3x$

$f(0) = 7$

$$\int f'(x)\,dx = \int 3x\,dx$$

$$f(x) = \frac{3x^2}{2} + C$$

$$f(0) = \frac{3(0)^2}{2} + C$$

$$7 = 0 + C$$

$\underline{C = 7}$

$$\boxed{f(x) = \frac{3}{2}x^2 + 7}$$

$f'(x) = 6x^2 - 5$

$f(1) = 4$

$$\int f'(x)\,dx = \int (6x^2 - 5)\,dx$$

$$f(x) = \frac{6x^3}{3} - 5x + C$$

$$f(x) = 2x^3 - 5x + C$$

$$f(1) = 2(1)^3 - 5(1) + C$$

$$4 = 2 - 5 + C$$

$\underline{C = 7}$

$$\boxed{f(x) = 2x^3 - 5x + 7}$$

$f''(x) = 2x - 3$

$f'(1) = 2$

$f(0) = 3$

$$\int f''(x)\,dx = \int (2x - 3)\,dx$$

$$f'(x) = \frac{2x^2}{2} - 3x + C$$

$$f'(x) = x^2 - 3x + C$$

$$f'(1) = 1^2 - 3(1) + C$$
$$2 = 1 - 3 + C$$
$$C = 4 \qquad \underline{f'(x) = x^2 - 3x + 4}$$

$$\int f'(x)\,dx = \int (x^2 - 3x + 4)\,dx$$
$$f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 4x + D$$
$$f(0) = \frac{0^3}{3} - \frac{3(0)^2}{2} + 4(0) + D$$
$$3 = 0 - 0 + 0 + D$$
$$D = 3 \qquad \boxed{f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 4x + 3}$$

$f''(x) = x^2 - 4$

$f'(2) = 3$

$f(1) = -4$

$$\int f''(x)\,dx = \int (x^2 - 4)\,dx$$
$$f'(x) = \frac{x^3}{3} - 4x + C$$
$$f'(2) = \frac{2^3}{3} - 4(2) + C$$
$$3 = \frac{8}{3} - 8 + C$$
$$C = 25/3 \qquad \underline{f'(x) = \frac{x^3}{3} - 4x + \frac{25}{3}}$$

$$\int f'(x)\,dx = \int \left(\frac{x^3}{3} - 4x + \frac{25}{3}\right) dx$$
$$f(x) = \frac{1x^4}{3\cdot 4} - \frac{4x^2}{2} + \frac{25x}{3} + D$$
$$f(1) = \frac{1^4}{12} - 2(1)^2 + \frac{25(1)}{3} + D$$
$$-4 = \frac{1}{12} - 2 + \frac{25}{3} + D$$
$$-48 = 1 - 24 + 100 + 12D$$
$$-125 = 12D$$
$$D = -\frac{125}{12} \qquad \boxed{f(x) = \frac{x^4}{12} - 2x^2 + \frac{25x}{3} - \frac{125}{12}}$$

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separable 1^{st} order derivative equations

$$\frac{dy}{dx} = \frac{x^2}{y^2} \quad \Rightarrow \int y^2\,dy = \int x^2\,dx$$
$$\frac{y^3}{3} = \frac{x^3}{3} + C \quad |\cdot 3$$

$$y^3 = x^3 + 3C$$ 3C is constant so $3C \to C$

$$y^3 = x^3 + C$$

$$y = \sqrt[3]{x^3 + C}$$

if we get a value ($y(1)=2$) we have to solve for C:

$y(1) = 2$

$$2^3 = 1^3 + C$$

$$8 = 1 + C$$

$C = 7$ $\qquad y = \sqrt[3]{x^3 + 7}$

$y' = xy \qquad y(0) = 5$

$$\frac{dy}{dx} = xy \implies \int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$e^{\ln y} = e^{x^2/2 + C}$$

$$y = e^{x^2/2 + C}$$

$$y = e^{x^2/2} \cdot e^{c}$$ → constant

$$y = C \cdot e^{x^2/2}$$ → general solution

$y(0)=5$:

$5 = C \cdot e^{0^2/2} \qquad 5 = C \cdot e^0 \qquad 5 = C$

$$y = 5e^{x^2/2}$$ → particular solution

$$\frac{dy}{dx} = y^2 + 1 \qquad \int \frac{dy}{y^2+1} = \int dx \qquad y(1) = 0$$

$$\tan^{-1}(y) = x + C$$

$$\tan(\tan^{-1} y) = \tan(x + C)$$

$$y = \tan(x + C)$$

$$0 = \tan(1 + C)$$

$$\boxed{C = -1} \longrightarrow \underline{\underline{y = \tan(x-1)}}$$

Homogeneous Differential Eq.

$$2x\,dy = (x + y)\,dx$$

→ homogeneous = degree has to be same

$$y = vx$$

$$dy = dv(x) + v(dx)$$

$$dy = x\,dv + v\,dx$$

$$2x(x\,dv + v\,dx) = (x + vx)\,dx$$

$$2\cancel{x}(x\,dv + v\,dx) = \cancel{x}(1+v)\,dx$$

$$2x\,dv + \cancel{2}v\,dx = dx + \cancel{v\,dx}$$

$$2x\,dv + v\,dx = dx$$

$$2x\,dv = dx - v\,dx$$

$$2x\,dv = (1-v)\,dx \quad \Big|\cdot \frac{1}{2x(1-v)}$$

$$\frac{\cancel{2x}\,dv}{\cancel{2x}(1-v)} = \frac{\cancel{(1-v)}\,dx}{2x\cancel{(1-v)}}$$

$$\frac{1}{1-v}\,dv = \frac{1}{2x}\,dx$$

$$\int \frac{1}{1-v}\,dv = \frac{1}{2}\int \frac{1}{x}\,dx$$

$$[illegible] = \frac{1}{[illegible]} \ln|x| + C$$

$-1 \cdot \ln(1-v)$ $\quad 2 \ln|x|$

$$\ln(1-v)^{-1} = \ln(x^{1/2}) + \ln C$$

$\ln A + \ln B = \ln(A \cdot B)$

$$\ln\left(\frac{1}{1-v}\right) = \ln\sqrt{x} + \ln C$$

$$\ln\left(\frac{1}{1-v}\right) = \ln\left(C\sqrt{x}\right)$$

$$\frac{1}{1-v} = C\sqrt{x}$$

$\boxed{v = \frac{y}{x}}$

$$\frac{1}{1-\frac{y}{x}} = C\sqrt{x}$$

$$\frac{x}{x-y} = C\sqrt{x} \quad \Big/ (x-y)$$

$$x = C\sqrt{x}(x-y) \quad \Big/ ^2$$

$$x^2 = C^2 x (x-y)^2 \quad \Big/ \div x$$

$$x = C^2 (x-y)^2$$

$$\boxed{|x| = C(x-y)^2}$$

$$y' = \frac{xy}{x^2-y^2}$$

$$\frac{dy}{dx} = \frac{xy}{x^2-y^2}$$

$$xy\,dx = (x^2-y^2)\,dy$$

replace

$y = vx$

$dy = x\,dv + v\,dx$

$$x(vx)\,dx = (x^2 - v^2x^2)(x\,dv + v\,dx)$$

$$v\cancel{x^2}\,dx = \cancel{x^2}(1-v^2)(x\,dv + v\,dx)$$

$$\cancel{v\,dx} = x\,dv \cancel{+ v\,dx} - xv^2\,dv - v^3\,dx$$

$$v^3 dx = x(1-v^2)\,dv \qquad \int \cdot \left(\frac{1}{v^3 x}\right)$$

$$\frac{\cancel{v^3}\,dx}{\cancel{v^3}\,x} = \frac{\cancel{x}(1-v^2)\,dv}{v^3\,\cancel{x}}$$

$$\frac{1}{x}\,dx = \frac{1-v^2}{v^3}\,dv \qquad \int \frac{1}{x}\,dx = \int \frac{1-v^2}{v^3}\,dv$$

$$\int \frac{1}{x}\,dx = \int \frac{1}{v^3}\,dv - \int \frac{1}{v}\,dv$$

$$\ln|x| = -\frac{1}{2v^2} - \ln|v| + C$$

replace $v = \frac{y}{x}$

$$\ln|x| = -\frac{1}{2\left(\frac{y}{x}\right)^2} - \ln\left|\frac{y}{x}\right| + C$$

$$\ln|x| = -\frac{1 \cdot x^2}{2\,\frac{y^2}{x^2} \cdot x^2} - \left(\ln|y| - \ln|x|\right) + C$$

$$\cancel{\ln|x|} = -\frac{x^2}{2y^2} - \ln|y| \cancel{+ \ln|x|} + C$$

$$\ln|y| = -\frac{x^2}{2y^2} + C$$

$$e^{\ln|y|} = e^{-\frac{x^2}{2y^2} + C} \qquad y = e^{-\frac{x^2}{2y^2}} \cdot e^{C} \to C$$

$$\boxed{y = C \cdot e^{-\frac{x^2}{2y^2}}}$$

$$y' = \frac{2x e^{-y/x} + y}{x} \qquad \text{for } (1,0)$$

$$\frac{dy}{dx} = \frac{2xe^{-y/x}+y}{x}$$

$$x\,dy = (2xe^{-y/x}+y)\,dx$$

$$y = vx$$
$$dy = x\,dv + v\,dx$$

$$x(x\,dv + v\,dx) = (2xe^{-v\cancel{x}/\cancel{x}} + vx)\,dx$$

$$\cancel{x}(x\,dv + v\,dx) = \cancel{x}(2e^{-v}+v)\,dx$$

$$x\,dv \cancel{+v\,dx} = 2e^{-v}dx \cancel{+v\,dx}$$

$$x\,dv = 2e^{-v}dx \qquad \Big/ \frac{1}{x(2e^{-v})}$$

$$\frac{1}{2e^{-v}}dv = \frac{1}{x}dx \qquad \int \frac{e^v}{2}dv = \int \frac{1}{x}dx$$

$$\frac{1}{2}\int e^v dv = \int \frac{1}{x}dx \qquad \frac{1}{2}e^v = \ln|x| + C \qquad v = \frac{y}{x}$$

$$\frac{1}{2}e^{y/x} = \ln|x| + C \qquad e^{y/x} = 2\ln|x| + C$$

$$\boxed{e^{y/x} = \ln(x^2) + C} \rightarrow \text{general solution}$$

solution for $(1, 0)$ $\qquad x = 1 \quad y = 0$

$$e^{0/1} = \ln(1^2) + C$$

$$1 = 0 + C$$

$$\boxed{C = 1}$$

$$\boxed{e^{y/x} = \ln(x^2) + 1}$$

First order linear differential eq.

standard form: $y' + P(x)y = Q(x)$

integrating factor: $I(x) = e^{\int P(x)dx}$

general solution: $y = \frac{1}{I(x)}\left(\int I(x)\,Q(x)\,dx + C\right)$

EX

$$y' + 2y = 2e^x$$

$$y' + P(x)y = Q(x)$$

$P(x) = 2$

$Q(x) = 2e^x$

$I(x) = e^{2x}$

$$I(x) = e^{\int P(x)\,dx} = e^{\int 2dx} = e^{2x}$$

$$y = \frac{1}{e^{2x}}\left(\int e^{2x} \cdot 2e^x\,dx + C\right)$$

$$y = \frac{1}{e^{2x}}\left(2\int e^{3x}dx + C\right)$$

$$y = \frac{1}{e^{2x}}\left(2 \cdot \frac{e^{3x}}{3} + C\right)$$

$$y = \frac{1}{\cancel{e^{2x}}} \cdot \frac{2e^{\cancel{3}x}}{3} + \frac{C}{e^{2x}}$$

$$\boxed{y = \frac{1}{2}e^x + Ce^{-2x}}$$

$$xy' + 4y = 2x^3 \quad \Big/ \div x$$

$$y' + \frac{4}{x}y = 2x^2$$

$$y' + P(x)y = Q(x)$$

$$\boxed{\begin{aligned} P(x) &= \frac{4}{x} \\ Q(x) &= 2x^2 \\ I(x) &= x^4 \end{aligned}}$$

$$I(x) = e^{\int P(x)dx} = e^{\int \frac{4}{x}dx} = e^{4\ln|x|} = \cancel{e}^{\cancel{\ln} x^4} = x^4$$

$$y = \frac{1}{I(x)} \cdot \left(\int I(x)\,Q(x)\,dx + C \right)$$

$$y = \frac{1}{x^4}\left(\int x^4 \cdot 2x^2\,dx + C \right)$$

$$y = \frac{1}{x^4}\left(2\int x^6\,dx + C \right) \qquad y = \frac{1}{x^4}\left(2 \cdot \frac{x^7}{7} + C \right)$$

$$y = \frac{1}{\cancel{x^4}} \cdot \frac{2x^{\cancel{7}\,3}}{7} + \frac{C}{x^4} \qquad \boxed{y = \frac{2x^3}{7} + \frac{C}{x^4}}$$

check: $$y = \frac{2x^3}{7} + \frac{C}{x^4} \Rightarrow y' = \frac{6x^2}{7} + C\left(\frac{-4}{x^5}\right)$$

$$xy' + 4y = 2x^3$$

$$x\left(\frac{6x^2}{7} - \frac{4C}{x^5}\right) + 4\left(\frac{2x^3}{7} + \frac{C}{x^4}\right) = 2x^3$$

$$\frac{6x^3}{7} - \cancel{\frac{4C}{\;}} + \frac{8x^3}{\;} + \cancel{\frac{4C}{\;}} = 2x^3$$

$$\frac{14x^3}{7} = 2x^3 \qquad 2x^3 = 2x^3 \qquad \checkmark$$

$$(x-2)\,y' + y = x^2 - 4$$

$$y' + \frac{1}{x-2}\,y = x+2$$

$$y' + P(x)\,y = Q(x)$$

$$\boxed{\begin{aligned} P(x) &= \frac{1}{x-2} \\ Q(x) &= x+2 \\ I(x) &= x-2 \end{aligned}}$$

$$I(x) = e^{\int P(x)\,dx} = e^{\int (x-2)^{-1}\,dx} = e^{\ln(x-2)} = x-2$$

$$y = \frac{1}{I(x)}\left(\int I(x)\,Q(x)\,dx + C\right)$$

$$y = \frac{1}{x-2}\left(\int (x-2)(x+2)\,dx + C\right)$$

$$y = \frac{1}{x-2}\left(\int (x^2-4)\,dx + C\right)$$

$$\boxed{y = \frac{1}{x-2}\left(\frac{x^3}{3} - 4x + C\right)}$$

$$(4y - 3x)\,dx + 5x\,dy = 0 \quad \Big/ \ \frac{1}{dx}$$

$$4y - 3x + 5x\,y' = 0$$

$$5x\,y' + 4y = 3x \quad \Big/ \div 5x$$

$$y' + \frac{4}{5x}\,y = \frac{3}{5}$$

$$\boxed{\begin{aligned} P(x) &= \frac{4}{5x} \\ Q(x) &= \frac{3}{5} \\ I(x) &= x^{4/5} \end{aligned}}$$

$$I(x) = e^{\int P(x)\,dx} = e^{\int 4/5x\,dx} = e^{\frac{4}{5}\int \frac{1}{x}\,dx} = e^{4/5\ln(x)} = e^{\ln(x^{4/5})}$$

$I(x) = e \quad = e \quad = e \quad = e \quad = e$

$= x^{4/5}$

$$y = \frac{1}{x^{4/5}} \cdot \left(\int x^{4/5} \cdot \frac{3}{5}\, dx + C \right)$$

$$y = \frac{1}{x^{4/5}} \left(\frac{3}{5} \int x^{4/5}\, dx + C \right) \qquad y = \frac{1}{x^{4/5}} \cdot \left(\frac{\cancel{3}}{\cancel{5}} \cdot \frac{\cancel{5}\, x^{9/5}}{\cancel{9}\; 3} + C \right)$$

$$y = \frac{x^{\cancel{9/5}\;1}}{3\, \cancel{x^{4/5}}} + \frac{C}{x^{4/5}} \qquad \boxed{y = \frac{1}{3} x + \frac{C}{x^{4/5}}}$$

$$\frac{dy}{dx} = \frac{x^2 - y}{x} \qquad x\, dy = (x^2 - y)\, dx \quad | \div dx$$

$$x y' = x^2 - y \quad | + y \qquad x y' + y = x^2 \quad | \div x$$

$$y' + \frac{1}{x} y = x$$

$$I(x) = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x$$

$$\boxed{\begin{array}{l} P(x) = \frac{1}{x} \\ Q(x) = x \\ I(x) = x \end{array}}$$

$$y = \frac{1}{x} \left(\int x \cdot x\, dx + c \right) \qquad y = \frac{1}{x} \left(\frac{x^3}{3} + c \right)$$

$$\boxed{y = \frac{1}{3} x^2 + \frac{c}{x}}$$

Bernoulli's Eq.

$$y' + \underline{P(x)}\, y = \underline{Q(x)}\, y^{\boxed{n}}$$

$$I(x) = e^{\int (1-n) P(x)\,dx}$$

$$y^{1-n} = \frac{1}{I(x)} \left(\int (1-n)\, Q(x)\, I(x)\, dx + C \right)$$

Ex $\quad y' + \frac{2}{x} y = x^2 y^3$

$$\boxed{\begin{array}{ll} P(x) = \frac{2}{x} & I(x) = \frac{1}{x^4} \\ Q(x) = x^2 & \\ n = 3 & \end{array}}$$

$$I(x) = e^{\int (1-n) P(x)\,dx} = e^{\int -2 \cdot \frac{2}{x} dx} = e^{-4 \int \frac{1}{x} dx} = e^{-4 \ln|x|} =$$

$$= e^{\ln x^{-4}} = x^{-4} = \frac{1}{x^4}$$

$$y^{-2} = \frac{1}{1/x^4} \left(\int (1-3)\, x^2 \cdot \frac{1}{x^4}\, dx + C \right)$$

$$y^{-2} = x^4 \left(\int -2 \cdot x^{-2}\, dx + C \right)$$

$$y^{-2} = x^4 \left(-2 \cdot \frac{x^{-1}}{-1} + C \right) \qquad y^{-2} = x^4 \cdot \frac{2}{x} + Cx^4$$

$$y^{-2} = 2x^3 + Cx^4 \qquad \boxed{\frac{1}{y^2} = 2x^3 + Cx^4}$$

$$y' + 3x^2 y = 4x^2 y^2$$

$$\begin{array}{ll} P(x) = 3x^2 & n = 2 \\ Q(x) = 4x^2 & I(x) = e^{-x^3} \end{array}$$

$Q(x) = 4x \qquad I(x) = C$

$$I(x) = e^{\int (1-n) P(x) dx} = e^{\int -3x^2 dx} = e^{-3\int x^2 dx} = e^{-\cancel{3}\frac{x^3}{\cancel{3}}} = e^{-x^3}$$

$$y^{-1} = \frac{1}{e^{-x^3}} \left(\int -1 \cdot 4x^2 \cdot e^{-x^3} dx + C \right)$$

$$\int -4x^2 e^{-x^3} \qquad u = -x^3 \qquad dx = \frac{du}{-3x^2}$$
$$du = -3x^2 dx$$

$$\int -4\cancel{x^2}\, e^u \frac{du}{-3\cancel{x^2}} \qquad \frac{4}{3}\int e^u du = \boxed{\frac{4}{3} e^{-x^3}}$$

$$y^{-1} = e^{x^3}\left(\frac{4}{3} e^{-x^3} + C\right)$$

$$\boxed{\frac{1}{y} = \frac{4}{3} + C e^{x^3}} \qquad \boxed{y = \frac{3}{4 + Ce^{x^3}}}$$

$$(x^2y^4 - 1)dx + x^3y^3 dy = 0 \qquad / \div dx$$

$$x^2y^4 - 1 + x^3y^3 y' = 0 \qquad / \div x^3y^3$$

$$\frac{1}{x} y - \frac{1}{x^3y^3} + y' = 0 \qquad y' + \frac{1}{x} y = \frac{1}{x^3} y^{-3}$$

$$\boxed{\begin{array}{ll} P(x) = \frac{1}{x} & n = -3 \\ Q(x) = \frac{1}{x^3} & I(x) = x^4 \end{array}}$$

$$= e^{\int (1-n) P(x) dx} = e^{\int 4 \cdot \frac{1}{x} dx} = e^{4\int \frac{1}{x} dx} = e^{4 \ln|x|} = e^{\ln x^4}$$

$$I(x) = e \quad = e \quad = e \quad = e \quad = e \quad = \underline{x^4}$$

$$y^4 = \frac{1}{x^4}\left(\int 4 \cdot \frac{1}{x^3} \cdot x^4\,dx + C\right)$$

$$y^4 = \frac{1}{x^4}\left(\int 4x\,dx + C\right) \qquad y^4 = \frac{1}{x^4}\left(\int 4x\,dx + C\right)$$

$$y^4 = \frac{1}{x^4}\left(4\,\frac{x^2}{2} + C\right) \qquad y^4 = \frac{2x^2}{x^4} + \frac{C}{x^4}$$

$$y^4 = \frac{2}{x^2} + \frac{C}{x^4} \quad \Big/ \cdot x^4 \qquad x^4 y^4 = 2x^2 + C$$

$$\boxed{x^4 y^4 - 2x^2 = C}$$