① for which values a does the following eq. has no solutions?

$$\begin{cases} ax + ay = a \\ ax + y = 1 \end{cases}$$

$$\begin{bmatrix} a & a \\ a & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 1 \end{bmatrix}$$

no solutions only if $\det(A) = 0$

$\det(A) = a - a^2$

$a - a^2 = 0$

$\underline{\underline{a = 0}}$ or $\underline{\underline{a = 1}}$

$\rightarrow a = 0$ $\begin{cases} 0 = 0 \\ y = 1 \end{cases}$ ✓

$\rightarrow a = 1$ $\begin{cases} x + y = 1 \\ x + y = 1 \end{cases}$ ✓

b) $\begin{cases} ax + y = a^2 \\ x + ay = a + 1 \end{cases}$ $\begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a^2 \\ a+1 \end{bmatrix}$

$\det \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} = a^2 - 1$

$a^2 - 1 = 0$

$a^2 = 1$

$\boxed{a = \pm 1}$

$\rightarrow a = 1$ $\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$ ✓

$\rightarrow a = -1$ $\begin{cases} -x + y = 1 \end{cases}$

$$\begin{cases} x + y = 1 \\ x - y = 0 \end{cases} \quad \times \qquad \text{no solution}$$

(2) following vectors are orthogonal; determine y

$$\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}, \begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ y \\ z \end{bmatrix}$$

the dot product of all vectors must be 0:

$$0 = \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix} = x + b$$

$$0 = \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} \cdot \begin{bmatrix} a \\ y \\ z \end{bmatrix} = a + ay + bz$$

$$0 = \begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ y \\ z \end{bmatrix} = ax + z$$

(3) $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ 2 eigenvalues a, b where a < b

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(4-\lambda) + 2 \\ &= 4 - \lambda - 4\lambda + \lambda^2 + 2 \\ &= \lambda^2 - 5\lambda + 6 \end{aligned}$$

$$0 = \lambda^2 - 5x + 6$$

$$\lambda_1 = 2 \qquad \lambda_2 = 3$$

for $\lambda_1 = 2$

$$\text{Null}(A - 2I) = \text{Nul}\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} x_1 = -x_2 \\ x_2 \text{ free} \end{cases}$$

so, when $\begin{pmatrix} 1 \\ x \end{pmatrix}$: to satisfy this eq. $\boxed{x = -1}$

for $\lambda_2 = 3$

$$\text{Null}(A - 3I) = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = -\tfrac{1}{2} x_2 \\ x_2 \text{ free} \end{cases} \qquad \begin{pmatrix} 1 \\ y \end{pmatrix} \rightarrow \boxed{y = -2}$$