(1)

$$\begin{bmatrix}1\\1\\1\end{bmatrix} + s\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}3\\5\\7\end{bmatrix} + t\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$$s\begin{bmatrix}1\\2\\3\end{bmatrix} - t\begin{bmatrix}2\\0\\1\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix} \qquad \boxed{\begin{matrix}s=2\\t=0\end{matrix}}$$

eq: $3x + 5y + 7z = 0$

(2) $A = \left[\begin{array}{ccc|c}1 & p & 0 & 1\\ p & 1 & 0 & 1\\ 0 & 1 & 1 & 0\end{array}\right]$

a) $A\underline{x} = b$ inconsistent? for all p where $p \neq 0$
b) $A\underline{x} = b$ unique sol? for $p = 0$
c) prove that $\lambda = 1$ for each p

- if $p = 0$ $\Rightarrow$ A is triangular $\rightarrow$ $\lambda = 1$ (with mult. 3)

- if $p \neq 0$

we can divide by p, because $p \neq 0$

$$A - \lambda I = A - I = \begin{bmatrix}0 & p & 0\\ p & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \sim \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

$$\sim \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$ $\leftarrow$ free var, so $\lambda = 1$

is true

(3) $T: \mathbb{R}^4 \to \mathbb{R}^4$

$$T(\underline{x}) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \\ x_4 \end{bmatrix}$$

---

(1)

$$[A \,\vdots\, T(\underline{x})] = \left[\begin{array}{cccc|c} 2 & 0 & 2 & 4 & 8 \\ 1 & 2 & 1 & -2 & 2 \\ 2 & 0 & 1 & 2 & 5 \\ 1 & 0 & 1 & 2 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underline{x} = \begin{bmatrix} 1 \\ 2x_4 - 1 \\ 3 - 2x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

(2)

x_3 free var

$$x_1 + a x_2 + b x_3 = 14$$
$$c x_2 + d x_3 = -40$$

pivots

solution set:
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ \vdots \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -2 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ x_3 \end{bmatrix} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix} \quad \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 = 1+2\lambda \\ x_2 = 3-2\lambda \\ x_3 = 1+\lambda \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & a & b & 14 \\ 0 & c & d & -40 \end{array}\right]$$

$$1+2\lambda + a(3-2\lambda) + b(1+\lambda) = 14$$
$$c(3-2\lambda) + d(1+\lambda) = -40$$

? IDK

$$\underline{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{array}{cccc} & & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \end{array}\right] \rightarrow \text{note: } \begin{aligned} x_1 &= -1 + 2x_3 \\ x_2 &= 5 - 2x_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \end{array}\right] \underset{R1: R1+3R2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & 1 & 2 & 5 \end{array}\right] \underset{R_2 : R_2(-8)}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -8 & -16 & -40 \end{array}\right]$$

so, $a=3$, $b=4$, $c=-8$, $d=-16$.

(4)
$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & -2 \\ 0 & -6 & 3 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\dim \mathrm{Nul}(A) = 0$ → no free var

$\dim \mathrm{Col}(A) = 3$

$\dim \mathrm{Row}(A) = 3$

$\dim \mathrm{Nul}(A^T) = 1$ → 1 free var

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 4 & -6 & 0 \\ 0 & -2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 3 & 0 \end{bmatrix}$$

$$Col(A) = Span \cancel{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\}}$$

not correct, because we dont know the original matrix

$$Row(A) = \{ [1 \ 3 \ 0], [0 \ 4 \ -2], [0 \ 0 \ 1] \}$$

5

$$A - 3I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$(A-3I)\underline{x} = 0 \quad \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

← free var, so 3 is an eigenvalue

$$A - 0I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

→ free var, so 0 is an eigenvalue

$$\underline{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(6) $\underline{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\underline{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

~~NO~~ CLUE ! ✓

$\text{proj}_u(\underline{y}): \hat{\underline{y}} = \frac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} \underline{u}$

$$\hat{\underline{y}} = \frac{20}{50} \underline{u} = 2/5 \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} \in \text{Span}\{\underline{u}\}$$

error:

$$\underline{y} - \hat{\underline{y}} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

$$\underline{y} = \underbrace{\begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}}_{\in \text{Span}\{\underline{u}\}} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

$\hookrightarrow$ orthogonal to $\underline{u}$

(7) $\underline{u} \cdot \underline{v} = (\underline{x} + \underline{y}) \cdot (\underline{x} - \underline{y}) = \underline{x} \cdot \underline{x} - \underline{y} \cdot \underline{y} = \|\underline{x}\|^2 - \|\underline{y}\|^2$

but $\|\underline{x}\| = \|\underline{y}\|$ so

$\|\underline{x}\|^2 - \|\underline{y}\|^2 = 0$, so they are orthogonal

(8)

a) T

b) F

c) F (counterexample $\underline{0}$)

d) T

e) F

② Different approach:

$$\underline{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad (\lambda = 0) \qquad \underline{x} = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix} \quad (\lambda = -1)$$

$$\begin{cases} 1 + 3a + b = 14 \\ 3c + d = -40 \end{cases} \qquad \begin{cases} -1 + 5a + 0 = 14 \\ 5c + 0 = -40 \end{cases} \begin{matrix} \Rightarrow \\ \Rightarrow \end{matrix} \boxed{\begin{matrix} a = 3 \\ c = -8 \end{matrix}}$$

$$\begin{matrix} 1 + 3(3) + b = 14 \\ 3(-8) + d = -40 \end{matrix} \quad \begin{matrix} \Rightarrow \\ \Rightarrow \end{matrix} \boxed{\begin{matrix} b = 4 \\ d = -16 \end{matrix}}$$