1

p	q	r	$\overset{a}{q \lor r}$	$\overset{b}{\neg p \land q}$	$a \Leftrightarrow b$	$r \Rightarrow (a \Leftrightarrow b)$
T	T	T	T	F	F	F
T	T	F	T	F	F	T
T	F	T	T	F	F	F
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	F	F	T	T

2a $(\forall x \in \mathbb{Z})(x \geq 1)$

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

* BASE CASE:

$P(1)$

$$LHS = \frac{1}{(2-1)(2+1)} = \frac{1}{1 \cdot 3} = \frac{1}{3}$$

$P(1)$

$LHS = RHS$

$$RHS = \frac{1}{2+1} = \frac{1}{3}$$

* INDUCTION STEP:

assume $P(n)$ holds,

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1} \rightarrow \text{holds}$$

• we want to show $P(n+1)$:

$$\sum_{i=1}^{n+1} \frac{1}{(2i-1)(2i+1)} = \frac{n+1}{2(n+1)+1} = \frac{n+1}{2n+3}$$

• we know:

$$\sum_{i=1}^{n+1} \frac{1}{(2i-1)(2i+1)} = \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} + \frac{1}{(2n+1)(2n+3)}$$

$$= \frac{n}{(2n+1)} + \frac{1}{(2n+1)(2n+3)}$$

$$= \frac{n(2n+3) + 1}{(2n+1)(2n+3)} = \frac{2n^2+3n+1}{(2n+1)(2n+3)} =$$

$$= \frac{2n(n+1)+n+1}{(2n+1)(2n+3)} = \frac{(n+1)\cancel{(2n+1)}}{\cancel{(2n+1)}(2n+3)} = \frac{n+1}{2n+3} \checkmark$$

2b $(\forall n \in \mathbb{Z})(n \geq 1)$

$7^{2n-1}+5$ is divisible by 12

3 2 4

* BASE CASE

$P(1)$ $\quad 7^{2-1}+5 = 7+5 = 12$ $\quad$ 12 is divisible by 12 ✓

* INDUCTION STEP:

- assume $P(n) = 7^{2n-1}+5$ holds
- we want to show that $P(n+1)$ holds too

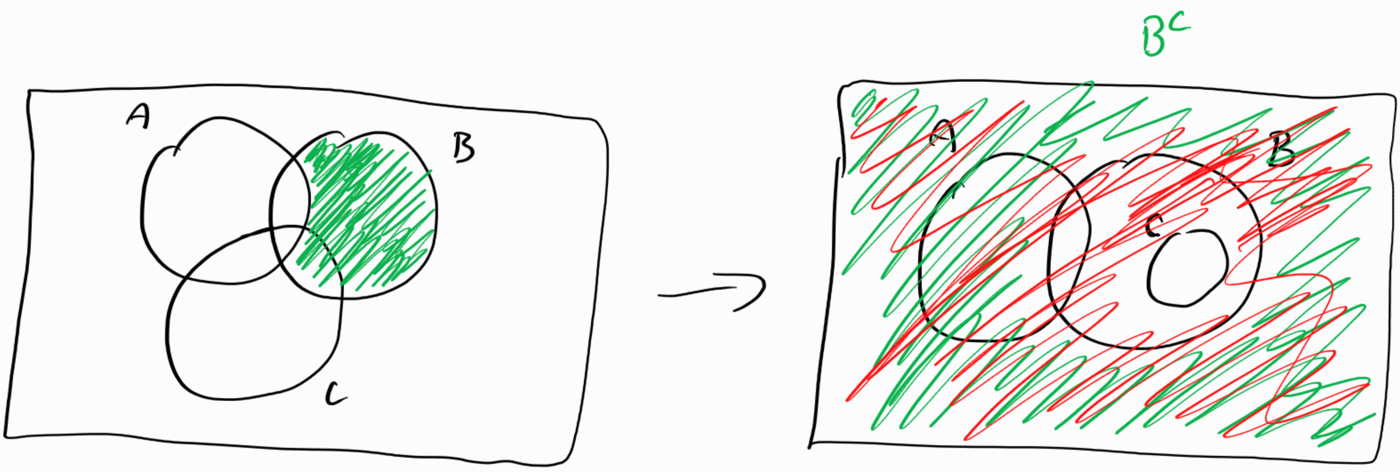
$$7^{2(n+1)-1}+5 = 7^{2n+2-1}+5 = 7^{2n+1}+5$$

$$7^{2n+1}+5 = 12k \quad (k \in \mathbb{Z}) = 7^2 \cdot 7^{2n-1}+5 = 49 \cdot 7^{2n-1}+5$$

$$(48+1)\cdot 7^{2n-1}+5 = \underbrace{48\cdot 7^{2n-1}}_{48/2 \checkmark} + \underbrace{7^{2n-1}+5}_{/12}$$

(3a) $\forall$ sets A, B, C

$$(C \subseteq B \setminus A) \Longleftrightarrow ((A \cap C = \emptyset) \wedge (B^c \subseteq C^c))$$



seems T:

Proof:

"$\Rightarrow$" • take an arbitrary set C, assume $C \subseteq B \setminus A$

• contrapositive: $(A \cap C \neq \emptyset) \vee (B^c \not\subseteq C^c) \Rightarrow (C \not\subseteq B \setminus A)$

case 1: $A \cap C \neq \emptyset$

there exist an element x,

$x \in A \cap C$, so $x \in A$ and $x \in C$

$x \in A$, so $x \notin B \setminus A$

$x \in A$ and $x \notin B \setminus A$ so $C \subseteq B \setminus A$

case 2:

there exist an element y, $y \in B^c$ and $y \notin C^c$,

$y \notin B$ and $y \in C$

so $y \notin B \setminus A$ and $y \in C$, so $C \not\subseteq B \setminus A$

"$\Leftarrow$" • take arbitrary sets A, B, C

• assume $A \cap C = \emptyset$ and $B^c \subseteq C^c \rightarrow C \subseteq B$

• we want to prove $C \subseteq B \setminus A$

• let $x \in C$, we need to show $x \in B \setminus A$

• since $C \subseteq B$, we know $x \in B$ too

• since $A \cap C = \emptyset$, we know $x \notin A$, hence $C \subseteq B \setminus A$

$\square$

3b $\forall$ sets A, B

$$(A \times A) \setminus (B \times B) = (A \setminus B) \times (A \setminus B)$$

counterexample:

take: $A = \{1, 2\}$

$B = \{2, 3\}$

$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

$B \times B = \{(2,2), (2,3), (3,2), (3,3)\}$

$(A \times A) \setminus (B \times B) = \{(1,1), (1,2), (2,1)\}$

$A \setminus B = \{1\}$ LHS $\neq$ RHS

$(A \setminus B) \times (A \setminus B) = \{(1,1)\}$ $\square$

(4a) xRy means "$x + |y| \geq 0$" $x, y \in \mathbb{Z}$

* REFLEXIVE: ✓

$x + |x| \overset{?}{\geq} 0$

if $x < 0$ ✓ because $-x + |-x| \geq 0$, $-x + x \geq 0$, $0 \geq 0$ ✓

if $x = 0$ ✓ because $0 + 0 = 0$

if $x > 0$ ✓ $x + x \geq 0$ ✓ $2x \geq 0$

* SYMMETRIC $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(xRy \Rightarrow yRx)$ ✗

xRy $x + |y| \geq 0$

yRx $y + |x| \geq 0$

take $x = 1$ $y = -5$

xRy $1 + |-5| \geq 0$ ✓

yRx ? $-5 + |1| \not\geq 0$ ✗

* TRANSITIVE: $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\forall z \in \mathbb{Z})(xRy \wedge yRz \Rightarrow xRz)$ ✗

xRy $x + |y| \geq 0$

yRz $y + |z| \geq 0$

xRz ? $x + |z| \geq 0$

$x = -5$ $y = 6$ $z = 2$

xRy $-5 + |6| \geq 0$ $1 \geq 0$ ✓

yRz $6 + |2| \geq 0$ $8 \geq 0$ ✓

xRz ? $-5 + |2| \not\geq 0$ $-3 \not\geq 0$ ✗

(4b) A is a finite set $|A| =$ at least 2

- assume R is complete
- we want to show $(\forall x \in A)(xRx)$
- let $x \in A$
- Since A contains at least 2 elements, there is another element $y \in A$, $x \neq y$

• from completeness we know : xRy or yRx

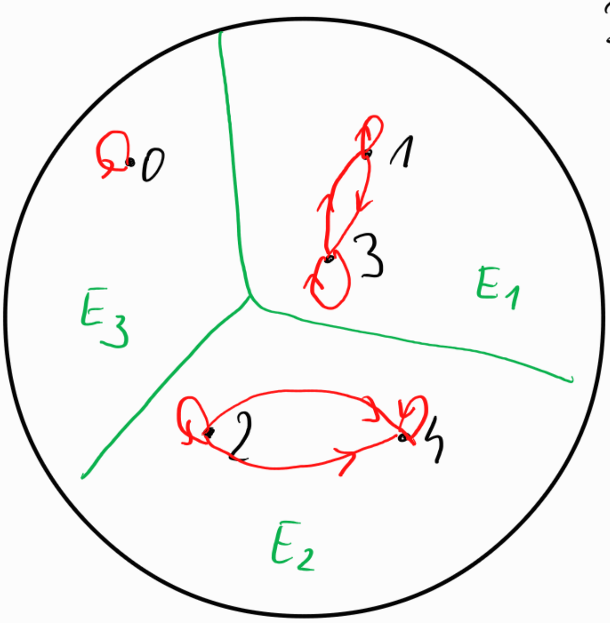
From symm. it follows that yRx,
so from transitivity it follows xRy ✓

symmetry : xRy
transitivity : yRx

□

(4c) $A = \{0, 1, 2, 3, 4\}$

equivalence relation
3 equivalence classes



(5a) 20 members   4 roles

$$\underline{20} \cdot \underline{19} \cdot \underline{18} \cdot \underline{17} = \underline{\underline{116280}}$$

(5b) 20 overs   4 different types
(4 sided dice throw 20 times)

$$\binom{4+20-1}{4} = \binom{23}{4} = \underline{8855}$$

5c) 26 letters   10 num   len 6

(36 char)   but there has to be at least 1 num
or at least 1 letter

$$\left(36\right)^6 - \underbrace{26^6}_{\text{passwords with only letters}} - \underbrace{10^6}_{\text{passwords with only numbers}} = \underline{1\ 866\ 866\ 560}$$

5d) $\binom{n}{k}$ is the number of subsets of size k of a sign n set

So the summation split subsets into groups according their cardinality and count them separately.

6a) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{N})((x+y=0) \vee (x \geq 0))$ $\boxed{T}$

- let x be an arbitrary integer

Case 1: $x < 0$:

take $y = |x|$ (or $-(x)$)

in this case $x+y=0$ alwas holds,

so $T \vee ?$ is always T

Case 2: $x \geq 0$:

in this case the second part is T,

so the whole statement is T

So $(x+y=0) \vee (x \geq 0)$ holds. $\square$

(6b) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(x - y \in \mathbb{N})$ $\boxed{F}$

neg. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x - y \notin \mathbb{N})$

- let x be an arbitrary natural num.
- take $x = y$, so $y - y = 0$ and $0 \notin \mathbb{N}$
- negation is proven, so the statement is F

$\square$

(6c) $\neg((\exists x \in \mathbb{R})(\forall y \in \mathbb{Z})(x \neq y+1))$ $\boxed{F}$

$[(\forall x \in \mathbb{R})(\exists y \in \mathbb{Z})(x = y+1)]$

- take $x = \frac{1}{2}$ • let y be an arbitrary integer
- then we know $y+1 \in \mathbb{Z}$, because $y \in \mathbb{Z}$
- since $\frac{1}{2} \notin \mathbb{Z}$, $x \neq y+1$

↳ the negation is T, so the statement is F

$\square$

(6d) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{Z})(\exists z \in \mathbb{N})(y - z < x)$ $\boxed{T}$

- let $x = 1$ $y < x+2$
- let y be an arbitrary integer

↳ if $y \leq 1$, take $z = 1$ (clearly $y - z \leq 0$ ✓)

↳ if $y > 1$ ($y \geq 2$), take $z = y$ (clearly $y - z \leq 0$ ✓)

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(7) [illegible]

7a $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2 - 9$

$g: \mathbb{R} \to \mathbb{R}$ $g(x) = \begin{cases} 5x+2 & \text{if } x<0 \\ x^3 & \text{if } x \geq 0 \end{cases}$ $(-\infty; 2)$ $[0; \infty)$

give definition $(g \circ f)(x) = g(f(x))$

$x^2 - 9 < 0 \iff -3 < x < 3$

$(g \circ f)$: $\begin{cases} 5(x^2-9)+2 & \text{if } -3<x<3 \\ (x^2-9)^3 & \text{if } x \leq -3 \text{ or } x \geq 3 \end{cases}$

7b $f: \mathbb{R}_0^+ \to \mathbb{R}_0^+$

$f(x) \begin{cases} (x+3)^2 - 9 & \text{if } 0 \leq x \leq c \\ \sqrt[+]{(5x)} + 50 & \text{if } x > c \end{cases}$ $[0; 55]$ $(55; \infty)$

$c \in \{3, 4, 5, 6\}$

$f(x) \begin{cases} (x+3)^2 - 9 & \text{if } 0 \leq x \leq 5 & (1) \\ \sqrt[+]{5x} + 50 & \text{if } x > 5 & (2) \end{cases}$

* INJECTIVE
  - x and y are from $\mathbb{R}^+$ (domain)
  - assume $f(x) = f(y)$

(1) x, y are from (1)

$(x+3)^2 - 9 = (y+3)^2 - 9$

$(x+3)^2 - 9 = (y+3)^2 - 9$

$(x+3)^2 = (y+3)^2$ (since $x, y \in \mathbb{R}_0^+$)

$x+3 = y+3$

$x = y$ ✓

2 x, y are from 2

$+\sqrt{5x} + 50 = +\sqrt{5y} + 50$

$\sqrt{5x} = \sqrt{5y}$

$5x = 5y$

$x = y$ ✓

* SURJECTIVE

- take $y \in \mathbb{R}_0^+$ (codomain)

1 $y = (x+3)^2 - 9$ $\qquad$ $y+9 = (x+3)^2$ $\qquad$ $\sqrt{y+9} = x+3$

$\underbrace{\sqrt{y+9} - 3 = x}$

take $x = \sqrt{y+9} - 3$

sanity check:

$y \in \mathbb{R}_0^+$; $\sqrt{y+9} - 3 \in \mathbb{R}_0^+$; $x \in \mathbb{R}_0^+$

$y \in [0; 55]$; $\sqrt{y+9} - 3 \in [0; 5]$; $x \in [0; 5]$ ✓

$f(x) = f(\sqrt{y+9} - 3) = (\sqrt{y+9} - 3 + 3)^2 - 9 = (\sqrt{y+9})^2 - 9 = y + 9 - 9 = y$ ✓

2 $y = +\sqrt{5x} + 50$ $\qquad$ $y - 50 = +\sqrt{5x}$ $\qquad$ $(y-50)^2 = 5x$

$\underbrace{x = \frac{(y-50)^2}{5}}$

take $x = \frac{(y-50)^2}{5}$

sanity check:

$y \in \mathbb{R}_0^+$; $\frac{(y-50)^2}{5} \in \mathbb{R}_0^+$; $x \in \mathbb{R}_0^+$

$y \in (55; \infty)$; $\frac{(y-50)^2}{5} \in (5; \infty)$; $x \in (5; \infty)$

$$f(x) = f\left(\frac{(y-50)^2}{5}\right) = \sqrt[+]{5 \cdot \frac{(y-50)^2}{5}} + 50 = \sqrt[+]{(y-50)^2} + 50 =$$

$$= y - 50 + 50 = y \checkmark$$

* INVERSE

$$f^{-1}(x) \begin{cases} \sqrt{x+9} - 3 & \text{if } 0 \leq x \leq 55 \\ \frac{(x-50)^2}{5} & \text{if } x > 55 \end{cases}$$

8 $A = \{\emptyset, \{2\}, 1\}$ $B = \{1, \{2,3\}, \emptyset, 2\}$

a) $P(A \cap B) = \{\emptyset; \{\emptyset\}, \{1\}, \{\emptyset; 1\}\}$

$A \cap B = \{\emptyset; 1\}$

b) write all partitions of set $\{7, 8, 9\}$

$\{\{7\}, \{8\}, \{9\}\}$ $\{\{9\}, \{7, 8\}\}$

$\{\{7\}, \{8, 9\}\}$ $\{\{7, 8, 9\}\}$

$\{\{8\}, \{7, 9\}\}$