RELATIONS

- a relation describes the relationship between different elements of a given set A
- xRy $\quad$ x is related to y

EX. $\quad A = \{0, 1, 2\}$ $\quad$ let xRy means "$x \leq y$"

$0R0$? Yes $\quad 0 \leq 0$

$0R1$? Yes $\quad 0 \leq 1$

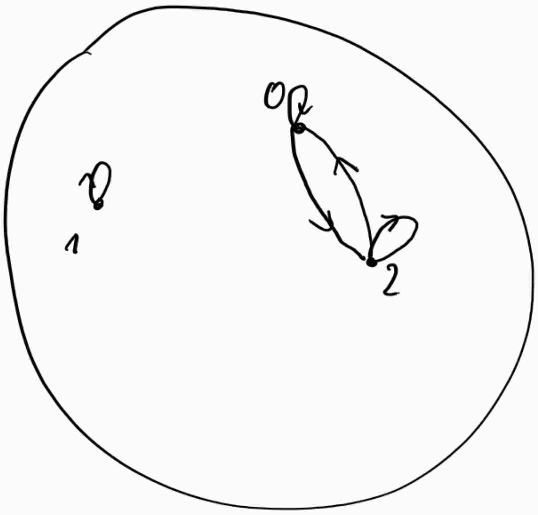
$1R0$? No $\quad 1 \not\leq 0$

$$\boxed{R \subseteq A \times A}$$

$A = \{0,1,2\}$ $\qquad A \times A = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$

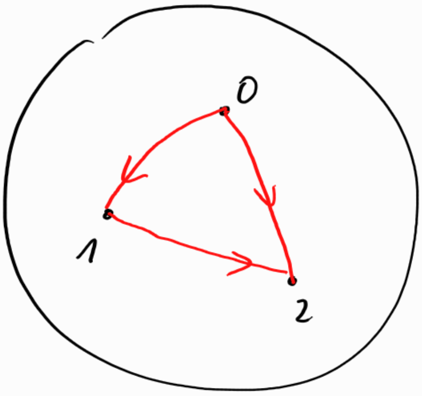
$R = \{(0,0), (0,1), (0,2), (1,1), (1,2), (2,2)\}$

$A = \{0,1,2\}$ $\qquad xRy$ means "$x - y$ is even"



$A = \{0,1,2\}$

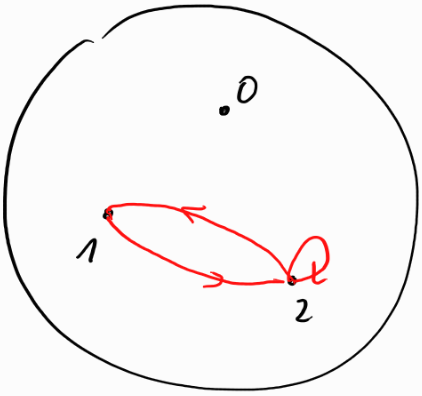
xRy "$x < y$"



refl. ✗
symm. ✗
trans. ✓

$A = \{0,1,2\}$

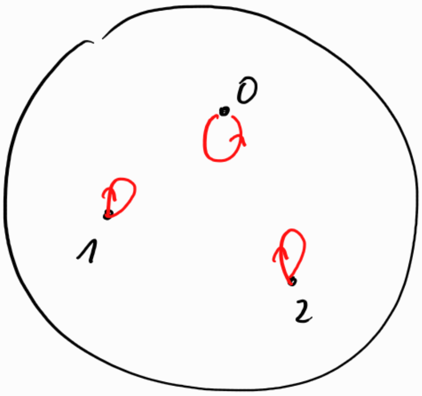
xRy "$x + y \geq 3$"



refl. ✗
symm. ✓
trans. ✗

$A = \{0,1,2\}$

xRy "$x = y$"



refl. ✓
symm. ✓
trans. ✓

Reflexivity

- if every element is related to itself
- $(\forall x \in A)(xRx)$

negation $(\exists x \in A)(x \not R x)$

Symmetry

- whenever there is an arrow from x to y, there is also an arrow from y to x
- $(\forall x \in A)(\forall y \in A)(xRy \Rightarrow yRx)$

• $(\forall x \in A)(\forall y \in A)(xRy \Rightarrow yRx)$

neg. $(\exists x \in A)(\exists y \in A)(xRy \wedge y \not R x)$

Transitivity

• whenever you see
$$\begin{tikzcd} x \arrow[r, bend left] & y \arrow[r, bend left] & z \end{tikzcd}$$
, then you also have an arrow
$$\begin{tikzcd} x \arrow[rr, bend left] & y & z \end{tikzcd}$$

• $(\forall x \in A)(\forall y \in A)(\forall z \in A)(xRy \wedge yRz \Rightarrow xRz)$

neg. $(\exists x \in A)(\exists y \in A)(\exists z \in A)(xRy \wedge yRz \Rightarrow x \not R z)$

Equivalence relations

• if it is → reflexive
→ transitive
→ symmetric

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let $A = \mathbb{N}$

let R be the relation where xRy means "$x-y$ is even"

Prove that it's equivalence relation

Proof:

(1) Reflexive:

• let $x \in \mathbb{N}$

• $x - x = 0$ and 0 is even ✓

(2) symmetric:

• let $x, y \in \mathbb{N}$

• assume xRy ✓

• $x - y$ is even

- $x - y = 2k \quad (k \in \mathbb{Z})$
- $-(x-y) = -2k$
- $y - x = 2(-k)$
- $y - x$ is also even
- yRx

③ transitivity:

- let $x, y, z \in \mathbb{N}$
- assume xRy and yRz

$x - y = 2k$ $\qquad\qquad\qquad\qquad$ $y - z = 2l$ ✓

- add those equations

$$x - y + y - z = 2k + 2l$$
$$x - z = 2k + 2l$$
$$x - z = 2(k + l)$$

- $x - z$ is also even
- xRz

$\square$

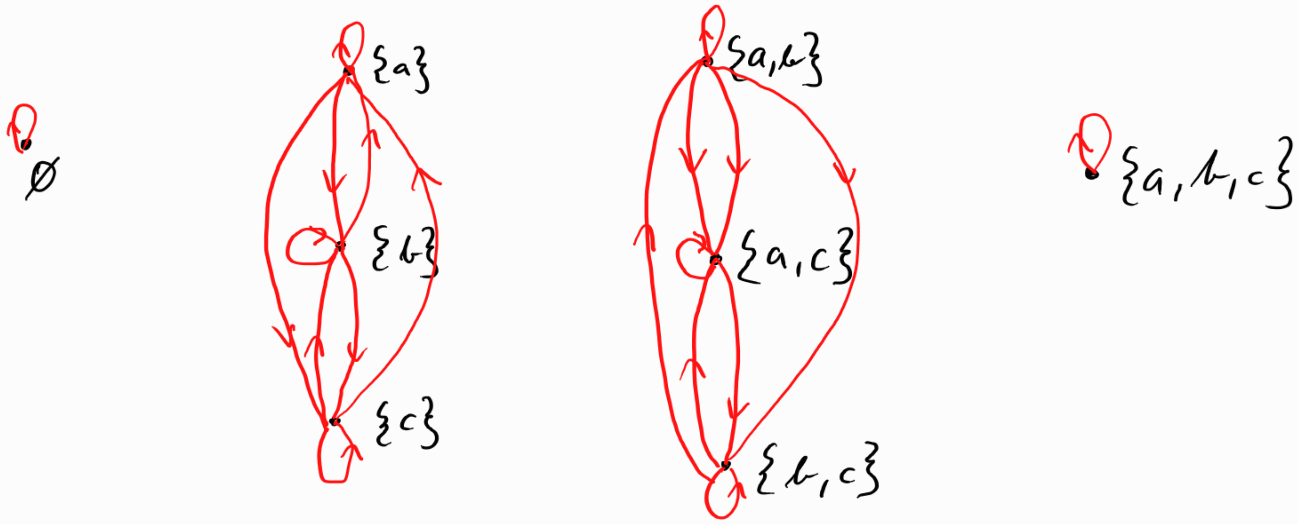
$A = \{1, 2\}$ $\qquad$ draw a relation diagram that is reflexive + transitive, but not symetric

$A = P(\{a, b, c\})$ $\qquad$ XRY means "$|X| = |Y|$"

How many equivalence classes?

$A = \mathbb{P}(\{a,b,c\}) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

There are $\underline{4}$ equivalence classes.



1st class (cardinality 0) $= \phi$

2nd class (cardinality 1) $= \{a\}, \{b\}, \{c\}$

3rd class (cardinality 2) $= \{a,b\}, \{a,c\}, \{b,c\}$

4th class (cardinality 3) $= \{a,b,c\}$

$A = \mathbb{R}$ $\quad xRy$ means "$y > x-1$"

* Reflexive: YES

$x > x-1 \quad (x \in \mathbb{R})$

* Symmetric: NO

counterexample: $x=-2 \quad y=2$

xRy $\quad$ $y \not R x$

$2 > -3$ $\quad$ $-2 \not> 1$ $\quad$ so $x \not R y$

* Transitive: NO

$$\left.\begin{array}{ll} xRy & y > x-1 \\ yRz & z > y-1 \end{array}\right\} \overset{?}{\Rightarrow} z > x-1$$

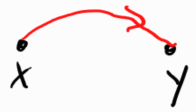
$x=2 \quad y=1.5 \quad z=1$

$xRy \quad 1.5>1$

$yRz \quad 1>0.5$

$x \not R z \,? \quad 1 \not> 1$

Anti-symmetry



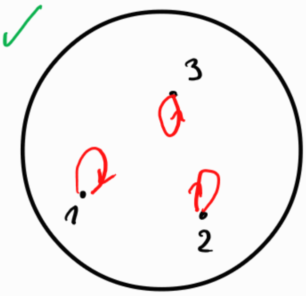
$x \neq y \implies$

$$(\forall x \in A)(\forall y \in A)(xRy \wedge x \neq y \implies y \not R x)$$

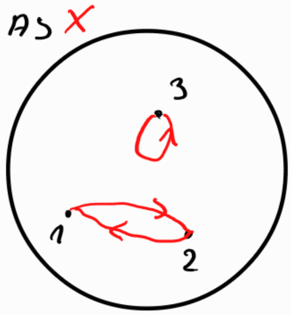
$$(\forall x \in A)(\forall y \in A)(xRy \wedge yRx \implies x = y)$$

neg. $(\exists x \in A)(\exists y \in A)(xRy \wedge x \neq y \implies yRx)$

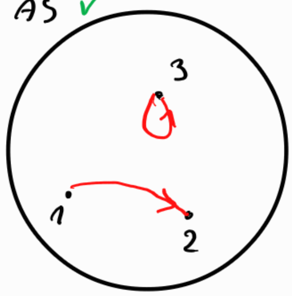
S ✓
AS ✓



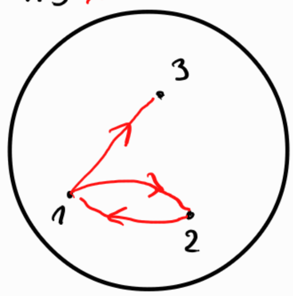
S ✓
AS ✗

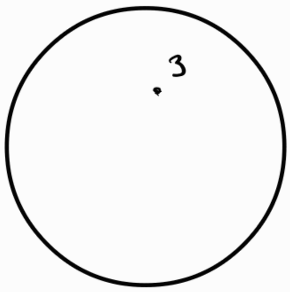


S ✗
AS ✓



S ✗
AS ✗





S ✓
AS ✓

$\emptyset$ S ✓
AS ✓

Partial order

a relation is a partial order if:

- reflexive
- transitive
- anti-symmetric

"$\leq$" on $\mathbb{N}$

- Reflexive: YES

  $x \leq x$ ✓

- Transitive? YES

  $$\left.\begin{matrix} xRy & x \leq y \\ yRz & y \leq z \end{matrix}\right\} \Rightarrow x \leq z \quad \text{so } xRz \;✓$$

- Anti-symmetric? YES
  - let $x, y \in A$
  - assume $\left.\begin{matrix} xRy \Rightarrow x \leq y \\ yRx \Rightarrow y \leq x \end{matrix}\right\} x = y$ ✓

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let $A = \mathbb{P}(\mathbb{N}) \longrightarrow \{\emptyset, \{1\}, \{2\}, \ldots, \{1,2\}, \{1,3\}, \ldots\}$

let xRy mean "$x \subseteq y$" $\qquad \{1\} \subseteq \{1,2\}$ so $\{1\} R \{1,2\}$

Proof:

- Reflexive: YES
  - let $x \in \mathbb{P}(\mathbb{N})$
  - since $x \subseteq x$, so xRx ✓

- Transitive: YES
  - let $x, y, z \in \mathbb{P}(\mathbb{N})$
  - assume $xRy \Rightarrow x \subseteq y$

    $yRz \Rightarrow y \subseteq z$

  - suppose $x = \emptyset$, then clearly $x \subseteq z$, so xRz
  - suppose $x \neq \emptyset$
    - ↳ let $x \in X$
    - ↳ then also $x \in Y$ (because $X \subseteq Y$) ✓
    - ↳ and then also $x \in Z$ (because $Y \subseteq Z$)

↳ hence $X \subseteq Z$, so XRZ ✓

- Anti-symmetric YES
  - let $X, Y \in P(N)$
  - assume YRX, so $Y \subseteq X$

$\Big\} \Rightarrow X = Y$ ✓

$\square$