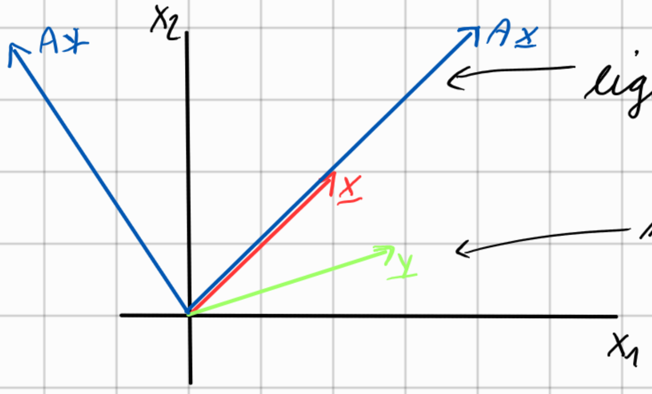
$n \times n$ matrix A. $\quad T: \underline{x} \to A\underline{x}$



eigenvector of A

not eigenvector of A

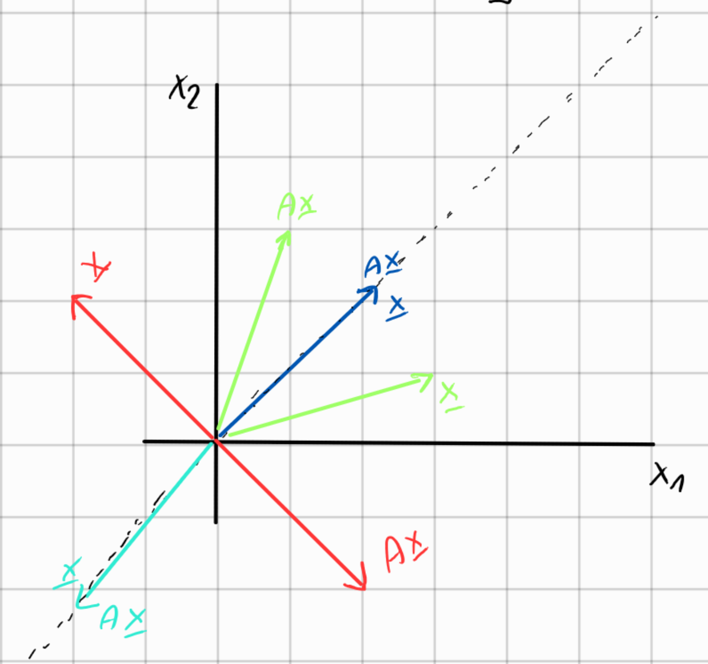
A nonzero $x \in \mathbb{R}^n$ is an eigenvector of when
$A\underline{x} = \lambda \underline{x}$ for some scalar λ

↳ eigenvalue

A produces a scalar multiple of $\underline{x}$ (the direction does NOT change)

↓ apart from a minus sign

EX $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A\underline{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$



A doesn't change the direction of the vectors on the line $x_2 = x_1$

$$A\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

so, $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is an eigenvector with eigenvalue 1

$\boxed{\lambda = 1}$

↳ any vector of the form $\begin{bmatrix} t \\ t \end{bmatrix}$ where $t \neq 0$

Each vector perpendicular to the line $x_2 = x_1$ is also an eigenvector

$$A \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = (-1) \cdot \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

so, $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue -1

$\boxed{\lambda = -1}$

↳ any vector of the form $\begin{bmatrix} -t \\ t \end{bmatrix}$ where $t \neq 0$

Suppose $A\underline{x} = \underline{0}$ has a non-trivial sols.

$\Rightarrow \exists \underline{x} \neq 0 : A\underline{x} = 0\,\underline{x}$

So, each non-trivial sol. is an eigenvector with eigenvalue 0

A is invertible $\Leftrightarrow A\underline{x} = \underline{0}$ has only the trivial sol.

$\Leftrightarrow 0$ is not an eigenvalue of A

Is $\underline{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

$A\underline{u} = \lambda \underline{u}$ for some scalar λ?

$$A\underline{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$ NO

Is a scalar p an eigenvalue of A?

$A\underline{x} = p\underline{x}$ for some vector $\underline{x} \neq \underline{0}$?

$\iff A\underline{x} - p\underline{x} = 0 \iff (A - pI)\underline{x} = \underline{0}$
has a non-trivial sol?
$\iff A - pI$ has a free var.

EX $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Is 1 an eigenvalue?

$$A - 1\cdot I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

x_2 is free var. so 1 is an eigenvalue ✓

What are the corresponding eigenvectors?

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenspace of λ is $\text{Nul}(A - \lambda I) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

→ a vector space that consists of $\underline{0}$ and all eigenvalues corresponding to λ

How to find the eigenvalues?

λ is an eigenvalue $\iff (A-\lambda I)\underline{x}=\underline{0}$ has non-trivial sols

$\iff A-\lambda I$ is not invertible

$\iff \det(A-\lambda I)=0$

So, solve $\underline{\det(A-\lambda I)=0}$ for λ

↳ polynomial of degree
(characteristic equation / polynomial)

(EX) find the eigenvalues:

$$A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \qquad A-\lambda I=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=$$

$$=\begin{bmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{bmatrix}$$

$$\det(A-\lambda I)=(2-\lambda)(2-\lambda)-1\cdot 1$$
$$=\lambda^2-4\lambda+3$$

$$\det(A-\lambda I)=0$$
$$\lambda^2-4\lambda+3=0 \iff (\lambda-1)(\lambda-3)=0$$
$$\underline{\lambda_1=1} \qquad \underline{\lambda_2=3}$$

find the corresponding eigenvectors:

$$\lambda_1 = 1 \qquad A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{with} \quad x_2 \neq 0$$

$$\lambda_2 = 3 \qquad A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad x_2 \neq 0$$

EX $\quad A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$

$$\underline{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad A\underline{u} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \overset{\lambda_1}{0} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad A\underline{v} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \underset{\lambda_2}{10} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

EX $\quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(= \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \right.$ with $\varphi = \pi/2$
counterclockwise rotation of $\pi/2$

eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$$\lambda^2 + 1 = 0 \iff \lambda = \pm i$$

So, $\lambda_1 = i$ and $\lambda_2 = -i$

eigenvalues can be complex numbers

EX $A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ upper triangular matrix

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 1 & 0 \\ 0 & -3-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (5-\lambda)(-3-\lambda)(-3-\lambda) \quad (=0)$$

$\lambda_1 = 5$ and $\lambda_2 = -3$ (with multiplicity 2)

So, for triangular or diagonal matrices the eigenvalues are the entries on the main diagonal !

Properties:

* A is invertible $\iff$ 0 is not an eigenvalue of A
* $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$
* $\operatorname{trace}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$
  ↳ where trace(A): sum of the entries on the main diagonal

* If $\underline{v_1}, \ldots, \underline{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of a matrix A,

then $\{\underline{v}_1, \dots, \underline{v}_r\}$ is lin. independent