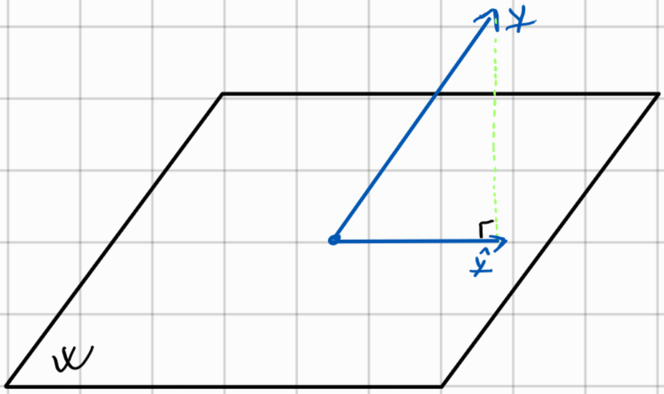
Let $\underline{y} \in \mathbb{R}^n$ and let W be a subspace of $\mathbb{R}^n$

$\text{proj}_W(\underline{y})$: Orthogonal projection of $\underline{y}$ onto W
(the vector in W that is the closest to $\underline{y}$)



$\hat{\underline{y}} = \text{proj}_W(\underline{y})$

(Note: if $\underline{y} \in W$, then $\text{proj}_W(\underline{y}) = \underline{y}$)

If $\{\underline{u}_1, \dots, \underline{u}_k\}$ is an orthogonal basis for W, then

$$\text{proj}_W(\underline{y}) = \underbrace{\frac{\underline{y}\cdot\underline{u}_1}{\underline{u}_1\cdot\underline{u}_1}\underline{u}_1}_{\text{orth. proj. of } \underline{y} \text{ onto } \underline{u}_1} + \frac{\underline{y}\cdot\underline{u}_2}{\underline{u}_2\cdot\underline{u}_2}\underline{u}_2 + \dots + \frac{\underline{y}\cdot\underline{u}_k}{\underline{u}_k\cdot\underline{u}_k}\underline{u}_k$$

Hence, if $\{\underline{u}_1, \dots, \underline{u}_k\}$ is an orthonormal basis for W, then

$$\text{proj}_W(\underline{y}) = (\underline{y}\cdot\underline{u}_1)\underline{u}_1 + \dots + (\underline{y}\cdot\underline{u}_k)\underline{u}_k$$

$$= \begin{bmatrix} | & | & & | \\ \underline{u}_1 & \underline{u}_2 & \dots & \underline{u}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} \underline{y}\cdot\underline{u}_1 \\ \underline{y}\cdot\underline{u}_2 \\ \vdots \\ \underline{y}\cdot\underline{u}_k \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \underline{u}_1 & \underline{u}_2 & \dots & \underline{u}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & \underline{u}_1^T & - \\ - & \underline{u}_2^T & - \\ & \vdots & \\ - & \underline{u}_k^T & - \end{bmatrix} \underline{y}$$

$$= UU^T x$$

projection matrix, where U is an orthogonal matrix

↓ matrix with orthonormal cols.

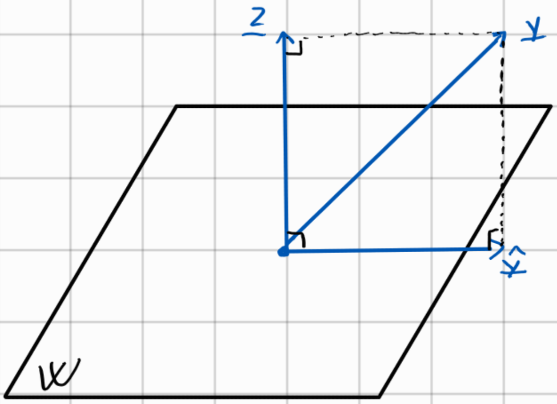
$\underline{y}\cdot\underline{u}_1 = \underline{y}^T\underline{u}_1 = \underline{u}_1^T\underline{y}$

$\underline{y}\cdot\underline{u}_2 = \underline{y}^T\underline{u}_2 = \underline{u}_2^T\underline{y}$

Recall: for an orthogonal matrix U we have $U^TU = I_k$

What if we apply orthogonal projection twice?

$$(UU^T)(UU^T\underline{y}) = U(U^TU)U^T\underline{y} = UI_kU^T\underline{y} = UU^T\underline{y} = \mathrm{proj}_W(\underline{y})$$



$$\underline{y} = \underbrace{\hat{\underline{y}}}_{\in W} + \underbrace{\underline{z}}_{\in W^\perp}$$

$$\mathrm{proj}_{W^\perp}(\underline{y}) = z = \underline{y} - \hat{\underline{y}} = \underline{y} - UU^T\underline{y} = (I_k - UU^T)\underline{y}$$

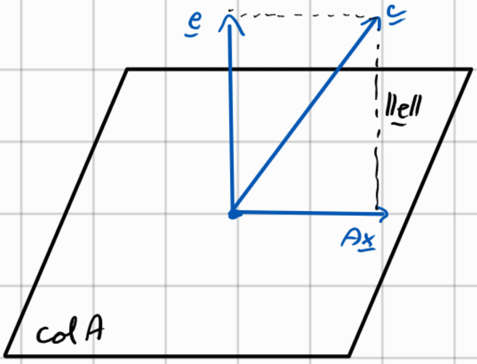
projection matrix onto $W^\perp$

we want to find a lin. combination of the cols of A to get as close as possible to $\underline{c}$.

error vector: $\underline{e} = \underline{c} - A\underline{x}$

try to minimize $\|\underline{e}\|^2 = e_1^2 + e_2^2 + e_3^2 + \dots + e_n^2$

$\Leftrightarrow \min \|\underline{e}\|$



So, take $\underline{x}$ such that

$A\underline{x} = \mathrm{proj}_{\mathrm{col}A}(\underline{c})$

But what is $\underline{x}$?

Note: $\underline{e} \in (\mathrm{Col}\, A)^{\perp} \Rightarrow \underline{e} \in \mathrm{Nul}\,(A^T) \Rightarrow A^T \underline{e} = \underline{0} \Rightarrow A^T(\underline{c} - A\underline{x}) = \underline{0}$

$\Rightarrow A^T \underline{c} - A^T A \underline{x} = \underline{0} \Rightarrow A^T \underline{c} = A^T A \underline{x}$

$\Rightarrow$ Solve $A^T A \underline{x} = A^T \underline{c}$

($A^T A$ invertible $\Leftrightarrow$ $\underline{x}$ unique — Else, ∞ many choices for $\underline{x}$

↳ $\Leftrightarrow$ cols of A are lin indep.)

Recall: A is diagonalizable $\Leftrightarrow$ the sum of the dimensions of the eigenspaces equals n

Symmetric matrix: $A = A^T$

$$\begin{bmatrix} -1 & 6 & -4 \\ 6 & 2 & 0 \\ -4 & 0 & 3 \end{bmatrix}$$

For an $n \times n$ symmetric matrix:

* all eigenvalues are real numbers
* eigenvectors from different eigenspaces are orthogonal
* A is diagonalizable !

Proof: Assume $A\underline{x}_1 = \lambda_1 \underline{x}_1$ and $A\underline{x}_2 = \lambda_2 \underline{x}_2$

$\lambda_1 \neq \lambda_2$

Then, $\lambda_1 (x_1 \cdot x_2) = \lambda_1 (\underline{x}_1^T \underline{x}_2) = (\lambda_1 \underline{x}_1)^T \underline{x}_2 = (A x_1)^T \underline{x}_2$

$= \underline{x}_1^T A^T \underline{x}_2 \overset{A \text{ is symm.}}{=} \underline{x}_1^T A \underline{x}_2 = \underline{x}_1^T (\lambda_2 \underline{x}_2) = \lambda_2 (\underline{x}_1^T x_2)$

$= \lambda_2 (x_1 \cdot \underline{x}_2)$

So, $\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} (x_1 \cdot x_2) = 0$

↳

so, $x_1 \cdot x_2 = 0$ so, $x_1 \perp x_2$ $\square$

A is called orthogonaly diagonalizable if there is an orthogonal matrix P and a diagonal matrix D such that

$A = PDP^{-1}$

$A = PDP^T$

A is orthogonally diagonalizable $\Leftrightarrow$ A is symmetric

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EX: $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ with eigenvalues -2 and 7

orthogonally diagonalize A:

$A-(-2)I = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$\underline{x} = x_3 \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$ $\underline{v_1} = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$

$A - 7I = \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\underline{x} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix}$ $\underline{v_2} = \begin{bmatrix} -1/2 \\ 1 \\ \end{bmatrix}$ $\underline{v_3} = \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix}$

$0 \qquad 1 \qquad 0 \qquad 1$

We need to make $\underline{v}_2$ and $\underline{v}_3$ orthogonal

projection of $\underline{v}_2$ onto $\underline{v}_3$:
$$\hat{\underline{v}}_2 = \frac{\underline{v}_2 \cdot \underline{v}_3}{\underline{v}_3 \cdot \underline{v}_3}\,\underline{v}_3 = \frac{-1/2}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-1/4\\0\\-1/4\end{bmatrix}$$

Component of $\underline{v}_2$ orthogonal to $\underline{v}_3$:

$$\underline{z}_2 = \underline{v}_2 - \hat{\underline{v}}_2 = \begin{bmatrix}-1/2\\1\\0\end{bmatrix} - \begin{bmatrix}-1/4\\0\\-1/4\end{bmatrix} = \begin{bmatrix}-1/4\\1\\1/4\end{bmatrix}$$

Note: $\underline{z}_2$ is also an eigenvector because $\underline{z}_2$ is a linear combination of $\underline{v}_2$ and $\underline{v}_3$

Moreover $\underline{z}_2 \perp \underline{v}_3$

So, $\{\underline{z}_2, \underline{v}_3\} = \left\{\begin{bmatrix}-1/4\\1\\1/4\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ forms an orthogonal basis for the eigenspace