Homogeneous SLE: $Ax = 0$

→ always consistent, as there is a trivial solution $x = 0$

→ no free variables = no nontrivial solution

→ ≥ 1 free variables = nontrivial solution

$$\begin{aligned} 2x_1 + 4x_2 &= 0 \\ x_1 + 2x_2 &= 0 \end{aligned}$$ ← homogeneous SLE

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right] \underset{R_1 : R_1 \cdot 1/2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \underset{R_2 : R_2 - R_1}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{w}}$$

So, sol. set is Span $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

$$\begin{aligned} 2x_1 + 4x_2 &= 6 \\ x_1 + 2x_2 &= 3 \end{aligned}$$ ← non-homogeneous SLE

same row operations

$$\left[\begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \end{array}\right] \underset{R_1 : R_1 \cdot 1/2}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array}\right] \underset{R_2 : R_2 - R_1}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 - 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{w}}$$

EX.

$$x_1 + 4x_2 - 5x_3 = 0$$
$$2x_1 - x_2 + 8x_3 = 9$$

$A = \begin{bmatrix} 1 & 4 & -5 \\ 2 & -1 & 8 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$

Sol. set of $Ax = 0$: $\vec{x} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

Particular sol. of $Ax = b$

$\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ sol set of $Ax = b$:

$$\vec{x} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The set $\{\vec{v_1}, \dots, \vec{v_p}\}$ is linearly independent if
$c_1 \vec{v_1} + \dots + c_p \vec{v_p} = 0$ implies $c_1 = \dots = c_p = 0$
(it has only the trivial solution)

Otherwise: lineary dependent

EX: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ lin. independent? NO

for example: $5 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-3) \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

So, lin. dependent

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ lin dependent?

$c_1 \cdot \begin{bmatrix}1\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$

- we need $c_1 = 0$ and $c_2 = 0$
- So there are NO nontrivial sol.
- only trivial sol.
- lin. independent

→ Consider corresponding homogeneous SLE and reduce it to REF:

* no free var. → unique sol. (only the trivial sol.)
  → lin. independent
* some free var. → infinitely many sols.
  → lin dependent

If a set contains more vectors than there are entries in each vector (more cols than rows)
→ there must be a col without a pivot
→ some free var.
→ lin. dependent

What about a set containing only 1 vector?
- is $\{\vec{v}\}$ linearly dependent?

$c \cdot \vec{v} = 0$

- if $\vec{v} \neq 0$, then we need $c = 0$ (only trivial sol),
  lin. independent
- if $\vec{v} = 0$, then c can be anything (also nontrivial sol) lin. dependent

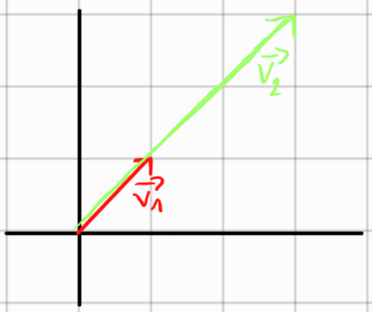
What about a set containing the zero vector?
Is $\{\vec{v_1}, \dots, \vec{v_p}, 0\}$ lin. dependent?

$$c_1 \cdot \vec{v_1} + \dots + c_p \vec{v_p} + c_{p+1} \cdot 0 = 0$$

e.g. $c_1 = c_p = 0$, $c_{p+1} = 8$ is nontrivial sol

- so the set containing the zero vector is ALWAYS lin. dep.

What about a set with 2 vectors?
is $\{\vec{v_1}, \vec{v_2}\}$ lin dep? (suppose $v_1, v_2 \neq 0$)

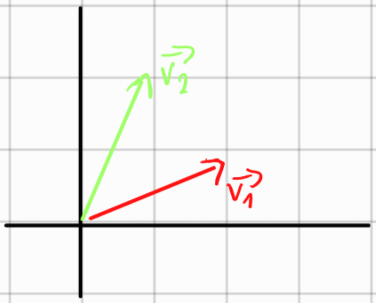


1) they lie on the same line:

$$\vec{v_2} = 3\vec{v_1}$$

$$3\vec{v_1} + (-1)\vec{v_2} = 0$$

→ we found a nontrivial sol, so $\{\vec{v_1}, \vec{v_2}\}$ is lin. dep.



2) they do not lie on the same line (not multiples of eachother)

→ lin. independent

- by contradiction:
  - ↳ suppose $\{\vec{v_1}, \vec{v_2}\}$ is lin. dependent
  - ↳ then there is a nontrivial sol.

$$c_1 \vec{v_1} + c_2 \vec{v_2} = \vec{0}$$

↳ suppose $c_1 \neq 0$

$$c_1\vec{v_1} + c_2\vec{v_2} = \vec{0}$$
$$c_1\vec{v_1} = -c_2\vec{v_2}$$
$$\vec{v_1} = \frac{-c_2}{c_1}\vec{v_2}$$

← ↯ this is not possible because $\vec{v_2}$ is not a multiple of $\vec{v_1}$

So, c_1 must be 0

So, then we need $c_2 \neq 0$

but then $c_1\vec{v_1} + c_2\vec{v_2} = \vec{0}$ $\Rightarrow$ $\underset{c_2 \neq 0}{\underset{\uparrow}{c_2}}\,\underset{\vec{v_2} \neq 0}{\underset{\uparrow}{\vec{v_2}}} = 0$ ↯

$\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}\}$

$\vec{v_4}$ is lin. combination $\vec{v_1}, \vec{v_2}, \vec{v_3}$

$$\vec{v_4} = 2\vec{v_1} - 8\vec{v_2} + 3.5\vec{v_3}$$

$$\underset{2}{\underset{\uparrow}{c_1}}\vec{v_1} + \underset{-8}{\underset{\uparrow}{c_2}}\vec{v_2} + \underset{3.5}{\underset{\uparrow}{c_3}}\vec{v_3} + \underset{-1}{\underset{\uparrow}{c_4}}\vec{v_4} + \underset{0}{\underset{\uparrow}{c_5}}\vec{v_5} = \vec{0}$$

non. trivial sol.

lin. dep.