Operations with vectors:

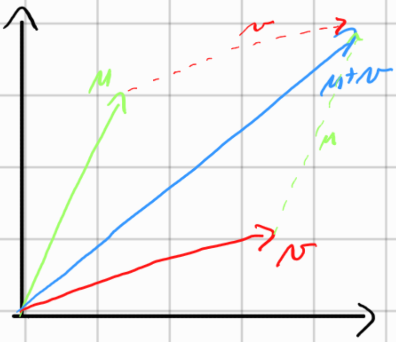
* <u>scaling</u>: $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\quad 2\vec{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$-\vec{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

* <u>addition</u>: $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{u} + \vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

(the vectors have to have the same length)



* Combination of operations:

eg. $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Q: can $\vec{b}$ be written as a linear combination of $\vec{u}$ and $\vec{v}$?

- can we find c_1 and c_2 such that $c_1 \cdot \vec{u} + c_2 \cdot \vec{v} = \vec{b}$?

$$c_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \end{bmatrix} \quad \begin{bmatrix} 3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

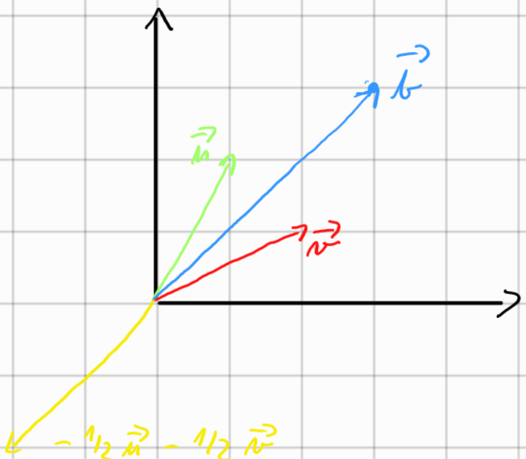
$$\begin{bmatrix} c_1 + 2c_2 \\ 2c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad \text{SLE} \begin{cases} c_1 + 2c_2 = 3 \\ 2c_1 + c_2 = 3 \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Observation:

- every SLE can be written as a vector eq. and the other way around
- solving an SLE means investigating whether $\vec{b}$ can be written as a linear combination of the columns of A

eg. (continued):



eg. (no solution):

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \\ \;\;\vec{u} \;\;\; \vec{v} \end{array} \qquad \vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Hence there is no way to obtain $\vec{b}$ by taking a linear combination of $\vec{u}$ and $\vec{v}$

eg. (∞ many solutions in $\mathbb{R}^2$): $\begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases}$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{u} \quad \vec{v}$

There are ∞ many ways to lineary combine $\vec{u}$ and $\vec{v}$ to get $\vec{b}$

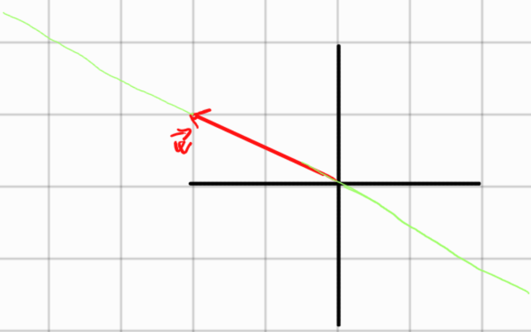
free var.

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \underset{R_2 : R_2 - 2R_1}{\sim} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} x_1 = -2x_2 \\ x_2 \text{ is free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Hence the solution set is $x_2 \cdot \vec{w}$ where $\vec{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

i.e., span$\{\vec{w}\}$: any scalar multiple of $\vec{w}$



any point on this line is a solution to the SLE

eg. (∞ many solutions in $\mathbb{R}^3$) $\begin{cases} x_1 - 3x_2 + 2x^3 = 0 \\ 2x_1 - 6x_2 + 4x_3 = 0 \end{cases}$

$$\begin{bmatrix} 1 & -3 & 2 & | & 0 \\ 2 & -6 & 4 & | & 0 \end{bmatrix} \underset{R_2 : R_2 - 2R_1}{\sim} \begin{bmatrix} 1 & -3 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \begin{cases} x_1 = 3x_2 - 2x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}_1} + x_3 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_2}$$

Hence, the solution is any linear combination of $\vec{w}_1$ and $\vec{w}_2$. span $\{\vec{w}_1, \vec{w}_2\}$
So, the solution set is a plane in $\mathbb{R}^3$.

linear combination of vectors $\rightarrow$ product of a matrix (A) and a vector $(\vec{x})$

Def of $A\vec{x}$: linear combination of the columns of A with the entries of $\vec{x}$ being the weights.

$$A\vec{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Three things with the same solution set:

* the SLE with augmented matrix $[\vec{a}_1, \vec{a}_2, \dots \vec{a}_n \,|\, \vec{b}]$
* the vector equation $x_1 \cdot \vec{a}_1 + x_2 \cdot \vec{a}_2 + \dots + x_n \vec{a}_n = \vec{b}$
* the matrix equation $[\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \vec{b}$ $\quad A\vec{x} = \vec{b}$

Example: $A = \begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -2 \end{bmatrix}$ Is $A\vec{x} = \vec{b}$ consistent for every $\vec{b} \in \mathbb{R}^3$?

$$\begin{bmatrix} -2 & -10 & 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & b_1 \\ 0 & 1 & -2 & b_2 \\ -2 & -10 & 6 & b_3 \end{array}\right] \underset{R_3 : R_3 + 2R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 5 & -3 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 0 & 0 & b_3 + 2b_1 \end{array}\right]$$

NO!

for example: inconsistent if $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

consistent if: $\vec{b} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$