(6a) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})\left(\frac{x+y}{3} \notin \mathbb{N}\right)$ $\boxed{T}$

- let $x, y \in \mathbb{N}$
- let x be an arbitary $\mathbb{N}$
- x can be divisible by 3 or not
- x divisible by 3

  ↳ $x = 3k$ $(k \in \mathbb{N})$ → take $y = 1$

  $\frac{3k+1}{3} \notin \mathbb{N}$

- x not divisible by 3

  ↳ $x = 3k+1$ $(k \in \mathbb{N})$ → take $y = 1$

  $\frac{3k+1+1}{3} = \frac{3k+2}{3} \notin \mathbb{N}$

  ↳ $x = 3k+2$ $(k \in \mathbb{N})$ → take $y = 2$

  $\frac{3k+2+2}{3} = \frac{3k+4}{3} \notin \mathbb{N}$

$\square$

(6b) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{Z})(3x + 4y \neq 20)$

- let $x \in \mathbb{N}$ and $y \in \mathbb{Z}$
- take $x = 1$
- let y be an arbitrary $\mathbb{Z}$

  $3 + 4y \neq 20$ because $4y$ is always even and odd + even is never even, so it can't be 20

(7c) $\neg((\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x^2 - x + y \text{ is odd}))$ $\boxed{F}$

• let $x, y \in \mathbb{Z}$
• let x be a $\mathbb{Z}$
• x is even $\rightarrow x = 2k$ $(k \in \mathbb{Z})$ take $y = 1$

$$(2k)^2 - 2k + 1 = 4k^2 - 2k + 1 \text{ is odd}$$

• x is odd $\rightarrow x = 2k+1$ $(k \in \mathbb{Z})$ take $y = 1$

$$(2k+1)^2 - 2k - 1 + 2 = 4k^2 + \overset{2}{\cancel{4}}k \cancel{+1} \cancel{-2k} \cancel{-1} + 1 = 4k^2 + 2k + 1$$

is odd

□

(6d) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{Z})(\exists z \in \mathbb{N})\ (x + y + z \geq -32)$

• let $x, z \in \mathbb{N}$ and $y \in \mathbb{Z}$
• take $x = 1$
• let y be an arbitrary $\mathbb{Z}$

↳ $y \geq -32$ $\rightarrow$ take $z = 1$

$$1 - 32 + 1 \geq -32 \qquad -30 \geq -32$$

↳ $y < -32$ $\rightarrow$ take $z = |y|$

(8b) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{N})(\forall z \in \mathbb{N})\ (x - y < z - 10)$

• let x be a
• if $x \leq 10$ take $y =$

$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{N})(\exists z \in \mathbb{N})\ (x + y = z)$

$$x = z - y$$

$x = 0$ take $y = z = 1$ $\qquad 0 = 1 - 1$ ✓

$x \geq 1$ take $y = 1$ and ...

take $y = 1$ and $z = x + 1$ ✓

$x < 0$ take $y = |x| + 1$ and $z = 1$ ✓

$(\forall x \in \mathbb{R})(x \in \mathbb{N} \leftrightarrow (2x \in \mathbb{N} \wedge 3x \in \mathbb{N}))$

$\Leftarrow$ assume $2x \in \mathbb{N}$ and $3x \in \mathbb{N}$

$\forall n \in \mathbb{Z}$ $n \geq 1$

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Base case

$P(1)$ $LHS = \frac{1}{1(1+1)} = \frac{1}{2}$ $LHS = RHS$ ✓

$RHS = \frac{1}{1+1} = \frac{1}{2}$

Induction:

assume $P(n)$ holds

proof $P(n+1)$ holds too

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{n+2}$$

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{n+1}\left(n + \frac{1}{n+2}\right)$$

$$= \frac{1}{n+1}\left(\frac{n(n+2)+1}{n+2}\right)$$

$$= \frac{1}{n+1}\left(\frac{n^2+2n+1}{n+2}\right)$$

$$= \frac{1}{\cancel{n+1}} \cdot \frac{(n+1)^{\cancel{2}}}{n+2} = \boxed{\frac{n+1}{n+2}}$$

Base case:

$P(1)$ $\quad 7-4=3 \quad 3/3$ ✓

Induction

assume $P(n)$ holds proof $P(n+1)$ holds

$$7^{n+1} - 4^{n+1} = 7^n \cdot 7 - 4^n \cdot 4$$

$$= 7 \cdot 7^n - 7 \cdot 4^n + 3 \cdot 4^n$$

$$= 7\underbrace{\left(7^n - 4^n\right)}_{\text{div by } 3} + \underbrace{3}_{\text{div by } 3} \cdot 4^n$$

Base:

LHS $= 1 \times 1! = 1$ $\qquad$ LHS = RHS ✓

RHS $= (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$

Induction:

$$\boxed{(n+2)!-1}$$

$$\sum_{i=1}^{n+1} (i \times (i!)) = (n+1)!-1 + (n+1)\cdot(n+1)!$$

$$(n+1)!\left[1 + n+1\right] - 1$$

$$(n+1)!\,(n+2) - 1 = \boxed{(n+2)!-1}$$

$\square$

Base:

$$2^3 - 3 = 8-3 = 5$$

$5/5$ ✓

Induction

$$2^{3(n+1)} - 3^{n+1} = 2^{3n+3} - 3^{n+1} = 2^{3n}\cdot 8 - 3^n \cdot 3 =$$

$$= 8\cdot 2^{3n} - 8\cdot 3^n + 5\cdot 3^n = \underbrace{\underbrace{8(2^{3n}-3^n)}_{\text{div } 5} + \underbrace{5\cdot 3^n}_{\text{div } 5}}_{\text{div } 5}$$

$\square$

Base:

$$LHS = 1(1+2) = 3$$

$$RHS = \frac{1(2)(9)}{6} = \frac{18}{6} = 3$$

$RHS = LHS$

Induction:

$$\frac{(n+1)(n+2)(2n+9)}{} \qquad \frac{n+1}{}\,(2n^2+13n+18)$$

$$\boxed{\frac{[illegible]}{6}} = \frac{n+1}{6}\left(\frac{2n\ [illegible]}{6}\right)$$

$$= \frac{n(n+1)(2n+7)}{6} + (n+1)(n+3) = \frac{n(n+1)(2n+7) + 6(n+1)(n+3)}{6}$$

$$= \frac{n+1}{6}\left(\frac{n(2n+7)+6(n+3)}{6}\right) = \frac{n+1}{6}\left(\frac{2n^2+7n+6n+9}{6}\right)$$

$$= \frac{n+1}{6} \cdot \frac{2n^2+13n+9}{6}$$

$\square$

Base:

$$LHS = (2-1)(2) = 2$$

$$RHS = \frac{1 \cdot 2 \cdot 3}{3} = \frac{6}{3} = 2 \qquad RHS = LHS \checkmark$$

Induction

$$4n^3+3n^2+12n^2+9n+8n+6$$

$$\frac{(n+1)(n+2)(4n+3)}{3} = \frac{4n^3+15n^2+17n+6}{3}$$

$$\frac{n(n+1)(4n-1)}{3} + (2n+1)(2n+2) =$$

$$= \frac{n(n+1)(4n-1)+3(2n+1)(2n+2)}{3} =$$

$$= \frac{(n^2+n)(4n-1) + 3(4n^2+6n+2)}{[illegible]} =$$

3

$$= \frac{4n^3 - n^2 + 4n^2 - n + 12n^2 + 18n + 6}{3}$$

$$= \frac{4n^3 + 15n^2 + 17n + 6}{3}$$ ✓

$\square$

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a) F b) T c) F d) T e) T f) F g) F

---

a) T b) F c) T d) T e) F f) F

$$(B \cap (A^c \cup C)^c = \emptyset) \longleftrightarrow (A \subseteq B^c \cup C)$$

$\rightarrow$ $(A \not\subseteq B^c \cup C) \rightarrow (B \cap (A^c \cup C)^c \neq \emptyset)$

assume $(A \not\subseteq B^c \cup C)$

$x \in A$ such that $x \in B$ and $x \notin C$

so $x \notin A^c \cup C$ $(x \in A$ and $x \notin C)$

so $x \in (A^c \cup C)^c$ and $x \in B$ so

$B \cap (A^c \cup C)^c \neq \emptyset$ ✓

$\leftarrow$ $(A \subseteq B^c \cup C) \rightarrow (B \cap (A^c \cup C)^c = \emptyset)$

assume $A \subseteq B^c \cup C$

$x \in A$, $x \in (B^c \cup C)$

if $x \in B^c$ or $x \notin C$ (written above: and $x \notin C$), $x \notin B$ so $B \cap (A^c \cup C)^c = \phi$

if $x \in C$ and $x \in B$, so $x \notin (A^c \cup C)^c$; $x \notin (A \cap C^c)$

$x \notin C^c$ so ϕ

if $x \in B^c$ and $x \in C$

$x \notin B$ and $x \in C$

so $B \cap (A^c \cup C)^c = \phi$ because $x \notin B$

$(B \subseteq A^c \cup C) \longleftrightarrow ((A \cap B) \setminus (A \cap C) = \phi)$

$\rightarrow$ assume $B \subseteq A^c \cup C$

$x \in B$, so $x \in A^c$ or $x \in C$

- $x \in A^c$ and $x \notin C$

  $x \in B$, $x \notin A$, $x \notin C$

  $x \notin (A \cap B)$ and $x \notin (A \cap C)$ so T

- $x \in C$ and $x \notin A^c$

  $x \in B$, $x \in A$, $x \in C$

  $x \in (A \cap B)$ and $x \in (A \cap C)$

  so $(A \cap B) \setminus (A \cap C) = \phi$ T

- $x \in A^c$ and $x \in C$

  $x \notin A$, $x \in B$, $x \in C$

$x \notin (A \cap B)$ and $x \notin (A \cap C)$

so $(A \cap B) \setminus (A \cap C) = \emptyset$ T

$$(A \cup (C^c \setminus B)) = ((A \cup C^c) \setminus B)$$

$A = \{1\}$

$B = \{1\}$

$C = \emptyset$

$(C^c \setminus B) = \mathcal{U} - \{1\}$

$\mathcal{U} - \{1\} \cup \{1\} = \mathcal{U}$

$(A \cup C^c) = \mathcal{U}$

$\mathcal{U} \setminus B = \boxed{\mathcal{U} - \{1\}}$

---

a) $\{(4,2), (4,3), (5,2), (5,3)\}$

b) $\emptyset$

c) $\{\emptyset, \{6\}\} \times \{2,3\} = \{(\emptyset,2), (\emptyset,3), (\{6\},2), (\{6\},3)\}$

d) $\{\emptyset, \{4\}, \{5\}, \{4,5\}\}$

e) $\{\emptyset, \{4\}, \{5\}, \{6\}, \{4,5\}, \{4,6\}, \{5,6\}, \{4,5,6\}\}$

f) $\{((2,4),6), ((2,5)6), ((3,4),6), ((3,5),6)\}$

g) $\{\emptyset, \{4\}, \{5\}, \{4,5\}\} \times \{\emptyset, \{0\}\}$

$= \{(\emptyset,\emptyset), \cancel{(\emptyset,\{\emptyset\})}, (\{4\},\emptyset), \cancel{(\{4\},\{\emptyset\})}, (\{5\},\emptyset), \cancel{(\{5\},\{\emptyset\})},$
$(\{4,5\},\emptyset), \cancel{(\{4,5\},\{\emptyset\})}$

h) $\{2,\emptyset\} \times \{2,3\} = \{(2,2), (2,3), (\emptyset,2), (\emptyset,3)\}$

$\{\emptyset\} \times \{1, \{3\}\} = \{(\emptyset,1), (\emptyset,\{3\})\}$

$A \cap D = \emptyset \qquad P(\emptyset) = \{\emptyset\}$

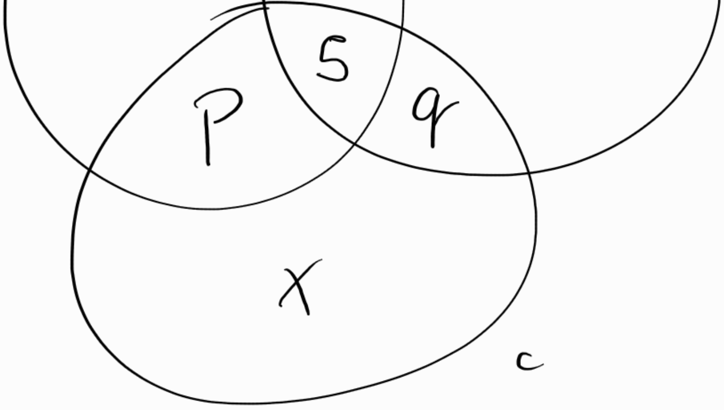
$B \setminus A = \{\emptyset\} \qquad P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} \cup A \qquad \{\emptyset, \{\emptyset\}\}$

$12|4 \qquad 1|2|4$

$1|24 \qquad 14|2$

124

200



$$x+p+r+5=179$$

$$x+p+q+5=17$$

$$x+r+q+5=179$$

$$\boxed{3x+2p+2r+2q=360}$$

$$3x+p+q+r+5=200$$

$$\boxed{3x+p+q+r=195}$$

$$p+r+q=\underline{\underline{165}}$$