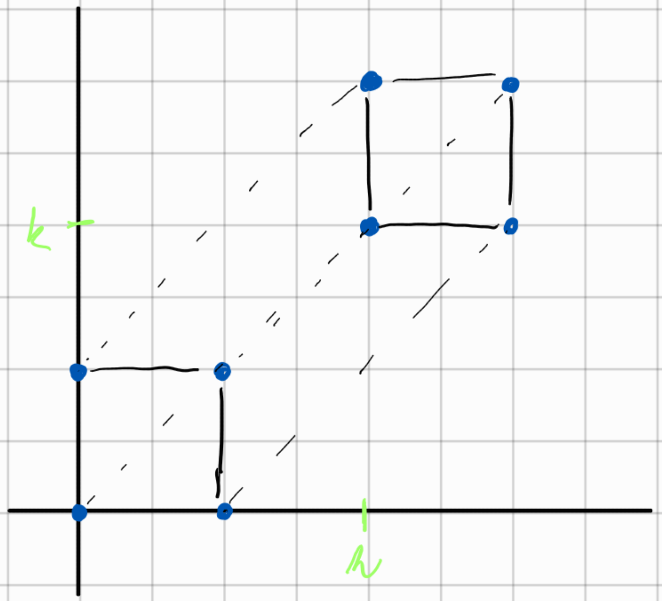


translation, but non-linear because $T(\underline{0}) \neq \underline{0}$

we cannot find a 2×2 matrix A such that

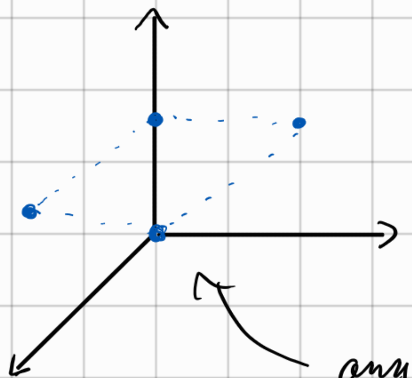
$$A\underline{x} = T(\underline{x}) = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$$

NOT POSSIBLE

$$\begin{bmatrix} 1 & ? \\ ? & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$$

Homogeneous coordinates: work on a specific plane in $\mathbb{R}^3$, eg. plane with $x_3 = 1$

(note: plane in $\mathbb{R}^3$ not $\mathbb{R}^2$)



any point on this plane has coordinate $\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + h \\ x_2 + k \\ 1 \end{bmatrix}$$

Can we find elementary matrix E such that $EA_1 = A_2$

$EA = I_2$

inverse matrix of A

$\frac{1}{5} \cdot 5 = 1$

$5 \cdot \frac{1}{5} = 1$

An $n \times n$ matrix A is called invertible if there exists $n \times n$ matrix C such that $CA = I_n$ and $AC = I_n$

This C is an inverse of A

The inverse of a matrix is unique

NOTE:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad D = ad - bc$$

$$A^{-1} = \frac{1}{D} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \text{inverse matrix}$$

If a matrix is not invertible, it's called singular

For a matrix A to be invertible:

* A must be a square ($n \times n$)
* RREF of a must be an identity matrix

How to find the inverse of invertible matrix A?

→ we need to find E such that $EA = I_n$

$$[A \mid I_n] \sim E[A \mid I_n] = [EA \mid EI_n] = [I_n \mid E]$$

1) row reduce $[A \mid I_n]$
2) if this leads to $[I_n \mid E]$, that A is invertible and $A^{-1} = E$
3) otherwise A is not invertible

EX

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ inverse of A?

$D = ad - bc$
$= 4 - 4 = 0$
$\Downarrow$
not invertible

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \underset{R_2 : R_2 - 2R_1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

A is not invertible because we cannot reduce it to I_2

EX $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ inverse of A?

$D = ad - bc = 2$

$$A^{-1} = 2 \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \underset{R_1 : R_1 - 2R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Theorem: Let A be $n \times n$ invertible. Then, for each $\underline{b} \in \mathbb{R}^n$, the equation $A\underline{x} = \underline{b}$ has a unique solution $\underline{x} = A^{-1}\underline{b}$

Proof: * show that $\underline{x} = A^{-1}\underline{b}$ is a sol to $A\underline{x} = \underline{b}$:

$A\underline{x} = A(A^{-1}\underline{b}) = AA^{-1}\underline{b} = I_n \underline{b} = \underline{b}$ ✓

$A\underline{x} = A(A^{-1}\underline{b}) = AA^{-1}\underline{b} = I_n\underline{b} = \underline{b}$

* uniqueness: A is invertible → A has a pivot in every col → no free var → sol. is unique □

$$A\underline{x} = \underline{b}$$
$$A^{-1}(A\underline{x}) = A^{-1}\underline{b}$$
$$(A^{-1}A)\underline{x} = A^{-1}\underline{b}$$
$$I_n\underline{x} = A^{-1}\underline{b}$$
$$\underline{x} = A^{-1}\underline{b}$$

Properties of invertible matrices:

* $(A^{-1})^{-1} = A$
* if A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$
  (in general $(ABCD\ldots)^{-1} = \ldots D^{-1}C^{-1}B^{-1}A^{-1}$)
* $(A^T)^{-1} = (A^{-1})^T$

Inverse of a 2×2 matrix: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

* if $ad-bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

↓ determinant

* if $ad-bc = 0$, then A is not invertible

SUMMARY:

Let A be a $m \times n$ matrix with cols $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$

$m \geq n$: $\begin{bmatrix} A \end{bmatrix}$

1 → A has pivot in every col
2 → A has n pivot positions
3 → There are no free var.
4 → $A\underline{x} = \underline{0}$ has only a trivial sol.
5 → $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ is linearly indep.
6 → $T: \underline{x} \mapsto A\underline{x}$ is one-to-one / injective

$m \leq n$ $\begin{bmatrix} A \end{bmatrix}$

a → A has a pivot in every row
b → A has m pivot positions
c → The echelon form of A does not contain a row of all zeros
d → $A\underline{x} = \underline{b}$ is consistent for every $\underline{b}$ in $\mathbb{R}^m$
e → span $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\} = \mathbb{R}^m$
f → $T: \underline{x} \mapsto A\underline{x}$ is onto / surjective

The Invertible Matrix Theorem

If A is a square $(m=n)$, then statements 2 and b are equivalent. Hence the following statements are equivalent for square matrices

* 1 - 6 , a - f
* A is invertible
* There is a matrix C such that $CA = I_n$ and $AC = I_n$
* A is row equivalent to I_n
* A^T is invertible
* $\det A \neq 0$