sum of infinite geometric series:

$$\sum_{n=0}^{\infty} a(r)^n \qquad \text{sum} = \frac{a_1}{1-r}$$

sum of finite geometric series:

$$\sum_{n=0}^{x} a(r)^n \qquad \text{sum} = \frac{a_1 (1-r)^n}{1-r}$$

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converging / diverging series

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n$$

$$\lim_{n \to \infty} S_n = L \quad \rightarrow \text{converges}$$

$$\lim_{n \to \infty} S_n = \pm\infty \,/\, DNE \quad \rightarrow \text{diverges}$$

(EX)

$$\sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + 8 + \dots = \infty$$

diverges

$a_n = 2n$

$$S_n = \left(\frac{a_1 + a_n}{2}\right) n = \left(\frac{2+2n}{2}\right) n = \underline{(n+1)\, n}$$

$$\lim_{n \to \infty} (n+1)\, n = \infty \quad \rightarrow \text{diverges}$$

divergence test:

$a_n = 2n$

$$\lim_{n \to \infty} a_n = $$

$$= \lim_{n \to \infty} 2n = \infty$$

$\downarrow$

diverges

DIVERGENCE TEST

if $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ → diverges

if $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n$ → converges or diverges

EX $\sum_{n=1}^{\infty} \frac{5n+3}{7n-4}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n+3}{7n-4} = \frac{5}{7} \neq 0$ → series diverges

(we keep adding 5/7)

EX $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \ldots$ → diverges

→ Harmonic Series

$\lim_{n\to\infty} \frac{1}{n} = \boxed{0}$ divergence test failed!

EX $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$\lim_{n\to\infty} \frac{1}{2^n} = 0$ divergence test failed

$|r| < 1$ → converge

$S = \frac{a_1}{1-r} = \frac{1/2}{1-1/2} = \underline{\underline{1}}$

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

INTEGRAL TEST:

$$\int_1^{\infty} f(x)\,dx \rightarrow \sum_{n \to 1}^{\infty} a_n$$

convergent $\rightarrow$ convergent

divergent $\rightarrow$ divergent

$$f(x) = \frac{1}{x}$$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{a \to \infty} \ln(x)\Big|_1^a = \lim_{a \to \infty} \ln(a) - \ln(1)$$

$$= \infty - 0 = \underline{\underline{\infty}} \rightarrow \underline{\text{divergent}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$$

for integral test:

$$a_n = f(n)$$

f $\rightarrow$ positive, $\rightarrow$ continuous, $\rightarrow$ decreasing $\Big\}$ for $x \geq 1$

$$f(x) = \frac{1}{(x+2)^2}$$

$\rightarrow$ positive ✓

$\rightarrow$ continuous ✓

$\rightarrow$ decreasing ✓ $\rightarrow f'(x) = \frac{-2}{(x+2)^3}$

$\Downarrow$

decreasing on $(-2; \infty)$ ✓

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{(x+2)^2}\,dx =$$

$$\lim_{a\to\infty} \int_1^a \frac{1}{(x+2)^2}\,dx = \begin{array}{l} u = x+2 \\ du = dx \end{array} = \lim_{a\to\infty} \int_1^a u^{-2}\,du =$$

$$= \lim_{a\to\infty} \frac{u^{-1}}{-1}\Big|_1^a = \lim_{a\to\infty} -\frac{1}{u}\Big|_1^a = \lim_{a\to\infty} -\frac{1}{x+2}\Big|_1^a =$$

$$= \lim_{a\to\infty} \frac{-1}{a+2} - \frac{-1}{1+2} = 0 + \frac{1}{3} = \underline{\underline{\frac{1}{3}}}$$

$\to$ series converges

$$\sum_{n=1}^{\infty} \frac{2n}{3n^2+4} \to f(x) = \frac{2x}{3x^2+4}$$

$\to$ positive ✓

$\to$ continuous ✓

$\to$ decreasing:

$$\frac{d}{dx}\frac{2x}{3x^2+4} = \frac{(3x^2+4)\cdot 2 - (2x)(6x)}{(3x^2+4)^2}$$

$$= \frac{6x^2+8-12x^2}{(3x^2+4)^2} = \frac{-6x^2+8}{(3x^2+4)^2}$$

$$-\frac{6x^2+8}{(3x^2+4)^2} \le 0$$

$$8-6x^2 \le 0$$

$$-6x^2 \le -8$$

$$x^2 \ge \frac{8}{6} \qquad x = \pm\frac{2}{\sqrt{3}}$$

$-2/\sqrt{3}$ $\quad$ $+2/\sqrt{3}$

$-$ $\quad$ $+$ $\quad$ $-$

$$D\left[\frac{2}{\sqrt{3}}; \infty\right)$$

$$D\,[2;\infty)$$

$$\int_2^{\infty} \frac{2x}{3x^2+4}\,dx = \begin{array}{l} u = 3x^2+4 \\ du = 6x\,dx \\ dx = \frac{du}{6x} \end{array}$$

$$\int_2^{\infty} \frac{2\cancel{x}}{u}\,\frac{du}{6\cancel{x}} = \frac{1}{3}\int_2^{\infty} \frac{1}{u}\,du =$$

$$= \frac{1}{3}\ln|u|\Big|_2^{\infty} = \frac{1}{3}\ln(3x^2+4)\Big|_2^{\infty}$$

$$\lim_{a\to\infty} \frac{1}{3}\ln(3x^2+4)\Big|_2^a =$$

$$= \lim_{a\to\infty} \underbrace{\frac{1}{3}\ln(3a^2+4)}_{\infty} - \frac{1}{3}\ln(16) = \underline{\underline{\infty}}$$

$\Downarrow$

series diverges

P-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$p > 1 \rightarrow$ converge

$p \leq 1 \rightarrow$ diverge

$$\sum_{n=1}^{\infty} \left(n^{-2.4} + 8n^{-1.6}\right) = \sum_{n=1}^{\infty} \frac{1}{n^{2.4}} + 8 \sum_{n=1}^{\infty} \frac{1}{n^{1.6}}$$

$\Downarrow$ converge $\quad \Downarrow$ converge $\quad$ = converge

Direct comparison Test

$b_n \geq a_n > 0$

big $\quad$ small

1) $\sum b_n \rightarrow$ converges, $\sum a_n$ converges

2) $\sum a_n \rightarrow$ diverges, $\sum b_n$ diverges

$$\sum_{n=1}^{\infty} \frac{1}{4+3^n} \leq \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \rightarrow \text{geom. series}$$

$$\underset{a_n}{\frac{1}{4+3^n}} \leq \underset{b_n}{\frac{1}{3^n}}$$

$|r| < 1$

$\left|\frac{1}{3}\right| < 1 \quad \rightarrow$ convergent

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3+5} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \nearrow p \qquad \frac{1}{n^3+5} \leq \frac{1}{n^3}$$

a_n $\qquad$ b_n $\qquad$ a_n $\qquad$ b_n

$p = 3$
$p > 1$ $\rightarrow$ converges

$$\sum_{n=1}^{\infty} \frac{1}{4+\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \nearrow \text{p-series} \qquad \frac{1}{4+\sqrt[3]{n}} \leq \frac{1}{n^{1/3}}$$

a_n $\qquad$ b_n

$p = \frac{1}{3}$
$p < 1$ $\rightarrow$ diverges, but inconclusive!

if small series D, big series diverges
BUT NOT THE OPPOSITE

$$\sum_{n=1}^{\infty} \frac{2^n}{7^n+8} \leq \sum_{n=1}^{\infty} \frac{2^n}{7^n} \longrightarrow \sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^n \nearrow \text{geom. series}$$

$\downarrow$ a_n $\qquad$ $\downarrow$ b_n

$$|r| = \left|\frac{2}{7}\right| < 1$$

$\Downarrow$
converges

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n} \nearrow \text{p-series} \qquad \ln(n) > 1 \qquad n \geq 3$$

$\downarrow$ b_n $\qquad$ $\downarrow$ a_n

$p = 1$ $\rightarrow$ diverges

$$\sum_{n=1}^{\infty} \frac{n+5}{n^2} \geq \sum_{n=1}^{\infty} \frac{n}{n^2} \to \sum_{n=1}^{\infty} \frac{1}{n} \to p \text{ series}$$

$\downarrow$ b_n $\qquad$ $\downarrow$ a_n

$p = 1 \to a_n$ diverges, so b_n diverges

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+5}} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \to p \text{ series}$$

$\downarrow$ a_n $\qquad$ $\downarrow$ b_n

$p = 2 \quad p > 1$

b_n converges, a_n converges

Limit Comparison Test

$a_n > 0 \qquad b_n > 0$

$$\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = L$$

$\sum a_n$ & $\sum b_n$ $\to$ converge

$\to$ diverge

$$\sum_{n=1}^{\infty} \frac{n^2}{n^5+8} \leq \sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3} \to p \text{ series}$$

$\downarrow$ a_n $\qquad$ $\downarrow$ b_n

$p = 3 \quad p > 1$

b_n converges

$$\lim_{n\to\infty}\left(\frac{n^2\cdot n^3}{n^5+8}\right)=\lim_{n\to\infty}\left(\frac{n^5}{n^5+8}\right)=\underline{1} \rightarrow a_n \text{ converges}$$

$$\underbrace{\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+2}}}_{a_n} \leq \underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{b_n} \rightarrow p\text{-series}$$

$p=1$

b_n diverges

$$\lim_{n\to\infty}\left(\frac{1\ \ n}{\sqrt{n^2+2}}\right)=\lim_{n\to\infty}\frac{n}{n}=1$$

both diverge

$$\underbrace{\sum_{n=1}^{\infty}\frac{1}{3^n+5}}_{a_n} \leq \underbrace{\sum_{n=1}^{\infty}\frac{1}{3^n}}_{b_n}=\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n \rightarrow \text{geom. series}$$

$$|r|=\left|\frac{1}{3}\right|<1$$

$\rightarrow$ b_n converges

$$\lim_{n\to\infty}\left(\frac{1\cdot 3^n}{3^n+5}\right)=\lim_{n\to\infty}\left(\frac{3^n}{3^n}\right)=1 \rightarrow a_n \text{ converges}$$

Alternating Series Test

$$\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\dots$$

forms:

$$\sum^{\infty}(-1)^n a_n \quad \text{or} \quad \sum^{\infty}(-1)^{n+1}a_n$$

$a_n>0$

$n=1 \qquad\qquad n=1$

conditions for the test:

1) $\lim\limits_{n\to\infty} a_n = 0 \to$ converges if $\neq 0 \to$ diverges

2) $a_{n+1} \leq a_n$

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

$a_n = \frac{1}{n}$

1) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ ✓

2) $a_{n+1} \leq a_n$

$\frac{1}{n+1} \leq \frac{1}{n}$ ✓

$\to$ converges

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{5n+3}{2n-7}$$

$\lim\limits_{n\to\infty} \left(\frac{5n+3}{2n-7}\right) = \lim\limits_{n\to\infty} \left(\frac{5n}{2n}\right) = \frac{5}{2} \neq 0 \to$ diverges

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$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!}$$

$\lim\limits_{n\to\infty} \left(\frac{1}{n!}\right) = 0$

$\frac{1}{(n+1)!} \leq \frac{1}{n}$ ✓

converges

$$\sum^{\infty} (-1)^{n+1} \frac{1}{5^n}$$

$\lim \left(\frac{1}{5^n}\right) = 0$

$n=1$ $\qquad\qquad$ $n\to\infty$

$$\frac{1}{5^{n+1}} \le \frac{1}{5^n} \quad \checkmark \qquad\qquad \text{converges}$$

Absolute Convergence, Conditional Convergence

if $\sum |a_n| \to$ converges, then $\sum a_n$ converges

"absolutely convergent"

if $\sum |a_n| \to$ diverges and $\sum a_n$ converges

"conditionally convergent"

if $\sum |a_n|$ diverges and $\sum a_n$ diverges

"divergent"

EX

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^3} = \frac{-1}{1^3} + \frac{1}{2^3} - \frac{1}{3^3} + \dots \quad \to \text{converges}$$

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n\pi)}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \to p=3 \quad p>1 \to \text{converges}$$

$$|\cos(n\pi)| = 1 \qquad n \ge 1 \qquad |a_n| = \frac{1}{n^3}$$

"absolutely convergent"

$$\sum^{\infty} (-1)^{n+1} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \dots \qquad \to \text{converges}$$

$\sum_{n=1}$ $\quad n^{1/3} \quad \sqrt[3]{1} \quad \sqrt[3]{2} \quad \sqrt[3]{3} \quad \dots$

$|a_n| = \frac{1}{n^{1/3}} \qquad |(-1)^{n+1}| = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \qquad p = 1/3 \leq 1 \quad \rightarrow$ diverges

$\rightarrow$ divergence test

$a_n = \frac{1}{\sqrt[3]{n}} \qquad \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0 \ \checkmark \qquad \frac{1}{\sqrt[3]{n+1}} \leq \frac{1}{\sqrt[3]{n}} \ \checkmark$

"conditionally convergent"

$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^3+5} \longrightarrow \sum |a_n| = \sum_{n=1}^{\infty} \frac{n^3}{n^3+5}$

$\downarrow$

divergence test

$\downarrow$

$\lim_{n\to\infty} \frac{(-1)^n \cdot n^3}{n^3+5} = \text{DNE} \neq 0$

$\downarrow$

fail!

$\downarrow$

diverges

$\downarrow$

divergence test

$\lim_{n\to\infty} \frac{n^3}{n^3+5} = \underline{\underline{1}} \neq 0 \rightarrow$ fail!

$\downarrow$

diverges

"divergent"

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1 \qquad \sum a_n \to \text{converges}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1 \qquad \to \sum a_n \to \text{diverges}$$

$$= \infty$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \qquad \to \sum a_n \to \text{inconclusive}$$

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} \qquad a_n = \frac{3^n}{n!} \qquad a_{n+1} = \frac{3^{n+1}}{(n+1)!}$$

$\to$ converges absolutely

$$\lim_{n\to\infty} \left|\frac{3^{n+1}\cdot n!}{(n+1)!\; 3^n}\right| = \lim_{n\to\infty} \left|\frac{\cancel{3^n}\cdot 3\cdot \cancel{n!}}{(n+1)\cancel{n!}\;\cancel{3^n}}\right| =$$

$$\lim_{n\to\infty} \left|\frac{3}{(n+1)}\right| = 0 < 1$$

$$\sum_{n=1}^{\infty} \frac{n}{4^n} \qquad a_n = \frac{n}{4^n} \qquad a_{n+1} = \frac{n+1}{4^{n+1}}$$

$\to$ converges

$$\lim_{n\to\infty} \left|\frac{(n+1)\cdot \cancel{4^n}}{\cancel{4^n}\cdot 4\cdot n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{4n}\right| = \frac{1}{4}\lim_{n\to\infty} \left|\frac{n+1}{n}\right| = \underline{\underline{\frac{1}{4}}} < 1$$

$$\sum^{\infty} \frac{3^{n+2}\cdot n^2}{\;} \qquad a_{n+1} = \frac{3^{n+3}\cdot (n+1)^2}{\;}$$

$$\sum_{n=1} \quad 4^n \qquad \qquad 4^{n+1}$$

$\Rightarrow$ converges

$$\lim_{n\to\infty}\left|\frac{\cancel{3^n}\cdot 3^{\cancel{3}}\cdot(n+1)^2\cdot\cancel{4^n}}{\cancel{4^n}\cdot 4\cdot\cancel{3^n}\cdot\cancel{3^2}\cdot n^2}\right| = \lim_{n\to\infty}\left|\frac{3(n+1)^2}{4n^2}\right| =$$

$$= \frac{3}{4}\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\right| = \underline{\underline{\frac{3}{4}}} < 1$$

Root Test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}$$

$< 1 \rightarrow$ converges

> 1 or $\infty \rightarrow$ diverges

$= 1 \rightarrow$ inconclusive

$$\sum_{n=1}^{\infty}\frac{1}{4^n}$$

$$\lim_{n\to\infty}\left(\frac{1}{4^n}\right)^{1/n} = \lim_{n\to\infty}\frac{1}{4} = \frac{1}{4} < 1 \rightarrow \text{converges}$$

$$\sum_{n=1}^{\infty}\left(\frac{3+5n}{2+3n}\right)^n$$

$$\lim_{n\to\infty}\left[\left(\frac{3+5n}{2+3n}\right)^n\right]^{1/n} = \lim_{n\to\infty}\left(\frac{3+5n}{2+3n}\right) = \frac{5}{3} > 1 \rightarrow \text{diverges}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \left[\left(\frac{1}{n^2} + \frac{1}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{n}\right) = 0+0 = 0 < 1$$

converges