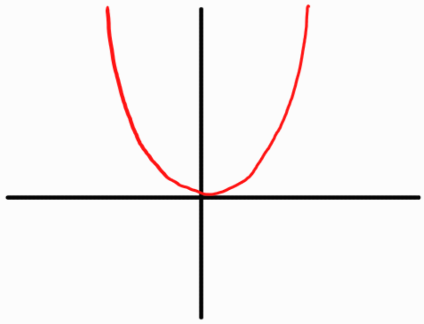
FUNCTIONS

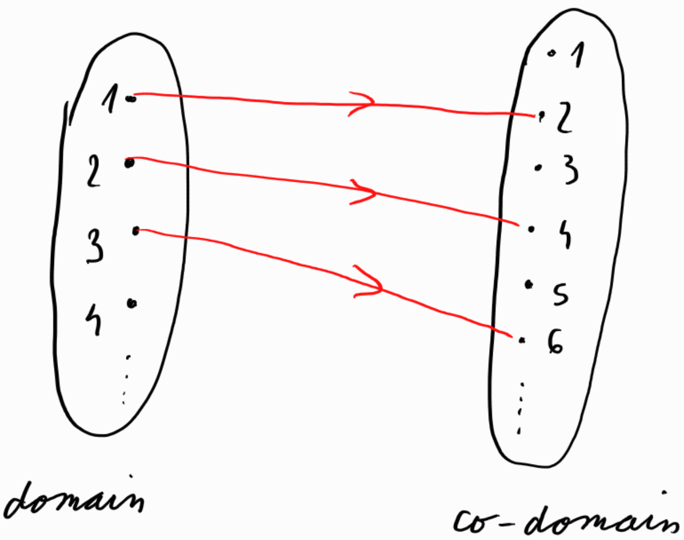
$f(x) = x^2$



$f: \underbrace{A}_{\text{domain}} \to \underbrace{B}_{\text{co-domain}}$

- a function is a mapping from a set A to a set B
- To be a function, $f(x)$ has to be defined for every $x \in A$ (and $f(x) \in B$) and $f(x)$ is unambiguously defined
  ↳ the function value is unique

$f: \mathbb{N} \to \mathbb{N}$ $\qquad$ $f(x) = 2x$



$f: \mathbb{N} \to E$, where $E = \{x \in \mathbb{N} : x \text{ is even}\}$

$f: \mathbb{N} \to [2; \infty)$

$\cancel{f: \mathbb{N} \to [4; \infty)}$

$f: \mathbb{N} \to \mathbb{R}$

- To be a function:
  - ↳ $f(x)$ has to be defined for all $x \in A$
  - ↳ $f(x) \in B$ for all $x \in A$
  - ↳ $f(x)$ has only one outcome

- $f: \mathbb{R} \to \mathbb{R}$, where $f(x) = \frac{1}{x}$

  NO, $f(0)$ not defined

- $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where $f(x) = \frac{1}{x}$

  YES

- $f: \mathbb{N} \to \mathbb{N}$, where $f(x) = 2x$

  YES

- $f: \mathbb{N} \to \mathbb{N}$, where $f(x) = \frac{x}{2}$

  NO, $f(1) = \frac{1}{2} \notin \mathbb{N}$

- $f: \mathbb{N} \to \mathbb{R}$, where $f(x) = \frac{x}{2}$

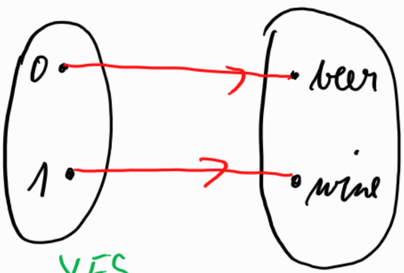
  YES

- $f: A \to B$, with $A = \{0, 1, 2, 3, 4, 5\}$, $B = \mathbb{N}$ and $f(x) = 3x$

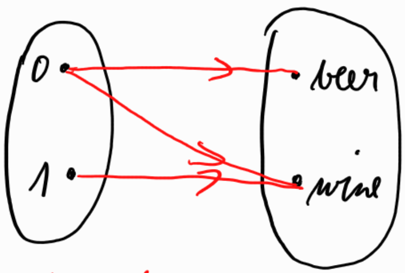
  NO, $f(0) = 0 \notin \mathbb{N}$

Is this a function?

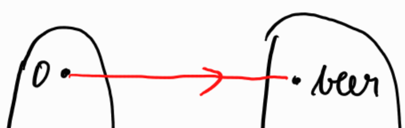
$A = \{0, 1\}$ $\quad$ $B = \{\text{beer}, \text{wine}\}$ $\quad$ $f: A \to B$



YES

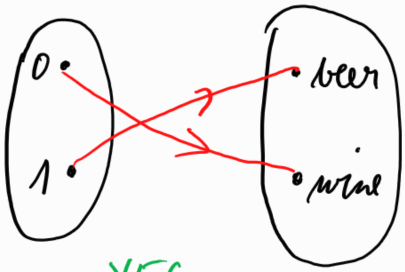


NO, $f(0)$ has 2 outcomes

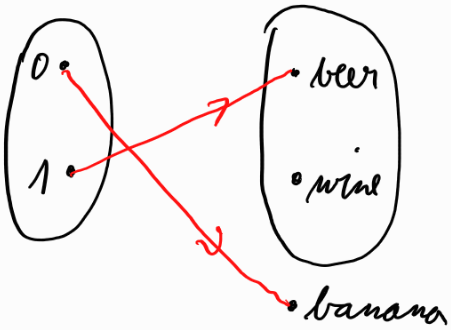


YES

NO, $f(1)$ is not defined



YES



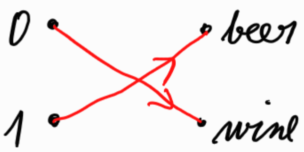
NO, $f(0) = \text{banana} \notin B$

Injectivity

$$(\forall x \in A)(\forall y \in A)(x \neq y \Rightarrow f(x) \neq f(y))$$

$$(\forall x \in A)(\forall y \in A)(f(x) = f(y) \Rightarrow x = y)$$

neg. $(\exists \in A)(\exists y \in A)(x \neq y \ \wedge \ f(x) = f(y))$



injective ✓

injective ✗

---

let $f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$

Injective? NO because take $x = 3$ and $y = -3$
then $f(x) = 9$ and $f(y) = 9$

---

let $f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = 2x - 1$

Injective? YES

Proof: • let $x, y \in \mathbb{Z}$
• assume $f(x) = f(y)$

• so $2x-1=2y-1$

$$2x=2y$$

$$x=y$$

• so $x=y$

$\square$

$f: \mathbb{N} \to \mathbb{N}$ where $f(x) \begin{cases} 2x & \text{if } x \text{ is even} \\ x & \text{if } x \text{ is odd} \end{cases}$

Injective? YES

• let $x, y \in \mathbb{N}$

• 4 cases $\begin{cases} x & \text{even/odd} \\ y & \text{even/odd} \end{cases}$

1) x and y are both even

• assume $f(x)=f(y)$

$$2x=2y$$

$$x=y$$

2) x and y are both odd

• assume $f(x)=f(y)$

$$x=y$$

3) x is even, y is odd

• so $f(x)=\underbrace{2x}_{\text{even}}$ and $f(y)=\underbrace{y}_{\text{odd}}$

• so $f(x) \neq f(y)$

• so $f(x)=f(y) \Rightarrow x=y$ is always true

4) x is odd, y is even

• $f(x)=x$ and $f(y)=2y$

• so $f(x) \neq f(y)$

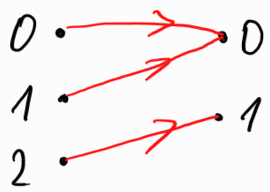
• so $f(x)=f(y) \Rightarrow x=y$ is always true

$\square$

Surjectivity

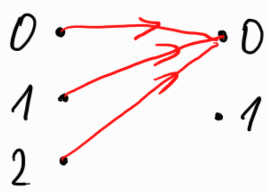
$$(\forall y \in B)(\exists x \in A)(f(x) = y)$$

neg. $(\exists y \in B)(\forall x \in A)(f(x) \neq y)$



inj: ✗

surj: ✓



inj: ✗

surj: ✗

Bijectivity

- both surjective and injective

$f: \mathbb{N} \to \mathbb{N}$ where $f(x) = x^2$

Surjective? NO — no $\mathbb{N}$ number such that $x^2 = 3$

3 is a lonely point

$f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$

Surjective? YES

Proof:
- let $y \in \mathbb{Z}$
- choose $x = y + 1$

Sanity check: $x \in \mathbb{Z}$? $(y+1) \in \mathbb{Z}$? Yes, because $y \in \mathbb{Z}$

so $f(x) = f(y+1) = (y+1) - 1 = y$ ✓

□

$f: \mathbb{N} \to \mathbb{Z}$ $\quad f(x) = x - 1$

Surjective? NO

Surjective: NO

Take $y = -5$

$f(x) = y$

$x - 1 = -5$

$x = -4 \notin \mathbb{N}$

$\square$

$A = \{x \in \mathbb{R} : x \geq 1\}$ $B = \{y \in \mathbb{R} : y \geq -1\}$

$f : A \to B$ $f(x) = x^2 - 2x$

Surjective

Proof: • let $y \in \{y \in \mathbb{R} : y \geq -1\}$, let $y \in B$

• $f(x) = y$

$x^2 - 2x = y$

$x^2 - 2x + 1 - 1 = y$

$(x-1)^2 - 1 = y$

$(x-1)^2 = y + 1$

$x - 1 = \pm\sqrt{y+1}$

$x = 1 \pm \sqrt{y+1}$

• take $x = 1 + \sqrt{y+1}$

because $x \geq 1$

Sanity check:

(1) Does $\sqrt{y+1}$ exist? YES, because $y \geq -1$

(2) $1 + \sqrt{y+1} \geq 1$?

$1 + \sqrt{-1+1} = 1 + 0 = 0$ $1 = 1$

$f(x) = f(1 + \sqrt{y+1})$

$= (1 + \sqrt{y+1})^2 - 2(1 + \sqrt{y+1})$

$= \cancel{1} + \cancel{2\sqrt{y+1}} + y + \cancel{1} \cancel{-2} \cancel{-2\sqrt{y+1}}$

$= \underline{y}$

$\square$

---

let $f: \mathbb{N} \to \mathbb{Z}$ where $f(x) \begin{cases} 1/2 x - 1/2 & \text{if } x \text{ is odd} \quad (a) \\ -1/2 x & \text{if } x \text{ is even} \quad (b) \end{cases}$

Prove that this is surjection ✓

note:

(a) $1/2 x - 1/2 \geq 1/2 - 1/2 = 0$ (a) values ≥ 0

(b) $-1/2 x \leq -1/2 \cdot 1 = -1/2 < 0$ (b) values < 0

Proof: let $y \in \mathbb{Z}$

Case 1: $y < 0$

$f(x) = y \Rightarrow -1/2 = y \Rightarrow \underline{x = -2y}$

sanity check:

↳ $-2y \in \mathbb{N}$ because $y < 0$ and thus $-2y > 0$ and $y \in \mathbb{Z}$ and thus $-2y \in \mathbb{N}$

↳ $-2y$ is even because $y \in \mathbb{Z}$, so we can apply rule (b)

$f(x) = f(-2y) = -1/2 \cdot (-2y) = y$ ✓

Case 2: $y \geq 0$

$f(x) = y \Rightarrow 1/2 x - 1/2 = y \Rightarrow \underline{x = 2y + 1}$

sanity check:

↳ $2y+1 \in \mathbb{N}$ because $y \geq 0$ and thus $2y+1 \geq 1$ and $y \in \mathbb{Z}$ thus $2y+1 \in \mathbb{N}$

↳ $2y+1$ is odd because $y \geq 0$, so we can apply rule ⓐ

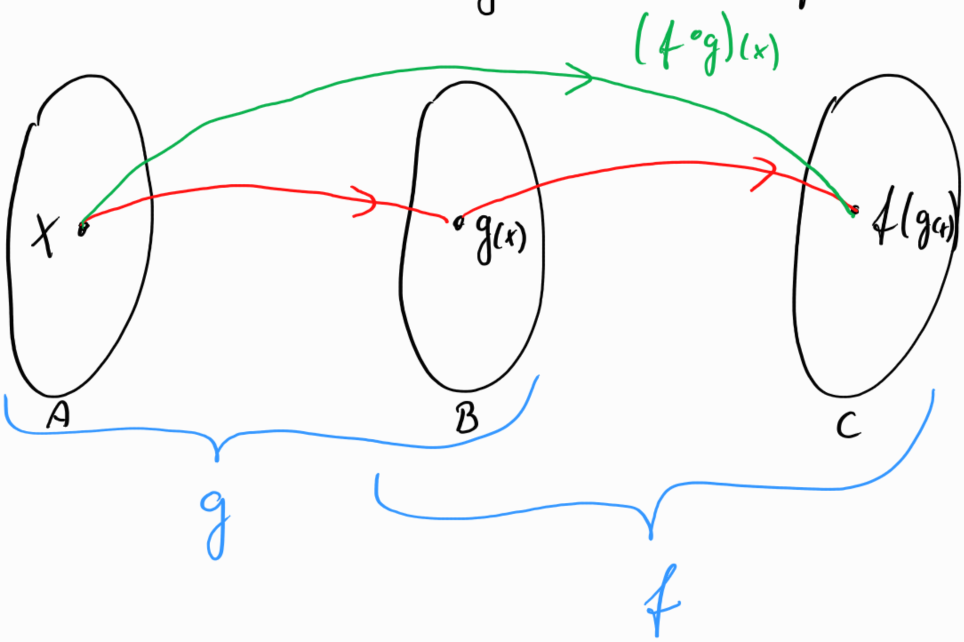
$$f(x) = f(2y+1) = \frac{1}{2}(2y+1) - \frac{1}{2} = y + \frac{1}{2} - \frac{1}{2} = y \checkmark$$

$\square$

Composition

$g: A \to B$ $\quad\quad$ $f: B \to C$

then $(f \circ g)(x) = f(g(x))$ → the function obtained by first using g and then f



Domain of $f \circ g$ is the domain of g, which is A
Co-domain of $f \circ g$ is the co-domain of f, which is C

→ $(f \circ g): A \to C$

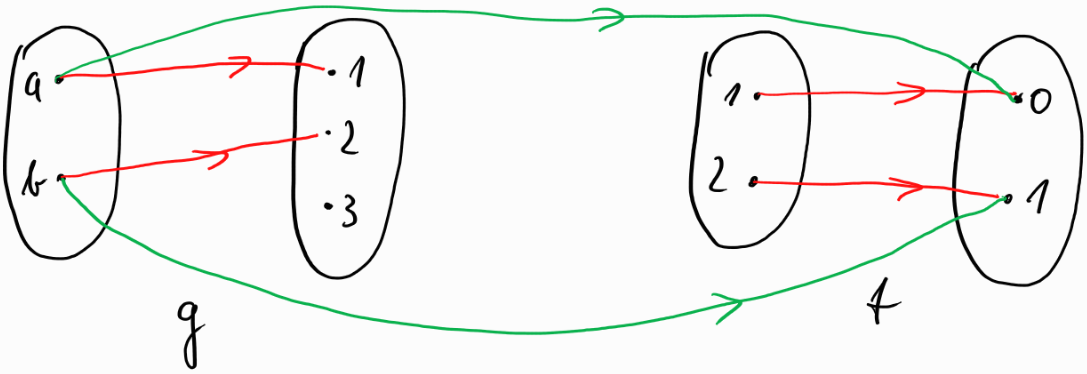
$g: A \to B$ $\quad\quad$ $f: C \to D$

Is $f \circ g$ well-defined?

NO, there might exist an element $x \in A$ such that $g(x) \notin C$.

Composition is well-defined if:

* $(\forall x \in A)\ (g(x) \in C)$
* $\text{range}(g) \subseteq \text{domain}(f)$
* $\text{range}(g) \subseteq C$



Is $f \circ g$ well-defined? YES, because $\left.\begin{array}{l}\text{range}(g) = \{1,2\} \\ \text{domain}(f) = \{1,2\}\end{array}\right\} =$

$f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 4$

$g: \mathbb{N} \to \mathbb{Z}$ where $g(x) = x^2 + 1$

- Is $f \circ g$ well-defined? YES
  $\text{codomain}(g) = \mathbb{Z} = \text{domain}(f)$
- Is $g \circ f$ well-defined? NO
  $f(-2) = -2 - 4 = -6 \notin \mathbb{N} = \text{domain}(g)$
- Is $f \circ f$ well-defined? YES
  $\text{codomain}(f) = \mathbb{Z} = \text{domain}(f)$
- Is $g \circ g$ well-defined? YES
  $\text{range}(g) \subseteq \mathbb{N} = \text{domain}(g)$

$f: \mathbb{N} \to \mathbb{N}$ $\qquad\qquad\qquad\qquad$ $g: \mathbb{N} \to \mathbb{N}$

$$f(x) \begin{cases} 2x & \text{if } x \leq 10 \\ 3x-20 & \text{if } x > 10 \end{cases}$$

$$g(x) \begin{cases} x^2 & \text{if } x \leq 24 \\ x+3 & \text{if } x > 24 \end{cases}$$

Given $g \circ f$

x	1	2	3	...	9	10	11	12	13	14	15	16	...
$f(x)$	2	4	6	...	18	20	13	16	19	22	25	28	...
$g(f(x))$	4	16	36	...	18^2	20^2	13^2	16^2	19^2	22^2	28	31	...

$x \to f(x)$: $\underbrace{1 \; 2 \; 3 \ldots 9 \; 10}_{2x}$ $\quad$ $\underbrace{11 \; 12 \; 13 \; 14 \; 15 \; 16}_{3x-20}$

$f(x) \to g(f(x))$: $\underbrace{2 \; 4 \; 6 \ldots 18 \; 20}_{(2x)^2}$ $\quad$ $\underbrace{13 \; 16 \; 19 \; 22}_{(3x-20)^2}$ $\quad$ $\underbrace{25 \; 28 \ldots}_{(3x-20)+3}$

so $g \circ f: \mathbb{N} \to \mathbb{N}$ where

$$(g \circ f)(x) = \begin{cases} 4x^2 & \text{if } x \leq 10 \\ (3x-20)^2 & \text{if } 10 < x \leq 14 \\ (3x-20)+3 & \text{if } x > 14 \end{cases}$$

$f: \mathbb{R}^0_+ \to \mathbb{R}^0_+$ $\qquad\qquad$ $g: \mathbb{R}^0_+ \to \mathbb{R}^0_+$

$$f(x) \begin{cases} 2x & \text{if } x \leq 10 \\ 3x-20 & \text{if } x > 10 \end{cases}$$

$$g(x) \begin{cases} x^2 & \text{if } x \leq 24 \\ x+3 & \text{if } x > 24 \end{cases}$$

$$x \leq 10 \implies f(x) = 2x \leq 20 < 24 \implies g(f(x)) = (2x)^2 = \underline{4x^2}$$

$x > 10 \implies f(x) = 3x - 20$

$$3x - 20 \leq 24$$
$$3x \leq 44$$
$$x \leq \frac{44}{3}$$

if $10 < x \leq \frac{44}{3} \implies g(f(x)) = \underline{(3x-20)^2}$

if $x > \frac{44}{3} \implies g(f(x)) = |3x-20| + 3 = \underline{3x - 17}$

$g \circ f : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ where

$$(g \circ f)(x) = \begin{cases} 4x^2 & \text{if } x \leq 10 \\ (3x-20)^2 & \text{if } 10 < x \leq \frac{44}{3} \\ 3x - 17 & \text{if } x > \frac{44}{3} \end{cases}$$

Inversion

a function is invertible $\iff$ the function is bijection (surjective and injective)

definitions of invertible:

* f is a bijection
* Reversing the arrows from A to B, to be from B to A, gives a well-defined function
* There exist a function $f^{-1} : B \to A$ such that $(\forall x \in A)(\forall y \in B)(f(x) = y \iff f^{-1}(y) = x)$

---

let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 1$

Prove that f is invertible and give f^{-1}

* INJECTIVITY ✓
  - let $x, y \in \mathbb{R}$ (domain)
  - assume $f(x) = f(y)$

$$2x - 1 = 2y - 1$$
$$x = y \quad ✓$$

* SURJECTIVE ✓
  - let $y \in \mathbb{R}$ (codomain)

$$\left(f(x) = y \Rightarrow 2x + 1 = y \Rightarrow \boxed{x = \frac{y-1}{2}} \right)$$

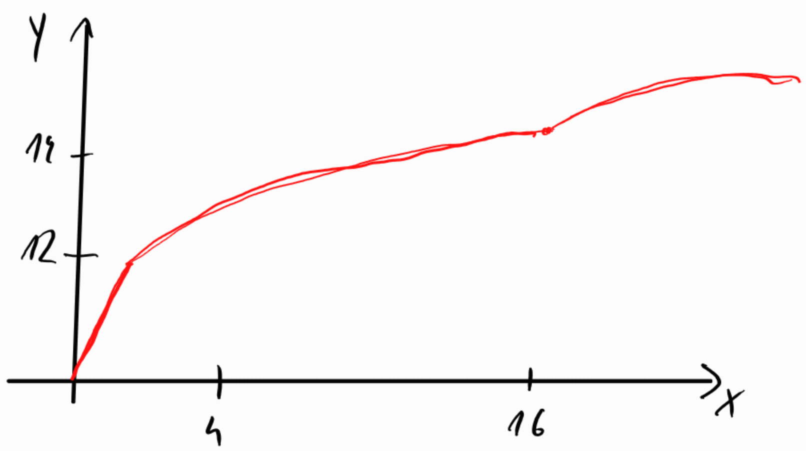
  - take $x = \frac{y-1}{2}$
  - $f(x) = f\left(\frac{y-1}{2}\right) = 2\left(\frac{y-1}{2}\right) + 1 = y - 1 + 1 = y$ ✓

* f^{-1}:

$$f^{-1} : \mathbb{R} \to \mathbb{R} \quad \text{where} \quad \underline{f^{-1}(x) = \frac{x-1}{2}}$$

---

let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ where
$$\begin{cases} 3x & \text{if } 0 \leq x < 4 \quad (1) \\ \sqrt{x} + 10 & \text{if } 4 \leq x < 16 \quad (2) \\ x/8 + 12 & \text{if } x \geq 16 \quad (3) \end{cases}$$



if $x = [0;4) \rightarrow f(x) = [0;12)$

if $x = [4, 16) \longrightarrow f(x) = [12, 14)$

if $x = [16; \infty) \longrightarrow f(x) = [14; \infty)$

prove that $f(x)$ is invertable and give $f^{-1}(x)$:

* INJECTIVITY: ✓

- let $x, y \in \mathbb{R}_{\geq 0}$ (domain)
- assume $f(x) = f(y)$
- since we know x, y are from the same subdomain

1 x and y are from 1

$$3x = 3y \qquad x = y \quad ✓$$

2 x, y are from 2

$$\sqrt{x} + 10 = \sqrt{y} + 10$$
$$\sqrt{x} = \sqrt{y}$$
$$x = y \quad ✓$$

3 x, y are from 3

$$x/8 + 12 = y/8 + 12$$
$$x/8 = y/8$$
$$x = y \quad ✓$$

*SURJECTIVITY:

- let $y \in \mathbb{R}_{\geq 0}$ (codomain)

1 $y = 3x \rightsquigarrow x = \frac{y}{3}$

take $x = \frac{y}{3}$

sanity check:

- $y \in \mathbb{R}_{\geq 0}$, so $y/3 \in \mathbb{R}_{\geq 0}$ so $x \in \mathbb{R}_{\geq 0}$
- $y \in [0; 12)$, so $\frac{y}{3} \in [0; 4)$, so $x \in [0; 4)$ 1

$f(x) = f(y/3) = 3(y/3) = y$ ✓

② $y = \sqrt{10} + x \qquad x = y - \sqrt{10} = (y-10)^2$

take: $x = (y-10)^2$

sanity check:

$y \in \mathbb{R}_{\geq 0}$; $(y-10)^2 \in \mathbb{R}_{\geq 0}$, so $x \in \mathbb{R}_{\geq 0}$

$y \in [12, 14)$; $(y-10)^2 \in [4;16)$; $x \in [4;16)$

$f(x) = f((y-10)^2) = \sqrt{(y-10)^2} + 10 = y - 10 + 10 = y$ ✓

③ $y = \frac{x}{8} + 12 \qquad x = 8(y-12)$

take $x = 8y - 96$

sanity check:

$y \in \mathbb{R}_{\geq 0}$; $\overset{y \geq 14}{8y - 96} \in \mathbb{R}_{\geq 0}$; $x \in \mathbb{R}_{\geq 0}$

$y \in [14;\infty)$; $8y - 96 \in [16;\infty)$; $x \in [16;\infty)$

$f(x) = f(8y-96) = \frac{8(y-12)}{8} + 12 = y$ ✓

* INVERSE

① $y = 3x \xrightarrow{f^{(-1)}_{(x)}} x = 3y \qquad y = \frac{3}{x} \rightarrow \underbrace{f^{-1}_{(x)} = \frac{3}{x}}$

② $y = \sqrt{x} + 10 \rightarrow x = \sqrt{y} + 10 \qquad x - 10 = \sqrt{y}$

$y = (x+10)^2 \rightarrow \underline{f^{-1}_{(x)} = (x+10)^2}$

③ $y = \frac{x}{8} + 12 \rightarrow x = \frac{y}{8} + 12 \qquad x - 12 = \frac{y}{8}$

$y = 8(x-12) \rightarrow f^{-1} = 8(x-12)$

$y = 8(x-12) \longrightarrow \boxed{f_{(x)} = 8(x-12)}$

$f^{-1}_{(x)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ where

$$f^{-1}_{(x)} \begin{cases} x/3 & \text{if } 0 \leq x < 12 \\ (x+10)^2 & \text{if } 12 \leq x < 14 \\ 8(x-12) & \text{if } 14 \leq x \end{cases}$$

---

let $f(x) : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$

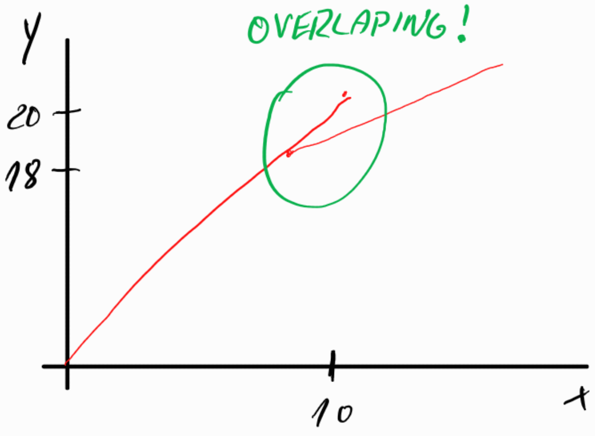
Prove or disprove injectivity: NOT INJECTIVE

- let $x, y \in \mathbb{R}$ (domain)
- assume $f(x) = f(y)$

$$x^2 = y^2$$
$$x = y$$
$$x = -y \quad !$$

---

let $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ where $f(x) \begin{cases} 2x & \text{if } 0 \leq x < 10 \quad (1) \\ 2(x-1) & \text{if } 10 \leq x \quad (2) \end{cases}$



Prove or disprove injectivity:

- let $x, y \in \mathbb{R}_{\geq 0}$ (domain)
- assume $f(x) = f(y)$

~~(1) x, y are from (1)~~

~~$2x = 2y$~~ $\quad x = y$

~~(2) $2(x-1) = 2(y-1)$~~ $\quad x = y$

BECAUSE

WRONG BECAUSE
THEY OVERLAP (and have
different values