(1) $f(x) = x(x^2-2)-1$ $\qquad c = \frac{a+b}{2}$ $\qquad [0,2]$

if $f(c)=0$ then root

$c = \frac{0+2}{2} = 1$

$f(1) = 1(1^2-2)-1 = -2$
$f(0) = -1$
$f(2) = 2(2^2-2)-1 = 3$

updated interval $[1,2]$

Newtons Method: $\quad p_{n+1} = p_n - f(p_n) / f'(p_n)$

$p_0 = \frac{1+2}{2} = 1.5$

$f'(x) = 3x^2-2$

$f(p_0) = f(1.5) = 1.5(1.5^2-2)-1 = -0.625$
$f'(p_0) = f'(1.5) = 3 \cdot 1.5^2 - 2 = 4.75$

$p_1 = 1.5 - \frac{-0.625}{4.75} = \underline{1.632}$ (4sf)

(2) $\dot{y} = 1/(1+ty)$ $\qquad y(0) = w_0 = 2$ $\qquad t \in [0; 1.5]$ $\qquad h = 0.5$

2^{nd} order Ralston $y(0.5) = w_1 = 2.33593750$

$f(0.5; 2.3359) \approx 0.4613$
$f(0,2) = 1$

$w_2 = y_{(1.0)} = 2.3359 + \frac{0.5}{2}(3 \cdot 0.4613 - 1) \approx \underline{2.4319}$

$f(1, 2.4319) \approx 0.2914$

$w = y = 2.4319 + \frac{0.5}{2}(3 \cdot 0.2914 - 0.4613) \approx 2.5351$

$w_3 = y(1.5) = 2.151\ldots \frac{\ldots}{2}(3 \cdot 0.\ldots \; 0.1613) = 2.5351$

abs. error $= |2.63577585 - 2.5351| = 0.1$

relative error $\approx 0.03 \approx 3.82\%$

(3) $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{0.509 - 0.838}{2.25 - 2} = -1.316$

$f[x_1, x_2] = -1.3160$

$f[x_2, x_3] = -0.5560$

$f[x_3, x_4] = \dfrac{0.020 - 0.041}{3 - 2.75} \approx -0.084$

$f[x_0, x_1, x_2] = 0$ $\qquad f[x_1, x_2, x_3] = 1.5200$

$f[x_2, x_3, x_4] = \dfrac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2} = \dfrac{-0.084 - (-0.5560)}{3 - 2.5} = 0.944$

$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \dfrac{1.5200 - 0}{2.75 - 2} = 2.0267$

$f[x_1, x_2, x_3, x_4] = \dfrac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1} = \dfrac{0.944 - 1.5200}{3 - 2.25} = -0.768$

$f[x_0] = 0.838$

$P(x) = 0.838 + (x-2)\left(-1.316 + (x-2.25)\left(0 + (x-2.5) \cdot 2.0267\right)\right)$

$P(2.333) = 0.390$ (3dp)

$f''''(x) \leq 140$

$|f(x) - P(x)| \leq \dfrac{\prod_{i=0}^{n} |x - x_i|}{(n+1)!} \left|f^{(n+1)}(\varepsilon)\right|$ $\qquad n = 3$, $x = 2.333$

$|f(x) - P(x)| \leq \dfrac{(2.333-2)(2.333-2.25)(2.333-2.5)(2.333-2.75)}{4!} \cdot 140 = \underline{\underline{0.011}}$

(4) $n = 4$ $\qquad \displaystyle\int_2^3 \frac{x}{x^3 - 6}\, dx$ $\qquad f(x) = \dfrac{x}{x^3 - 6}$ $\qquad h = \dfrac{3-2}{4} = 0.25$

$$T_4(f,2,3) = 0.25\left(\frac{1}{2}f(2) + f(2.25) + f(2.5) + f(2.75) + \frac{1}{2}f(3)\right)$$

$f(2) = 1$ $\quad f(2.25) \approx 0.4174$ $\quad f(2.5) \approx 0.2597$ $\quad f(2.75) \approx 0.1859$

$f(3) = 0.1429$

$$T_4(f,2,3) = 0.3586$$

$[2.0, 2.5]$

$$T_1(f,2,2.5) = \frac{f(2)+f(2.5)}{2} = 0.5\,\frac{1+0.2597}{2} = \underline{0.3149}$$

$$T_2(f,2,2.5) = 0.25\left(\frac{1}{2}f(2) + f(2.25) + \frac{1}{2}f(2.5)\right) = \underline{0.2618}$$

$$E_1 = \frac{1}{3}\left|T_1 - T_2\right| = \underline{0.0177}$$

$[2.5, 3]$

$$T_1(f,2.5,3) = 0.5\,\frac{f(2.5)+f(3)}{2} = 0.1000$$

$$T_2(f,2.5,3) = 0.25\left(\frac{1}{2}f(2.5) + f(2.75) + \frac{1}{2}f(3)\right) = 0.0968$$

$$E_2 = \frac{1}{3}\left|T_1 - T_2\right| = \underline{0.00107}$$

Total error $\quad E_1 + E_2 = 0.019$

5. $n = 6$ $\qquad [0; 2\pi]$

$m = 3$

$$a_0 = \frac{1}{3}\sum_{i=-3}^{2} f(x_i)\cdot \underbrace{\cancel{\cos 0}}_{1}$$

$$a_0 = \frac{1}{3}(2.08 + 2.26 + 1.63 + 0.18 + 0.43 + 1.53) = \underline{\underline{2.703}}$$

$$a_1 = \frac{1}{3}\sum_{i=-3}^{2} f(x_i)\cos(x_i)$$

$$a_1 = \frac{1}{3}\Big(2.08\cdot\underbrace{\cancel{\cos(0)}}_{1} + 2.26\cdot\underbrace{\cancel{\cos(\tfrac{\pi}{3})}}_{0.5} + 1.63\cdot\underbrace{\cancel{\cos(\tfrac{2\pi}{3})}}_{-0.5} + 0.18\cdot\underbrace{\cancel{\cos(\pi)}}_{-1} + 0.43\cdot\underbrace{\cancel{\cos(\tfrac{4\pi}{3})}}_{-0.5} + 1.53\cdot\underbrace{\cancel{\cos(\tfrac{5\pi}{3})}}_{0.5}\Big)$$

$$a_1 = 0.922$$

$$b_1 = \frac{1}{3} \sum_{i=-3}^{2} f(x_i) \sin(x_i)$$

$$b_1 = \frac{1}{3}\Big(2.08 \cdot \underbrace{\sin(0)}_{0} + 2.26 \cdot \underbrace{\sin(\tfrac{\pi}{3})}_{0.8660} + 1.63 \cdot \underbrace{\sin(\tfrac{2\pi}{3})}_{0.8660} + 0.18 \cdot \underbrace{\sin(\pi)}_{0} + 0.43 \cdot \underbrace{\sin(\tfrac{4\pi}{3})}_{-0.8660} + 1.53 \cdot \underbrace{\sin(\tfrac{5\pi}{3})}_{-0.8660}\Big)$$

$$b_1 = 0.557$$

$$S_1(t) = 1.352 + 0.922 \cos(t) + 0.557 \sin(t)$$

$$S_1(\pi/2) = 1.909$$

estimate $\int_0^{2\pi} f(t)^2 \, dt = \int_0^{2\pi} S_1(t)^2 \, dt$

$$S_1(t)^2 = \left(1.352 + 0.922 \cos(t) + 0.557 \sin(t)\right)^2$$

6 $A = \begin{bmatrix} 5 & 1 & 2 \\ -1 & 2 & 0 \\ 2 & 0 & -3 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1^{(1)} = \left(b_1 - a_{12} x_2^{(0)} - a_{13} x_3^{(0)}\right) / a_{11} = (1 - 1 \cdot 0 - 2 \cdot 0)/5 = 1/5 = 0.2$$
$$x_2^{(1)} = \left(b_2 - a_{21} x_1^{(1)} - a_{23} x_3^{(0)}\right) / a_{22} = (3 - (-1) \cdot 0.2 - 0 \cdot 0)/2 = 1.6$$
$$x_3^{(1)} = \left(b_3 - a_{31} x_1^{(1)} - a_{32} x_2^{(1)}\right) / a_{33} = (2 - 2 \cdot 0.2 - 0 \cdot 1.6)/(-3) = -0.5333$$

$$x^{(1)} = \begin{bmatrix} 0.2 \\ 1.6 \\ -0.533 \end{bmatrix}$$

$$x_1^{(2)} = \left(b_1 - a_{12} x_2^{(1)} - a_{13} x_3^{(1)}\right) / a_{11} = (1 - 1 \cdot 1.6 - 2 \cdot (-0.5333))/5 = 0.0933$$
$$x_2^{(2)} = \left(b_2 - a_{21} x_1^{(2)} - a_{23} x_3^{(1)}\right) / a_{22} = (3 - (-1) \cdot 0.0933 - 0 \cdot (-0.5333))/2 = 1.5467$$
$$x_3^{(2)} = \left(b_3 - a_{31} x_1^{(2)} - a_{32} x_2^{(2)}\right) / a_{33} = (2 - 2 \cdot 0.0933 - 0 \cdot 1.5467)/(-3) = -0.6045$$

$$x^{(2)} = \begin{bmatrix} 0.0933 \\ 1.5467 \\ -0.6045 \end{bmatrix}$$

Residual: $Ax^{(2)} - b$

$$Ax^{(2)} = \begin{bmatrix} 5 & 1 & 2 \\ -1 & 2 & 0 \\ 2 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0.0933 \\ 1.5467 \\ -0.6045 \end{bmatrix} = \begin{bmatrix} 0.8042 \\ 3.0001 \\ 2.0001 \end{bmatrix}$$

$$Ax^{(2)} - b = \begin{bmatrix} 0.8042 \\ 3.0001 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.1958 \\ 0.0001 \\ 0.0001 \end{bmatrix}$$