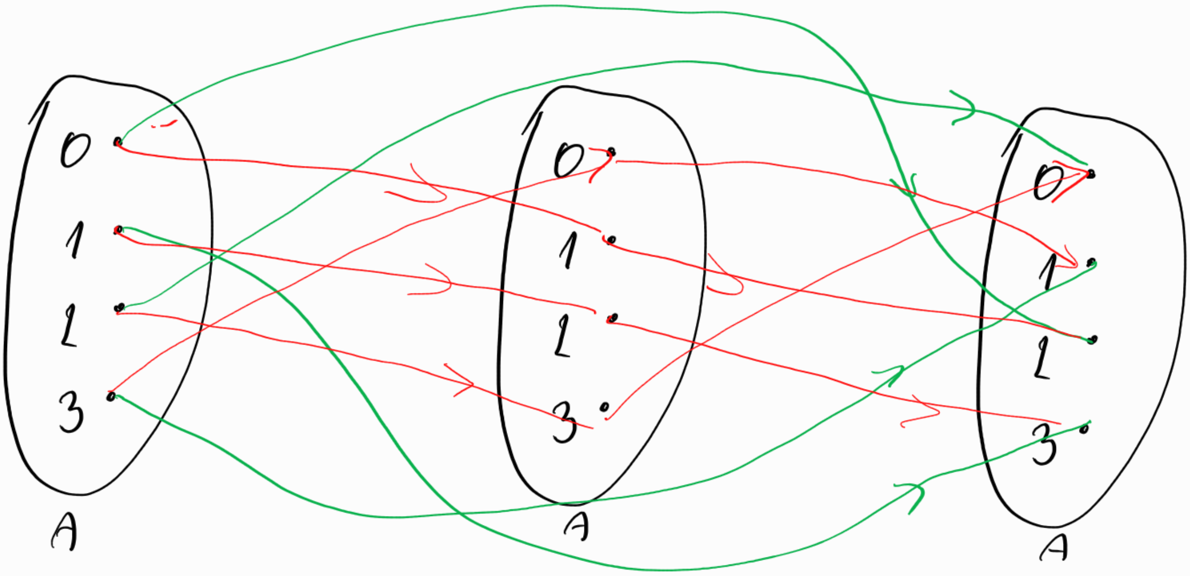
(1a) $A=\{0,1,2,3\}$ $\quad f: A \to A$ such that

$(f\circ f)(0)=2 \quad (f\circ f)(1)=3 \quad (f\circ f)(2)=0 \quad (f\circ f)(3)=1$



$$(f\circ f)\quad \begin{cases} x=0 & f(x)=1 \\ x=1 & f(x)=2 \\ x=2 & f(x)=3 \\ x=3 & f(x)=0 \end{cases}$$

(1b) let $\mathbb{R}^+=\{x\in\mathbb{R} : x\geq 0\}$ $\quad f:\mathbb{R}^+ \to \mathbb{R}^+$

$$f(x)\begin{cases} 6x & \text{if } 0\leq x\leq c \quad (1) \\ (x+1)^2-1 & \text{if } x>c \quad (2) \end{cases} \qquad c\in\{2,3,4,5\}$$

try $c=2$ $\begin{cases} 6x & \text{if } 0\leq x\leq 2 \\ (x+1)^2-1 & \text{if } x>2 \end{cases}$ $\quad y=[0;12)$, $y=(8;\infty)$ overlap

try $c=3$ $\begin{cases} y=[0;18] \\ y=(15;\infty) \end{cases}$ overlap

try $c=4$ $\begin{cases} y=[0;24] \\ y=(24;\infty) \end{cases}$ ✓

try $c=5$ $\begin{cases} y=[0;30] \\ y=(35;\infty) \end{cases}$

* INJECTIVITY:

- let $x, y \in \mathbb{R}^+$ (domain)
- assume $f(x) = f(y)$

(1) x, y are from (1)

$$6x = 6y \qquad x = y$$

(2) x, y are from (2)

$$(x+1)^2 - 1 = (y+1)^2 - 1$$

$$(x+1)^2 = (y+1)^2$$

$$\pm(x+1) = \pm(y+1)$$

$$x = y$$

* SURJECTIVITY

- let $y \in \mathbb{R}^+$ (codomain)

(1) $y = 6x \longrightarrow x = \frac{y}{6}$

take $x = y/6$

sanity check:

- $y \in \mathbb{R}^+$, so $y/6 \in \mathbb{R}^+$, so $x \in R^+$
- $y = [0;24]$, so $y/6 = [0;4]$, $x = [0;4]$

$$f(x) = f\left(\frac{y}{6}\right) = 6\left(\frac{y}{6}\right) = y \ \checkmark$$

(2) $y \in (x+1)^2 - 1$ $\qquad y + 1 = (x+1)^2$

$\sqrt{y+1} = x + 1$ $\qquad x = \sqrt{y+1} - 1$

take $x = \sqrt{y+1} - 1$

sanity check:

- $y \in \mathbb{R}^+$, so $\sqrt{y+1} - 1 \overset{y>24}{\in} \mathbb{R}^+$; $x \in \mathbb{R}^+$
- $y \in (24;\infty)$; $\sqrt{y+1} - 1 \in (4 \; ;\infty)$

$$f(x) = f\left(\sqrt{y+1} - 1\right) = \left(\sqrt{y+1} - 1 + 1\right)^2 - 1 = \left(\sqrt{y+1}\right)^2 - 1 =$$

$$= y + 1 - 1 = y \ \checkmark$$

$f^{-1}(x) : \mathbb{R}^+ \to \mathbb{R}^+$

$$f^{-1}(x) \begin{cases} x/6 & \text{if } 0 \le x \le 24 \\ \sqrt{x+1} - 1 & \text{if } x > 24 \end{cases}$$

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(2) $f: \mathbb{N} \to \mathbb{N}$ $\qquad g: \mathbb{N} \to \mathbb{N}$

$$f(x) = \begin{cases} x^2 & \text{if } 1 \le x \le 10 \\ 2x+1 & \text{if } x > 10 \end{cases}$$

$$g(x) = \begin{cases} x+2 & \text{if } 1 \le x \le 50 \\ 3x & \text{if } x > 50 \end{cases}$$

x	1	2	3	...	7	8	...	10	11	12	13	...	24
$f(x)$	1	4	9		49	64		100	23	25	27		49
$g(x)$	3	6	11		51	192		300	25	27	29		51

1 2 3

x	25	26	...
$f(x)$	51	53	
$g(x)$	153	159	

4

$$(g \circ f)(x) = \begin{cases} x^2 + 2 & \text{if } 1 \le x \le 7 \\ 3x^2 & \text{if } 8 \le x \le 10 \\ 2x+3 & \text{if } 11 \le x \le 24 \\ 6x+3 & \text{if } x > 24 \end{cases}$$

(3a) $\mathbb{R}^+ = \{x \in \mathbb{R} : x \ge 0\}$ $\quad f : \mathbb{R}^+ \to \mathbb{R}^+$

$$f(x) \begin{cases} x^2 & \text{if } 0 \le x \le 6 \quad (1) \\ \dots & \text{if } x > 6 \quad (2) \end{cases}$$

$c \in \{31, 32, 33\}$

$\left({}^{+}\sqrt{(t+3)} + C \quad \text{if } x > 6 \quad (2)$

(1) $y \in [0;36]$

(2) $y \in (34.73, \infty)$

$$6^2 = {}^{+}\sqrt{6+3} + C$$

$$36 = 3 + C$$

$$\boxed{C = 33} \qquad \text{choose } C = 33$$

Proof:

$*$ INJECTIVITY

- let $x, y \in \mathbb{R}^+$ (domain)
- assume $f(x) = f(y)$

(1) x, y are from (1)

$$x^2 = y^2$$

$$x = y \ \checkmark \quad (x, y \in \mathbb{R}^+)$$

(2) x, y are from (2)

$${}^{+}\sqrt{(x+3)} + 33 = {}^{+}\sqrt{(y+3)} + 33$$

$$\sqrt{(x+3)} = \sqrt{(y+3)}$$

$$x + 3 = y + 3$$

$$x = y \ \checkmark$$

$*$ SURJECTIVITY:

- let $y \in \mathbb{R}^+$ (codomain)

(1) $y = x^2 \qquad \underbrace{x = {}^{+}\sqrt{y}}$

take $x = {}^{+}\sqrt{y}$

sanity check:

$y \in \mathbb{R}^+$, ${}^+\sqrt{y} \in R^+$, $x \in R^+$

$y \in [0;36]$; ${}^+\sqrt{y} \in [0;6]$, $x \in [0;6)$

$f(x) = f({}^+\sqrt{y}) = ({}^+\sqrt{y})^2 = y$ ✓

2 $y = {}^+\sqrt{(x+3)} + 33$ $\qquad$ $y - 33 = \sqrt{(x+3)}$

$(y-33)^2 = x+3$ $\qquad$ $\underbrace{x = (y-33)^2 - 3}$

check: for such y, $y - 33 > 3$

$(y-33)^2 > 9$

$9 - 3 > 6$

$x > 6$

$f(x) = f((y-33)^2 - 3) = \sqrt{(y-33)^2 - 3 + 3} + 33 =$

$= \sqrt{(y-33)^2} + 33 = y - 33 + 33 = y$

*INVERSE:

$$f^{-1}_{(x)} \begin{cases} {}^+\sqrt{y} & \text{if } 0 \leq x \leq 36 \\ (y-33)^2 - 3 & \text{if } x > 36 \end{cases}$$

3b $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2 - 10$

$g: \mathbb{R} \to \mathbb{R}$ $g(x) = \begin{cases} 4x+7 & \text{if } x<0 \\ 5x^3 & \text{if } x \geq 0 \end{cases}$

$(g \circ f)(3) = g(f(3)) = g(-1) = \underline{\underline{3}}$

$f(3) = -1$

$(g \circ f)(4) = g(f(4)) = g(6) = \underline{\underline{1080}}$

$f(4) = 6$

4a $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ $f: \mathbb{R}^+ \to \mathbb{R}^+$

$f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } 0 \leq x < 6 \\ 2x+c & \text{if } x \geq 6 \end{cases}$ $c \in \{3, 4, 5, 6, 7, 8\}$

$\begin{cases} y = [0; 18) \\ y = (18; \infty) \end{cases}$ so I think $c = 6$

$f(x) \begin{cases} \frac{1}{2}x^2 & \text{if } 0 \leq x < 6 \\ 2x+6 & \text{if } 6 \geq x \end{cases}$ $y = [0; 18)$ ① $y = [18; \infty)$ ②

* INJECTIVE

• let $x, y \in \mathbb{R}^+$ (domain)

- assume $f(x) = f(y)$

(1) x, y are from (1)

$$\frac{1}{2}x^2 = \frac{1}{2}y^2$$
$$x^2 = y^2$$
$$x = y \quad (\text{because } x, y \in \mathbb{R}^+)$$

(2) x, y are from (2)

$$2x+6 = 2y+6$$
$$2x = 2y$$
$$x = y$$

* SURJECTIVE

- let $y \in \mathbb{R}^+$ (codomain)

(1) $y = \frac{1}{2}x^2 \qquad x^2 = 2y \qquad \boxed{x = \sqrt{2y}}$

take $x = \sqrt{2y}$

check:

$y \in \mathbb{R}^+$, $\sqrt{2y} \in \mathbb{R}^+$, $x \in \mathbb{R}^+$

$y \in [0;18)$; $\sqrt{2y} = [0;6)$; $x \in [0;6]$

$$f(x) = f(\sqrt{2y}) = \frac{1}{2}(\sqrt{2y})^2 = \frac{1}{2}2y = y$$

(2) $y = 2x+6 \qquad y-6 = 2x \qquad \boxed{x = \frac{y-6}{2}}$

take $x = \frac{y-6}{\ }$

check:

$y \in \mathbb{R}^+, \quad \frac{y-6}{2} \in \mathbb{R}^+, \quad x \in \mathbb{R}^+$

$y \in [18;\infty) \; ; \quad \frac{y-6}{2} \in [6;\infty) \; ; \quad x = [6;\infty)$

$$f(x) = f\left(\frac{y-6}{2}\right) = 2\left(\frac{y-6}{2}\right) + 6 = y - 6 + 6 = y$$

* INVERSION

$$f^{-1}(x) \begin{cases} \sqrt{2x} & \text{if } \quad 0 \leq x < 18 \\ \frac{x-6}{2} & \text{if } \quad x \geq 18 \end{cases}$$

1b $\quad A = \{0,1,2,3\} \qquad B = \{0,1,2,3,4\}$

$g: B \to B \qquad g(x) = 4 - x$

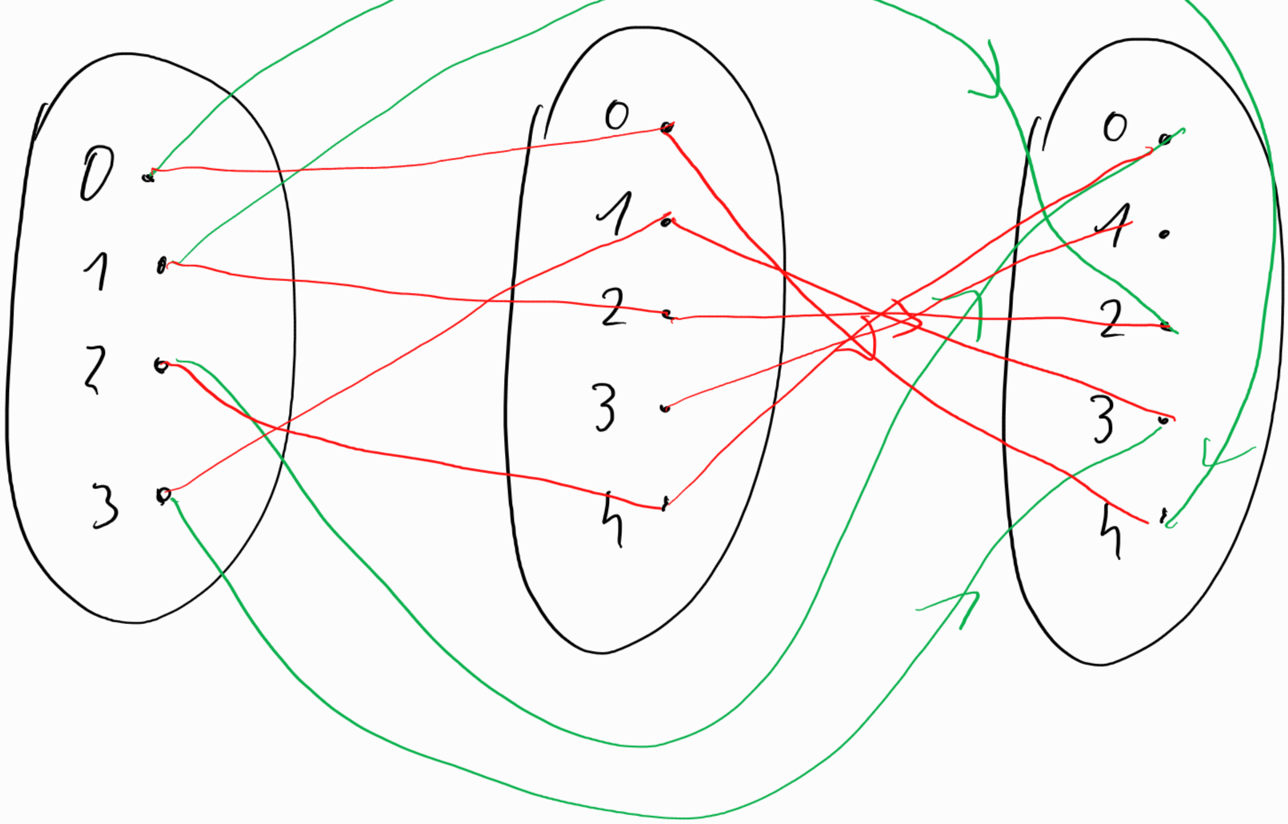
$f: A \to B \qquad (g \circ f)(0) = 4 \qquad (g \circ f)(1) = 2$

$\qquad (g \circ f)(2) = 0 \qquad (g \circ f)(3) = 3$

$g(0) = 4$
$g(1) = 3$
$g(2) = 2$
$g(3) = 1$
$g(4) = 0$

take

$f(0) = 0$
$f(1) = 2$
$f(2) = 4$
$f(3) = 1$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3
4