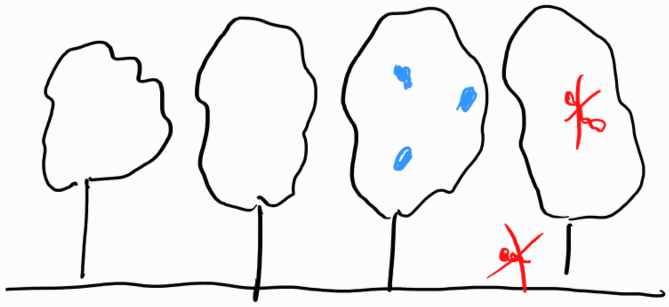
Proofs

- $\forall$ → for all
- $\exists$ → there exists
- $\neg$ → not / negation
- negation = flip the quantifiers and negate the body

Higher depth quantifiers

$(\exists \text{ tree } t)\,(\forall \text{ monkey } m)\,(m \text{ climbs } t)$

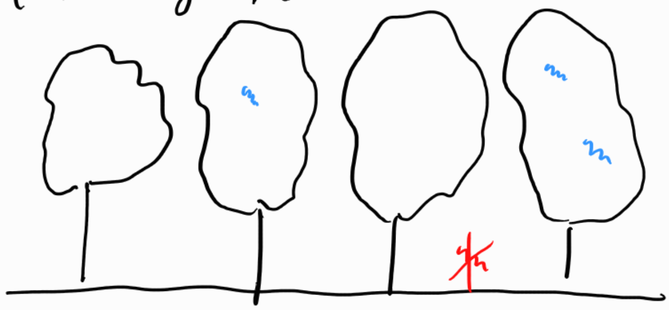


neg.

$(\forall \text{ tree } t)\,(\exists \text{ monkey } m)$

$(m \text{ doesn't climb } t)$

$(\forall \text{ monkey } m)\,(\exists \text{ tree } t)\,(m \text{ climbs } t)$



neg.

$(\exists \text{ monkey } m)\,(\forall \text{ tree } t)$

$(m \text{ doesn't climb } t)$

$(\forall \text{ tree } t)\,(\exists \text{ monkey } m)\,(m \text{ climbs } t)$

neg.

$(\exists \text{ tree } t)\,(\forall \text{ monkey } m)\,(m \text{ doesn't climb } t)$

the order of the quantifiers matter!

Counterexample

- $(\forall x \in \mathbb{N})(x^2 > 4)$ F

  counterexample: take $x=1$ $1^2 \not> 4$

- $(\forall x \in \mathbb{R})(x^2 > x)$ F

  counterexample: take $x=1$ $1^2 \not> 1$

  take $x = 1/10$ $(1/10)^2 \not> 1/10$ $1/100 \not> 1/10$

Direct Proof

- $(\forall x \in \mathbb{N})(x^2 \geq x)$

  direct proof:
  - let $x \in \mathbb{N}$
  - then $x^2 = x \cdot x \geq 1$ (because $x \in \mathbb{N}$, so $x \geq 1$)

    $1x = x$

  $\square$

- $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(3x + y \leq 4)$

  direct proof:
  - let x be an arbitrary integer
  - let $x \in \mathbb{Z}$
  - take $y = 4 - 3x$ ($y \in \mathbb{Z}$, because $x \in \mathbb{Z}$)
  - so, $3x + 4 - 3x \leq 4$

    $4 \leq 4$ ✓

$\square$

Proof by contradiction

- Theorem: The interval $(0,1)$ doesn't contain a biggest number

Proof by contradiction:

- suppose the interval contains a biggest number
- call this number x
- consider now $y = \frac{x+1}{2}$ (the average of x and 1)
- $y \in (0,1)$? Yes, because $y = \frac{x+1}{2} < \frac{1+1}{2} = 1$

  because $x < 1$

  and because $y = \frac{x+1}{2} > \frac{1}{2} > 0$

  because $x > 0$

- moreover $y = \frac{x+1}{2} > \frac{x+x}{2} = x$ ↯

  because $x < 1$, $1 > x$

- Hence, our assumption is wrong and thus $(0,1)$ doesn't contain a highest number

$\square$

Proof by contrapositive

$$p \Rightarrow q \iff \neg q \Rightarrow \neg p$$

- ($\forall$ natural number $p > 2$)(p is prime $\Rightarrow$ p is odd) $\boxed{T}$

  contrapositive:

  ($\forall$ natural number $p > 2$) (p is even $\Rightarrow$ p is not a prime)

  - let $p \in \mathbb{N}$ and $p > 2$
  - assume p is even
  - so p is divisible by 2 and $p \neq 2$
  - so p is divisible by some number (namely 2) other then 1 and itself

and vosely

• so p is not prime

$\square$

• Theorem: $(\forall x \in \mathbb{Z})(x^2 \neq 3)$

Proof by contrapositive: $(x^2 = 3) \Rightarrow (x \notin \mathbb{Z})$

• assume $x^2 = 3$

• then $x = \pm\sqrt{3}$

• either way $x \notin \mathbb{Z}$

$\square$

Biconditional proofs

if and only if / iff

$p \Leftrightarrow q$

$(p \Rightarrow q) \land (q \Rightarrow p)$

• Theorem: $(\forall x \in \mathbb{Z})((x^2 \text{ is even}) \Leftrightarrow (x \text{ is even}))$ $\boxed{T}$

Proof: • let $x \in \mathbb{Z}$

"$\Leftarrow$" • Assume x is even

• so $x = 2k \quad (k \in \mathbb{Z})$

• $x^2 = (2k)^2 = 4k^2 = \underline{2}(2k^2)$

even

"$\Rightarrow$"

$\begin{pmatrix} x^2 \text{ is even} \Rightarrow x \text{ even} \\ x \text{ odd} \Rightarrow x^2 \text{ odd} \end{pmatrix}$

• x^2 is even

• assume x is odd

• so $x = 2k+1 \quad (k \in \mathbb{Z})$

• $x^2 = (2k+1)^2 = 4k^2 + 4k + 1$

odd

• so x^2 is odd

$\square$

• Theorem $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})((x \geq y+1) \Leftrightarrow (x^2 \geq y^2 + 3))$

Proof: • let $x, y \in \mathbb{N}$

"$\Rightarrow$" • assume $x \geq y+1$ $\quad (x \geq 0$ and $y+1 \geq 0)$

• then $x^2 \geq (y+1)^2$

$x^2 \geq y^2 + 2y + 1$

$x^2 \geq y^2 + 2 \cdot 1 + 1$

$x^2 \geq y^2 + 3$

"$\Leftarrow$" • assume $x < y+1$, so $x \leq y$

contrapositive

$(x < y+1) \Leftrightarrow (x^2 < y^2 + 3)$ • then $x^2 \leq y^2 < y^2 + 3$

$\square$

Induction Proof:

• for all natural numbers "$\sum_{i=1}^{n} i = \frac{n}{2}(n+1)$" holds

* BASE CASE:

Does $P(1)$ holds?

$$\sum_{i=1}^{1} i \stackrel{?}{=} \frac{1}{2}(1+1)$$

$$1 \stackrel{?}{=} 1 \quad \checkmark$$

* INDUCTIVE STEP

$P(n) \Rightarrow P(n+1)$

our goal is to prove $P(n+1)$:

$$\sum_{i=1}^{n+1} i = \frac{n+1}{2}(n+1+1)$$

• let $n \in \mathbb{N}$

- we assume $P(n)$ holds
- consider:

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$$

$$= \frac{n}{2}(n+1) + (n+1)$$

$$= (n+1)\left(\frac{n}{2}+1\right)$$

$$= (n+1)\left(\frac{n+2}{2}\right) = \underline{\underline{\frac{1}{2}(n+1)(n+2)}}$$

$\square$

- Proove $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$

* BASE CASE

$P(1)$ $\quad 4^1 - 1 = 3 \quad$ 3 is divisible by 3 ✓

* INDUCTIVE STEP:

- let $n \in \mathbb{N}$
- assume $P(n)$ holds
- we want to proove $P(n+1)$
- observe:

$$4^{n+1} - 1 = 4^n \cdot 4 - 1$$

$$= 4 \cdot 4^n - 4 + 3$$

$$= 4(\underbrace{4^n - 1}_{\text{divisible by } 3}) + \underbrace{3}_{\text{divisible by } 3}$$

- so the whole expression is divisible by 3

$\square$

- Proove for all $n \in \mathbb{N}$, $n \geq 3$

$$n^2 > 2n + 1$$

* BASE CASE:

  $P(3)$: $3^2 > 2 \cdot 3 + 1$ $9 > 7$ ✓

* INDUCTIVE STEP

  - let $n \in \mathbb{N}$, $n \geq 3$
  - assume $P(n)$ holds
  - we want to proove $P(n+1)$
  - $P(n+1)$: $(n+1)^2 > 2(n+1) + 1$

  - note that $\underbrace{(n+1)^2}_{\text{new LHS}} - \underbrace{n^2}_{\text{old LHS}} = 2n + 1$ (by algebra)

  - new RHS $2(n+1)+1$
    old RHS $2n+1$
    $\Big\}$ new RHS − old RHS $= 2$
    so increase of RHS is 2

  - if I can show:

    increase LHS $\geq$ increase RHS, we are done

  - $2n + 1 \overset{?}{\geq} 2$ Yes, because $n \geq 3$