Incognito mock exam

(2)

$$\sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

* BASE CASE:

$$P(1) \quad (-1)^1 \cdot 1^2 = \frac{(-1)^1 \cdot 1 \cdot (1+1)}{2}$$

$$-1 = \frac{-1 \cdot 2}{2}$$

$$-1 = -1 \quad \checkmark$$

* INDUCTION STEP:

- $P(n)$ holds
- we want to show $P(n+1)$ holds too:

$$\sum_{i=1}^{n+1} (-1)^i i^2 = \frac{(-1)^{n+1} \cdot (n+1) \cdot (n+2)}{2}$$

$$LHSN + (-1) \cdot (n+1)$$

$$\sum_{i=1}^{n+1} (-1)^i i^2 = \sum_{i=1}^{n} (-1)^i i^2 + (-1)^{n+1} \cdot (n+1)^2$$

$$\underbrace{\frac{(-1)^{n+1} \cdot (n+1) \cdot (n+2)}{2} - \frac{(-1)^n n (n+1)}{2}}$$

$$=$$

$$(-1)^{n+1} \cdot (n+1)^2$$

$$\frac{1}{2} \left[(-1)^{n+1} \cdot \cancel{(n+1)} \cdot (n+2) - (-1)^n \cdot n \cancel{(n+1)} \right] = (-1)^{n+1} \cdot (n+1)^{\cancel{2}}$$

$$(-1)^{n+1} \cdot (n+2) - (-1)^n \cdot n = 2(-1)^{n+1} \cdot (n+1)$$

$(\text{if } n+1 \neq 0,\ n \geq 1)$

1 n is even

$(-1)^n = 1$, $(-1)^{n+1} = -1$ gives:

$$-(n+2) - n = -2(n+1)$$

$$-2n-2=-2n-2 \quad \checkmark$$

(2) n is odd

$(-1)^n=-1$, $(-1)^{n+1}=1$ gives

$$(n+2)+n = 2(n+1)$$

$$2n+2=2n+2 \quad \checkmark$$

$n \in \mathbb{N}$, so n is either even or odd

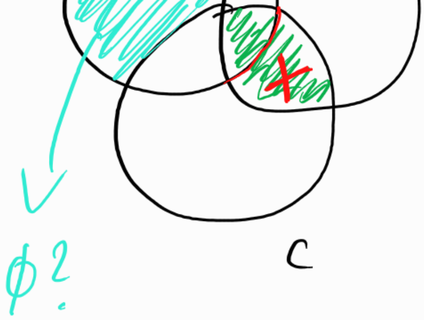
this shows $LHS + \Delta = RHS + \Delta$, because $\Delta L = \Delta R$, so $P(n+1)$

This concludes induction

$\square$

(3a) forall sets A, B, C:

$$(B \cap C \subseteq A) \Rightarrow ((A \setminus B) \cap (A \setminus C) = \emptyset)$$



Probably F

negation:

there exist sets A, B, C:

$$((A \setminus B) \cap (A \setminus C) \neq \emptyset) \wedge (B \cap C \subseteq A)$$

let:

$A = \{1, 2, 3\}$

$B = \{3, 4\}$

$C = \{3, 5\}$

$B \cap C \subseteq A$? ✓

$\{3\} \subseteq A$

$(A \setminus B) \cap (A \setminus C) \neq \emptyset$? ✓

$\{1, 2\} \cap \{1, 2\} = \{1, 2\} \neq \emptyset$

So the statement is <u>False</u>

(3b) Prove or disprove:

$(\forall \text{ sets } A, B, C, D)$

$$((A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)))$$

$\big((A\times B)\cup(C\times D) \subseteq (A\cup C)\times(B\cup D)\big)$

tryout:

$A=\{1,2,3\}$

$B=\{2,3,4\}$

$C=\{5,6\}$

$D=\{7\}$

$(A\cup C)=\{1,2,3,5,6\}$

$\times$

$(B\cup D)=\{2,3,4,7\}$

$=$

$\{(1,2),(1,3)\ \ldots\ (6,7)\}$

$(A\times B)\cup(C\times D)$

$\{(1,2),(1,3),\ \ldots,(3,4)\}\cup\{(5,7),(6,7)\}$

Prove by direct proof:

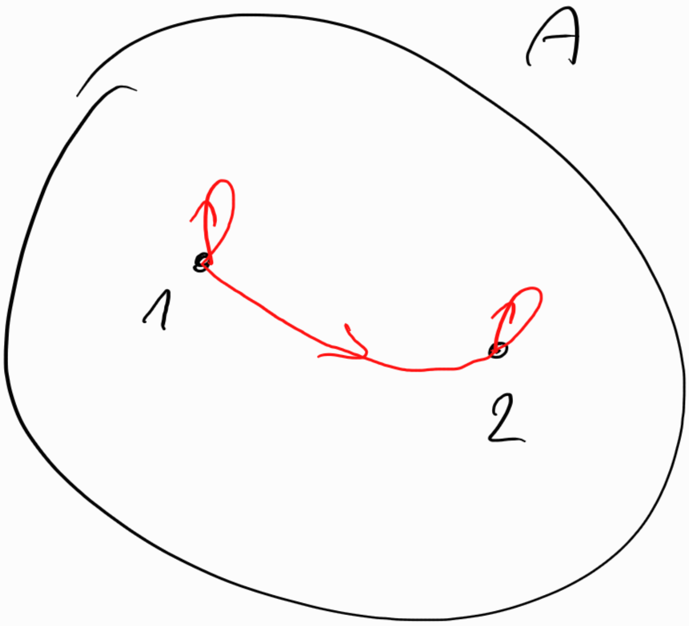
- let A,B,C,D be arbitary sets
- want to show $(A\times B)\cup(C\times D)\subseteq(A\cup C)\times(B\cup D)$
- statement: $\forall x\in(A\times B)\cup(C\times D):\big(x\in(A\cup C)\times(B\cup D)\big)$
- let $(x,y)\in(A\times B)\cup(C\times D)$
- we know $(x\in A\land y\in B)\lor(x\in C\land y\in D)$
- want to show $x\in A\cup C$ and $y\in B\cup D$
- because $(x,y)\in(A\cup C)\times(B\cup D)$

$x \in (A \cup C)$ and $y \in (B \cup D)$

- we know that $x \in A$ or $x \in C$, therefore $x \in A \cup C$
- we know that $y \in B \cup D$
- thus $(x, y) \in (A \cup C) \times (B \cup D)$ TRUE

$\square$

4a



4b $A = P(\{a, b, c\})$

XRY means "$|X| = |Y|$"

$$P = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

1st	2nd	3rd	4th
$\emptyset$	$\subset\{a\}$	$\subset\{a,b\}$	$\subset\{a,b,c\}$
	$\subset\{b\}$	$\subset\{a,c\}$	
	$\subset\{c\}$	$\subset\{b,c\}$	

4 equivalence classes

6b

$$(\forall x\in\mathbb{R})(\forall y\in\mathbb{R})(\exists z\in\mathbb{N})\big((z^2\geq x^2+y^2)\wedge(z<5)\big)$$

counter example:

let $x=5$, $y=6$,

$$x^2+y^2=5^2+6^2=61$$

$z=1 \rightarrow z^2=1 \ngeq 61$

$z=2 \rightarrow z^2=4 \ngeq 61$

$z=3 \rightarrow z^2=9 \ngeq 61$

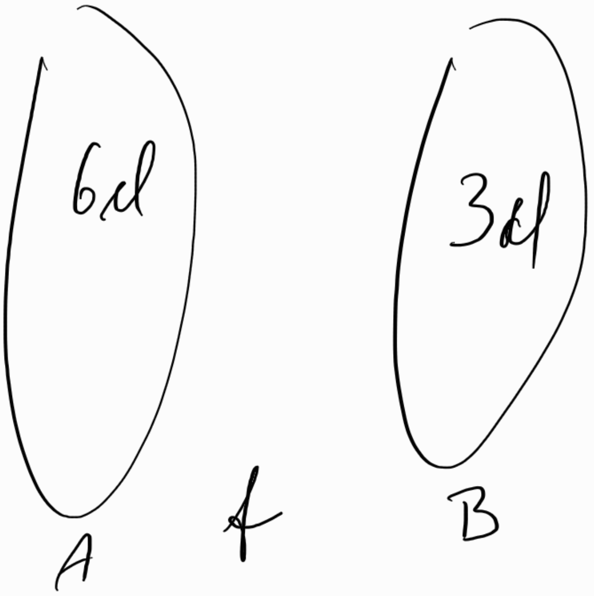
$z=4 \rightarrow z^2=16 \ngeq 61$

<u>doesn't hold</u>

covers all possible $x \in \mathbb{N}$: $x<5$

neg. $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{N})[(z^2 < x^2+y^2) \vee (z \geq 5)]$

5a



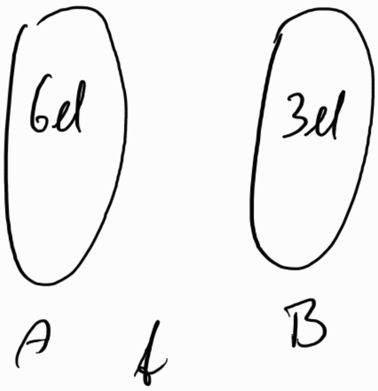
$f: A \to B$

$|A| = 6$

$|B| = 3$

$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6 = \underline{\underline{729}}$

5b



$f: A \to B$

$|A| = 6$

$|B| = 3$

answer: 0

because:

invertible $\iff$ injection $\wedge$ surjection

f can't be injective because $B \subset A$

5c

40 grandchildren

20 chocolates