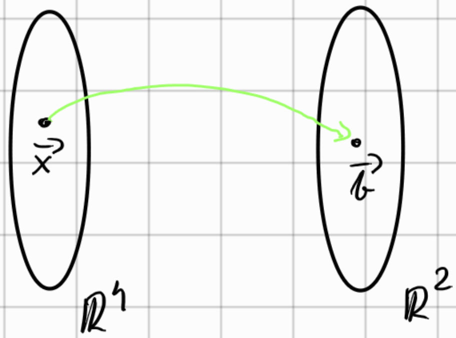
matrix vector product:

$$\underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 13 \\ 21 \end{bmatrix}}_{\vec{b}}$$

properties:

- $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$
- $A(c\cdot\vec{u}) = c\cdot(A\vec{u})$

multiplication by $A$ transforms $\vec{x}$ into $\vec{b}$




Transformation / function / mapping

$$\underset{\text{transformation operator}}{T}(\overset{\text{input}}{\vec{x}}) = \vec{y} \leftarrow \text{output}$$

$$T: \underset{\text{domain}}{\mathbb{R}^n} \rightarrow \underset{\text{codomain}}{\mathbb{R}^m}$$

image: $T(\vec{x})$

range: the set of all images

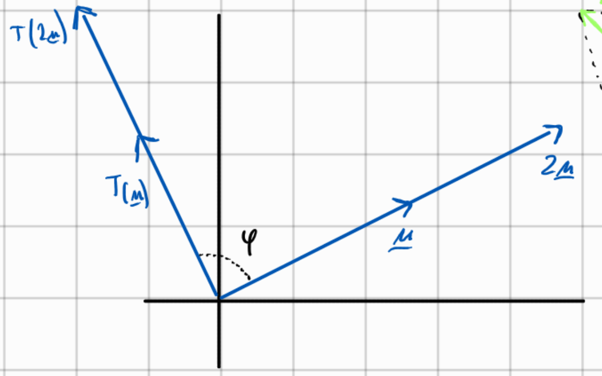
A transformation is linear if:

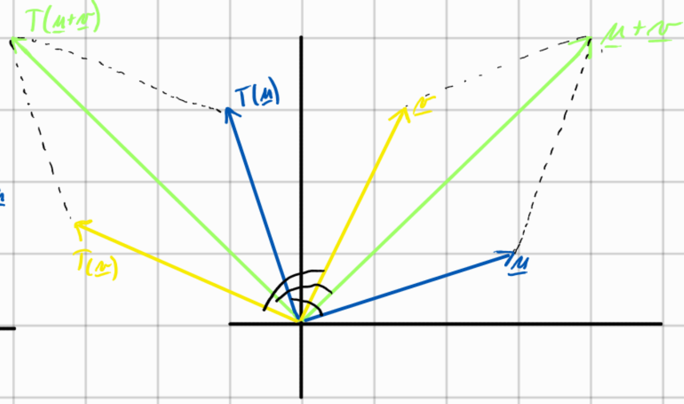
* $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
* $T(c\cdot\vec{u}) = c\cdot T(\vec{u})$

If $T$ is linear, then $T(\vec{0}) = \vec{0}$

Rotation: (around the origin through an angle $\varphi$)

## Rotation: (around the origin through an angle $\varphi$)




So, it's a linear transformation

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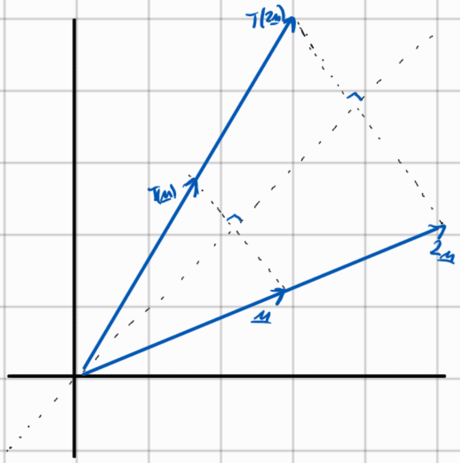
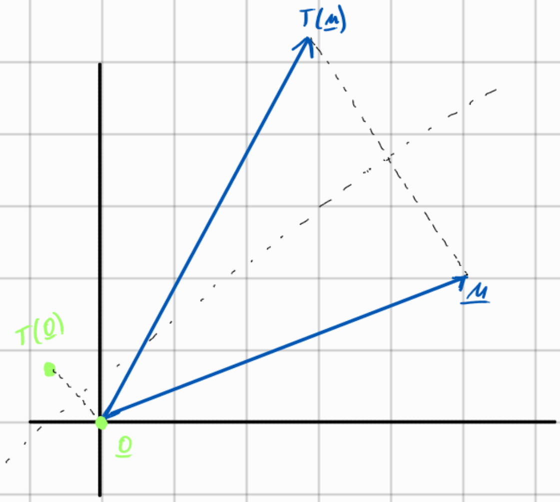
## Reflection (in a line through the origin)




So, it is a linear transformation

---

## Reflection (in a line NOT through the origin)




NOT a linear transf. (because $T(\underline{0}) \neq \underline{0}$)

- A transformation is linear if :

$\rightarrow T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
$\rightarrow T(c\cdot\vec{u}) = c\cdot T(\vec{u})$

Alternative definition:
$\rightarrow T(c\cdot\vec{u} + d\cdot\vec{v}) = c\cdot T(\vec{u}) + d\cdot T(\vec{v})$

If $T$ is linear, then $T(\vec{0}) = \vec{0}$

- transformation of the matrix product:

$$\underbrace{A}_{m\cdot n}\underbrace{\vec{x}}_{n\cdot 1} = \underbrace{b}_{m\cdot 1}$$

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$T(\vec{u}) = A\cdot(\vec{u})$$

properties of vector matrix:

* $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$
$T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A(\vec{u}) + A(\vec{v}) = T(\vec{u}) + T(\vec{v})$

* $A(c\cdot\vec{u}) = c\cdot A(\vec{u})$
$T(c\cdot\vec{u}) = A(c\cdot\vec{u}) = c(A\vec{u}) = c\,T(\vec{u})$

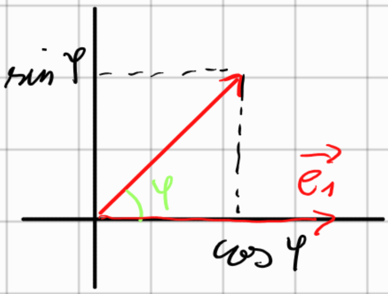
$\Rightarrow$ Every matrix transformation is a linear transformation

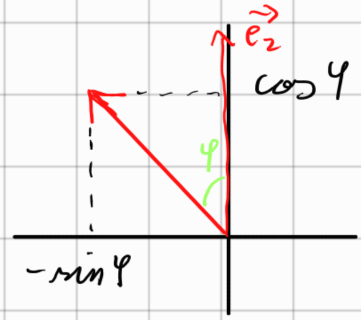
The opposite is also true
(at least in the context $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$)

Theorem: let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a lin. transform.
There is a unique matrix $A$ such that for $\vec{x} \in \mathbb{R}^n$, $A\vec{x} = T(x)$

(EX) rotation around the origin through an angle $\varphi$

$T: \mathbb{R}^2 \to \mathbb{R}^2$




So $T\vec{e_1} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$ So $T\vec{e_2} = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$

So $A = \left[ T(\vec{e_1}) \;\; T(\vec{e_2}) \right] = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

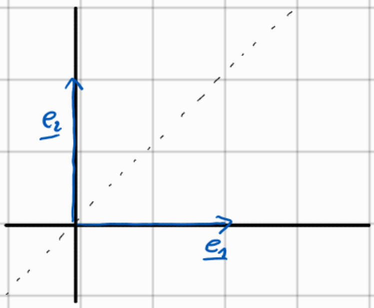
---

suppose the standard matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

what is the geometric interpretation?

$T(\underline{e_1}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{e_2}$

$T(\underline{e_2}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{e_1}$

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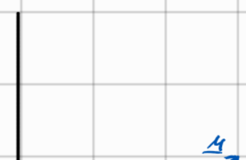
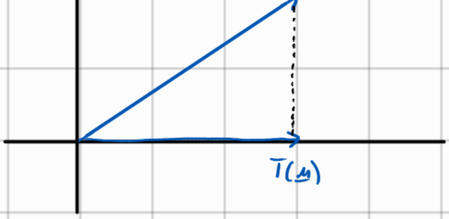
Projection (onto the x-axis)



This is a linear transformation with standard matrix




$$A = \left[ T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \right] = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

So, $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$

---

## SURJECTIVITY:

a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is surjective/onto if each $\underline{b} \in \mathbb{R}^m$ is the image of at least one $\underline{x} \in \mathbb{R}^n$
( range = codomain )

## INJECTIVITY:

a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is injective/one-to-one if each $\underline{b} \in \mathbb{R}^m$ is the image of at most one $\underline{x} \in \mathbb{R}^n$

---

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with a standard matrix $A$

$A$ has pivot in every column:
$\iff$ columns of $A$ are linearly independent
$\iff$ $A\underline{x} = \underline{0}$ has only the trivial solution
$\iff$ $T(\underline{x}) = \underline{0}$ has only the trivial solution
$\iff$ $T$ is injective

$A$ has pivot in every row:
$\iff$ columns of $A$ span $\mathbb{R}^m$
$\iff$ for each $\underline{b} \in \mathbb{R}^m$, $A\underline{x} = \underline{b}$ has a solution
$\iff$ for each $\underline{b} \in \mathbb{R}^m$, $T(\underline{x}) = \underline{b}$ has a solution

$\Longleftrightarrow T$ is surjective

## Matrix algebra

- equality: same size and the corresponding entries are equal
- addition: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 4 \\ 7 & 5 \end{bmatrix}$ same size !
- scaling: $2 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{bmatrix}$
- matrix product: $\underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}}_{A} \cdot \begin{array}{ccc} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \\ \underline{b}_1 \quad \underline{b}_2 \quad \underline{b}_3 \end{array} = \begin{bmatrix} A\underline{b}_1 & A\underline{b}_2 & A\underline{b}_3 \end{bmatrix}$

$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Each col of AB is a linear combination of the cols of A with the entries of the corresponding col of B being the weights

$\rightarrow$ we need # cols of A = # rows of B

$$\left.\begin{array}{l} A: m \times n \\ B: n \times p \end{array}\right\} \quad C = AB \quad m \times p$$

In general $AB \neq BA$

## Composition of linear transformations

$T_1 : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix $A$: $m \times n$
$T_2 : \mathbb{R}^m \to \mathbb{R}^q$ — " — $B$: $q \times m$

$T = T_2 \circ T_1 = T_2(T_1)$

Then, $T : \mathbb{R}^n \to \mathbb{R}^q$ and standard matrix $C$: $q \times n$ where $C = BA$

$$T(\underline{x}) = T_2(T_1(x)) = B(A\underline{x}) = \underbrace{BA}_{C}\,\underline{x}$$

**Transpose:** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

A matrix is **symmetric** if $A^T = A$ eg: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

**identity matrices** are **symmetric** $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Power** of a **square** $(n \times n)$ matrix $A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k\text{-times}}$

$A^0 = I_n$