$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \qquad \varphi = \pi/2 \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A$$

$\mathbb{R}^2 \to \mathbb{R}^2$ $\qquad \lambda^2 + 1 = 0 \qquad \lambda_1 = i \qquad \lambda_2 = -i$

$\mathbb{C}^2 \to \mathbb{C}^2$

$$A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \sim \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= \cos\varphi + i \cdot \sin\varphi \\ \lambda_2 &= \cos\varphi - i \cdot \sin\varphi \end{aligned}$$

$$\underline{x}_1 = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \underline{x}_2 = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let A and B be 2 $n \times n$ matrices

A and B are similar $\iff \exists$ invertible matrix P

s.t. $A = PBP^{-1}$

$B = P^{-1}AP$

Theorem: If A and B are similar, then they have the same eigenvalues

Proof: $|B-\lambda I| = |P^{-1}AP - \lambda I| = |P^{-1}AP - \lambda P^{-1}P| =$

$= |P^{-1}(AP - \lambda P)| = |P^{-1}(A-\lambda I)P| =$

$= |P^{-1}|\cdot|A-\lambda I|\cdot|P| = \frac{1}{\cancel{|P|}}\cdot|A-\lambda I|\cdot\cancel{|P|} = |A-\lambda I|$

So, A and B have the same characteristic polynomial

$\Rightarrow$ same eigenvalues

$$A^k = \underbrace{A\cdot A\cdot A\cdots A}_{k \text{ times}}$$

For diagonal matrix ex:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad D^2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

$$D^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$$

If A is similar to a diagonal matrix D, then ex:

$\hookrightarrow A = PDP^{-1}$

$$A^k = \underbrace{A\cdot A\cdot A\cdots A}_{k \text{ times}} = PD\cancel{P^{-1}}\cdot\cancel{P}D\cancel{P^{-1}}\cdots\cancel{P}DP^{-1}$$

$$= PD^kP^{-1}$$

A is called diagonalizable if A is similar to a diagonal matrix $\hookrightarrow A = PDP^{-1}$

* How to build D?

- A is similar to D $\Rightarrow$ A and D have the same eigenvalues: $\lambda_1, \dots, \lambda_n$
- D is a diagonal matrix $\Rightarrow$ the eigenvalues are on the diagonal

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \; \lambda_n \end{bmatrix}$$

* How to build P?

- $P = \{\underline{v_1}, \dots, \underline{v_n}\}$ where $\{\underline{v_1}, \dots, \underline{v_n}\}$ are lin. indep. in $\mathbb{R}^n$ (because P is invertible)
- $A = PDP^{-1} \Rightarrow AP = PD$, where

$AP = A[\underline{v_1}, \dots, \underline{v_n}] = [A\underline{v_1}, \dots, A\underline{v_n}]$

and $PD = [\underline{v_1}, \dots \underline{v_n}] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \; \lambda_n \end{bmatrix} = [\lambda_1 \underline{v_1}, \dots, \lambda_n v_n]$

So, for $AP = PD$, we need $A\underline{v_i} = \lambda_i \underline{v_i}$ $\forall i \in \{1, \dots, n\}$

$\Rightarrow \underline{v_1}, \dots, \underline{v_n}$ are the eigenvectors correspon.

so the eigenvalues $\lambda_1, \dots, \lambda_n$

So, A is diagonalizable $\Leftrightarrow$ A has n lin. indep. eigenvectors

* if there are n distinct eigenvalues $\Rightarrow$ n lin. indep. eigenvectors

$\Rightarrow A$ is diagonalizable

eigenvectors from diff. eigenspaces are lin. indep.

EX $\quad A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\qquad \lambda_1 = 1 \,;\; \lambda_2 = 2$

diagonalizable ✓ $(\lambda_1 \neq \lambda_2)$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

eigenspace of $\lambda_1 = 1$: $A - 1I = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So, pick, for example, $\underline{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

eigenspace of $\lambda_2 = 2$: $A - 2I = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Pick, for ex., $\underline{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Then $P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Check: $A = PDP^{-1}$ $\quad\quad AP = PD$

$$AP = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

$$\parallel$$

$$PD = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \quad \checkmark$$

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* What if the eigenvalues of A are NOT all distinct?
(some have multi > 1)

Is A diagonalizable? It depends ...

# of lin. indep. eigenvectors corresponding to λ $=$ dim. of the eigenspace of λ $=$ $\dim(\text{Nul}(A - \lambda I))$

$\leq$ mult. of λ

$\uparrow$

if strictly $<$ for some λ, then A is NOT diagonalizable

Theorem:

an $n \times n$ matrix A is diagonalizable $\iff$ the sum of the dimensions of the eigenspaces equals n

EX $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Is A diag.?

$\lambda = 0$ with multiplicity 2

$A - 0I = A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 1 free var

so, the $\dim(\mathrm{Nul}(A - \lambda I)) = 1$
$< 2 =$ mult. of λ

and thus A is not diagonalizable

EX $A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$

Is -5 an eigenvalue?

$A - (-5)I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ It has 2 free var, so $\lambda = -5$ is an eigenvalue with mult. 2 or 3

- if mult. $= 2$ $\to$ then diag.
- if mult. $= 3$ $\to$ then not diag.

$$\left.\begin{array}{l} \operatorname{trace}(A) = (-4)+(-3)+(-2) = -9 \\ \operatorname{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 = (-5)+(-5)+\lambda_3 \end{array}\right\}$$

$$-10 + \lambda_3 = -9 \quad \rightarrow \quad \underline{\underline{\lambda_3 = 1}}$$

So, A is diag.

$$D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

eigenspace of -5: $\underline{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\underline{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

eigenspace of 1: $A - 1I = \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$

$$\underline{x} = x_3 \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix} \qquad \underline{v_3} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Then $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$