$\boxed{6.1}$ (1) $\underline{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\underline{u} \cdot \underline{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \end{bmatrix} = 5$$

$$\underline{v} \cdot \underline{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \end{bmatrix} = 4$$

$$\frac{\underline{v} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} = \frac{4}{5}$$

(9) $\begin{bmatrix} -30 \\ 40 \end{bmatrix}$ find the unit vector:

$\frac{1}{len} \cdot \underline{v}$ → formula for unit vector

$$len \begin{bmatrix} -30 \\ 40 \end{bmatrix} = \sqrt{(-30)^2 + 40^2} = 50$$

$$\frac{1}{50} \cdot \begin{bmatrix} -30 \\ 40 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$

(13) distance between $x = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$

$$\|x - y\|^2 = [10 - (-1)]^2 + [(-3) - (-5)]^2 = \underline{\underline{125}}$$

$$dist(x, y) = \sqrt{125} = \underline{\underline{5\sqrt{5}}}$$

(15) orthogonal?

$$a = \begin{bmatrix} 8 \\ -5 \end{bmatrix} \qquad b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$a \cdot b = 8(-2) + (-5)(-3) = -1 \neq 0$$ not orthogonal

(31) $$\underline{u} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} -8 \\ -7 \\ 4 \end{bmatrix}$$

$$\underline{u} \cdot \underline{v} = 3(-8) + (-4)(-7) + (-1)(4) =$$
$$-24 + 28 - 4 = 0$$

$$\|u\|^2 = \underline{u} \cdot \underline{u} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} \begin{bmatrix} 3 & -4 & -1 \end{bmatrix} = 9 + 16 + 1 = \underline{\underline{25}}$$

$$\|v\|^2 = v \cdot v = \begin{bmatrix} -8 \\ -7 \\ 4 \end{bmatrix} \begin{bmatrix} -8 & -7 & 4 \end{bmatrix} = 64 + 49 + 16 = \underline{\underline{129}}$$

$$\|u + v\|^2 = (u + v) \cdot (u + v) = \begin{bmatrix} -5 \\ -11 \\ 3 \end{bmatrix} \begin{bmatrix} -5 & -11 & 3 \end{bmatrix}$$

$$= 25 + 121 + 9 = \underline{\underline{155}}$$

$\boxed{6.2}$

1 is set are orthogonal

$$\left\{ \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 \end{bmatrix} = -5 + 8 - 3 = 0$$

$$\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & -4 & -7 \end{bmatrix} = 15 - 8 - 7 = 0$$

$$\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \begin{bmatrix} 3 & -4 & -7 \end{bmatrix} = -3 - 16 + 21 = 2 \neq 0$$

set not orthogonal

5 is set orthogonal ?

$$\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 4 \end{bmatrix} = -3 - 6 - 3 + 12 = 0$$

$$\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 8 & 7 & 0 \end{bmatrix} = 9 - 16 + 7 + 0 = 0$$

$$\begin{bmatrix} \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 8 & 7 & 0 \end{bmatrix} = -3 + 24 - 21 + 0 = 0$$

set is orthogonal

(9) $$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 1 \end{bmatrix} = 1 + 0 - 1 = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 4 \end{bmatrix} = 4 + 0 - 4 = 0$$

$$\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 4 \end{bmatrix} = 4 - 8 + 4 = 0$$

set $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is orthogonal

since $\underline{u}_1, \underline{u}_2, \underline{u}_3$ are $\neq \underline{0}$ $\rightarrow$ lin. indep.

2 vectors automatically form a basis for $\mathbb{R}^2$

$$\underline{x} = \frac{\underline{x} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 + \frac{\underline{x} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \underline{u}_2 = -\frac{9}{10} \underline{u}_1 + \frac{13}{20} \underline{u}_2$$

(11) compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

$$\underline{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\hat{\underline{y}} = \frac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} \underline{u} = \frac{10}{20} \underline{u} = \frac{1}{2} \underline{u} = \frac{1}{2} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}}$$

$$\underline{y} \cdot \underline{u} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \begin{bmatrix} -4 & 2 \end{bmatrix} = -4 + 14 = \underline{\underline{10}}$$

$$\underline{u} \cdot \underline{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \begin{bmatrix} -4 & 2 \end{bmatrix} = 16 + 4 = \underline{\underline{20}}$$

(13) $\underline{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

write $\underline{y}$ as the sum of 2 orthogonal vectors, one in Span$\{\underline{u}\}$ and one orthogonal to $\underline{u}$

$$\hat{\underline{y}} = \frac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} \underline{u} = -\frac{13}{65} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix}}}$$

$$\underline{y} \cdot \underline{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & -7 \end{bmatrix} = 8 - 21 = \underline{-13}$$

$$\underline{u} \cdot \underline{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix} \begin{bmatrix} 4 & -7 \end{bmatrix} = 16 + 49 = \underline{65}$$

$\underline{y}$ orthogonal to $\underline{u}$:

$$\underline{y} - \hat{\underline{y}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$$

$$\underline{y} = \hat{\underline{y}} + (\underline{y} - \hat{\underline{y}}) = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$$

(17) $$\underline{u} \cdot \underline{v} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \begin{bmatrix} -1/2 & 0 & 1/2 \end{bmatrix} = -1/6 + 0 + 1/6 = 0$$

set $\{\underline{u}, \underline{v}\}$ is orthogonal

$$\|\underline{u}\|^2 = \underline{u} \cdot \underline{u} = 1/3$$

$$\|\underline{v}\|^2 = \underline{v} \cdot \underline{v} = 1/2$$

set $\{\underline{u}, \underline{v}\}$ is not orthonormal

normalization:

$$\left\{ \frac{\underline{u}}{\|\underline{u}\|}, \frac{\underline{v}}{\|\underline{v}\|} \right\} = \left\{ \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \right.$$

(18)

$$\underline{u} \cdot \underline{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} = 0$$

$\{\underline{u}, \underline{v}\}$ is an orthogonal set

$$\|\underline{u}\|^2 = \underline{u}\cdot\underline{u} = \begin{bmatrix}0\\0\\1\end{bmatrix}[0\ 0\ 1] = 1$$

$$\|\underline{v}\|^2 = \underline{v}\cdot\underline{v} = \begin{bmatrix}0\\-1\\0\end{bmatrix}[0\ -1\ 0] = 1$$

so $\{\underline{u}, \underline{v}\}$ is orthonormal set

$\boxed{7.1}$

rows become cols

① symm? $A = \begin{bmatrix}4 & 3\\3 & -8\end{bmatrix}$ $A^T = \begin{bmatrix}4 & 3\\3 & -8\end{bmatrix}$

$A = A^T \rightarrow$ symm. ✓

③ $A = \begin{bmatrix}3 & 5\\3 & 7\end{bmatrix}$ $A^T = \begin{bmatrix}3 & 3\\5 & 7\end{bmatrix}$

$A \neq A^T \rightarrow$ not symm.

⑦ $A = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$ $A^T = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\-1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

$$A^T A = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\-1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2$$

it's orthogonal ✓

⑪ $A = \begin{bmatrix}-2/3 & 1/3 & 2/3\\ \ldots\end{bmatrix}$ $A^T = \begin{bmatrix}-2/3 & 0 & 5/3\\ \ldots\end{bmatrix}$

$$\begin{bmatrix} 0 & 2/3 & -1/3 \\ 5/3 & 2/3 & 4/3 \end{bmatrix} \qquad \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 4/3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -2/3 & 0 & 5/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 4/3 \end{bmatrix} \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 0 & 2/3 & -1/3 \\ 5/3 & 2/3 & 4/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \neq I_3$$

not orthogonal

(13) $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ $\qquad \lambda = 5$

$$\det \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = (4-\lambda)^2 - 1 \quad \begin{cases} \lambda_1 = 5 \\ \lambda_2 = 3 \end{cases}$$

for $\lambda_1 = 5$:

$$(A - 5I) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{cases} x_1 = x_2 \\ x_2 \text{ free} \end{cases}$$

$$\underline{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{eigenvector} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda = 3$

$$(A - 3I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{cases} x_1 = -x_2 \\ x_2 \text{ free} \end{cases}$$

$$\underline{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{eigenvector} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

normalize $v_1 = [1$ and $v_2 = [-1$

normalize $\underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\underline{v_1}: \quad \frac{1}{\|\underline{v_1}\|}\underline{v_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\underline{v_2}: \quad \frac{1}{\|\underline{v_2}\|}\underline{v_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$P = [\underline{v_1} \quad \underline{v_2}] \qquad P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

15 $\begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix} = A \qquad A = PDP^{-1}$

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 6 \\ 6 & 10-\lambda \end{bmatrix} = (5-\lambda)(10-\lambda) - 36 = \lambda^2 - 15\lambda + 14 \quad \begin{cases} \lambda_1 = 14 \\ \lambda_2 = 1 \end{cases}$$

$\Rightarrow$ for $\lambda = 14$:

$$A - 14I = \begin{bmatrix} -9 & 6 \\ 6 & -4 \end{bmatrix} \sim \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$$

$\begin{cases} x_1 = 2/3\, x_2 \\ x_2 \text{ free} \end{cases}$ $\quad \underline{x} = x_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ $\quad \underline{e_1}$ eigenvector $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

normalize $\underline{e_1}$: $\dfrac{1}{\|\underline{e_1}\|}\underline{e_1} = \dfrac{1}{\sqrt{(2/3)^2+1}}\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \dfrac{\sqrt{13}}{3}\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} \sqrt{13}/9 \\ \sqrt{13}/3 \end{bmatrix}$$

$\rightarrow$ for $\lambda = 1$

$$A - 1I = \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = -3/2\, x_2 \\ x_2 \text{ free} \end{cases} \qquad \underline{x} = x_2 \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} \qquad \underline{e_2} = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

normalize $\underline{e_2} = \frac{1}{\|e_2\|} \underline{e_2} = \frac{1}{\sqrt{(-3/2)^2 + 1^2}} \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} = \frac{\sqrt{13}}{2} \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -3\sqrt{13}/4 \\ \sqrt{13}/2 \end{bmatrix}$$

$$P = \begin{bmatrix} \sqrt{13}/9 & -3\sqrt{13}/4 \\ \sqrt{13}/3 & \sqrt{13}/2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 14 \end{bmatrix}$$

(18) $A = \begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix} \qquad \lambda = -3, -6, 9$

for $\lambda = -3$

$$A - (-3)I = \begin{bmatrix} 4 & -6 & 4 \\ -6 & 6 & -2 \\ 4 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim$$