Differential equation = equation that includes derivatives of an unknown function

↳ solution: explicit form $y(x)$

$$F = m \cdot a$$

$$F(t) = m \frac{d^2 x}{d t^2} \quad \Rightarrow \text{solution } x(t)$$

$$\frac{d I(t)}{dt} = S(t) I(t)$$ → some covid prediction formula

↳ usually: easy to set up, hard to solve

EX:

$$\frac{dy}{dx} = 2y(3-y)$$

$$y' = y + x$$

$$y'' + 2y' + 3y = \cos(x)$$

$$y' = yx$$

find $y(x)$

→ order of differential equation

$$F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x) = 0$$

↓ dependent variable

→ independent variable

"ordinary" differential equation (ODE)

↳ PDE (partial differential equation)

↳ PDE (partial diff. equations)

SDE (stochastic —"—)

DDE (delay —"—)

↳ usually solved numerically (see Numerical Math.)

$$y' = \sin(x) \longrightarrow \int \frac{dy}{dx}\, dx = \int \sin(x)\, dx$$

$$y(x) = -\cos(x) + C$$

↳ solution is NOT unique!

$y(0) = 1 \qquad y(0) = -\cos(0) + C = 1 \quad \Longrightarrow \quad C = 2$

↳ unique solution: $y(x) = 2 - \cos(x)$

$\begin{cases} y' = \sin x \\ y(0) = 1 \end{cases}$ | initial value problems (IVP)

$$y'' = a \qquad \int \frac{d^2y}{d^2x}\, dx = \int a\, dx$$

$$\int \frac{dy}{dx} = \int (ax + C_1)\, dx$$

$$y(x) = \frac{1}{2} a x^2 + C_1 x + C_2$$

$y(0) = 1 = C_2 \quad \Longrightarrow \quad C_2 = 1$

$y'(0) = 2 = C_1 \quad \Longrightarrow \quad C_1 = 2$

$y'' = a$ |

$y(0) = 1$ | solution: (unique) $y(x) = \frac{1}{2} a x^2 + 2x + 1$
$y'(0) = 2$

↳ you need as many initial values as the order of a ODE

LINEAR DIFFERENTIAL EQUATIONS:

- an ODE is linear, if it's linear in $y(x)$ and in all derivatives $y^{(n)}$

$$a_n(x)\, y^{(n)}(x) + \ldots + a_1(x)\, y'(x) + a_0(x)\, y(x) = f(x)$$

↳ general format linear OED

→ a linear ODE is HOMOGENEOUS if $f(x) = 0$
$y(x) = 0$ is a solution

$$a_2(x)\, y''(x) + a_1(x)\, y'(x) + a_0(0)\, y(x) = 0$$

↳ $y_1(x)$, $y_2(x)$ 2 different solutions
→ then $y_1(x) + y_2(x)$ is also a solution

↳ $$a_2(x)\, y_1''(x) + a_1(x)\, y_1'(x) + a_0(0)\, y_1(x) = 0$$
$$a_2(x)\, y_2''(x) + a_1(x)\, y_2'(x) + a_0(0)\, y_2(x) = 0$$

$$+\ a_2(x)\left(y_1'' + y_2''\right) + a_1(x)\left(y_1' + y_2'\right) + a_0(x)\left(y_1 + y_2\right) = 0$$

$\frac{d^2 y}{}$

$$\frac{}{d x^2} = -y$$

$$y_1 = \sin(x) \to y_1' = \cos(x) \to y_1'' = -\sin(x) = -y_1$$

$$y_2 = \cos(x) \to y_2' = -\sin(x) \to y_2'' = -\cos(x) = -y_2$$

$\Rightarrow$ so $A\sin(x) + B\cos(x)$ solves $y'' = -y$

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non homogeneous equations (first order)

$$y' = p(x)\, y = q(x)$$

$y_H(x)$ of the homogeneous equation

$$y_H'(x) + p(x)\, y_H(x) = 0$$

$$y_P'(x) + p(x)\, y_P(x) = q(x)$$

$$\left(y_H'(x) + y_P'(x)\right) + p(x)\left(y_P + y_H\right) = q(x)$$

$$y' + 2y = 3$$

↳ homogeneous equation: $y' + 2y = 0$

$$y_H' = -2y_H$$

$$= \frac{dy_H}{dx} = -2y_H$$

$$= \int \frac{dy_H}{y_H} = -2\int dx$$

$$= \ln|y| + C = -2x$$

$$= k \cdot y = e^{-2x} \iff y_H(x) = Ae^{-2x}$$

↳ $y_p = \frac{3}{}$ $\to y_p' = 0$

$$y_p = \frac{3}{2}$$

$$\cancel{y'_p} + 2\left(\frac{3}{2}\right) = 3$$
0

$\Rightarrow$ total solution:

$$y(x) = y_p + y_H(x) = Ae^{-2x} + \frac{3}{2}$$

Solving OED:

1) separable OED (1st order)

$$\frac{dy}{dx} = f(x) \cdot g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)\, dx$$

EX: $y' = xy$

$y' = \frac{x}{y}$

$y' = \sin(x) \cos(y)$

NOT: $\cancel{y' = x^2 + y^2}$

EX1 $\frac{dy}{dx} = \frac{x}{y}$ $\left(\frac{x}{y} = x \cdot \frac{1}{y}\right)$

$$\int y\, dy = \int x\, dx \Rightarrow \frac{y^2}{2} + C = \frac{x^2}{2}$$

$\rightarrow$ we want : $y(x)$

$$\frac{y^2}{2} + C_1 = \frac{x^2}{2} \Leftrightarrow y^2 = x^2 + C_2$$

$$\Rightarrow \underline{\underline{y(x) = \pm\sqrt{x^2 + C_2}}}$$

$y(0) = 2 \Rightarrow y(0) = \pm\sqrt{C_2} \Rightarrow C_2 = 4$

$y(x) = \sqrt{x^2 + 4}$

$y(x) = \sqrt{x^2+4}$

$$y'(x) = \frac{2x}{2\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+4}} \qquad \frac{x}{y} = \frac{x}{\sqrt{x^2+4}} \quad ✓$$

Ex2 $\begin{cases} \dfrac{dy}{dx} = xy \\ y(0) = 3 \end{cases} \rightarrow$ find $y(x)$

$$\frac{dy}{dx} = xy \implies \int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \frac{x^2}{2} + c$$

$$|y| = e^{x^2/2} \cdot e^c \qquad (e^c = k)$$

$$|y| = e^{x^2/2} \cdot k$$

$$y = ke^{x^2/2}$$

$y(0) = k \cdot e^{0^2/2}$

$y(0) = k = 3 \implies k = 3$

$\hookrightarrow y(x) = 3e^{x^2/2}$

check: $y'(x) = 3x\,e^{x^2/2} \stackrel{?}{=} xy = x \cdot 3 \cdot e^{x^2/2}$ ✓

2) homogeneous linear equations (1st order)

$$y' + p(x)y = q(x)$$

total solution: $y_P(x) + y_H(x)$

step 1: homogeneous solution:

$$y' + p(x)\,y = 0 \iff \frac{dy}{dx} = -p(x)\,y$$

$$\iff \int \frac{dy}{y} = \int -p(x)\,dx$$

$$\Rightarrow \ln|y| = \mu(x) + C$$

$$y_H(x) = k \cdot e^{-\mu(x)}$$

step 2: particular solution

$$y_P(x) = k(x)\,e^{-\mu(x)}$$

$$y'_P(x) = k'(x)\,e^{-\mu(x)} - \underbrace{k(x)\,\mu'(x)\,e^{-\mu(x)}}_{p(x)\cdot y_P(x)}$$

$$= -p(x)\,y + q(x)$$

$$\Rightarrow k'(x)\,e^{-\mu(x)} = q(x)$$

$$\Rightarrow k(x) = \int q(x)\,e^{\mu(x)}\,dx$$

step 3: insert initial value

EX1

$$y' + \frac{y}{x} = 1 \qquad (x > 0)$$

$$p(x) = \frac{1}{x} \quad ; \quad q(x) = 1$$

1) homogeneous equation

$$y' + \frac{y}{x} = 0$$

$$\Rightarrow y' = \frac{dy}{dx} = -\frac{y}{x} \quad \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln|y| = -\ln|x| + C$$

$$\Rightarrow |y| = \frac{1}{x} + C$$

$$\left(|y| = \frac{C_2}{x} \right)$$

2) parameter variation

$$y_p(x) = \frac{C_2(x)}{x}$$

$$y_p'(x) = \frac{C_2'(x)}{x} - \frac{C_2(x)}{x^2}$$

$$\frac{y_p}{x} = \frac{C_2(x)}{x^2} \quad \Rightarrow y_p'(x) + \frac{y_p}{x} = \boxed{\frac{C_2'(x)}{x} = 1}$$

$$\Rightarrow C_2'(x) = x \quad \Rightarrow C_2(x) = \frac{x^2}{2} + C_3$$

$$y(x) = \frac{C_2(x)}{x} = \frac{x}{2} + \frac{C_3}{x}$$

3) check:

$$y' = \frac{1}{2} - \cancel{\frac{C_3}{x^2}}$$

$$\frac{y}{x} = \frac{1}{2} + \cancel{\frac{C_3}{x^2}}$$

$$y' + \frac{y}{x} = 1$$

Integral equation:

$$y(x) = a + b \int_{e}^{x} F(y(t), t)\, dt$$

$$\text{IVP} \begin{cases} y'(x) = F(y, x) \\ y(c) = a \end{cases}$$