SET THEORY

- sets are UNORDERED $\{1,2,3,4\} = \{2,1,4,3\}$
- sets have NO DUPLICATES $\{1,2,3,3,3\} = \{1,2,3\}$
- sets can be EMPTY ϕ or $\{\}$
- sets can be FINITE or INFINITE
- cardinality of set (size) $|A|$ $|\{1,2,3,4\}| = 4$

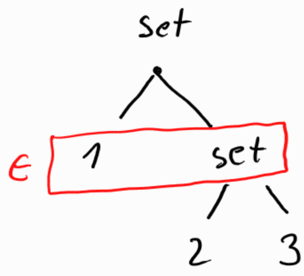
$A = \{1, \{2,3\}\}$ $|A| = 2$

$1 \in A$ $2 \notin A$

$\{2,3\} \in A$ $3 \notin A$

$\{1\} \notin A$ $\phi \notin A$

$\{2\} \notin A$



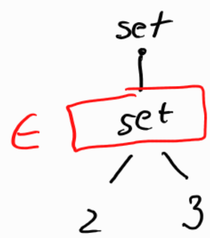
$A = \{\{2,3\}\}$ $|A| = 1$

$2 \notin A$ $\{2,3\} \in A$

$3 \notin A$

$\{2\} \notin A$

$\phi \in A$



Subsets $\subseteq$

$$\boxed{\begin{aligned} B \subseteq A &\iff \text{for all } x \in B,\ x \in A \\ &\iff (\forall x \in B)(x \in A) \\ &\iff \text{if } x \in B \text{ then } x \in A \end{aligned}}$$

- $A \subseteq A$ ✓

$$\boxed{\phi \neq \{\phi\}}$$

- $\phi \subseteq A$ ✓
- $\phi \subseteq \phi$ ✓

$$\boxed{|\phi| = 0 \qquad |\{\phi\}| = 1}$$

- $A = \{1, 2, \{3\}\}$ $\qquad \{1,2,3\} \subseteq A$ ✗ no because $3 \notin A$

Set equality

$A = B$

$$\boxed{\begin{aligned} A = B &\iff (A \subseteq B) \wedge (B \subseteq A) \\ &\iff (x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A) \end{aligned}}$$

- To prove $A = B$, prove $A \subseteq B$ and also $B \subseteq A$

- $A \cup B$ union $= \{x \in A \vee x \in B\}$
- $A \cap B$ intersection $= \{x \in A \wedge x \in B\}$
- $A \setminus B$ difference $= \{x \in A \wedge x \notin B\}$

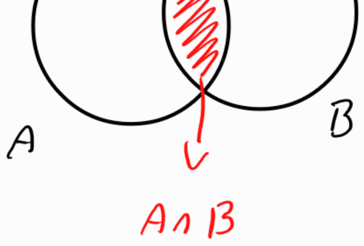
$R = \{1, 3, 5\}$ $\qquad S = \{2, 5\}$ $\qquad T = \{2, 4\}$ $\qquad W = \{2, 4, 5\}$

$R \cup T = \{1, 3, 5, 2, 4\}$ $\qquad R \cup S = \{1, 3, 5, 2\}$

$R \cap S = \{5\}$ $\qquad R \setminus T = \{1, 3, 5\}$ $\qquad R \setminus S = \{1, 3\}$

$S \setminus W = \phi$

Venn Diagrams



- Prove: for all sets A and B
  if $A \subseteq B$ then $A \cup B = B$

- Proof:
  - let A, B be arbitrary sets
  - assume $A \subseteq B$
  - we want to prove $A \cup B = B$

  i) $A \cup B \subseteq B$
  - let x be an element from $A \cup B$
  - suppose $x \in B \rightarrow$ then $x \in B$
  - suppose $x \in A \rightarrow$ then $x \in B$ $(A \subseteq B)$
  - so $A \cup B \subseteq B$ ✓

  ii) $B \subseteq A \cup B$
  - let x be an element from B
  - then by definition $x \in A \cup B$
  - so $B \subseteq A \cup B$ ✓

  so $A \cup B = B$

  $\square$

- Prove: for all sets A, B, C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- let A, B, C be arbitrary sets

i) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

- let x be an element from $A \cap (B \cup C)$
- so $x \in A$ and $x \in B \cup C$
- suppose $x \in B$ then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$
- suppose $x \in C$ then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$
- so the statement is true

ii) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

- let x be an element from $(A \cap B) \cup (A \cap C)$
- so $x \in A$ and $x \in B$ or $x \in C$ or both B, C
- suppose $x \in B$, then $x \in B \cup C$, so $x \in A \cap (B \cup C)$
- suppose $x \in C$, then $x \in B \cup C$, so $x \in A \cap (B \cup C)$
- so the statement is true

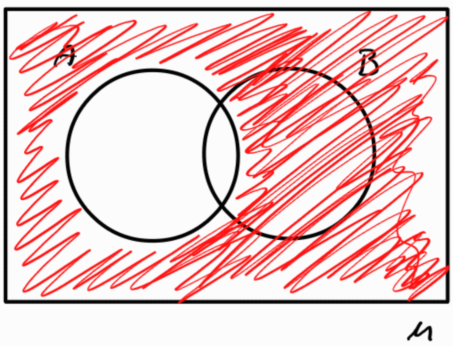
$\square$

$$A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$$

Complement

A^c "everything not in A"



$A^c = \{ x \in U : x \notin A \}$

$$= \mathcal{U} \setminus A$$

Basic properties

$$\emptyset^c = \mathcal{U} \qquad (A^c)^c = A$$

$$\mathcal{U}^c = \emptyset \qquad \text{if } A \subseteq B \text{ then } B^c \subseteq A^c$$

De Morgans Law

$$(A \cup B)^c \equiv A^c \cap B^c$$

$$(A \cap B)^c \equiv A^c \cup B^c$$

Power Sets

$\mathbb{P}(A)$ = set of all subsets of A

$= \{ B : B \subseteq A \}$

$A \{1,2,3\}$ $\quad \mathbb{P}(A) = \{ \emptyset; \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$

$A = \emptyset \quad \mathbb{P}(A) = \{\emptyset\} \quad |\mathbb{P}(A)| = 1$

$\mathbb{P}(\mathbb{P}(\emptyset)) = \{ \emptyset, \{\emptyset\} \} \quad |\mathbb{P}(\mathbb{P}(\emptyset))| = 2$

A is a finite set with n elements $\quad |A| = n$, then $|\mathbb{P}(A)| = 2^n$

$A = \{1\} \quad \mathbb{P}(A) = \{ \emptyset, \{1\} \}$

$A = \{1, 2, \emptyset\} \quad P(A) = \{ \emptyset; \{1\}, \{2\}, \{\emptyset\}, \{1,2\}, \{1,\emptyset\}, \{2,\emptyset\},$

$\{1, 2, \emptyset\} \}$

$\{1,2,4\}$

$A=\{1,3,\{1,2,3\}\}$ $\quad P(A)=\{\phi; \{1\}, \{2\}, \{\{1,2,3\}\}, \{1,\{1,2,3\}\},$
$\{1,3\}, \{3,\{1,2,3\}\}, \{1,2,\{1,2,3\}\}\}$

Product

$$A\times B=\{(a,b): a\in A \text{ and } b\in B\}$$

$$|A\times B| = |A| * |B|$$

$A=\{\text{red}, \text{green}, \text{blue}\}$ $\qquad B=\{1,2\}$

$A\times B=\{(\text{red},1), (\text{red},2), (\text{green},1), (\text{green},2), (\text{blue},1), (\text{blue},2)\}$

Is it true that $(A\times B)\times C \cancel{=} A\times B\times C$? NO!

$A=\{r,b\}$ $\qquad B=\{\text{beer}, \text{wine}, \text{water}\}$ $\qquad C=\{0,1\}$

$((r,\text{beer}),0) \in (A\times B)\times C$

$(r,\text{beer},0) \in A\times B\times C$

true only when:

- $A=B$ $\qquad \left.\begin{array}{l} A\times B=A\times A \\ B\times A=A\times A \end{array}\right\}$ same ✓

- $A=\phi$ $\qquad \left.\begin{array}{l} A\times B=\phi \\ B\times A=\phi \end{array}\right\}$ same ✓

- $B=\phi$

Set partition

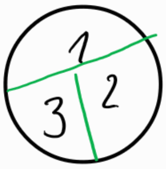
A

a partition of a set A is a set of non-empty subsets of A, $\{A_1, A_2, A_3, \dots\}$ such that:

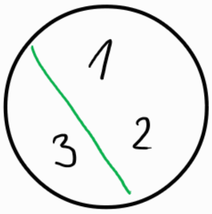
① $A_i \cap A_j = \emptyset$ for all $i \neq j$

② $A_1 \cup A_2 \cup \dots = A$

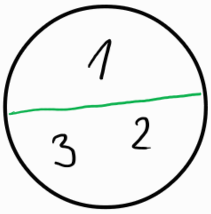
$A = \{1, 2, 3\}$ what are possible partitions?



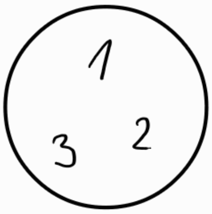
$\{\{1\}, \{2\}, \{3\}\}$



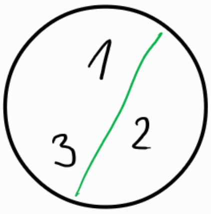
$\{\{1,2\}, \{3\}\}$



$\{\{1\}, \{2,3\}\}$



$\{\{1,2,3\}\}$



$\{\{1,3\}, \{2\}\}$

These are NOT valid partitions:

$\{\emptyset, \{1,2\}, \{3\}\}$ ✗ empty blocks aren't allowed

$\{\{1,2\}, \{1,2\}\}$ ✗ blocks are overlaping

$\{\{1\}, \{2\}\}$ ✗ doesn't cover A: 3 is missing

What is partition of $\{\underbrace{\{1,2\}}_{a}, \underbrace{\{3\}}_{b}\}$

$\{\{\{1,2\}\}, \{\{3\}\}\}$