Systems of linear equations, Gaussian elimination

x: chickens
y: cows

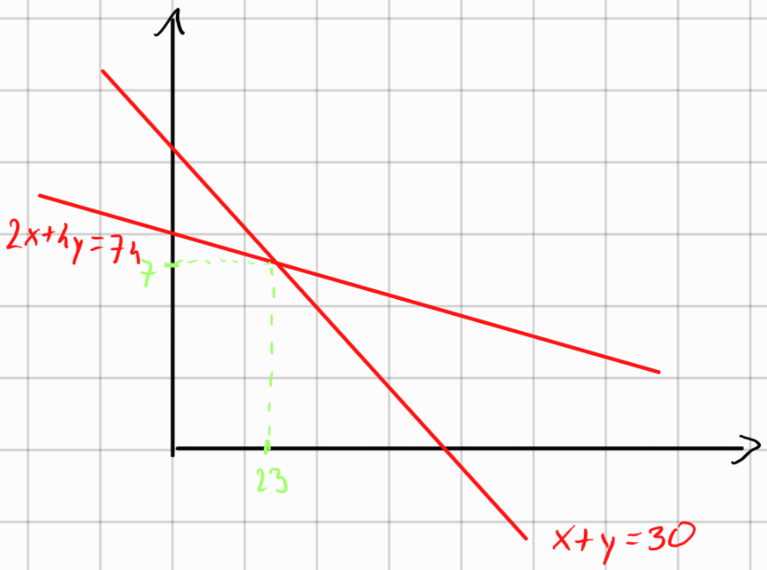
$$\begin{aligned} x+y&=30 \\ 2x+4y&=74 \end{aligned}$$

This is the system of linear equations (SLE)

2 equations in 2 variables
we need a solution (a pair (x,y)) that works for both equations

We have 2 variables so we "live" in $\mathbb{R}^2$
Each equation of this SLE, represents a line in $\mathbb{R}^2$

From a geometric / row point of view: we're looking for an intersection of these two lines



In general, solving an SLE means:
* $\mathbb{R}^2$: find intersection of lines in a plane
* $\mathbb{R}^3$: —— " —— planes in a space
* $\mathbb{R}^4$: —— " —— hyperplanes in a hyperspace

($x+y=30$ $\qquad$ $-2x-2y=-60$)

$$\begin{cases} x+y=30 \\ 2x+4y=74 \end{cases} \longrightarrow \begin{cases} -\cancel{2x} - 2y = -60 \\ \cancel{2x} + 4y = 74 \end{cases} +$$

$$2y = 14 \quad / \div 2$$

$$y = 7$$

$$x = 30 - y$$

$$x = 23$$

So, the SLE has a solution and the solution is unique

Efficent procedure to solve SLE: Gaussian elliminotion / row reduction

An SLE can be summarized by:

* coefficient matrix A
* vector $\vec{b}$ of RHS numbers

} augmented matrix $[A \vdots \vec{b}]$

$$\begin{cases} x+y=30 \\ 2x+4y=74 \end{cases} \qquad A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 30 \\ 74 \end{bmatrix}$$

2x2 matrix

rows = equations, cols = variables

$$[A \vdots \vec{b}] = \left[\begin{array}{cc:c} 1 & 1 & 30 \\ 2 & 4 & 74 \end{array}\right] \underset{R_2 : R_2 - 2 \cdot R_1}{\sim} \left[\begin{array}{cc:c} 1 & 1 & 30 \\ 0 & 2 & 14 \end{array}\right] \underset{R_2 : R_2 \cdot 1/2}{\sim}$$

$$\left[\begin{array}{cc:c} 1 & 1 & 30 \\ 0 & 1 & 7 \end{array}\right] \underset{R_1 : R_1 - R_2}{\sim} \left[\begin{array}{cc:c} 1 & 0 & 23 \\ 0 & 1 & 7 \end{array}\right]$$

$$\begin{cases} 1 \cdot x + 0 \cdot y = 23 \\ 0 \cdot x + 1 \cdot y = 7 \end{cases} \qquad \text{solution: } \begin{cases} x = 23 \\ y = 7 \end{cases}$$

The matrices are row equivalent if one matrix can be changed into the other by means of a row operation:

① Replacement: add a scalar multiple of a row to another row

② Scaling: multyplying a row by non-zero scalar

③ Interchange: swap rows

If augmented matrices of two SLEs are row equivalent then the two matrices are equivallent

An SLE can have:

* no solution — SLE inconsistent
* one unique
* infinitely many

$\left.\begin{matrix} \text{one unique} \\ \text{infinitely many} \end{matrix}\right\}$ SLE consistent

$x+y=1$

$x+y=2$

two parallel lines

→ no intersection

→ no solution

→ SLE inconsistent

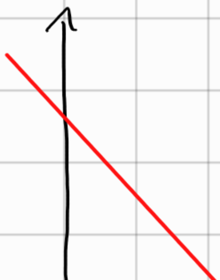
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \underset{R_2 : R_2 - R_1}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \quad \begin{cases} x+y=1 \\ 0=1 \quad ↯ \end{cases}$$

such a row $\left[\begin{array}{ccc|c} 0 & \dots & 0 & x \end{array}\right]$ $x \neq 0$

is typical for inconsistent SLE

$x+y=3$

$2x+2y=6$



they are the same line

→ indefinitely many

points of intersection
→ infinitely many solutions

$$\begin{bmatrix} 1 & 1 & \vdots & 3 \\ 2 & 2 & \vdots & 6 \end{bmatrix} \underset{R_2 : R_2 - 2R_1}{\sim} \begin{bmatrix} 1 & 1 & \vdots & 3 \\ 0 & 0 & \vdots & 0 \end{bmatrix} \quad \begin{cases} x + y = 3 \\ 0 = 0 \end{cases}$$

Solution is in parametric form : $\begin{cases} x = 3 - y \\ y \text{ is free} \end{cases}$

- y is a free variable (we can choose any value)
- x is a basic variable (we cannot choose)

Note: having a zero row, does <u>not</u> necessarily mean that there are indefinitely many solutions

eg:

$$\begin{bmatrix} 1 & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix} \quad \begin{cases} x = 1 \\ y = 0 \\ 0 = 0 \end{cases}$$

So, SLE has a unique solution

Row reduction:

Row reduction form (REF) (not unique for a matrix)

$$\begin{bmatrix} -2 & 1 & 0 & \vdots & 3 \\ 0 & 2 & 1 & \vdots & 0 \\ 0 & 0 & -4 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & -3 & 2 & | & 1 \\ 0 & 0 & 4 & | & 8 \\ 0 & 0 & 0 & | & 5 \end{bmatrix}$$

one unique solution

no solution

pivot (leading entry) : leftmost nonzero element in a row

① all nonzero rows are above any zero row
② every pivot in a row is a column to the right of the pivot of the row above it
③ all entries below a pivot are zero

It tells you:
* where there is a solution (existing question)
  → is there a row $[0 \dots 0 | \alpha]$ $\alpha \neq 0$?
  (last column is a pivot column)
* wheather the solution is unique (uniqueness question)
  → free variables?
  → all basic variables?
  → does every column have a pivot?

RREF (unique for matrix)

$$x_1 = 2,\; x_2 = 0,\; x_3 = -1 \leftarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow x_1 = -1,\; x_2 = -2,\; x_3 \text{ is free}$$

one unique solution — indefinitely many solutions

④ all pivots are $= 1$
⑤ each pivot is the only nonzero entry in its col

It tells you:
* the solution

Ex. <u>Gausian Ellimination</u>

$$\begin{cases} 2x_2 - 8x_3 = 8 \\ x_1 - 2x_2 + x_3 = 0 \\ \dots \end{cases}$$

$$-4x_1 + 5x_2 + 9x_3 = -9$$

$$\left[\begin{array}{ccc|c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \underset{R_3 : R_3 + 4R_1}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \underset{R_3 : R_3 + 3/2 R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \underset{\substack{R_1 : R_1 - R_3 \\ R_2 : R_2 + 8R_3}}{\sim}$$

REF: the SLE is consistend
the solution is unique

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \underset{R_2 : R_2 \cdot 1/2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \underset{R_1 : R_1 + 2R_2}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

RREF

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$