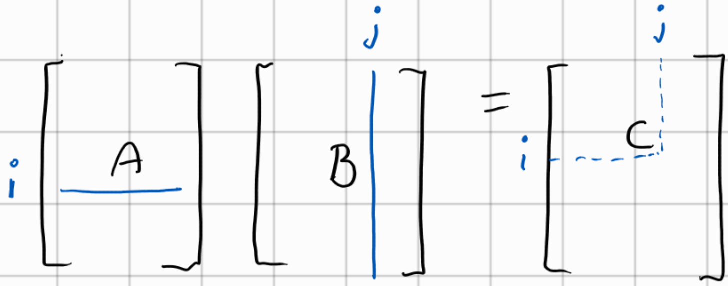
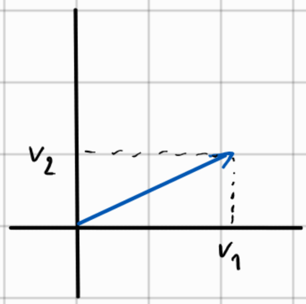
inner/dot product $\underline{u}, \underline{v} \in \mathbb{R}^n$

$$\underline{u} \bullet \underline{v} = \underline{u}^T \underline{v} = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \dots + u_n v_n$$



$$C_{ij} = row_i(A)^T \bullet col_j(B)$$

Length of a vector



$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \text{length} = \sqrt{v_1^2 + v_2^2} = \sqrt{\underline{v} \cdot \underline{v}}$$

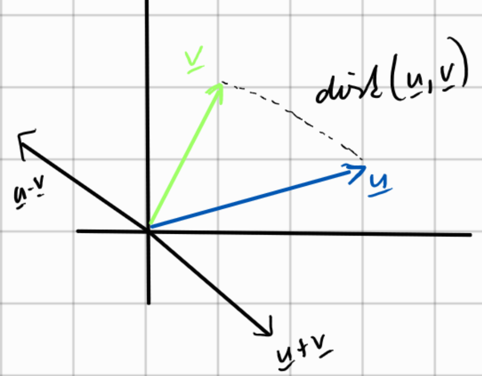
in $\mathbb{R}^3$:

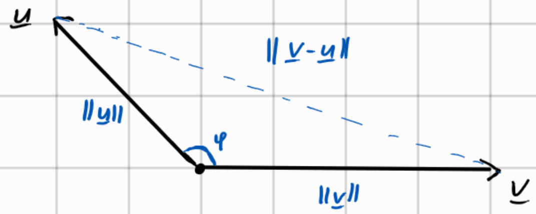
$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \text{length} = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\underline{v} \cdot \underline{v}}$$

length/norm of a vector $\underline{v}$ is defined by

$$\|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}}$$

Distance between 2 vectors





law of cosine

$$\underline{\|\underline{v} - \underline{u}\|^2} = \|\underline{u}^2\| + \|\underline{v}^2\| - 2 \cdot \|\underline{u}\| \cdot \|\underline{v}\| \cdot \cos\varphi$$

$$\hookrightarrow (\underline{v} - \underline{u}) \bullet (\underline{v} - \underline{u})$$

$$=$$

$$\underline{v} \cdot (\underline{v} - \underline{u}) - \underline{u}(\underline{v} - \underline{u})$$

$$=$$

$$\underline{v} \cdot \underline{v} - \cancel{\underline{v} \cdot \underline{u}} - \cancel{\underline{u} \cdot \underline{v}} + \underline{u} \bullet \underline{u}$$

$$=$$

$$\cancel{-2}\, \underline{u} \bullet \underline{v} + \|\cancel{\underline{v}}\|^2 + \|\cancel{\underline{u}}\|^2$$

so, $\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos\varphi$ (note $\cos\varphi = 0$ iff $\varphi = \frac{\pi}{2} + 2k\pi \quad k \in R$, $= \frac{3\pi}{2} + 2k\pi$)

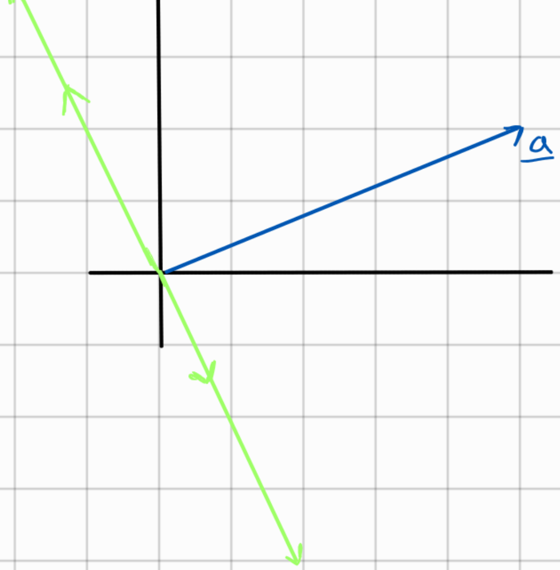
so, if $\underline{u} \neq \underline{0}$ and $\underline{v} \neq \underline{0}$, then $\underline{u} \cdot \underline{v} = 0 \iff \varphi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$
$\iff$ $\underline{u}$ and $\underline{v}$ are perpendicular

Two vectors are orthogonal ($\underline{u} \perp \underline{v}$) if $\underline{u} \cdot \underline{v} = 0$

EX $\quad 2x_1 + x_2 = 0 \qquad \underline{a} \cdot \underline{x} = 0$ with $\underline{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

solution set $\quad 2x_1 + x_2 = 0 \Rightarrow x_1 = -1/2\, x_2$

$$\Rightarrow \underline{x} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



sol set: $x_2\begin{bmatrix}-1\\2\end{bmatrix}$

indeed all vectors orthogonal to $\underline{a}$

In general: a hyperplane through the origin (homo. eq.)

$$a_1x_1 + \dots + a_nx_n = 0 \quad \text{or} \quad \underline{a} \cdot \underline{x} = 0$$

$\downarrow$

all vectors $x \in \mathbb{R}^n$ that are orthogonal to $\underline{a}$

* in $\mathbb{R}^2$: you get a line
* in $\mathbb{R}^3$: you get a plane
* in $\mathbb{R}^n$: you get a hyperplane (subspace of dim $n-1$)

Recall the Null space of a matrix A $m \times n$

$$\mathrm{Nul}(A) = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{0} \}$$

$$\begin{bmatrix} — \underline{r_1} — \\ — \underline{r_2} — \\ \vdots \\ — \underline{r_m} — \end{bmatrix} \begin{bmatrix} | \\ \underline{x} \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\left\{ \begin{array}{l} \underline{r_1}^T \cdot \underline{x} = 0 \\ \underline{r_2}^T \cdot \underline{x} = 0 \end{array} \right.$

So every $\underline{x} \in \mathrm{Nul}(A)$ is

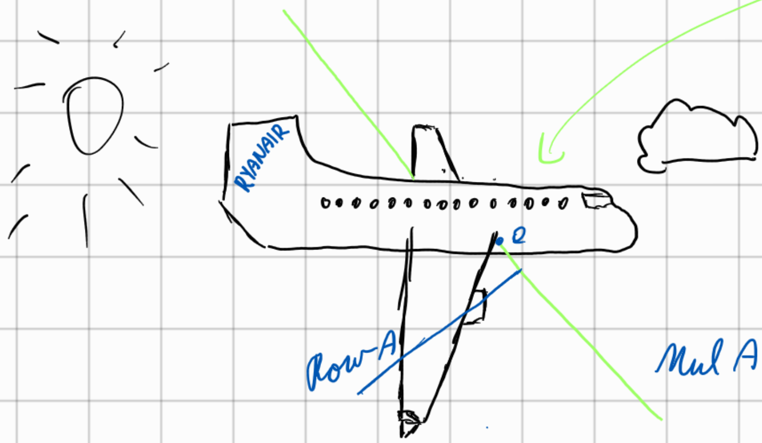
$$\begin{cases} \underline{r_2} \cdot \underline{x} = 0 \\ \vdots \\ \underline{r_n} \cdot \underline{x} = 0 \end{cases}$$ orthogonal to each of the rows of A

$$(c_1 \underline{r_1}^T + c_2 \underline{r_2}^T + \dots + c_n \underline{r_n}^T)\, \underline{x}$$
$$=$$
$$c_1(\underline{r_1}^T \underline{x}) + c_2(\underline{r_2}^T \underline{x}) + \dots + c_n(\underline{r_n}^T \underline{x}) = 0 + 0 + \dots + 0$$

So every vector in $Nul(A)$ is orthogonal to every vector in $Row(A)$

$\Rightarrow Nul(A) \perp Row(A)$

plane in $\mathbb{R}^3$



$$\underset{\substack{\downarrow \\ \text{in } \mathbb{R}^n \\ \dim = \#\text{free var}}}{Nul(A)} \perp \underset{\substack{\downarrow \\ \text{in } \mathbb{R}^n \\ \dim = \#\text{pivot}}}{Row(A)}$$

$\sum \dim = n$

---

and since $Col(A) = Row(A^T)$ we also have
$Nul(A^T) \perp Col(A)$

W: subspace of $\mathbb{R}^n$

$W^\perp$ ("W perpendicular"): orthogonal complement of all vectors in $\mathbb{R}^n$ that are orthogonal to W

$$(\text{Row}(A))^{\perp} = \text{Nul}(A) \quad \text{and} \quad (\text{Col}(A))^{\perp} = \text{Nul}(A^T)$$

$W^{\perp}$ is also a subspace of $\mathbb{R}^n$

In general $\dim(W) + \dim(W^T) = n$

$\{\underline{v}_1, \dots, \underline{v}_m\}$ is an orthogonal set if $\underline{v}_i \cdot \underline{v}_j = 0$ for all $i \neq j$

Theorem: If $S = \{\underline{v}_1, \dots, \underline{v}_m\}$ is an orthogonal set and $\underline{0} \notin S$ then S is lin. indep. and thus S forms a basis for $\text{Span}\{\underline{v}_1, \dots, \underline{v}_m\}$

Orthogonal basis for a subspace W of $\mathbb{R}^n$: it is a basis of W and orthogonal set

$\{\underline{u}_1, \dots, \underline{u}_k\}$ orth. basis for W. Let $\underline{y} \in W$.

$$\underline{y} = c_1 \underline{u}_1 + \dots + c_k \underline{u}_k$$

What are the weights $c_1, \dots, c_k$?

$$\underline{y} \cdot u_1 = (c_1 \underline{u}_1, \dots, c_k \underline{u}_k) \cdot \underline{u}_1 = c_1 \underline{u}_1 \cdot \underline{u}_1 + \cancel{\dots + c_k \underline{u}_k \cdot \underline{u}_1}$$

$$= c_1 \underline{u}_1 \cdot \underline{u}_1$$

$\underline{y} \cdot \underline{u}_1$ $\quad$ $\underline{y} \cdot u_2$

$\Rightarrow \quad c_1 = \frac{\quad}{\underline{u}_1 \cdot \underline{u}_1} \qquad c_2 = \frac{\quad}{\underline{u}_2 \cdot \underline{u}_2} \qquad$ etc.

So, it's easy to find the weights
(non-orthogonal basis: solving an SLE)

$\{\underline{v}_1, \dots, \underline{v}_k\}$ is an orthonormal set if it is orthogonal unit vectors
↳ vectors of lenght 1

EX $\left\{ \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \right\}$ forms an orthonormal basis for $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ forms an orthogonal basis for $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ forms an orthonormal basis for $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^2$

How to test whether $\{\underline{v}_1, \dots, \underline{v}_n\}$ is orthogonal or orthonormal?

$$A = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k \\ | & | & & | \end{bmatrix}$$

Compute $A^T A = \begin{bmatrix} — \underline{v}_1^T — \\ — \underline{v}_2^T — \\ \vdots \end{bmatrix} \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k \\ | & | & & | \end{bmatrix} = \begin{array}{cc} & j \\ i & \begin{bmatrix} & \\ & \underline{v}_i \cdot \underline{v}_j \end{bmatrix} \end{array}$

$$\begin{bmatrix} - & \underline{v}_k^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix}$$

$\{\underline{v}_1, \dots, \underline{v}_k\}$ is orthogonal $\iff A^T A$ is diagonal

$\{\underline{v}_1, \dots, \underline{v}_k\}$ is orthonormal $\iff A^T A$ is identity matrix

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A square A is an orthogonal matrix

$\iff A^T A = I_n \iff A^{-1} = A^T$

Watch out: an orthogonal matrix has orthonormal cols

Properties: $T: \underline{x} \mapsto A\underline{x}$, where A is orthogonal:

$$* \ |A\underline{x}| = \sqrt{(A\underline{x}) \cdot (A\underline{x})} = \sqrt{(A\underline{x})^T (A\underline{x})} =$$

$$= \sqrt{\underline{x}^T A^T A \underline{x}} = \sqrt{\underline{x}^T I_n \underline{x}} = \sqrt{\underline{x}^T \underline{x}}$$