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Online Learning under Adversarial Nonlinear Constraints

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Overview

In many applications, learning systems are required to process continuous non-stationary data streams. We study this problem in an online learning framework [1] and propose an algorithm that can deal with adversarial time-varying and nonlinear constraints.

- Our algorithm, called Constraint Violation Velocity Projection (CVV-Pro), achieves $\sqrt{T}$ regret and converges to the feasible set at a rate of $1/\sqrt{T}$, despite the fact that the feasible set is slowly time-varying and a priori unknown to the learner.
- CVV-Pro relies only on local sparse linear approximations of the feasible set and therefore avoids optimizing over the entire set at each iteration, which is in sharp contrast to projected gradients or Frank-Wolfe methods.

Applications:

The setting of online learning with time-varying constraints has applications in adversarial contextual bandits with sequential risk constraints, network resource allocation, logistic and ridge regression, job scheduling, two-player game with shared resource constraints, system identification and optimal control.

Assumptions

Convexity and Compactness:

There exist constants $R, L_\mathcal{F}, L_\mathcal{G} > 0$ such that
1) $\mathcal{F}$ is a class of convex functions, where every $f \in \mathcal{F}$ satisfies $||\nabla f(x)|| \leq L_\mathcal{F}, \forall x \in \mathcal{B}_{1R}$, with $\mathcal{B}_R$ a hypersphere of radius R centered at the origin;
2) $\mathcal{G}$ is a class of concave $\beta_\mathcal{G}$-smooth functions, where every g satisfies $||\nabla g(x)|| \leq L_\mathcal{G}, \forall x \in \mathcal{B}_{4R}$;
3) The feasible set $\mathcal{C}_t$ is non-empty and contained in $\mathcal{B}_R$ for all t.

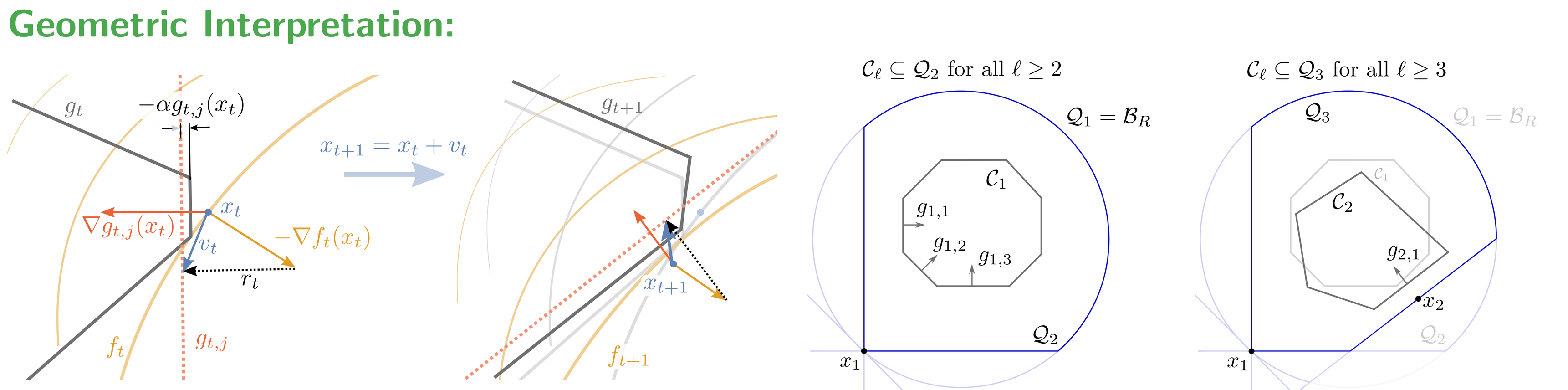
Slowly Time-Varying & Polyhedral Intersection:

The environment chooses $f_t \in \mathcal{F}$ and $g_t \in \mathcal{G}$ such that
i) the constraints are slowly time-varying $||g_t(x) - g_{t-1}(x)||_\infty = \mathcal{O}(1/t)$ for all $x \in \mathcal{B}_{4R}$; and
ii) the feasible set $\mathcal{C}_t$ is contained in polyhedral intersection $\mathcal{Q}_t := \cap_{\ell=0}^{t-1} \mathcal{S}_\ell$, where $\mathcal{S}_0 = \mathbb{R}^n$ and $\mathcal{S}_t := \{x \in \mathbb{R}^n \mid G(x_t)^\top (x - x_t) \geq 0\}$ is a cone centered at x_t for $t \geq 1$

Notation:

The index set $I(x_t) := \{i \in \{1, \dots, m\} \mid g_{t,i}(x_t) \leq 0\}$ contains the indices of all not strictly satisfied constraints at x_t. The columns of matrix $G(x_t) := [\nabla g_{t,i}(x_t)]_{i \in I(x_t)}$ store the corresponding gradients.

Geometric Interpretation:



Algorithm

Algorithm 1 Constraint Violation Velocity Projection (CVV-Pro)

1: **Input:** $L_\mathcal{F}, R > 0$ and $x_1 \in \mathcal{B}_R$
2: **Initialization:** Scale $\alpha = L_\mathcal{F}/R$, step sizes $\left\{\eta_t = \frac{1}{\alpha\sqrt{t+15}}\right\}_{t\geq 1}$ and constraint $g_{m+1}(x) = \frac{1}{2}\left[R^2 - \|x\|^2\right]$
3: **for** $t = 1$ **to** T **do**
4: **Play** x_t and **observe** cost information $f_t(x_t), \nabla f_t(x_t)$ and constraint information $\left\{\left(g_i(x_t), \nabla g_i(x_t)\right)\right\}_{i \in I(x_t)}$
5: **Construct a velocity polytope** [2]

$$V_\alpha(x_t) := \left\{v \in \mathbb{R}^n \mid [\nabla g_{t,i}(x_t)]^\top v \geq -\alpha g_{t,i}(x_t),\ \forall i \in I(x_t)\right\}$$

6: **Augment** $V_\alpha(x_t)$ **with a bounded hyper-sphere**

$$V'_\alpha(x_t) := \left\{v \in V_\alpha(x_t) \mid [\nabla g_{m+1}(x_t)]^\top v \geq -\alpha g_{m+1}(x_t)\right\} \text{ if } \|x_t\| > R, \text{ otherwise } V'_\alpha(x_t) = V_\alpha(x_t)$$

7: **Solve the velocity projection problem**

$$v_t = \arg\min_{v \in V'_\alpha(x_t)} \frac{1}{2}\|v + \nabla f_t(x_t)\|^2$$

8: **Update** $x_{t+1} = x_t + \eta_t v_t$
9: **end for**

Geometric Properties

Time Invariant Constraints:

Lemma 1 ($V'_\alpha(x_t)$ is non-empty). *$\alpha(x - x_t) \in V'_\alpha(x_t)$ for all $x \in \mathcal{C}$.*

Lemma 2 (Normal Cone). *$-r_t^\top (x - x_t) \leq 0$ for all $x \in \mathcal{C}$.*

Time Varying Constraints:

Lemma 3 (Normal Cone). *$\mathcal{C}_T \subseteq \mathcal{Q}_T$ implies for any $x \in \mathcal{C}_T$ that $-r_t^\top (x - x_t) \leq 0$ for all $t \in \{1, \dots, T\}$.*

Bounded Decisions and Velocities:

Lemma 4 (Main). *$g_{m+1}(x_t) \geq -27\frac{R^2}{\sqrt{t}}$, $\|x_t\| \leq 4R$ and $\|v_t\| \leq 7L_\mathcal{F}$.*

Main Result

Theorem 1 (Time-Varying Constraints). *Suppose the functions $\{f_t, g_t\}_{t\geq 1}$ satisfy the Assumptions. Then on input $R, L_\mathcal{F} > 0$ and $x_1 \in \mathcal{B}_R$, CVV-Pro guarantees the following for all $T \geq 1$:*

(regret) $\sum_{t=1}^T f_t(x_t) - \min_{x \in \mathcal{C}_T} \sum_{t=1}^T f_t(x) \leq 246 L_\mathcal{F} R\sqrt{T}$;

(feasibility) $g_{t,i}(x_t) \geq -265\left[\frac{L_\mathcal{G}}{R} + 4\beta_\mathcal{G}\right]\frac{R^2}{\sqrt{t+15}}$, *for all $t \in \{1, \dots, T\}$ and $i \in \{1, \dots, m\}$;*

(attraction) $g_{m+1}(x_t) \geq -27\frac{R^2}{\sqrt{t+15}}$, *for all $t \in \{1, \dots, T\}$.*

Length Unit Consistency: $R \to \ell$ implies $L_\mathcal{F}, L_\mathcal{G} \to 1/\ell$ and $\beta_\mathcal{G} \to 1/\ell^2$

Proof Sketch

(Regret) Standard OGD analysis bounds $\sum_{t=1}^T f_t(x_t) - f_t(x_T^\star) \leq \sum_{t=1}^T [\nabla f_t(x_t)]^\top (x_t - x_T^\star)$. Furthermore,

$$[\nabla f_t(x_t)]^\top (x_t - x_T^\star) = -r_t^\top (x_T^\star - x_t) + \frac{\eta_t}{2}\|v_t\|^2 + \frac{\|x_t - x_T^\star\|^2 - \|x_{t+1} - x_T^\star\|^2}{2\eta_t}$$

Lem.3 bounds the inner product, Lem.4 bounds the velocity, the regret follows by telescoping argument.

(Feasibility) We give bounds on the slowly time-varying constraints $g_{t,i}(x)$ from below. Let $\mathcal{V}_\alpha = 7L_\mathcal{F}$, by Lem.4. In particular, we show that $g_{t+1,i}(x_{t+1}) \geq (1 - \alpha\eta_t) g_{t,i}(x_t) - \eta_t^2\left[2\frac{L_\mathcal{G}}{R} + 7\beta_\mathcal{G}\right]\mathcal{V}_\alpha^2$ for all $i \in I(x_t)$ and $g_{t+1,i}(x_{t+1}) \geq -\eta_{t+1} 7\mathcal{V}_\alpha\left[L_\mathcal{G} + \frac{\beta_\mathcal{G}\mathcal{V}_\alpha}{4\alpha}\right]$ for all $i \in \{1, \dots, m\} \backslash I(x_t)$. Using a non-trivial inductive argument, we establish a lower bound of the form $g_{t,i}(x_t) \geq -c\eta_t$.

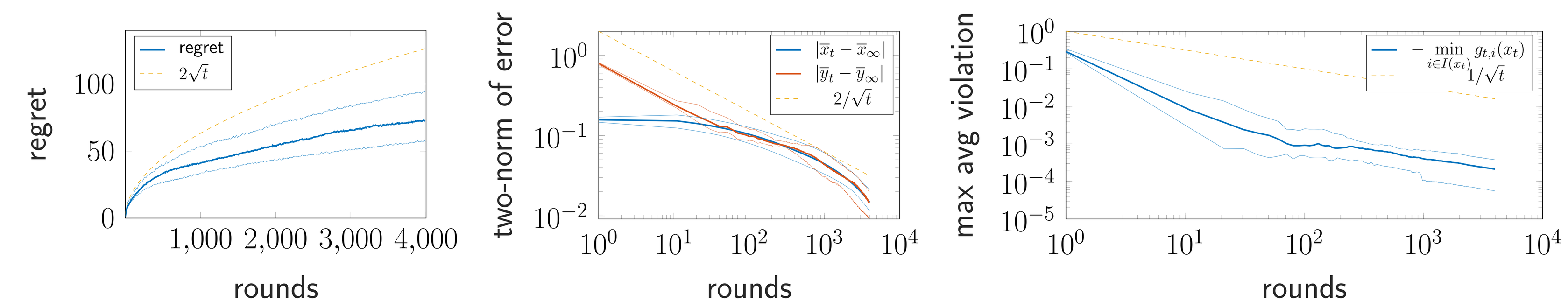
Two-Player Game With Shared Resource Constraints

Problem Formulation:

$$\min_{x \in \triangle_n} \max_{y \in \triangle_n} x^\top A y \quad \text{subject to} \quad C_x x + C_y y \leq b$$

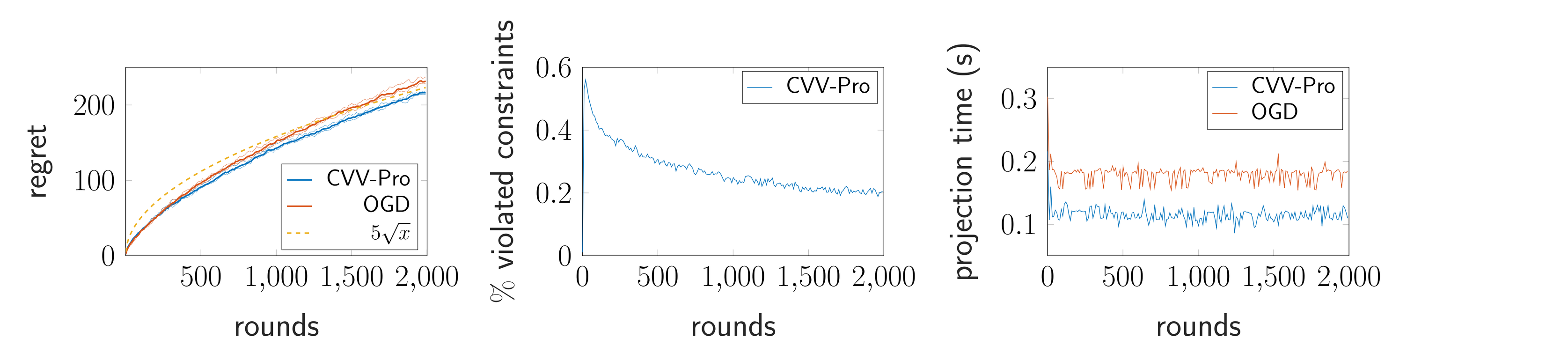
Initialization: A, C_x, C_y are sampled u.a.r. from normal distribution; $b = 1$ is a capacity vector
Best response and randomization: $\hat{y}_t = \arg\max_{y \in \triangle_n} x_t^\top A y$ and $y_t = 0.8\hat{y}_t + 0.2 r_t$ where $r_t \overset{\text{u.a.r.}}{\sim} \triangle_n$
Time-averaged constraints: $g_T(x) = \frac{1}{T}\sum_{t=1}^T \widetilde{g}_t(x)$, where $\widetilde{g}_t(x) = b - C_x x - C_y y_t$
$g_T(x)$ satisfies the slowly time-varying assumption; Cost function: $f_t(x) = x^\top A y_t$

Simulation:



The CVV-Pro algorithm is executed on five random instances of the two-player game with shared resource constraints. Decision dimension $n = 100$, $m = 10$ shared resource constraints, capacity $b = 1$, $T = 4000$ iterations. The thick line is the mean and the thin lines are the minimum and maximum over the five runs. The time-averaged decisions $\overline{x}_t := \frac{1}{t}\sum_{\ell=1}^t x_\ell$ converge towards $\overline{x}_\infty := \overline{x}_{10000}$. Similar behavior is reported for the decisions $\overline{y}_t$.

Comparison between CVV-Pro and OGD:



Decision dimension $n = 1000$, $m = 100$ shared resource constraints, capacity $b = 1.3$, $T = 2000$ iterations.

Conclusion

We give an online algorithm that handles unknown and time-varying constraints. It has a regret bound of $\mathcal{O}(\sqrt{T})$ and guarantees convergence of violated constraints at a rate of $\mathcal{O}(1/\sqrt{T})$. Our method offers computational efficiency by projecting over a local sparse linear approximation set, reducing the computational cost compared to the expensive projection over the entire feasible set.

Paper
arXiv:2306.03655

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References

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[2] Michael Muehlebach and Michael I. Jordan. On constraints in first-order optimization: A view from non-smooth dynamical systems. Journal of Machine Learning Research, 23(256):1–47, 2022