Variance-Reduction Methods: SGD(+SWA) vs Nesterov vs SVRG

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Problem: SGD does not converge to the minimum, but instead oscillates around it.

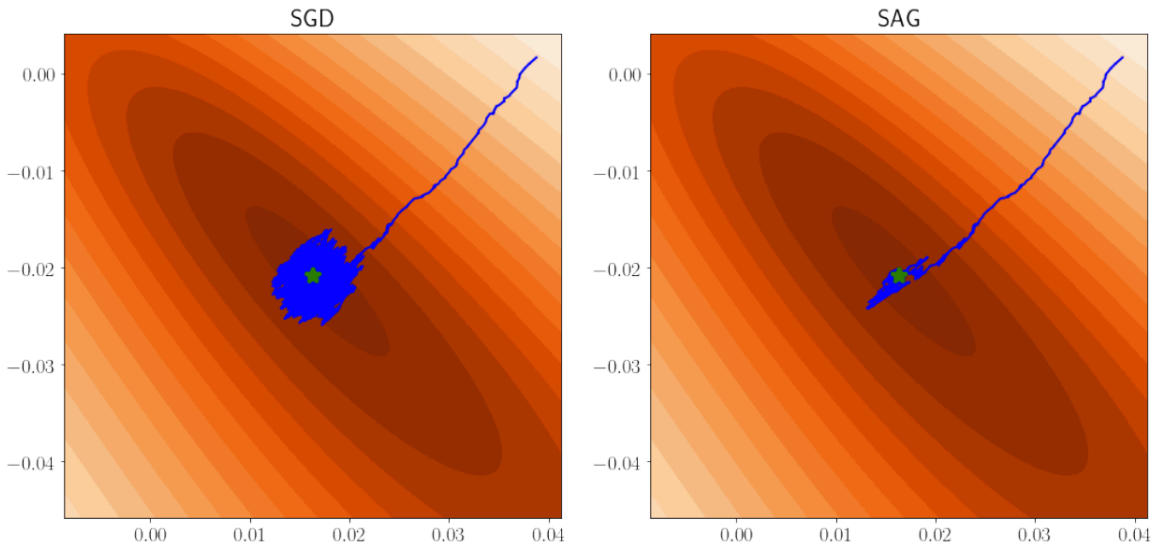


Fig. 2. Level set plot of 2D logistic regression with the iterates of `SGD` (left) and `SAG` (right) with constant stepsize. The green star is the x_* solution.

The authors of the paper review and criticize common solutions to this problem:

- *Scheduling LR* – but it is difficult to tune
- *Momentum* – but it does not converge to the full gradient whatever
- *Mini-batching* – but the cost of this iteration increases proportionally to the batch size.

Authors' Solution: **Variance Reduction Methods**

Let's use estimate $g_k \in \mathbb{R}^d$ gradient such that $g_k \approx \nabla f(x_k)$.

Then iteration step looks like: $x_{k+1} = x_k - \gamma g_k$,

To make such algorithm converge with a *constant step size*, we need to ensure that the variance of our gradient estimate g_k converges to zero (VR-property):

$$\mathbf{E}\left[\|g_k - \nabla f(x_k)\|^2\right] \quad \underset{k\to\infty}{\longrightarrow} \quad 0,$$

Ideal (unreal) **VR-method**: $\texttt{SGD}_\star$

Algorithm: $x_{k+1} = x_k - \gamma\left(\nabla f_{i_k}(x_k) - \nabla f_{i_k}(x_\star)\right),$

This algorithm is unreal because we don't know $\nabla f_i(x_\star),$ but we can think that real VR-methods is "approximation" of $\texttt{SGD}_\star$.

Of course it satisfies main VR-property:

$$\begin{aligned}\mathbf{E}\left[\|g_k - \nabla f(x_k)\|^2\right] &= \mathbf{E}\left[\|\nabla f_i(x_k) - \nabla f_i(x_\star) - \nabla f(x_k)\|^2\right] \\ &\le \mathbf{E}\left[\|\nabla f_i(x_k) - \nabla f_i(x_\star)\|^2\right],\end{aligned}$$

SVRG: Stochastic Variance-Reduced Gradient method

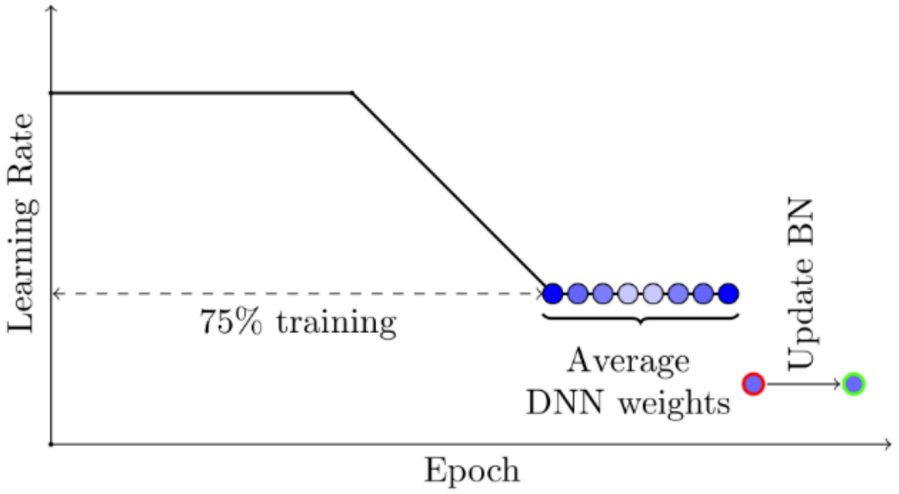
Parameters stepsize $\gamma > 0$
Initialization $\bar{x}_0 = x_0 \in \mathbb{R}^d$
for $s = 1, 2, \ldots$ **do**
    Compute and store $\nabla f(\bar{x}_{s-1})$
    $x_0 = \bar{x}_{s-1}$
    Choose the number of inner-loop iterations t
    for $k = 0, 1, \ldots, t-1$ **do**
        Sample $i_k \in \{1, \ldots, n\}$
        $g_k = \nabla f_{i_k}(x_k) - \nabla f_{i_k}(\bar{x}_{s-1}) + \nabla f(\bar{x}_{s-1})$
        $x_{k+1} = x_k - \gamma g_k$
    $\bar{x}_s = x_t.$

Properties of SVRG:

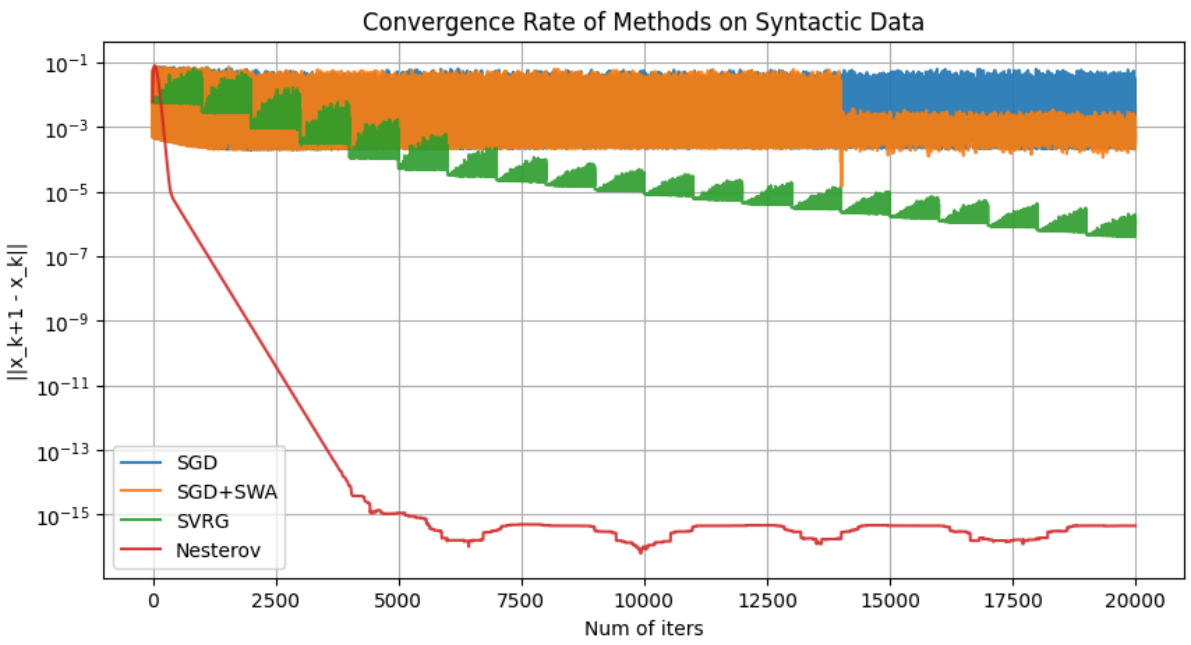
- Requires only $\mathcal{O}(d)$ memory, less that other VR methods
- Has iteration complexity $\mathcal{O}((\kappa_{\max} + n)\log(1/\varepsilon))$, similar to other VR methods
- Gradient estimate g_k is bounded:

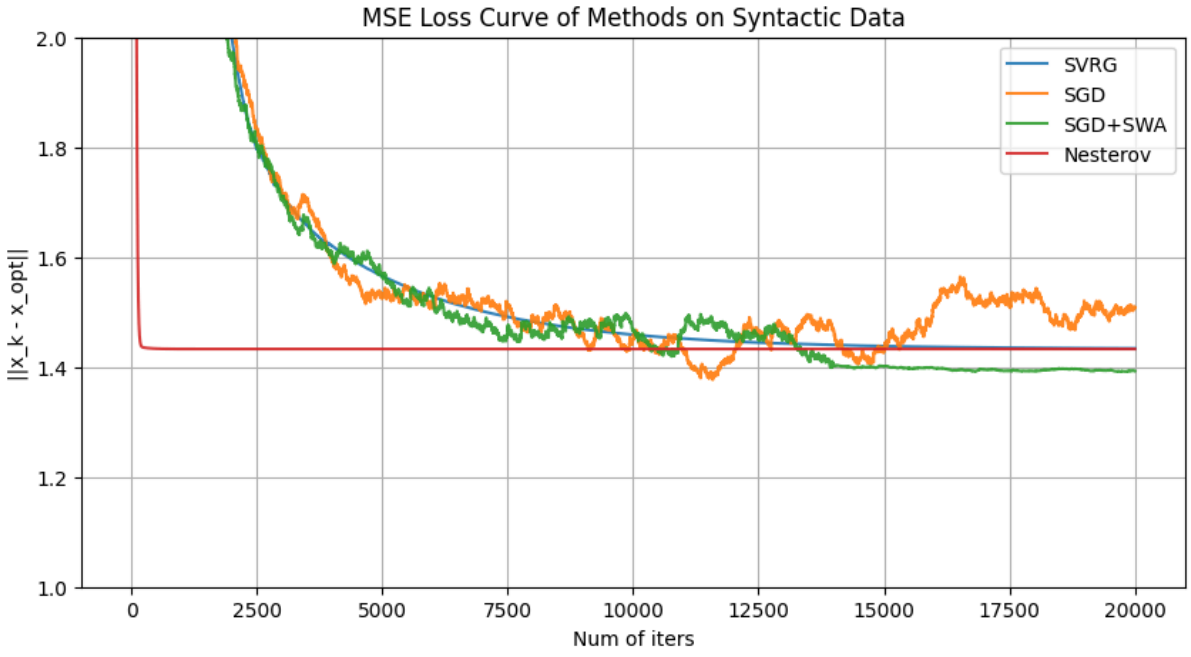
$$\begin{aligned}\mathbf{E}\left[\|g_k - \nabla f(x_k)\|^2\right] &\leq \mathbf{E}\left[\|\nabla f_i(x_k) - \nabla f_i(\bar{x})\|^2\right] \\ &\leq L_{\max}^2 \|x_k - \bar{x}\|^2,\end{aligned}$$

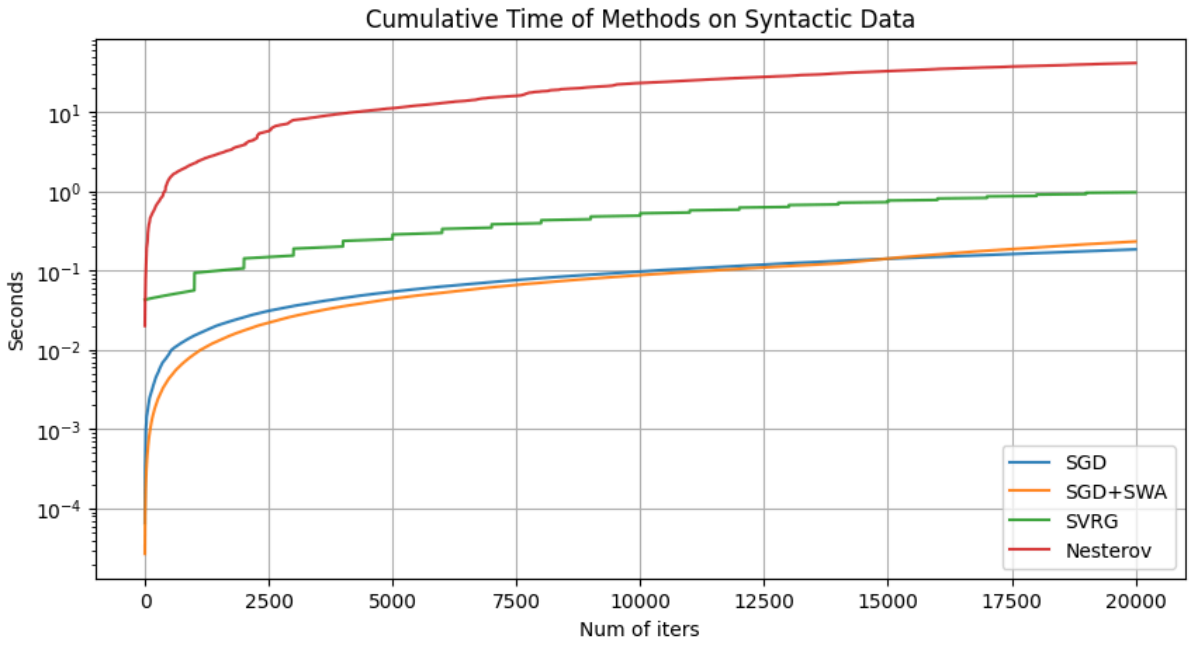
For a more interesting baseline, I try to used **SWA** for **SGD**



I **implemented the SVRG algorithm** in Python and compared it on synthetic data with **SGD** and **Nesterov GD**. I also tested **SWA for SGD.**


Convergence Rate of Methods on Syntactic Data
||x_k+1 - x_k||
Num of iters
SGD
SGD+SWA
SVRG
Nesterov


MSE Loss Curve of Methods on Syntactic Data
||x_k - x_opt||
Num of iters
SVRG
SGD
SGD+SWA
Nesterov



Real Data: **Student Depression Dataset**

- **Binary classification**, 27k samples, 18 features (categorical & numerical)
- **Basic preprocessing**: drop NaNs, One-Hot encoded, standard scaled
- Set **same LR** and **number of iterations** for each method

ROC-AUC Score on test set for methods:

SGD	SGD + SWA	Nesterov	SVRG
0.731	0.900	0.920	0.917