Метод сеток решения задачи Дирихле для уравнения Пуассона

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$$\frac{\partial^2 U(x,y)}{\partial x^2}+\frac{\partial^2 U(x,y)}{\partial y^2} = \phi(x, y), 0\le x\le 1, 0\le t\le 1.$$

$U|_\Gamma = \psi(x, y)$

$\Gamma : x^2 + y^2 = 4$

In[1]:=
```
h = 0.4
l = 0.4
α = 0.3 * 16
mMax = 5
nMax = 5
mMin = -5
nMin = -5
ϕ[x_, y_] := (α^2 + 1) x + (1 - α^2) y
ψ[x_, y_] := (1 - α^2) x + α y^2 + 2 α
```

Out[1]= 0.4

Out[2]= 0.4

Out[3]= 4.8

Out[4]= 5

Out[5]= 5

Out[6]= −5

Out[7]= −5

In[16]:=
```
ClearAll[X]
ClearAll[Y]
X = (# h) & /@ Range[mMin, mMax]
Y = (# h) & /@ Range[nMin, nMax]
γ[x_, y_] := Sqrt[4 - y^2] - Abs[x]
```

Out[18]=
```
{-2., -1.6, -1.2, -0.8, -0.4, 0., 0.4, 0.8, 1.2, 1.6, 2.}
```

Out[19]=
```
{-2., -1.6, -1.2, -0.8, -0.4, 0., 0.4, 0.8, 1.2, 1.6, 2.}
```

In[21]:=
```
conditions = {U_{-5, 0} == ψ[X[[-5 + 6]], Y[[0 + 6]]],
  U_{-4, 3} == ψ[X[[-4 + 6]], Y[[3 + 6]]],
  U_{-3, 4} == ψ[X[[-3 + 6]], Y[[4 + 6]]],
  U_{0, 5} == ψ[X[[0 + 6]], Y[[5 + 6]]],
  U_{3, 4} == ψ[X[[3 + 6]], Y[[4 + 6]]],
  U_{4, 3} == ψ[X[[4 + 6]], Y[[3 + 6]]],
  U_{5, 0} == ψ[X[[5 + 6]], Y[[0 + 6]]],
  U_{4, -3} == ψ[X[[4 + 6]], Y[[-3 + 6]]],
  U_{3, -4} == ψ[X[[3 + 6]], Y[[-4 + 6]]],
  U_{0, -5} == ψ[X[[0 + 6]], Y[[-5 + 6]]],
  U_{-3, -4} == ψ[X[[-3 + 6]], Y[[-4 + 6]]],
  U_{-4, -3} == ψ[X[[-4 + 6]], Y[[-3 + 6]]],
  U_{-4,1} == (h ψ[X[[-4 + 6]] + γ[X[[-4 + 6]], Y[[1 + 6]]], Y[[1 + 6]]] + γ[X[[-4 + 6]], Y[[1 + 6]]] U_{-3,1}) / (h + γ[X[[-4 + 6]], Y[[1 + 6]]]),
  U_{-4,2} == (h ψ[X[[-4 + 6]] + γ[X[[-4 + 6]], Y[[2 + 6]]], Y[[2 + 6]]] + γ[X[[-4 + 6]], Y[[2 + 6]]] U_{-3,2}) / (h + γ[X[[-4 + 6]], Y[[2 + 6]]]),
  U_{-2,4} == (h ψ[X[[-2 + 6]], Y[[4 + 6]] + γ[Y[[4 + 6]], X[[-2 + 6]]]] + γ[Y[[4 + 6]], X[[-2 + 6]]] U_{-2,3}) / (h + γ[Y[[4 + 6]], X[[-2 + 6]]]),
  U_{-1,4} == (h ψ[X[[-1 + 6]], Y[[4 + 6]] + γ[Y[[4 + 6]], X[[-1 + 6]]]] + γ[Y[[4 + 6]], X[[-1 + 6]]] U_{-1,3}) / (h + γ[Y[[4 + 6]], X[[-1 + 6]]]),
  U_{1,4} == (h ψ[X[[1 + 6]], Y[[4 + 6]] + γ[Y[[4 + 6]], X[[1 + 6]]]] + γ[Y[[4 + 6]], X[[1 + 6]]] U_{-1,3}) / (h + γ[Y[[4 + 6]], X[[1 + 6]]]),
  U_{2,4} == (h ψ[X[[2 + 6]], Y[[4 + 6]] + γ[Y[[4 + 6]], X[[2 + 6]]]] + γ[Y[[4 + 6]], X[[2 + 6]]] U_{-2,3}) / (h + γ[Y[[4 + 6]], X[[2 + 6]]]),
```

$$U_{4,2} == \frac{h\,\psi[X[\![4+6]\!]+\gamma[X[\![4+6]\!],Y[\![2+6]\!]],Y[\![2+6]\!]]+\gamma[X[\![4+6]\!],Y[\![2+6]\!]]\,U_{-3,2}}{h+\gamma[X[\![4+6]\!],Y[\![2+6]\!]]},$$

$$U_{4,1} == \frac{h\,\psi[X[\![4+6]\!]+\gamma[X[\![4+6]\!],Y[\![1+6]\!]],Y[\![1+6]\!]]+\gamma[X[\![4+6]\!],Y[\![1+6]\!]]\,U_{-3,1}}{h+\gamma[X[\![4+6]\!],Y[\![1+6]\!]]},$$

$$U_{4,-1} == \frac{h\,\psi[X[\![4+6]\!]+\gamma[X[\![4+6]\!],Y[\![-1+6]\!]],Y[\![-1+6]\!]]+\gamma[X[\![4+6]\!],Y[\![-1+6]\!]]\,U_{-3,1}}{h+\gamma[X[\![4+6]\!],Y[\![-1+6]\!]]},$$

$$U_{4,-2} == \frac{h\,\psi[X[\![4+6]\!]+\gamma[X[\![4+6]\!],Y[\![-2+6]\!]],Y[\![-2+6]\!]]+\gamma[X[\![4+6]\!],Y[\![-2+6]\!]]\,U_{-3,2}}{h+\gamma[X[\![4+6]\!],Y[\![-2+6]\!]]},$$

$$U_{2,-4} == \frac{h\,\psi[X[\![2+6]\!],Y[\![-4+6]\!]+\gamma[Y[\![-4+6]\!],X[\![2+6]\!]]]+\gamma[Y[\![-4+6]\!],X[\![2+6]\!]]\,U_{-2,3}}{h+\gamma[Y[\![-4+6]\!],X[\![2+6]\!]]},$$

$$U_{1,-4} == \frac{h\,\psi[X[\![1+6]\!],Y[\![-4+6]\!]+\gamma[Y[\![-4+6]\!],X[\![1+6]\!]]]+\gamma[Y[\![-4+6]\!],X[\![1+6]\!]]\,U_{-1,3}}{h+\gamma[Y[\![-4+6]\!],X[\![1+6]\!]]},$$

$$U_{-1,-4} == \left(h\,\psi[X[\![-1+6]\!],Y[\![-4+6]\!]+\gamma[Y[\![-4+6]\!],X[\![-1+6]\!]]]+\gamma[Y[\![-4+6]\!],X[\![-1+6]\!]]\,U_{-1,3}\right)\Big/ (h+\gamma[Y[\![-4+6]\!],X[\![-1+6]\!]]),$$

$$U_{-2,-4} == \left(h\,\psi[X[\![-2+6]\!],Y[\![-4+6]\!]+\gamma[Y[\![-4+6]\!],X[\![-2+6]\!]]]+\gamma[Y[\![-4+6]\!],X[\![-2+6]\!]]\,U_{-2,3}\right)\Big/ (h+\gamma[Y[\![-4+6]\!],X[\![-2+6]\!]]),$$

$$U_{-4,-2} == \left(h\,\psi[X[\![-4+6]\!]+\gamma[X[\![-4+6]\!],Y[\![-2+6]\!]],Y[\![-2+6]\!]]+\gamma[X[\![-4+6]\!],Y[\![-2+6]\!]]\,U_{-3,2}\right)\Big/ (h+\gamma[X[\![-4+6]\!],Y[\![-2+6]\!]]),$$

$$U_{-4,-1} == \left(h\,\psi[X[\![-4+6]\!]+\gamma[X[\![-4+6]\!],Y[\![-1+6]\!]],Y[\![-1+6]\!]]+\gamma[X[\![-4+6]\!],Y[\![-1+6]\!]]\,U_{-3,1}\right)\Big/ (h+\gamma[X[\![-4+6]\!],Y[\![-1+6]\!]])\Big\};$$

In[22]:=
```
Ui = U_{Round[#[[1]]/0.4],Round[#[[2]]/0.4]} & /@
  Select[Flatten[Outer[{#1, #2} &, X, Y], 1], #[[1]]^2 + #[[2]]^2 <= 2^2 &]
```

Out[22]=

$$\{U_{-5,0}, U_{-4,-3}, U_{-4,-2}, U_{-4,-1}, U_{-4,0}, U_{-4,1}, U_{-4,2}, U_{-4,3}, U_{-3,-4}, U_{-3,-3}, U_{-3,-2}, U_{-3,-1}, U_{-3,0},$$
$$U_{-3,1}, U_{-3,2}, U_{-3,3}, U_{-3,4}, U_{-2,-4}, U_{-2,-3}, U_{-2,-2}, U_{-2,-1}, U_{-2,0}, U_{-2,1}, U_{-2,2}, U_{-2,3}, U_{-2,4},$$
$$U_{-1,-4}, U_{-1,-3}, U_{-1,-2}, U_{-1,-1}, U_{-1,0}, U_{-1,1}, U_{-1,2}, U_{-1,3}, U_{-1,4}, U_{0,-5}, U_{0,-4}, U_{0,-3}, U_{0,-2},$$
$$U_{0,-1}, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4}, U_{0,5}, U_{1,-4}, U_{1,-3}, U_{1,-2}, U_{1,-1}, U_{1,0}, U_{1,1}, U_{1,2},$$
$$U_{1,3}, U_{1,4}, U_{2,-4}, U_{2,-3}, U_{2,-2}, U_{2,-1}, U_{2,0}, U_{2,1}, U_{2,2}, U_{2,3}, U_{2,4}, U_{3,-4}, U_{3,-3}, U_{3,-2},$$
$$U_{3,-1}, U_{3,0}, U_{3,1}, U_{3,2}, U_{3,3}, U_{3,4}, U_{4,-3}, U_{4,-2}, U_{4,-1}, U_{4,0}, U_{4,1}, U_{4,2}, U_{4,3}, U_{5,0}\}$$

In[23]:=
```
equations =
  DeleteCases[Flatten[Table[If[Intersection[Ui, {U_{m+1,n}, U_{m,n}, U_{m,n+1}, U_{m-1,n}, U_{m,n-1}}] ==
        Sort[{U_{m+1,n}, U_{m,n}, U_{m,n+1}, U_{m-1,n}, U_{m,n-1}}], (U_{m+1,n} - U_{m,n} - U_{m-1,n})/h^2 +
      (U_{m,n+1} - 2 U_{m,n} + U_{m,n-1})/l^2 - ϕ[X[[m + 6]], Y[[n + 6]]] == 0], {m, -4, 4}, {n, -4, 4}], 1], Null];
```

In[24]:=
```
solution = First@Solve[Join[equations, conditions], U_{Round[#[[1]]/0.4],Round[#[[2]]/0.4]} & /@
    Select[Flatten[Outer[{#1, #2} &, X, Y], 1], #[[1]]^2 + #[[2]]^2 ≤ 2^2 &]]
```

Out[24]=
$$\{U_{-5,0} \to 53.68,\ U_{-4,-3} \to 51.776,\ U_{-4,-2} \to 26.3522,\ U_{-4,-1} \to 18.5176,\ U_{-4,0} \to -3.37149,$$
$$U_{-4,1} \to 18.5176,\ U_{-4,2} \to 26.3522,\ U_{-4,3} \to 51.776,\ U_{-3,-4} \to 48.336,\ U_{-3,-3} \to -0.327889,$$
$$U_{-3,-2} \to -8.58567,\ U_{-3,-1} \to -6.62053,\ U_{-3,0} \to 0.376003,\ U_{-3,1} \to -2.82763,$$
$$U_{-3,2} \to -1.8808,\ U_{-3,3} \to 6.55694,\ U_{-3,4} \to 48.336,\ U_{-2,-4} \to 28.8179,\ U_{-2,-3} \to 10.658,$$
$$U_{-2,-2} \to 5.74903,\ U_{-2,-1} \to 3.6606,\ U_{-2,0} \to 2.589,\ U_{-2,1} \to 5.51329,\ U_{-2,2} \to 9.54365,$$
$$U_{-2,3} \to 16.1443,\ U_{-2,4} \to 33.3414,\ U_{-1,-4} \to 15.8573,\ U_{-1,-3} \to -1.76629,$$
$$U_{-1,-2} \to -5.91319,\ U_{-1,-1} \to -5.64331,\ U_{-1,0} \to -4.10802,\ U_{-1,1} \to -2.90808,$$
$$U_{-1,2} \to -0.80564,\ U_{-1,3} \to 4.7959,\ U_{-1,4} \to 21.6745,\ U_{0,-5} \to 28.8,\ U_{0,-4} \to 3.99362,$$
$$U_{0,-3} \to -1.8919,\ U_{0,-2} \to -3.29837,\ U_{0,-1} \to -3.37613,\ U_{0,0} \to -2.72223,\ U_{0,1} \to -1.24641,$$
$$U_{0,2} \to 0.879241,\ U_{0,3} \to 3.89286,\ U_{0,4} \to 9.68337,\ U_{0,5} \to 28.8,\ U_{1,-4} \to 6.57235,$$
$$U_{1,-3} \to -3.90557,\ U_{1,-2} \to -7.71916,\ U_{1,-1} \to -8.34054,\ U_{1,0} \to -7.65217,$$
$$U_{1,1} \to -6.21488,\ U_{1,2} \to -3.63549,\ U_{1,3} \to 1.68018,\ U_{1,4} \to 12.3895,\ U_{2,-4} \to 6.53527,$$
$$U_{2,-3} \to -6.69156,\ U_{2,-2} \to -9.85005,\ U_{2,-1} \to -10.0773,\ U_{2,0} \to -9.58475,\ U_{2,1} \to -8.4754,$$
$$U_{2,2} \to -6.77507,\ U_{2,3} \to -2.51375,\ U_{2,4} \to 11.0587,\ U_{3,-4} \to -4.56,\ U_{3,-3} \to -13.3567,$$
$$U_{3,-2} \to -14.6022,\ U_{3,-1} \to -14.65,\ U_{3,0} \to -14.7766,\ U_{3,1} \to -13.6147,\ U_{3,2} \to -12.7155,$$
$$U_{3,3} \to -11.2993,\ U_{3,4} \to -4.56,\ U_{4,-3} \to -18.752,\ U_{4,-2} \to -18.2132,\ U_{4,-1} \to -18.6223,$$
$$U_{4,0} \to -21.0341,\ U_{4,1} \to -18.6223,\ U_{4,2} \to -18.2132,\ U_{4,3} \to -18.752,\ U_{5,0} \to -34.48\}$$

In[61]:=
```
ClearAll[U]
```

In[75]:=
```
u = Outer[{#1, #2} &, X, Y];
answer = u /. {p_, q_} :> If[p^2 + q^2 <= 2^2, U_{Round[p/0.4], Round[q/0.4]}, ""] /. solution
```

Out[76]=

```
{{, , , , , 53.68, , , , , },
 {, , 51.776, 26.3522, 18.5176, -3.37149, 18.5176, 26.3522, 51.776, , }, {, 48.336,
  -0.327889, -8.58567, -6.62053, 0.376003, -2.82763, -1.8808, 6.55694, 48.336, },
 {, 28.8179, 10.658, 5.74903, 3.6606, 2.589, 5.51329, 9.54365, 16.1443, 33.3414, },
 {, 15.8573, -1.76629, -5.91319, -5.64331, -4.10802, -2.90808,
  -0.80564, 4.7959, 21.6745, }, {28.8, 3.99362, -1.8919, -3.29837,
  -3.37613, -2.72223, -1.24641, 0.879241, 3.89286, 9.68337, 28.8},
 {, 6.57235, -3.90557, -7.71916, -8.34054, -7.65217, -6.21488, -3.63549,
  1.68018, 12.3895, }, {, 6.53527, -6.69156, -9.85005, -10.0773,
  -9.58475, -8.4754, -6.77507, -2.51375, 11.0587, }, {, -4.56, -13.3567,
  -14.6022, -14.65, -14.7766, -13.6147, -12.7155, -11.2993, -4.56, },
 {, , -18.752, -18.2132, -18.6223, -21.0341, -18.6223, -18.2132, -18.752, , },
 {, , , , , -34.48, , , , , }}
```

In[91]:=
```
Grid[Join[{Join[{"y_i/x_i"}, Range[-2., 2., 0.4]]},
  MapThread[Prepend[#2, #1] &, {Range[-2., 2., 0.4], answer}]],
 Spacings -> {2, 1}, Frame -> True]
```

Out[91]=

y_i/x_j	−2.	−1.6	−1.2	−0.8	−0.4	0.	0.4	0.8	1.2	1.6	2.
−2.						53.68					
−1.6			51.776	26.3522	18.5176	−3.37149	18.5176	26.3522	51.776		
−1.2		48.336	−0.327889	−8.58567	−6.62053	0.376003	−2.82763	−1.8808	6.55694	48.336	
−0.8		28.8179	10.658	5.74903	3.6606	2.589	5.51329	9.54365	16.1443	33.3414	

−0.4		15.˙8˙5˙73	−1.˙7˙6˙6˙2˙9	−5.˙9˙1˙3˙1˙9	−5.˙6˙4˙3˙3˙1	−4.˙1˙0˙8˙0˙2	−2.˙9˙0˙8˙0˙8	−0.˙8˙0˙5˙6˙4	4.7˙9˙59	21.˙6˙7˙45	
0.	28.8	3.9˙9˙3˙62	−1.˙8˙9˙1˙9	−3.˙2˙9˙8˙3˙7	−3.˙3˙7˙6˙1˙3	−2.˙7˙2˙2˙2˙3	−1.˙2˙4˙6˙4˙1	0.8˙7˙9˙2˙41	3.8˙9˙2˙86	9.6˙8˙3˙37	28.8
0.4		6.5˙7˙2˙35	−3.˙9˙0˙5˙5˙7	−7.˙7˙1˙9˙1˙6	−8.˙3˙4˙0˙5˙4	−7.˙6˙5˙2˙1˙7	−6.˙2˙1˙4˙8˙8	−3.˙6˙3˙5˙4˙9	1.6˙8˙0˙18	12.˙3˙8˙95	
0.8		6.5˙3˙5˙27	−6.˙6˙9˙1˙5˙6	−9.˙8˙5˙0˙0˙5	−1˙0˙.˙0˙7˙7˙3	−9.˙5˙8˙4˙7˙5	−8.˙4˙7˙5˙4	−6.˙7˙7˙5˙0˙7	−2.˙5˙1˙3˙7˙5	11.˙0˙5˙87	
1.2		−4.˙5˙6	−1˙3˙.˙3˙5˙6˙7	−1˙4˙.˙6˙0˙2˙2	−1˙4˙.˙6˙5	−1˙4˙.˙7˙7˙6˙6	−1˙3˙.˙6˙1˙4˙7	−1˙2˙.˙7˙1˙5˙5	−1˙1˙.˙2˙9˙9˙3	−4.˙5˙6	
1.6			−1˙8˙.˙7˙5˙2	−1˙8˙.˙2˙1˙3˙2	−1˙8˙.˙6˙2˙2˙3	−2˙1˙.˙0˙3˙4˙1	−1˙8˙.˙6˙2˙2˙3	−1˙8˙.˙2˙1˙3˙2	−1˙8˙.˙7˙5˙2		

2.

$$-3\ddot{\cdot}.$$
$$4\ddot{\cdot}.$$
$$\therefore\ddot{\cdot}$$
$$4\ddot{\cdot}.$$
$$8$$