N 1

1) $\int_{-\infty}^{+\infty} \frac{x^4}{1+x^6}\,dx = \oint_{C_R} \frac{z^4}{1+z^6}\,dz = 2\pi\cdot \operatorname*{res}_{z=i}\left(\frac{z^4}{1+z^6}\right) =$

$= 2\pi\cdot\left(\frac{z^4}{(1+z^6)'}\right)\Bigg|_{z=i} = 2\pi i\left(\frac{z^4}{6z^5}\right)\Bigg|_{z=i} = \frac{2\pi i}{6i} = \frac{2\pi}{3}$

2)

$$\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}\, d\theta \overset{z=e^{i\theta}}{=} 2\pi i \sum_k{}' \operatorname*{res}_k \left(\frac{\frac{z^2+\frac{1}{z^2}}{2}}{\left(2+\frac{z+\frac{1}{z}}{2}\right)\cdot zi} \right) =$$

$$= 2\pi i \sum_k{}' \operatorname*{res}_k \left(\frac{(z^4+1)(-i)}{z^2(z^2+4z+1)} \right) = 2\pi i \sum_k{}' \operatorname*{res}_k \left(\frac{(z^4+1)(-i)}{z^2(z+\sqrt{3}+2)\cdot(z+2-\sqrt{3})} \right)$$

$$\operatorname*{res}_{z=-2+\sqrt{3}} = \left(\frac{(-i)(z^4+1)}{z^2(z+\sqrt{3}+2)} \right)\Bigg|_{z=-2+\sqrt{3}} = (-i)\frac{7}{\sqrt{3}}$$

$$\operatorname*{res}_{z=0} = \lim_{z\to 0} \frac{d}{dz}\left(\frac{(-i)(z^4+1)}{z^2+4z+1} \right) = 4i$$

$$\ominus\ 2\pi i\left(4i - i\frac{7}{\sqrt{3}}\right) = 2\pi \cdot \frac{7-4\sqrt{3}}{\sqrt{3}}$$

3)

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = 2\pi i \sum_k \underset{k}{res}\left(\frac{1}{(z-ia)(z+ia)\cdot(z-ib)^2(z+ib)^2}\right) =$$

$$\underset{z=ia}{res} = \left.\frac{1}{(z+ia)(z^2+b^2)^2}\right|_{z=ia} = \frac{-i}{2a(b^2-a^2)^2}$$

$$\underset{z=ib}{res} = \lim_{z\to ib} \frac{d}{dz}\left(\frac{1}{(z^2+a^2)(z+ib)^2}\right) =$$

$$= \lim_{z\to ib}\left[-2\cdot\frac{izb+2z^2+a^2}{(z^2+a^2)^2(z+ib)^3}\right] = -i\cdot\frac{a^2-3b^2}{4b^3(a^2-b^2)^2}$$

$$= \pi\cdot\left(\frac{1}{a(b^2-a^2)^2} + \frac{a^2-3b^2}{2b^3(a^2-b^2)^2}\right) =$$

$$= \pi\cdot\left(\frac{2b^3+a^3-3ab^2}{2ab^3\cdot(b-a)^2(b+a)^2}\right) = \frac{2b+a}{2ab^3(b+a)^3}\pi$$

N 6

$$\int_0^{\infty} \frac{x \cdot \sin(ax)}{x^2+k^2}\, dx = \operatorname{Im} \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x}{x^2+k^2} e^{iax} dx = \frac{1}{2} \operatorname{Im} \oint_{C_R} \frac{z}{z^2+k^2} e^{iaz} dz$$

$$= \frac{1}{2} \operatorname{Im}\left[2\pi i \sum_k \operatorname*{res}_k f(z) \cdot e^{iaz} \right] \ominus$$

$$\operatorname*{res}_{z=ik} = \left(\frac{z \cdot e^{iaz}}{z+ik} \right)\Bigg|_{z=ik} = \frac{ik\, e^{-ak}}{2ik} = \frac{e^{-ak}}{2}$$

$$\ominus \frac{1}{2} \operatorname{Im}\left[2\pi i \, \frac{e^{-ak}}{2} \right] = \frac{\pi}{2} e^{-|a||k|} \cdot \operatorname{sign}|a|$$

N 7

$$2) \quad pV \int_0^{\infty} \frac{x\, dx}{(x^2+a^2) \sin bx} = \qquad a > 0 \quad b > 0$$

$$= pV \ \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x\, dx}{(x^2+a^2) \sin bx} = pV \ \frac{1}{2} \oint_{C_R} \frac{z\, dz}{(z^2+a^2) \sin bx} =$$

$$= \frac{1}{2} \cdot 2\pi i \sum_k \operatorname*{res}_k f(z) \ominus$$

$$\operatorname*{res}_{z=ai} = \left(\frac{z}{(z+ai) \sin bz} \right)\Bigg|_{z=ai} = \frac{1}{2 \cdot \sin(abi)} = \frac{1}{2 i \operatorname{sh}(ab)}$$

$$\ominus \frac{\pi}{2 \cdot \operatorname{sh}(ab)}$$

N4

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2(x^2+1)}\,dx = \operatorname{Im} \oint_{C_R} \frac{\sin z}{z^2(z^2+1)} e^{iz}\,dz =$$

$$\operatorname{Im} \frac{\sin x}{x^2(x^2+1)} e^{ix}$$

$$= \operatorname{Im}\left[2\pi i \sum_k \operatorname{res}_k f(z)\, e^{iz} \right] \ominus$$

$$\operatorname{res}_{z=i} = \left(\frac{\sin z \cdot e^{iz}}{z^2(z+i)} \right)\Bigg|_{z=i} = -\frac{sh 1 \cdot e^{-1}}{2}$$

$$\operatorname{res}_{z=0} = \frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left(\frac{e^{iz} \cdot \sin z}{z^2+1} \right) = \frac{1}{2} \lim_{z\to 0} \left[ie^{iz} \frac{\sin z}{z^2+1} + e^{iz} \frac{\cos z\cdot(z^2+1) - 2z\sin z}{(z^2+1)^2} \right]$$

$$= \frac{1}{2}$$

$$\ominus \operatorname{Im}\left[2\pi i \left(\frac{1}{2} - \frac{\frac{e^1 - e^{-1}}{2} \cdot e^{-1}}{2} \right) \right] = \frac{\pi}{2}\left(1 + \frac{1}{e^2} \right)$$

N2

$$\underset{z=\infty}{\mathrm{res}}\, f(z) = 0 - \underset{z=2}{\mathrm{res}}\, f(z) \qquad f(z) = z^3 \cdot \cos\left(\frac{1}{z-2}\right)$$

$t = z-2$:

$$(t+2)^3 \cdot \cos\frac{1}{t} = (t^3 + 6t^2 + 12t + 8) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\, t^{2n}} =$$

$$= (t^3 + 6t^2 + 12t + 8) \cdot \left(1 - \frac{1}{2t^2} + \frac{1}{24t^4} - \ldots\right)$$

$$c_{-1} = -6 + \frac{1}{24} = -\frac{143}{24}$$

$$\underset{z=\infty}{\mathrm{res}}\, f(z) = \frac{143}{24}$$