4.1

a) $\varphi(z) = \sqrt[3]{z}$ $\quad \varphi = |z|^{\frac{1}{3}} e^{\frac{\varphi i}{3} + \frac{2}{3}\pi i k}$ $\quad k \in \mathbb{Z}$

$\arg(z) \in [\frac{\pi}{2} ; \frac{5\pi}{2}] \Rightarrow \arg \varphi \in [\frac{\pi}{6} , \frac{5\pi}{6}]$

$\varphi(1) = e^{\frac{2}{3}\pi i}$

$\varphi(i+0) = e^{\frac{5\pi}{6} i}$

$z = e^{\frac{5\pi - 0}{2} i} = i + 0$

$\varphi(i-0) = e^{\frac{\pi}{6} i}$

$z = e^{\frac{\pi + 0}{2} i}$

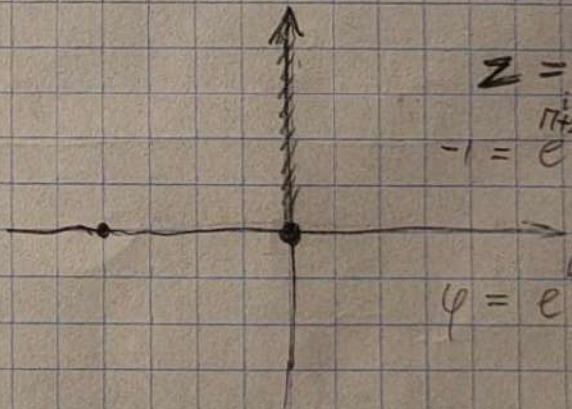
$[\frac{\pi}{2} ; \frac{5\pi}{2}]$

$[\frac{\pi}{6} ; \frac{5\pi}{6}]$

$z = e^{i(\varphi + 2\pi k)}$

$-1 = e^{\pi i + 2\pi i k}$

$\varphi(z) = \sqrt[3]{z}$

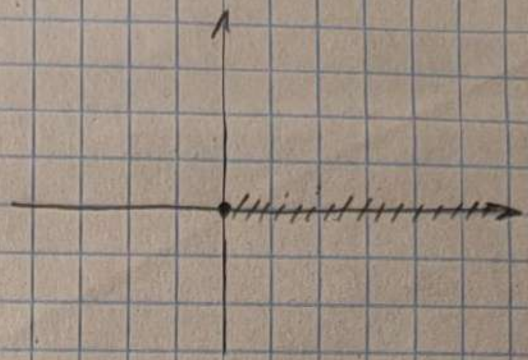


$\varphi = e^{i(\frac{\pi}{3} + \frac{\pi}{3} k)}$

в) $\varphi(z) = \ln z$

$z = |z| \cdot e^{i(\psi + 2\pi k)}$

$\ln z = \ln|z| + i(\psi + 2\pi k)$



$\varphi(1 - i0) = \ln 1 + i(\psi + 2\pi k)$ $\quad \psi = 0 \quad k = 0$

$\Rightarrow \psi \in [-2\pi; 0]$

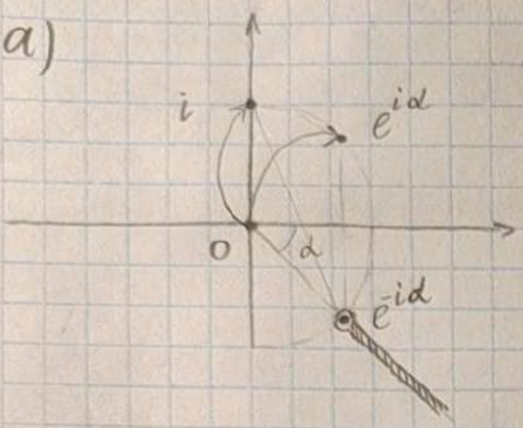
$\varphi(1 + i0) = \ln e^{(-2\pi + 0)i} = -2\pi i$

$\varphi(i) = -\frac{3\pi}{2} i$

$\varphi(-i) = -\frac{\pi}{2} i$

4.2 $\varphi_1(z)=\sqrt{z-e^{-i\alpha}}$ $\varphi_1(0)=i\,e^{-i\frac{\alpha}{2}}$

a)



$$\varphi_1(i)=\left|\frac{i-e^{-i\alpha}}{-e^{-i\alpha}}\right|^{\frac{1}{2}}\cdot e^{-\frac{i}{4}\cdot(2\pi-(\pi+\alpha))}\cdot i\cdot e^{-i\frac{\alpha}{2}}=$$

$$=\left|e^{i\frac{\frac{\pi}{2}-\alpha}{2}}\left(e^{i\frac{\frac{\pi}{2}+\alpha}{2}}-e^{i\frac{-\frac{\pi}{2}-\alpha}{2}}\right)\right|^{\frac{1}{2}}\cdot i\cdot e^{-\frac{i}{4}(\pi+\alpha)}=$$

$$=\sqrt{\left|2i\cdot\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right|}\cdot i\cdot e^{-\frac{i}{4}(\pi+\alpha)}=$$

$$=\sqrt{2\cdot\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}\cdot i\cdot e^{-\frac{i}{4}(\pi+\alpha)}=$$

$$=\sqrt{2\cdot\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}\cdot e^{\frac{i}{4}(\pi-\alpha)}$$

$$\varphi_1(e^{i\alpha})=\left|\frac{e^{i\alpha}-e^{-i\alpha}}{-e^{-i\alpha}}\right|^{\frac{1}{2}}\cdot e^{-\frac{i}{2}\left(\frac{\pi}{2}-\alpha\right)}\cdot i\cdot e^{-i\frac{\alpha}{2}}=\sqrt{|2i\cdot\sin\alpha|}\cdot e^{i\frac{\pi}{4}}=$$

$$=\sqrt{2\cdot\sin\alpha}\cdot e^{i\frac{\pi}{4}}$$

$\varphi_2(z)=\ln(z-e^{-i\alpha})$ $\varphi_2(0)=-i\pi-i\alpha$

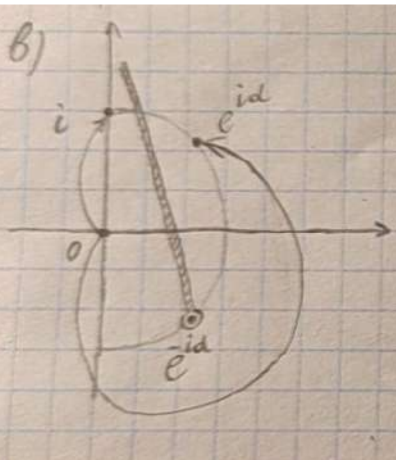
$$\varphi_2(i)=\ln\left|\frac{i-e^{-i\alpha}}{-e^{-i\alpha}}\right|-\frac{i}{2}(\pi-\alpha)-i\pi-i\alpha=$$

$$=\ln\left(2\cdot\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right)-\frac{3i}{2}\pi-\frac{i}{2}\alpha$$

$$\varphi_2(e^{i\alpha}) = \ln\left|\frac{e^{i\alpha} - e^{-i\alpha}}{-e^{-i\alpha}}\right| - i\cdot\left(\frac{\pi}{2} - \alpha\right) - i\pi - i\alpha =$$

$$= \ln\left(2\cdot\sin\alpha\right) - i\frac{3\pi}{2}$$

в)



$$\varphi_1(i) = \sqrt{2\cdot\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}\cdot e^{\frac{i}{4}(\pi-\alpha)}$$

$$\varphi_2(i) = \ln\left(2\cdot\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right) - \frac{3i}{2}\pi - \frac{i}{2}\alpha$$

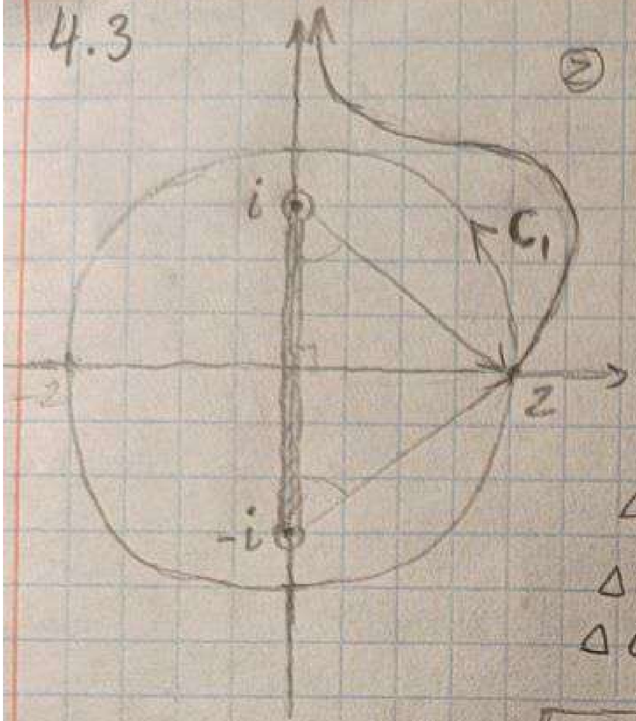
$$\varphi_1(e^{i\alpha}) = \sqrt{2\cdot\sin\alpha}\cdot e^{\frac{i}{2}\left(\frac{3\pi}{2}+\alpha\right)}\cdot i\cdot e^{-i\frac{\alpha}{2}} =$$

$$= \sqrt{2\cdot\sin\alpha}\cdot$$

$$\varphi_2(e^{i\alpha}) = \ln\left|\frac{e^{i\alpha} - e^{-i\alpha}}{-e^{-i\alpha}}\right| + i\left(\frac{3\pi}{2} + \alpha\right) - i\pi - i\alpha =$$

$$= \ln\left|2i\, e^{i\alpha} \cdot \sin\alpha\right| + i\frac{\pi}{2} = \ln(2\cdot\sin\alpha) + i\frac{\pi}{2}$$

4.3



②

$$f(z) = \sqrt{1+z^2} = \sqrt{(z-i)(z+i)}$$

$$f_0 = f(2) = \sqrt{5}$$

$$\Delta \arg(z-i) = 2\pi$$
$$\Delta \arg(z+i) = 2\pi$$
$$\Delta \arg(g) = 4\pi$$

$$f(2) = f_0 \cdot \sqrt{\left|\frac{1+2^2}{1+2^2}\right|} \cdot e^{\frac{i}{2}4\pi} = \sqrt{5} \cdot e^{2\pi i} = \sqrt{5}$$

$$I_1 = \oint\limits_{C_1} f(z)\,dz = 2\pi i \sum_j \operatorname{res}_j (f(z)) = -2\pi i \cdot \operatorname{res}_{z=\infty} f(z)$$

$z = iy$

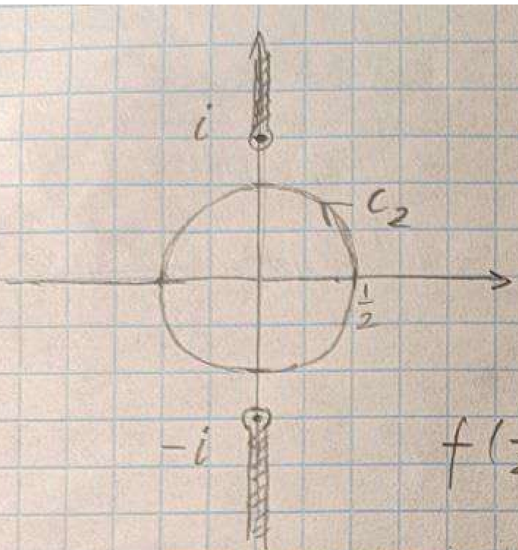
$$f(z) = \left|\frac{1-y^2}{5}\right|^{\frac{1}{2}} \cdot e^{\frac{i}{2}(\pi - \operatorname{arctg}\frac{2}{1} + \operatorname{arctg}\frac{2}{1})} \cdot \sqrt{5} =$$
$$= \sqrt{|1-y^2|} \cdot e^{\frac{i}{2}\pi} = i\sqrt{y^2-1} = iy\sqrt{1-\frac{1}{y^2}} =$$
$$= iy \cdot \left(1 - \frac{1}{2}\frac{1}{y^2} - \frac{1}{8}\frac{1}{y^4} + \ldots\right) = iy - \frac{i}{2}\frac{1}{y} + \ldots$$

$$f(z) = z + \frac{1}{2}\frac{1}{z} + \ldots$$

$$C_{-1} = \frac{1}{2} \quad \Rightarrow \operatorname{res}_{\infty} f(z) = -\frac{1}{2}$$

$$I_1 = \pi i$$

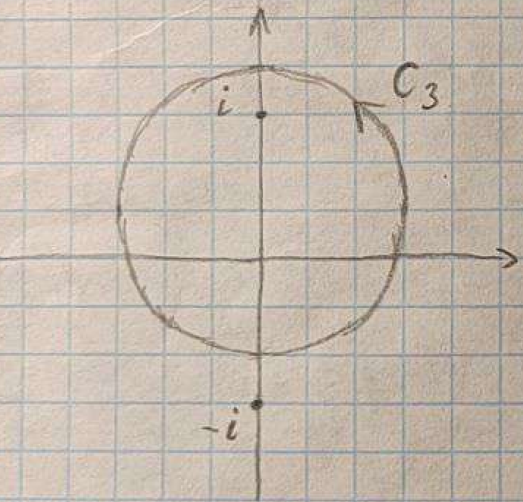
$$f_0 = f\left(\tfrac{1}{2}\right) = \frac{\sqrt{3}}{2}$$



$$\Delta \arg (z-i) = 0$$
$$\Delta \arg (z+i) = 0$$
$$\Delta \arg (g) = 0$$

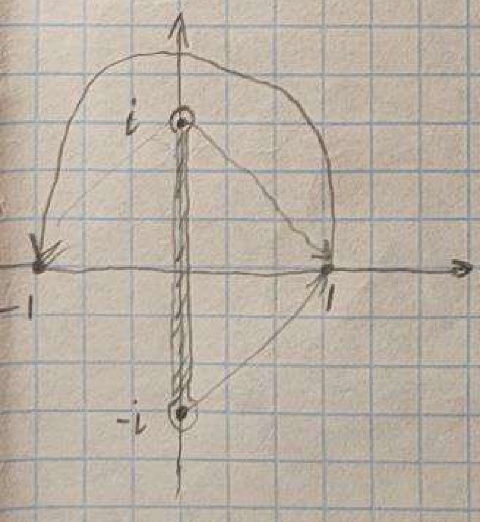
$$f\left(\tfrac{1}{2}\right) = \sqrt{\left|\frac{1+(\frac{1}{2})^2}{1+(\frac{1}{2})^2}\right|} \cdot e^{\frac{i}{2}\cdot 0} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$I_2 = \oint\limits_{C_2} f(z)\,dz = 2\pi i \cdot \sum_j \operatorname{res}_j f(z) = 0$$



$$\Delta \arg (z-i) = 2\pi$$
$$\Delta \arg (z+i) = 0$$
$$\Delta \arg (g) = 2\pi$$

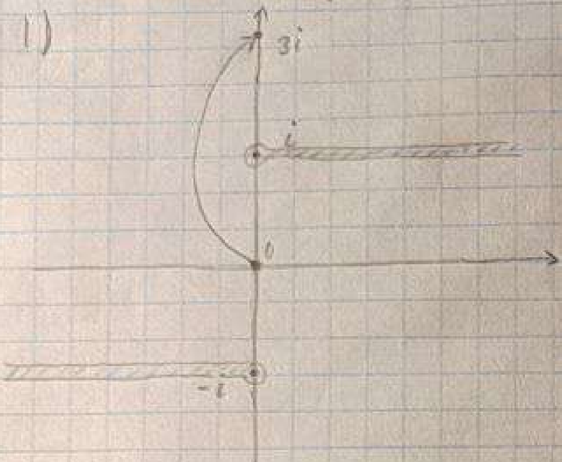
$$f(z) = \sqrt{\left|\frac{g(z)}{g(z_0)}\right|} \cdot e^{i\pi} \cdot f_0 =$$
$$= f_0 \cdot e^{i\pi} = -f_0$$



$$f(-1) = \left|\frac{1-i(-1)}{1-i\cdot 1}\right|^{\frac{1}{2}} \left|\frac{1+i(-1)}{1+i\cdot 1}\right|^{\frac{1}{2}} \cdot e^{\frac{i}{2}\left(\frac{3\pi}{2}+\frac{\pi}{2}\right)} \cdot \sqrt{2}$$
$$= \sqrt{2} \cdot e^{i\pi} = -\sqrt{2}$$

4.6 $\varphi = \sqrt[3]{1+z^2} = \sqrt[3]{(z-i)(z+i)}$ $\varphi(0) = 1$

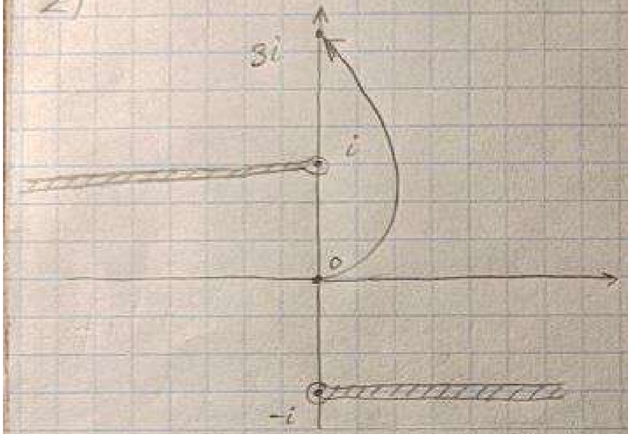
1)



$\Delta \arg(z-i) = -\pi$
$\Delta \arg(z+i) = 0$
$\Delta \arg(g) = -\pi$

$$f(3i) = \sqrt[3]{\left|\frac{1+(3i)^2}{1}\right|} \cdot e^{\frac{i}{3}(-\pi)} \cdot 1 =$$
$$= 2 \cdot e^{-i\frac{\pi}{3}} = 1 - i\sqrt{3}$$

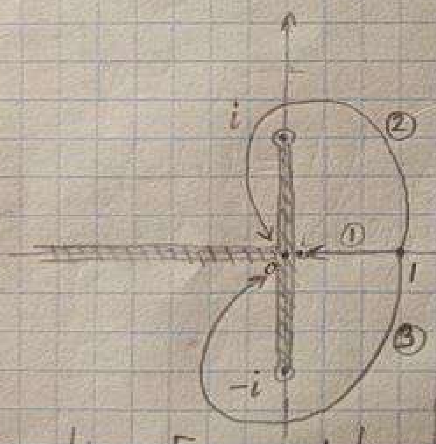
2)



$\Delta \arg(z-i) = \pi$
$\Delta \arg(z+i) = 0$
$\Delta \arg(g) = \pi$

$$f(3i) = \sqrt[3]{\left|\frac{1+(3i)^2}{1}\right|} \cdot e^{\frac{i}{3}\pi} \cdot 1 =$$
$$= 1 + i\sqrt{3}$$

4.7 $f(x) = \ln\sqrt{1+x^2}$ A: $\{x : -\infty < x < 0\} \cup \{ix : -1 < x < 1\}$



$f(x) = \frac{1}{2}\ln(1+x^2)$
$F(z) = \frac{1}{2}\ln[(z-i)(z+i)]$
$F_0 = F(1)$

1) $$\lim_{\varepsilon \to 0} F(\varepsilon) = \frac{1}{2} \lim_{\varepsilon \to 0}\left[\ln\left|\frac{g(\varepsilon)}{g_0}\right| + i \cdot \Delta\arg(g) + F_0\right] =$$
$$= \frac{1}{2}\lim_{\varepsilon \to 0}\left[\ln|g(\varepsilon)| + i \cdot 0\right] = \frac{1}{2}\lim_{\varepsilon \to 0} \ln(1+\varepsilon^2) = 0$$

2) $$\lim_{\varepsilon \to 0} F(\varepsilon \cdot e^{i\frac{3\pi}{4}}) = \frac{1}{2}\lim_{\varepsilon \to 0}\left[\ln|g(\varepsilon \cdot e^{i\frac{3\pi}{4}})| + i \cdot \left(\frac{7\pi}{4} + \frac{\pi}{4}\right)\right] = \pi i$$

3) $$\lim_{\varepsilon \to 0} F(\varepsilon \cdot e^{-i\frac{3\pi}{4}}) = \frac{1}{2}\lim_{\varepsilon \to 0}\left[\ln|g(\varepsilon \cdot e^{-i\frac{3\pi}{4}})| + i \cdot \left(-\frac{7\pi}{4} - \frac{\pi}{4}\right)\right] = -\pi i$$

4.9

$$f(z) = z^a (z-1)^b$$

- $N(1,1)$: $f = z(z-1)$

$$N(1,1) = 0$$

- $N(1, \frac{1}{2})$: $f = z\sqrt{z-1}$ $\qquad f=0$: $z=1$

$N(1, \frac{1}{2}) = 2$ $\qquad f=\infty$: $z=\infty$

- $N(\frac{1}{2}, \frac{1}{3}) = 3$
- $N(\frac{2}{3}; \frac{1}{3}) = 2$

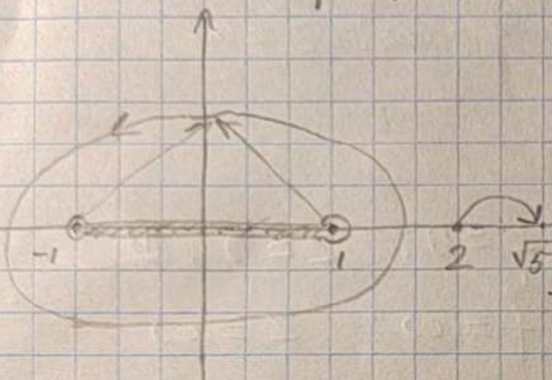
4.10

$$f^2(z) - 2\cdot f(z) + z^2 = 0$$

$$f(z) = 1 \pm \sqrt{1-z^2} = 1 \pm \sqrt{(1-z)(1+z)} = 1 \pm i\sqrt{(z-1)(z+1)}$$

$$\varphi = \sqrt{(z-1)(z+1)} \qquad \omega = (z-1)(z+1) \qquad z^2-1$$

$$\Delta \arg(z-1) = 2\pi$$
$$\Delta \arg(z+1) = 2\pi$$
$$\Delta \arg(\omega) = 4\pi$$



$$\varphi(z_0) = \sqrt{\frac{\omega(z_0)}{\omega(z_0)}} \cdot e^{i2\pi} \cdot \varphi_0 = \varphi_0$$

$$f_0 = 1 \pm i\varphi_0$$

$$g(z) = \ln(1 + f(z)) = \ln(2 \pm i\varphi) = \ln\left(2 \pm i\sqrt{(z-1)(z+1)}\right)$$

$$g_0 = g(2) = \ln(2 \pm i\sqrt{3})$$

$$g(\sqrt{5}) = \ln\left|\frac{2 \pm i2}{2 \pm i\sqrt{3}}\right| + i\cdot 0 + \ln(2 \pm i\sqrt{3}) = \ln(2 \pm i2)$$

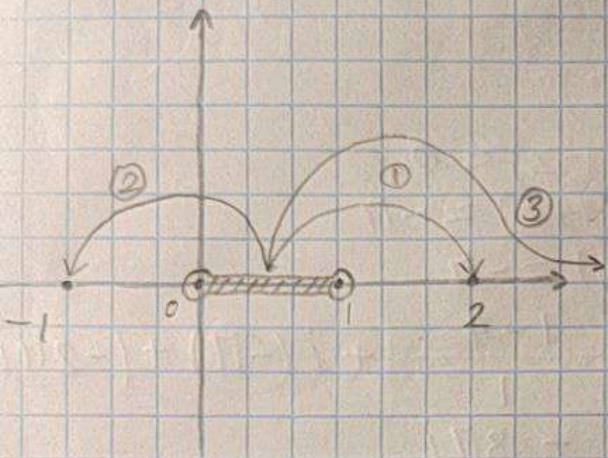
$$2 \pm i2 = \sqrt{2}\cdot e^{\pm i\frac{\pi}{4} + 2\pi n}$$

$$g(\sqrt{5}) = \frac{1}{2}\ln 2 \pm i\frac{\pi}{4} + 2\pi n$$

4.4 $\varphi(z) = z^{\mu} \cdot (1-z)^{1-\mu}$ $\qquad \varphi(\tfrac{1}{2}+i0) = \tfrac{1}{2}$

$\mu \in \mathbb{R}$

$$\varphi(z) = \varphi(z_0) \cdot \left|\frac{g_1(z)}{g_1(z_0)}\right|^{\mu} \cdot \left|\frac{g_2(z)}{g_2(z_0)}\right|^{1-\mu} \cdot e^{i\Delta \arg \varphi}$$



1) $\varphi(2)$:

$\Delta \arg(z) = 0$

$\Delta \arg(1-z) = -\pi$

$\Delta \arg(\varphi) = -\pi(1-\mu)$

$$\varphi(2) = \frac{1}{2} \cdot \left|\frac{2}{\frac{1}{2}+i0}\right|^{\mu} \cdot \left|\frac{1-2}{-\frac{1}{2}-i0}\right|^{1-\mu} \cdot e^{-i\pi(1-\mu)} =$$

$$= 2^{\mu} \cdot e^{(\mu-1)\pi i}$$

2) $\varphi(-1)$: $\qquad \Delta \arg(z) = \pi$

$\Delta \arg(1-z) = 0$

$\Delta \arg(\varphi) = \pi\mu$

$$\varphi(-1) = \frac{1}{2}\left|\frac{-1}{\frac{1}{2}+i0}\right|^{\mu} \cdot \left|\frac{2}{-\frac{1}{2}-i0}\right|^{1-\mu} \cdot e^{i\pi\mu} =$$

$$= 2^{1-\mu} \cdot e^{\pi\mu i}$$

3) $$\lim_{z\to\infty} \frac{\varphi(z)}{z} = \lim_{z\to\infty}\left[z^{\mu-1} \cdot (1-z)^{1-\mu}\right] \qquad \left(\frac{\varphi(z)}{z}\right)\Big|_{z_0} = 1$$

$\Delta \arg(z) = 0$

$\Delta \arg(1-z) = -\pi$

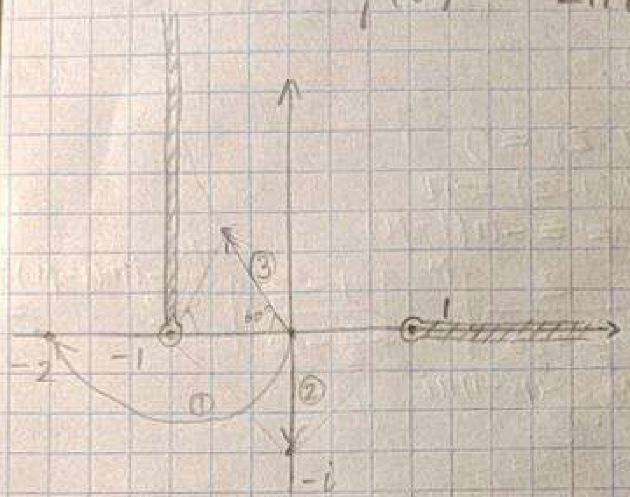
$\Delta \arg(\varphi) = -\pi(1-\mu) = \pi \cdot (\mu-1)$

$$\frac{\varphi(z)}{z}\Big|_{z\to\infty} = 1 \cdot \frac{|z|^{\mu-1}}{|\frac{1}{2}+i0|^{\mu-1}} \cdot \frac{|1-z|^{1-\mu}}{|-\frac{1}{2}-i0|^{1-\mu}} \cdot e^{i\pi(\mu-1)} =$$

$$= 1 \cdot e^{i\pi(\mu-1)} \cdot \left|\frac{1-z}{z}\right|^{1-\mu} = e^{i\pi(\mu-1)}$$

4.5 $\varphi(z) = \ln(1-z^2) = \ln\big((1-z)(1+z)\big)$

$\varphi(0) = -2\pi i$



1) $\varphi(-2)$:

$\Delta \arg(1-z) = 0$

$\Delta \arg(1+z) = -\pi$

$\Delta \arg(g) = -\pi$

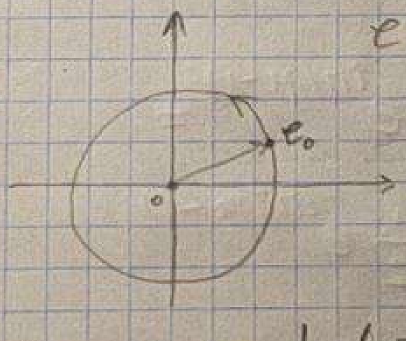
$$\varphi(-2) = \ln\left|\frac{1-4}{1}\right| + i(-\pi) + (-2\pi i) =$$
$$= \ln 3 - 3\pi i$$

2) $$\varphi(-i) = \ln\left|\frac{1-(-1)}{1}\right| + i\left(-\frac{\pi}{3}+\frac{\pi}{3}\right) - 2\pi i =$$
$$= \ln 2 - 2\pi i$$

3) $$\varphi\left(\frac{-1+\sqrt{3}i}{2}\right) = \ln\left|\frac{1+\frac{1}{2}+i\frac{\sqrt{3}}{2}}{1}\right| + i\left(\frac{\pi}{3} - \operatorname{arctg}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}+1}\right)\right) - 2\pi i =$$
$$= \ln\sqrt{3} + i\left(\frac{\pi}{3}-\frac{\pi}{6}\right) - 2\pi i =$$
$$= \frac{1}{2}\ln 3 - \frac{11}{6}\pi i$$

4.8

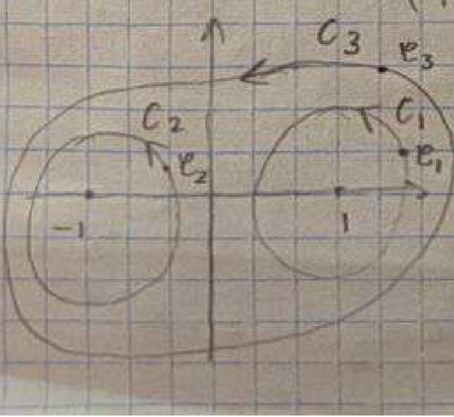
1) $$f(z) = \ln z = \left| z = \frac{1}{\varepsilon} \right| = -\ln\varepsilon = f(\varepsilon)$$



$\Delta \arg(g) = 2\pi$

$$\ln(\varepsilon_0) = \ln\left|\frac{g(\varepsilon_0)}{g(\varepsilon_0)}\right| + i(2\pi) + \ln(\varepsilon_0) =$$
$$= \ln(\varepsilon_0) + 2\pi i$$

2) $$f(z) = \ln\left(\frac{z-1}{z+1}\right) = \left| z = \frac{1}{\varepsilon} \right| = \ln\left(\frac{\frac{1}{\varepsilon}-1}{\frac{1}{\varepsilon}+1}\right) =$$
$$= \ln\left(\frac{1-\varepsilon}{1+\varepsilon}\right) = f(\varepsilon)$$



① $$f(\varepsilon_1) = \ln\left|\frac{g(\varepsilon_1)}{g(\varepsilon_1)}\right| + i(2\pi - 0) + \ln(g(\varepsilon_1))$$
$$= \ln\left(\frac{1-\varepsilon_1}{1+\varepsilon_1}\right) + 2\pi i$$

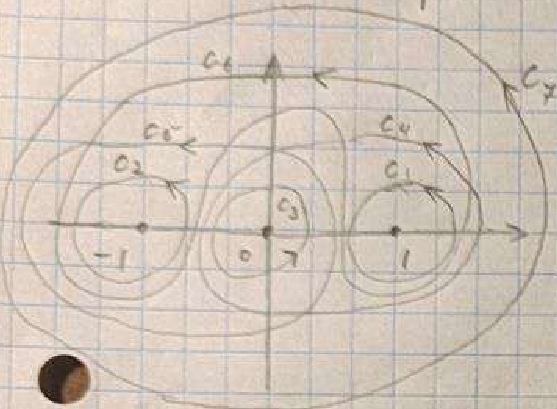
② $$f(\varepsilon_2) = \ln\left|\frac{g(\varepsilon_2)}{g(\varepsilon_2)}\right| + i(0-2\pi) + \ln(g(\varepsilon_2)) =$$
$$= \ln\left(\frac{1-\varepsilon_2}{1+\varepsilon_2}\right) - 2\pi i$$

③ $f(\tau_3) = \ln\left(\frac{1-\tau_3}{1+\tau_3}\right) + i(2\pi - 2\pi) = \ln\left(\frac{1-\tau_3}{1+\tau_3}\right)$

3) $f(z) = \ln(z^2-1) = \ln((z-1)(z+1)) = \left| z = \frac{1}{\tau} \right| =$

$= \ln\left(\frac{1}{\tau^2}(1-\tau)(1+\tau)\right) = \ln((1-\tau)(1+\tau)) - 2\ln\tau =$

$= f(\tau)$



① $f(\tau_1) = \ln\left(\frac{1-\tau_1^2}{\tau_1^2}\right) + 2\pi i$

② $f(\tau_2) = \ln\left(\frac{1-\tau_2^2}{\tau_2^2}\right) - 4\pi i$

③ $f(\tau_3) = \ln\left(\frac{1-\tau_3^2}{\tau_3^2}\right) + 2\pi i$

④ $f(\tau_4) = \ln\left(\frac{1-\tau_4^2}{\tau_4^2}\right) - 2\pi i$

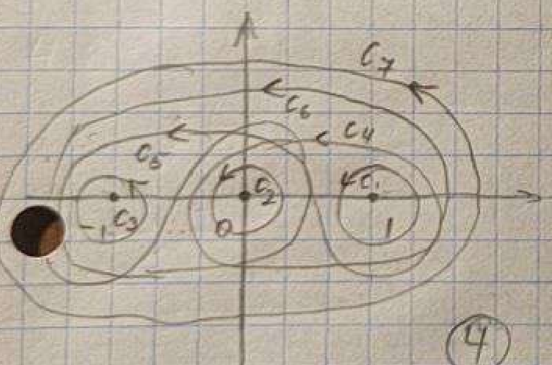
⑤ $f(\tau_5) = \ln\left(\frac{1-\tau_5^2}{\tau_5^2}\right) - 2\pi i$

⑥ $f(\tau_6) = \ln\left(\frac{1-\tau_6^2}{\tau_6^2}\right) + 4\pi i$

⑦ $f(\tau_7) = \ln\left(\frac{1-\tau_7^2}{\tau_7^2}\right)$

4) $f(z) = \sqrt{z^2-1} = \sqrt{(z-1)(z+1)} = \left| z = \frac{1}{\tau} \right| =$

$= i\sqrt{\frac{(\tau-1)(\tau+1)}{\tau^2}} = f(\tau)$



① $f(\tau_1) = -i\sqrt{\frac{\tau_1^2-1}{\tau_1^2}}$

② $f(\tau_2) = i\sqrt{\frac{\tau_2^2-1}{\tau_2^2}}$

③ $f(\tau_3) = -i\sqrt{\frac{\tau_3^2-1}{\tau_3^2}}$

④ $f(\tau_4) = -i\sqrt{\frac{\tau_4^2-1}{\tau_4^2}}$

⑤ $f(\tau_5) = -i\sqrt{\frac{\tau_5^2-1}{\tau_5^2}}$

⑥ $f(\tau_6) = i\sqrt{\frac{\tau_6^2-1}{\tau_6^2}}$

⑦ $f(\tau_7) = i\sqrt{\frac{\tau_7^2-1}{\tau_7^2}}$