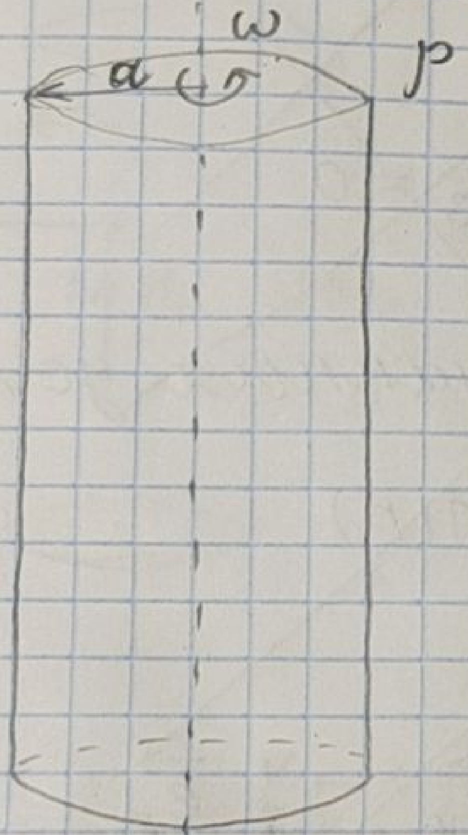
Задача №1



$$\vec{H} = \nabla \times \vec{A}$$

$$\nabla_{cy} = \frac{\partial}{H_r \partial r}\vec{e}_r + \frac{\partial}{H_\varphi \partial \varphi}\vec{e}_\varphi + \frac{\partial}{H_z \partial z}\vec{e}_z =$$

$$= \frac{\partial}{\partial r}\vec{r} + \frac{\partial}{r\partial \varphi}\vec{\varphi} + \frac{\partial}{\partial z}\vec{z}$$

$$\nabla_{cy}^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\Delta \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c^2}\vec{J}$$

in:
$$\Delta_{cy}\vec{A} = -\frac{4\pi}{c}\vec{J}_{cy}$$

$$\vec{J}_{cy} = \begin{pmatrix} 0 \\ \rho\omega r \\ 0 \end{pmatrix}$$

$$\Delta_{cy}\vec{A} = \left(\nabla_{cy}^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\varphi}{\partial \varphi}\right)\vec{r} + \left(\nabla_{cy}^2 A_\varphi - \frac{A_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \varphi}\right)\vec{\varphi} +$$
$$+\left(\nabla_{cy}^2 A_z\right)\vec{z}$$

$$\begin{cases} \nabla_{cy}^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\varphi}{\partial \varphi} = 0 \\ \nabla_{cy}^2 A_\varphi - \frac{A_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \varphi} = -\frac{4\pi}{c}\rho\omega \cdot r \\ \nabla_{cy}^2 A_z = 0 \end{cases}$$

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial \varphi^2} + \frac{\partial^2 A_r}{\partial z^2} - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\varphi}{\partial \varphi} = -\frac{4\pi}{c}\rho\omega \cdot r \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_\varphi}{\partial \varphi^2} + \frac{\partial^2 A_\varphi}{\partial z^2} - \frac{A_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \varphi} = 0 \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \varphi^2} + \frac{\partial^2 A_z}{\partial z^2} = 0 \end{cases}$$

в силу симметрии задачи:

$$\frac{\partial^2 A_i}{\partial x_j^2} = 0 \qquad \frac{\partial A_i}{\partial \varphi} = 0$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial A_\varphi}{\partial r}\right)-\frac{A_\varphi}{r}=-\frac{4\pi}{c}\rho\omega\cdot r^2$$

о.р. $$\frac{\partial}{\partial r}\left(r\frac{\partial A_\varphi}{\partial r}\right)-\frac{A_\varphi}{r}=0$$

$$\frac{\partial A_\varphi}{\partial r}+r\frac{\partial^2 A_\varphi}{\partial r^2}-\frac{A_\varphi}{r}=0$$

$$A_\varphi^0=C_1\cdot r+\frac{C_2}{r}$$

ч.р. $$A_\varphi=C_1(r)\cdot r+\frac{C_2(r)}{r}$$

$$A_\varphi \to 0 \quad r \to \infty \Rightarrow A_\varphi = \frac{C_2(r)}{r}$$

$$C_2''(r) - \frac{C_2'(r)}{r} + \frac{C_2(r)}{r^2} - \frac{C_2(r)}{r^2} = -\frac{4\pi}{c} \rho\omega \cdot r^2$$

$$C_2''(r) - \frac{C_2'(r)}{r} = -\frac{4\pi}{c} \rho\omega \cdot r^2$$

$$C_2(r) = -\frac{\pi\rho\omega}{2c} r^4 + C_{21} r^2 + C_{22}$$

$$A_\varphi = -\frac{\pi\rho\omega}{2c} r^3 + C_{21} r + \frac{C_{22}}{r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_r}{\partial r}\right)-\frac{A_r}{r^2}=0 \qquad r\cdot\frac{\partial^2 A_r}{\partial r^2}+\frac{\partial A_r}{\partial r}-\frac{A_r}{r}=0$$

$$A_r = C_3\cdot r+\frac{C_4}{r}$$

$$A_r\to 0 \quad r\to\infty \quad \Rightarrow \quad A_r=\frac{C_4}{r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial A_z}{\partial r}\right)=0$$

$$A_z = C_5\cdot \ln r + C_6$$

$$A_z\to 0 \quad r\to\infty \quad \Rightarrow \quad A_z = C_6$$

$$\begin{cases} A_r = \frac{C_4}{r} \\ A_\varphi = -\frac{\pi p w}{2c}r^3 + C_{21} r + \frac{C_{22}}{r} \\ A_z = C_6 \end{cases}$$

out: $\quad \Delta_{cy}\vec{A}=\vec{0}$

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_r}{\partial r}\right)-\frac{A_r}{r^2}=0 \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\varphi}{\partial r}\right)-\frac{A_\varphi}{r^2}=0 \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right)=0 \end{cases}$$

$$\begin{cases} A_r = C_1 r + \frac{C_2}{r} \\ A_\varphi = C_3\cdot r+\frac{C_4}{r} \\ A_z = C_5\cdot\ln r + C_6 \end{cases} \quad \begin{matrix} A_i\to 0 \\ r\to\infty \end{matrix} \quad \Rightarrow \begin{cases} A_r = \frac{C_2}{r} \\ A_\varphi = \frac{C_4}{r} \\ A_z = C_6 \end{cases}$$

$$\vec{A}_{in} = \vec{A}_{out} \qquad r=a$$

$$\begin{cases} \frac{C_{4in}}{a} = \frac{C_{2out}}{a} \\ -\frac{\pi p w}{2c}a^3 + C_{21in}a + \frac{C_{22in}}{a} = \frac{C_{4out}}{a} \\ C_{6in} = C_{6out} \end{cases} \Rightarrow \begin{cases} C_{4in} = C_{2out} = C_2 \\ C_{22in} = C_{4out} = C_4 \\ C_{6in} = C_{6out} = C_6 \\ C_{21in} = \frac{\pi p w}{2c}a^2 \end{cases}$$

$$\vec{H}_{in} = \nabla_{cy} \times \vec{A}_{in}$$

$$H_{in\,r} = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} = 0$$

$$H_{in\,\varphi} = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = 0$$

$$H_{in\,z} = \frac{1}{r}\frac{\partial (r \cdot A_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \varphi} = \frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{\pi p \omega}{2c} r^4 + \right.$$
$$\left. + \frac{\pi p \omega}{2c} a^2 r^2 + C_4\right) =$$
$$= -\frac{2\pi p \omega}{c} r^2 + \frac{\pi p \omega}{c} a^2$$

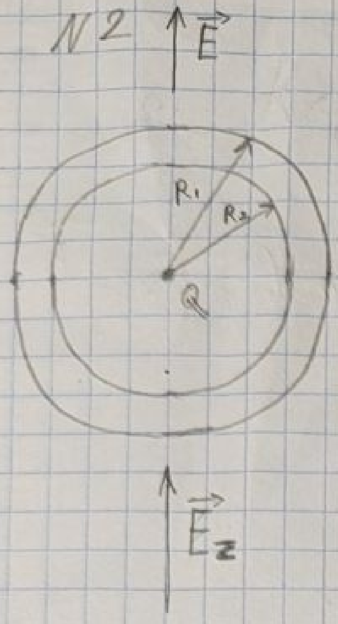
$$\vec{H}_{out} = \nabla_{cy} \times \vec{A}_{out}$$

$$H_{out\,r} = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} = 0$$

$$H_{out\,\varphi} = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = 0$$

$$H_{out\,z} = \frac{1}{r}\frac{\partial (r \cdot A_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \varphi} = 0$$

N2



$$\vec{E}_{sp} = -\nabla_{sp}\varphi$$

$$\Delta_{sp}\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho = -4\pi Q\cdot\delta(\vec{r})$$

в связи с симметрией задачи

$$\varphi = \varphi(r, \theta)$$

$$-\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)$$

$$\varphi: \quad \varphi(r,\theta) = R(r)\cdot\Phi(\theta)$$

$$-\Phi(\theta)\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) = \frac{1}{\Phi(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi(\theta)}{\partial\theta}\right)$$

$$-\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) = \frac{\Phi(\theta)}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi(\theta)}{\partial\theta}\right) = \lambda$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) = -\lambda\cdot R(r)$$

$$2rR' + r^2R'' + \lambda R = 0 \qquad R = r^\mu$$

$$r^2\mu(\mu-1)r^{\mu-2} + 2r\mu r^{\mu-1} + \lambda r^\mu = 0$$

$$\mu(\mu-1) + 2\mu + \lambda = 0$$

$$\mu^2 + \mu = -\lambda \qquad \lambda = -\mu(\mu+1)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi(\theta)}{\partial\theta}\right) = \lambda\cdot\Phi(\theta)$$

$$\frac{\partial}{\partial\cos\theta}\left(\sin^2\theta\frac{\partial\Phi(\theta)}{\partial\cos\theta}\right) = \lambda\cdot\Phi(\theta) \qquad \cos\theta = x$$

$$\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial\Phi}{\partial x}\right) = \lambda\Phi$$

$$(1-x^2)\Phi'' - 2x\Phi' - \lambda\Phi = 0$$

$$\Phi = P_n(x) = P_n(\cos\theta) \qquad \lambda = -n(n+1) \qquad n\in\mathbb{Z}^+$$

$$-n(n+1) = -\mu(\mu+1) \qquad \mu = n$$

$$\mu = -(n+1)$$

$$\varphi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta) \sim$$

$$\sim \left(A_0 + \frac{B_0}{r}\right) + \left(A_1 r + \frac{B_1}{r^2}\right)\cdot\cos\theta$$

$$\vec{E} = -\nabla_{sp}\varphi \qquad \nabla_{sp} = \frac{\partial}{\partial r}\vec{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{\varphi} + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{\theta}$$

$$\nabla_{sp}\varphi = \left[-\frac{B_0}{r^2} + \left(A_1 - 2\frac{B_1}{r^3}\right)\cdot\cos\theta\right]\vec{r} +$$

$$+\left[-\left(A_1 + \frac{B_1}{r^3}\right)\cdot\sin\theta\right]\vec{\theta}$$

$r < R_2$: (в силу симметрии)

$$\vec{E} = \frac{B_0}{r^2}\vec{r} \qquad B_0 = Q$$

$$\vec{E} = \frac{Q}{r^2}\vec{r}$$

$R_2 \leq r < R_1$:

$$\vec{E} = \vec{0}$$

$r \geq R_1$:

$$\begin{cases} \vec{E} = \left[\frac{B_0}{r^2} + \left(2\frac{B_1}{r^3} - A_1\right)\cos\theta\right]\vec{r} + \left[\left(A_1 + \frac{B_1}{r^3}\right)\cdot\sin\theta\right]\vec{\theta} \\ \lim\limits_{r\to\infty}\vec{E} = E_z\cos\theta\cdot\vec{r} - E_z\sin\theta\cdot\vec{\theta} \\ \vec{E}\Big|_{r=R_1} = \vec{0} \\ B_0 = Q \end{cases}$$

$$\vec{E} = \left[\frac{Q}{r^2} + E_z\left(1 + \frac{2R_2^3}{r^3}\right)\cdot\cos\theta\right]\vec{r} + \left[E_z\left(\frac{R_2^3}{r^3} - 1\right)\cdot\sin\theta\right]\vec{\theta}$$

N 3 $$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell = \frac{1}{2^\ell}\oint_G \frac{(t^2-1)^\ell}{(t-x)^{\ell+1}}\,dt$$

$$(1-x^2)P_\ell''(x) - 2x P_\ell'(x) + \ell(\ell+1)P_\ell(x) = 0$$

$$\frac{d}{dx}P_\ell(x) = \frac{d}{dx}\frac{1}{2^\ell}\oint_G \frac{(t^2-1)^\ell}{(t-x)^{\ell+1}}\,dt = \frac{1}{2^\ell}\oint_G \frac{d}{dx}\left(\frac{(t^2-1)^\ell}{(t-x)^{\ell+1}}\right)dt =$$

$$= \frac{1}{2^\ell}\oint_G (\ell+1)\frac{(t^2-1)^\ell}{(t-x)^{\ell+2}}\,dt$$ (формула Лейбница)

$$\frac{d^2}{dx^2}P_\ell(x) = \frac{1}{2^\ell}\oint_G (\ell+1)(\ell+2)\frac{(t^2-1)^\ell}{(t-x)^{\ell+3}}\,dt$$

$$\frac{1}{2^\ell}\oint_G \frac{(t^2-1)^\ell}{(t-x)^{\ell+1}}\cdot\left[\frac{(1-x^2)(\ell^2+3\ell+2)}{(t-x)^2} - \frac{2x(\ell+1)}{t-x} + \ell(\ell+1)\right] =$$

$$= \frac{1}{2^\ell}\oint_G \frac{(t^2-1)^\ell}{(t-x)^{\ell+3}}(\ell+1)\left[(1-x^2)(\ell+2) - 2x(t-x) + \ell(t-x)^2\right]dt =$$

$$= \frac{1}{2^\ell}\oint_G (\ell+1)\frac{(t^2-1)^\ell}{(t-x)^{\ell+3}}\left[\ell t^2 - 2(\ell+1)xt + \ell+2\right]dt = 0$$

$$F(t,x) = \ell t^2 - 2(\ell+1)xt + \ell + 2$$

$$\alpha = \ell + 3$$

$$\frac{(t^2-1)^\ell(\ell t^2 - 2(\ell+1)xt + \ell+2)}{(t-x)^{\ell+3}} = \frac{d}{dt}\frac{(t^2-1)^{\ell+1}}{(t-x)^{\ell+2}}$$