20.21 (1, 2, 3, 4, 5, 6, 7, 8, 9)

1) $\frac{z}{(z+2)^2}$ $\quad z_0=-2$

$t=z+2:\quad \frac{t-2}{t^2}=\frac{1}{t}-\frac{2}{t^2}=\frac{-2}{(z+2)^2}+\frac{1}{z+2}$

2) $\frac{e^z+1}{e^z-1}$ $\quad z_0=2\pi i k \quad k\in\mathbb{Z}$

$z'=z+2\pi i k$

$f(z')=\frac{g(z')}{h(z')} \qquad g(z')=f(z')\cdot h(z')$

$g(z')=2+z'+\frac{z'^2}{2}+\frac{z'^3}{6}+\dots$

$h(z')=z'+\frac{z'^2}{2}+\frac{z'^3}{6}+\dots$

$f(z')=\frac{2}{z'}+\dots$

$\frac{2}{z'}=\frac{2}{z+2\pi i k} \quad k\in\mathbb{Z}$

3) $\frac{z-1}{\sin^2 z}=(z-1)\cdot \operatorname{cosec}^2 z \qquad z=0$

$(z-1)\cdot\operatorname{cosec}^2 z=(z-1)\left(\frac{1}{z}+\frac{1}{6}z+\frac{7}{360}z^3+\dots\right)^2$

Главная часть: $\frac{-1}{z^2}+\frac{1}{z}$

4) $\frac{e^{iz}}{(z^2+b^2)^2}$ $\qquad z=ib \quad b>0$

$\frac{e^{i(z-ib)}\cdot e^{-b}}{(z-ib)^2(z+ib)^2}$

$t=z-ib:\quad \frac{e^{it}\cdot e^{-b}}{t^2(t+2bi)^2}$

$e^{it}=\sum_{n=0}^{\infty}\frac{i^n t^n}{n!}$

$$\frac{e^{-b}}{t^2(t+2bi)^2} = \frac{e^{-b}}{-4b^2 t^2 \cdot \left(1-\frac{it}{2b}\right)^2} = \frac{-e^{-b}}{4b^2} \frac{1}{t^2} \sum_{n=0}^{\infty} \frac{i^n t^n}{(2b)^n}(n+1) =$$

$$= -e^{-b} \sum_{n=0}^{\infty} i^n \frac{t^{n-2}}{(2b)^{n+2}} (n+1)$$

$$-e^{-b}\left(\frac{1}{(2b)^2}\frac{1}{t^2} + \frac{i}{4b^3}\frac{1}{t} + \dots\right)\left(1 + it + \dots\right)$$

Главная часть: $-\frac{e^{-b}}{4b^2(z-ib)^2} - \frac{e^{-b}(i+bi)}{4b^2}\frac{1}{z-ib}$

5) $\frac{(z^2+1)^2}{z^2+b^2}$ $\qquad z=\infty$

$t=\frac{1}{z}$: $\quad \frac{\left(\frac{1}{t^2}+1\right)^2}{\frac{1}{t^2}+b^2} = \frac{(1+t^2)^2}{t^2(1+b^2t^2)} =$

$$=\left(\frac{1}{t^2}+2+t^2\right)\frac{1}{1-(-b^2t^2)} = \left(\frac{1}{t^2}+2+t^2\right)\cdot\sum_{n=0}^{\infty}(-1)^n b^{2n} t^{2n}$$

$$\left(\frac{1}{t^2}+2+t^2\right)\left(1-b^2t^2+\dots\right)$$

Главная часть: $\quad z^2$

6) $\frac{z \cdot e^{iz}}{(z^2+b^2)^2}$ $\qquad z=ib \qquad b>0$

$t=z-ib$: $\quad \frac{(t+ib)\cdot e^{it}\cdot e^{-b}}{t^2(t+2bi)^2}$

$$e^{it} = \sum i^n \frac{t^n}{n!}$$

$$\frac{1}{t^2(t+2bi)^2} = \frac{-1}{4b^2t^2}\cdot\frac{1}{\left(1-\frac{it}{2b}\right)^2} = \frac{-1}{4b^2t^2}\sum_{n=0}^{\infty}(n+1)\frac{i^n}{2^n b^n}t^n =$$

$$= -\sum_{n=0}^{\infty}(n+1)\frac{i^n}{(2b)^{n+2}}t^{n-2}$$

7) $\dfrac{z}{(z^2+b^2)^2}$ $\qquad z = ib$

$$t = z - ib: \quad \frac{t+ib}{t^2(t+2ib)^2} = \frac{t+ib}{-4b^2t^2\left(1-\frac{it}{2b}\right)^2}$$

$$\frac{1}{4b^2t^2}\frac{1}{\left(1-\frac{it}{2b}\right)^2} = -\sum_{n=0}^{\infty}(n+1)\frac{i^n}{(2b)^{n+2}}t^{n-2}$$

$$-(t+ib)\sum_{n=0}^{\infty}(n+1)\frac{i^n}{(2b)^{n+2}}t^{n-2} =$$

$$= -(t+ib)\left(\frac{1}{4b^2}\frac{1}{t^2} + \frac{i}{4b^3}\frac{1}{t} + \ldots\right)$$

Главная часть: $-\dfrac{i}{4b}\dfrac{1}{(z-ib)^2}$

$$\frac{i}{4b}\frac{1}{t^2} - \frac{1}{4b^2}\frac{1}{t} + \frac{1}{4b^2}\frac{1}{t}$$

$$-e^{-b}(t+ib)\left(\sum_{n=0}^{\infty} i^n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(n+1)\frac{i^n}{(2b)^{n+2}}t^{n-2}\right)=$$

$$=-e^{-b}(t+ib)(1+it+\ldots)\left(\frac{1}{4b^2}\frac{1}{t^2}+\frac{i}{4b^3}\frac{1}{t}+\ldots\right)$$

Главная часть: $-\frac{ie^{-b}}{4b}\frac{1}{(z-ib)^2}+\frac{e^{-b}}{4b}\frac{1}{z-ib}$

8) $\operatorname{ctg} \pi z \qquad z \in \mathbb{Z}$

Главная часть: $\dfrac{1}{\pi(z-k)} \qquad k \in \mathbb{Z}$

9) $\dfrac{1}{\sin \pi z} = \operatorname{cosec} \pi z \qquad z \in \mathbb{Z}$

Главная часть: $\dfrac{1}{\pi(z-k)} \qquad k \in \mathbb{Z}$

21.02

1) $\frac{1}{z+z^3} = \frac{1}{z}\frac{1}{1+z^2} = \frac{1}{z}\frac{1}{(z-i)(z+i)}$

$\underset{z=0}{res} = 1$

$\underset{z=i}{res} = -\frac{1}{2}$

$\underset{z=-i}{res} = -\frac{1}{2}$

2) $\frac{z^2}{1+z^4}$ $\qquad z^4 = -1$

$z = e^{i(\frac{\pi}{4}+\frac{\pi}{2}k)} \quad k = \mathbb{Z}$

$\underset{z=e^{i\frac{\pi}{4}}}{res} = \frac{i}{(e^{i\frac{\pi}{4}}-e^{i\frac{3\pi}{4}})(e^{i\frac{\pi}{4}}-e^{i\frac{5\pi}{4}})(e^{i\frac{\pi}{4}}-e^{i\frac{7\pi}{4}})}$ $\quad z = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$

$\underset{z=e^{i\frac{3\pi}{4}}}{res} = \frac{-i}{(e^{i\frac{3\pi}{4}}-e^{i\frac{\pi}{4}})(e^{i\frac{3\pi}{4}}-e^{i\frac{5\pi}{4}})(e^{i\frac{3\pi}{4}}-e^{i\frac{7\pi}{4}})}$

$\underset{z=e^{i\frac{5\pi}{4}}}{res} = \frac{i}{(e^{i\frac{5\pi}{4}}-e^{i\frac{\pi}{4}})(e^{i\frac{5\pi}{4}}-e^{i\frac{3\pi}{4}})(e^{i\frac{5\pi}{4}}-e^{i\frac{7\pi}{4}})}$

$\underset{z=e^{i\frac{7\pi}{4}}}{res} = \frac{-i}{(e^{i\frac{7\pi}{4}}-e^{i\frac{\pi}{4}})(e^{i\frac{7\pi}{4}}-e^{i\frac{3\pi}{4}})(e^{i\frac{7\pi}{4}}-e^{i\frac{5\pi}{4}})}$

3) $\frac{z^2}{(1+z)^3}$ $\qquad \underset{z=-1}{res} = c_{-1}$

$t = z+1: \quad \frac{(t-1)^2}{t^3} = \frac{t^2-2t+1}{t^3} = \frac{1}{t} - \frac{2}{t^2} + \frac{1}{t^3} =$

$= \frac{1}{z+1} - \frac{2}{(z+1)^2} + \frac{1}{(z+1)^3}$

$\underset{z=-1}{res} = 1$

4) $\frac{1}{(z^2+1)^3} = \frac{1}{(z-i)^3(z+i)^3}$

$t_1 = z - i: \quad \frac{1}{t_1^3(t_1+2i)^3} = \frac{1}{t_1^3}\frac{1}{(2i)^3}\frac{1}{(1-\frac{it_1}{2})^3} = \sum_{n=2}^{\infty} \frac{(n-1)n}{2} \frac{i^{n-5}}{2^{n-1}} t_1^{n-5}$

$\underset{z=i}{res} = -12i$

$t_2 = z + i$: $\dfrac{1}{t^3 (t-2i)^3} = \ldots$

$\operatorname*{res}_{z=-i} = 12i$

5) $\dfrac{1}{(z^2+1)(z-1)^2} = \dfrac{1}{(z-i)(z+i)(z-1)^2}$

$\operatorname*{res}_{z=i} = \dfrac{1}{2i(i-1)^2} = \dfrac{1}{4}$

$\operatorname*{res}_{z=-i} = \dfrac{1}{-2i(-i-1)^2} = \dfrac{1}{4}$

$\operatorname*{res}_{z=1} = 0 - \operatorname*{res}_{z=i} - \operatorname*{res}_{z=-i} = -\dfrac{1}{2}$