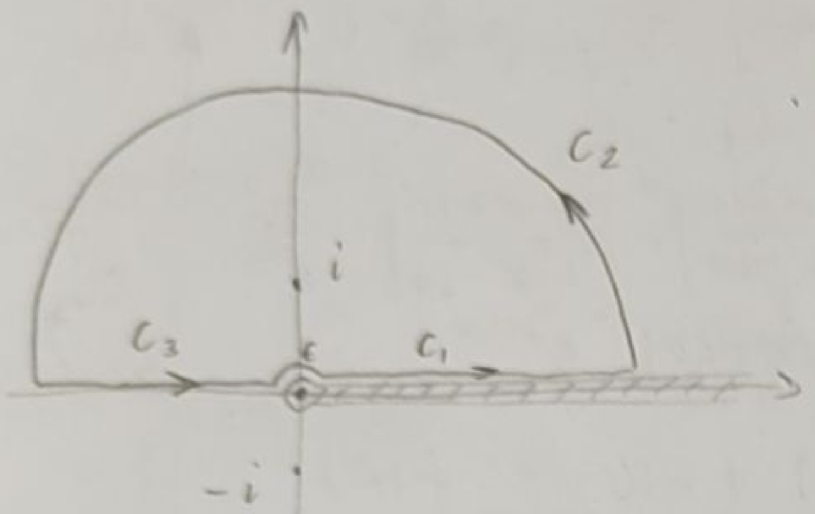
6.1 $I = \int_0^{\infty} \frac{\ln x}{x^2+1} dx$

$I_{1+2+3} = \oint \frac{\ln z}{z^2+1} dz = 2\pi i \cdot \operatorname{res}_{z=i} f(z)$

$\operatorname{res}_{z=i} f(z) = \left.\frac{\ln z}{z+i}\right|_{z=i} = \frac{\ln(1) + i\frac{\pi}{2}}{2i} = \frac{\pi}{4}$

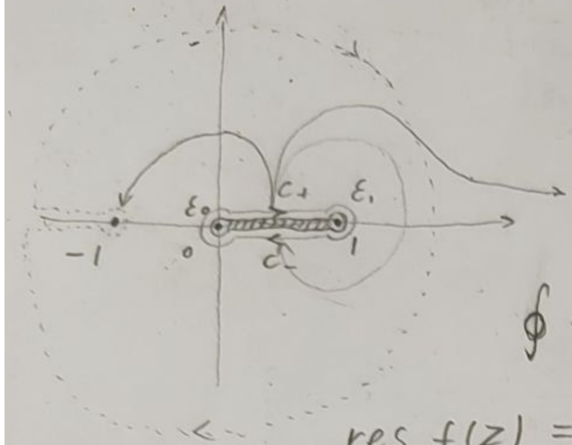
$I_{1+2+3} = i\frac{\pi^2}{2}$



$I_3 = \int_{-\infty}^{0} \frac{\ln z + i\pi}{z^2+1} dz = I_1 + i\pi \int_0^{\infty} \frac{dx}{x^2+1}$

$I_{1+2+3} = 2 I_1 + i\pi \int_0 \frac{dx}{x^2+1}$ $\Rightarrow I_1 = 0$ $\quad i\frac{\pi^2}{2} = i\pi \int_0^{\infty} \frac{dx}{x^2+1}$

6.2 $I(\alpha) = \int_0^1 \frac{x^\alpha (1-x)^{2-\alpha}}{x+1}\, dx \qquad \mathrm{Re}\,\alpha \in (-1;3)$



$$\oint = \underset{\varepsilon_0}{\cancel{\int}}^{0} + \underset{\varepsilon_1}{\cancel{\int}}^{0} + \underset{C_+}{\cancel{\int}}^{I} + \int_{C_-} = (1-e^{2\pi\alpha i})\cdot I \qquad -e^{i\pi\alpha}\cdot 2i\cdot\sin(\pi\alpha)$$

$$f(x-i0) = \left|\frac{x-i0}{x+i0}\right|^\alpha \cdot \left|\frac{1-x+i0}{1-x-i0}\right|^{2-\alpha} \cdot e^{2\pi(\alpha-2)i} \cdot f(x+i0)$$

$$= f(x+i0)\cdot e^{2\pi\alpha i}$$

$$\oint = 2\pi i \cdot \operatorname*{res}_{z=-1} f(z) + 2\pi i \cdot \operatorname*{res}_{z=\infty} f(z)$$

$$\operatorname*{res}_{z=-1} f(z) = \left(z^\alpha (1-z)^{2-\alpha}\right)\Big|_{z=-1} = e^{\alpha\pi i}\cdot(2)^{2-\alpha} = e^{\alpha\pi i}\cdot 2^{2-\alpha}$$

$$f(x) = \frac{x^\alpha(1-x)^{2-\alpha}}{x+1} = x\cdot\left(1-\frac{1}{x}\right)^{2-\alpha}\cdot\frac{1}{1-\left(-\frac{1}{x}\right)}\cdot e^{i\pi\alpha} =$$

$$= e^{i\pi\alpha}\cdot x\cdot\left(1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}+\dots\right)\cdot\left(1-\frac{2-a}{x}+\frac{(2-a)(1-a)}{x^2}+\dots\right) \doteq$$

$$= e^{i\pi\alpha}\cdot\left(x-1+\frac{1}{x}-\frac{1}{x^2}+\dots\right)\cdot\left(1-\frac{2-a}{x}+\frac{(2-a)(1-a)}{2\cdot x^2}+\dots\right) =$$

$$= e^{i\pi\alpha}\cdot\left(\frac{(2-a)(1-a)}{2\cdot x}+\frac{2-a}{x}+\frac{1}{x}+\dots\right)$$

a^2-3a+2

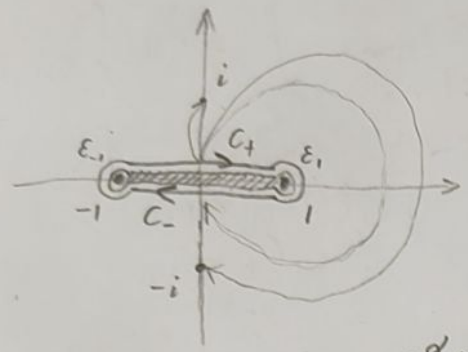
$$\operatorname*{res}_{z=\infty} f(z) = -e^{i\pi\alpha}\cdot\left[\tfrac{1}{2}(2-a)(1-a)-a+3\right] = -e^{i\pi\alpha}\left[\tfrac{1}{2}a^2-\tfrac{5}{2}a+4\right]$$

$$-e^{i\pi\alpha}\cdot 2i\cdot\sin(\pi\alpha)\cdot I = 2\pi i\cdot e^{\alpha\pi i}\cdot\left(-\tfrac{1}{2}a^2+\tfrac{5}{2}a-4+2^{2-\alpha}\right)$$

$$I = \frac{\pi}{\sin(\pi\alpha)}\left(\tfrac{1}{2}a^2-\tfrac{5}{2}a+4-2^{2-\alpha}\right)$$

6.3 $\operatorname{Re}\alpha \in (-1;\,2)$

$$I(\alpha) = \int_{-1}^{1} \frac{(1-x)^{\alpha}(1+x)^{1-\alpha}}{(x^2+1)}\,dx$$



$$f_-(z) = f_+(z)\cdot e^{-2\pi\alpha i}$$

$$\oint = \int_{\varepsilon_{-1}}^{\to 0} + \int_{\varepsilon_1}^{\to 0} + \int_{C_+}^{\to I} + \int_{C_-} = \left(1 - e^{-2\pi\alpha i}\right) I$$

$$\oint = 2\pi i \sum_j \operatorname*{res}_j f(z) = -2\pi i \sum_k \operatorname*{res}_k f(z)$$

$$\operatorname*{res}_{z=i} f(z) = \left.\frac{(1-z)^{\alpha}(1+z)^{1-\alpha}}{z+i}\right|_{z=i} = \frac{1}{2i}\cdot\left|\frac{1-i}{1-0-i0}\right|^{\alpha}\cdot\left|\frac{1+i}{1+0+i0}\right|^{1-\alpha}\cdot e^{\frac{\pi}{4}\alpha i}\cdot e^{-\frac{\pi}{4}(1-\alpha)i} =$$

$$= \frac{1}{2i}\sqrt{2}^{\alpha}\cdot\sqrt{2}^{1-\alpha}\cdot e^{\frac{\pi}{2}\alpha i}\cdot e^{-\frac{\pi}{4}i} = \frac{e^{\frac{\pi}{2}\alpha i}\cdot e^{-\frac{\pi}{4}i}}{\sqrt{2}\cdot i}$$

$$\operatorname*{res}_{z=-i} f(z) = \frac{-1}{2i}\sqrt{2}^{\alpha}\cdot\sqrt{2}^{1-\alpha}\cdot e^{-\frac{\pi}{4}\alpha i}\cdot e^{-\frac{7\pi}{4}(1-\alpha)i} = \frac{-1}{\sqrt{2}\cdot i}\, e^{i\frac{3}{2}\pi\alpha}\cdot e^{-\frac{7\pi}{4}i}$$

$$\operatorname*{res}_{z=\infty} f(z) = -e^{i\pi\alpha}$$

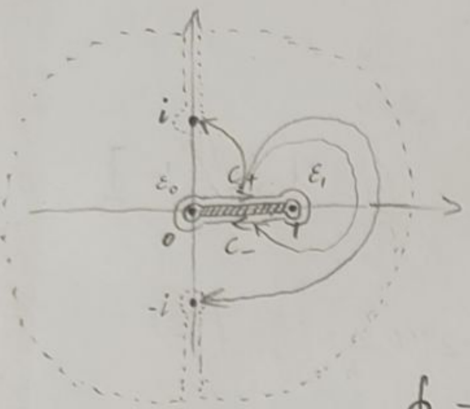
$$\oint = \frac{2\pi i}{\sqrt{2}\cdot i}\left(e^{\frac{\pi}{2}\alpha i}\cdot e^{-\frac{\pi}{4}i} - e^{\frac{3\pi}{2}\alpha i}\cdot e^{-\frac{7\pi}{4}i} - \sqrt{2}\cdot i\cdot e^{i\pi\alpha}\right)$$

$$= 2\pi i\cdot e^{\pi\alpha i}\cdot\left(-1 + \cos\frac{\pi\alpha}{2} + \sin\frac{\pi\alpha}{2}\right)$$

$$e^{-\pi\alpha i}\cdot 2i\cdot\sin\pi\alpha\cdot I = 2\pi i\cdot e^{\pi\alpha i}\cdot\left(-1 + \cos\frac{\pi\alpha}{2} + \sin\frac{\pi\alpha}{2}\right)$$

$$I = \frac{\pi}{\sin\pi\alpha}\left(-1 + \cos\frac{\pi\alpha}{2} + \sin\frac{\pi\alpha}{2}\right)$$

6.5 $I = \int_0^1 \ln\frac{1-x}{x}\,\frac{dx}{x^2+1}$ $\qquad I^* = \int_0^1 \ln^2\left(\frac{1-x}{x}\right)\frac{dx}{x^2+1}$



$$\oint = \overset{\to 0}{\int_{\varepsilon_0}} + \overset{\to 0}{\int_{\varepsilon_1}} + \int_{C_+} + \int_{C_-}$$

$$\int_{C_-}\left(\ln\frac{1-x}{x} - i\pi\right)^2\frac{dx}{x^2+1} = -\int_{C_+} + 2\pi i\overbrace{\int_{C_+}\ln\frac{1-x}{x}\frac{dx}{x^2+1}}^{I} + \pi^2\int_{C_+}\frac{dx}{x^2+1}$$

$$\oint = 2\pi i\cdot I + \pi^2\int_{C_+}\frac{dx}{x^2+1}$$

$$\oint = 2\pi i\sum_k \operatorname*{res}_k f(z) \qquad g(z) := \ln\frac{1-z}{z} \qquad g_0\left(\tfrac{1}{2}+i0\right) = 0$$

$$\operatorname*{res}_{z=i} f(z) = \ln^2\frac{1-z}{z}\cdot\frac{1}{z+i}\bigg|_{z=i} = \frac{1}{2i}\left(\frac{1}{2}\ln 2 - i\frac{3\pi}{4}\right)^2$$

$$g(i) = \ln\left|\frac{1-i}{\frac{1}{2}}\right| - \ln\left|\frac{i}{\frac{1}{2}}\right| - i\frac{\pi}{4} - i\frac{\pi}{2} = \frac{1}{2}\ln 2 - i\frac{3\pi}{4}$$

$$\operatorname*{res}_{z=-i} f(z) = \ln^2\frac{1-z}{z}\cdot\frac{1}{z-i}\bigg|_{z=-i} = \frac{1}{-2i}\left(\frac{1}{2}\ln 2 - i\frac{5\pi}{4}\right)^2$$

$$g(-i) = \ln\left|\frac{1+i}{\frac{1}{2}}\right| - \ln\left|\frac{-i}{\frac{1}{2}}\right| - i\frac{7\pi}{4} + i\frac{\pi}{2} = \frac{1}{2}\ln 2 - i\frac{5\pi}{4}$$

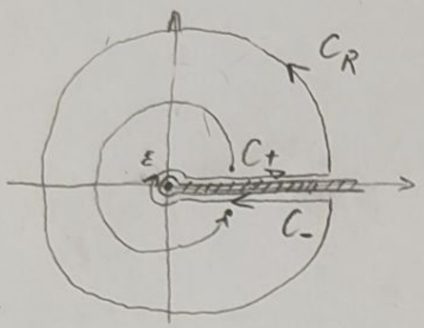
$$\oint = 2\pi i\cdot\frac{1}{2i}\left[\frac{1}{4}\ln^2 2 - i\frac{3\pi}{4}\ln 2 - \frac{9\pi^2}{16} - \frac{1}{4}\ln^2 2 + i\frac{5\pi}{4}\ln 2 + \frac{25\pi^2}{16}\right] =$$

$$= \pi\cdot\left[i\frac{\pi}{4}\ln 2 - \pi^2\right]$$

$$2\pi i\cdot I = i\frac{\pi^2}{4}\cdot\ln 2 \qquad I = \frac{\pi}{8}\ln 2$$

6.4 $I = \int\limits_0^{+\infty} \frac{\ln x \, dx}{\sqrt[3]{x}\,(x+1)^2}$ $\qquad I^* = \int\limits_0^{+\infty} \frac{x^\alpha \, dx}{\sqrt[3]{x}\,(x+1)^2} = \int\limits_0^{+\infty} \frac{x^{\alpha-\frac{1}{3}}}{(x+1)^2}\,dx$

$$I = \left(\frac{d}{d\alpha} I^*\right)\Bigg|_{\alpha=0} \qquad \frac{d}{d\alpha}(x^\alpha) = x^\alpha \cdot \ln x$$



$$\oint = \overset{\to 0}{\int_\varepsilon} + \overset{\to 0}{\int_{C_R}} + \overset{\to I^*}{\int_{C_+}} + \int_{C_-}$$

$$g(x-i0) = \left|\frac{x-i0}{x+i0}\right|^{\alpha-\frac{1}{3}} \cdot e^{2\pi(\alpha-\frac{1}{3})i} \cdot (x+i0)^{\alpha-\frac{1}{3}} =$$

$$= x^{\alpha-\frac{1}{3}} \cdot e^{2\pi(\alpha-\frac{1}{3})i}$$

$$\oint = \left(1 - e^{2\pi(\alpha-\frac{1}{3})i}\right) I^*$$

$$\oint = 2\pi i \cdot \operatorname*{res}_{z=-1} f(z) = 2\pi i\left(\alpha-\frac{1}{3}\right)(-1)^{\alpha-\frac{4}{3}}$$

$$\operatorname*{res}_{z=-1} f(z) = \lim_{z\to-1} \frac{d}{dz}\left(z^{\alpha-\frac{1}{3}}\right) = \lim_{z\to-1}\left(\left(\alpha-\frac{1}{3}\right)\cdot z^{\alpha-\frac{4}{3}}\right) = \left(\alpha-\frac{1}{3}\right)(-1)^{\alpha-\frac{4}{3}}$$

$$I^* = \frac{2\pi i\left(\alpha-\frac{1}{3}\right)(-1)^{\alpha-\frac{4}{3}}}{e^{\pi(\alpha-\frac{1}{3})i}\cdot 2i\cdot\sin\left(\pi\left(\frac{1}{3}-\alpha\right)\right)} = \frac{\pi\left(\alpha-\frac{1}{3}\right)\cdot e^{\pi(\alpha-\frac{4}{3})i}}{e^{\pi(\alpha-\frac{1}{3})i}\cdot\sin\left(\pi\left(\frac{1}{3}-\alpha\right)\right)} =$$

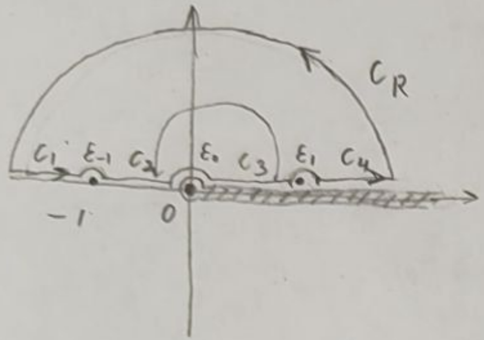
$$= \frac{\pi\left(\frac{1}{3}-\alpha\right)}{\sin\left(\pi\left(\frac{1}{3}-\alpha\right)\right)} \qquad \frac{d}{d\alpha} I^* = -\frac{\pi}{\sin\left(\pi\left(\frac{1}{3}-\alpha\right)\right)} + \frac{\cos\left(\pi\left(\frac{1}{3}-\alpha\right)\right)\cdot\pi^2\left(\frac{1}{3}-\alpha\right)}{\sin^2\left(\pi\left(\frac{1}{3}-\alpha\right)\right)}$$

$$I = \left(\frac{d}{d\alpha} I^*\right)\Bigg|_{\alpha=0} = -\frac{\pi}{\sin\frac{\pi}{3}} + \frac{\pi^2\cdot\cos\frac{\pi}{3}}{3\cdot\sin^2\frac{\pi}{3}} = -\frac{2\pi}{\sqrt{3}} + \frac{2\pi}{9} = \frac{2\pi}{9} - \frac{2\pi}{\sqrt{3}}$$

6.8

$$I = pv\int_0^\infty \frac{\sqrt{x}\,dx}{x^2-1}$$

$$\oint = \int_{\varepsilon_{-1}} + \int_{\varepsilon_1} + \cancel{\int_{\varepsilon_0}}^{\;0} + \cancel{\int_{C_R}}^{\;0} + \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{4}$$



$$\int_{C_r}\left(\ldots+\frac{C_{-1}}{z}+c_0+\ldots\right)dz = \left|\begin{matrix} z=p\cdot e^{i\varphi} \\ dz = ipe^{i\varphi}d\varphi\end{matrix}\right| =$$

$$=-i\int_0^\pi\left(\ldots+\frac{C_{-1}}{p\cdot e^{i\varphi}}+c_0+\ldots\right)p\cdot e^{i\varphi}d\varphi =$$

$$=-i\int_0^\pi\left(\ldots+C_{-1}+c_0\cdot r\cdot e^{i\varphi}+\ldots\right)d\varphi =$$

$$=-i\cdot\pi\cdot C_{-1}+\ldots \qquad \lim_{r\to 0}\left(-i\pi C_{-1}+\ldots\right) = -i\pi C_{-1}$$

$$\Rightarrow \frac{1}{2}\operatorname*{res}_{z=z_0} f(z)$$

$$\int_{\varepsilon_{-1}} = -\pi i\cdot\operatorname*{res}_{z=-1} f(z) = -\pi i\cdot\left.\frac{\sqrt{z}}{z-1}\right|_{z=-1} = -\pi i\,\frac{e^{\frac{\pi}{2}i}}{-2} = -\frac{\pi}{2}$$

$$\int_{\varepsilon_1} = -\pi i\cdot\operatorname*{res}_{z=1} f(z) = -\pi i\cdot\left.\frac{\sqrt{z}}{z+1}\right|_{z=1} = -\frac{\pi i}{2}$$

$$\int_{C_3}+\int_{C_4} = I \qquad \int_{C_1}+\int_{C_2} = iI \qquad \oint = 2\pi i\sum_k \operatorname*{res}_k f(z) = 0$$

$$I + iI = \frac{\pi}{2}+\frac{\pi}{2}i \qquad \Rightarrow \qquad I = \frac{\pi}{2}$$

5.10

$$f(z) = \int_1^z \left(\frac{1}{\omega} + \frac{\alpha}{\omega^3}\right)\cdot \cos\omega \; d\omega = \underbrace{\int_1^z \frac{1}{\omega} \cos\omega \, d\omega}_{= I_1} + \underbrace{\int_1^z \frac{\alpha}{\omega^3}\cdot\cos\omega \, d\omega}_{= I_2}$$

$$I_1 = \ln\omega\cdot\cos\omega \Big|_1^z + \int_1^z \ln\omega\cdot\sin\omega \, d\omega = \ln z\cdot\cos z + \int_1^z \ln\omega\cdot\sin\omega \, d\omega$$

$$I_2 = -\frac{1}{2}\frac{\alpha}{\omega^2}\cdot\cos\omega \Big|_1^z - \frac{\alpha}{2}\int_1^z \frac{\sin\omega}{\omega^2} d\omega = -\frac{1}{2}\frac{\alpha}{z^2}\cdot\cos z + \frac{\alpha}{2}\cos(1) +$$

$$+ \frac{\alpha}{2\omega}\cdot\sin\omega \Big|_1^z - \frac{\alpha}{2}\int_1^z \frac{1}{\omega}\cos\omega \, d\omega = -\frac{1}{2}\frac{\alpha}{z^2}\cos z + \frac{\alpha}{2}\cos(1) + \frac{\alpha}{2z}\sin z -$$

$$-\frac{\alpha}{2}\cdot\sin(1) - \frac{\alpha}{2}\ln\omega\cdot\cos\omega \Big|_1^z - \frac{\alpha}{2}\int_1^z \ln\omega\cdot\sin\omega \, d\omega$$

$$I_1 + I_2 = \ln z\cdot\cos z - \frac{\alpha}{2}\ln z\cdot\cos z - \frac{1}{2}\frac{\alpha}{z^2}\cos z + \frac{\alpha}{2}\cos(1) + \frac{\alpha}{2z}\sin z -$$

$$-\frac{\alpha}{2}\cdot\sin(1)$$

$\ln z\cdot\cos z\cdot\left(1 - \frac{\alpha}{2}\right)$ – должно быть однозначным

$$\Rightarrow \underline{\alpha = 2}$$