19.07

2) e^{-z^2} $(a=\infty)$

$z=x: \lim\limits_{x\to\infty} e^{-x^2} = 0$

$\Rightarrow$ существенно особая

$z=iy: \lim\limits_{y\to\infty} e^{y^2} = \infty$

3) $\sin\frac{\pi}{z^2}$ $(a=0)$

$\lim\limits_{z\to 0} \sin\frac{\pi}{z^2} = \nexists$ $\Rightarrow$ существенно особая

19.15

1) $$f(z) = \frac{(1+z^2)^2}{1-z^2}$$

Нули: $z = \pm i$ 2 порядок

Полюсы: $z = \pm 1$ 1 порядок

$z = \pm \infty$ 2 порядок

2) $$f(z) = \operatorname{ctg} z$$

Нули: $z = \frac{\pi}{2} + \pi k \quad k \in \mathbb{Z}$ 1 порядка

Полюсы: $z = \pi k \quad k \in \mathbb{Z}$ 1 порядка

3) $$f(z) = z \cdot \operatorname{tg}^2 z$$

Нули: $z = 0$ 3 порядка

$z = \cancel{\frac{\pi}{2}} + \pi k \quad k \in \mathbb{Z} \not\ni \{0\}$ 2 порядка

Полюсы: $z = \frac{\pi}{2} + \pi k \quad k \in \mathbb{Z}$ 2 порядка

20.01

1) $\sum\limits_{n=-\infty}^{\infty} 2^{-|n|} z^n = \sum\limits_{n=-\infty}^{0} 2^n z^n + \sum\limits_{n=1}^{\infty} 2^{-n} z^n = \sum\limits_{n=0}^{\infty} \frac{2^{-n}}{z^n} + \sum\limits_{n=1}^{\infty} 2^{-n} z^n$

$r = \lim\limits_{n\to\infty} \left| \frac{2^{-n-1}}{2^{-n}} \right| = \frac{1}{2} \qquad R = \lim\limits_{n\to\infty} \left| \frac{2^{-n}}{2^{-n-1}} \right| = 2$

$\frac{1}{2} < |z| < 2$

2) $\sum\limits_{n=-\infty}^{\infty} \frac{z^n}{3^n+1} = \sum\limits_{n=-\infty}^{0} \frac{z^n}{3^n+1} + \sum\limits_{n=1}^{\infty} \frac{z^n}{3^n+1} = \sum\limits_{n=0}^{\infty} \frac{1}{\frac{1}{3^n}+1} \frac{1}{z^n} + \sum\limits_{n=1}^{\infty} \frac{1}{3^n+1} z^n$

$r = \lim\limits_{n\to\infty} \sqrt[n]{\left|\frac{1}{\frac{1}{3^n}+1}\right|} = 1 \qquad R = \left(\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{3^n+1}}\right)^{-1} = \left(\frac{1}{3}\right)^{-1} = 3$

$1 < |z| < 3$

4) $\sum\limits_{n=-\infty}^{\infty} 2^{-n^2} \cdot (z+1)^n = \sum\limits_{n=0}^{\infty} \frac{2^{-n^2}}{(z+1)^n} + \sum\limits_{n=1}^{\infty} 2^{-n^2} \cdot (z+1)^n$

$r = \lim\limits_{n\to\infty} \sqrt[n]{|2^{-n^2}|} = \lim\limits_{n\to\infty} 2^{-n} = 0$

$R = \left(\lim\limits_{n\to\infty} \sqrt[n]{|2^{-n^2}|}\right)^{-1} = \left(\lim\limits_{n\to\infty} 2^{-n}\right)^{-1} = \lim\limits_{n\to\infty} 2^n = \infty$

$|z+1| \in \mathbb{R}$

20.09

1) $\frac{1}{z(z-3)^2} \qquad a=1 \quad D:\ 1 < |z-1| < 2$

$t = z-1:\ \frac{1}{(t+1)(t-2)^2} = \frac{A}{t+1} + \frac{B}{t-2} + \frac{C}{(t-2)^2} \ominus$

$\begin{cases} A\cdot(t^2-4t+4) \\ B\cdot(t^2-t-2) \\ C\cdot(t+1) \end{cases} \quad \begin{cases} t^2:\ A+B=0 \\ t:\ -4A-B=0 \\ 1:\ 4A-2B+C=1 \end{cases} \quad \begin{cases} A=-B \\ -3B=C \\ -4B-2B-3B=1 \end{cases} \quad \begin{cases} B=-\frac{1}{9} \\ C=\frac{1}{3} \\ A=\frac{1}{9} \end{cases}$

$\ominus\ \frac{1}{9}\frac{1}{t+1} - \frac{1}{9}\frac{1}{t-2} + \frac{1}{3}\frac{1}{(t-2)^2} = \frac{1}{9}\frac{1}{t}\frac{1}{1-(-\frac{1}{t})} + \frac{1}{9}\frac{1}{2}\frac{1}{1-\frac{t}{2}} + \frac{1}{6}\frac{1}{2}\frac{1}{(1-\frac{t}{2})^2} =$

$= \frac{1}{9}\frac{1}{t}\sum\limits_{n=0}^{\infty}(-1)^n\frac{1}{t^n} + \frac{1}{9}\frac{1}{2}\sum\limits_{n=0}^{\infty}\left(\frac{t}{2}\right)^n + \frac{1}{6}\frac{1}{2}\sum\limits_{n=0}^{\infty}(n+1)\left(\frac{t}{2}\right)^n =$

$= \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{9}\cdot\frac{1}{t^{n+1}} + \sum\limits_{n=0}^{\infty}\left[\frac{1}{9}+\frac{n+1}{6}\right]\frac{t^n}{2^{n+1}} = \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{9}\cdot\frac{1}{(z-1)^{n+1}} + \sum\limits_{n=0}^{\infty}\left[\frac{1}{9}+\frac{n+1}{6}\right]\cdot\frac{(z-1)^n}{2^{n+1}}$

5) $\frac{1}{z(z-1)(z-2)}$ $\qquad \alpha=0 \qquad -\frac{3}{2}\in D$

$$\frac{1}{z}\cdot\left(\frac{1}{z-2}-\frac{1}{z-1}\right)=\frac{1}{z}\left(\frac{-1}{2}\,\frac{1}{1-\frac{z}{2}}-\frac{1}{z}\,\frac{1}{1-\frac{1}{z}}\right)=\frac{1}{z}\left(\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n-\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right)=$$

$$=-\sum_{n=0}^{\infty}\frac{1}{z^{n+2}}-\sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n+1}}$$

7) $\frac{z^3}{(z+1)(z-2)}$ $\qquad \alpha=-1 \qquad D:\ 0<|z+1|<3$

$t=z+1$

$$\frac{(t-1)^3}{t\cdot(t-3)}=\frac{1}{t}\cdot\frac{(t-1)^3}{t-3}=\frac{1}{t}\left(1+\frac{2}{t-3}\right)(t-1)^2=$$

$$=\frac{1}{t}\left[(t-1)^2+\frac{2(t-1)}{t-3}(t-1)\right]=\frac{1}{t}\left(t^2-2t+1+\left(2+\frac{4}{t-3}\right)(t-1)\right]=$$

$$=\frac{1}{t}\left(t^2-1+\frac{4t}{t-3}-\frac{4}{t-3}\right)=t-\frac{1}{t}+\frac{4}{t-3}-\frac{4}{t(t-3)}=t-\frac{1}{t}+\frac{4}{t-3}+\frac{4}{3}\left(\frac{1}{t}+\frac{1}{3-t}\right)$$

$$=t+\frac{1}{3}\cdot\frac{1}{t}+\frac{8}{3}\,\frac{1}{t-3}=t+\frac{1}{3}\,\frac{1}{t}-\frac{8}{9}\,\frac{1}{1-\frac{t}{3}}=t+\frac{1}{3}\,\frac{1}{t}-\frac{8}{9}\sum_{n=0}^{\infty}\left(\frac{t}{3}\right)^n=$$

$$=\frac{1}{3}\,\frac{1}{t}-\frac{8}{9}+\frac{19}{27}t-\frac{8}{9}\sum_{n=2}^{\infty}\left(\frac{t}{3}\right)^n=$$

$$=\frac{1}{3}\,\frac{1}{z+1}-\frac{8}{9}+\frac{19}{27}(z+1)-\frac{8}{9}\sum_{n=2}^{\infty}\frac{(z+1)^n}{3^n}$$

8) $a=0 \quad D: |z|>2$

$$\frac{1}{5}\left(\frac{1}{z^2-1}-\frac{1}{z^2+4}\right)=\frac{1}{5}\left[\frac{1}{2}\left(\frac{1}{z-1}-\frac{1}{z+1}\right)-\frac{1}{4i}\left(\frac{1}{z-2i}-\frac{1}{z+2i}\right)\right]$$

$$=\frac{1}{10}\frac{1}{z-1}-\frac{1}{10}\frac{1}{z+1}+\frac{i}{20}\frac{1}{z-2i}-\frac{i}{20}\frac{1}{z+2i}=$$

$$=\frac{1}{10}\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}-\frac{1}{10}\frac{1}{z}\frac{1}{1-(-\frac{1}{z})}+\frac{i}{20}\frac{1}{z}\frac{1}{1-\frac{2i}{z}}-$$
$$-\frac{i}{20}\frac{1}{z}\frac{1}{1-(-\frac{2i}{z})}=$$

$$=\frac{1}{10}\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}-\frac{1}{10}\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n}+$$
$$+\frac{i}{20}\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2i}{z}\right)^n-\frac{i}{20}\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2i}{z}\right)^n=$$

$$=\frac{1}{10}\sum_{n=0}^{\infty}\left[1+(-1)^{n+1}\right]\frac{1}{z^{n+1}}+\frac{i}{20}\sum_{n=0}^{\infty}\left[1+(-1)^{n+1}\right]\frac{(2i)^n}{z^{n+1}}=$$

$$=\frac{1}{10}\sum_{n=0}^{\infty}\left[1+(-1)^{n+1}\right]\cdot\left[1+2^{n-1}i^{n+1}\right]\frac{1}{z^{n+1}}$$

20.16

1) $z^3 e^{\frac{1}{z}}$ $\quad a=0 \quad 0<|z|<\infty$

$$z^3 \cdot e^{\frac{1}{z}} = z^3 \cdot \sum_{n=0}^{\infty} \frac{1}{n! \cdot z^n} = \sum_{n=0}^{\infty} \frac{1}{n! \cdot z^{n-3}}$$

2) $z^2 \cdot \sin\left(\pi \frac{z+1}{z}\right)$ $\quad a=0 \quad 0<|z|<\infty$

$$z^2 \cdot \sin\left(\pi + \frac{\pi}{z}\right) = -z^2 \cdot \sin\frac{\pi}{z} = -z^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{z}\right)^{2n+1} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot \pi^{2n+1}}{(2n+1)!} (z)^{-2n+1}$$

3) $z^3 \cdot \cos\frac{1}{z-2}$ $\quad a=2 \quad 0<|z-2|<\infty$

$t=z-2:$ $\quad (t+2)^3 \cdot \cos\frac{1}{t} = \left(t^3+6t^2+12t+8\right) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{-2n} =$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{-2n+3} + 6\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{-2n+2} + 12\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{-2n+1} + 8\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{-2n}$$

5) $\dfrac{e^{\frac{1}{z-1}}}{z(z+1)}$ $\quad a=1 \quad 1<|z-1|<2$

$t=z-1:$ $\quad \dfrac{e^{\frac{1}{t}}}{(t+1)(t+2)} = e^{\frac{1}{t}} \cdot \left(\dfrac{1}{t+1} - \dfrac{1}{t+2}\right) = e^{\frac{1}{t}} \cdot \left(\dfrac{1}{t} \dfrac{1}{1-\left(-\frac{1}{t}\right)} - \dfrac{1}{2} \dfrac{1}{1-\left(-\frac{t}{2}\right)}\right) =$

$$= e^{\frac{1}{t}} \cdot \left(\frac{1}{t}\sum_{n=0}^{\infty} (-1)^n \frac{1}{t^n} - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{2^n}\right) = \sum_{k=0}^{\infty} \frac{1}{k!\, t^k} \sum_{n=0}^{\infty} (-1)^n \frac{1}{t^{n+1}} +$$

$$+ \sum_{k=0}^{\infty} \frac{1}{k!\, t^k} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{t^n}{2^{n+1}} =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!\,(z-1)^k} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{1}{(z-1)^{n+1}} + \sum_{k=0}^{\infty} \frac{1}{k!\,(z-1)^k} \cdot \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{2^{n+1}}$$

6) $e^{\frac{t}{2}\left(z-\frac{1}{z}\right)}$ $\quad a=0 \quad 0<|z|<\infty$

$$e^{\frac{t}{2}z} \cdot e^{-\frac{t}{2}\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{(tz)^n}{2^n \cdot n!} \cdot \sum_{m=0}^{\infty} (-1)^m \frac{t^m}{2^m\, m!} z^{-m}$$

20.21

1) $\dfrac{z}{(z+2)^2}$ $\qquad z_0 = -2$

$$t = z+2 : \quad \frac{t-2}{t^2} = \underbrace{\frac{1}{t} - \frac{2}{t^2}}_{\text{главная часть}} = \underbrace{\frac{1}{z+2} - \frac{2}{(z+2)^2}}$$

20.21

$$f(z) = \sum_{n=f_1}^{f_2}{}' c_n z^n \qquad f_1 < f_2$$

$f(z)$ - неизвестна

$g(z)$, $h(z)$ - известны

$$g(z) = \sum_{n=g_1}^{g_2}{}' a_n z^n \qquad h(z) = \sum_{n=h_1}^{h_2}{}' b_n z^n$$

$$g_1 < g_2 \qquad h_1 < h_2$$

$$f_1 + h_1 = g_1$$

$$f_2 + h_2 = g_2$$

$$f(z) = \frac{g(z)}{h(z)} \qquad f(z) \cdot h(z) = g(z)$$

$$a_n = \sum_{\substack{i=f_1 \\ j=h_1}}^{\substack{h_2 \\ f_2}} \delta(i+j-n)\, f_i \cdot h_j$$

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0, & x \neq 0 \end{cases}$$

2) $f(z) = \dfrac{e^z + 1}{e^z - 1} = \dfrac{g(z)}{h(z)}$ $z_0 = 2\pi i k \quad k \in \mathbb{Z}$

$$\left.\begin{aligned} g(z) &= e^z + 1 = 2 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots \\ h(z) &= e^z - 1 = z + \frac{z^2}{2} + \frac{z^3}{6} + \dots \end{aligned}\right| \Rightarrow \begin{aligned} f_1 &= -1 \\ f_2 &= +\infty \end{aligned}$$

$a_0 = 2 = 1 \cdot f_{-1} \qquad f_{-1} = 2$

$a_1 = 1 = \frac{1}{2} \cdot f_{-1} + 1 \cdot f_0 \quad f_0 = 0$

$\dots$

Главная часть $f(z)$: $\dfrac{2}{z}$

т.к. e^z периодическая, то

$$\frac{2}{z + 2\pi i k} \qquad k \in \mathbb{Z}$$

3) $f(z) = \frac{z-1}{\sin^2 z} = \frac{z}{\sin^2 z} - \frac{1}{\sin^2 z}$ $\quad z_0 = 0$

$$\sin^2 z = z^2\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \dots\right)^2$$

$$f(z) = \frac{1}{z}(a_0 + a_1 z + a_2 z^2 + \dots) - \frac{1}{z^2}(a_0 + a_1 z + a_2 z^2 + \dots) =$$

$$= \left(\frac{a_0}{z} + a_1 + a_2 z + \dots\right) - \left(\frac{a_0}{z^2} + \frac{a_1}{z} + a_2 + \dots\right)$$

$$a_0 = 1 \qquad a_1 = 0$$

Главная часть $f(z)$: $-\frac{1}{z^2} + \frac{1}{z}$

4) $f(z) = \frac{e^{iz}}{z^2+b^2}$ $\qquad z_0 = ib \qquad b>0$

$$f(z) = e^{iz}\frac{1}{(z-ib)(z+ib)} \ominus$$

$$\frac{1}{z-ib}\cdot\frac{1}{z+ib} = \frac{1}{z-ib}\cdot\left(\frac{1}{2ib} + a(z-ib)+\ldots\right) =$$

$$= \frac{1}{2ib}\,\frac{1}{z-ib} + \ldots$$

$$e^{iz} = e^{-b}\left(1 + (z+ib)+\ldots\right)$$

$$\ominus\ e^{-b}\left(1+(z+ib)+\ldots\right)\cdot\left(\frac{1}{z-ib}\,\frac{1}{2ib}+\ldots\right)$$

Главная часть $f(z)$: $\frac{e^{-b}}{2ib}\,\frac{1}{z-ib}$

5) $f(z) = \frac{(z^2+1)^2}{z^2+b^2}$ $\qquad z_0 = \infty$

$$f(z) = \frac{z^4+2z^2+1}{z^2+b^2} =$$

$$\begin{array}{r|l} -\;z^4+2z^2+1 & z^2+b^2 \\ z^4+b^2z^2 & z^2+(2-b^2)+\ldots \\ -\;(2-b^2)z^2+1 & \\ (2-b^2)z^2+b^2(2-b^2) & \\ \ddots & \end{array}$$

$$= z^2+(2-b^2)+\ldots$$

Главная часть $f(z)$: z^2

6) $f(z) = \frac{z\cdot e^{iz}}{(z^2+b^2)^2}$ $\qquad z_0 = ib \qquad b>0$

$$\ominus\ (-ib)\cdot e^{-b}\cdot\left(1+(z+ib)+\ldots\right)\cdot\frac{-1}{4b^2}\left(\frac{1}{z-ib}+\ldots\right)^2$$

Главная часть $f(z)$:

$$-\frac{ie^{-b}}{4b}\cdot\frac{1}{(z-ib)^2} + \frac{e^{-b}}{4b}\,\frac{1}{z-ib}$$

7) $f(z)=\dfrac{z}{(z^2+b^2)^2}$ $\quad z_0=ib$

$$(-ib)\cdot\frac{-1}{4b^2}\cdot\left(\frac{1}{z-ib}+\ldots\right)^2$$

Главная часть $f(z)$:

$$-\frac{i}{4b}\,\frac{1}{(z-ib)^2}$$

8) $f(z)=ctg\,\pi z \qquad z_0=k \qquad k\in\mathbb{Z}$

$$f(z)=\frac{1}{\pi(z-k)}-\frac{1}{3}\pi(z-k)-\frac{1}{45}\pi^3(z-k)^3-\ldots$$

Главная часть $f(z)$:

$$\frac{1}{\pi(z-k)} \qquad k\in\mathbb{Z}$$

9) $f(z) = \frac{1}{\sin \pi z}$ $\quad z_0 = k \quad k \in \mathbb{Z}$

$f(z) = \operatorname{cosec}(\pi z)$

$f(z) = \frac{(-1)^k}{\pi(z-k)} + \frac{(-1)^k}{6}\pi(z-k) + \frac{(-1)^k \cdot 7}{360}\pi^3(z-k)^3 + \ldots$

Главная часть $f(z)$:

$\frac{(-1)^k}{\pi(z-k)}$