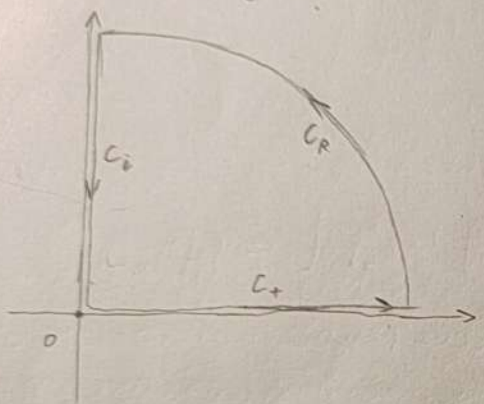
(1.1)

$$C = \int_0^\infty \frac{\ln t}{t} \sin t \, dt$$

$$f(a) = \int_0^\infty t^{a-1} \cdot \sin t \, dt$$

$$f'(a) = \int_0^\infty \frac{\sin t}{t} (t^a \cdot \ln t) \, dt \implies C = f'(a)\Big|_{a=0}$$

$$f(a) = \mathrm{Im} \int_0^\infty t^{a-1} \cdot e^{it} \, dt \qquad F(a) = \int_0^\infty z^{a-1} \cdot e^{iz} \, dz$$



$$\oint_C = \int_{C_+} + \int_{C_i} + \int_{C_R} \to 0 = 2\pi i \sum_j \mathrm{res}_j = 0$$

$$\int_{C_+} = F(a) \qquad \int_{C_i} = |z = ik| = i \int_\infty^0 (ik)^{a-1} \cdot e^{-k} \, dk = -(i)^a \int_0^\infty k^{a-1} \cdot e^{-k} \, dk = -e^{i\frac{\pi}{2}a} \cdot \Gamma(a)$$

$$F(a) = e^{i\frac{\pi}{2}a} \cdot \Gamma(a)$$

$$f(a) = \mathrm{Im}\left(e^{i\frac{\pi}{2}a} \cdot \Gamma(a)\right) = \sin\frac{\pi a}{2} \cdot \Gamma(a)$$

$$f'(a) = \frac{\pi}{2} \cdot \cos\frac{\pi a}{2} \Gamma(a) + \sin\frac{\pi a}{2} \cdot \Gamma'(a) = \frac{\pi^2}{2} \frac{\cos\frac{\pi a}{2}}{\sin \pi a} \frac{1}{\Gamma(1-a)} + \sin\frac{\pi a}{2} \Gamma'(a)$$

$$f'(a)\Big|_{a \to 0} = \frac{\pi^2}{2\pi a} + \frac{\pi a}{2} \cdot \frac{a\gamma - 1}{a^2} = \frac{\pi}{2}\left(\frac{1}{a} + \frac{a\gamma - 1}{a}\right) = \frac{\pi\gamma}{2}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \qquad \Gamma(z) = \frac{\pi}{\sin \pi z} \frac{1}{\Gamma(1-z)}$$

$$\Gamma'(z)\Gamma(1-z) + \Gamma(z)\Gamma'(1-z) = -\frac{\pi^2}{\sin^2 \pi z} \cdot \cos \pi z$$

$$\lim_{\varepsilon \to 0} \Gamma(\varepsilon) = \lim_{\varepsilon \to 0}\left[\frac{\pi}{\sin \pi\varepsilon} \frac{1}{\Gamma(1-\varepsilon)}\right] = \frac{1}{\varepsilon} \frac{1}{\Gamma(1)} = \frac{1}{\varepsilon}$$

$$\lim_{\varepsilon \to 0} \Gamma'(1-\varepsilon) = \Gamma'(1) = -\gamma$$ (γ – постоянная Эйлера-Маскерони)

и тогда, при $z \to 0$:

$$\Gamma'(z) \cdot \Gamma(1-z) + \Gamma(z) \cdot \Gamma'(1-z) = \Gamma'(z) - \frac{\gamma}{z} = -\frac{1}{z^2}$$

$$\lim_{z \to 0} \Gamma'(z) = \frac{z\gamma - 1}{z^2}$$

(1.2)

$$I(\nu)=\int_0^{\frac{\pi}{2}} \sin^\nu x\, dx$$

$\nu \in \mathbb{C}$: $|\sin x = t|$ $t^\nu = t^{x+iy} = t^x \cdot t^{iy} = t^x \cdot e^{iy\cdot \ln t}$

$e^{iy\cdot\ln t} = \cos(y\cdot \ln t) + i\cdot \sin(y\cdot\ln t)$ при $t\to 0$ $e^{iy\ln y}$ – ограничена

$\Rightarrow$ интеграл $I(\nu)$ сходится при $Re(\nu) > -1$

$$I(\nu)=\int_0^{\frac{\pi}{2}} \sin^\nu x\, dx = \left| \begin{array}{l} t=\sin x \\ dt=\cos x\, dx \\ dx = \frac{dt}{\cos x} = \frac{dt}{\sqrt{1-t^2}} \end{array}\right| = \int_0^1 \frac{t^\nu}{\sqrt{1-t^2}}\, dt =$$

$$= \left| \begin{array}{ll} z=t^2 & t=\sqrt{z} \\ dt = \frac{1}{2}\frac{dz}{\sqrt{z}} & \end{array}\right| = \frac{1}{2}\int_0^1 \frac{z^{\frac{\nu}{2}}}{\sqrt{1-z}} \frac{dz}{\sqrt{z}} = \frac{1}{2}\int_0^1 \frac{z^{\frac{\nu-1}{2}}}{\sqrt{1-z}}\, dz = \frac{1}{2}\int_0^1 z^{\frac{\nu-1}{2}}\cdot(1-z)^{-\frac{1}{2}}\, dz \ominus$$

$$\boxed{B(x,y)=\int_0^1 t^{x-1}\cdot(1-t)^{y-1}\, dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}}$$

$$\ominus\ \frac{1}{2}\cdot B\left(\frac{\nu+1}{2}, \frac{1}{2}\right) = \frac{1}{2}\,\frac{\Gamma(\frac{\nu+1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{\nu}{2}+1)} = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2}+1)}$$

(1.3)

$$h(a)=\int_0^1 \frac{t^{a-1}}{1+t}\, dt = \int_0^1 t^{a-1}\cdot\left(\sum_{n=0}^{\infty}(-1)^n t^n\right) dt =$$

$$\boxed{\begin{array}{l} \frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n\cdot t^n \\ \psi(a+1) = -\gamma + \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+a}\right] \\ \psi(\frac{a}{2}+1) = -\gamma + \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{2}{2n+a}\right] \\ \psi(\frac{a+1}{2}+1) = -\gamma + \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{2}{2n+1+a}\right] \end{array}}$$

$$\int_0^1 \sum_{n=0}^{\infty}(-1)^n\cdot t^{n+a-1}\, dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+a} =$$

$$= \frac{1}{a} + \sum_{n=1}^{\infty}\frac{1}{2n+a} - \frac{1}{1+a} - \sum_{n=1}^{\infty}\frac{1}{2n+1+a} =$$

$$= -\frac{1}{2}\psi\left(\frac{a}{2}+1\right) + \frac{1}{2}\psi\left(\frac{a+1}{2}+1\right) + \frac{1}{a(1+a)} =$$

$$= -\frac{1}{2}\psi\left(\frac{a-2}{2}+1\right) + \frac{1}{2}\psi\left(\frac{a-1}{2}+1\right) =$$

$$= -\frac{1}{2}\psi\left(\frac{a}{2}\right) + \frac{1}{2}\psi\left(\frac{a+1}{2}\right)$$

1.4

$$H(a,b)=\int_0^1 \frac{t^{a-1}-t^{b-1}}{1+t}\frac{dt}{\ln t} \qquad a>0 \quad b>0$$

$$\frac{d}{da}H(a,b)=\frac{d}{da}\int_0^1 \frac{t^{a-1}-t^{b-1}}{1+t}\frac{dt}{\ln t}=\int_0^1\frac{t^{a-1}}{1+t}dt=h(a)$$

$$\frac{d}{db}H(a,b)=\frac{d}{db}\int_0^1 \frac{t^{a-1}-t^{b-1}}{1+t}\frac{dt}{\ln t}=\int_0^1\frac{-t^{b-1}}{1+t}dt=-h(b)$$

$$H(1,1)=\int_0^1\frac{1-1}{1+t}\frac{dt}{\ln t}=0$$

$$\boxed{\psi(z)=\frac{d}{dz}\left(\ln(\Gamma(z))\right) \qquad h(z)=\frac{1}{2}\psi\left(\frac{1+z}{2}\right)-\frac{1}{2}\psi\left(\frac{a}{2}\right)}$$

$$H(a,b)=\int h(a)\,da=-\int h(b)\,db$$

$$\int h(a)\,da=\int\left[\frac{1}{2}\psi\left(\frac{1+a}{2}\right)-\frac{1}{2}\psi\left(\frac{a}{2}\right)\right]da=\frac{1}{2}\int\psi\left(\frac{1+a}{2}\right)da-\frac{1}{2}\int\psi\left(\frac{a}{2}\right)da=$$

$$=\int\psi\left(\frac{1+a}{2}\right)d\left(\frac{1+a}{2}\right)-\int\psi\left(\frac{a}{2}\right)d\left(\frac{a}{2}\right)=$$

$$=\int\frac{d\left(\ln\Gamma\left(\frac{1+a}{2}\right)\right)}{da}da-\int\frac{d\left(\ln\Gamma\left(\frac{a}{2}\right)\right)}{da}da=\ln\Gamma\left(\frac{1+a}{2}\right)-\ln\Gamma\left(\frac{a}{2}\right)+C(b)$$

$$-\int h(b)\,db=\ln\Gamma\left(\frac{b}{2}\right)-\ln\Gamma\left(\frac{1+b}{2}\right)+C(a)$$

$$H(a,b)=\ln\frac{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)}+C$$

$$H(1,1)=\ln\frac{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(1)}+C=\ln\frac{1\cdot\sqrt{\pi}}{\sqrt{\pi}\cdot 1}+C=\ln\frac{\sqrt{\pi}}{\sqrt{\pi}}+C=C=0$$

$$\Downarrow$$

$$H(a,b)=\ln\frac{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1+b}{b}\right)}$$

(2)

$$I(\alpha) = \frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^{\infty} x^{\alpha}(1+x)^{2\alpha}(2+x)^{3\alpha} e^{-x}\,dx$$

При $x \to 0$: $x^{\alpha}(1+x)^{2\alpha}(2+x)^{3\alpha} e^{-x} \to 2^{3\alpha} x^{\alpha}$

$$I(\alpha) = \underbrace{\frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^{+\infty} x^{\alpha}\left[(1+x)^{2\alpha}(2+x)^{3\alpha} - 2^{3\alpha}\right] e^{-x}\,dx}_{\mathrm{I}} + \underbrace{\frac{2^{3\alpha}}{\Gamma\left(\frac{\alpha+1}{2}\right)} \underbrace{\int_0^{+\infty} x^{\alpha} e^{-x}\,dx}_{\mathrm{II}}}_{\mathrm{II}'}$$

$$\mathrm{II}: \int_0^{+\infty} x^{\alpha} e^{-x}\,dx = \cancel{-x^{\alpha} e^{-x}\Big|_0^{+\infty}} + \alpha \int_0^{+\infty} x^{\alpha-1} e^{-x}\,dx = \alpha \cdot \Gamma(\alpha)$$

$$\mathrm{II}': 2^{3\alpha} \frac{\alpha \cdot \Gamma(\alpha)}{\Gamma\left(\frac{\alpha+1}{2}\right)} = 2^{3\alpha} \alpha \cdot \frac{\Gamma(\alpha)\,\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)} = 2^{3\alpha} \alpha \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)} B\left(\frac{1}{2}, \frac{\alpha}{2}\right) \ominus$$

$$B\left(\frac{1}{2}, \frac{\alpha}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{\alpha}{2}-1}\,dx = \left|\begin{matrix} x = t^2 \\ dx = 2t\,dt \end{matrix}\right| = 2\int_0^1 (1-t^2)^{\frac{\alpha}{2}-1}\,dt =$$

$$= \int_{-1}^{1} (1-t^2)^{\frac{\alpha}{2}-1}\,dt = \left|\begin{matrix} u = \frac{t+1}{2} \\ dt = 2\,du \end{matrix}\right| = 2^{\alpha-1}\int_0^1 u^{\frac{\alpha}{2}-1}(1-u)^{\frac{\alpha}{2}-1}\,du =$$

$$= 2^{\alpha-1} \cdot B\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right) = 2^{\alpha-1} \frac{\Gamma^2\left(\frac{\alpha}{2}\right)}{\Gamma(\alpha)}$$

$$\ominus \frac{\alpha \cdot 2^{4\alpha-1}}{\sqrt{\pi}} \Gamma\left(\frac{\alpha}{2}\right)$$

- $I_{con}(-1) = -\frac{1}{32\sqrt{\pi}}\Gamma\left(-\frac{1}{2}\right) = \frac{1}{16}$

- $I_{con}(z) = \frac{z \cdot 2^{4z-1}}{\sqrt{\pi}} \Gamma\left(\frac{z}{2}\right) \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{n!}{2^k} \binom{z+n}{n} \binom{2z}{n+k} \binom{3z}{k}$

$$\binom{z}{n} = \begin{cases} 1 & n = 0 \\ \prod_{k=1}^{n} \frac{z-k+1}{k}, & n \neq 0 \end{cases}$$

$$\underset{z=-2}{\mathrm{res}}\, I_{con}(z) = -\frac{1}{32\sqrt{\pi}} \underset{z=-2}{\mathrm{res}}\, \Gamma\left(\frac{z}{2}\right) = \frac{1}{16\sqrt{\pi}}$$

(3)

$$G(in) = \sum_{k=1}^{\infty} \left(\frac{1}{-a+ik+in} - \frac{1}{-a-ik+in} + \frac{2i}{k} \right)$$

1) $$\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+z} \right]$$

$$i\cdot\psi(z+1+ia) = -i\gamma + \sum_{n=1}^{\infty} \left[\frac{i}{n} + \frac{1}{in+iz-a} \right]$$

$$i\psi(-z+1-ia) = -i\gamma + \sum_{n=1}^{\infty} \left[\frac{i}{n} - \frac{1}{-in+iz-a} \right]$$

$$G(in) = 2i\gamma + i\cdot\psi(n+1+ia) + i\cdot\psi(-n+1-ia) = i\left(2\gamma + \psi(n+1+ia) + \psi(-n+1-ia)\right)$$

2) $G(in) \to G(z)$ $\quad \operatorname{Im} z > 0$

$$G(z) = \sum_{k=1}^{\infty} \left(\frac{1}{-a+ik+z} - \frac{1}{-a-ik+z} + \frac{2i}{k} \right)$$

$$G(z) = i\cdot\left(2\gamma + \psi(-i\cdot z+1+ia) + \psi(i\cdot z+1-ia)\right)$$ — прямое, неверное, продолжение

$$-\gamma + \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n-iz+ia} \right] \quad -\gamma + \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+iz-ia} \right]$$

poles: $\quad z = a - in \quad$ $\boxed{z = a + in}$

$\quad \operatorname{Im} z < 0 \quad$ $\boxed{\operatorname{Im} z > 0}$

$$\boxed{\psi(z) = \psi(1-z) - \pi\cdot \operatorname{ctg} \pi z}$$

$$G(z) = i\left(2\gamma + \psi(-iz+1+ia) + \psi(ia-iz) - \pi\cdot\operatorname{ctg}(\pi + \pi i z - \pi i a)\right)$$

④

$$L(z)=\frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}e^{-x}}{1+e^{-2x}}dx$$

• $$L(1)=\frac{1}{\Gamma(1)}\int_0^\infty \frac{e^{-x}}{1+e^{-2x}}dx=\int_0^\infty \frac{e^{-x}}{1+e^{-2x}}dx=\left|\begin{matrix} t=e^{x} \\ dx=\frac{dt}{t}\end{matrix}\right|=\int_1^\infty \frac{\frac{1}{t}}{1-\frac{1}{t^2}}\frac{dt}{t}=$$

$$=\int_1^\infty \frac{dt}{t^2-1}$$

• $$\frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}e^{-x}}{1+e^{-2x}}dx=\left|\begin{matrix} u=\frac{e^{-x}}{1+e^{-2x}} & du=\left(\frac{e^{-x}}{1+e^{-2x}}\right)'dx \\ dv=x^{z-1}dx & V=\frac{1}{z}\cdot x^{z}\end{matrix}\right|=$$

$$=\frac{1}{\Gamma(z)}\left[\frac{1}{z}x^{z}\frac{e^{-x}}{1+e^{-2x}}\Big|_0^\infty-\frac{1}{z}\int_0^\infty x^{z}\left(\frac{e^{-x}}{1+e^{-2x}}\right)dx\right]=$$

$$\left|\begin{matrix} \frac{1}{z}x^{z}\frac{e^{-x}}{1+e^{-2x}}\Big|_{x=0}=\frac{1}{z}\cdot(0)^{z}\cdot\frac{1}{2}=0 \\ \frac{1}{z}x^{z}\frac{e^{-x}}{1+e^{-2x}}\Big|_{x\to\infty}=\lim\limits_{x\to\infty}\frac{1}{z}\frac{x^{z}}{e^{x}}=0\end{matrix}\right|=-\frac{1}{z\cdot\Gamma(z)}\int_0^\infty x^{z}\left(\frac{e^{-x}}{1+e^{-2x}}\right)'dx$$

$$\lim_{z\to0}L(z)=\lim_{z\to0}\left[-\frac{1}{z\cdot\Gamma(z)}\int_0^\infty x^{z}\left(\frac{e^{-x}}{1+e^{-2x}}\right)'dx\right]=-\int_0^\infty\left(\frac{e^{-x}}{1+e^{-2x}}\right)'dx$$

• $$L(z)=\sum_{n=0}^{\infty}(-1)^n(1+2n)^{-z}\qquad S=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$$

$$S=L(1)=\frac{\pi}{4}$$

• $$F_1=\ln\prod_{n=0}^{N-1}\frac{4n+3}{4n+5}=\ln\prod_{n=0}^{N-1}\frac{n+\frac{3}{4}}{n+\frac{5}{4}}=\ln\left[\frac{(N-1)!\,N^{\frac{5}{4}}}{\prod_{n=0}^{N-1}(n+\frac{5}{4})}\Big/\frac{(N-1)!\,N^{\frac{3}{4}}}{\prod_{n=0}^{N-1}(n+\frac{3}{4})}\cdot N^{-\frac{1}{2}}\right]=$$

• $\ln\Gamma(n+1)=\ln(n!)=n\cdot\ln n-n$

Определение гамма-функции по Гауссу:

$$\Gamma(z)=\lim_{n\to\infty}\frac{(n-1)!\,n^{z}}{z(z+1)\dots(z+n-1)}$$

$$=\ln\left[\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}N^{-\frac{1}{2}}\right]$$

• $$F_2=\int_N^\infty\left[\frac{\ln(4n+3)}{(4n+3)^2}-\frac{\ln(4n+5)}{(4n+5)^2}\right]dn=\int_N^{+\infty}\frac{\ln(4n+3)}{(4n+3)^2}dn-\int_N^{+\infty}\frac{\ln(4n+5)}{(4n+5)^2}dn=\left|\begin{matrix} \mathrm{I}: x=4n+3 \\ dn=\frac{1}{4}dx \\ \mathrm{II}: x=4n+5 \\ dn=\frac{1}{4}dx\end{matrix}\right|=$$

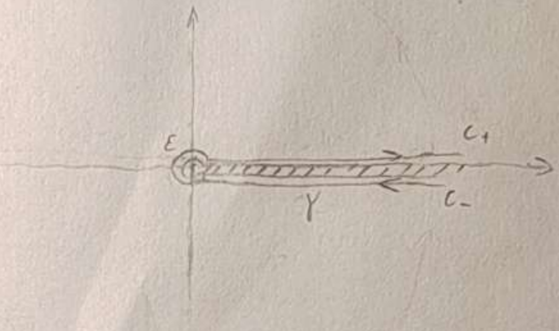
$$=\frac{1}{4}\int_{4N+3}^{+\infty}\frac{\ln(x)}{x^{z}}dx-\frac{1}{4}\int_{4N+5}^{+\infty}\frac{\ln(x)}{x^{z}}dx=\frac{1}{4}\int_{4N+3}^{4N+5}\frac{\ln(x)}{x^{z}}dx=$$

$$=\frac{1}{4}\left[\frac{1}{1-z}(4N+5)^{1-z}\cdot\ln(4N+5)-\frac{1}{1-z}(4N+3)^{1-z}\cdot\ln(4N+3)-\frac{1}{1-z}(4N+5)^{1-z}+\frac{1}{1-z}(4N+3)^{1-z}\right]$$

$$\lim_{z\to0}F_2=\frac{1}{4}(4N+5)\cdot(\ln(4N+5)-1)+\frac{1}{4}(4N+3)(1-\ln(4N+3))\approx\frac{1}{2}\ln N+\ln 2$$

• $$L'(0)=F_1+F_2=\ln\left[\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}N^{-\frac{1}{2}}\right]+\frac{1}{2}\ln N+\ln 2=\ln\left[2\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}\right]$$

(5)



$$\int_\gamma = \int_{C+} + \int_{C-} + \cancel{\int_\varepsilon}^{\to 0} = \int_{C_+} + \int_{C_-}$$

$$\int_{C_+} = L(z)$$

$$\int_{C+} = \frac{1}{\Gamma(z)}\int_0^{+\infty} \frac{t^{z-1}e^{-t}}{1+e^{-2t}}\,dt \qquad \int_{C-} = -e^{2\pi i z}\cdot\int_{\gamma+}$$

$$\int_\gamma = (1-e^{2\pi i z})\int_{C+}$$

$$L(z) = \frac{1}{1-e^{i2\pi z}}\frac{1}{\Gamma(z)}\int_\gamma \frac{t^{z-1}e^{-t}}{1+e^{-2t}}\,dt$$

$$1+e^{-2t}=0$$

$$e^{-2t} = -1 = e^{i(\pi+2\pi n)} \qquad n\in\mathbb{Z}$$

$$-2t = i(\pi+2\pi n)$$

$$t = -i\frac{\pi}{2} - i\pi n$$

$n\in\mathbb{Z}_-$ — positive poles $\qquad$ $n\in\mathbb{Z}_+$ — negative poles

$$\underset{\text{positive}}{\operatorname{res}} \frac{t^{z-1}e^{-t}}{1+e^{-2t}} = \left(-\frac{t^{z-1}\cdot e^{-t}}{2e^{-2t}}\right)\Bigg|_{t=\frac{\pi}{2}(2n+1)e^{i\frac{\pi}{2}}} = \frac{e^{-i\frac{\pi}{2}}}{2i}\left(\frac{\pi}{2}\right)^{z-1}(-1)^n(2n+1)^{z-1}\cdot e^{i\frac{\pi}{2}z} \quad (n\in\mathbb{N})$$

$$\underset{\text{negative}}{\operatorname{res}} \frac{t^{z-1}\cdot e^{-t}}{1+e^{-2t}} = \left(-\frac{t^{z-1}\cdot e^{-t}}{2e^{-2t}}\right)\Bigg|_{t=\frac{\pi}{2}(2n+1)e^{-i\frac{\pi}{2}}} = \frac{e^{-i\frac{\pi}{2}}}{2i}\left(\frac{\pi}{2}\right)^{z-1}(-1)^n(2n+1)^{z-1}\, e^{-i\frac{\pi}{2}z} \quad (n\in\mathbb{N})$$

$$L(z) = \frac{1}{1-e^{2i\pi z}}\frac{1}{\Gamma(z)}\sum_k \operatorname{res}_k f(z) = \left(\frac{\pi}{2}\right)^{z}\cdot\operatorname{cosec}\left(\frac{\pi}{2}z\right)\cdot\frac{L(1-z)}{\Gamma(z)}$$

$$\Rightarrow L(1-z) = \left(\frac{\pi}{2}\right)^{-z}\cdot\sin\frac{\pi}{2}z\cdot\Gamma(z)\cdot L(z)$$

$$L(-2k-1) = L(1-(2k+2)) = \left(\frac{\pi}{2}\right)^{-(2k+2)}\cdot\underbrace{\sin[\pi(k+1)]}_{0}\cdot\Gamma(2k+2)\cdot L(2k+2) = 0, \quad k\in\mathbb{N}$$

(6)

$$C = \int_0^{+\infty} \frac{\ln x \, dx}{\cosh x}$$

$$L(z) = \frac{1}{2} \frac{1}{\Gamma(z)} \int_0^{+\infty} \frac{x^{z-1}}{\cosh x} dx$$

$$2 \frac{d}{dz} \left[\Gamma(z) L(z) \right] = \int_0^{+\infty} \frac{x^{z-1} \ln x}{\cosh x} dx$$

$$C = 2 \frac{d}{dz} \left[\Gamma(z) L(z) \right] \Big|_{z=1}$$

$$C = 2 \frac{d}{dz} \left[\left(\frac{\pi}{2}\right)^z \cdot \operatorname{cosec}\left(\frac{\pi}{2} z\right) L(1-z) \right] \Big|_{z=1} =$$

$$= 2 \cdot \left[\left(\frac{\pi}{2}\right)^z \cdot \ln \frac{\pi}{2} \cdot \operatorname{cosec}\left(\frac{\pi}{2} z\right) - \left(\frac{\pi}{2}\right)^z \cdot \operatorname{ctg}\left(\frac{\pi}{2} z\right) \operatorname{cosec}\left(\frac{\pi}{2} z\right) \cdot \frac{\pi}{2} \right] L(1-z) -$$

$$- 2 \cdot \left(\frac{\pi}{2}\right)^z \cdot \operatorname{cosec}\left(\frac{\pi}{2} z\right) \cdot L'(1-z) \Big|_{z=1} =$$

$$= \pi \cdot \ln \frac{\pi}{2} \cdot L(0) - \pi \cdot L'(0) = \pi \cdot \ln \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}}$$