1. (10 points) Consider the function $f(x, y) = x^3 + 3y^2x^2$ and the point $A(1, 1)$.
   (a) Calculate the gradient of the function f at the point A.
   (b) Find the second order Taylor approximation in the neighborhood of A.
2. (10 points) Consider the equation $3x^3 + 5y^5 + z^3 + z = 10$.
   (a) Check whether the equation defines the function $z(x, y)$ at a point $A(1, 1, 1)$.
   (b) Find dz at the point A.
3. (10 points) Find all local extrema of the function $f(x, y) = x^2y - 3xy^2 + 5xy + 2$ such that $x \neq 0$ and $y \neq 0$. Classify them.
4. (10 points) Find all local constrained extrema of the function $f(x, y, z) = x + 2y + 3z$ subject to $\ln x + \ln y + \ln z = 0$. Do not forget to classify extrema.
5. (10 points) Consider the function $f(x, y) = h(x)g(y) + ax^3$, where h and g are twice differentiable and a is a parameter. Let's denote the maximum point by $x^*(a)$ and $y^*(a)$ and assume that second order conditions for maximization are met.

   Find the sign of dx^*/da.
6. (10 points) The level curves of the function $f(x, y)$ are given by the equation $y - x^2 = c$.

   Draw two level curves of the function $g(x, y) = f(x - 2, |y| + 1)$.

7. Consider a problem

$$\begin{cases} xyz \to \max \\ \text{s.t. } x+y+z=c \\ x,y,z>0 \end{cases}$$

where c is a parameter and $c>0$.

   (a) (15 points) Solve this problem using first-order conditions. Use bordered Hessian for sufficiency.

   (b) (5 points) Use the result of part a) to show that arithmetic mean $(x+y+z)/3$ is no less than the geometric mean $(xyz)^{1/3}$.

8. (a) (10 points) Consider a utility maximization problem

$$\begin{cases} u(x,y) \to \max \\ \text{s.t. } p_x x + p_y y = I \\ x,y>0 \end{cases},$$

   where $u \in C^1$ and parameters p_x, p_y and I are positive. Let (x^*, y^*) be the solution of this problem. Form the value function $V(p_x, p_y, I) = u(x^*, y^*)$.

   Using appropriate envelope theorem show that

$$x^* = -\frac{\partial V}{\partial p_x} \Big/ \frac{\partial V}{\partial I}, \quad y^* = -\frac{\partial V}{\partial p_y} \Big/ \frac{\partial V}{\partial I}.$$

   (b) (10 points) Let $F(x,y)$ be a function such that $F \in C^2$ and $F'_y \neq 0$. The equation $F(x,y)=0$ defines the implicit function $y(x)$.

   Find the expression for d^2y/dx^2.

   The expression should include only derivatives of $F(x,y)$.