Retake 24.01.2015. Good luck!
Part A

1. The function $f(x, y)$ is given by $f(x, y) = u^2(x, y) + v^3(x, y)$. The values of u and v and their gradients at the point $(x, y) = (1, 1)$ are also known, $u(1, 1) = 3$, $v(1, 1) = -2$, $\operatorname{grad} u = (1, 4)$, $\operatorname{grad} v = (-1, 1)$. Find $\operatorname{grad} f$ at the point $(1, 1)$ if $u, v \in C^1$.

2. Find the local maxima and minima of the function $f(x, y) = x^4 + 2y^4 - xy$. Determine whether the extrema you have found are global or local.

3. For the function $f(x, y) = x^3y^5 + x^2 - y^3 + xy$ find first order Taylor approximation at the point $(1, 1)$ and second order Taylor approximation at the same point.

4. Use Lagrange multipliers to find the height and radius of a cylinder with the least possible surface area among those with a volume of 6π m^3. Make sure you check the second order condition for minimisation.

5. Find the equation of the tangent plane to the surface given by $x^3 + z^3 - 3xz = y - 1$ at the point $(1, 4, 2)$.

6. Suppose $f(x, y)$ is a twice differentiable function. Let x and y be defined in terms of u, v as follows: $x(u, v) = ue^{2v}$, $y(u, v) = u^2 - v^2$. Let $F(u, v) = f(x(u, v), y(u, v))$. Calculate F''_{uu} and F''_{uv}.

Part B

7. A firm's inventory $I(t)$ is depleted at a constant rate per unit time, i.e. $I(t) = x - \delta t$, where x is an amount of good reordered by the firm whenever the level of inventory is zero. The order is fulfilled immediately. The annual requirement for the commodity is 200 units and the firm orders the commodity n times a year where $200 = nx$. The firm incurs two types of inventory costs: a holding cost and an ordering cost. Since the average stock of inventory is $x/2$, the holding cost equals $C_hx/2$, the cost of placing one order is C_o, and with n orders a year the annual ordering cost equals C_on.

   (a) Minimize the cost of inventory $C = C_hx/2 + C_on$ by choice of x and n subject to the constraint $nx = 200$ by the Lagrange multiplier method.

   (b) Use the envelope theorem to approximate the change in the minimal cost if the requirement for the commodity rises to 204 units.

8. A two-product firm produces outputs y_1 and y_2 from a single factor of production which is labor, in other words, there is a function f, such that $f(y_1, y_2) \le \bar{L}$. Output prices are p_1 and p_2. The firm has a fixed amount of labor supply $\bar{L} > 0$ that should be utilized in full.

   (a) (10 points) Set the problem of the revenue maximization under the labor constraint mathematically and derive first-order conditions. Assume that both outputs should be produced in positive amounts.

   (b) (10 points) Let the maximum value of the total revenue under the labor constraint be $TR(p_1, p_2, \bar{L})$. What are its derivatives with respect to the prices?

1. $\operatorname{grad} f = 2u \operatorname{grad} u + 3v^2 \operatorname{grad} v = 6 \cdot (1, 4) + 12 \cdot (-1, 1) = (-6, 36)$. Maybe solved by computing f'_x (formula + value, 3+1 pts) and f'_y (3+1 pts) and putting them in a vector (2 pts).

2. FOC statement — 1 pt, FOC solution — 4 pts, SOC — 3 pts, globality — 2 pts.

   Critical points: $(x, y) = (0, 0)$, $(2^{-9/8}, 2^{-11/8})$, $(-2^{-9/8}, -2^{-11/8})$

3. 1 pt for each derivative (5 pts total f_x, f_y, f_{xx}, f_{yy}, f_{xy}), 2 pts for first order, 3 pts for second order approximation

4. problem formulation (target function, constraint) — 2 pts, NDCQ — 1 pt, FOC formulation — 1 pt, FOC solution 4, SOC check — 2 pts

5. $\operatorname{grad} f = (3x^2 - 3z, -1, 3z^2 - 3x) = (-3, -1, 9)$ (4 pts). So tangent plane equation is $-3x - y + 9z = c_0$ (4 pts). Plugging in the coordinates of the point we obtain $c_0 = 11$ (2 pts).

6. F_u 4 pts, F_{uu} — 3 pts, F_{uv} — 3pts

7. a) NDCQ — 2 pts, Langrangean — 1 pt, FOC statement — 1 pt, FOC solution — 5 pts, SOC — 5 pts, minimum value — 2 pts, b) 4 pts

8. formulation — 5, NDCQ — 2 pts, FOC statement — 3 points