Caltech Division of the Humanities and Social Sciences

Euler's Theorem for Homogeneous Functions

KC Border

October 2000

v. 2015.11.11::14.24

Let $f\colon \boldsymbol{R}^n_+ \to \boldsymbol{R}$. We say that f is **homogeneous of degree k** if for all $x \in \boldsymbol{R}^n_+$ and all $\lambda > 0$,

$$f(\lambda x) = \lambda^k f(x).$$

1 Euler's theorem *Let $f\colon \boldsymbol{R}^n_+ \to \boldsymbol{R}$ be continuous, and also differentiable on $\boldsymbol{R}^n_{++}$. Then f is homogeneous of degree k if and only if for all $x \in \boldsymbol{R}^n_{++}$,*

$$kf(x) = \sum_{i=1}^{n} D_i f(x) x_i. \tag{$*$}$$

Proof: ($\Longrightarrow$) Suppose f is homogeneous of degree k. Fix $x \in \boldsymbol{R}^n_{++}$, and define the function $g\colon [0,\infty) \to \boldsymbol{R}$ (depending on x) by

$$g(\lambda) = f(\lambda x) - \lambda^k f(x),$$

and note that for all $\lambda \geqslant 0$,

$$g(\lambda) = 0.$$

Therefore

$$g'(\lambda) = 0$$

for all $\lambda > 0$. But by the chain rule,

$$g'(\lambda) = \sum_{i=1}^{n} D_i f(\lambda x) x_i - k\lambda^{k-1} f(x).$$

Evaluate this at $\lambda = 1$ to obtain ($*$).

($\Longleftarrow$) Suppose

$$kf(x) = \sum_{i=1}^{n} D_i f(x) x_i$$

for all $x \in \boldsymbol{R}^n_{++}$. Fix any $x \gg 0$ and again define $g\colon [0,\infty) \to \boldsymbol{R}$ (depending on x) by

$$g(\lambda) = f(\lambda x) - \lambda^k f(x)$$

and note that $g(1) = 0$. Then for $\lambda > 0$,

$$\begin{aligned} g'(\lambda) &= \sum_{i=1}^{n} D_i f(\lambda x) x_i - k\lambda^{k-1} f(x) \\ &= \lambda^{-1} \left(\sum_{i=1}^{n} D_i f(\lambda x) \lambda x_i \right) - k\lambda^{k-1} f(x) \\ &= \lambda^{-1} k f(\lambda x) - k\lambda^{k-1} f(x), \end{aligned}$$

so

$$\begin{aligned} \lambda g'(\lambda) &= k\big(f(\lambda x) - \lambda^k f(x)\big) \\ &= k g(\lambda). \end{aligned}$$

Since λ is arbitrary, g satisfies the following differential equation:

$$g'(\lambda) - \frac{k}{\lambda} g(\lambda) = 0$$

and the initial condition $g(1) = 0$. By Theorem 5 below,

$$g(\lambda) = 0 \cdot e^{A(\lambda)} + e^{-A(\lambda)} \int_1^{\lambda} 0 \cdot e^{A(t)}\, dt = 0$$

where, irrelevantly, $A(\lambda) = -\int_1^{\lambda} \frac{k}{t}\, dt = -k \ln \lambda$. This implies g is identically zero, so f is homogeneous on $\boldsymbol{R}^{\mathrm{n}}_{++}$. Continuity guarantees that f is homogeneous on $\boldsymbol{R}^{\mathrm{n}}_{+}$. ∎

2 Corollary *Let $f\colon \boldsymbol{R}^{\mathrm{n}}_{+} \to \boldsymbol{R}$ be continuous and differentiable on $\boldsymbol{R}^{\mathrm{n}}_{++}$. If f is homogeneous of degree k, then $D_j f(x)$ is homogeneous of degree $k-1$.*

Proof if f is twice differentiable: By the first half of Euler's theorem,

$$\sum_{i=1}^{n} D_i f(x) x_i = k f(x)$$

so differentiating both sides with respect to the j^{th} variable,

$$D_j \left(\sum_{i=1}^{n} D_i f(x) x_i \right) = k D_j f(x)$$

or

$$\sum_{i=1}^{n} D_{ij} f(x) x_i + D_j f(x) = k D_j f(x)$$

or

$$\sum_{i=1}^{n} D_{ij} f(x) x_i = (k-1) D_j f(x). \tag{1}$$

Thus $D_j f(x)$ is homogeneous of degree $(k-1)$ by second half of Euler's theorem. ∎

Proof without twice differentiability: The difference quotients satisfy

$$\frac{f(\lambda x + \lambda h) - f(\lambda x)}{\|\lambda h\|} = \frac{\lambda^k f(x+h) - \lambda^k f(x)}{\lambda \|h\|} = \lambda^{k-1} \frac{f(x+h) - f(x)}{\|h\|}$$

whenever $\lambda > 0$. Thus f is differentiable at λx if and only if it is differentiable at x and $D_i f(\lambda x) = \lambda^{k-1} D_i f(x)$ for all $i = 1, \ldots, n$.

∎

3 Corollary *If f is homogeneous of degree k, then*

$$\frac{D_i f(\lambda x)}{D_j f(\lambda x)} = \frac{D_i f(x)}{D_j f(x)}$$

for $\lambda > 0$ and $x \in \boldsymbol{R}^{\mathrm{n}}_{++}$.

Proof: By Corollary 2 each f_i satisfies $D_i f(\lambda x) = \lambda^{k-1} D_i f(x)$, so

$$\frac{D_i f(\lambda x)}{D_j f(\lambda x)} = \frac{\lambda^{k-1} D_i f(x)}{\lambda^{k-1} D_j f(x)} = \frac{D_i f(x)}{D_j f(x)}.$$

∎

4 Corollary *If f is homogeneous of degree 1 and twice differentiable, then the Hessian matrix $[D_{ij} f(x)]$ is singular for all $x \in \boldsymbol{R}^{\mathrm{n}}_{++}$.*

Proof: By (1),

$$\sum_{i=1}^{n} D_{ij} f(x) x_i = (k-1) D_j f(x).$$

When $k = 1$ this becomes $[D_{ij} f(x)] x = 0$ in matrix terms, so for $x \neq 0$ we conclude that $[D_{ij} f(x)]$ is singular. ∎

5 Theorem (Solution of first order linear differential equations) *Assume P, Q are continuous on the open interval I. Let $a \in I$, $b \in \boldsymbol{R}$.*

Then there is one and only one function $y = f(x)$ that satisfies the initial value problem

$$y' + P(x)y = Q(x)$$

with $f(a) = b$. It is given by

$$f(x) = be^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)}\,dt$$

where

$$A(x) = \int_a^x P(t)\,dt.$$

For a proof see [1, Theorems 8.2 and 8.3, pp. 309–310].

References

[1] Apostol, T. M. 1967. *Calculus*, 2d. ed., volume 1. Waltham, Massachusetts: Blaisdell.