1. (10 points) Consider the function $f(x, y) = 2x^3 + 7y^2x^2$ and the point $A(1, 1)$.
   (a) Calculate the gradient of the function f at the point A.
   (b) Find the second order Taylor approximation in the neighborhood of A.

2. (10 points) Consider the equation $3x^3 + 5y^5 + z^3 + 3z = 12$.
   (a) Check whether the equation defines the function $z(x, y)$ at a point $A(1, 1, 1)$.
   (b) Find dz at the point A.

3. (10 points) Find all local extrema of the function $f(x, y) = x^2y - 4xy^2 + 5xy + 2$ such that $x \neq 0$ and $y \neq 0$. Classify them.

4. (10 points) Find all local constrained extrema of the function $f(x, y, z) = x + 2y + 5z$ subject to $\ln x + \ln y + \ln z = 0$. Do not forget to classify extrema.

5. (10 points) Consider the function $f(x, y) = h(x)g(y) + ax^7$, where h and g are twice differentiable and a is a parameter. Let's denote the maximum point by $x^*(a)$ and $y^*(a)$ and assume that second order conditions for maximization are met.

   Find the sign of dx^*/da.

6. (10 points) The level curves of the function $f(x, y)$ are given by the equation $y - x^2 = c$.

   Draw two level curves of the function $g(x, y) = f(x - 5, |y| + 4)$.

7. Consider a problem

$$\begin{cases} xyz \to \max \\ \text{s.t. } x + y + z = c \\ x, y, z > 0 \end{cases}$$

where c is a parameter and $c > 0$.

(a) (15 points) Solve this problem using first-order conditions. Use bordered Hessian for sufficiency.

(b) (5 points) Use the result of part a) to show that arithmetic mean $(x + y + z)/3$ is no less than the geometric mean $(xyz)^{1/3}$.

8. (a) (10 points) Consider a utility maximization problem

$$\begin{cases} u(x, y) \to \max \\ \text{s.t. } p_x x + p_y y = I \\ x, y > 0 \end{cases},$$

where $u \in C^1$ and parameters p_x, p_y and I are positive. Let (x^*, y^*) be the solution of this problem. Form the value function $V(p_x, p_y, I) = u(x^*, y^*)$.
Using appropriate envelope theorem show that

$$x^* = -\frac{\partial V}{\partial p_x} / \frac{\partial V}{\partial I}, \quad y^* = -\frac{\partial V}{\partial p_y} / \frac{\partial V}{\partial I}.$$

(b) (10 points) Let $F(x, y)$ be a function such that $F \in C^2$ and $F'_y \neq 0$. The equation $F(x, y) = 0$ defines the implicit function $y(x)$.
Find the expression for d^2y/dx^2.
The expression should include only derivatives of $F(x, y)$.