Exams collection. Growing. Mathematics for economists (MFE), Methods of optimal solution (MOS)

October 28, 2019

Contents

1 2008-2009

1.1 MFE, mock, ??.11.08

Calculators are not allowed.

Candidates should attempt:
- all questions from Part A
- two of three questions from Part B.

Part A.

Problem 1. [10]
Find and sketch on the plane the sets:
a) $([1;3) \times [-1;4)) \cap ((2;5] \times (2;5])$
b) $([1;3) \times [-1;4)) \cup ((2;5] \times (2;5])$
c) $([1;3) \times [-1;4)) \setminus ((2;5] \times (2;5])$

Problem 2. [10]
Calculate the directional derivative of the function $f(x,y) = 2x^3 + 2y^2$ at the point $A(1;2)$ in the following directions:
a) $\vec{l} = (1;3)$
b) $\vec{l}$ which is orthogonal to the curve given by the equation $x^2 + y^2 = 5$
c) Direction of the fastest growth of $f(x,y)$

Problem 3. [10]
Consider the following system of equation:
$$\begin{cases} xyzw + 2x^3y^3z^3 + 4w^3 = 7 \\ x + y + z^3 + w^3 + w^2z^2 = 5 \end{cases}$$
a) Does this system define functions $z(x,y)$ and $w(x,y)$ at a point $x = 1$, $y = 1$, $z = 1$, $w = 1$?
b) If it's possible find $\frac{\partial z}{\partial x}$ and $\frac{\partial w}{\partial y}$ at that point

Problem 4. [10]
Find the Hesse matrix of the function $f(x,y) = (2 + cos(x))^{sin(y)+5}$

Comment: clearly state Young theorem if you use it

Problem 5. [10]
Find the total differential for the function $f(x, y) = x^2y^2 + xy^2 + 2x + 4y$
Using the total differential find approximately $f(1.001, 1.999)$

Problem 6. [10]
Using the chain rule find all the first derivatives of $p(a, b)$ where $p(a, b) = f(x(a, b), y(a, b))$, $x(a, b) = a^2+ab$ and $y(a, b) = b^3 + ab$

Part B.

Problem 7. [20]
Find the critical points of the function $f(x, y, z) = x^3 - y^3 + 9xy - z^2e^{-z^2}$
Classify them using the Hesse matrix.

Problem 8. [20]
The individual lives for two periods. He has a utility function $U(c_1, c_2) = u(c_1) + \beta u(c_2)$. In each period his wage is equal to w. His budget constraint requires that his period 1 consumption be his wage w minus any savings, $c_1 = w - s$. The government taxes savings at the rate q per dollar saved. So his second period consumption will be $c_2 = w + (1 + r)s(1 - q)$. The individual takes q, r, w as given. The individual seeks to maximise his utility U.
a) Find the first order condition for the optimal savings amount
b) Using the implicit function theorem find $\frac{\partial s}{\partial q}$
c) Find $\frac{\partial s}{\partial q}$ explicitly if $u(c) = ln(c)$

Problem 9. [20]
A firm sells its output into a perfectly competitive market and faces a fixed price p. It hires labor in a competitive labor market at a wage w, and rents capital in a competitive capital market at rental rate r. The production function is $f(L, K)$. The firm seeks to maximize its profits. Both factors are used in positive amounts in the optimal point.
a) Express the profit π as a function of L and K
b) Find necessary first order conditions for a profit-maximizing point (L^*, K^*)
c) Find the second order conditions sufficient for maximum (two inequalities)
d) Using the implicit function theorem find $\frac{\partial K^*}{\partial w}$. Is it positive or negative if you additionally know that $\frac{\partial^2 \pi}{\partial L \partial K} > 0$?

1.2 MFE, fall semester exam, 21.01.2009

Lecturer: K.A. Bukin, Classteachers: B.B. Demeshev, I.O. Kachkovski
SECTION A. Answer all FOUR questions from this section (60 marks in total)

1. Find the gradient of $f(x, y) = \frac{x^2y}{\sqrt{x^2+y^2}}$ at the point $M(1, 3)$. Compute the derivative of f at M in direction of the vector $\{-1, 1\}$.

2. Calculate all partial derivatives of the first and second order of u with respect to x and y if $u = f(\xi, \eta)$ and $\xi = x + xy$, $\eta = x/y$.

3. Consider the system of equations

$$\begin{cases} x_1^2 + x_1 y_1 + x_2 y_2 = 3 \\ x_2^2 + x_1 y_2 - x_2 y_1 = 1 \end{cases}$$

Are implicit functions $y = y(x)$ and $x = x(y)$ defined around the point $M(1, 1, 1, 1)$? Here y denotes the vector with components y_1 and y_2 and the same notation holds for x. If your answer is affirmative, find $\partial y_1/\partial x_1$ and $\partial x_2/\partial y_2$ at the point M.

4. Find all stationary points of $f(x, y) = x^4 + y^4 - 4xy + 1$. Classify them as local minimum, maximum or saddle point.

SECTION B Answer TWO of the three questions from this section (20 marks each)

5. Consider a problem of identifying Pareto–optimal allocations in the two goods economy with the two agents having identical utility functions $U_1(x_1, y_1) = x_1 y_1$ and $U_2(x_2, y_2) = x_2 y_2$, where x_i and y_i denote respectively the amounts of goods x and y that are allocated to agent $i = 1, 2$. Given a weight $\alpha \in (0, 1)$, the solution of that problem can be found by maximizing the weighted utility function $\alpha U_1 + (1 - \alpha) U_2$ with respect to all the variables subject to the resource constraints $x_1 + x_2 = w_1$, $y_1 + y_2 = w_2$. In this economy w_1, w_2 are positive endowments.

   (a) Using the Lagrangean derive the necessary conditions for extremum and solve the system of equations.

   (b) Without applying the second–order conditions explain why the solution $(x_1^*, x_2^*, y_1^*, y_2^*)$ is the maximum of the problem stated above.

   (c) Calculate the value of the weighted utility function at the Pareto-optimal allocation and compare it with the case when all the endowment in x and y is given to a single agent. How would you explain a difficulty you may find here?

6. A firm's inventory $I(t)$ is depleted at a constant rate per unit time, i.e. $I(t) = x - \delta t$, where x is an amount of good reordered by the firm whenever the level of inventory is zero. The order is fulfilled immediately. The annual requirement for the commodity is 200 units and the firm orders the commodity n times a year where $200 = nx$. The firm incurs two types of inventory costs: a holding cost and an ordering cost. Since the average stock of inventory is $x/2$, the holding cost equals $C_h x/2$, the cost of placing one order is C_o, and with n orders a year the annual ordering cost equals $C_o n$.

   (a) Minimize the cost of inventory $C = C_h x/2 + C_o n$ by choice of x and n subject to the constraint $nx = 200$ by the Lagrange multiplier method.

   (b) Use the envelope theorem to approximate the change in the minimal cost if the requirement for the commodity rises to 204 units.

7. Use the Lagrange multiplier method to prove that the triangle with maximum area that has a given perimeter $2p$ is equilateral. Be sure to justify that the extreme you find is indeed maximum.

   Hint 1: Use Heron's formula for the area:

$$A = \sqrt{p(p - x)(p - y)(p - z)}$$

where x, y and z are the lengths of the sides.

Hint 2: You may find that maximizing A^2 instead of just A will require somewhat more pleasant calculations while giving the same answer.

2 2009-2010

2.1 MFE, fall semester exam, 19.01.2010

Lecturer: K. A. Bukin, Classteachers: A. Arlashin, I. Kachkovski, M. Martinovic
Marks will be deducted for insufficient explanation within your answers. Total duration of the exam is 120 min.
Variant I
SECTION A. Answer all six questions from this section (10 marks each)

1. Consider the function $f(x, y) = (1 + x)/y^2$. On the (x, y)-plane, sketch several level curves of f. Calculate and sketch unit length vectors indicating directions of the most rapid growth of f at points $(1, 1)$ and $(3, -1)$.

2. Use the first-order differential to approximate $\sqrt[3]{4 \cdot 0.9^2 + 2.2^2}$

3. Find the equation of the tangent plane to the surface given by $x^3 + z^3 - 3xz = y - 1$ at the point $(1, 4, 2)$.

4. Find $\partial^2 u/\partial x \partial y$ and $\partial^3 u/\partial x \partial z^2$ if $u = f(s, t)$ and $s = x/y$, $t = y/z$. Assume that f has continuous third-order partial derivatives.

5. Given the system
$$\begin{cases} xe^{u+v} + 2uv = 1 \\ ye^{u-v} - \frac{u}{1+v} = 2x \end{cases}$$
find du and dv at $x_0 = 1$, $y_0 = 2$, $u_0 = 0$, $v_0 = 0$.

6. Use Lagrange multipliers to find the dimension of a rectangular box with the least possible surface area among those with a volume of 27 m^3. Make sure you check the second order condition for minimisation.

SECTION B. Answer both questions from this section (20 marks each)

7. Dr. Cooper is a two-period consumer and his utility is $u(c_0, c_1) = \ln c_0 + 0.8 \ln c_1$, with c_0 the initial period consumption and c_1 the second period consumption. He gets income of Y_0 and Y_1 in these two periods respectively, and can borrow and lend money at the same interest rate of r.

   That means, if Dr. Cooper consumes less than Y_0 in the initial period, then the difference is saved and results in the second period additional income, equal to $(Y_0 - c_0)(1 + r)$, of course. If he decides to consume more than Y_0 in the initial period, he has to borrow and repay the borrowed amount together with interest payments in the second period, thus lowering his second-period consumption (to what amount?).

   (a) Show that irrespective of whether Dr. Cooper goes borrowing or lending, his budget constraint is
   $$c_0(1 + r) + c_1 = Y_0(1 + r) + Y_1$$

(b) Sketch the budget constraint and several indifference curves on the (c_0, c_1)-plane.

(c) Suppose now that $r = 0.4$ and $Y_0 = Y_1 = 10$. Set up the utility optimisation problem with variables c_0 and c_1. Use Lagrange multipliers to solve it. Justify that the point you have found is the point of maximum.

(d) Using the Envelope Theorem, approximate the change in Dr. Cooper's maximum utility if the interest rate rises to 0.45 with unchanged income in both periods.

8. Consider the Edgeworth exchange economy with two agents, Robinson and Friday, and two goods, wheat and fish. Robinson has the utility of possessing w units of wheat and f units of fish equal to $u_R(w, f) = wf$, Friday's utility is $u_F(w, f) = w^2 f$. There is one unit of wheat and one unit of fish in the economy.

(a) Set up (but do not solve) the problem of optimisation of Friday's utility given the fixed level of Robinson's utility, say, $u_R(w, f) = 1/3$. There should be 4 variables (the product levels, or allocations, for Robinson and Friday) and 3 equality constraints.

(b) The problem above can also be set up in 2 variables with 1 constraint: maximize $(1-w)^2(1-f)$ subject to $wf = 1/3$. Use Lagrange multipliers to solve it. Find the optimal product levels of both agents. Do not check the second-order condition.

(c) In the context, explain the geometric significance of the equation

$$\nabla u_R(w, f) = \lambda \cdot \nabla u_F(1-w, 1-f)$$

where λ is some real number.

(d) The optimal levels in (ii) are also the solution of the problem of optimising Robinson's utility given some fixed level of Friday's utility, $u_F(w, f) = a$. What is the value of a? What is the optimal value of Robinson's utility under the given constraint?

2.2 MFE, fall semester exam, solution, 19.01.2010

1.

2.

3.

4.

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial s}\frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial^2 f}{\partial s^2}\frac{-x}{y^2} + \frac{\partial^2 f}{\partial s \partial t}\frac{1}{z}\right)\frac{1}{y} - \frac{1}{y^2}\frac{\partial f}{\partial s}$$

$$\frac{\partial^3 u}{\partial x \partial z^2} = \frac{1}{z^4}\frac{\partial^3 f}{\partial s \partial t^2} + \frac{2}{z^3}\frac{\partial^2 f}{\partial s \partial t}$$

3 2010-2011

3.1 MFE, fall semester exam, 18.01.2011

Variant 4, Classteachers: A.Arlashin, G.Sharygin, S.Provornikov
Section A. Answer all 6 questions from this section (10 marks each)

1. Let $\bar{n}$ be the unit normal at a point $M_0(x_0, y_0, z_0)$ to the sphere represented by the equation $x^2+y^2+z^2 = 16$ in R^3. The point M_0 belongs to that sphere.
   (a) Find the components of $\bar{n}$ as the functions of M_0.
   (b) Find the set of points on the sphere at which the directional derivative of the function $u(x, y, z) = x - z$ in the direction of $\bar{n}$ turns to zero.
2. Suppose $f(x, y)$ is a twice differentiable function. Let x and y be defined in terms of u, v as follows: $x(u, v) = ue^{2v}$, $y(u, v) = u^2 - v^2$. Let $F(u, v) = f(x(u, v), y(u, v))$. Calculate F''_{uu} and F''_{uv}.
3. Find the linear approximation of the function $\cos(e^x+e^y)$ in the neighborhood of the point $(\ln(\pi/4), \ln(\pi/4))$.
4. In the economy producing two goods x and y , the production possibilities set (PPS) is given by the system of inequalities $\left\{x^2 - xy + 2.5y^2 \leq 450,\ x \geq 0,\ y \geq 0\right\}$ By definition the marginal rate of transformation is defined as $MRT = -\frac{dy}{dx}$, where derivative (if it exists) is taken at a point on the boundary of PPS which is called Production Possibilities Frontier.
   (a) By completing perfect squares prove that PPS is a bounded set.
   (b) Using the Implicit Function Theorem find MRT for that economy in terms of x, y. Consider $x > 0$ and $y > 0$.
   (c) Find point(s) at which $MRT = 1$.
5. A two goods producer is a monopolist and it faces the inverse demand functions $p_x = 60 - 2x$ and $p_y = 50 - 3y + x$ where x, y are produced positive quantities. Let the total cost function be $C(x, y) = x^2 + 2y^2 + xy$. Find production levels that maximize firm's profit. Use second-order conditions to verify maximization.
6. In the macroeconomic linear IS-LM model for the closed economy $Y = \bar{C} + m(Y - T) + G + \bar{I} - ar$ and $\bar{L} + bY - cr = M_s$, where M_s is money supply, r — interest rate, G — government expenditures, T — lump sum tax and the constant parameters $\bar{C} > 0$, $0 < m < 1$, $\bar{I} > 0$, $a > 0$, $\bar{L} > 0$, $b > 0$, $c > 0$. Find the formulas for dr/dT, dY/dT. Assume that government expenditures and money supply are fixed exogenous variables.

Section B Answer both questions from this section (20 marks each)

7. In the economy described by the production possibilities set in question four from section A, the world prices on goods are set as follows: $p_x = 5, p_y = 4$. In order to maximize its national income the economy maximizes the objective function $5x+4y$ on the constraint set $\left\{x^2 - xy + 2.5y^2 \leq 450,\ x \geq 0,\ y \geq 0\right\}$. Find the produced quantities in that economy. You may not use the Kuhn-Tucker formulation here. Can the Weierstrass Theorem on attainment of the greatest and the least values be applicable in that maximization problem? Explain.

8. A consumer splits her time $\bar{L}$ hours a week between labor and leisure. Her utility function is represented by $u(c, l) = c^{\alpha}l^{1-\alpha}$, where c is the amount of consumption and l is leisure in hours and $0 < \alpha < 1$. The weekly budget constraint is written as $pc + wl = w\bar{L}$, where p is the price of consumption, w is the hourly wage rate.

   (a) Find the consumer's optimal bundle (c^*, l^*). Justify your answer by checking second-order conditions or otherwise.

   (b) Let $\alpha = 3/4$, $\bar{L} = 168$, $p = 16$, $w = 8$. Using Envelope Theorem estimate the change in the maximum value of her utility if the wage rate has decreased by 0.5.

4 2011-2012

4.1 MFE, mock, 31.10.11

Marks will be deducted for insufficient explanation within your answers. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min.

SECTION A
Answer SIX of the six questions from this section.

1. Estimate value of $\sin^2(61^o) \cdot \tan(31^o)$ by linear approximation using derivatives at 60^o and 30^o. Convert degrees into radians first.

   (a) Degrees to radians. 2 points.

   (b) Appoximation. 8 points.

2. Let D be the domain of the function $f(x, y) = \ln(x) + \sqrt{y - x}$. Find D, the set D^o of internal points of D, the set ∂D of boundary points of D.

   (a) The set D. 4 points.

   (b) Internal points. 3 points.

   (c) Boundary points. 3 points.

3. Consider the function $f(x, y) = x^2 + y^3 - xy + 3y$ at the point $(2; 1)$. Find all the directions in which the growth rate of the function constitutes 60% of the maximal possible growth rate at that point.

   (a) Gradient and its length. 3 points.

   (b) Equation for direction. 3 points.

   (c) Solution of the equation. 4 points. One missing solution implies penalty 2 point.

4. Find the Hesse matrix of $f(x, y) = y \ln(x^2 + y)$. Clearly state Young's theorem if you use it.

   (a) First derivatives. 2 points.

   (b) Second derivatives without the use of Young's theorem. 8 points .

   (c) Second derivatives with the use of Young's theorem. 6 points for derivatives. 2 points for the statement of the theorem.

5. Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, \text{if}(x,y) \neq (0,0) \\ 0, \text{if}(x,y) = (0,0) \end{cases} \tag{1}$$

   Is this function continuous at $(0;0)$?

   (a) 10 points. Answer without proof = 0 points.

6. The value of y is determined as a function of t by the equation

$$\int_0^t f(x,y)\,dx = 1 \tag{2}$$

   Find dy/dt

   (a) Mention of the formula $dy/dt = -\frac{G_t}{G_y}$. 2 points.

   (b) All the rest. 8 points. Answer: $-f(t,y)/\int_0^t f'_y(x,y)\,dx$.

PLEASE TURN OVER

SECTION B

Answer TWO of the two questions from this section.

1. Find all critical points of the function $z = z(x,y)$ implicitly defined by the equation

$$x^2 + y^2 + z^2 - xz - yz + x + y + 4z + 1 = 0 \tag{3}$$

2. Let the production function $q = F(K,L)$ be twice continuously differentiable. Marginal rate of technical substitution is defined by the formula

$$MRTS = \frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}} \tag{4}$$

   The derivatives in the denominator and numerator are taken while the same amount of the output q is fixed. Show that under the conditions $F_L > 0$, $F_K > 0$, $F_{LL} < 0$, $F_{KK} < 0$, $F_{LK} \geq 0$, the marginal rate monotonously declines with the growth of the factor L. In other words $\frac{\partial MRTS}{\partial L} < 0$.

4.2 MFE, fall semester exam, 29.12.11

Lecturer: K. Bukin

Classteachers: B. Demeshev, A. Kalchenko, S. Slavnov, D. Yesaulov

Marks will be deducted fro insufficient explanations within your solutions. Exam lasts for 120 minutes.

Section A. Answer all 6 questions from this section (10 marks each)

1. The function $f(x,y)$ is given by $f(x,y) = u^2(x,y) + v^3(x,y)$. The values of u and v and their gradients at the point $(x,y) = (1,1)$ are also known, $u(1,1) = 3$, $v(1,1) = -2$, $\text{grad}\,u = (1,4)$, $\text{grad}\,v = (-1,1)$. Find $\text{grad}\,f$ if $u,v \in C^1$.

2. Find the Hesse matrix of the function $f(x,y) = \int_{x-3y}^{2x+y} h(t)\ \mathrm{d}t$ if $h \in C^1$.

3. Determine the values of a for which the quadratic form $x^2 + 2axy + 2xz + z^2$ is positive definite, negative definite, positive semidefinite, negative semidefinite, and indefinite.

4. Given the system of two equations $x^2+y^2 = \frac{1}{2}z^2$ and $x+y+z = 2$, find $\frac{\mathrm{d}x}{\mathrm{d}z}$ and $\frac{\mathrm{d}y}{\mathrm{d}z}$ in the neighborhood of the point $(1, -1, 2)$.

5. Find the maxima and minima of the function $f(x,y,z) = xyz$ subject to the constraints $x+y+z = 5$ and $xy+yz+xz = 8$, $x > 0$, $y > 0$, $z > 0$.

6. If the function y is given by $y(x) = \frac{1}{2}(e^x + e^{-x})$, check that $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\sqrt{y^2-1}}$.

Section B. Answer both questions from this section (20 marks each)

7. Consider the following maximization problem $f(x,y,a,b) = ax^2 - x + by^2 - y$, where a and b are real numbers.

   (a) Derive (x^*, y^*) — the point that satisfies the first order conditions.

   (b) Specify the conditions for a and b to ensure that the point (x^*, y^*) is the maximizer.

   (c) Use the Hessian to specify conditions for concavity and convexity of f depending on the values of parameters.

   (d) Solve the comparative statics problem: compute $\frac{\partial y^*}{\partial a}$ and $\frac{\partial y^*}{\partial b}$ as a and b marginally change.

   (e) Using one of the envelope theorems find the rate of change of the value function $f(x^*, y^*, a, b)$ as the result of the marginal change in a and b.

8. Robinson Crusoe splits his time $\bar{L}$ hours a week between labor and leisure. His utility function is represented by $u(c,l) = \alpha \ln c + (1-\alpha)\ln l$, where c is the amount of food and l is leisure in hours and $0 < \alpha < 1$. The food is produced by him in accordance with the production function $c = \sqrt{\bar{L} - l}$.

   (a) Find Robinson's optimal bundle (c^*, l^*) that provides him with the maximum utility (welfare) possible. Justify your answer by checking second-order conditions or otherwise.

   (b) Let $\alpha = 1/4$, $\bar{L} = 168$. Using the envelope theorem, estimate the change in the maximum value of his welfare if his utility function has slightly changed to become $u(c,l) = \frac{1}{5}\ln c + \frac{4}{5}\ln l$.

4.3 MFE, fall semester sols

2 $f_{xx} = 4h'(2x+y) - h'(x-3y)$, $f_{xy} = 2h'(2x+y) + 3h'(x-3y)$, $f_{yy} = h'(2x+y) - 9h'(x-3y)$,

7 (a) $x^* = 1/2a$, $y^* = 1/2b$

   (b) SOC for maximum: $a < 0, b < 0$

   (c) Convex if $a \geq 0, b \geq 0$, concave if $a \leq 0, b \leq 0$.

   (d) $\partial x^*/\partial b = 0$, $\partial x^*/\partial a = -1/2a^2$,

   (e) $\partial f^*/\partial a = x^{*2} = 1/4a^2$

4.4 MFE, fall semester retake, 25.01.12

Lecturer: K. Bukin. Classteachers: B. Demeshev, A. Kalchenko, S. Slavnov, D. Yesaulov

Section A. Answer all 6 questions from this section (10 marks each)

1. For the function $f(x, y) = 2xy + 3$ find the level curves and the equations for their tangents at the points $(1, 2)$ and $(2, 2)$.
2. The population of a certain country grows exponentially, $N_t = N_{1990} \cdot \exp(r(t-1990))$. The population was 70 million in 1990 and 80 million in 2000, what will be the population in 2013?
3. Use the chain rule to find $f'(x)$ and $f''(x)$ for $f(x) = u(a, b, x)$ where $a = \cos(x)$ and $b = x^3$.
4. The system of equations defines $x(z)$ and $y(z)$:

$$\begin{cases} x^2 + zxy + y^2 + 6z + y^3 = 10 \\ y^3x^2 + 3x + 2y + z = 7 \end{cases}$$

   Find $x'(z)$ and $y'(z)$ at the point $x = 1$ and $y = 1$.

5. Consider the function $f(x, y) = x^2 + y^3 - xy + 3y$ at the point $(2; 1)$. Find all the directions in which the growth rate of the function constitutes 60% of the maximal possible growth rate at that point.
6. Consider the objective function $f(x, y) = 4kx^3 + k^2xy + 3ky^4 - 13x - 13y$. The point $(x, y) = (1, 1)$ is the maximum of the function.
   (a) Find the value of k
   (b) Find the approximate increase of the maximum value if k will change by $\Delta k = 0.01$

Section B. Answer both questions from this section (20 marks each)

7. We wish to build a picnic zone for the travellers along a highway. The picnic zone should be rectangular with an area of 1000 m^2 and should have a fence on the three sides not adjacent to the highway. The price of one meter of fence is equal to $ 20.
   (a) Find the dimensions of the picnic area that minimize the fencing costs.
   (b) Using hessian or otherwise check that you have found the costs-minimizing solution.
   (c) Using the Envelope theorem estimate the change in the costs if we increase the area of the picnic zone by 1 m^2.
8. A monopolistic firm with the cost function $TC(Q) = 30 + 15Q + Q^2$ sells a single product in two separate markets. The demand functions for these markets are given by $Q_1 = 25 - P_1$, $Q_2 = 29 - P_2$.
   (a) Find the optimal quantities Q_1, Q_2 to be supplied to the respective markets in order to maximize the profit. Using hessian or otherwise check the second order condition.
   (b) Calculate the point elasticity of demand for each of the three markets. Is it true that the optimal price is negatively related to the absolute value of elasticity at the optimal levels of output?
   (c) Using the Envelope theorem estimate the change in the optimal profit if the demand on the second market changes to $Q_2 = 29.2 - 1.1P_2$.

4.5 MFE, mock exam, 02.04.12

Time allowed 120 minutes.
Students should answer all of the following eight questions. Calculators are not permitted in the exam. Marks will be deducted for insufficient explanations within your answers.
Section A: 10 points each question.

1. Determine whether the function $f(x, y) = \ln(5x + y) - 5(x + y)^2$ is convex (concave up), concave (concave down), strictly convex, strictly concave or neither.

2. Solve the differential equation $y^{(4)} - y = \cos(x)$. The $y^{(4)}$ denotes the forth derivative of y.

3. Consider the monopolist producing two distinct goods. The cost function is given by $TC(q_1, q_2) = q_1 + kq_2$, where the constant $k \in (0; 1)$. And the demand functions are given by $q_1(p_1, p_2) = q_2(p_1, p_2) = (p_1 p_2)^{-3}$.

   (a) Find the optimal production bundle for the monopolist.

   (b) For which values of k one of the product is priced under marginal costs?

4. The density of a standard normal random variable X is given by $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$. Taking the fact that $\int_{-\infty}^{\infty} f(x)\, dx = 1$ for granted calculate $E(X^2)$, $E(X^4)$.

   Hint: if you don't remember, $E(X^n) = \int_{-\infty}^{+\infty} x^n f(x)\, dx$

5. The point elasticity of demand for a good is given by $\varepsilon = p^2/(p^2 + 4p + 3)$. Find the demand function $q(p)$ given the initial condition $q(1) = 1$.

6. Find the values a and b such that the function f is homogeneous:

$$f(x, y) = 2x^{b-a}y^{b+2} + y^{a+1}x^{-3b} + y^{7b}x^{-2a}$$

   For the values of a and b you have found expand the function

$$h(x) = \sqrt{1 + f(x, x)} \cdot (1 - \cos(f(x, x))$$

   as a power series up to x^4. State the range for x where your expansion is correct.

Section B: 20 points each question.

1. It is known that $x_0 = 0$, $x_{100} = 100k$ where $k \in \mathbb{Z}$ is constant and for any $n \in \{2, 3, \dots 100\}$ the following difference equation is satisfied:

$$x_n - 2x_{n-1} + x_{n-2} = -1$$

   (a) Find the particular solution

   (b) Find the maximum value of x_n for $n \in \{0, \dots, 100\}$ as a function of k.

2. Using the Lagrange multiplier method without reducing the number of variables by substitution find the minimum of the function

$$f(x, y, z) = 2x^2 + 4y^2 + xy + 8z^2 + 2yz$$

   subject to $x + y + 1.5z \geq 1.2$ and $x + y + z = 1$.

4.6 MOR, 22.05.12

Section A. Solve **two** of the following **two** problems

1. Use Lagrange multipliers method to solve optimization problem
$$xy \to \max$$
subject to $x \geq 0$, $0 \leq y \leq 3$, $x + 2y \leq 8$, $y \geq \frac{x^2}{16} + 1$.

2. Solve the linear program depending on the parameter β,
$$2x_1 + 4x_2 + 5x_3 - x_4 \to \min$$
subject to $x_1 + x_3 - x_4 \geq 0$, $-x_1 + x_2 + \frac{1}{2}x_3 + \beta x_4 \geq 1$, $x_i \geq 0$.
For what values of β the minimum of the objective function equals 3?

Section B. Solve **two** of the following **three** problems

3. Find the general solution of the differential equation $y'' + 2y' + y = xe^{-x} + \cos(x)$

4. Solve the initial-value problem for the system of difference equations
$\begin{cases} x_{t+1} = x_t + y_t \\ y_{t+1} = 3x_t - y_t - 5 \end{cases}$, where $x_0 = y_0 = 0$.

5. In the model of interacting inflation and unemployment based on the Phillips relation, both unemployment rate U and expected inflation π are the solutions of the system $\dot{\pi} = \frac{3}{4}(p - \pi)$, $\dot{U} = -\frac{1}{2}(m - p)$, where m is exogenously defined positive rate of nominal money growth and p is the posteriori observed inflation satisfying equation $p = \frac{1}{6} - 3U + \pi$. Find the steady-state solutions for the inflation both expected and observed as well as the unemployment rate in terms of m. Explore the dynamic stability of solutions. What is the natural rate of unemployment?

Section C. Solve **two** of the following **three** problems

6. Find all pure and mixed Nash equilibria in the following bimatrix game:

	D	E	F
A	3;4	1;3	1;0
B	2;7	3;6	0;3
C	0;2	2;1	5;6

7. Two players play a version of Rock-Paper-Scissor game. Paper beats Rock, Rock beats Scissors, Scissors beats Paper. The two players simultaneously make their choice. The first player can choose any object. The second player can choose Rock or Paper. The winner receives 1 rouble from the loser. In case of a draw the wealth of a player does not change.

   (a) Construct the payoff matrix of the game.

(b) Find all pure and mixed Nash equilibria

8. Two players are trying to bribe the judge. The possible amount of bribe is any real number between 0 and 1 million roubles. The player who gives the biggest bribe is announced as the winner of the affair by the judge. The winner receives 1 million roubles. The loser gets nothing. Obviously bribes are not returned by the judge. In the case of equal bribes each player gets nothing.

   (a) Are there any pure Nash equilibria in this game?

   (b) Find at least one mixed Nash equilibrium.

   Short tips:

   (a) No

   (b) If the second player choses his move according continuous distribution function F then the expected payoff of the first player for the bribe b is equal to

   $$1 \cdot P(b_2 \leq b) - b = 1 \cdot F(b) - b$$

   If a rational player uses mixed strategies he is indifferend between the corresponding pure strategies. That means that $F(b) - b = const$ for pure strategies that are mixed. For pure strategies that are mixed the density function $f(b) = F'(b) = 1$. Do I know such a random variable? Yes, I know! A uniform on $[0; 1]$.

4.7 MFE, retake exam, 19.09.2012

Time allowed 120 minutes.
Students should answer all of the following eight questions. Calculators are not permitted in the exam. Marks will be deducted for insufficient explanations within your answers.
Section A: 10 points each question.

1. It is known that the functions $f_1(x)$ and $f_2(x)$ are concave up. Is it possible that the function $h(x) = \max\{f_1(x), f_2(x)\}$ is concave down?

2. The implicit function $z(x, y)$ is given by the equation $x - z = f(y - z)$ where f is some unknown differentiable function. Find $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$.

3. Consider the monopolist producing two distinct goods. The cost function is given by $TC(q_1, q_2) = q_1 + kq_2$, where the constant $k \in (0; 1)$. And the demand functions are given by $q_1(p_1, p_2) = q_2(p_1, p_2) = (p_1 p_2)^{-3}$.

   (a) Find the optimal production bundle for the monopolist.

   (b) For which values of k one of the product is priced under marginal costs?

4. The density of an exponential random variable X is given by $f(x) = \exp(-x)$ for $x > 0$. Calculate $E(X)$, $E(X^2)$.

   Hint: if you don't remember, $E(X^n) = \int_{-\infty}^{+\infty} x^n f(x)\, dx$

5. The point elasticity of demand for a good is given by $\varepsilon = p^2/(p^2 + 4p + 3)$. Find the demand function $q(p)$ given the initial condition $q(1) = 1$.

6. Find the values a and b such that the function f is homogeneous:

$$f(x, y) = 2x^{b-a}y^{b+2} + y^{a+1}x^{-3b} + y^{7b}x^{-2a}$$

For the values of a and b you have found expand the function

$$h(x) = \sqrt{1 + f(x, x)} \cdot (1 - \cos(f(x, x))$$

as a power series up to x^4. State the range for x where your expansion is correct.

Section B: 20 points each question.

1. Use Lagrange multipliers method to solve optimization problem

$$x + \ln y \to \max$$

subject to $x \geq 0$, $x + y \leq 4$, $x + 2y \leq 6$.

2. Let Y_t, C_t, I_t denote national income, consumption, and investment in period t respectively. The economy is described by the system

$$\begin{cases} Y_t = C_t + I_t \\ C_t = c + mY_t \\ Y_{t+1} = Y_t + rI_t \end{cases} \tag{5}$$

, where c, m and r are positive constants.

(a) Find the function Y_t

(b) Find the asymptote of $\ln(Y(t))$ as t tends to infinity.

4.8 MOR, retake exam, 11.09.12

Section A. Solve **two** of the following **two** problems

1. Use Lagrange multipliers method to solve optimization problem

$$x + \ln y \to \max$$

subject to $x \geq 0$, $x + y \leq 4$, $x + 2y \leq 6$.

2. Solve the linear program depending on the parameter β,

$$2x_1 + 4x_2 + 5x_3 + x_4 \to \min$$

subject to $x_1 + x_3 + x_4 \geq 0$, $-x_1 + x_2 + \frac{1}{2}x_3 - \beta x_4 \geq 1$, $x_i \geq 0$.
For what values of β the minimum of the objective function equals 3?

Section B. Solve **two** of the following **three** problems

3. Find the general solution of the differential equation $y'' + 4y' + 4y = xe^{-3x} + \cos(x)$

4. Consider the system of difference equations

$$\begin{cases} x_{t+1} = 2x_t - 4y_t \\ y_{t+1} = x_t - 3y_t + 3 \end{cases}$$

   (a) Solve the system

   (b) Find the equilibrium solution and check whether it's stable

5. A policymaker desires to double in 10 periods of time the value of GDP y_t produced in period t. Evolution of GDP over time is given by equation $4y_{t+2} - 4y_{t+1} + y_t = 2^t + t^2$. Is doubling of GDP feasible? If the answer is positive, is it possible to find the period t when the value of y_t will first exceed $2y_0$, where y_0 is the initial GDP?

Section C. Solve **two** of the following **three** problems

6. Find all pure and mixed Nash equilibria in the following bimatrix game:

	D	E	F
A	5;5	2;4	2;1
B	3;8	4;7	1;4
C	1;3	3;2	6;7

7. A man has two sons. When he dies, the value of his estate after tax is \$1000. In his will it states that the sons must specify the sum of money s_i that they are willing to accept. If $s_1 + 2s_2 \leq 1000$, then each gets the sum he asked for and the rest goes the cats' shelter. If $s_1 + 2s_2 > 1000$, then neither of them gets any money and the entire sum goes to the cats' shelter. Assume that the sons only care about the money they will inherit and they ask for the whole dollars. Find the pure strategies Nash equlibria of this game.

8. Two players are trying to bribe the judge. The possible amount of bribe is any real number between 0 and 1 million roubles. The probability that the player will win is proportional to the amount of the bribe. In the case of zero bribes the probability is equal $1/2$ for each player. The winner receives 1 million roubles. The loser gets nothing. Obviously bribes are not returned by the judge.

   (a) Are there any pure Nash equilibria in this game?

   (b) Find at least one mixed Nash equilibrium.

5 2012-2013

5.1 MFE, mock, 25.10.12

Marks will be deducted for insufficient explanation within your answers. All problems are mandatory. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min.

SECTION A

1. Consider the function $f(x,y) = x^3 + y^2 + xy - y^5$. Using the total differential find the approximate value of $f(1.01, 0.98)$.

2. Consider the function $f(x,y) = x^3 + y^2 + xy - y^5$ at the point $A = (1,1)$. I have two choices:

   (a) move from A in the direction $\vec{l} = (2,1)$ by a small number ε

   (b) move from A in the direction $\vec{m} = (1,2)$ by 2ε

   Using the directional derivative find which choice will give me a bigger value of the function f at the destination point.

3. Consider the following system of equations:
$$\begin{cases} xyz + 2x^3y^3z^3 + 4x^3 = 7 \\ x + y^3 + z^3 + xy^3 + 2x^2z^2 = 6 \end{cases}$$

   (a) Does this system define functions $z(x)$ and $y(x)$ at a point $x = 1$, $y = 1$, $z = 1$?

   (b) If it's possible find $y'(x)$ and $z'(x)$ at that point

4. Determine whether the following limit exists
$$\lim_{x,y \to 0} \frac{x^3 + y^3}{x^2 + y^2}$$

5. Consider the function $g(u) = f(x,y)$, $x = 2u$ and $y = u - u^2$. Find $g'(u)$ and $g''(u)$. Assume that f has continuous second partial derivatives at any point.

6. Find the equation of a tangent plane to the surface $z = x + y^2$ at the point $(0,1,1)$.

SECTION B

7. Find all the stationary points of the implicit function $z(x,y)$ given by the equation $x^2 + y^2 + z^2 + xy + xz + 2 = 4x + 3y$.

8. Consider a market with the demand curve $q^d = f(p)$ and the supply curve $q^s = g(p,a)$ where the parameter a describes the available technology. The goal is to find how will the equilibrium price p^* and quantity q^* react to the change of the technology parameter a. We assume that $f'(p) < 0$, $\partial g/\partial p > 0$ and $\partial g/\partial a > 0$.

   (a) Find dp^*/da, dq^*/da

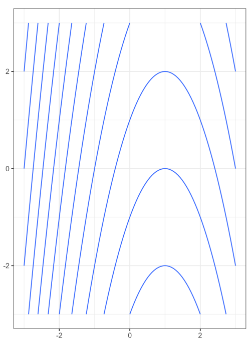
   (b) If possible determine the sign of the derivatives dp^*/da and dq^*/da.

5.2 MFE, fall semester exam, 27.12.2012

Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min.

SECTION A:

1. The level curves of a function $f(x, y)$ are shown below. Draw the level curves of the functions $g(x, y) = f(x + 1, y)$ and $h(x, y) = f(-x, |y|)$.



2. Find the local maxima and minima of the function $f(x, y) = x^4 + 2y^4 - xy$. Determine whether the extrema you have found are global or local.

3. Find the gradient of the function $h(x, y) = f(x, y) \cdot g(x, y)$ at the point $A = (1, 7)$. It is known that at the point A: grad $f = (1, 1)$, grad $g = (3, 3)$, $f(A) = 4$, $g(A) = 5$.

4. Consider the following system of equations as defining functions $y_1(x_1, x_2)$ and $y_2(x_1, x_2)$

$$\begin{cases} x_1^3 + x_1 y_1^3 + x_2 y_1 y_2 + y_2^3 = 4 \\ x_2 + x_2^3 + y_1 y_2^2 + y_2^3 = 4 \end{cases}$$

   (a) If possible find dy_1 at the point $(x_1, x_2, y_1, y_2) = (1, 1, 1, 1)$.

   (b) Find approximately y_1 for $x_1 = 1.01$ and $x_2 = 0.98$.

5. Find the constrained extrema of the function $f(x, y) = x + 2y$ subject to $2x^2 + y^2 = 10$.

6. Find the Hesse matrix of the function $h(x, y) = \P(Z \in [2x; 3y])$ where Z is a standard normal random variable with probability density function given by $f(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$. It is supposed that $3y > 2x$.

SECTION B:

1. A firm's production function is $Q = K + L + 2\sqrt{KL}$, where $K > 0$ and $L > 0$ are capital and labor, respectively. The firm is perfectly competitive and seeks to maximize its output, but the firm is run by accountants who have imposed a fixed budget on the production of C dollars per hour, which means satisfying constraint $wL + rK = C$, where w and r are hourly wage rate and rental rate of capital, respectively.

   (a) State the constrained optimization problem associated with that production and solve it by the Lagrange multiplier method (it is sufficient to find the optimal values of capital and labor alone).

   (b) Check the concavity of the production function and use it to classify the critical point.

(c) Explain the economic meaning of the Lagrange multiplier in this problem. Use the appropriate envelope theorem.

(d) Let $w = r$. Would it be right to conclude that if the wage rate goes up by 1% and the rental rate goes down by the same 1% under the fixed budget the output will not change? How can you prove this mathematically?

2. It is well known that a perfectly competitive firm operating in the long-run produces at the minimum point of its average costs curve, where $p = AC = MC$. Let the AC curve be U-shaped. If we assume that this particular firm uses only labor, equation $AC(y, w) = MC(y, w)$ may be used to find $y = y(w)$ as an implicit function of wages (y denotes the output). Moreover, the second-order condition that guarantees the profit maximization is supposed to hold.

   (a) Show that Implicit Function Theorem can be applied, the function $y = y(w)$ exists and find its derivative.

   (b) If we know that at long-run equilibrium $\frac{\partial MC}{\partial w} > \frac{\partial AC}{\partial w}$ what can we say about new long-run equilibrium when the market adjusts to a small rise in wage w? Will the equilibrium price go up? What about the output of the firm?

5.3 MFE, fall retake 23.01.13

Part A, 10 points for each problem.

1. It is known that $f'_x(x, y) > 0$ and $f'_y(x, y) > 0$. Sketch possible level curves for $f(x, y)$. What are the possible values of the angle between the gradient of the function f and the x-axis?

2. Find the local maxima and minima of the function $f(x, y) = x^4 + 4y^4 - xy$. Determine whether the extrema you have found are global or local.

3. Find and classify the constrained extrema of the function $f(x, y) = x + 5y$ subject to $2x^2 + y^2 = 10$.

4. Given the system
$$\begin{cases} xe^{u+v} + 2uv = 1 \\ ye^{u-v} - \frac{u}{1+v} = 2x \end{cases}$$
find du and dv at $x_0 = 1$, $y_0 = 2$, $u_0 = 0$, $v_0 = 0$.

5. Find the total differential for the function $f(x, y) = x^2y^2 + xy^2 + 2x + 4y$. Using the total differential find approximately $f(1.001, 1.999)$

6. Use the chain rule to find $f'(x)$ and $f''(x)$ for $f(x) = u(a, b, x)$ where $a = \cos(x)$ and $b = x^3$.

Part B, 20 points for each problem.

7. A consumer maximizes the quasilinear utility function $u(x, y) = v(x) + y$, where $v' > 0$, $v'' < 0$, subject to the budget constraint $px + y = I$.

(a) (10 points) Denote the demand on x by x^*. Show that $\frac{\partial x^*}{\partial I} = 0$ and $\frac{\partial x^*}{\partial p} < 0$.

(b) (10 points) Let $V = u(x^*, y^*)$, where (x^*, y^*) is the optimal bundle. Assuming that $y^* > 0$ and using the appropriate envelope theorem, show that $\frac{\partial V}{\partial I}$ is constant.

8. A two-product firm produces outputs y_1 and y_2 from a single factor of production which is labor, in other words, there is a function f, such that $f(y_1, y_2) \leq \bar{L}$. Output prices are p_1 and p_2. The firm has a fixed amount of labor supply $\bar{L} > 0$ that should be utilized in full.

   (a) (10 points) Set the problem of the revenue maximization under the labor constraint mathematically and derive first-order conditions. Assume that both outputs should be produced in positive amounts.

   (b) (10 points) Let the maximum value of the total revenue under the labor constraint be $TR(p_1, p_2, \bar{L})$. What are its derivatives with respect to the prices?

5.4 MFE, mock, 28.03.2013

Marks will be deducted for insufficient explanation within your answers. All problems are mandatory. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min.

SECTION A

1. Compute all the roots of the complex number, $\sqrt[5]{-1+i}$

2. The matrix A has the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = -2$.

   (a) Compute the eigenvalues of the following matrices: $B = A^2$, $C = A + 3I$, $D = A^{-1}$ where I is the corresponding identity matrix.

   (b) What can be said about the definiteness of these matrices?

3. Consider the implicit function $y(x)$ given by the equation

$$y^3 + y + 3x^3 + x^2 = 14.$$

   (a) Does this equation define the implicit function $y(x)$ in the neighborhood of the point $(1, 2)$?

   (b) If the implicit function is defined find the Taylor series for $y(x)$ up to the second order term.

4. Consider the difference equation $y_{t+1}(2 + 3y_t) = 4y_t$ with initial condition $y_0 = 2/3$.

   (a) Using the substitution $z_t = 1/y_t$ solve the difference equation.

   (b) What is the limit of y_t as $t \to \infty$?

5. The homogeneous function f is given by the equation

$$f(x, y) = \int_0^{xy^a} t^3 + xt \, dt.$$

   Find the value of the parameter a and the degree of homogeneity of $\partial f / \partial x$.

6. Find the two indefinite integrals $\int e^x \cos(2x)\, dx$, $\int \frac{x+1}{x^2-5x+4}\, dx$

SECTION B

7. Suppose you enter a casino with k dollars in your pocket. You decide to play a game in which you win \$1 with the probability $2/3$ and lose \$1 dollar with the probability $1/3$. The game is over when $k = 0$ (no money left) or $k = 100$.

   Denote the probability to win the game, i.e. reaching $k = 100$, starting with k dollars as $x_k = \P(win|k)$. Using the total probability formula

   $$\P(win|k) = \frac{1}{3} \cdot \P(win|k-1) + \frac{2}{3} \cdot \P(win|k+1)$$

   derive the difference equation for x_k and solve the boundary-value problem with $x_0 = 0$ and $x_{100} = 1$.

8. Let $N(t)$ denote the size of population, $X(t) = \sqrt{N}$ the total output in the economy. Consider the following model

   $$\frac{\dot{N}}{N} = \alpha - \beta \frac{N}{X}$$

   where $\alpha > 0$, $\beta > 0$. Find N and explore its behavior as $t \to \infty$.

5.5 MOR, exam, 22.05.2013

You need to solve exactly FIVE problems out of 7. At least ONE problem from each section should be chosen for solving. Each question is worth 20 points.

SECTION A

1. A two-product monopoly seeks to maximize its profit. The revenue follows the formula $R(x, y) = 6x - x^2 + y - y^2$. The total costs function is given by $C(x, y) = x^2 + y^2 + 4x + 3y$, where x and y are the outputs.

   State the problem of the monopoly, given condition that its profit $\pi = R - C$ should always remain nonnegative. Apply the Kuhn-Tucker conditions. Use Weierstrass theorem to confirm sufficiency.

2. For any real number λ, find the minimal value of the objective function $x_1 + 6x_2 + 2x_3 + 4x_4$ subject to the constraints $\lambda x_1 + x_2 - x_3 + x_4 \geq -1$, $x_1 + 1.5x_2 + x_3 + x_4 \geq 5$, all choice variables are nonnegative.

SECTION B

3. Solve the system of differential equations:

   $$\begin{cases} \dot{x} = x - y \\ \dot{y} = 2x - y \end{cases}$$

4. Given the Metzler equation of inventory cycles $y_t = 3by_{t-1} - 2by_{t-2} + N$, where $0 < b < 1$ and N is a number, find all such values of b for which solution represents convergent stepped time path (the complex roots case). Does the value of N affect your conclusion?

5. Show that Chebyshev's equation $(1-x^2)y'' - xy' + y = 0$, where $|x| < 1$, can be reduced to equation $\ddot{y} + y = 0$ by substituting $x = \cos t$. Hence find the general solution of Chebyshev's equation.

SECTION C

6. Find all pure and mixed Nash equilibria in the following bimatrix game:

	d	e	f
a	4;5	1;4	1;1
b	2;8	5;0	0;4
c	0;3	2;2	5;7

7. There is an auction of a painting with two players. The value of the painting for the first player is a random variable v_1, for the second player — v_2. The random variables v_1 and v_2 are independent and uniformly distributed from 0 to 1 million dollars. Each player makes the bid b_i knowing only his own value of the painting. The player who makes the highest bid gets the painting and pays the arithmetic mean of the two bids.

   Find a Nash equilibrium where each player uses linear strategy of the form $b_i = k \cdot v_i$.

5.6 MOR, marking scheme, 22.05.2013

1. 5 points for setting Kuhn-Tucker Lagrangian correctly and writing down first-order conditions. Another 3 points for showing that the only constraint in the problem is binding. 5 points for proving that an internal critical point ($x > 0$, $y > 0$) does not exist. Plus 3 points for finding corner critical point. And 4 points for showing that Weierstrass theorem is applicable.

2. 10 points for conversion to the dual program and correct analysis of the feasible region. The rest 10 points for maximizing objective function.

3. 10 points for either finding eigen values +eigen vectors or rewarding the students who managed to reduce the system to one equation and solved it successfully. 10 points for finding general solution.

4. 5 points for finding roots of the characteristic equation. Another 10 points for finding interval for b that provides convergent stepped time path. 5 points for conclusion that for these b values the particular integral is represented by a constant and thus the value of N does not affect the answer.

5. 10 points for correct chain rule differentiation that reduces Chebyshev equation to the linear equation with the constant coefficients. Another 10 points for solving the latter and finding after substitution $y = c_1\sqrt{1-x^2} + c_2 x$.

6. Correct elimination of strictly dominated strategies — 4 points. Correct picture of best response functions — 12 points. The rest — 4 points. If only pure Nash Equilibria are found — the total is 4 points.

7. 10 points for the payoff function of the first player given that his valuation of the painting is v_1 and his bid is b_1. If we denote by W the win of the first player than the expected utility of the first player is given by:
$$P(W)\left(v_1 - E[(b_1+b_2)/2 \mid W]\right) = \frac{b_1}{k}\left(v_1 - \frac{b_1+0.5b_1}{2}\right)$$
5 points for stating FOC and 5 points for obtaining strategy from FOC. The final answer is $b_i = \frac{2}{3}v_i$.

5.7 MOR, retake exam, 14.09.2013

You need to solve exactly FIVE problems out of 7. At least ONE problem from each section should be chosen for solving. Each question is worth 20 points.

SECTION A

1. Maximise the function $f(x,y) = x + ay$, subject to $1 - x^2 \geq y^2$, $x + y \geq 0$ for all values of a using the Lagrange multipliers approach.

2. For any real number λ, find the minimal value of the objective function $x_1 + 6x_2 + 2x_3 + 4x_4$ subject to the constraints $\lambda x_1 + x_2 - x_3 + x_4 \geq -1$, $2x_1 + 3x_2 + 2x_3 + 2x_4 \geq 10$, all the choice variables are nonnegative.

SECTION B

3. Solve the system of differential equations:
$$\begin{cases} \dot{x} = 2x + y + 1 \\ \dot{y} = -2x + 2y \end{cases}$$

4. Solve the difference equation $y_{t+2} - 3y_{t+1} + 2y_t = \sin(\pi t/2)$ and determine whether the solution paths are convergent or divergent.

5. In the model of interacting inflation and unemployment based on the Phillips relation, both unemployment rate U and expected inflation π are the solutions of the system $\dot{\pi} = \frac{3}{4}(p - \pi)$, $\dot{U} = -\frac{1}{2}(m - p)$, where m is exogenously defined positive rate of nominal money growth and p is the posteriori observed inflation satisfying equation $p = \frac{1}{6} - 3U + \pi$. Find the steady-state solutions for the inflation both expected and observed as well as the unemployment rate in terms of m. Explore the dynamic stability of solutions. What is the natural rate of unemployment?

SECTION C

6. Find all pure and mixed Nash equilibria in the following bimatrix game:

	d	e	f
a	5;5	2;6	0;4
b	2;1	0;2	4;3
c	0;4	3;5	1;0

7. Two players have found The Magic Box. The Box has two holes. Simulteneously each of the two players may put any amount of money from 0 to 100 euros into his hole. Then the Magic Box will multiply the total sum by $a > 1$, divide the resulting sum into two equal parts and give them back to the players. The value of a is known. Find all the pure and mixed Nash Equilibria of this game for all values of the parameter a.

6 2013-2014

6.1 MFE, mock, 31.10.13

Marks will be deducted for insufficient explanation within your answers. All problems are mandatory. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min.

SECTION A

1. Find the angle between the gradients of the function $f(x, y, z) = x^3 + xyz - 2z^3$ at the points $(1, 2, -1)$ and $(0, 1, 2)$.

2. Consider the function $f(x, y, z) = x^3 + xyz - 2z^3$. Using the total differential find the approximate value of $f(1.02, 0.99, -0.98)$.

3. Consider the function $f(x, y, z) = x^4 + (x + y)^2 + (x + z)^3$.

   (a) Find the Hesse matrix. Clearly state the Young theorem if you use it.

   (b) Find all the points where the Hesse matrix is positive definite.

4. Consider the following system of equation:

$$\begin{cases} 3xyzw + 2x^3y^3z^3 + 4w^3 = 9 \\ 5x + y + z^3 + w^3 + w^2z^2 = 9 \end{cases}$$

   (a) Does this system define functions $z(x, y)$ and $w(x, y)$ at a point $x = 1$, $y = 1$, $z = 1$, $w = 1$?

   (b) If it's possible find $\frac{\partial z}{\partial x}$ and $\frac{\partial w}{\partial y}$ at that point

5. Function $z(x, y)$ is given by the equation $z(x, y) = f(x^2 + y^2)$. Simplify $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y}$.

6. A production function $y = f(x_1, x_2)$ exhibits constant returns to scale, that is $f(tx_1, tx_2) = tf(x_1, x_2)$ for every $t > 0$, where $x_1, x_2 \geq 0$. Let $f_1(x_1, x_2) = \frac{\partial f}{\partial x_1}$ and $f_2(x_1, x_2) = \frac{\partial f}{\partial x_2}$. Find $\frac{\partial f_1(tx_1, tx_2)}{\partial t}$ and $\frac{\partial f_2(tx_1, tx_2)}{\partial t}$.

SECTION B

7. A person has an option to buy a x_0 units of good at a price p_0 per unit. She can use her leisure time seeking a lower price $p(t) < p_0$, where t is time spent on search. Her gain of finding a lower price provided the cost of search is wt, where w is the wage rate, can be evaluated by the formula $g(t) = (p_0 - p(t))x_0 - wt$.

   (a) (10 points) Set the problem for maximizing the gain. Find the first-order condition. Assume that $p' < 0$ and $p'' > 0$.

   (b) (10 points) Let t^* be optimal time of search. Using IFT find the formula for $\frac{\partial t^*}{\partial x_0}$ and determine its sign.

8. (20 points) Find dz and d^2z if the function z is defined by the equation $F(x/z, y/z) = 2013$. You may denote the derivatives of F with respect to the first and the second arguments by F_1 and F_2 correspondingly.

6.2 MFE, mock, 31.10.13, Solution and marking scheme, A. Kalchenko, D. Esaulov

1. Three partial derivatives — 4 points, Two gradient vectors — 3 points, cosine of the angle — 2 points, angle itself — 1 point.

   Solution.

$$\nabla f = (6x^2 + 3yz, 3xz, 3xy - 3z^2)$$
$$g_1 = \nabla f(2, 1, -1) = (21, -6, 3)$$
$$g_2 = \nabla f(0, 1, -2) = (-6, 0, -12)$$
$$\cos(\widehat{g_1, g_2}) = \frac{(g_1, g_2)}{\|g_1\|\|g_2\|} = \frac{-162}{\sqrt{486} \cdot \sqrt{180}} = -\sqrt{\frac{3}{10}}$$
$$(\widehat{g_1, g_2}) = \arccos(-\sqrt{\frac{3}{10}}) \approx 123^\circ$$

2. Three partial derivatives — 4 points (answers from the previous question may be used). Approximate Δf — 4 points. Approximate f — 2 points.

   Solution.

$$f(1, 1, -1) = 0$$
$$\nabla f = (6x^2 + 3yz, 3xz, 3xy - 3z^2)$$
$$\nabla f(1, 1, -1) = (3, -3, 0)$$
$$\Delta f \approx (\nabla f)^T \cdot \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = (3, -3, 0) \cdot \begin{pmatrix} 0.01 \\ -0.01 \\ 0.02 \end{pmatrix} = 0.06$$
$$f(1.01, 0.99, -0.98) \approx f(1, 1, -1) + \Delta f = 0.06$$

3. Hesse matrix — 5 points. Young theorem penalty — (-1) point. Small b — 5 points.

   Solution.

$$\nabla f = \left(8x^3 + 2(x + y) + 3(x + z)^2, 2(x + y), 3(x + z)^2\right)$$

$$D^2f = \begin{pmatrix} 2+24x^2+6(x+z) & 2 & 6(x+z) \\ 2 & 2 & 0 \\ 6(x+z) & 0 & 6(x+z) \end{pmatrix}$$

According to the Sylvester's criterion the matrix is positive definite iff $\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0$:

$$\begin{cases} \Delta_1 &= 2+24x^2+6(x+z) > 0 \\ \Delta_2 &= 48x^2+12(x+z) > 0 \\ \Delta_3 &= 288x^3+288x^2z > 0 \end{cases} \iff \begin{cases} 12x^2+3(x+z)+1 > 0 \\ 4x^2+(x+z) > 0 \\ x^2(x+z) > 0 \end{cases}$$

The Hesse matrix is positive definite at the points (x, y, z) s.t. $x \neq 0, x+z > 0$

4. Sufficient conditions for the existence of the implicit function — 2 points. Two derivatives — 8 points.

   Solution.

   Let $f_1 = 4xyzw + 2x^3y^3z^3 + 4w^3 - 10$, $f_2 = 5x + y + z^3 + w^3 + w^2z^2 - 9$.

   (a) Check the conditions of the IFT:

      i. $f_1(1,1,1,1) = 0$, $f_2(1,1,1,1) = 0$

      ii. $f_1, f_2 \in C^1$

      iii. $$\left|\frac{\partial(f_1,f_2)}{\partial(z,w)}\right| = \begin{vmatrix} 4wxy+6x^3y^3z^2 & 12w^2+4xyz \\ 2w^2z+3z^2 & 3w^2+2wz^2 \end{vmatrix} = \begin{vmatrix} 10 & 16 \\ 5 & 5 \end{vmatrix} = -30 \neq 0$$

   (b) $$\frac{\partial z}{\partial x} = -\frac{\left|\frac{\partial(f_1,f_2)}{\partial(x,w)}\right|}{\left|\frac{\partial(f_1,f_2)}{\partial(z,w)}\right|} = -\frac{\begin{vmatrix} 4wyz+6x^2y^3z^3 & 12w^2+4xyz \\ 5 & 3w^2+2wz^2 \end{vmatrix}}{\begin{vmatrix} 4wxy+6x^3y^3z^2 & 12w^2+4xyz \\ 2w^2z+3z^2 & 3w^2+2wz^2 \end{vmatrix}} = -\frac{\begin{vmatrix} 10 & 16 \\ 5 & 5 \end{vmatrix}}{\begin{vmatrix} 10 & 16 \\ 5 & 5 \end{vmatrix}} = -1$$

   $$\frac{\partial w}{\partial y} = -\frac{\left|\frac{\partial(f_1,f_2)}{\partial(z,y)}\right|}{\left|\frac{\partial(f_1,f_2)}{\partial(z,w)}\right|} = -\frac{\begin{vmatrix} 4wxy+6x^3y^3z^2 & 4wxz+6x^3y^2z^3 \\ 2w^2z+3z^2 & 1 \end{vmatrix}}{\begin{vmatrix} 4wxy+6x^3y^3z^2 & 12w^2+4xyz \\ 2w^2z+3z^2 & 3w^2+2wz^2 \end{vmatrix}} = -\frac{\begin{vmatrix} 10 & 10 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 10 & 16 \\ 5 & 5 \end{vmatrix}} = -\frac{4}{3}$$

5. Probably the easiest ;) Two derivatives — 8 points, simplify — 2 points.

   Solution.

   $$\frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot 2x$$
   $$\frac{\partial z}{\partial y} = f'(x^2+y^2)\cdot 2y$$
   $$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y\cdot 2xf'(x^2+y^2) - x\cdot 2yf'(x^2+y^2) = 0$$

6. Showing that $f_i(tx_1, tx_2) = f_i(x_1, x_2)$ — 6 points, obtaining two zeros — 4 points. Typical given solutions: Using the fact that $f_i(tx_1, tx_2) = f_i(x_1, x_2)$ without proof — penalty (-4). Correct expression for $\frac{\partial f_i(tx_1,tx_2)}{\partial t}$ ignoring the fact that f has constant returns to scale — 4 points in total.

Solution. Notice that $f_1(tx_1, tx_2) = \frac{\partial f}{\partial x_1}(tx_1, tx_2)$. It means that you differentiate w.r.t. x_1 in the first place and only then you substitute the point (tx_1, tx_2)

Method 1. Differentiate the identity $f(tx_1, tx_2) = tf(x_1, x_2)$ w.r.t. x_1 by chain rule. You get

$$f_1(tx_1, tx_2) \cdot \underbrace{\frac{\partial(tx_1)}{\partial x_1}}_{=t} + f_2(tx_1, tx_2) \cdot \underbrace{\frac{\partial(tx_2)}{\partial x_1}}_{=0} = tf_1(x_1, x_2).$$

Divide both sides by t. You get

$$f_1(tx_1, tx_2) = f_1(x_1, x_2).$$

By analogy $f_2(tx_1, tx_2) = f_2(x_1, x_2)$. Thus $f_i(tx_1, tx_2),\ i = 1, 2$, are independent of t and

$$\frac{\partial f_1(tx_1, tx_2)}{\partial t} = \frac{\partial f_2(tx_1, tx_2)}{\partial t} = 0$$

Method 2. By given conditions $f(x_1, x_2) = 1/tf(tx_1, tx_2)$. Therefore by chain rule

$$f_1(x_1, x_2) = 1/t\big(f_1(tx_1, tx_2) \cdot \underbrace{\frac{\partial(tx_1)}{\partial x_1}}_{=t} + f_2(tx_1, tx_2) \cdot \underbrace{\frac{\partial(tx_2)}{\partial x_1}}_{=0}\big) = f_1(tx_1, tx_2).$$

The conclusion is the same as in Method 1.

7. FOC — 10 points, derivative — 5 points, sign — 5 points. *Solution.*

(a) $g(t) \to \max_{t \geq 0}$
FOC:
$g'(t) = -p'(t)x_0 - w = 0$

(b) Optimal time t^* satisfies the equation $F(t^*, x_0, w) = p'(t^*)x_0 + w = 0$. By IFT:

$$\frac{\partial t^*}{\partial x_0} = -\frac{\frac{\partial F}{\partial x_0}}{\frac{\partial F}{\partial t^*}} = -\frac{p'(t^*)}{x_0 p''(t^*)} > 0$$

8. $\frac{\partial z}{\partial x}$ — 3 points, $\frac{\partial z}{\partial y}$ — 3 points, expression for dz — 2 points, second derivatives — 3 points each, expression for $d^2 z$ — 3 points.

Solution. Denote $F_1 = \frac{\partial F}{\partial x}, F_2 = \frac{\partial F}{\partial y}$. Notice that in fact $F_i = F_i(x/z, y/z), i = 1, 2$. Firstly, we should mention that we can use IFT for $z = z(x, y)$ if the points (x,y,z) satisfy the condition $\frac{\partial F}{\partial z} = F_1 \cdot (-x/z^2) + F_2 \cdot (-y/z^2) \neq 0$.

Next,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

By IFT

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} = -\frac{F_1 \cdot (1/z)}{F_1 \cdot (-x/z^2) + F_2 \cdot (-y/z^2)} = \frac{F_1 z}{F_1 x + F_2 y},$$

$$\frac{\partial z}{\partial y} = \cdots = \frac{F_2 z}{F_1 x + F_2 y}.$$

Next,

$$d^2z = \frac{\partial^2 z}{\partial x^2}(dx)^2 + (\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial^2 z}{\partial x \partial y})dxdy + \frac{\partial^2 z}{\partial y^2}(dy)^2.$$

Let's find second derivatives:

$$\frac{\partial^2 z}{\partial x^2} = \frac{(F_1x + F_2y)(\frac{\partial F_1}{\partial x}z + F_1\frac{\partial z}{\partial x}) - F_1z(F_1 + \frac{\partial F_1}{\partial x}x + \frac{\partial F_2}{\partial x}y)}{(F_1x + F_2y)^2},$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(F_1x + F_2y)(\frac{\partial F_1}{\partial y}z + F_1\frac{\partial z}{\partial y}) - F_1z(F_2 + \frac{\partial F_1}{\partial y}x + \frac{\partial F_2}{\partial y}y)}{(F_1x + F_2y)^2},$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(F_1x + F_2y)(\frac{\partial F_2}{\partial x}z + F_2\frac{\partial z}{\partial x}) - F_2z(F_1 + \frac{\partial F_1}{\partial x}x + \frac{\partial F_2}{\partial x}y)}{(F_1x + F_2y)^2},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(F_1x + F_2y)(\frac{\partial F_2}{\partial y}z + F_2\frac{\partial z}{\partial y}) - F_2z(F_2 + \frac{\partial F_2}{\partial y}y + \frac{\partial F_1}{\partial y}x)}{(F_1x + F_2y)^2}.$$

6.3 MFE, fall semester exam, 25.12.2013

SECTION A

1. Find the gradient of $f(x, y) = \frac{x^2y}{\sqrt{x^2+y^2}}$ at the point $M(2, 1)$. Compute the derivative of f at M in direction of the vector $\{-1, 1\}$.

2. The system of equations defines $x(z)$ and $y(z)$:

$$\begin{cases} x^2 + zxy + y^2 + 5z + y^3 = 9 \\ y^3x^2 + 3x + 2y + z = 7 \end{cases}$$

   Find $x'(z)$ and $y'(z)$ at the point $x = 1$ and $y = 1$. State the implicit function theorem.

3. For the function $f(x, y) = x^3y^5 + x^2 - y^3 + xy$ find first order Taylor approximation at the point $(1, 1)$ and second order Taylor approximation at the same point.

4. Find all stationary points of $f(x, y) = -2y^3 + 24y - x^2e^y$. Classify them as local minimum, maximum or saddle point.

5. Find the constrained extrema of the function $f(x, y) = 2x^2 + x + y + y^2$ subject to $2x^2 + y^2 = 5$.

6. For each value of a determine whether the function $f(x, y, z) = x^2 + xz + ayz + z^2$ is concave, convex, strictly concave, strictly convex.

SECTION B

7. Robinson Crusoe produces nuts (good x) and corn (good y) using two factors of production: labor L and land T in accordance with the production functions, namely $x = \sqrt{L_xT_x}$ and $y = \sqrt{L_yT_y}$, where the L_x, L_y, T_x, T_y are the quantities of labor and land employed by Crusoe in the production processes. The overall supply of labor and supply of the cultivated land equal 1. In order to find the **Production Possibilities Frontier** in the Crusoe's economy the following problem should be solved:

$$(A) \begin{cases} \sqrt{L_xT_x} \to \max \\ \sqrt{L_yT_y} = y = const \\ L_x + L_y = 1,\ T_x + T_y = 1 \end{cases}$$

In this problem all variables take nonnegative values and $0 \le y \le 1$.

(a) (15 points) Solve this maximization problem by the Lagrange multiplier method. You may consider only the case when $L_x,\ L_y,\ T_x,\ T_y > 0$ and $0 < y < 1$.

(b) (5 points) Let $L_x^*,\ L_y^*,\ T_x^*,\ T_y^*$ be the solution of problem (A) (optimal combination of factors). Then the produced quantities of nuts and corn equal $x = \sqrt{L_x^* T_x^*}$ and $y = \sqrt{L_y^* T_y^*}$. Find the relationship between x and y in the form of some equation $G(x, y) = 0$.

8. (continuation) Crusoe does not care about his leisure time and consumes nuts and corn. His utility function is $u(x, y) = \frac{1}{3} \ln x + \frac{2}{3} \ln y$.

   (a) (10 points) By solving utility maximization problem: $\begin{cases} u(x,y) \to \max \\ G(x,y) = 0 \end{cases}$, find his optimal consumption bundle $(\tilde{x}, \tilde{y})$.

   (b) (10 points) Crusoe has plowed more land in the amount of dT (a small number). By using Envelope theorem find $\partial \tilde{x} / \partial T$. Hint. Firstly apply the theorem to problem (A).

6.4 MFE, 25.12.2013, marking scheme

1. f'_x — 3 points (2 for formula, 1 for value at the point), f'_y — 3 points, the rest — 4 points

2. Statement of IFT — 2 points. Each derivative — 4 points (3 for formula, 1 for calculations)

3. 4 points — first order (1 pt for each first derivative, 2 pt for final formula), 6 points — second order (1 pts for each second derivative, 3 pts for final formula)

4. Correct FOC — 2 points, Solution of FOC — 4 points. Check SOC — 4 points.

5. NDCQ — 1 pt, Correct FOC — 2 pts, Solution of FOC — 4 pts, check SOC — 3 pts.

6. Hesse matrix — 2 pts. Concavity and convexity — 5 pts. Considering particular value of a for strict concavity and strict convexity — 3 pts.

7. Point a. NDCQ — 2 pts. Writing FOC — 3 pts. Solving FOC — 5 pts. Checking SOC — 5 pts.

   First, simplify problem:

$$\begin{cases} L_x T_x \to \max_{L_x, T_x} \\ (1 - L_x)(1 - T_x) = y^2 \\ 0 < L_x < 1,\ 0 < T_x < 1,\ y > 0 \end{cases}$$

   Optimal point: $L_x = 1 - y$, $T_x = 1 - y$, $\lambda = (y - 1)/y$

   Hesse: $\Delta = (1 - L_x)(1 - T_x)(1 - \lambda) > 0$

   Point b — 5 pts:

$$x = \sqrt{L_x T_x} = \sqrt{(1 - y)^2} = 1 - y$$

8. Point a. NDCQ — 1 pt. Writing FOC — 2 pts. Solving FOC — 4 pts. Checking SOC — 3 pts. Point b. Solution 1. Applying Envelope theorem to problem (A) — 5 pts, conclusion — 5 pts. Solution 2. Applying IFT to the FOC — 10 pts.

6.5 MFE, retake, 24.01.2014

Part A.

1. Give an example of

   (a) a function $f(x, y)$ that has a non-zero gradient in all the points except the point $(0, 5)$ and zero gradient in the point $(0, 5)$

   (b) a function $g(x, y)$ that has no gradient on the line $x = 3y$ and a non-zero gradient when $x \neq 3y$

2. The population of a certain country grows exponentially, $N_t = N_{1990} \cdot \exp(r(t-1990))$. The population was 70 million in 1990 and 80 million in 2000, what will be the population in 2014?

3. Find and classify the extrema of the function $f(x, y) = x^2 - y^2$ subject to $x^2 + y^2 = 1$.

4. Given the system

$$\begin{cases} xe^{u+v} + 2uv = 1 \\ ye^{u-v} - \frac{u}{1+v} = 2x \end{cases}$$

   find du and dv at $x_0 = 1$, $y_0 = 2$, $u_0 = 0$, $v_0 = 0$.

5. Consider the objective function $f(x, y) = 4kx^3 + k^2xy + 3ky^4 - 13x - 13y$. The point $(x, y) = (1, 1)$ is the maximum of the function. Find the value of k

6. In the macroeconomic linear IS-LM model for the closed economy $Y = \bar{C} + m(Y - T) + G + \bar{I} - ar$ and $\bar{L} + bY - cr = M_s$, where M_s is money supply, r — interest rate, G — government expenditures, T — lump sum tax and the constant parameters $\bar{C} > 0$, $0 < m < 1$, $\bar{I} > 0$, $a > 0$, $\bar{L} > 0$, $b > 0$, $c > 0$. Find the formulas for dr/dT, dY/dT. Assume that government expenditures and money supply are fixed exogenous variables.

Part B.

7. The production function of a firm is given by $y = \sqrt{x_1} + \sqrt{x_2}$, where x_1 and x_2 are the factors of production. Given the factor prices $w_1 = 5w$, $w_2 = w > 0$ find the total costs function of the firm.

8. A consumer splits her time $\bar{L}$ hours a week between labor and leisure. Her utility function is represented by $u(c, l) = c^{\alpha}l^{1-\alpha}$, where c is the amount of consumption and l is leisure in hours and $0 < \alpha < 1$. The weekly budget constraint is written as $pc + wl = w\bar{L}$, where p is the price of consumption, w is the hourly wage rate.

   (a) Find the consumer's optimal bundle (c^*, l^*). Justify your answer by checking second-order conditions or otherwise.

   (b) Let $\alpha = 1/4$, $\bar{L} = 168$, $p = 10$, $w = 5$. Using Envelope Theorem estimate the change in the maximum value of her utility if the wage rate has decreased by 0.5.

6.6 MFE, retake, marking scheme

1. $5 + 5 = 10$, example $f(x, y) = x^2 + (y - 5)^2$, $f(x, y) = 1/(x - 3y)$
2. equation for r 3 pts, the rest — 7 pts. $e^r = (8/7)^{0.1}$
3. NDCQ — 1 pt, formulated FOC — 2 pts, solution — 4 pts, SOC — 3 pts, $(1, 0)$, $(-1, 0)$ — maxima, $(0, 1)$, $(0, -1)$ — minima
4. Use of IFT — 1 pt, four partial derivatives — 6 pts, differentials — 3 pts.
5. FOC, 2 values of k, $k_1 = 1$, $k_2 = -13$ — 6 pts, choice of $k = -13$ using SOC — 4 pts.
6. 5 pts each derivative, $Y'(T) = cm/(cm - c - ab)$, $r'(T) = bm/(cm - c - ab)$
7. formulation of maximization problem — 5 pts, NDCQ — 2 pts, FOC — 3 pts, solution of FOC — 5 pts, SOC — 5 pts
8. NDCQ — 2 pts, FOC — 3 pts, solution of FOC — 5 pts, SOC — 5 pts, question b — 5 pts. $pc = \alpha w\bar{L}$, $wl = (1 - \alpha)w\bar{L}$

6.7 MFE, mock exam, 24.03.14

SECTION A

1. Solve the differential equation $y'' - 8y' + 7y = x$ with initial conditions $y(0) = 0$, $y'(0) = 1$.
2. Find all the complex roots of the equation $z^3 + 3z^2 + (3 - i)z = 0$.
3. Expand the function $f(x) = x^2 \sin(1 - \cos(\ln(1 + 5x)))$ as a power series in terms up to x^6. State the range for which your expansion is valid.
4. Determine the value of the following integrals

$$\int \frac{\ln(5x)}{x^2}\, dx, \qquad \int \frac{1}{e^{5x} - e^{-5x}}\, dx$$

5. Use the Lagrange multiplier method to find the maximum value of $f(x, y, z) = (5+\sqrt{x})^2(1+\sqrt{y})^2(5+\sqrt{z})^2$ among positive numbers with $x + 25y + z = 10$.
6. Derive the explicit formula (without dots or sum sign) for the sum $S_n = 1\cdot3+2\cdot4+3\cdot5+\ldots+n\cdot(n+2)$.

   Hint: You may obtain and solve a difference equation for S_n

SECTION B

7. A function $f(x)$ defined on $\mathbb{R}$ is called bounded if there exists a number $M > 0$ such that $|f(x)| \leq M$ for all x. Consider the equation $y' + ay = f(x)$, where $a > 0$ is a number and $f(x)$ is a bounded continuous function defined on $\mathbb{R}$.
   (a) Using variation of constant method or otherwise find the general solution
   (b) Prove that there exists a bounded particular solution

(c) Let $\tilde{y}(x)$ be a bounded solution existence of which was proven in b). Prove that $\tilde{y}(x)$ is a unique solution with such property.

Hint: parts b) and c) can be treated separately.

8. Consider a problem of maximizing an output under the budget constraint $F(K, L) \to \max$ subject to $wL + rK = B$, where w, r are fixed factor prices, B is firm's budget and $F(K, L)$ is a continuously differentiable homogeneous production function. A set of points $(L(B), K(B))$ forms a so-called firm's expansion path, where B take all positive values and $L(B)$, $K(B)$ are the solutions of the constrained maximization problem.

   Find the equation of the firm's expansion path if the point $(8, 16)$ lies on this path.

6.8 MFE, mock marking scheme

1. General solution of homogeneous — 4 pts, particular solution — 3 pts, constants — 3 pts.
2. $z = 0$ — 2 pts, correct discriminant — 2 pts, take root of complex number — 6 pts.
3. knowledge of Taylor expansion of sin, cos and ln — 3pts, correct expansion 4 pts, valid range — 3 pts.
4. Each integral — 5 pts
5. NDCQ — 1 pt, FOC — 2 pts, solution of FOC — 4 pts, SOC — 3 pts
6. Equation — 1 pt, General solution of homogeneous — 2 pts, particular solution — 5 pts, constant — 2 pts.
7. a — 8 pts (if only the general solution of homegeneous is obtained then 4 pts), b — 6 pts, c — 6 pts.
8. NDCQ — 2 pts, Langrange function — 1 pt, FOC — 2 pts. Proof that expansion path is of the form $K = aL$ — 13 pts, value of a — 2 pts.

6.9 MOR, exam, 23.05.14

You need to solve all the problems from Sections A, B, C.
SECTION A

1. (15 points) A problem with the mixed constraints is given: $3x_1x_2 - x_2^3 \to \max$

   subject to $2x_1 + 5x_2 \geq 20$, $x_1 - 2x_2 = 5$, $x_1, x_2 \geq 0$.

   (a) Check NDCQ conditions.
   (b) Form the Langrangian function.
   (c) Solve the maximization problem by the use of Langrange method or otherwise.
   (d) Justify that the maximum point was found.

2. (20 points) Consider the linear programming problem with parameter β and nonnegative x_i:

$$\begin{aligned} 3x_1 + x_2 - x_3 + 4x_4 &\to \max \\ x_1 + x_2 - x_3 + x_4 &\leq 1 \\ \beta x_1 + x_2 + x_3 + 2x_4 &\leq 2 \end{aligned}$$

(a) Find the optimal values of the primal variables for $\beta = 6$.

(b) Find the function $\phi(\beta)$, where $\phi(\beta)$ is the maximum value of the objective function for fixed value of β.

(c) Sketch the graph of $\phi(\beta)$.

SECTION B

3. (15 points) Consider the system of difference equation:

$$\begin{cases} x_{t+1} = x_t - y_t + 6 \\ y_{t+1} = 2x_t - y_t + 3 \end{cases}$$

(a) Solve the system of difference equations

(b) Explore the stability of its solutions.

4. (15 points) Solve the Euler's equation $x^2y'' - 4xy' + 6y = 0$ on the interval $(0, \infty)$ by using the substitution $x = e^t$ or otherwise. The answer should be written as a function of x.

SECTION C

5. (15 points) Consider the following bimatrix game:

	D	E	F
A	4;3	2;2	2;1
B	-2;8	4;7	2;4
C	1;2	3;1	3;3

(a) Find all the pure and mixed Nash equilibria

(b) State whether the equilibria are Pareto-optimal

6. (20 points) Three players play the following game. Simultenuously each of them chooses one of three numbers: 1, 2 and 3. If all players choose the same number, then everyone gets nothing. Otherwise the player with smallest unique number receives two rubles, and other players pay one ruble each. Example: if 1, 1 and 3 are chosen, then the winner is the player who chose 3. She receives two rubles, and each of the other two players pays one ruble.

(a) Find all the pure and mixed Nash equilibria

(b) State whether the equilibria are Pareto-optimal

7 2014-2015

7.1 MFE, mock, 30.10.14

Variant 1. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)
SECTION A

1. Consider the function $f(x, y, z) = x^5 + 2xyz - 3z^3$. Using the total differential find the approximate value of $f(1.02, 0.99, 1)$.

2. Consider the function $f(x, y, z) = x^4 + (x + y)^2 + (x + z)^3$.

   (a) Find the Hesse matrix. Clearly state the Young theorem even if you don't use it.

   (b) Find all the points where the Hesse matrix is positive definite.

3. Consider the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{7x^2+3y^2}}, \text{ if } (x, y) \neq (0, 0) \\ a, \text{ if } (x, y) = (0, 0) \end{cases}$$

   Find all the values of a such that the function f will be continuous.

4. The microbe Veniamin is staying on the surface of the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(1, 2, 1)$. All coordinates are measured in centimetres. He digs into the ellipsoid perdpendicularly to the surface by 0.02 cm. Find new approximate coordinates of Veniamin.

5. Consider the function $u(x, y) = 2f(r)$ where $r = \sqrt{x^2 + y^2}$. Is it possible to represent the function $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ as a function of r alone, i.e. $g(r)$? If yes, then find $g(r)$.

6. Given the system

$$\begin{cases} u^2 - w^2 + x^2 + y^2 = 0 \\ uw + xy = 0 \end{cases}$$

   (a) Define a sufficient condition for functions $u(x, y)$ and $w(x, y)$ to be differentiable

   (b) Find $\frac{\partial u}{\partial x}$

SECTION B

7. (20 points) Let the demand and supply for an ice-cream on the sunny day be $q_D = D(p, T, d)$ and $q_S = S(p, T)$ correspondingly. Here p is the price, T is the temperature on this day, d – distance of the selling place from the center of the park, $D_p < 0$, $D_T > 0$, $S_p > 0$, $S_T < 0$, $D_d < 0$.

   (a) Find analytically how the equilibrium price p^* changes with the increase of T. How does it change with the increase of d?

   (b) Let q^* be the equilibrium supply quantity. Find $\frac{\partial q^*}{\partial T}$. Find the condition when $\frac{\partial q^*}{\partial T} < 0$.

8. (20 points) Consider the utility function

$$U(x_1, x_2) = \left(x_1^{\frac{\sigma-1}{\sigma}} + x_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

   where $\sigma \geq 0$ is a parameter.

   (a) Sketch the indifference curves corresponding to $\sigma = 2$.

   (b) Find the limiting utility function and sketch the corresponding indifference curves for two cases: $\sigma \to \infty$ and $\sigma \to 0$

7.2 MFE, mock, 30.10.14-marking

1. $f'_x = 5x^4 + 2yz = 7$, $f'_y = 2xz = 2$, $f'_z = 2xy - 9z^2 = -7$, $df = 7 \cdot 0.02 - 2 \cdot 0.01 - 7 \cdot 0 = 0.12$, $f(1.02, 0.99, 1) \approx f(1,1,1) + df = 0 + 0.12 = 0.12$. each derivative — 2 points, formula for differential — 3 points, calculation of new f — 1 point

2. Statement of the theorem — 2 points, Hesse matrix — 4 points, condition for positive definiteness — 4 points (2 points — inequalities, 2 points — solution).

$$\begin{pmatrix} 12x^2 + 2 + 6(x+z) & 2 & 6(x+z) \\ 2 & 2 & 0 \\ 6(x+z) & 0 & 6(x+z) \end{pmatrix}$$

   Condition: $x + z > 0$ and $x \neq 0$.

3. $\lim_{x,y \to 0} \frac{xy}{\sqrt{7x^2+3y^2}} = \lim_{x,y \to 0} \frac{sign(xy)}{\sqrt{7/y^2+3/x^2}} = 0$ (6 points), conclusion that $a = 0$ (4 points)

4. grad $f = (8x, 2y, 2z) = (8, 4, 2)$ (5 points). To dig into we need the direction $(-8, -4, -2)$ (1 point). The length of gradient, $|$ grad $f| = \sqrt{84}$ (2 points). New coordinates are approximately equal $(1, 2, 1) + \frac{0.02}{\sqrt{84}}(-8, -4, -2) = (1, 2, 1) + \frac{0.02}{\sqrt{21}}(-4, -2, -1)$ (2 points).

5. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2f'\frac{x^2}{(x^2+y^2)^{1/2}} + 2f'\frac{y^2}{(x^2+y^2)^{1/2}} = 2f'(r)r$. Derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ — 3 points each. The remaining part — 4 points.

6. statement — 3 points, correct formula with determinants for the derivative — 4 points, calculations of all derivatives in determinants — 3 points,

$$\frac{\partial u}{\partial x} = -\frac{2xu + 2wy}{2u^2 + 2w^2}$$

7. a) Each of the two derivatives — 5 points b) the derivative — 5 points, condition — 5 points

$$\frac{\partial p^*}{\partial T} = \frac{S_T - D_T}{D_p - S_p}$$

$$\frac{\partial p^*}{\partial d} = \frac{D_d}{S_p - D_p}$$

$$\frac{\partial q^*}{\partial T} = S_p\frac{S_T - D_T}{D_p - S_p} + S_T$$

8. 4 points for each plot of indifference curves, 4 points for each limiting utility function

   limiting cases: $\sigma \to \infty$, $U(x_1, x_2) = x_1 + x_2$, $\sigma \to 0$, $U(x_1, x_2) = \min(x_1, x_2)$

7.3 Fall-exam, 24.12.14

Variant 1. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)

SECTION A

1. Consider the set $A = \cup_{i=1}^{\infty} \left\{1 - \frac{1}{i}\right\} \subset \mathbb{R}$.
   (a) Is the set A bounded? Open? Closed? Compact?
   (b) Roughly sketch the set $A \times A$

2. Consider the system of equations

$$\begin{cases} xyz^2 + x^3y + 3y^3z^3 - 2x = 3 \\ 2x^2yz + z^2 + 3xy + 5yx^3 = -5 \end{cases}$$

   (a) Are the function $x(z)$ and $y(z)$ defined around the point $A = (-1, 1, 1)$?
   (b) Find df/dz, where $f(z) = x(z)y(z)$

3. Consider the functions $f(x, y) = x^2 + 2x + y^2 + 6y + 7$ and $g(x, y) = x^2 - 8x + y^2 - 10y + 9$. Find all the points where the gradients are parallel.

4. Find and classify all the local extrema of the function $g(x, y) = y^3 - 12y + x^2e^y$. Which of them are global?

5. It is known that the point $(1, 0)$ is the constrained local maximum of the function $f(x, y) = 5x - ky - 3x^2 + 2xy - 5y^2$ subject to $x + y = 1$.
   (a) Find the value of k and the maximum value of the function f
   (b) Using Envelope theorem find the new value of maximum if k will increase by 0.1

6. Use Lagrange multipliers to find the height and radius of a cylinder with the least possible surface area among those with a volume of 6π m^3. Make sure you check the second order condition for minimisation.

SECTION B

7. In perfectly competitive agricultural industry a typical firm uses labor, capital and land. Its short-run costs can be found by the formula $C_{sr}(y, K^*, T^*) = \frac{y^3}{K^*T^*} + K^* + T^*$, where y is the output, K^* and T^* are fixed quantities of capital and land, respectively. The long run costs $C_{lr}(y)$ can be found by minimizing C_{sr} with respect to K^* and T^*.
   (a) Find $C_{lr}(y)$
   (b) The profits of the firm in both short-run and long-run can be found by $\pi_{sr} = py - C_{sr}$ and $\pi_{lr} = py - C_{lr}$, where p is price. When the profits are maximized with respect to the output, the maximum values of these are denoted by $\tilde{\pi}_{sr}(p, K^*, T^*)$ and $\tilde{\pi}_{lr}(p)$, respectively. Let $p = 3$. Show that $\tilde{\pi}_{sr}(3, K^*, T^*) \leq 0$ for all values of $K^* > 0$ and $T^* > 0$.
   (c) Use Envelope Theorem to evaluate $\frac{\partial \tilde{\pi}_{sr}}{\partial K^*}$ and $\frac{\partial \tilde{\pi}_{sr}}{\partial T^*}$ for $p = 3$. Is it possible that these derivatives turn zero simultaneously for $T^* \neq K^*$?

8. Let $f(x)$ be twice continuously differentiable function whose second-order derivative is $f''(x) > 0$ for all x. Consider a constrained minimization problem $F = \sum_{i=1}^{n} f(x_i) \to \min$ subject to $\sum_{i=1}^{n} x_i = 1$, $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Using the first-order conditions find the critical point. By checking bordered Hessian or otherwise show that the found point is minimum.

7.4 Fall-exam, 24.12.14-marking

1. bounded — 2 pts, not closed — 2 pts, not open — 2 pts, not compact — 2 pts, graph — 2 pts
2. the function is defined — 2 pts, x' and y' — $2 \times 3 = 6$ pts, df/dz — 2 pts.
3. grad f — 2 pts, grad g — 2 pts, condition — 2 pts, solution of condition — 4 pts
4. FOC statement — 1 pt, FOC solution — 4 pts, SOC check — 3 pts, globality — 2 pts
5. NDCQ — 1 pt, FOC statement — 1 pt, max and k — 3 pts, SOC checking — 3 pts, Envelope — 2 pts
6. problem formulation (target function, constraint) — 2 pts, NDCQ — 1 pt, FOC formulation — 1 pt, FOC solution 4, SOC check — 2 pts
7. (a) FOC statement — 1 pt, FOC solution — 4 pts, SOC check — 3 pts
   (b) proof — 6 pts
   (c) derivatives — 2 pts each, inequality check — 2 pts
8. FOC statement — 2 pt, FOC solution — 8 pts, SOC check — 10 pts

7.5 fall-Retake 24.01.2015

Part A

1. The function $f(x, y)$ is given by $f(x, y) = u^2(x, y) + v^3(x, y)$. The values of u and v and their gradients at the point $(x, y) = (1, 1)$ are also known, $u(1, 1) = 3$, $v(1, 1) = -2$, grad $u = (1, 4)$, grad $v = (-1, 1)$. Find grad f at the point $(1, 1)$ if $u, v \in C^1$.
2. Find the local maxima and minima of the function $f(x, y) = x^4 + 2y^4 - xy$. Determine whether the extrema you have found are global or local.
3. For the function $f(x, y) = x^3y^5 + x^2 - y^3 + xy$ find first order Taylor approximation at the point $(1, 1)$ and second order Taylor approximation at the same point.
4. Use Lagrange multipliers to find the height and radius of a cylinder with the least possible surface area among those with a volume of 6π m^3. Make sure you check the second order condition for minimisation.
5. Find the equation of the tangent plane to the surface given by $x^3 + z^3 - 3xz = y - 1$ at the point $(1, 4, 2)$.
6. Suppose $f(x, y)$ is a twice differentiable function. Let x and y be defined in terms of u, v as follows: $x(u, v) = ue^{2v}$, $y(u, v) = u^2 - v^2$. Let $F(u, v) = f(x(u, v), y(u, v))$. Calculate F''_{uu} and F''_{uv}.

Part B

7. A firm's inventory $I(t)$ is depleted at a constant rate per unit time, i.e. $I(t) = x - \delta t$, where x is an amount of good reordered by the firm whenever the level of inventory is zero. The order is fulfilled immediately. The annual requirement for the commodity is 200 units and the firm orders the commodity n times a year where $200 = nx$. The firm incurs two types of inventory costs: a holding cost and an ordering cost. Since the average stock of inventory is $x/2$, the holding cost equals $C_h x/2$, the cost of placing one order is C_o, and with n orders a year the annual ordering cost equals $C_o n$.

   (a) Minimize the cost of inventory $C = C_h x/2 + C_o n$ by choice of x and n subject to the constraint $nx = 200$ by the Lagrange multiplier method.

   (b) Use the envelope theorem to approximate the change in the minimal cost if the requirement for the commodity rises to 204 units.

8. A two-product firm produces outputs y_1 and y_2 from a single factor of production which is labor, in other words, there is a function f, such that $f(y_1, y_2) \leq \bar{L}$. Output prices are p_1 and p_2. The firm has a fixed amount of labor supply $\bar{L} > 0$ that should be utilized in full.

   (a) (10 points) Set the problem of the revenue maximization under the labor constraint mathematically and derive first-order conditions. Assume that both outputs should be produced in positive amounts.

   (b) (10 points) Let the maximum value of the total revenue under the labor constraint be $TR(p_1, p_2, \bar{L})$. What are its derivatives with respect to the prices?

7.6 fall-retake. marking

1. $\operatorname{grad} f = 2u \operatorname{grad} u + 3v^2 \operatorname{grad} v = 6 \cdot (1,4) + 12 \cdot (-1,1) = (-6, 36)$. Maybe solved by computing f'_x (formula + value, 3+1 pts) and f'_y (3+1 pts) and putting them in a vector (2 pts).

2. FOC statement — 1 pt, FOC solution — 4 pts, SOC — 3 pts, globality — 2 pts.

   Critical points: $(x, y) = (0, 0)$ (saddle), $(2^{-9/8}, 2^{-11/8})$ (global min), $(-2^{-9/8}, -2^{-11/8})$ (global min)

3. 1 pt for each derivative (5 pts total f_x, f_y, f_{xx}, f_{yy}, f_{xy}), 2 pts for first order, 3 pts for second order approximation

$$f(x, y) \approx 2 + 6(x-1) + 3(y-1) + \frac{1}{2}(8(x-1)^2 + 2 \cdot 16(x-1)(y-1) + 14(y-1)^2)$$

4. problem formulation (target function, constraint) — 2 pts, NDCQ — 1 pt, FOC formulation — 1 pt, FOC solution 4, SOC check — 2 pts

$$R^* = \sqrt[3]{3},\ \lambda^* = 2/\sqrt[3]{3},\ h^* = 2\sqrt[3]{3}$$

5. $\operatorname{grad} f = (3x^2 - 3z, -1, 3z^2 - 3x) = (-3, -1, 9)$ (4 pts). So tangent plane equation is $-3x - y + 9z = c_0$ (4 pts). Plugging in the coordinates of the point we obtain $c_0 = 11$ (2 pts).

6. F_u 4 pts, F_{uu} — 3 pts, F_{uv} — 3pts

7. a) NDCQ — 2 pts, Langrangean — 1 pt, FOC statement — 1 pt, FOC solution — 5 pts, SOC — 5 pts, minimum value — 2 pts, b) 4 pts

$$n^* = 10\sqrt{\frac{C_h}{C_o}},\ x^* = 20\sqrt{\frac{C_o}{C_h}}$$

8. formulation — 5, NDCQ — 2 pts, FOC statement — 3 points

7.7 mock. 25.03.2015

Variant 1. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)
SECTION A

1. Find all the complex roots of the equation $(z+i)^3 = 1+i$.

2. Solve the differential equation $-2x^2y' = x^2 + y^2$ with initial condition $y(e) = e$.

3. Consider the equation $y^3 + xy + 3x^2 + 2x^3 = 7$.

   (a) Does this equation define the implicit function $y(x)$ at a point $(x = 1, y = 1)$?

   (b) If the function $y(x)$ is defined find its second order Taylor expansion

4. The function $f(x, y)$ for positive x and y is defined as

$$f(x,y) = x^{42}y^a + x^{b+1}\sqrt{y+x} + \frac{1}{y^a x^b}$$

   (a) Find the values of a and b such that f is homogeneous

   (b) For the values of a and b you have found find the degree of homegeneity of $x\frac{\partial^2 f}{\partial y^2} + y\frac{\partial^2 f}{\partial x^2}$

5. Consider two vectors, $\vec{x} = (1, 0, -1)$ and $\vec{y} = (1, 1, -2)$. Find a vector $\vec{z}$ with maximal length (called *first principal component*) such that $\vec{z}$ is a linear combination of $\vec{x}$ and $\vec{y}$, i.e. $\vec{z} = a\vec{x} + b\vec{y}$ with weights satisfying the condition $a^2 + b^2 = 1$.

6. The Fibonacci sequence is defined as $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0$ and $F_1 = 1$.

   (a) Find explicit formula for F_n

   (b) Find the "golden ratio", $\phi = \lim_{n\to\infty} F_{n+1}/F_n$

   (c) Is it true that F_n is the closest integer to $\phi^n/\sqrt{5}$?

SECTION B

7. It is known that functions 1, x and x^2 are particular solutions of the second-order linear differential equation $a(x)y'' + b(x)y' + y = 1$, where $a(x)$ and $b(x)$ are continuous functions.

   (a) Find the general solution of this equation

   (b) Find $a(x)$ and $b(x)$

8. Let $f(x)$ be a concave function defined on $[0; \infty)$ and $f(0) = 0$. Is it true that for $k \geq 1$ the following inequality holds: $kf(x) \geq f(kx)$?

7.8 mock. marking. some sols

1. Find all the complex roots of the equation
   - **2 point** for polar form of the right-hand side
   - **3 points** if only one root found
   - **3 points** for introduction of $+2k\pi$
   - **2 points** for explicitly finding 2 more roots
2. Solve the differential equation
   - **1 point** for spotting that the equation is homogeneous
   - **1 point** for correct change of variables
   - **7 points** for general solution
   - **1 point** for finding the constant using initial condition
3. Consider the equation
   - **3 points** for part (a)
   - **2 points** for $y'(x)$
   - **3 points** for $y''(x)$
   - **2 points** for correct series
4. Homogeneous function
   - **6 points** for part (a): a - 3, b - 3
   - **4 points** for part (b)
5. Find a vector with maximal length
   - **2 points** for correct formulation of the maximization problem
   - **3 points** for NDCQ, Lagrangian and FOC (1 point each)
   - **3 points** for finding all 4 critical points
   - **2 points** for the SOC
6. Fibonacci sequence
   - **4 points** for (a): characteristic polynomial - 1, roots - 1, general solution - 1, constants - 1
   - **3 points** for (b)
   - **3 points** for (c)
7. Second-order equation
   - **13 points** for (a)
   - **7 points** for (b): $a(x)$ - 4, $b(x)$ - 3

Solution

(a) Let $z(x) = y-1$, then $z' = y'$, $z'' = y''$ and three particular solutions become $z = 0, z = x-1, z = x^2 - 1$. The equation then becomes a homogeneous second-order equation:

$$a(x)z'' + b(x)z' + z = 0$$

The general solution of the homogeneous second-order linear equation can be expressed as $z^{gen}(x) = C_1 z_1 + C_2 z_2$, where z_1 and z_2 are linearly independent particular solutions to the equation. As long as $z = x - 1$ and $z = x^2 - 1$ are linearly independent particular solutions, the general solution of the homogeneous equation is

$$z(x) = C_1(x-1) + C_2(x^2-1)$$

Therefore the solution of the original equation is

$$y(x) = z(x) + 1 = C_1(x-1) + C_2(x^2-1) + 1$$

(b) Putting $y = x$ into the equation one gets

$$b(x) + x = 1 \Longrightarrow b(x) = 1 - x$$

Putting $y = x^2$ one gets

$$2a(x) + 2(1-x)x + x^2 = 1 \Longrightarrow a(x) = \frac{(x-1)^2}{2}$$

Another Solution

Please note that question (b) can be answered independently from (a). After finding $a(x)$ and $b(x)$ the equation becomes:

$$\frac{(x-1)^2}{2}y'' + (1-x)y' + y = 1$$

$y = 1, y = x$ and $y = x^2$ are particular solutions so the general solution has the following form:

$$y = Ax^2 + Bx + C$$

By putting this into the equation one gets:

$$\frac{(x-1)^2}{2}(2A) + (1-x)(2Ax+B) + Ax^2 + Bx + C = 1$$

$$Ax^2 - 2Ax + A + 2Ax + B - 2Ax^2 - Bx + Ax^2 + Bx + C = 1$$

$$\Longrightarrow A + B + C = 1 \Longrightarrow C = 1 - A - B$$

$$\Longrightarrow y = Ax^2 + Bx + 1 - A - B = A(x^2-1) + B(x-1) + 1$$

8. Concave function

- **5 points** for the definition of concave function
- **15 points** for proving the inequality

- **OR 3 points** for showing an example

Solution
By definition the function is concave if

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2) \ \ \forall x_1, x_2, \forall \alpha \in [0;1]$$

As long as $k \geq 1, 0 \leq \frac{1}{k} \leq 1$
By putting $x_1 = kx, x_2 = 0$ and $\alpha = 1/k$ into the definition of the concave function one gets

$$f(x+0) \geq \frac{1}{k}f(kx) + (1-\frac{1}{k})f(0)$$

$$\Longrightarrow kf(x) \geq f(kx), \quad QED$$

7.9 MOR, Final exam 2015, 26.05.2015

Variant 1. 26 May 2015. Please, don't forget to write you variant number. Total duration of the exam is 120 min. Good luck! :)

SECTION A. You need to solve BOTH problems.

1. Using Lagrange multipliers maximize the function $f(x_1, x_2, x_3) = -x_1 - 2x_2 + 3x_3$ subject to constraints: $2(x_1+1)^2 + x_2^2 + 3(x_3-1)^2 \leq 5$ and $x_1, x_2, x_3 \geq 0$. Find the point(s) of maximum and maximum value of f. Justify your answer by reference to Weierstrass theorem if it is relevant.

2. For all real values of parameter β that lies within the range $-1 < \beta < 0$ maximize linear function $2x_1 - x_2 + 8x_3 - 19$ subject to constraints $x_1 \geq x_2 + x_3 + \beta$, $2x_1 + x_2 + 4x_3 \leq \beta + 1$ and $x_1, x_2, x_3 \geq 0$. You are not asked to find the maximizer.

SECTION B. You need to solve TWO problems of your choice.

3. Find the general solution of equation $y'' - 4y' + 8y = \sin 2x + e^{-2x}$.

4. Solve the initial value problem for the system of difference equations

$$\begin{cases} x_{t+1} = -x_t + 2y_t + 7 \\ y_{t+1} = -2x_t + 2y_t \end{cases}$$

   where $x_0 = y_0 = 0$.

5. By variation of parameters solve $y'' + y = \frac{1}{\sin^2 x}$.

SECTION C. You need to solve BOTH problems

6. Andrey and Boris play the following game. Each player throws a fair coin and observes the result of his own toss. Then simultaneously Andrey guesses the result of Boris' toss and Boris guesses the result of Andrey's toss. They receive one dollar each if at least one guess was correct and receive nothing otherwise.

(a) Find all pure Nash equilibria of this game

(b) Are the Nash equilibria Pareto-optimal?

(c) What is the probability of at least one correct guess in the Nash equilibria?

7. Anna and Bella play the simplified version of Battleship game. Anna places a two-decker destroyer ship on the 1×4 grid. Then Bella has one shot. Bella does not know where the Anna's ship is located. If Bella hits the Anna's ship then Bella wins the game, otherwise Anna wins.

   (a) Find at least one Nash equilibria of this game

   (b) What is the probability that Anna wins in the Nash equilibria?

7.10 MOR, Final marking scheme

1. Lagrange function — 1 pt, NDCQ — 2 pts, correct FOC — 2, solution – 3, soc — 2

2. Correct dual — 2 pts, feasible set drawn — 3 pts, all cases considered — 5 pts

3. homogeneous equation solved — 4 pts (characteristic equation — 1 pt, roots — 1 pt, solution — 2 pts); particular solution — 5 pts (2 for exp, 3 for cos and sin), adding them up — 1 pt

4. eliminating one variable — 2 pts, roots of characteristic equation — 1 pt, solution of homogeneous equation — 2 pts, solution for eliminated variable — 2 pts, particular solution — 1 pt, constants — 2 pts

5. homogeneous equation solved — 4 pts (characteristic equation — 1 pt, roots — 1 pt, solution — 2 pts); variation of constants — 6 pts (2 pts given for replacing constants c_1 and c_2 by functions)

6. correct strategy sets — 3 pts

   Каждый игрок знает, как выпала его монетка и по правилам игры должен попытаться предсказать, как выпала монетка у другого игрока. Следовательно, у каждого игрока 4 стратегии:

   A . Предсказать, что чужая монетка выпала орлом

   B . Предсказать, что чужая монетка выпала решкой

   C . Предсказать, что чужая монетка выпала так, как своя

   D . Предсказать, что чужая монетка выпала не так, как своя

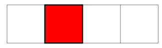
   Составляем матрицу 4×4 и заполняем её вероятностями выигрыша игроков. Игроки выигрывают одновременно, поэтому можно писать одну вероятность.

	A	B	C	D
A	3/4	3/4	3/4	3/4
B	3/4	3/4	3/4	3/4
C	3/4	3/4	1/2	1
D	3/4	3/4	1	1/2

   Равновесиями Нэша будут профили (A,A), (A,B), (B,A), (B,B), (C,D), (D,C). В равновесии Нэша вероятность выигрыша игроков равна 1 или 3/4. matrix — 4 pts, pure NE — 1 pt, Pareto-optimality — 1 pt, probability — 1 pt

7. correct strategy sets — 3 pts (Anna has 3 strategies, Bella — 4 strategies); matrix and pure Nash — 2 pts, elimination of non-strictly dominated strategies — 2 pts, mixed Nash in 2x2 matrix — 2 pts, probability that Anna wins — 1 pt

   Составляем матрицу игры:

				
	$-1, 1$	$-1, 1$	$1, -1$	$1, -1$
	$1, -1$	$-1, 1$	$-1, 1$	$1, -1$
	$1, -1$	$1, -1$	$-1, 1$	$-1, 1$

   Поскольку надо найти хотя бы одно равновесие Нэша, мы имеем право вычеркивать нестрого доминируемые или эквивалентные стратегии. Сводим к матрице 2×2.

8 2015-2016

8.1 Mock 28.10.2015

Variant 1. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)
SECTION A

1. Consider the function $f(x, y, z) = x^5 + 2xyz - 3z^3$. Using the total differential find the approximate value of $f(1.01, 0.99, 1.01)$.

2. Consider the function $f(x, y) = 2x^4 - (x + y)^3$.

   (a) Find the Hesse matrix. Clearly state the Young theorem even if you don't use it.

   (b) Find the definiteness (positive definite, positive semidefinite, etc) of the Hesse matrix at the point $(1, 2)$.

3. Let the function $f(x, y)$ be defined by the formula

$$f(x,y) = \begin{cases} -1, \text{ if } x > y \\ 1, \text{ if } x \le y \end{cases}$$

   (a) Find the limits $\lim_{x\to\infty} \lim_{y\to\infty} f(x, y)$ and $\lim_{y\to\infty} \lim_{x\to\infty} f(x, y)$

   (b) Does the limit $\lim_{x\to\infty, y\to\infty} f(x, y)$ exist?

4. The functions f and g are given: $f(x, y) = x^2 + 2xy + y^4$, $g(x, y) = -5x^2 - xy - 2y^4$. Find at least one direction from the point $(1, 1)$ in which both functions will grow.

5. The function z is defined by the formula $z(x, y) = f(x^3 - y^2)$. Simplify the expression $2y\frac{\partial z}{\partial x} + 3x^2\frac{\partial z}{\partial y}$.

6. Consider the function $f(x, y) = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \ldots}}}}.$

   (a) Find the value of $(f^2(x, y) - x)^2 - y - f(x, y)$

(b) Find $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(1, 1)$

SECTION B

7. (20 points) Let S_1 and S_2 be two sets from $\mathbb{R}^2$: $S_1 = \{(x, y) \in \mathbb{R}^2 | xy = 1\}$, $S_2 = \{(x, y) \in \mathbb{R}^2 | xy = -1\}$ and $S = S_1 + S_2$. We denote by the sum $S_1 + S_2$ the set

$$S_1 + S_2 = \{(x, y) \in \mathbb{R}^2 | (x, y) = (x_1, y_1) + (x_2, y_2),\ (x_1, y_1) \in S_1,\ (x_2, y_2) \in S_2\}$$

   (a) Are the sets S_1 and S_2 closed? Justify your answer.

   (b) Does the origin belongs to the set S?

   (c) Is the set S closed?

8. (20 points) Consider a Cournot duopoly of the two identical firms that compete by choosing outputs y_1 and y_2 simultaneously. Marginal costs of these firms are constant $MC_1 = MC_2 = c > 0$. When the outputs y_1 and y_2 are set, the price of a good can be found by the formula $p = a - b(y_1 + y_2)$, where $a > c$, $b > 0$.

   (a) Find equations of the level curves for the profits of the firms $\pi_1(y_1, y_2)$ and $\pi_2(y_1, y_2)$.

   (b) It is known that the point of equilibrium outputs (y_1^*, y_2^*) in the coordinate plane (y_1, y_2) can be found by drawing tangent lines to the level curves and these tangents should be parallel to the axes. Then (y_1^*, y_2^*) is the point of intersection of the tangents. By finding corresponding gradients of $\pi_1(y_1, y_2)$, $\pi_2(y_1, y_2)$ and using the hint stated above, find (y_1^*, y_2^*) in terms of a, b and c.

8.2 Mock 28.10.2015. Marking scheme

1. Three partial derivatives = 1 pt for formula, 1 pt for value (6 pts). Value of function in the initial point = 1 pt, formula of differential = 1 pt, answer = 1 pt

2. First derivatives = 2 pts (1 pt for each), Hesse matrix = 3 pts, statement of the Young theorem = 2 pts, definiteness = 3 pts

3. 3 pts + 3 pts + 4 pts

4. Two gradients = 4 pts (2 pts each).

   Directional derivative approach: two directional derivatives = 2 pts, condition = 2 pts, correct direction = 2 pts.

   Visual approach on plane: observation that both gradients are above $y = x$ = 2 pts, observations that $\cos\alpha > 0$ if $\alpha < \pi/2$ = 2 pts, correct direction = 2 pts

5. Two derivatives = 8 pts (4 pts each). Final answer = 2 pts

6. Evaluation of expression = 2 pts, two derivatives = 8 pts (4 pts each)

7. a — 8 pts, b — 4 pts, c — 8 pts

8. a — 6 pts (3 pts + 3 pts), gradients of profit functions = 6 pts (3 pts each), remaining part of the proof = 8 pts

8.3 Fall Exam, 27.12.2015

Variant 1. Section A

1. At the beginning James Bond is located at the point $(1, 1)$. To choose his new location he calculates the gradient of the function $f(x, y) = x^2 + y^2 - 3xy + x$ from his current location and moves in the direction given by the gradient by its length. Where he will be after two movements?

2. Consider the system of equations

$$\begin{cases} x^4 + y^4 + z^4 = 3 \\ x + x^3 + y + 2y^3 + z + 3z^3 = 5 \end{cases}$$

   (a) Are the function $x(z)$ and $y(z)$ defined around the point $A = (-1, 1, 1)$?

   (b) Find dx/dz and dy/dz

3. Find and classify unconstrained extrema of the function $f(x, y) = x^4 + y^8 - 2xy$

4. Find and classify constrained extrema of the function $f(x, y) = xy$ subject to $x^2 + 4y^2 = 9$

5. The function u is defined by the equation $u^3(t) + u(t) = f(x, y)$, where $x = 2 - t$ and $y = 1 + 2t$ and f is in C^2. Find du/dt and d^2u/dt^2

6. Consider the function $f(x) = h(x) - ax$, where the function h is twice differentiable and $h''(x) < 0$ for all x. The global maximum of f is denoted by $x^*(a)$.

   (a) Find dx^*/da

   (b) It is known that for $a = 1$ the optimal point is $x^* = 3$ and the value of maximum is 2015. What is the approximate value of maximum for $a = 1.01$?

Variant 1. Section B. Problems 7 and 8 can be solved separately.

7. A risk-averse Alex possesses w dollars of wealth in money and property. His house worth $L < w$ dollars can be completely destroyed by a landslide with the probability p, $0 < p < 1$. Let's denote his wealth if landslide occurs by x_L and x_{NL} otherwise. Then his expected utility can be calculated by $E(u) = p \ln x_L + (1 - p) \ln x_{NL}$, where $x_L, x_{NL} > 0$.

   (a) Show that $E(u)$ is a concave function in its domain.

   (b) Show that the set in (x_L, x_{NL}) plane defined by the inequality $E(u) \geq const$ is convex.

8. In order to reduce risk Alex buys insurance from a perfectly competitive company. By doing that he maximizes his expected utility $E(u)$ with respect to (x_L, x_{NL}) subject to constraint imposed by the company $p(w - L - x_L) + (1 - p)(w - x_{NL}) = 0$.

   (a) Find his optimal bundle (x_L^*, x_{NL}^*). Use bordered Hessian to check sufficiency. Is Alex better-off with the insurance? Explain.

   (b) Let $E(u)^*$ be the maximum value of $E(u)$ with insurance. By applying Envelope Theorem find $\partial E(u)^*/\partial p$. Express your answer in terms of p, w and L alone.

8.4 Solution to Fall Exam, 27.12.2015

1. Marking

   - first movement 6 pt: 1 pt for the formula of gradient, 2 pt for the calculation of derivatives, 1 pt for the substitution of initial point, 2 pt for the location after first movement
   - second movement 4 pt: 2 pt for the calculation of gradient at the new point, 2 pt for the final location.
   - common mistake: movement by 2*(first gradient) — 6 pt.

   Solution:

   Calculate the gradient of the function

   $$\nabla f(x,y) := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x - 3y + 1, 2y - 3x).$$

   Then substitute the initial point (1, 1) to the formula above: $\nabla f(1,1) = (0,-1)$. This is the direction of James's first movement. He moves exactly by the length of this vector (problem conditions) so we don't need to normalize obtained result. Thus Bond's location after the first movement will be $(1,1) + (0,-1) = (1,0)$.

   Now calculate the gradient at the new point:$\nabla f(1,0) = (3,-3)$. This is the direction of the second movement. The final location will be $(1,0) + (3,-3) = (4,-3)$. This is the final answer.

2. Marking

   - 2pt for IFT checking
   - each derivative 4 pt: 3 pt for the formula, 1 pt for the calculations

   Solution:

   Let $f_1 = x^4 + y^4 + z^4 - 3$, $f_2 = x + x^3 + y + 2y^3 + z + 3z^3 - 5$.

   (a) Check the conditions of the IFT:

      i. $f_1(-1,1,1) = 0$, $f_2(-1,1,1) = 0$,

      ii. $f_1, f_2 \in C^1$,

      iii.
      $$\left|\frac{\partial(f_1,f_2)}{\partial(x,y)}\right| = \begin{vmatrix} 4x^3 & 4y^3 \\ 1+3x^2 & 1+6y^2 \end{vmatrix} = \begin{vmatrix} -4 & 4 \\ 4 & 7 \end{vmatrix} = -44 \neq 0.$$

   (b)
   $$\frac{dx}{dz} = -\frac{\left|\frac{\partial(f_1,f_2)}{\partial(z,y)}\right|}{\left|\frac{\partial(f_1,f_2)}{\partial(x,y)}\right|} = -\frac{\begin{vmatrix} 4z^3 & 4y^3 \\ 1+9z^2 & 1+6y^2 \end{vmatrix}}{\begin{vmatrix} 44x^3 & 4y^3 \\ 1+3x^2 & 1+6y^2 \end{vmatrix}} = -\frac{\begin{vmatrix} 4 & 4 \\ 10 & 7 \end{vmatrix}}{\begin{vmatrix} -4 & 4 \\ 4 & 7 \end{vmatrix}} = -\frac{3}{11};$$

   $$\frac{dy}{dz} = -\frac{\left|\frac{\partial(f_1,f_2)}{\partial(x,z)}\right|}{\left|\frac{\partial(f_1,f_2)}{\partial(z,y)}\right|} = -\frac{\begin{vmatrix} 4x^3 & 4z^3 \\ 1+3x^2 & 1+9z^2 \end{vmatrix}}{\begin{vmatrix} 4x^3 & 4y^3 \\ 1+3x^2 & 1+6y^2 \end{vmatrix}} = -\frac{\begin{vmatrix} -4 & 4 \\ 4 & 10 \end{vmatrix}}{\begin{vmatrix} -4 & 4 \\ 4 & 7 \end{vmatrix}} = -\frac{14}{11}.$$

3. FOC - 2 pts, all the critical points - 4 pts, SOC - 4 pts.

4. NDCQ - 1 pt, FOC - 2 pts, correct solution - 4 pts, SOC - 3 pts.

5. marking

   - first derivative: 3 pt for differentiating w.r.t. t OR applying IFT. 2 pt for correct answer
   - second derivative: 1 pt for the idea of differentiating the expression for the first derivative, 2 pt for correct application of chain rule, 2 pt for correct answer

   Solution:

   Differentiate both sides of the equality w.r.t. t:

$$3u^2\frac{du}{dt} + \frac{du}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{du}{dt} = \frac{\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}{3u^2+1} = \frac{-\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y}}{3u^2+1}$$

   In order to find $\frac{d^2u}{dt^2}$ differentiate the latter expression w.r.t. t (don't forget to apply the chain rule for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$):

$$\frac{d^2u}{dt^2} = \frac{(-\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial t} - \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial t} + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial t} + 2\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial t})(3u^2+1) - 6u\frac{\partial u}{\partial t}(-\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y})}{(3u^2+1)^2}$$

   Finally substitute $\frac{du}{dt}$:

$$\frac{d^2u}{dt^2} = \frac{(\frac{\partial^2 f}{\partial x^2} - 4\frac{\partial^2 f}{\partial \partial dx}\frac{\partial y}{\partial t} + 4\frac{\partial^2 f}{\partial y^2})(3u^2+1)^2 - 6u(-\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y})^2}{(3u^2+1)^3}$$

6. marking

   (a) 1 pt for FOC, 1 pt for showing that FOC has unique solution, 1 pt for SOC, 2 pt for derivative

   (b) 3 pt for envelope theorem (idea + correct value) OR finding $f^*(a)$ and differentiating, 2 pt for linear approximation (idea and correct f^*)

   Solution:

   (a) Write down first order condition for maximization problem:

$$f'(x) = h'(x) - a = 0$$

   Let's denote $g(x) = h'(x)$. Then $g'(x) = h''(x) < 0$ therefore $g(x)$ is a strictly decreasing function and the FOC equation has the only solution:

$$x^* = g^{-1}(a)$$

   As long as $f''(x) = h''(x) < 0$ $f(x)$ is a concave function and this point is indeed a global maximum.

$$\frac{dx^*}{da} = \frac{dg^{-1}(a)}{da} = \frac{1}{\frac{dg}{dx}(x^*)} = \frac{1}{h''(x^*)}$$

In fact there is no need to find x^*, but find it's derivative by differentiating FOC w.r.t. a:

$$h''(x^*) \cdot \frac{dx^*}{da} - 1 = 0 \implies \frac{dx^*}{da} = \frac{1}{h''(x^*)}$$

(b) Apply envelope theorem to the maximization problem:

$$\frac{df^*}{da} = \frac{\partial f^*}{\partial a} = -x^* = -3$$

$$f^*(1.01) \approx 2015 - 3 \cdot 0.01 = 2014.97$$

7. marking: a — 15 pts, b — 5 pts

$$H = \begin{pmatrix} \frac{-p}{x_l^2} & 0 \\ 0 & \frac{p-1}{x_{NL}^2} \end{pmatrix}$$

Here $\Delta_1 < 0$, $\Delta_2 > 0$, the function is concave.

8. marking: NDCQ — 2 pts, Lagrangian function — 1 pt, FOC — 1 pt, solution of FOC — 3 pts, SOC — 4 pts, better-off — 4 pts, Envelope theorem — 5 pts

Short answers:

Optimal bundle: $(x_L, x_{NL}) = (w - pL, w - pL)$

SOC: det $H = \frac{p(1-p)}{(w-pL)^2} > 0$

Envelope theorem: $\partial E(u)^*/\partial p = \frac{-L}{w-pL} < 0$

Alex is better-off. Proof:

Utility without insurance: $E(u_0) = p \ln(w - L) + (1 - p) \ln w$.

Utility with insurance: $E(u)^* = \ln(w - pL)$

Using concavity of ln: $E(u)^* > E(u_0)$.

8.5 Fall-exam retake, 25.01.2016

1. Find the total differential for the function $f(x, y) = x^2y^2 + xy^2 + 2x + 4y$. Using the total differential find approximately $f(1.001, 1.999)$

2. The system of equations defines $x(z)$ and $y(z)$:

$$\begin{cases} x^2 + zxy + y^2 + 6z + y^3 = 10 \\ y^3x^2 + 3x + 2y + z = 7 \end{cases}$$

Find $x'(z)$ at the point $x = 1$ and $y = 1$.

3. Find the local maxima and minima of the function $f(x, y) = x^4 + 2y^4 - xy$. Determine whether the extrema you have found are global or local.

4. Calculate all partial derivatives of the first and second order of u with respect to x and y if $u = f(\xi, \eta)$ and $\xi = x + xy$, $\eta = x/y$.

5. Use Lagrange multipliers to find the height and radius of a cylinder with the least possible surface area among those with a volume of 6π m^3. Make sure you check the second order condition for minimisation.

6. Consider the function $f(x) = h(x) - ax$, where the function h is twice differentiable and $h''(x) < 0$ for all x. The global maximum of f is denoted by $x^*(a)$.

   (a) Find dx^*/da

   (b) It is known that for $a = 1$ the optimal point is $x^* = 3$ and the value of maximum is 2016. What is the approximate value of maximum for $a = 1.01$?

Part B.

7. Two simple independent problems :)

   (a) **(10 points)** Find all values of the parameter λ such that the function $f = 2x^2 + 3y^2 + z^2 + 4xy - 2xz - 2\lambda yz$ is convex.

   (b) **(10 points)** Write down the equation of the tangent plane to a surface $z = x^3 - y^3$ at the point $(-1; 1; -2)$.

8. **(20 points)** Consider a problem of finding the extremal values of the function $f(x, y) = e^x + e^y + cx + cy$ under the constraint $x + y = c$, where c is a positive parameter.

   (a) Find out what kind of a problem you need to set: a problem of maximization or minimization?

   (b) Let $f(x*(c), y*(c))$ be the *value function* of the problem. If c slightly increases and becomes $c + \Delta c$, estimate the change in $f(x*(c), y*(c))$. Your answer should contain c and Δc only.

8.6 Final exam, 02 april 2016

1. Solve differential equation $y'' + 2y' + y = 2x + 3e^{-x}\sqrt{x+1}$.

2. Given the difference equation $3y_{t+2} + 2y_{t+1} + \gamma y_t = 5$ with the real parameter γ, find all the values of γ for whose the time path of this equation is convergent. Choose some value of γ beyond the found range and show that the corresponding time path is divergent.

3. Find all mixed Nash equilibria of the following game

	d	e	f
a	0;0	0;-1	6;-2
b	0;0	-1;1	7;0
c	-1;5	-1;4	5;9

4. For non-negative y_i minimize the function $11y_1 + 5y_2 + 4y_3 + 4y_4$ subject to constraints $y_1 + y_2 + y_3 + 0.1y_4 \geq 2$ and $2y_1 + y_2 + 0.1y_3 + y_4 \geq 7$.

5. Use the Lagrange multiplier method to find the maximum value of $f(x, y, z) = (7+x)^2(1+y)^2(7+z)^2$ among positive numbers subject to $x^2 + 49y^2 + z^2 = 100$.

6. James Bond added $x^3 + 7x - 8y - 6y^3$ to the unknown homogeneous function $f(x, y)$. The new function was homogeneous once again! Please help the Secret Service agent recover the function f if $f(1, 1) = 1$.

7. Solve initial-value problem for the difference equation

$$y_{t+3} - 4y_{t+2} + 5y_{t+1} - 2y_t = 1$$

with the initial values $y_0 = y_1 = y_2 = 0$.

Hint: one of the characteristic roots is 1.

8. Using Lagrange multipliers maximize the function $f(x_1, x_2, x_3) = 2x_1 - x_2 - 3x_3$ subject to constraints: $3(x_1 - 1)^2 + (x_2 + 1)^2 + 2x_3^2 \leq 4$ and $x_1, x_2, x_3 \geq 0$. Find the point(s) of maximum and maximum value of f.

8.7 Final exam, 02 april 2016 - Marking scheme

1. A1

   - 5pt: characteristic equation 2pt, characteristic roots 1pt, complementary function 2pt
   - 2pt: linear part of particular solution (1pt for the form, 1pt for coefficients)
   - 3pt: second part of particular solution by variation of parameters method

2. A2

   - 2pt: characteristic roots 1pt, for convergence their absolute values are less than one 1pt
   - 5pt: distinct real roots case 2pt, same roots case 1pt, complex roots case 2pts
   - 3pt: example of time path (particular solution 1pt, general solution 1pt, divergence 1pt)

3. A3:

   - 2pts: elimination
   - 2pts: $E(u_1)$, $E(u_2)$
   - 2pts: best responses
   - 1pt: plot
   - 3pts: all NE

4. A4:

   - 2 pts: dual
   - 2 pts: plot of constraints
   - 1 pt: optimal point on the graph
   - 1 pt: value of max
   - 2 pts: non-binding ($y_i = 0$)
   - 2 pts: binding constraints

5. A5

   - 2 pt: NDCQ
   - 1 pt: Lagrangean

- 2 pt: First-order condition
- 2 pt: solution of FOC
- 2 pt: Second-order condition and

6. A6

   - 3 pt: any attempts
   - 2 pt: homogeneity condition
   - 5 pt: correct answer

7. B7

   - order of equation 1pt, characteristic equation 3pt, solutions 3pt
   - complementary function 4pt, particular solution 4pt: 2pt for the form + 2pt for finding the coefficient
   - general solution 2pt, definite solution 3pt

8. B8

   - NDCQ 2pt, Lagrangean 2pt
   - FOC 4pt, solution of FOC 6pt, max value of function 2pt
   - SOC 4pt

9 2016-2017

9.1 MFE, mock, 2016-10-28

SECTION A

1. Consider the function $f(x, y, z) = x^3 + 2xz - 3z^3 - y^2$. Using the total differential find the approximate value of $f(1.01, 0.99, 1.02)$.

2. Consider the system

$$\begin{cases} x^3 + y^3 + z^3 = 3 \\ x + x^3 + 2y + 3y^2 + xyz + z^3 = 9 \end{cases}$$

   (a) Check whether the functions $y(z)$ and $x(z)$ are defined at a point $(1, 1, 1)$

   (b) Find $y'(z)$ and $x'(z)$

3. Consider the function $f(u) = g(x, y)$ where $x = u^2$ and $y = \cos u$. The function g has continuous second derivatives everywhere. Find $f'(u)$ and $f''(u)$.

4. The curve on the plane is defined by the equation $x^4 + y^2 + y^4 = 3$.

   (a) Find a vector that is orthogonal to the curve at the point $(1, 1)$

   (b) Find a vector that is parallel to the curve at the point $(1, 1)$

5. The function g is monotonic. The function f is the inverse of the function g. Find $f'(1)$ if it is known that $g(10) = 1$, $g'(10) = 5$, $g(1) = -2$, $g'(1) = 4$.

6. Let x be a vector, $x \in \mathbb{R}^n$, and A be $n \times n$ matrix of constants. Consider the function $f(x) = x^T Ax$.

   (a) Clearly state the Young's theorem

   (b) Express the Hesse matrix of f using A and A^T.

SECTION B

7. Consider the function $f(x, y) = \sqrt[3]{x^3 + y^3}$.

   (a) (7 points) Find $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$. Is the function $f(x, y)$ continuously differentiable everywhere?

   (b) (3 points) Find equation of the tangent plane to the graph of $z = f(x, y)$ at the origin.

   (c) (10 points) Let $\Delta f = f(x, y) - f(0, 0)$. Compare Δf with the df (total differential) at the origin. Base your comparison on the existence of the limit $\lim_{x\to0,y\to0} \frac{\Delta f - df}{\sqrt{x^2+y^2}}$ as $x \to 0$ and $y \to 0$.

8. Cournot duopoly produces good Y, where $Y = y_1 + y_2$. Here y_1 is the output of the first firm and y_2 is the output of the second firm. The inverse demand on good is given by the formula $p(Y) = 1/Y$, where is $p(Y)$ the price per unit. The total costs of the firms are $TC_1(y_1) = 2y_1$ and $TC_2(y_2) = y_2$, respectively. Let the profit of the first firm be $\pi_1 = p(y_1 + y_2)y_1 - 2y_1$ and the profit of the second firm be $\pi_2 = p(y_1 + y_2)y_2 - y_2$.

   (a) (8 points) Write down the system of the first-order conditions $\begin{cases} \frac{\partial \pi_1}{\partial y_1} = 0 \\ \frac{\partial \pi_2}{\partial y_2} = 0 \end{cases}$ and solve it.

   (b) (12 points) Government decides to impose a per unit tax t on both firms. It will increase costs for them by ty_1 and ty_2 respectively. Rewrite the system of first-order conditions accounting for the tax. Find $\frac{dy_1}{dt}$ and $\frac{dy_2}{dt}$ by referring to the appropriate IFT. Check that IFT conditions are verifiable here.

9.2 MFE, fall exam, 2016-12-27

SECTION A

1. Find the second order Taylor expansion of the function $f(x, y) = \sin(e^{2x} - e^{3y})$ at a point $x = 0, y = 0$.

2. Find the limit
$$\lim_{x\to0,y\to0} \frac{x^2y^2}{x^2 + 3y^8}$$

3. The function f is defined by $f(x, y) = x^3 + 5xy^2$. Consider the graph G of the function f

   (a) Find a vector that is orthogonal to the surface of G at the $x = 1, y = 1$.

   (b) Find a vector that is parallel to the surface of G at the $x = 1, y = 1$.

4. Using Lagrange multiplier method find and classify the constrained extrema of $f(x, y, z) = 2x+3y+9z$ subject to $x^2 + y^2 + z^2 = 1$.

5. Consider the sets $B_n = \big(-1/n, (n+1)/n\big)$, and the set $A = \cap_{n=1}^{\infty} B_n$.

   (a) Is the set A bounded? Open? Closed? Compact? Convex?

   (b) Sketch the set $A \times A$.

6. Find the local maxima of the function $f(x, y) = (12 - x)x \sin y + x^2 \sin y \cos y$. Check whether these local maxima are the global ones.

SECTION B

7. Let $u(c_t)$ be utility function of consumption c_t at time t which is discrete, $t \in \mathbb{N}$, ($\mathbb{N}$ – set of natural numbers). Function u is continuously differentiable and strictly concave for $c > 0$, $u(0) = 0$, $u'(c) > 0$, $\lim_{c \to 0+} u'(c) = +\infty$.

   (a) (5 points) Consider maximization problem: $\sum_{t=1}^{T} u(c_t) \to$ max subject to $\sum_{t=1}^{T} c_t = s$, $c_t \geq 0$, where the parameter s is positive. Let $T = 2$. Show that if (c_1^*, c_2^*) is the optimal bundle then $c_1^* = c_2^*$.

   (b) (7 points) Generalize this result for any natural T. You may refer to the Lagrange method.

   (c) (8 points) Let $(c_1^*, c_2^*, c_3^*, \ldots, c_T^*)$ be the optimal bundle. Find the limit of $\sum_{t=1}^{T} u(c_t^*)$ as $T \to \infty$ or show that it does not exist.

8. In the method of least squares the straight line $a + bx$ is fit to the data $\{(x_i, y_i),\ i \in 1, 2, \ldots, n\}$, by minimizing the sum $S = \sum_{i=1}^{n} (y_i - (a + bx_i))^2$ with respect to a and b.

   (a) (15 points) Using first-order conditions find optimal a and b. Under what conditions does the solution for a and b exist?

   Hint: you may find Cauchy-Schwartz inequality useful here.

   (b) (5 points) Show that the sufficient conditions are met.

9.3 MFE, fall retake, 2017-01-20

SECTION A

1. Find the second order Taylor expansion of the function $f(x, y) = \cos(e^{2x} - 1) - \cos(e^{3y} - 1)$ at a point $x = 0$, $y = 0$.

2. Find the limit or prove that it does not exist

$$\lim_{x \to 0, y \to 0} \frac{x^2 y^2}{x^4 + 3y^4}$$

3. Consider the sphere given by $x^2 + y^2 + z^2 = 1$. Find the equation of the tangent plane to the sphere at the point $x = 1/\sqrt{3}$, $y = 1/\sqrt{3}$, $z = -1/\sqrt{3}$.

4. Using Lagrange multiplier method find and classify the constrained extrema of $f(x, y, z) = 2x + 3y + 9z$ subject to $x^2 + y^2 + 4z^2 = 1$.

5. Consider the set A on the plane (x, y) given by the inequality

$$\frac{(x^2 + y^2 - 3)(x^2 + y^2 - 10)}{x^2 + y^2 - 10} \geq 0$$

   (a) Is the set A closed? open? bounded? convex? compact?

   (b) If possible represent the set A in the form $A = B_1 \times B_2$ where each set $B_i \subset \mathbb{R}$.

6. Find and classify the critical poinst of the function $f(x, y) = \exp(-x^2 - 6y^2 + 2xy + 2y)$. Check whether these local extrema are the global ones.

SECTION B

7. Short-run total costs of a firm are given by

$$STC(q, K) = q^2 + 3qK + 4K^2 - K + \frac{1}{16},$$

where q is the output and K is the amount of capital fixed in the short-run. In the long-run the firm can always adjust the capital in order to minimize costs. Use the appropriate envelope theorem to find $MC = (TC)'$ — long-run marginal costs.

8. Solve the constrained minimization problem in two variables: $x^2 + y^2 \to \min$ subject to constraint $(x - 1)^3 = y^2$. Check firstly whether the method of Lagrange multipliers is valid to apply.

9.4 28 March 2017

SECTION A

1. Find indefinite integrals

   (a) $\int e^{3x} \sin 2x \, dx$;

   (b) $\int \frac{x+3}{x^2-6x+9} \, dx$.

2. Solve the differential equation

$$y''' - y'' + 6y' - 6 = 42 \sin(x\sqrt{6}).$$

3. Solve the difference equation

$$y_{t+2} - 6y_{t+1} + 9y_t = 5t.$$

4. The function $f(x, y)$ is non-constant and homogeneous. It is also known that $h(x, y) = f'_x(x, y) + 3x^2 y$ is homogeneous of degree 3. Find the value of $\frac{x f'_x(x,y) + y f'_y(x,y)}{f(x,y)}$.

5. Solve the following linear programming problem:

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 \to \min \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \\ 3x_1 + 5x_2 + x_3 \geq 8 \\ 5x_1 + 3x_2 + x_3 \geq 9 \end{cases}.$$

6. Maximize the function
$$11 + 10x_1 - x_1^2 - 3x_2 + 8x_3 - x_3^2$$
subject to constraints $2x_1 - x_2 + 4x_3 \leq 10$ and $x_2 \leq 100$.

SECTION B

7. Consider the second-order differential equation
$$xy'' - y' - 4x^3y = 0.$$
It can be solved by an appropriate change of the variable $t = \phi(x)$.
   (a) (10 points) Find this function $\phi(x)$ by setting the task of cancellation the term with $\frac{dy}{dt}$.
   (b) (5 points) After the substitution the transformed equation has constant coefficients. Find its general solution.
   (c) (5 points) Solve the original equation.

8. Three players play the following game. Simulteneously each of them chooses one possible bet: either 1$ or 2$. A player is declared winner if his bet is unique and wins the amount of his bet. For example, if players have chosen 1, 2 and 1 then their corresponding payoffs are 0, 2 and 0.
   (a) Find all Nash equilibria in pure strategies.
   (b) Find symmetric Nash equilibrium in mixed strategies.

9.5 marking 28 March 2017

1. 1a = 5 points: 2 points - correct 1st integration by parts, 2 points - correct 2nd integration by parts, 1 point - calculations of the integral

   2a = 5 points: 1 point - correct choice of solving method (knowledge that it's somehow connected to partial fractions), 3 poiints - correct decomposition to the sum of partial fractions, 1 point - calculations of the integral

2. 4 points - correct complementary function (1 point - characteristic equation, 2 points - roots, 1 point - answer) 6 points - correct particular integral (2 points - correct form, 2 points - all necessary derivatives, 2 points - calculations)

3. 4 points - correct complementary function (2 points - equation and roots, 2 points - form and answer) 6 points - correct particular solution (3 points - correct form, 3 points - calculations)

4. 5 points - correct degree of homogenuity of $f(x, y)$ 1 point - understanding that the answer should be constant 1 point - recalling something about Euler's theorem 3 points - right answer

5. 2 points - Dual problem 2 points - graph 3 points - minimum 3 points - minimizer

6. 1 point - NDCQ 2 points - Lagrangean 3 points - FOC 4 points - solution of FOC

7. 7a = 10 points:

   5 points - correct calculations of dy/dt and d^2y/dt^2 for some function $t = \phi(x)$ or in general form using chain rule (2 points for the 1st derivative, 3 points - for the 2nd derivative) 5 points - correct guess that the substitution is $t = x^2$ and trying to do something with it (just correct guess without calculations - 2 points)

   7b = 5 points:

   1 point - characteristic equation 2 points - roots 2 points - solution

   7c = 5 points

8. 8a = 6 points

   Every player chooses 1 is not an equlibrium (every player wants to deviate) Every player chooses 2 is not an equlibrium (every player wants to deviate) Two players choose 1 and one player chooses 2 is an equilbrium. Two players choose 2 and one player chooses 1 is an equilbrium.

   8b = 14 points

   Step 1. Idea that in symmetric equilibrium every player chooses 1 with some probability p and 2 with probability (1-p).

   Step 2. First player should be indifferent between his pure strategies.

   payoff if I choose 1 = payoff if I choose 2: $(1-p)^2 = 2p^2$ (equation = 8 points)

   Step 3. two roots $p = -1 + \sqrt{2}$ и $p = -1 - \sqrt{2}$. one root is impossible, so $p = \sqrt{2} - 1$.

   Step 1 = 3 pts, step 2 = 6 pts, step 3 = 5 pts.

10 2017-2018

10.1 Mock

SECTION A

1. Consider the function $f(x, y) = x^3 + 2x - 3xy^3 - y^2$. Using the total differential find the approximate value of $f(0.98, 1.99)$.

2. Consider the system

$$\begin{cases} x^3 + y^3 + 2z^3 = 4 \\ x + x^3 + 2y + 3y^2 + xyz + z^3 = 9 \end{cases}$$

   (a) Check whether the functions $y(z)$ and $x(z)$ are defined at a point $(1, 1, 1)$;

   (b) Find $y'(z)$ if possible.

3. If possible find the limit

$$\lim_{x\to 0, y\to 0} \frac{\exp(x^2+y^2) - 1}{x^2 + y^2 + 3|x| + |y|};$$

4. The surface in $\mathbb{R}^3$ is defined by the equation $x^3 + 2y^3 + 3z^3 + zxy = 7$.

   (a) Find a unit vector that is orthogonal to the tangent plane at the point $(x = 1, y = 1, z = 1)$.

(b) Find the equation of the tangent plane.

5. Consider the function $f(x, y) = x^2 + y^3 + xy$, the vector $v = (1, 2)$ and the point $A = (-1, -1)$.

   (a) Find the gradient of f at the point A.

   (b) Find the directional derivative of f at the point A in the direction given by v.

6. Let x_n be a sequence in $\mathbb{R}^2$ given by

$$x_n = \begin{pmatrix} \cos\left(2\pi n/3\right) \\ \sin\left(2\pi(n^2 - 1)/3n\right) \end{pmatrix}$$

   (a) Find the accumulation points of this sequence.

   (b) Find the limit of this sequence if it exists.

SECTION B

7. The production function is given by $q(K, L) = (K^\rho + L^\rho)^{1/\rho}$, where $K > 0$, $L > 0$, $\rho \leq 1$ and $\rho \neq 0$.

   (a) MRTS (marginal rate of technical substitution) is defined as MRTS $= -\frac{dK}{dL}\ |_{q(K,L)=const}$. Using implicit function theorem find MRTS and express your answer as a function of K/L alone.

   (b) Let $t = K/L$. Find the derivative $\sigma = \frac{d \ln t}{d \ln \text{MRTS}}$.

   (c) Suggest at least one production function $q(K, L)$ such that $\sigma = 1$.

8. The closed first quadrant is denoted by $\bar{\mathbb{R}}^2_+$. Consider the function $F(x, y) = xy - (x^p/p + y^q/q)$ defined on $\bar{\mathbb{R}}^2_+$, where $p > 1$, $q > 1$ and $(p - 1)(q - 1) = 1$.

   (a) Find the set $S \subset \bar{\mathbb{R}}^2_+$ such that $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ at the same time. Sketch the set S.

   (b) What are the possible values of the function $F(x, y)$ for $(x, y) \in S$?

   (c) Is it true that the sign of $F(x, y)$ is the same for all $(x, y) \notin S$?

10.2 Midterm

1. [10 points] Check whether the function $f(x, y) = 4x^4 + y^2 + y^4 + 4x^2 + xy$ is concave, convex or neither.

2. [10 points] Find and classify the local extrema of $f(x, y) = 4 + x^3 + y^3 - 3xy$.

3. [10 points] Using Lagrange multiplier method find and classify the constrained extrema of $f(x, y, z) = 5x + 4y + 8z$ subject to $x^2 + y^2 + z^2 = 1$.

4. [10 points] Microbe Veniamin lives on the (x, y) plane. Veniamin likes to hop and likes the function $f(x, y) = 5x^2 + 2y^4$. From the point (x_t, y_t) he hops into the point

$$(x_{t+1}, y_{t+1}) = (x_t, y_t) - 0.001 \cdot \text{grad}\, f(x_t, y_t)$$

   Veniamin starts hopping from the point $(x = 1, y = 2)$.

   (a) What are the exact coordinates of Veniamin after one hop?

(b) Where he may find himself after 10^{2017} hops?

5. [10 points] Consider the function $p(x_1, x_2) = h(x_1 + x_2 a)$, where $h(t) = \exp(t)/(1 + \exp(t))$ and a is a fixed parameter. Find the second order Taylor expansion of p at $(x_1 = 0, x_2 = 0)$.

6. [10 points] Consider the function f defined for $x > 0$:

$$f(x) = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \dots}}}$$

   (a) Simplify the expression $f(x) - \frac{1}{f(x)}$;

   (b) Using implicit function theorem find $f'(1)$.

7. Let $f(x_1,\, x_2)$ be twice continuously differentiable function whose Hessian is negative definite. Consider long-run profit maximization problem

$$f(x_1,\, x_2) - w_1 x_1 - w_2 x_2 \to \max_{x_1, x_2},$$

   where $w_1,\, w_2 > 0$ are factor prices. The optimal bundle of factors consists of x_1^L, x_2^L which are called demand on factors.

   (a) [10 points] Write down first-order conditions for the problem and check that IFT is applicable here in order to find x_1^L, x_2^L.

   (b) [10 points] Prove that $\frac{\partial x_1^L}{\partial w_1} < 0$.

8. The previous problem is stated in the long-run. In the short-run the quantity of x_2 is fixed, i.e. $x_2 = b > 0$. The value function $\pi_L^*(w_1,\, w_2)$ for long-run problem is called profit function. It is clear that $\pi_L^*(w_1,\, w_2) \geqslant \pi_S^*$, where $\pi_S^*(w_1,\, w_2)$ is the profit function for the new short-run problem

$$f(x_1,\, b) - w_1 x_1 - w_2 b \to \max_{x_1}$$

   (a) [10 points] Let $g(x_1, x_2) \in C^2$ be an arbitrary function that takes the minimum value at $(\tilde{x}_1, \tilde{x}_2)$. Provide the argument justifying that $\frac{\partial^2 g}{\partial x_1^2} \geqslant 0$ at $(\tilde{x}_1, \tilde{x}_2)$.

   (b) [5 points] Let $z = \pi_L^* - \pi_S^*$. Explain why $\frac{\partial^2 z}{\partial w_1^2} \geqslant 0$.

   (c) [5 points] Using Envelope Theorem show that $\frac{\partial x_1^L}{\partial w_1} \leqslant \frac{\partial x_1^S}{\partial w_1}$, where x_1^L and x_1^S are factor demands in different periods.

10.3 Midterm marking

1. First derivatives - 2 points, Hessian matrix - 4 points, Hessian is positive definite - 2 points, Conclusion that the function is convex - 2 points

2. FOC - 2 points, Solution of FOC - 2 points, Hessian matrix - 2 points, Classification of critical point - 4 points: 2 points for each

3. NDCQ - 1 point, Lagrangean - 1 point, FOC - 2 points, Solution of FOC - 2 points, SOC - 4 points: 2 points for each extremum

4. Gradient - 2 points, Correct second step for x - 2 points, Correct second step for y - 2 points, Correct answer for coordinate x with explanation - 2 points, Correct answer for coordinate y with explanation - 2 points

5. general formula for second order Taylor expansion — 2 points, value at the initial point — 1 point, first order derivatives — 3 points, second order derivatives — 4 points.

6. point a — 2 points, f' expressed in terms of f — 5 points, exact value of f — 3 points.

7. a) FOC – 4 points; 1-st and 2-nd IFT conditions – 2 points; 3-d IFT conditions – 4 points; b) correct formula for derivative - 5 points

8. a) only positive definite Hessian – 5 points b) application of a) – 5 points c) only Envelope theorem – 3 points

10.4 Midterm retake

1. Find the local maxima and minima of the function $f(x, y) = x^4 + 2y^4 - xy$.

2. Find all critical points of the function $z = z(x, y)$ implicitly defined by the equation

$$x^2 + y^2 + z^2 - xz - yz + x + y + 4z + 1 = 0 \tag{6}$$

3. Using Lagrange multiplier method find and classify the constrained extrema of $f(x, y, z) = 5x+4y+8z$ subject to $x^2 + y^2 + z^2 = 1$.

4. For the function $f(x, y) = 2xy + 3$ find the level curve and the equation for its tangent at the point $(1, 2)$.

5. Use the chain rule to find $f'(x)$ and $f''(x)$ for $f(x) = u(a, b, x)$ where $a = \sin(x)$ and $b = x^3$.

6. Consider the function $f(x, y) = x^2 + y^3 - xy + 3y$ at the point $(2; 1)$. Find all the directions in which the growth rate of the function constitutes 80% of the maximal possible growth rate at that point.

7. Consider an expenditure minimization problem for the agent whose utility function is $u(x_1,\ x_2) = \sqrt{x_1 x_2}$. Let $\bar{u}$ be a prescribed level of utility. Then find solution to the problem

$$\begin{cases} p_1 x_1 + p_2 x_2 \to \min \\ \sqrt{x_1 x_2} = \bar{u} \end{cases},$$

where $x_1,\ x_2 \geq 0$. Let $e = p_1 \tilde{x}_1 + p_2 \tilde{x}_2$ be the expenditure function, $\tilde{x}_1$ and $\tilde{x}_2$ being the solutions of the minimization problem. By using the appropriate envelope theorem find $\frac{\partial e}{\partial p_1}$ and $\frac{\partial e}{\partial p_2}$.

8. For what values of p, q is the function $f(x, y) = x^p + y^q$ convex or concave. Consider only $x > 0$, $y > 0$.

10.5 Mitderm retake marking

1. formulate FOC = 2 pts, solve FOC = 4 pts, SOC = 4 pts;
2. system of 3 equations and one inequality = 5 pts; solution = 5 pts; Frequent incomplete solution: only two equations (with argument) = 3 pts;
3. NDCQ = 1 pt; Lagrangian function = 1 pt; formulate FOC = 2 pts; solve FOC = 3 pts; SOC = 3 pts;
4. level curve = 5 pts; tangent line = 5 pts;
5. first derivative = 4 pts; second derivative = 6 pts;
6. grad = 2 pts; maximum growth speed = 1 pt; equation for direction = 2 pt; solution = 5 pts;
7. NDCQ = 2 pts; formulate FOC = 2 pts; solve FOC = 6 pts; envelope theorem = 4 pts;
8. Hesse matrix = 4 pts, convexity = 8 pts, concavity = 8 pts.

10.6 March exam

1. Find the general solution of the differential equation $y'' + 4y' + 5y = 10x + 23$.
2. Find the general solution of the difference equation $y_{t+2} - 6y_{t+1} + 9y_t = 8$.
3. Find all pure and mixed Nash equilibria in the following game

	d	e	f
a	(5, 6)	(1, 0)	(2, 2)
b	(1, 1)	(4, 4)	(2, 2)
c	(2, 4)	(2, 2)	(1, 3)

4. Solve the following linear programming problem:

$$\begin{cases} 2x_1 + 2x_2 + 5x_3 \to \text{min} \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \\ 3x_1 + 5x_2 + x_3 \geq 9 \\ 5x_1 + 3x_2 + x_3 \geq 8 \end{cases} .$$

5. Expand the function $f(x) = \exp(1 - \cos^2(\ln(1 + 2x)))$ as a power series in terms up to x^5. State the range for which your expansion is valid.
6. Sketch the set $\Re(z \cdot (1 + i)) + z\bar{z} = 0$ on the complex plane.
7. A firm with the production function $y = x_1x_2 + x_1 + x_2$ employs factors $x_1, x_2 \geq 0$.
   a) [15 points] Minimize the function $100x_1 + x_2$ subject to constraint $x_1x_2 + x_1 + x_2 \geq y$, where $y \geq 0$ is the output. Justify the found optimal bundle(s).
   b) [5 points] Find the total costs function $TC(y)$.

8. Solve second-order differential equation $xy'' - (2x+1)y' + 2y = 0$ following hints:

   a) [5 points] Find a particular solution by substituting for $y(x)$ a polynomial $\tilde{y}(x)$ with the undetermined coefficients starting with the smallest degree possible.

   b) [15 points] Let $\tilde{y}(x)$ be the solution found in a), introduce new function $z(x) = y(x)/\tilde{y}(x)$. Derive equation for z and solve it. Then find y.

10.7 Marh marking scheme

1. 10 points

2. 10 points

3. Dual — 2 points, plot — 3 points, solution of dual — 2 points, original x_i — 3 points.

   Solution of dual $y = (0.25, 0.25)$, original $x = (13/16, 21/16, 0)$

4. NE in pure — 2 points, elimination with correct argumentation — 3 points, best response functions — 2 points, plot — 2 points, conclusion — 1 point.

   2 pure NEs, and $p = 1/3$, $q = 3/7$.

5. 10 points

6. 10 points

7. a. NDCQ - 2 points, Lagrangian - 2 points, FOC - 4 ponts, Solutions - 6 points (2 points each), 1 point for describing dependence on the y-value

   b. 2 points for each of the functions, 1 point for describing dependence on the y-value

8. a. Correct form of polynomial and its derivative - 2 points, plugging into initial equation and simplification - 2 points, result - 1 point

   b. The first derivative of y in terms of z and x - 2 points, he second derivative of y in terms of z and x - 2 points, differential equation with z' and z" - 2 points, substitution u=z' and the first order differential equation with u - 2 points, correct partial fractions and correct integral for u - 4 points, correct integral for z and final expression for y - 3 points.

11 2018-2019

11.1 MFE, 2018-10-26

1. (10 points) Consider the function $f(x, y) = x^3 + y^3 + 2xy$. Using the total differential find the approximate value of $f(1.98, 0.99)$.

2. (10 points) Consider the system

$$\begin{cases} x^3 + y^3 + z^2 = 3 \\ x + x^3 + 2y^3 x = 4 \end{cases}$$

   (a) Check whether the functions $z(y)$ and $x(y)$ are defined at a point $(1, 1, 1)$;

(b) Find $z'(y)$ if possible.

3. (10 points) Consider the function $f(x, y, z) = x^2 + 9y^2 + 2xy + \alpha z^2$.

   (a) Find the Hesse matrix. Clearly state the Young theorem if you use it.

   (b) For each value of α find the definiteness of Hesse matrix.

4. (10 points) Consider the function $u(x) = f(a, b, c)$, where $a = \alpha(q, r)$, $b = \beta(x)$, $c = \gamma(x, q)$, $q = x^2$ and $r = x^3$. All the functions are differentiable. Find $u'(x)$.

5. (10 points) Consider the function $f(x, y) = x^2+y^2+4y$. The microbe Veniamin is standing at $(1, 1)$ and is moving according to a simple rule. From a point (a, b) he jumps into the point $(a, b)-0.01 \operatorname{grad} f(a, b)$.

   (a) Where Veniamin will be after two jumps?

   (b) What will be the approximate location of Veniamin after 2018 jumps?

6. (10 points) Let $h(a, b) = \int_a^b \exp(-t^2) \cdot dt$. Find the grad $h(1, 2)$.

7. The domain of the function $z = xy-\frac{2}{3}x\sqrt{x}-\frac{1}{3}y^3+5x+3y$ is the nonnegative quadrant $\{x \geq 0, y \geq 0\}$.

   (a) (10 points) Find the equation of the tangent plane to the graph of z at $(1, 1, 8)$.

   (b) (10 points) Let grad $z(1, 1) = c$. Find all such points that grad $z(x, y) = c$.

8. Two drivers on a lonely island get utility from fast driving and money. Let $0 \leq x_1 \leq 1$ be the speed of the first car and $0 \leq x_2 \leq 1$ be the speed of the second car, respectively. They have the same amount of wealth $I > 1$. Utilities of the drivers are $U_1(x_1, x_2) = x_1 + I \cdot (1 - x_1x_2)$ and $U_2(x_1, x_2) = \ln x_2 + I \cdot (1 - x_1x_2)$.

   (a) (7 points) On (x_1, x_2)-plane draw the solutions of the equations $\frac{\partial U_1}{\partial x_1} = 0$ and $\frac{\partial U_2}{\partial x_2} = 0$.

   (b) (10 points) Let $(x_1^*, x_2^*) = (1, 1/I)$. Show that the system of inequalities hold $U_1(x_1^*, x_2^*) \geq U_1(x_1, x_2^*)$ and $U_2(x_1^*, x_2^*) \geq U_2(x_1^*, x_2)$.

   (c) (3 points) Explain why even the small bribe offered by the second driver will stop the first driver from using his car?

11.2 MFE, 2018-10-26 — marking

1. Correct formula of the total differential (2).

   Correct partial derivatives (2).

   Correct calculation (6).

2. All three IFT conditions (4).

   Correct formula for derivative (2).

   Correct calculations (4).

3. **Problem 3.**

Consider the function $f(x, y, z) = x^2 + 10y^2 + 2xy + \alpha z$
(a) Find the Hesse matrix. Clearly state the Young theorem if you use it.
(b) For each value of α find the definiteness of Hesse matrix.

Solution. (a)

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 20 & 0 \\ 0 & 0 & 2\alpha \end{pmatrix}$$

(b) $\Delta_1 = 2 > 0$, $\Delta_2 = 40 - 4 = 36 > 0$, $\Delta_3 = 72\alpha$.
1. If $\alpha > 0$ then all corner principal minors are greater than 0. According to Silvester criterion the Hesse matrix is positive definite.

2. If $\alpha < 0$ then $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 < 0$. According to Silvester criterion the matrix is indefinite.

3. If $\alpha = 0$ then we need to use generalized Silvester criterion (three corner determinants are not enough!).

- Minors of the 1st order: 2, 20, 2α. All are non-negative.
- Minors of the 2nd order: 36, 0, 0. All are non-negative.
- Minors of the 3rd order (the determinant of all matrix) is 0, also non-negative.

Thus, the matrix is positive semi-definite.

Marking scheme
(a)
2 pts for the Young theorem
2 pts for all partial derivatives
1 pt for the Hesse matrix

OR

4 pts for all partial derivatives if no Young theorem is used
1 pt for the Hesse matrix

(b)
1 pt for all three corner principal minors
1 pt for the case $\alpha > 0$
1 pt for the case $\alpha < 0$
2 pts for the case $\alpha = 0$

4. Consider the function $u(x) = f(a, b, c)$, where $a = \alpha(q, r)$, $b = \beta(x)$, $c = \gamma(x, q)$, $q = -x^2$ and $r = x^3$. All the functions are differentiable. Find $u'(x)$.

Solution+Marking scheme

$$u' = \frac{\partial f}{\partial a} \underbrace{\frac{\partial \alpha}{\partial q}}_{1pt} \underbrace{(-2x)}_{1pt} + \frac{\partial f}{\partial a} \underbrace{\frac{\partial \alpha}{\partial r}}_{1pt} \underbrace{3x^2}_{1pt} + \frac{\partial f}{\partial b} \underbrace{\frac{\partial \beta}{\partial x}}_{1pt} + \frac{\partial f}{\partial c} \underbrace{\frac{\partial \gamma}{\partial x}}_{1pt} + \frac{\partial f}{\partial c} \underbrace{\frac{\partial \gamma}{\partial q}}_{1pt} \underbrace{(-2x)}_{1pt}$$

+ 2 pts for the correct form of the answer

Penalties

-1 pt for writing down each extra term in the answer
-1 pt for using a instead of α in $\frac{\partial\alpha}{\partial q}$ (or $\frac{\partial\alpha}{\partial r}$). Not more than 1 pt penalty even if the mistake appeared twice.
-1 pt for using b instead of β (the same as above)
-1 pt for using instead of γ (the same as above)
-1 pt for using u instead of f (the same as above)

5. (a) Correct expressions of partial derivatives - 2 pts. Correct values of partial derivatives at the point (1,1) - 1 pt. Correct coordinates of the point after the first jump - 1 pt. Gradient at new point - 1pt. Coordinates of the point after the second jump - 1 pt.

   (b) 2 pts for each coordinate.

6. Formula of derivative of definite integral - 2 pts. Expression of each partial derivative - 2 pts. Correct value of each derivative - 2 pts.

7. (a) Derivatives = 6 pts (3 pts + 3 pts), formula of tangent plane = 4 pts.

   (b) Gradient at (1, 1) = 4 pts, system of equation = 3 pts, answer = 3 pts.
   Answer: all points (x, y) such that $y = \sqrt{x}$.

8. (a) Derivatives = 4 pts (2 pts + 2 pts), plot = 3 pts. $x_2 = 1/I$, $x_2 = 1/Ix_1$.

   (b) Inequality $I \geq I$ = 5 pts (4 pts for statement and 1 pt for proof), hard inequality = 5 pts (1 pt for statement and 4 pts for proof).

   (c) 3 pts.

11.3 Midterm, 2018-12-27

1. (10 points) Find the limit or prove that it does not exist

$$\lim_{x,y\to 0} \frac{1-\cos(x+2y)}{\sin(xy)}$$

2. (10 points) Using Lagrange multipliers find the extrema of the function $f(x, y) = x^2 + 4xy + y^2$ subject to $x^2 + 2y^2 = 16$.

3. (10 points) Consider the function $u(x, y) = x^2 - 4xy + ay^2 - \ln(xy)$ for $x > 0$ and $y > 0$. For which values of a the function u is convex?

4. (10 points) Find the second order Taylor approximation of a function $f(x, y) = x^5y^3 + 3x^2y$ at a point $x = 1, y = 2$.

5. (10 points) Use Lagrange multipliers to find the height and radius of a cylinder with the maximal volume among those with a surface $S = 10\pi$. Make sure you check the second order condition for maximisation.

6. (10 points) Let $h(x, y) = kx^2 + 6xy + 14y^2 + 4y + 10$.

(a) Find the minimal value of the function h for $k = 2$.

(b) Using envelope theorem find approximate minimal value of h for $k = 1.98$.

7. This is a road construction costs minimization problem. Let the terrain profile be represented by the function $y(t) = \begin{cases} 3 - 3|t|, & \text{if } |t| \le 1 \\ 0, & \text{otherwise} \end{cases}$. A road works start from the east of this hill (on the negative half-axis). Excavation costs can be found by the formula

$$I(a, b) = \int_{-b/a}^{0} (at + b - y(t))^2 \, dt$$

where $at + b$ is the road profile we need to find with the constants $a > 0$, $b > 0$, and $b/a \ge 1$.

(a) (15 points) Find the Hessian matrix of $I(a, b)$ and check its sign-definitness.

(b) (5 points) Let (a^*, b^*) be the solution of the first-order conditions for the minimization problem. Justify your choice for the (a^*, b^*) values.

8. (continuation of problem 7) (20 points)

Allowable grade of the road satisfies constraint $a \le 1$. Under this constraint solve the problem $I(a, b) \to \min$ with respect to b.

11.4 Midterm, 2018-12-27, marking

1. Polar coordinates — 1 point, Correct answer (limit does not exist) — 1 point, Correct Talylor expansion/ ideas about equivalency/L'Hopitale rule for the corresponding fucntion of ONE variable (after substitution) — 3 points, Everything else — 5 points

2. NDCQ — 1 point, Lagrange function — 2 points, FOC — 2 points, Solution of FOC — 3 points, SOC — 2 points

3. Each derivative 1 point (total 5 points), Hesse matrix – 1 point, conditions for positive (semi) definiteness – 1 points, correct answer (a greater than some real number) – 3 points

4. Each derivative and its value 1 point (total 5 points), correct formula for Taylor approximation – 5 points

5. NDCQ 1 point, Lagrange 1 point, FOC 3 points, Critical points 1 point, Hessian Matrix 2 points, Point classification 2 points

6. FOC 2 points, Critical point 3 points, Hessian 2 points, Envelope theorem 3 points

7. Many solutions are possible. First solution, without explicit $I(a, b)$. Each derivative I'_a, I'_b — 4 points. Second solution, with explicit $I(a, b)$. Function $I(a, b)$ — 4 points, each first derivative — 2 points. Three second derivatives — 1 point each. Hesse matrix — 1 point. Definitness — 3 points.

   Point b. FOC — 2 points, SOC — 3 points.

8. FOC for correct $I(a, b)$ — 10 points, SOC for correct $I(a, b)$ — 10 points.

   FOC for incorrect $I(a, b)$ — 5 points, SOC for incorrect $I(a, b)$ — 5 points.

11.5 March-exam, 2019-03-31

1. (10 points) Solve the equation

$$z^3 + iz^2 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z - i\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 0$$

   in complex numbers if one of the roots is $-i$.

2. (10 points) The function $f(x, y, z)$ is homogeneous of degree 9. Consider two functions, $h(x, y, z) = (x+3y+5z)\frac{\partial^2 f}{\partial x \partial y}$ and $q(x, y, z) = h(x, y, z)+(x^3+3xyz)\frac{\partial f}{\partial x}$. Check the homogeneity of these functions, for homogeneous functions state the degree.

3. (10 points) Find the minimal value of $x_1 + 2x_2 + 3x_3$ for non-negative values of all x_i given that $x_1 + 2x_2 + 2x_3 \geq 3$ and $3x_1 + 2x_2 + x_3 \geq 4$.

4. (10 points) Bill and John are relaxing in a pub. Bill decided to spend 10 dollars, John decided to spend 2 dollars. Money is infinitely divisible. They love the same music. Each can spend money on music or on drinks. The utility of each player is $u_i = (m_1 + m_2) \cdot d_i$, where $(m_1 + m_2)$ — is the total sum of money spent on music by both players and d_i — the personal expenses on drinks.

   Find all Nash Equilibria in pure strategies.

5. (10 points) Find the global maximum of $x + 2y + 3z$ given that $x^2 + y^2 + z^2 \leq 6$.

6. (10 points) Find the general solution of the difference equation $y_{t+3} - 2y_{t+2} + y_{t+1} - 2y_t = 2^t$.

7. In exchange economy of two agents and two goods all Pareto-optimal allocations can be found by solving the maximization problem

$$\begin{cases} u_1 = 2\sqrt{x_1} + y_1 \to \max \\ u_2 = 2x_2 + y_2 \geq \bar{u}_2; \\ x_1 + x_2 = 1; \\ y_1 + y_2 = 1, \end{cases}$$

   where $\bar{u}_2$ is a nonnegative parameter and all the amounts of goods x_1, x_2, y_1, y_2 are consumed by agents in nonnegative quantities.

   (a) (5 points) Write the Kuhn-Tucker Lagrangian of the problem and set the system of first-order Kuhn-Tucker conditions;

   (b) (5 points) Solve it in particular case when all goods are consumed in positive quantities;

   (c) (10 points) Solve the system completely and find all corner solutions.

8. (20 points)

   Consider the system of differential equations

$$\begin{cases} \dot{x} = x - y + \frac{\sin t + \cos t}{2 \sin t}; \\ \dot{y} = 2x - y. \end{cases}$$

(a) (4 points) Reduce the system to a single equation for $y(t)$.

(b) (16 points) By applying the variation of parameters method or otherwise find general solution for $y(t)$.

Note: you don't need to find $x(t)$.

12 2019-2020

12.1 MFE, October mock, 2019-10-25

The october exam was shorter than usual as it fell on the same day with linear algebra exam.

1. (15 points) Consider the function $f(x, y) = x^3 - 3y^3 + 2xy$. Using the total differential find the approximate value of $f(1.98, 0.99)$.

2. (15 points) Consider the system

$$\begin{cases} 3x^3 + y^3 + z^2 = 5 \\ x + x^3 + 2y^3 x = 4 \end{cases}$$

   (a) Check whether the functions $z(y)$ and $x(y)$ are defined at a point $(1, 1, 1)$;

   (b) Find $z'(y)$ if possible.

3. (15 points) Consider the function $h(b) = f(f(f(b \cdot f(b))))$. Find dh/db for $b = 1$ if it is known that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, $f'(1) = 3$, $f'(3) = 2$, $f'(2) = 1$.

4. (15 points) Consider the function $f(x, y) = xyz^3$, the vector $v = (1, 2)$ and the point $A = (-1, -1)$.

   (a) Find the gradient of f at the point A.

   (b) Find the directional derivative of f at the point A in the direction given by v.

5. (15 points) Provide an explicit example of a sequence in $\mathbb{R}^2$ that is unbounded and has exactly two accumulation points.

6. Two identical firms compete in a labor market with the supply function $w(L) = w_0 + aL$, where $w_0 > 0$, $a > 0$ and L is the labor amount supplied at the wage rate w.

   In order to find equilibrium one has to solve the system of equations

$$\begin{cases} f(L_1) - ME_1 = 0 \\ f(L_2) - ME_2 = 0 \end{cases},$$

   where $f'(L) < 0$ for all $L > 0$ and ME_1, ME_2 are marginal expenses which are found by differentiation, $ME_i = \partial(w(L)L_i)/\partial L_i$ for $i \in \{1, 2\}$ and $L = L_1 + L_2$.

   Suppose the equilibrium exists.

   (a) (10 points) Prove that $L_1^* = L_2^*$.

   (b) (15 points) Find $\partial L_1^*/\partial w_0$.

13 I wish for more problems...

Here we present more problems for practice.

1. A production function $y = f(x_1, x_2)$ exhibits constant returns to scale, that is $f(tx_1, tx_2) = tf(x_1, x_2)$ for every $t > 0$, where $x_1, x_2 \geq 0$.

   (a) Let $f_1(x_1, x_2) = \frac{\partial f}{\partial x_1}$ and $f_2(x_1, x_2) = \frac{\partial f}{\partial x_2}$. Using the chain rule show that both partial derivatives of f have the property $f_i(tx_1, tx_2) = f_i(x_1, x_2)$ for $i = 1, 2$.

   (b) Let $MRTS = f_1/f_2$. Show that its value remains the same along a straight line $x_2 = \lambda x_1$, where λ is a given positive number and $x_1, x_2 > 0$.

2. Consider the function $f(x, y, z) = 2x^5 + 2xyz - z^3$. Using the total differential find the approximate value of $f(1.02, 0.99, 1)$.

3. Consider the function $u(x, y) = 4f(r)$ where $r = \sqrt{x^2 + y^2}$. Is it possible to represent the function $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ as a function of r alone, i.e. $g(r)$? If yes, then find $g(r)$.

4. Given the system

$$\begin{cases} u^2 - w^2 + x^2 + y^2 = 0 \\ uw + xy = 0 \end{cases}$$

   (a) Define a sufficient condition for functions $u(x, y)$ and $w(x, y)$ to be differentiable

   (b) Find $\frac{\partial w}{\partial x}$