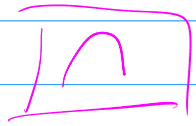
Hi !!

Kuhn-Tucker theorem under concavity !!

If

! f - concave, C^1



$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

! $h_1, h_2, \dots h_\ell$ - concave, C^1

$x \in \mathbb{R}^n$

! $D = U \cap \{ h_i(x) \geq 0,\ i \in \{1, \dots \ell\} \}$ U-open

$\hat{x}$ is not x^* | x^* may be on the border of D

[Slater's condition]: There is a point $\hat{x} \in D$ with all $h_i(\hat{x}) > 0$

" D is good set "

x^* maximizes f over D if and only if !!

F.O.C. $\dfrac{\partial L}{\partial x_i} = 0$

KT → $\dfrac{\partial L}{\partial \lambda_i} \geq 0 \quad \lambda_i \geq 0 \quad \lambda_i \cdot \dfrac{\partial L}{\partial \lambda_i} = 0$

Kuhn-Tucker

where $L = f + \lambda_1 \cdot h_1 + \dots + \lambda_\ell \cdot h_\ell$.

No need to check NDCQ, SOC !

Ex

$$\max_{x,y} \ -x^2 - x^4 - y^4 - 2y^2 + 2y + 6x$$

s.t.

$x \geq 0$

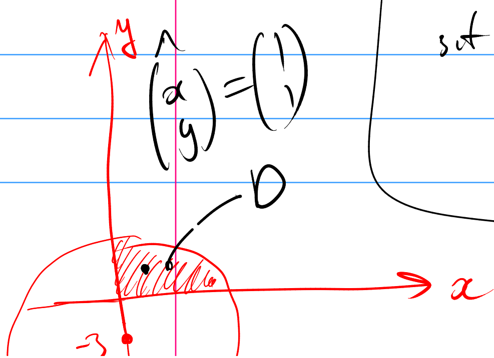
$y \geq 0$

$x^2 + y^2 + 6y \leq 100$

$x^2 + (y+3)^2 - 9 \leq 100$

$x^2 + (y+3)^2 \leq 109$

$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



1

$$f = -x^2 - x^4 - y^4 - 2y^2 + 2y + 6x$$

$$f'_x = -2x - 4x^3 + 6 \qquad f''_{xx} = -2 - 12x^2$$

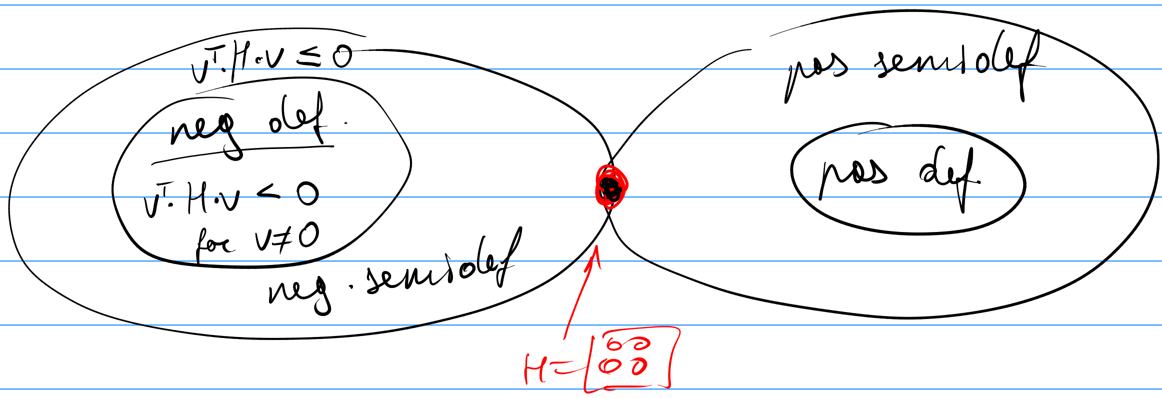
$$f'_y = -4y^3 - 4y + 2 \qquad f''_{yy} = -12y^2 - 4$$

$$f''_{xy} = 0 \qquad H = \boxed{\begin{vmatrix} -2-12x^2 & 0 \\ 0 & -12y^2-4 \end{vmatrix}}$$

$$\Delta_1 = -2 - 12x^2 < 0$$

$$\Delta_2 = (-2-12x^2)(-12y^2-4) > 0$$

H is neg def at every point $\Rightarrow$ H neg. semidef.



f - concave.

$$\begin{cases} x \ge 0 \\ y \ge 0 \\ x^2 + y^2 + 6y \le 100 \end{cases}$$

$H_{h_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ — $h_1 = x$

$\boxed{\alpha_1 \cdot x + \alpha_2 \cdot y + \alpha_3}$

linear

$\Downarrow$

h_1 – concave func.
convex

$h_2 = y$ — " —

$$100 - x^2 - y^2 - 6y \ge 0$$

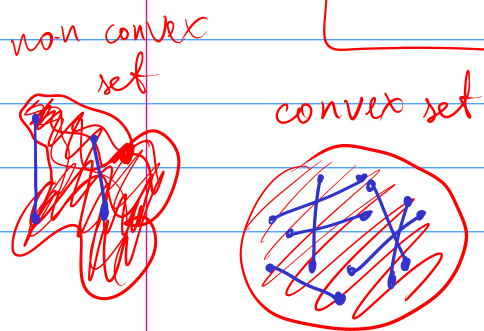
$$h_3 = 100 - x^2 - y^2 - 6y$$

$$H_{h_3} = \boxed{\begin{matrix} -2 & 0 \\ 0 & -2 \end{matrix}}$$

$$\Delta_1 = -2 < 0$$

$$\Delta_2 = (-2) \cdot (-2) = 4 > 0$$

H_{h_3} is neg def. $\Rightarrow$ H_{h_3} is neg. semi-def.



Remark:

self.

if $f, h_1 \dots h_e$ are concave, C^1 and x^* satisfies KT conditions then x^* is the maximizer.

if $f, h_1 \dots h_e$ are concave, C^1 and there are no x^* that satisfies KT cond $\Rightarrow$? some point may be the max-r.

necessary cond.

if $f, h_1 \dots h_e$ are concave, C^1 and Slater's cond is satisf, and there are no x^* that satisfies KT-cond $\Rightarrow$
$\Rightarrow$ no optimal points.

Non-negativity constraints.

$$\boxed{x \geq 0 \qquad y \geq 0 \qquad z \geq 0}$$

Ex. $x \in \mathbb{R}^2 \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$f \to \max$

$$\begin{cases} x_1 \geq 0 \\ x_2 \geq 0 \\ h_1(x_1, x_2) \geq 0 \end{cases}$$

original inequalities

let's focus on F.O.C.

$$L = f + \lambda_1 \cdot x_1 + \lambda_2 \cdot x_2 + \lambda_3 \cdot h_1$$

$$\frac{\partial L}{\partial x_1} = \boxed{\frac{\partial f}{\partial x_1} + \lambda_1 + \lambda_3 \cdot \frac{\partial h_1}{\partial x_1}} = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda_2 + \lambda_3 \cdot \frac{\partial h_1}{\partial x_2} = 0$$

$$\boxed{\frac{\partial L}{\partial \lambda_1} = x_1 \geq 0 \quad \frac{\partial L}{\partial \lambda_2} = x_2 \geq 0 \quad \frac{\partial L}{\partial \lambda_3} = h_1 \geq 0}$$

$$\lambda_1 \geq 0 \quad \lambda_2 \geq 0 \quad \lambda_3 \geq 0$$

$$\lambda_1 \cdot \frac{\partial L}{\partial \lambda_1} = 0 \qquad \lambda_2 \cdot \frac{\partial L}{\partial \lambda_2} = 0$$

$$\lambda_3 \cdot \frac{\partial L}{\partial \lambda_3} = 0$$

$$\lambda_1 = -\left(\frac{\partial f}{\partial x_1} + \lambda_3 \cdot \frac{\partial h_1}{\partial x_1}\right) \qquad \lambda_2 = -\left(\dots \right)$$

$$\frac{\partial f}{\partial x_1} + \lambda_3 \cdot \frac{\partial h_1}{\partial x_1} \leq 0 \qquad \frac{\partial f}{\partial x_2} + \lambda_3 \cdot \frac{\partial h_1}{\partial x_2} \leq 0$$

$$x_1 \cdot \left(\frac{\partial f}{\partial x_1} + \lambda_3 \cdot \frac{\partial h_1}{\partial x_1}\right) = 0$$

$$x_2 \cdot \left(\frac{\partial f}{\partial x_2} + \lambda_3 \cdot \frac{\partial h_1}{\partial x_2}\right) = 0$$

$$\tilde{L} = f + \lambda_3 \cdot h_1$$

$$\frac{\partial \tilde{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_3 \cdot \frac{\partial h_1}{\partial x_1} \leq 0 \qquad \frac{\partial \tilde{L}}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda_3 \cdot \frac{\partial h_1}{\partial x_2} \leq 0$$

$$\lambda_3 \geq 0 \qquad \frac{\partial \tilde{L}}{\partial \lambda_3} = h_1 \geq 0 \qquad \lambda_3 \cdot \frac{\partial \tilde{L}}{\partial \lambda_3} = 0$$

$$x_1 \cdot \frac{\partial \tilde{L}}{\partial x_1} = 0 \qquad x_2 \cdot \frac{\partial \tilde{L}}{\partial x_2} = 0 \qquad x_1 \geq 0 \qquad x_2 \geq 0.$$

$\boxed{\text{Theorem}}$ In the case $\boxed{x_1 \geq 0,\ x_2 \geq 0 \ldots x_n \geq 0}$

the F.O.C. may be rewritten as:

$$\frac{\partial \tilde{L}}{\partial x_i} \leq 0 \qquad x_i \geq 0 \qquad x_i \cdot \frac{\partial \tilde{L}}{\partial x_i} = 0$$

$$\frac{\partial \tilde{L}}{\partial \lambda_i} \geq 0 \qquad \lambda_i \geq 0 \qquad \lambda_i \cdot \frac{\partial \tilde{L}}{\partial \lambda_i} = 0$$

less unknowns to find

where $\tilde{L} = f + \lambda_1 \cdot h_1 + \ldots + \lambda_e \cdot h_e$ and no λ_i are introduced for non-negativity constraints ($x_i \geq 0$).

Old conditions:

$$\frac{\partial L}{\partial x_i} = 0$$

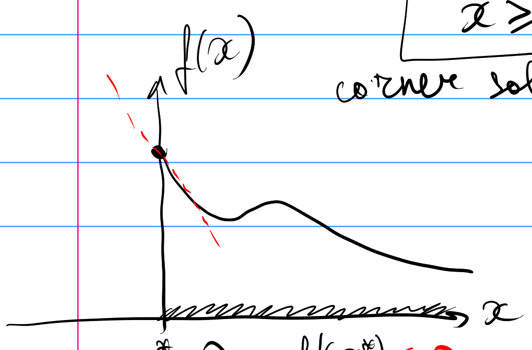
$$\frac{\partial L}{\partial \lambda_i} \geq 0 \qquad \lambda_i \geq 0 \qquad \lambda_i \cdot \frac{\partial L}{\partial \lambda_i} = 0$$

$\boxed{\text{Intuition}}$

$x \in \mathbb{R}$

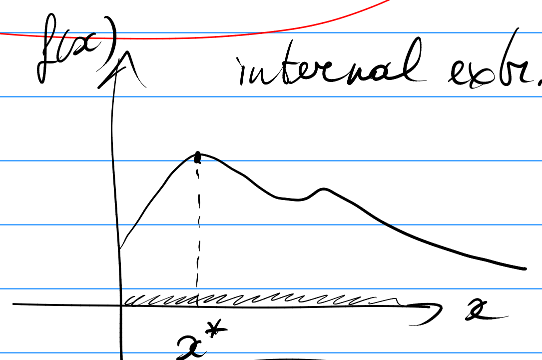
$$\boxed{\begin{array}{l} f(x) \to \max \\ x \geq 0 \end{array}}$$

corner solution



$x^* = 0 \qquad f'(x^*) < 0$

internal extr.



$$\boxed{\begin{array}{l} f'(x^*) = 0. \\ x > 0. \end{array}}$$

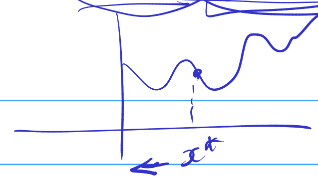
$x^* \geq 0$

$f'(x^*) \leq 0$

$x^* \cdot f'(x^*) = 0$

impossible

$x^* > 0$ $f'(x^*) < 0$

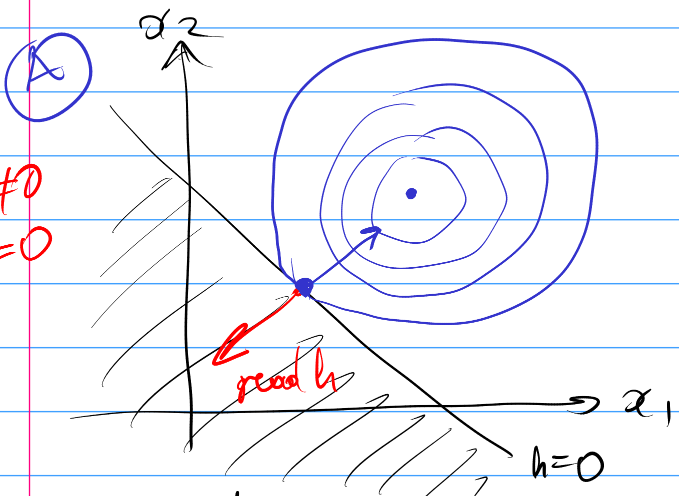


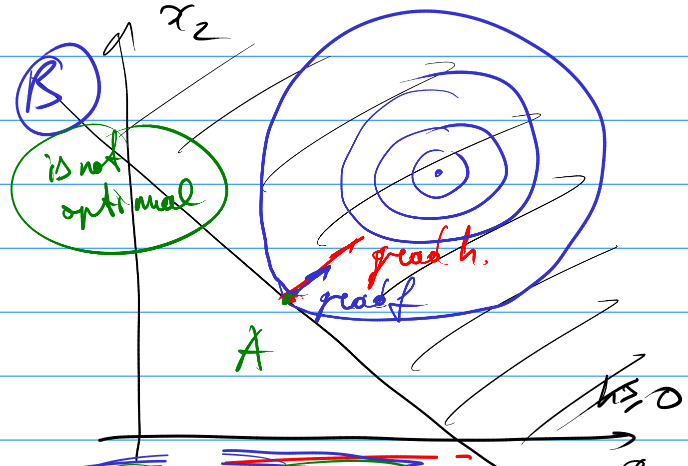
Int. $f(x_1, x_2)$ $h(x_1, x_2) \geq 0.$

$$L = f + \lambda \cdot h$$

$\lambda \neq 0$

$\frac{\partial L}{\partial \lambda} = 0$





$\boxed{\text{grad } f = -\lambda \cdot \text{grad } h}$

$\boxed{\lambda \geq 0}$

F.O.C

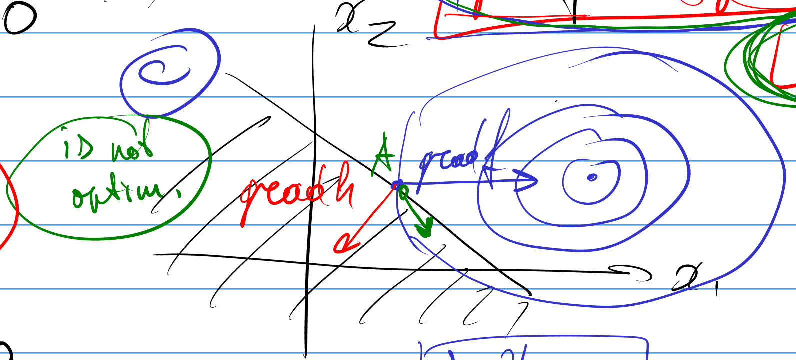
$\frac{\partial L}{\partial \lambda} = h \geq 0$

$\lambda \geq 0$ $\lambda \cdot \frac{\partial L}{\partial \lambda} = 0$

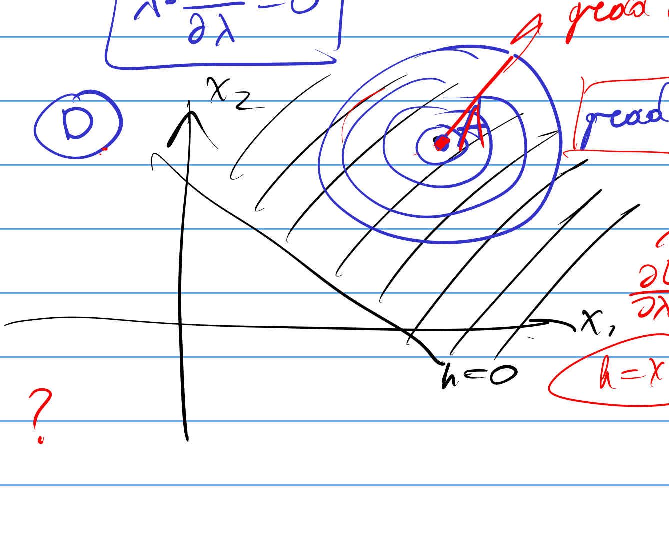
$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \lambda \cdot \frac{\partial h}{\partial x_i} = 0$$

$$\frac{\partial f}{\partial x_i} = -\lambda \frac{\partial h}{\partial x_i}$$

$$\boxed{\text{grad } f = -\lambda \cdot \text{grad } h.}$$



$\boxed{\lambda \cdot \frac{\partial L}{\partial \lambda} = 0}$



$\boxed{\text{grad } f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

$\lambda = 0$

$\frac{\partial L}{\partial \lambda} > 0$

$h = x_1 + x_2 - 2$

$\lambda \cdot \frac{\partial L}{\partial \lambda} = 0$?