Please, start every problem on a separate sheet, time allowed: 120 minutes.

1. (10 points) Solve the difference equation $x_n - 6x_{n-1} + 9x_{n-2} = 2^n$.

2. (10 points) Solve the differential equation $y'' - 6y' + 10y = \cos t$.

3. (10 points) Find the second order Taylor approximation at a point $(0, 0)$ of the function

$$f(x, y) = \cos(x + 2y) - \sin(2x + y).$$

4. (10 points) Solve the optimization problem

$$2x_1 + 3x_2 + 4x_3 \to \max$$

subject to

$$\begin{cases} x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \\ 2x_1 + 2x_2 + 3x_3 \leq 10 \\ 3x_1 + 3x_2 + 2x_3 \leq 10 \end{cases}.$$

5. (10 points) Solve the optimization problem

$$x^2 + y^2 + z^2 - 16x - 16y + 10z \to \min$$

subject to

$$x + y \leq 15.$$

6. (10 points) Sketch the set $A = \{z \mid z^2 + z \in \mathbb{R}\}$ on the complex plane $\mathbb{C}$.

7. Let $F(K, L)$ be twice continuously differentiable function with positive derivatives $F'_K > 0$, $F'_L > 0$ for all $K > 0$ and $L > 0$. The function F is homogeneous of degree 1.

   (a) (10 points) Prove that the determinant of its Hessian matrix is 0 for all $K > 0$ and $L > 0$.

   (b) (10 points) Let $Y = F(K, L)$. Denote the derivatives with respect to time by $\dot{Y}$, $\dot{K}$, $\dot{L}$. Prove that there exists a function $0 < \alpha(t) < 1$, such that $\dot{Y}/Y = \alpha(t)\dot{L}/L + (1 - \alpha(t))\dot{K}/K$.

8. (20 points) Consider the second order differential equation with constant coefficients $y'' + ay' + by = 0$. Let initial values $y(0)^2 + (y'(0))^2 \neq 0$.

   Find conditions (necessary and sufficient) on the coefficients a and b that guarantee that every solution of this equation in absolute value $|y(x)|$ will monotonically increase starting with some x_0.