

1. (10 points) Consider the function  $f(x, y) = x^3 - 2y^2x^2$  and the point  $A(1, 1)$ .
  - (a) Calculate the gradient of the function  $f$  at the point  $A$ .
  - (b) Find the second order Taylor approximation in the neighborhood of  $A$ .
2. (10 points) Consider the equation  $3x^3 + 5y^5 + z^3 + 4z = 13$ .
  - (a) Check whether the equation defines the function  $z(x, y)$  at a point  $A(1, 1, 1)$ .
  - (b) Find  $dz$  at the point  $A$ .
3. (10 points) Find all local extrema of the function  $f(x, y) = x^2y - 3xy^2 + 7xy + 2$  such that  $x \neq 0$  and  $y \neq 0$ . Classify them.
4. (10 points) Find all local constrained extrema of the function  $f(x, y, z) = x + 2y + 7z$  subject to  $\ln x + \ln y + \ln z = 0$ . Do not forget to classify extrema.
5. (10 points) Consider the function  $f(x, y) = h(x)g(y) + ax^9$ , where  $h$  and  $g$  are twice differentiable and  $a$  is a parameter. Let's denote the maximum point by  $x^*(a)$  and  $y^*(a)$  and assume that second order conditions for maximization are met.  
Find the sign of  $dx^*/da$ .
6. (10 points) The level curves of the function  $f(x, y)$  are given by the equation  $y - x^2 = c$ .  
Draw two level curves of the function  $g(x, y) = f(x - 6, |y| + 5)$ .

7. Consider a problem

$$\begin{cases} xyz \rightarrow \max \\ \text{s.t. } x + y + z = c \\ x, y, z > 0 \end{cases}$$

where  $c$  is a parameter and  $c > 0$ .

- (a) (15 points) Solve this problem using first-order conditions. Use bordered Hessian for sufficiency.  
 (b) (5 points) Use the result of part a) to show that arithmetic mean  $(x + y + z)/3$  is no less than the geometric mean  $(xyz)^{1/3}$ .

8. (a) (10 points) Consider a utility maximization problem

$$\begin{cases} u(x, y) \rightarrow \max \\ \text{s.t. } p_x x + p_y y = I \\ x, y > 0 \end{cases},$$

where  $u \in C^1$  and parameters  $p_x, p_y$  and  $I$  are positive. Let  $(x^*, y^*)$  be the solution of this problem. Form the value function  $V(p_x, p_y, I) = u(x^*, y^*)$ .

Using appropriate envelope theorem show that

$$x^* = -\frac{\partial V}{\partial p_x} / \frac{\partial V}{\partial I}, \quad y^* = -\frac{\partial V}{\partial p_y} / \frac{\partial V}{\partial I}.$$

- (b) (10 points) Let  $F(x, y)$  be a function such that  $F \in C^2$  and  $F'_y \neq 0$ . The equation  $F(x, y) = 0$  defines the implicit function  $y(x)$ .

Find the expression for  $d^2y/dx^2$ .

The expression should include only derivatives of  $F(x, y)$ .