1. (10 points) Find the limit or prove that it does not exist

$$\lim_{x\to 0, y\to 0} \frac{x^2y^2}{5x^4+2y^4}.$$

2. (10 points) Consider the function

$$f(x,y) = \int_0^{4x} 3e^{u^2}\,du + \int_0^{5y} 2\cos(u^2)\,du.$$

Find the gradient grad f at the point $(0,0)$.

3. (10 points) Consider the system

$$\begin{cases} x+y+z=2 \\ 2x^2+2y^2=z^2 \end{cases}.$$

   (a) Are the functions $x(z)$ and $y(z)$ defined in a neighborhood of the point $A(x=-1, y=1, z=2)$?
   (b) Find dx/dz at the point A if possible.

4. (10 points) The set S is defined by $S = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 2-x^2\}$. Two rectangles one on the top of the other are inscribed in S, thus they have the common side and the upper vertices lie on this parabola. Let $A_1 + A_2$ be the sum of their areas, where $A_1 > 0$, $A_2 > 0$.

   Consider the maximization problem $A_1 + A_2 \to \max$.

   (a) Solve the maximization problem or show that the maximum does not exist.
   (b) Check whether the Weierstrass theorem is applicable.

5. (10 points) Using Lagrange multiplier method find and classify the constrained extrema of $f(x,y,z) = 4x+5y+8z$ subject to $x^2+y^2+z^2=1$.

6. (10 points) Find all stationary points of $f(x,y) = -4xy + x^4 + y^4 + 21$. Classify them as local minimum, maximum or saddle point.

7. (20 points) Let the demand and supply for an ice-cream on the sunny day be $q_D = D(p,T,d)$ and $q_S = S(p,T)$ correspondingly. Here p is the price, T is the temperature on this day, d – distance of the selling place from the center of the park, $D_p < 0$, $D_T > 0$, $S_p > 0$, $S_T < 0$, $D_d < 0$.

   (a) Find analytically how the equilibrium price p^* changes with the increase of T. How does it change with the increase of d?
   (b) Let q^* be the equilibrium supply quantity. Find $\frac{\partial q^*}{\partial T}$. Find the condition when $\frac{\partial q^*}{\partial T} < 0$.

8. (20 points) We wish to build a picnic zone for the travellers along a highway. The picnic zone should be rectangular with an area of 2000 m^2 and should have a fence on the three sides not adjacent to the highway. The price of one meter of fence is equal to \$ 10.

   (a) Find the dimensions of the picnic area that minimize the fencing costs.
   (b) Using hessian or otherwise check that you have found the costs-minimizing solution.
   (c) Using the Envelope theorem estimate the change in the costs if we decrease the area of the picnic zone by 1 m^2.