Assignment 1 (Due on the week September 11 – 17)

1. Given the sets $A = \{1,\ 2,\ 3,\ 4,\ 6\}$, $B = \{2,\ 4,\ 6\}$, $C = \{1,\ 5,\ 6\}$. Find:

   (a) $(A \cap B) \cup C$,

   (b) $(A \times B) \cap (A \times C)$

2. Given the sets $A = \{a,\ b,\ c,\ d,\ e\}$, $B = \{f,\ c,\ d\}$, $C = \{a,\ f,\ c\}$. Find:

   (a) $(A \cup B) \cap C$,

   (b) $(A \cap B) \times (A \cap C)$

3. Find the direct product $A \times B \times C$, using the sets from the previous problem (problem 2).

4. If the domain of the function $y = 5 - 3x$ is the set $\{x \mid 1 \leq x \leq 4\}$, find the range of the function and express it as a set. Do the same for the function $y = x^2 - 6x + 13$.

5. Prove validity of the formula: $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$,

   (a) By using Venn diagrams (don't forget to show all intermediate calculations, not only the final picture).

   (b) By proving that every element of the set in left-hand side belongs to the set in the right-hand side and vice versa.

Assignment 2 (Due on the week September 18 – 24)

1. Prove that $|x + y + z| \leq |x| + |y| + |z|$ for all numbers x, y, and z.

2. Show that a *convergent* sequence in $\mathbb{R}^n$ can have only one accumulating point, and therefore only one limit.

3. Show that the positive orthant

$$\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \mid x_i > 0, i = 1, 2, \ldots, n\}$$

   is an open subset of $\mathbb{R}^n$ by finding a formula for ε in terms of the x_i's.

4. Prove that every convergent sequence in $\mathbb{R}^n$ is bounded.

5. Given two sets S_1 and S_2 in $\mathbb{R}^n$ define their sum by

$$S_1 + S_2 = \{x \in \mathbb{R}^n \colon x = x_1 + x_2,\ x_1 \in S_1, x_2 \in S_2\}.$$

   Prove that if S_1 and S_2 are compact, then $S_1 + S_2$ is also compact.

Assignment 3 (September 25 – October 1)

1. Prove that any intersection of closed sets is closed.

2. For each of the following subsets of $\mathbb{R}^2$,

   - Sketch the set.
   - Determine whether or not it is **open**, **closed** or **compact**. *Hint: a set is closed if and only if its complement is open.*
   - Give reasons for your negative answers to the previous part.

   (a) $\{(x, y) \colon x = 0, y \geq 0\}$,
   (b) $\{(x, y) \colon 1 \leq x^2 + y^2 \leq 2\}$,
   (c) $\{(x, y) \colon 1 \leq x \leq 2\}$,
   (d) $\{(x, y) \colon x = 0 \text{ or } y = 0, \text{ but not both}\}$.

3. Sketch level sets for each of the following functions from $\mathbb{R}^3$ to $\mathbb{R}^1$:

   (a) $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$,
   (b) $f(x_1, x_2, x_3) = x_1^2 + x_2^2$,
   (c) $f(x_1, x_2, x_3) = x_1^2 - x_2 - x_3$,
   (d) $f(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3$.

4. Does the following limit exist:
$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}?$$

5. Find points of discontinuity of the following functions:

   (a) $u = \sin \frac{1}{xy}$,
   (b) $u = \ln \frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$.

Assignment 4 (Due on the week October 2 – October 8)

1. Find $f'_x(x, b)$, if $f(x, y) = x + (y - 1) \arcsin \sqrt{\frac{x}{y}}$.

2. Find all partial derivatives of the following function: $u = \left(\frac{x}{y}\right)^z$.

3. Find all partial derivatives of the following function: $u = xyze^{x+y+z}$.

4. Find the total differential of the following function: $u = \ln(x^x + y^y + z^z)$.

5. Use differentials to approximate each of the following values of $f(x, y)$ at a given point. Show all necessary calculations that are to be done if no calculator is available.

   (a) $f(x, y) = x^4 + 2x^2y^2 + xy^4 + 10y$ at $x = 10.36$ and $y = 1.04$;
   (b) $f(x, y) = 6x^{2/3}y^{1/2}$ at $x = 998$ and $y = 101.5$;
   (c) $f(x, y, z) = \sqrt{x^{1/2} + y^{1/3} + 5z^2}$ at $x = 4.03$, $y = 7.95$ and $z = 1.02$.

Assignment 5 (Due on the week October 9 – 15)

1. Compute the directional derivative of the function $z = xy^2 - xy + x^3y$ at a point $M(4; -2)$ in the direction $\left(\frac{1}{\sqrt{10}}; \frac{3}{\sqrt{10}}\right)$

2. Find the derivative of the function $z = 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$ at a point $M\left(\frac{a}{\sqrt{2}}; \frac{b}{\sqrt{2}}\right)$ in the direction of the inward normal line at the point M to the curve line defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. Using the Chain Rule calculate $\frac{dz}{dt}$ at $t = 0$ if $z = \frac{5t^2+3xy}{2w^2y}$, $x = t^2 + 1$, $y = \sqrt{t^2 + 1}$ and $w = e^t + 1$.

4. Calculate all partial derivatives of the first order with respect to x and y, if $u = f(\xi, \eta, \zeta)$, where $\xi = x^2 + y^2$, $\eta = x^2 - y^2$, $\zeta = 2xy$.

5. Calculate the gradient function and Hesse matrix for the following functions:
   (a) $f(x, y) = xy - \ln(x^2 + 2y^2)$,
   (b) $f(x, y) = ax^2 + 2bxy + cy^2$.

Assignment 6 (Due on the week October 14 – 20)

1. Consider the equation $x^3 + 3y^2 + 4xz^2 - 3z^2y = 1$. Does this equation define z as a function of x and y? If so, compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at this point.
   (a) In a neighborhood of $x = 1$, $y = 1$,
   (b) In a neighborhood of $x = 1$, $y = 0$,
   (c) In a neighborhood of $x = 0.5$, $y = 0$.

2. Consider $3x^2yz + xyz^2 = 30$ as defining x as an implicit function of y and z around the point $x = 1$, $y = 3$, $z = 2$. If y increases to 3.2 and z remains at 2, use the Implicit Function Theorem to estimate the corresponding x.

3. (a) Find all points (x_0, y_0), where $y' = 0$, if $y(x)$ is an implicit function given by an equation $\ln\sqrt{x^2 + y^2} = \operatorname{arctg}(y/x)$.
   (b) Calculate $y''(x)$ at these points.

4. Prove that $y''(x) \equiv 0$ if $y(x)$ is an implicit function given by an equation $y = 2x \operatorname{arctg}(y/x)$. Where does $y(x)$ exist?

5. Find all partial derivatives of the first and second order of the composite function $w = f(x, y)$, where $x = u^2 + v^2$, $y = uv$.

Assignment 7 (Due on the week November 5 – 10)

1. The supply function of a certain commodity is: $Q = a + bP^2 + R^{1/2}$ ($a < 0, b > 0$), (here R is rainfall).
   (a) Find the price elasticity of supply and rainfall elasticity of supply.

(b) How do the two partial elasticities vary with P and R? In monotonic fashion (assuming positive P and R)?

2. Find all partial derivatives of the first and second order of the composite function $w = f(x, y, z)$, where $x = u + v^2$, $y = u - v$, $z = \ln u + \ln v$.

For each of the following functions find the critical points.

3. $z = x^2y^3(6 - x - y)$.

4. $u = x + \dfrac{y^2}{4x} + \dfrac{z^2}{y} + \dfrac{2}{z}$, $(x > 0, y > 0, z > 0)$.

5. Find the critical points (if any) of the implicit function z of variables x and y defined by $x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0$.

Assignment 8 (Due on the week November 11 – 17)

1. Find d^2u, if $u = x^3 + y^3 - 3xy(x - y)$.

2. Find dz and d^2z, if $xyz = x + y + z$.

3. Express the quadratic approximation of the following functions:

   (a) $f(x_1, x_2) = e^{x_1x_2-1}$ around the point $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$;

   (b) $f(x, y) = \dfrac{x}{y}$ around the point $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4. Determine the definiteness of the following symmetric matrices:
a) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$ c) $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$ d) $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$
e) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$ f) $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ g) $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$

5. Express the quadratic approximation of the function $f(x, y) = \tan^{-1} \dfrac{1 + x + y}{1 - x + y}$ around the point $a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Assignment 9 (Due on the week November 18 – 24)

For each of the following functions find the critical points and classify them as local max, local min, saddle point, or "can't tell". What can you say about their global properties (in other words are at least some of them global extrema)?

1. $f(x, y) = x^4 + x^2 - 6xy + 3y^2$,

2. $f(x, y) = x^2 - 6xy + 2y^2 + 10x + 2y - 5$,
3. $f(x, y) = xy^2 + x^3y - xy$,
4. $f(x, y) = 3x^4 + 3x^2y - y^3$,
5. $f(x, y, z) = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$,
6. $f(x, y, z) = (x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$.

Assignment 10 (Due on the week November 25 – 30)

1. Which of the following functions on $\mathbb{R}^n$ are concave or convex?
   (a) $f(x) = 3e^x + 5x^4 - \ln x$,
   (b) $f(x, y) = -3x^2 + 2xy - y^2 + 3x - 4y + 1$,
   (c) $f(x, y, z) = 3e^x + 5y^4 - \ln z$,
   (d) $f(x, y, z) = Ax^\alpha y^\beta z^\gamma$, $\alpha, \beta, \gamma > 0$.
2. Graph each of the following sets, and indicate whether it is convex:
   (a) $\{(x, y) \mid y = e^x\}$,
   (b) $\{(x, y) \mid y \geq e^x\}$,
   (c) $\{(x, y) \mid y \leq 13 - x^2\}$,
   (d) $\{(x, y) \mid xy \geq 1; x > 0,\ y > 0\}$.

Find critical points using the first-order conditions. To check whether a critical point is the optimal solution try the Weierstrass theorem where applicable.

3. $z = \dfrac{x}{a} + \dfrac{y}{b}$, if $x^2 + y^2 = 1$,
4. $z = x^2 + 12xy + 2y^2$, if $4x^2 + y^2 = 25$,
5. Maximize $u(x, y, z) = xy^2z^3$ subject to $x + 2y + 3z = a$, where $x,\ y,\ z,\ a > 0$.

Assignment 11 (Due on the week December 2 – 7)

1. Train services on a railway branch line cost \$ 1600 per month to operate. Passengers consist of the two cohorts: business passengers with the aggregated demand $Q_d = 2000 - 10P$, where Q_d is the number of journeys made per month and P is the price in cents charged for each journey and the retired holidaymakers with demand $Q_d = 4000 - 40P$. How much should the railway authority charge if:
   (a) the same price is charged for everyone?
   (b) prices for the cohorts are different?

   What price variant will the company choose?

2. Assume that $U = (x+2)(y+1)$, and we assign no specific numerical values to the positive price and income parameters in the budget constraint $p_x x + p_y y = I$.
   (a) Write down the Lagrangian function.
   (b) Find the optimal values of the choice variables and the Lagrange multiplier in terms of the price and income parameters, assuming that the optimal bundle includes both goods in positive quantities.
   (c) Check the second-order sufficient condition for maximum.

Find points of local extrema and classify them

3. $u = x_1^2 + x_2^2$, if $x_1^2 + x_2 = 1$,
4. $z = 2x^4 + y^4 - x^2 - 2y^2$,
5. $z = xy + \dfrac{50}{x} + \dfrac{20}{y}$, subject to $x > 0$, $y > 0$.

Assignment 12 (Due on the week December 9 – 14)

1. Use the Lagrange multiplier method to write the first-order conditions for the maximum of the function $f(x,y) = \sqrt{x} + \sqrt{y}$, subject to $ax + y = 1$, where a is a real parameter. For what values of a solution exists? Check sufficiency condition.
2. Find the points of relative optimum and classify them using second-order conditions: $u = x^2 + y + 2z \to extr$, s.t. $x^3yz^2 = w$, where w is a real parameter.
3. Find the points of relative optimum and classify them using second-order conditions: $u = (x+z)y \to extr$, s.t. $x^2 + y^2 = 2$, $y + z = 2$, where all the variables are positive.
4. A firm with the smooth production function $Q(x,y)$ wants to find the least-cost input combination for a production of a specified level output Q_0 representing, say, a customer's special order. Show that at the point of optimal input combination, the input-price-marginal-product ratio must be the same for each input.
5. Show that the function $z = (1 + e^y)\cos x - ye^y$ has an infinite number of points of maximum and no point of minimum.

Assignment 13 (Due on the week December 15 – 20)

1. Let $f(x,y) = -x^2 - y^2$, and we seek to maximize that function subject to constraint $(x-1)^3 = y^2$. Solve that problem with and without the additional Lagrange multiplier λ_0.
2. Find the critical points in the problem of constrained optimization and classify them using the second-order conditions: $f(x,y,z) = xyz \to extr$, subject to $x^2 + y^2 + z^2 = 1$, $x + y + z = 0$.
3. The weekly production of a factory depends on the amounts of capital and labor it employs by the formula $q(k,l) = \sqrt{kl}$. The cost of capital is \$4 per unit and the cost of labor is \$1. Find the minimum weekly cost of producing $q = 200$. How the cost of production changes if the factory has to produce $q = 202$?

4. A firm's inventory $I(t)$ is depleted at a constant rate per unit time, i.e. $I(t) = x - \delta t$, where x is an amount of good reordered by the firm whenever the level of inventory is zero. The order is fulfilled immediately. The annual requirement for the commodity is A and the firm orders the commodity n times a year where $A = nx$. The firm incurs two types of inventory costs: a holding cost and an ordering cost. Since the average stock of inventory is $x/2$ the holding cost equals $C_h x/2$, the cost of placing one order is C_0 and with n orders a year this cost equals $C_0 n$.
   (a) Minimize the cost of inventory $C = C_h x/2 + C_0 n$ by choice of x and n subject to the constraint $A = nx$ by the Lagrange multiplier method.
   (b) Using the envelope theorem interpret the Lagrange multiplier.
5. Use the Lagrange multipliers to find the dimensions of a rectangular box with the least possible surface area among those with a volume of 27 m^3. Check the second-order conditions. Evaluate the change in the minimal surface area if the volume drops by 0.5 m^3. Compare your estimate with the direct computation.

Assignment 14 (Due on the week January 20 – 26)

1. Determine whether the following functions are homogenous. If so, of what degree?
   (a) $f(x,y) = \sqrt{xy}$
   (b) $f(x,y) = (x^2 - y^2)^{1/2}$
   (c) $f(x,y) = x^3 - xy + y^3$
   (d) $f(x,y) = 2x + y + 3\sqrt[2]{xy}$
   (e) $f(x,y,w) = \dfrac{xy^2}{w} + 2xw$
   (f) $f(x,y,w) = x^4 - 5yw^3$
2. Deduce from Euler's theorem that, for production function with constant returns to scale:
   (a) If $\mathrm{MPP_K} = 0$, then $\mathrm{APP_L}$ is equal to $\mathrm{MPP_L}$.
   (b) If $\mathrm{MPP_L} = 0$, then $\mathrm{APP_K}$ is equal to $\mathrm{MPP_K}$.
3. Let the production function $Q = f(x)$ of a firm be a differentiable homogenous function of degree $k > 0$. The firm wants to maximize its revenue $pf(x)$ subject to $w^T x = C$, where w is a vector of input prices, p is the output price, C is the total cost of production. Assume that the firm buys all of the factors in positive quantities. Show that the cost function is a homogenous function $C(Q)$ of degree $\frac{1}{k}$. *Hint: use Euler's equation and derive necessary conditions for the minimization problem.*
4. Find the general solution and the definite solution, given
   (a) $\frac{dy}{dt} + 4y = 12,\ y(0) = 2;$
   (b) $\frac{dy}{dt} - 2y = 0,\ y(0) = 9;$
   (c) $y' = 3y^{2/3}, y(2) = 0;$
   (d) $(x^2 - 1)y' + 2xy^2 = 0, \quad y(0) = 1;$
5. Solve the following first-order linear differential equation:
$$(2x+1)y' = 4x + 2y.$$

Assignment 15 (Due on the week January 27 – Febuary 2)

1. Find the maximizer of $f(x,y) = x^2 + y^2$, subject to the constraints $2x + y \le 2$, $x \ge 0$, $y \ge 0$.
2. Solve Bernoulli equation, definitize the arbitrary constant:
$$\frac{dy}{dt} + 2y = y^2 e^t,\ y(0) = 6.$$
3. Show that the function
$$g(x,y) = 3xy^3 + 6x^4 - y(2x^{3/4} - y^{3/4})^4$$
is homogeneous. Verify Euler's Theorem for g.
4. Verify that each of the following differential equation is exact and solve it by step-by-step procedure:
   (a) $2yt^3 dy + 3y^2t^2 dt = 0$;
   (b) $3y^2 t dy + (y^3 + 2t)dt = 0$;
   (c) $t(1+2y)dy + y(1+y)dt = 0$;
5. Solve the following first-order differential equations:
   (a) $(t^2 + y^2)\dfrac{dy}{dt} = 2ty$;
   (b) $(t + y^2)dy = ydt$;

Assignment 16 (Due on the week Febuary 10 – 14)

1. Find the solution of the IVP in the Solow's growth model,
$$\dot{k} = A\sqrt{k} - (n+\delta)k,\ k(0) = k_0,$$
where $A, n, \delta > 0$.
2. Find the values a and b such that the function f is homogeneous:
$$f(x,y) = 2x^{b-a}y^{b+2} + y^{a+1}x^{-3b} + y^{7b}x^{-2a}.$$
For the values a and b you have found expand the function
$$h(x) = \sqrt{1 + f(x,x)}(1 - \cos(f(x,x))$$
as a power series up to x^4. State the range for x where your expansion is correct.
3. Maximize $3xy - x^3$ subject to the constraints
$$\begin{aligned} &2x - y = -5, \\ &5x + 2y \ge 37, \\ &x \ge 0,\ y \ge 0. \end{aligned}$$
4. Solve the following first-order differential equation
$$(y^2 - 2xy)dx + x^2 dy = 0.$$
5. For what values of (x_0, y_0) does the differential equation $xy' = \sqrt{x-y}$ have a unique solution $y = y(x)$ such that $y(x_0) = y_0$?

Assignment 17 (Due on the week Febuary 17 – 22)

1. A production function is given by $Q = 2KL + \sqrt{L}$, where Q is output, K is capital and L is labour. Given that the current levels of K and L are 9 and 4 respectively
   (a) Determine the value of the Marginal Rate of Technical Substitution, i. e. $\left(-\dfrac{dK}{dL}\right)$ where Q is kept constant.
   (b) Estimate the increase in labour needed to maintain the current level of output given a decrease in capital of half a unit.
   (c) Sketch the isoquant of the above function for $Q = 74$.
2. Solve the following differential equation:
$$y^2dx + (xy + \text{tg}(xy))dy = 0.$$
3. Minimize $x^2 - 2y$ subject to constraints $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.
4. The supply function for a commodity takes the form $Q = aP^2 + bP + c$ for some constants a, b, c, where P denotes the price of the commodity supplied. When $P = 1$, the quantity supplied is 3; when $P = 2$, the quantity supplied is 11; when $P = 3$, the quantity supplied is 25. Using matrices, find the constants a, b, c, and find the quantity supplied when the market price is 4.
5. Plot the phase line for each of the following equation and interpret dynamic behavior of the solution:
   (a) $\dfrac{dy}{dt} = (y+1)^2 - 16$ $(y \geq 0)$,
   (b) $\dfrac{dy}{dt} = \dfrac{1}{2}y - y^2$ $(y \geq 0)$.

Assignment 18 (Due on the week Febuary 24 – 29)

1. Find the polar and the exponential form of the following complex numbers:
   1). $1 + \sqrt{3}i$ 2). $2i$ 3). $-7i$
   4). $1 - \sqrt{3}i$ 5). $\sqrt{3}i$ 6). $3 + 4i$
   7). $-1 + \sqrt{3}i$ 8). $1 + i$ 9). $-5 + 12i$
2. Calculate the following 1). $(1 + \sqrt{3}i)^{\frac{1}{2}}$ 2). $(2i)^{\frac{1}{3}}$ 3). $(-7i)^{\frac{1}{6}}$
3. Find the particular integral and the complementary function, the general solution and the definite solution of the following:
   (a) $y''(t) - 4y'(t) + 8y = 0;\quad y(0) = 3,\ y'(0) = 7.$
   (b) $y''(t) + 4y'(t) + 8y = 2;\quad y(0) = 2.25,\ y'(0) = 4.$
   (c) $y''(t) + 3y'(t) + 4y = 12;\quad y(0) = 2,\ y'(0) = 2.$
   (d) $y''(t) - 2y'(t) + 10y = 12;\quad y(0) = 6,\ y'(0) = 8.5.$
   (e) $y''(t) + 9y = 3;\quad y(0) = 1,\ y'(0) = 3.$

4. Solve the following equations:

   (a) $y'' - 2y' - 3y = e^{4t}$,
   (b) $y'' + y = \sin t$,
   (c) $y'''' + 4y'' + 3y = 0$,
   (d) $y'' + 2y' - 3y = te^t$.

5. Expand $e^{-x} \ln(1 + 2x)$ in ascending powers of x as far as the term in x^4. State the range of values of x for which this series is valid.

Assignment 19 (Due on the week March 2 – 7)

1. Find the particular integral of each of the following equations by the method of undetermined coefficients:

   (a) $y''(t) + 2y' + y = t$,
   (b) $y''(t) + 4y' + y = 2t^2$,
   (c) $y''(t) + y' + 2y = e^t$,
   (d) $y''(t) + y' + 3y = \sin t$.

2. Without finding their characteristic roots, determine whether the following differential equations will give rise to convergent time paths:

   (a) $y'''(t) - 10y''(t) + 27y'(t) - 18y = 3$,
   (b) $y'''(t) + 11y''(t) + 34y'(t) + 24y = 5$,
   (c) $y'''(t) + 4y''(t) + 5y'(t) - 2y = -2$.

3. Find all a and b such that all solutions of the differential equation $\ddot{y} + a\dot{y} + by = 0$ converge to zero as t goes to ∞.

4. Find the polar form of the following complex numbers:

   1). $\dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i$ 2). $4(\sqrt{3} + i)$ 3). $1 + i$ 4). $1 - i$
   5). $-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ 6). $-2 - 3i$ 7). $3 - 4i$ 8) $-5 - i$

5. Using the method of Lagrange find points of extrema and classify them for the objective function $W = y^2 - 3x$ under constraints: $xy \leq 6$, $0 \leq x \leq 5$, $0 \leq y \leq 3$. How does the optimal value of the goal function change if the right side of each constraint decreases by $0,1$?

Assignment 20 (Due on the week March 10 – 14)

1. For each of the following difference equations find the complementary function, the particular integral and the definite solution:

   (a) $y_{t+1} + 3y_t = 4$ $(y_0 = 4)$

(b) $2y_{t+1} - y_t = 6\ (y_0 = 7)$

(c) $y_{t+1} = 0.2y_t + 4\ (y_0 = 4)$

2. Find the solutions of the following equations and determine whether the time paths are oscillatory and convergent:

   (a) $y_{t+1} - \frac{1}{3}y_t = 6\ (y_0 = 1)$

   (b) $y_{t+1} + 2y_t = 9\ (y_0 = 4)$

   (c) $y_{t+1} + \frac{1}{4}y_t = 5\ (y_0 = 2)$

   (d) $y_{t+1} - y_t = 3\ (y_0 = 5)$

3. Solve graphically the following linear program:

$$\begin{cases} \max(4x_1 + 2x_2) \\ 2x_1 + 3x_2 \leq 18, \\ -x_1 + 2x_2 \leq 9, \\ 2x_1 - 4x_2 \leq 10, \\ x_1 \geq 0,\ x_2 \geq 0. \end{cases}$$

4. Find the minimal value of the objective function in the linear program stated below, where λ is a real value parameter using the dual program

$$\begin{cases} \min(x_1 + 10x_2 + 2x_3 + 4x_4), \\ \lambda x_1 + 2x_2 - x_3 + x_4 \geq -1, \\ x_1 + 3x_2 + x_3 + x_4 \geq 5, \\ x_i \geq 0,\ i = 1, 2, 3, 4. \end{cases}$$

5. Consider the utility maximization problem in the economy of n-goods: $U(x_1, x_2, \ldots, x_n) = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \to$ max subject to the budget constraint $\sum\limits_{i=1}^{n} x_i \leq I$, $x_i \geq 0$, $a_i > 0$. Find the optimal bundle an check whether you found the maximum. Find the rate of change of the indirect utility function when the income slightly changes.

Assignment 21 (Due on the week March 16 – 21)

1. Given the demand and supply function for the cobweb model as follows, find the intertemporal equilibrium price, and determine whether the equilibrium is stable:

   (a) $Q_t^d = 18 - 3P_t,\ Q_t^s = -3 + 4P_{t-1}$

   (b) $Q_t^d = 22 - 3P_t,\ Q_t^s = -2 + P_{t-1}$

   (c) $Q_t^d = 19 - 6P_t,\ Q_t^s = -5 + 6P_{t-1}$

2. If a market model with inventory has the following numerical form

$$\begin{cases} Q_t^d = 21 - 2P_t, \\ Q_t^s = -3 + 6P_t, \\ P_{t+1} = P_t - 0.3(Q_t^s - Q_t^d) \end{cases}$$

find the time path Pt and determine whether it is convergent.

3. Solve the following linear program:

$$\begin{aligned} &\max 7x_1 - 6x_2 - x_3 - 7x_4 \\ &2x_1 - 3x_2 - 3x_3 - x_4 \leq 2, \\ &x_1 + 2x_2 + x_3 - x_4 \leq 1, \\ &x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

4. Solve the following linear program:

$$\begin{aligned} &\max -x_1 + 2x_2 + 3x_3 - x_4, \\ &2x_1 + x_2 + 2x_3 - x_4 \leq 2, \\ &-x_1 + x_2 + x_3 + 3x_4 \leq 2, \\ &x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

5. An economy produces an income Y_t at year t. In the absence of governmental expenditures the following identity is valid: $Y_t = C_t + I_t$, where C_t is the consumption in year t and I_t is the investment in year t. The investment I_t equals to 75% of the previous year income Y_{t-1} and there is no autonomous consumption. Let denote the marginal propensity to consume by k.

   Derive a first order difference equation for the consumption C_t. Solve this equation and find the condition upon k for which consumption will raise from year to year. If $k = 0.32$, $C_0 = 100$, sketch and describe the solution C_t for $t = 0, 1, 2, \ldots$ Determine when, if ever, the consumption in a year first falls bellow 10.

Bonus Assignment (Due before April 18)

There will be 5% of your score for this HA added to MathEcon earnings on 100-point scale.

1. Consider a bi-matrix game with the pay-off matrix

$$\begin{pmatrix} (4,3) & (5,1) & (6,2) & (5,1) \\ (2,1) & (8,4) & (3,6) & (3,4) \\ (3,0) & (9,6) & (2,8) & (4,6) \\ (1,1) & (4,2) & (7,3) & (2,2) \end{pmatrix}.$$

   Player A chooses a row and player B chooses a column. Left number represents the pay-off of player A and right number represents pay-off of player B. Find all Nash equilibria of this game both pure and mixed.

2. Find all solutions of this linear program for all values of parameter λ:

$$\begin{cases} x_1 + 10x_2 + 2x_3 + 4x_4 \to \min, \text{ subject to} \\ \lambda x_1 + 2x_2 - x_3 + x_4 \geq -1, \\ x_1 + 3x_2 + x_3 + x_4 \geq 5. \end{cases}$$

   All the variables here are nonnegative.

3. Solve the system of differential equations $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where matrix A is equal to $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$.

   Hint: one of the eigenvalues is 2.

4. Solve the system of difference equations

$$\begin{cases} x_{t+1} = 2x_t + y_t + 2^{t/2}, \\ y_{t+1} = -2x_t + 2t. \end{cases}$$

5. Form a difference equation of the smallest order possible whose linearly independent solutions in discrete time are represented by
$$2^t, \cos\left(\tfrac{\pi}{2}t\right), \sin\left(\tfrac{\pi}{2}t\right), t\cos\left(\tfrac{\pi}{2}t\right), t\sin\left(\tfrac{\pi}{2}t\right).$$

Some more old...

Assignment 23 (20 April)

1. Solve the following differential-equation system and analyse the time path:

$$\begin{cases} x'(t) - x(t) - 12y(t) = -60 \\ y'(t) + x(t) + 6y(t) = 36 \end{cases}$$

   with $x(0) = 13$, $y(0) = 4$.

2. Solve the following difference-equation system and analyse the time path

$$\begin{cases} x_{t+1} + x_t + 2y_t = 24 \\ y_{t+1} + 2x_t - 2y_t = 9 \end{cases}$$

   with $x_0 = 10$, $y_0 = 9$.

3. Find the eigenvalues of the matrix
$$A = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$
   and find the eigenvectors corresponding to each eigenvalue. Hence find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(a) Use your result to find the sequences x_t, y_t, z_t, such that $x_0 = 1$, $y_0 = 3$, $z_0 = 2$ and for $t > 0$,

$$\begin{cases} x_{t+1} = 4x_t - y_t - z_t \\ y_{t+1} = x_t + 2y_t - z_t \\ z_{t+1} = x_t - y_t + 2z_t \end{cases}$$

(b) Use your result to find the functions $x(t)$, $y(t)$, $z(t)$ such that $x(0) = 1$, $y(0) = 3$, $z(0) = 2$ and

$$\begin{cases} x' = 4x - y - z \\ y' = x + 2y - z \\ z' = x - y + 2z \end{cases}$$

4. Given the pay-off bi-matrix

	e	f	g	h
a	1,1	5,3	3,2	3,4
b	1,0	1,2	2,5	2,6
c	3,3	3,5	3,4	4,4
d	2,2	0,3	2,2	2,2

Find all Nash equilibria in pure and mixed strategies. Check whether they are Pareto-optimal.

5. Given the pay-off matrix of a zero-sum game

	c	d	e
a	-1	-3	0
b	-4	2	-5

Find all Nash equilibria in pure and mixed strategies. Check whether they are Pareto-optimal.

Assignment 24 (27 April)

1. A policymaker desires to double in 10 periods of time the value of GDP Y_t produced in period t. Evolution of GDP over time is given by equation $2Y_{t+2} - 3Y_{t+1} + Y_t = 2^t + t$. If it is possible find at least one value of Y_0 and Y_1 that the GDP will eventually double.

2. Solve the system of ODE
$$\begin{cases} \dot{x} = 3x - 2y \\ \dot{y} = 2x - y \end{cases}$$

3. Let production function $F(K, L)$ be twice continuous differentiable and homogeneous of the first degree. Show that its Hessian matrix has a zero determinant.

4. Two candidates, A and B, compete in an election. Of the 100 citizens, k support candidate A and $m = 100 - k$ support candidate B. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives the payoff of 2 if the candidate she supports wins, 1 if this candidate ties for first place, and 0 if this candidate loses. A citizen who votes receives the payoffs $2 - c$, $1 - c$, and $-c$ in these three cases, where $0 < c < 1$.

   (a) For $k = 50$, find the set of Nash equilibria in pure strategies. (Is the action profile in which everyone votes a Nash equilibrium? Is there any Nash equilibrium in which the candidates tie and not everyone votes? Is there any Nash equilibrium in which one of the candidates wins by one vote? Is there any Nash equilibrium in which one of the candidates wins by two or more votes?)

   (b) What is the set of Nash equilibria in pure strategies for $k < 50$?

5. General A is defending territory accessible by two mountain passes against an attack by general B. General A has three divisions at her disposal, and general B has two divisions. Each general allocates her divisions between the two passes. General A wins the battle at a pass if and only if she assigns at least as many divisions to the pass as does general B; she successfully defends her territory if and only if she wins the battle at both passes. Find all the mixed strategy equilibria.

Additional Computer Home Assignment (27 April)

You may use any open source software, R is recommended but not mandatory. Please provide not only the answers but also the code.

1. Find all the eigenvalues and eigenvectors of the following matrix

$$\begin{pmatrix} 5 & 2 & -1 \\ 2 & 3 & 6 \\ -1 & 6 & -2 \end{pmatrix}$$

2. Draw the solution of the second order equation

$$y'' + (x+1) \cdot y' + y = \arctan x$$

with initial conditions $y(0) = 0$, $y'(0) = 1$.

3. Find numerically the global minimum of the function $f(x, y) = x^4 + y^8 + 2xy - 4x + xy^2$.

4. Find at least one Nash Equilibrium of the following zero-sum game:

	e	f	g
a	2;-2	-3;3	0;0
b	0;0	3;-3	-5;5
c	-4;4	0;0	2;-2

5. Masha and Sasha play the following game. Masha writes two numbers on two small sheets of paper. The number on the first sheet is uniformly distributed on $[0; 1]$, the number on the second sheet is just the first number squared. Sasha selects one of the two sheets at random with equal probabilities. He looks at the number and may either keep it or discard it and take the other number. Sasha maximises his expected payoff. Find his optimal strategy and the maximal expected payoff.

6. Download stock prices of five selected stocks of your choice. Any time period is ok.

   (a) Estimate expected returns and covariance matrix for returns

   (b) For different values of λ solve the maximisation problem

   $$(\text{Expected portfolio return}) - \lambda(\text{MAD of portfolio return})$$

   (c) Plot the optimal portfolios and original stocks on a scatter plot using expected returns and MAD as axis.