Hi Lecture 4

Recap

$A = (x=1, y=1)$

$C_1: \quad x+y \leq 5$ (not active)

$C_2 \quad 2x+y \geq 3$ (active)

Q?

Plan

1. combination of $\geq$ and $=$. FOC?
2. SOC. Bordered Hessian matrix !!
3. concavity / convexity
4. Slater's condition. (no need to check SOC)

Combination of equality and ineq. constr-ts.

Short answer: combine F.O.C. for $=$, FOC for $\geq$.

Formal theorem.

If

$$D = U \cap \{x \in \mathbb{R}^n \mid \underbrace{h_i(x) \geq 0}_{l \text{ ineq. c}} \quad \underbrace{g_i(x) = 0}_{k \text{ eq. c.}} \quad \forall i\}$$

open set

ex

C_1 $x_1 + x_2 > 0$

C_2 $x_1 + x_2^2 = 7$

C_3 $2x_1 + 3x_2 \leq 10$

Which constr-ts may be active?

C_2, $\{C_2, C_3\}$

Sundaram ch 6.4/6.5

$\varphi_1 = g_1 \; \ldots \; \varphi_k = g_k \quad \varphi_{k+1} = h_1 \quad \varphi_{k+2} = h_2 \; \ldots \; \varphi_{k+l} = h_l$

$f: \mathbb{R}^n \to \mathbb{R}$, $\varphi_1, \varphi_2 \ldots \varphi_{k+l}, f$ are C^1 funct. $x^* \in D$

and

$$L = f + \sum_{i=1}^{k+l} \lambda_i \cdot \varphi_i$$

x^* maximizes f on D

rank of Jacobian matrix of active constr-ts at x^* is equal to the number of active constraints.

Then there is $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{k+l} \end{pmatrix}$ such that

F.O.C.

$$\begin{cases} \dfrac{\partial L}{\partial x_i} = 0 & i \in 1, \dots, n \\ \dfrac{\partial L}{\partial \lambda_i} = 0 & i \in \{1, \dots, k\} \quad \text{equality constr.} \\ \dfrac{\partial L}{\partial \lambda_i} \geq 0 & i \in \{k+1, \dots, k+l\} \quad \text{ineq. const.} \\ \lambda_i \geq 0 & i \in \{k+1, \dots, k+l\} \quad \text{ineq const.} \\ \lambda_i \cdot \dfrac{\partial L}{\partial \lambda_i} = 0 & i \in \{k+1, \dots, k+l\} \quad \text{ineq const.} \end{cases}$$

Cookbook procedure:

$$\begin{cases} \text{FOC} \\ x \in G \end{cases}$$

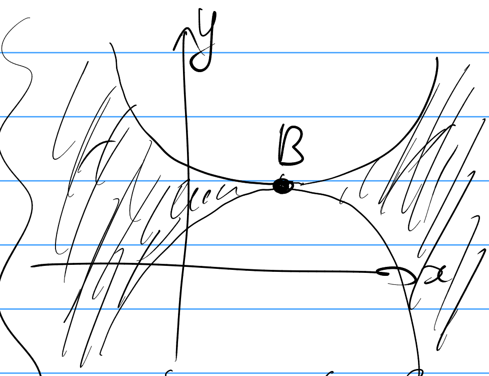
Kuhn-Tucker condition

NDCQ

$x_1 + x_2 > 0$ $\quad C_1$

$x_1 + x_2^2 = 7$ $\quad C_2$

$2x_1 + 3x_2 \leq 10$ $\quad C_3$



$$\begin{cases} y \leq (x-3)^2 + 5 \\ y \geq 5 - (x-3)^2 \end{cases}$$

Bad case

$\boxed{C_2 \text{ only}}$, C_2 and C_3

$J = (1, 2x_2)$

rank $J = 1$

no Bad point.

$J = \begin{pmatrix} 1 & 2x_2 \\ 2 & 3 \end{pmatrix}$

$$\begin{cases} \text{rank } J < 2 \\ x_1 + x_2 > 0 \\ x_1 + x_2^2 = 7 \\ 2x_1 + 3x_2 = 10 \end{cases}$$

$2x_2 = 1.5$

$x_2 = \frac{3}{4}$

no solutions.

Warning!

If x^* maximizes f on D.

$D = U \cap \{ g_i(x) = 0,\ h_i(x) \geq 0 \}$

Standardize the problem! Maybe twice! !!

(ex) find min $x^2 + 2y^2$ s.t. $x + y \geq 7$ $\longrightarrow$ max $-x^2 - 2y^2$ s.t. $x + y - 7 \geq 0$

(ex) find extr. $x^2 + 2y^2$ s.t. $x + y \geq 7$, $2x + 7y \leq 100$

$\nearrow$ max $x^2 + 2y^2$, $x + y - 7 \geq 0$, $100 - 2x - 7y \geq 0$

$\searrow$ max $-x^2 - 2y^2$, $x + y - 7 \geq 0$, $100 - 2x - 7y \geq 0$

max $x^2 + 2y^2$

s.t. $-x - y \leq -7$

$L = x^2 + 2y^2 + \lambda\left((-7) - (-x - y)\right)$

$$\boxed{\begin{array}{l} \max f \\ \text{s.t. } g(x,y) \leq b \end{array}}$$

$L = f + \lambda \cdot (b - g(x,y))$ $\boxed{\text{ok}}$

Sufficiency conditions. NDCQ, F.O.C

1. Use graphical approach.
2. Weierstrass theorem
   If f is continuous, set D is compact (closed bounded) then f should attain global min and global max on D.
3. Use bordered Hesse matrix.

(!) Include $\boxed{\text{only}}$ active constraints in the bordered Hesse matrix.

F.O.C $\nearrow$ A H_A [3×3]

F.O.C $\searrow$ B H_B [4×4]

Small recap.

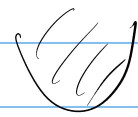
concave

„cave" 

def f is concave $\iff$ subgraph of f is convex $x \in \mathbb{R}^n$

$$\text{sub}\, f = \{(x,y) \mid y \le f(x)\}$$

$$\text{epi}\, f = \{(x,y) \mid y \ge f(x)\}$$

def f is convex $\iff$ epi f is convex. 

Theorem.

if f is C^2

matrix of second derivatives (Hesse)

f is concave $\iff$ H_f is negative semidef.
f is convex $\iff$ H_f is positive semidef.

Slater's condition.

$$D = \{h_i(x) \ge 0, \forall i\} \cap U$$

U open

There exists a point $\hat{x} \in D$ and $h_i(\hat{x}) > 0 \ \forall i$.

Sund 7.3.3

If $f, h_1, \dots, h_e$ are C^1, concave

$$D = \{h_i(x) \ge 0, \forall i\} \cap U$$

U open

Slater's cond is satisfied

then F.O.C. are necess. and sufficient.