Exams collection. Growing. Mathematics for economists (MFE), Methods of optimal solution (MOS).

2 января 2013 г.

Содержание

1 2008-2009

1.1 MFE, mock, ??.11.08

Calculators are not allowed.

Candidates should attempt:
- all questions from Part A
- two of three questions from Part B.

Part A.

Problem 1. [10]
Find and sketch on the plane the sets:
a) $([1;3)\times[-1;4))\cap((2;5]\times(2;5])$
b) $([1;3)\times[-1;4))\cup((2;5]\times(2;5])$
c) $([1;3)\times[-1;4))\backslash((2;5]\times(2;5])$

Problem 2. [10]
Calculate the directional derivative of the function $f(x,y)=2x^3+2y^2$ at the point $A(1;2)$ in the

following directions:
a) $\vec{l} = (1;3)$
b) $\vec{l}$ which is orthogonal to the curve given by the equation $x^2 + y^2 = 5$
c) Direction of the fastest growth of $f(x, y)$

Problem 3. [10]
Consider the following system of equation:
$$\begin{cases} xyzw + 2x^3y^3z^3 + 4w^3 = 7 \\ x + y + z^3 + w^3 + w^2z^2 = 5 \end{cases}$$
a) Does this system define functions $z(x, y)$ and $w(x, y)$ at a point $x = 1$, $y = 1$, $z = 1$, $w = 1$?
b) If it's possible find $\frac{\partial z}{\partial x}$ and $\frac{\partial w}{\partial y}$ at that point

Problem 4. [10]
Find the Hesse matrix of the function $f(x, y) = (2 + cos(x))^{sin(y)+5}$
Comment: clearly state Young theorem if you use it

Problem 5. [10]
Find the total differential for the function $f(x, y) = x^2y^2 + xy^2 + 2x + 4y$
Using the total differential find approximately $f(1.001, 1.999)$

Problem 6. [10]
Using the chain rule find all the first derivatives of $p(a, b)$ where $p(a, b) = f(x(a, b), y(a, b))$, $x(a, b) = a^2 + ab$ and $y(a, b) = b^3 + ab$

Part B.

Problem 7. [20]
Find the critical points of the function $f(x, y, z) = x^3 - y^3 + 9xy - z^2e^{-z^2}$
Classify them using the Hesse matrix.

Problem 8. [20]
The individual lives for two periods. He has a utility function $U(c_1, c_2) = u(c_1) + \beta u(c_2)$. In each period his wage is equal to w. His budget constraint requires that his period 1 consumption be his wage w minus any savings, $c_1 = w - s$. The government taxes savings at the rate q per dollar saved. So his second period consumption will be $c_2 = w + (1 + r)s(1 - q)$. The individual takes q, r, w as given. The individual seeks to maximise his utility U.
a) Find the first order condition for the optimal savings amount
b) Using the implicit function theorem find $\frac{\partial s}{\partial q}$
c) Find $\frac{\partial s}{\partial q}$ explicitly if $u(c) = ln(c)$

Problem 9. [20]
A firm sells its output into a perfectly competitive market and faces a fixed price p. It hires labor in a competitive labor market at a wage w, and rents capital in a competitive capital market at rental rate r. The production function is $f(L, K)$. The firm seeks to maximize its profits. Both factors are used in positive amounts in the optimal point.
a) Express the profit π as a function of L and K
b) Find necessary first order conditions for a profit-maximizing point (L^*, K^*)
c) Find the second order conditions sufficient for maximum (two inequalities)
d) Using the implicit function theorem find $\frac{\partial K^*}{\partial w}$. Is it positive or negative if you additionally know that $\frac{\partial^2 \pi}{\partial L \partial K} > 0$?

2 2011-2012

2.1 MFE, mock, 31.10.11

Marks will be deducted for insufficient explanation within your answers. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min.

SECTION A
Answer SIX of the six questions from this section.

1. Estimate value of $\sin^2(61^o) \cdot \tan(31^o)$ by linear approximation using derivatives at 60^o and 30^o. Convert degrees into radians first.
   (a) Degrees to radians. 2 points.
   (b) Appoximation. 8 points.
2. Let D be the domain of the function $f(x, y) = \ln(x) + \sqrt{y - x}$. Find D, the set D^o of internal points of D, the set ∂D of boundary points of D.
   (a) The set D. 4 points.
   (b) Internal points. 3 points.
   (c) Boundary points. 3 points.
3. Consider the function $f(x, y) = x^2 + y^3 - xy + 3y$ at the point $(2; 1)$. Find all the directions in which the growth rate of the function constitutes 60% of the maximal possible growth rate at that point.
   (a) Gradient and its length. 3 points.
   (b) Equation for direction. 3 points.
   (c) Solution of the equation. 4 points. One missing solution implies penalty 2 point.
4. Find the Hesse matrix of $f(x, y) = y \ln(x^2 + y)$. Clearly state Young's theorem if you use it.
   (a) First derivatives. 2 points.
   (b) Second derivatives without the use of Young's theorem. 8 points .
   (c) Second derivatives with the use of Young's theorem. 6 points for derivatives. 2 points for the statement of the theorem.
5. Consider the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, \text{if}(x, y) \neq (0, 0) \\ 0, \text{if}(x, y) = (0, 0) \end{cases} \tag{1}$$

   Is this function continuous at $(0; 0)$?
   (a) 10 points. Answer without proof = 0 points.
6. The value of y is determined as a function of t by the equation

$$\int_0^t f(x, y)dx = 1 \tag{2}$$

   Find dy/dt
   (a) Mention of the formula $dy/dt = -\frac{G_t}{G_y}$. 2 points.

(b) All the rest. 8 points

PLEASE TURN OVER

SECTION B

Answer TWO of the two questions from this section.

1. Find all critical points of the function $z = z(x, y)$ implicitly defined by the equation

$$x^2 + y^2 + z^2 - xz - yz + x + y + 4z + 1 = 0 \tag{3}$$

2. Let the production function $q = F(K, L)$ be twice continuously differentiable. Marginal rate of technical substitution is defined by the formula

$$MRTS = \frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}} \tag{4}$$

The derivatives in the denominator and numerator are taken while the same amount of the output q is fixed. Show that under the conditions $F_L > 0$, $F_K > 0$, $F_{LL} < 0$, $F_{KK} < 0$, $F_{LK} \geqslant 0$, the marginal rate monotonously declines with the growth of the factor L. In other words $\frac{\partial MRTS}{\partial L} < 0$.

2.2 MFE, fall semester exam, 29.12.11

Lecturer: K. Bukin

Classteachers: B. Demeshev, A. Kalchenko, S. Slavnov, D. Yesaulov

Marks will be deducted fro insufficient explanations within your solutions. Exam lasts for 120 minutes.

Section A. Answer all 6 questions from this section (10 marks each)

1. The function $f(x, y)$ is given by $f(x, y) = u^2(x, y) + v^3(x, y)$. The values of u and v and their gradients at the point $(x, y) = (1, 1)$ are also known, $u(1, 1) = 3$, $v(1, 1) = -2$, $\mathrm{grad}u = (1, 4)$, $\mathrm{grad}v = (-1, 1)$. Find $\mathrm{grad}f$ if $u, v \in C^1$.
2. Find the Hesse matrix of the function $f(x, y) = \int_{x-3y}^{2x+y} h(t)\,\mathrm{d}t$ if $h \in C^1$.
3. Determine the values of a for which the quadratic form $x^2 + 2axy + 2xz + z^2$ is positive definite, negative definite, positive semidefinite, negative semidefinite, and indefinite.
4. Given the system of two equations $x^2 + y^2 = \frac{1}{2}z^2$ and $x + y + z = 2$, find $\frac{\mathrm{d}x}{\mathrm{d}z}$ and $\frac{\mathrm{d}y}{\mathrm{d}z}$ in the neighborhood of the point $(1, -1, 2)$.
5. Find the maxima and minima of the function $f(x, y, z) = xyz$ subject to the constraints $x + y + z = 5$ and $xy + yz + xz = 8$, $x > 0$, $y > 0$, $z > 0$.
6. If the function y is given by $y(x) = \frac{1}{2}(e^x + e^{-x})$, check that $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\sqrt{y^2-1}}$.

Section B. Answer both questions from this section (20 marks each)

7. Consider the following maximization problem $f(x, y, a, b) = ax^2 - x + by^2 - y$, where a and b are real numbers.
   (a) Derive (x^*, y^*) — the point that satisfies the first order conditions.
   (b) Specify the conditions for a and b to ensure that the point (x^*, y^*) is the maximizer.
   (c) Use the Hessian to specify conditions for concavity and convexity of f depending on the values of parameters.
   (d) Solve the comparative statics problem: compute $\frac{\partial y^*}{\partial a}$ and $\frac{\partial y^*}{\partial b}$ as a and b marginally change.

(e) Using one of the envelope theorems find the rate of change of the value function $f(x^*, y^*, a, b)$ as the result of the marginal change in a and b.

8. Robinson Crusoe splits his time $\bar{L}$ hours a week between labor and leisure. His utility function is represented by $u(c, l) = \alpha \ln c + (1 - \alpha) \ln l$, where c is the amount of food and l is leisure in hours and $0 < \alpha < 1$. The food is produced by him in accordance with the production function $c = \sqrt{\bar{L} - l}$.

   (a) Find Robinson's optimal bundle (c^*, l^*) that provides him with the maximum utility (welfare) possible. Justify your answer by checking second-order conditions or otherwise.

   (b) Let $\alpha = 1/4$, $\bar{L} = 168$. Using the envelope theorem, estimate the change in the maximum value of his welfare if his utility function has slightly changed to become $u(c, l) = \frac{1}{5} \ln c + \frac{4}{5} \ln l$.

2.3 MFE, fall semester retake, 25.01.12

Lecturer: K. Bukin. Classteachers: B. Demeshev, A. Kalchenko, S. Slavnov, D. Yesaulov

Section A. Answer all 6 questions from this section (10 marks each)

1. For the function $f(x, y) = 2xy + 3$ find the level curves and the equations for their tangents at the points $(1, 2)$ and $(2, 2)$.
2. The population of a certain country grows exponentially, $N_t = N_{1990} \cdot \exp(r(t - 1990))$. The population was 70 million in 1990 and 80 million in 2000, what will be the population in 2013?
3. Use the chain rule to find $f'(x)$ and $f''(x)$ for $f(x) = u(a, b, x)$ where $a = \cos(x)$ and $b = x^3$.
4. The system of equations defines $x(z)$ and $y(z)$:

$$\begin{cases} x^2 + zxy + y^2 + 6z + y^3 = 10 \\ y^3x^2 + 3x + 2y + z = 7 \end{cases}$$

   Find $x'(z)$ and $y'(z)$ at the point $x = 1$ and $y = 1$.
5. Consider the function $f(x, y) = x^2 + y^3 - xy + 3y$ at the point $(2; 1)$. Find all the directions in which the growth rate of the function constitutes 60% of the maximal possible growth rate at that point.
6. Consider the objective function $f(x, y) = 4kx^3 + k^2xy + 3ky^4 - 13x - 13y$. The point $(x, y) = (1, 1)$ is the maximum of the function.

   (a) Find the value of k

   (b) Find the approximate increase of the maximum value if k will change by $\Delta k = 0.01$

Section B. Answer both questions from this section (20 marks each)

7. We wish to build a picnic zone for the travellers along a highway. The picnic zone should be rectangular with an area of 1000 m^2 and should have a fence on the three sides not adjacent to the highway. The price of one meter of fence is equal to \$ 20.

   (a) Find the dimensions of the picnic area that minimize the fencing costs.

   (b) Using hessian or otherwise check that you have found the costs-minimizing solution.

   (c) Using the Envelope theorem estimate the change in the costs if we increase the area of the picnic zone by 1 m^2.

8. A monopolistic firm with the cost function $TC(Q) = 30 + 15Q + Q^2$ sells a single product in two separate markets. The demand functions for these markets are given by $Q_1 = 25 - P_1$, $Q_2 = 29 - P_2$.

(a) Find the optimal quantities Q_1, Q_2 to be supplied to the respective markets in order to maximize the profit. Using hessian or otherwise check the second order condition.

(b) Calculate the point elasticity of demand for each of the three markets. Is it true that the optimal price is negatively related to the absolute value of elasticity at the optimal levels of output?

(c) Using the Envelope theorem estimate the change in the optimal profit if the demand on the second market changes to $Q_2 = 29.2 - 1.1P_2$.

2.4 MFE, mock exam, 02.04.12

Time allowed 120 minutes.
Students should answer all of the following eight questions. Calculators are not permitted in the exam. Marks will be deducted for insufficient explanations within your answers.
Section A: 10 points each question.

1. Determine whether the function $f(x, y) = \ln(5x + y) - 5(x + y)^2$ is convex (concave up), concave (concave down), strictly convex, strictly concave or neither.
2. Solve the differential equation $y^{(4)} - y = \cos(x)$. The $y^{(4)}$ denotes the forth derivative of y.
3. Consider the monopolist producing two distinct goods. The cost function is given by $TC(q_1, q_2) = q_1 + kq_2$, where the constant $k \in (0; 1)$. And the demand functions are given by $q_1(p_1, p_2) = q_2(p_1, p_2) = (p_1 p_2)^{-3}$.
   (a) Find the optimal production bundle for the monopolist.
   (b) For which values of k one of the product is priced under marginal costs?
4. The density of a standard normal random variable X is given by $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$. Taking the fact that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for granted calculate $E(X^2)$, $E(X^4)$.
   Hint: if you don't remember, $E(X^n) = \int_{-\infty}^{+\infty} x^n f(x)\,dx$
5. The point elasticity of demand for a good is given by $\varepsilon = p^2/(p^2 + 4p + 3)$. Find the demand function $q(p)$ given the initial condition $q(1) = 1$.
6. Find the values a and b such that the function f is homogeneous:
$$f(x, y) = 2x^{b-a}y^{b+2} + y^{a+1}x^{-3b} + y^{7b}x^{-2a}$$
   For the values of a and b you have found expand the function
$$h(x) = \sqrt{1 + f(x, x)} \cdot (1 - \cos(f(x, x))$$
   as a power series up to x^4. State the range for x where your expansion is correct.

Section B: 20 points each question.

1. It is known that $x_0 = 0$, $x_{100} = 100k$ where $k \in \mathbb{Z}$ is constant and for any $n \in \{2, 3, \ldots 100\}$ the following difference equation is satisfied:
$$x_n - 2x_{n-1} + x_{n-2} = -1$$
   (a) Find the particular solution
   (b) Find the maximum value of x_n for $n \in \{0, \ldots, 100\}$ as a function of k.
2. Using the Lagrange multiplier method without reducing the number of variables by substitution find the minimum of the function
$$f(x, y, z) = 2x^2 + 4y^2 + xy + 8z^2 + 2yz$$
   subject to $x + y + 1.5z \geqslant 1.2$ and $x + y + z = 1$.

2.5 MOR, 22.05.12

Section A. Solve **two** of the following **two** problems

1. Use Lagrange multipliers method to solve optimization problem

$$xy \to \max$$

subject to $x \geqslant 0$, $0 \leqslant y \leqslant 3$, $x + 2y \leqslant 8$, $y \geqslant \frac{x^2}{16} + 1$.

2. Solve the linear program depending on the parameter β,

$$2x_1 + 4x_2 + 5x_3 - x_4 \to \min$$

subject to $x_1 + x_3 - x_4 \geqslant 0$, $-x_1 + x_2 + \frac{1}{2}x_3 + \beta x_4 \geqslant 1$, $x_i \geqslant 0$.
For what values of β the minimum of the objective function equals 3?

Section B. Solve **two** of the following **three** problems

3. Find the general solution of the differential equation $y'' + 2y' + y = xe^{-x} + \cos(x)$
4. Solve the initial-value problem for the system of difference equations
$\begin{cases} x_{t+1} = x_t + y_t \\ y_{t+1} = 3x_t - y_t - 5 \end{cases}$, where $x_0 = y_0 = 0$.
5. In the model of interacting inflation and unemployment based on the Phillips relation, both unemployment rate U and expected inflation π are the solutions of the system $\dot{\pi} = \frac{3}{4}(p - \pi)$, $\dot{U} = -\frac{1}{2}(m - p)$, where m is exogenously defined positive rate of nominal money growth and p is the posteriori observed inflation satisfying equation $p = \frac{1}{6} - 3U + \pi$. Find the steady-state solutions for the inflation both expected and observed as well as the unemployment rate in terms of m. Explore the dynamic stability of solutions. What is the natural rate of unemployment?

Section C. Solve **two** of the following **three** problems

6. Find all pure and mixed Nash equilibria in the following bimatrix game:

	D	E	F
A	3;4	1;3	1;0
B	2;7	3;6	0;3
C	0;2	2;1	5;6

7. Two players play a version of Rock-Paper-Scissor game. Paper beats Rock, Rock beats Scissors, Scissors beats Paper. The two players simultaneously make their choice. The first player can choose any object. The second player can choose Rock or Paper. The winner receives 1 rouble from the loser. In case of a draw the wealth of a player does not change.
   (a) Construct the payoff matrix of the game.
   (b) Find all pure and mixed Nash equilibria
8. Two players are trying to bribe the judge. The possible amount of bribe is any real number between 0 and 1 million roubles. The player who gives the biggest bribe is announced as the winner of the affair by the judge. The winner receives 1 million roubles. The loser gets nothing. Obviously bribes are not returned by the judge. In the case of equal bribes each player gets nothing.
   (a) Are there any pure Nash equilibria in this game?
   (b) Find at least one mixed Nash equilibrium.

2.6 MFE, retake exam, 19.09.2012

Time allowed 120 minutes.
Students should answer all of the following eight questions. Calculators are not permitted in the exam. Marks will be deducted for insufficient explanations within your answers.
Section A: 10 points each question.

1. It is known that the functions $f_1(x)$ and $f_2(x)$ are concave up. Is it possible that the function $h(x) = \max\{f_1(x), f_2(x)\}$ is concave down?
2. The implicit function $z(x, y)$ is given by the equation $x - z = f(y - z)$ where f is some unknown differentiable function. Find $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$.
3. Consider the monopolist producing two distinct goods. The cost function is given by $TC(q_1, q_2) = q_1 + kq_2$, where the constant $k \in (0; 1)$. And the demand functions are given by $q_1(p_1, p_2) = q_2(p_1, p_2) = (p_1 p_2)^{-3}$.
   (a) Find the optimal production bundle for the monopolist.
   (b) For which values of k one of the product is priced under marginal costs?
4. The density of an exponential random variable X is given by $f(x) = \exp(-x)$ for $x > 0$. Calculate $E(X)$, $E(X^2)$.
   Hint: if you don't remember, $E(X^n) = \int_{-\infty}^{+\infty} x^n f(x)\, dx$
5. The point elasticity of demand for a good is given by $\varepsilon = p^2/(p^2 + 4p + 3)$. Find the demand function $q(p)$ given the initial condition $q(1) = 1$.
6. Find the values a and b such that the function f is homogeneous:

$$f(x, y) = 2x^{b-a}y^{b+2} + y^{a+1}x^{-3b} + y^{7b}x^{-2a}$$

   For the values of a and b you have found expand the function

$$h(x) = \sqrt{1 + f(x, x)} \cdot (1 - \cos(f(x, x))$$

   as a power series up to x^4. State the range for x where your expansion is correct.

Section B: 20 points each question.

1. Use Lagrange multipliers method to solve optimization problem

$$x + \ln y \to \max$$

   subject to $x \geqslant 0$, $x + y \leqslant 4$, $x + 2y \leqslant 6$.
2. Let Y_t, C_t, I_t denote national income, consumption, and investment in period t respectively. The economy is described by the system

$$\begin{cases} Y_t = C_t + I_t \\ C_t = c + mY_t \\ Y_{t+1} = Y_t + rI_t \end{cases} \tag{5}$$

   , where c, m and r are positive constants.
   (a) Find the function Y_t
   (b) Find the asymptote of $\ln(Y(t))$ as t tends to infinity.

2.7 MOR, retake exam, 11.09.12

Section A. Solve **two** of the following **two** problems

1. Use Lagrange multipliers method to solve optimization problem

$$x + \ln y \to \max$$

subject to $x \geqslant 0$, $x + y \leqslant 4$, $x + 2y \leqslant 6$.

2. Solve the linear program depending on the parameter β,

$$2x_1 + 4x_2 + 5x_3 + x_4 \to \min$$

subject to $x_1 + x_3 + x_4 \geqslant 0$, $-x_1 + x_2 + \frac{1}{2}x_3 - \beta x_4 \geqslant 1$, $x_i \geqslant 0$.
For what values of β the minimum of the objective function equals 3?

Section B. Solve **two** of the following **three** problems

3. Find the general solution of the differential equation $y'' + 4y' + 4y = xe^{-3x} + \cos(x)$
4. Consider the system of difference equations

$$\begin{cases} x_{t+1} = 2x_t - 4y_t \\ y_{t+1} = x_t - 3y_t + 3 \end{cases}$$

   (a) Solve the system

   (b) Find the equilibrium solution and check whether it's stable

5. A policymaker desires to double in 10 periods of time the value of GDP y_t produced in period t. Evolution of GDP over time is given by equation $4y_{t+2} - 4y_{t+1} + y_t = 2^t + t^2$. Is doubling of GDP feasible? If the answer is positive, is it possible to find the period t when the value of y_t will first exceed $2y_0$, where y_0 is the initial GDP?

Section C. Solve **two** of the following **three** problems

6. Find all pure and mixed Nash equilibria in the following bimatrix game:

	D	E	F
A	5;5	2;4	2;1
B	3;8	4;7	1;4
C	1;3	3;2	6;7

7. A man has two sons. When he dies, the value of his estate after tax is \$1000. In his will it states that the sons must specify the sum of money s_i that they are willing to accept. If $s_1 + 2s_2 \leqslant 1000$, then each gets the sum he asked for and the rest goes the cats' shelter. If $s_1 + 2s_2 > 1000$, then neither of them gets any money and the entire sum goes to the cats' shelter. Assume that the sons only care about the money they will inherit and they ask for the whole dollars. Find the pure strategies Nash equlibria of this game.
8. Two players are trying to bribe the judge. The possible amount of bribe is any real number between 0 and 1 million roubles. The probability that the player will win is proportional to the amount of the bribe. In the case of zero bribes the probability is equal 1/2 for each player. The winner receives 1 million roubles. The loser gets nothing. Obviously bribes are not returned by the judge.

   (a) Are there any pure Nash equilibria in this game?

   (b) Find at least one mixed Nash equilibrium.

3 2012-2013

3.1 MFE, mock, 25.10.12

Marks will be deducted for insufficient explanation within your answers. All problems are mandatory. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min.

SECTION A

1. Consider the function $f(x, y) = x^3+y^2+xy-y^5$. Using the total differential find the approximate value of $f(1.01, 0.98)$.
2. Consider the function $f(x, y) = x^3 + y^2 + xy - y^5$ at the point $A = (1, 1)$. I have two choices:
   (a) move from A in the direction $\vec{l} = (2, 1)$ by a small number ε
   (b) move from A in the direction $\vec{m} = (1, 2)$ by 2ε

   Using the directional derivative find which choice will give me a bigger value of the function f at the destination point.
3. Consider the following system of equations:
   $$\begin{cases} xyz + 2x^3y^3z^3 + 4x^3 = 7 \\ x + y^3 + z^3 + xy^3 + 2x^2z^2 = 6 \end{cases}$$
   (a) Does this system define functions $z(x)$ and $y(x)$ at a point $x = 1$, $y = 1$, $z = 1$?
   (b) If it's possible find $y'(x)$ and $z'(x)$ at that point
4. Determine whether the following limit exists
   $$\lim_{x,y\to 0} \frac{x^3 + y^3}{x^2 + y^2}$$
5. Consider the function $g(u) = f(x, y)$, $x = 2u$ and $y = u - u^2$. Find $g'(u)$ and $g''(u)$. Assume that f has continuous second partial derivatives at any point.
6. Find the equation of a tangent plane to the surface $z = x + y^2$ at the point $(0, 1, 1)$.

SECTION B

7. Find all the stationary points of the implicit function $z(x, y)$ given by the equation $x^2 + y^2 + z^2 + xy + xz + 2 = 4x + 3y$.
8. Consider a market with the demand curve $q^d = f(p)$ and the supply curve $q^s = g(p, a)$ where the parameter a describes the available technology. The goal is to find how will the equilibrium price p^* and quantity q^* react to the change of the technology parameter a. We assume that $f'(p) < 0$, $\partial g/\partial p > 0$ and $\partial g/\partial a > 0$.
   (a) Find dp^*/da, dq^*/da
   (b) If possible determine the sign of the derivatives dp^*/da and dq^*/da.

3.2 MFE, fall semester exam, 27.12.2012

Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min.
SECTION A:

1. The level curves of a function $f(x, y)$ are shown below. Draw the level curves of the functions $g(x, y) = f(x + 1, y)$ and $h(x, y) = f(-x, |y|)$.

`level_curves.pdf`

2. Find the local maxima and minima of the function $f(x, y) = x^4 + 2y^4 - xy$. Determine whether the extrema you have found are global or local.
3. Find the gradient of the function $h(x, y) = f(x, y) \cdot g(x, y)$ at the point $A = (1, 7)$. It is known that at the point A: grad$f = (1, 1)$, grad$g = (3, 3)$, $f(A) = 4$, $g(A) = 5$.
4. Consider the following system of equations as defining functions $y_1(x_1, x_2)$ and $y_2(x_1, x_2)$

$$\begin{cases} x_1^3 + x_1 y_1^3 + x_2 y_1 y_2 + y_2^3 = 4 \\ x_2 + x_2^3 + y_1 y_2^2 + y_2^3 = 4 \end{cases}$$

   (a) If possible find dy_1 at the point $(x_1, x_2, y_1, y_2) = (1, 1, 1, 1)$.

   (b) Find approximately y_1 for $x_1 = 1.01$ and $x_2 = 0.98$.

5. Find the constrained extrema of the function $f(x, y) = x + 2y$ subject to $2x^2 + y^2 = 10$.
6. Find the Hesse matrix of the function $h(x, y) = \mathbb{P}(Z \in [2x; 3y])$ where Z is a standard normal random variable with probability density function given by $f(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$. It is supposed that $3y > 2x$.

SECTION B:

1. A firm's production function is $Q = K + L + 2\sqrt{KL}$, where $K > 0$ and $L > 0$ are capital and labor, respectively. The firm is perfectly competitive and seeks to maximize its output, but the firm is run by accountants who have imposed a fixed budget on the production of C dollars per hour, which means satisfying constraint $wL + rK = C$, where w and r are hourly wage rate and rental rate of capital, respectively.

   (a) State the constrained optimization problem associated with that production and solve it by the Lagrange multiplier method (it is sufficient to find the optimal values of capital and labor alone).

   (b) Check the concavity of the production function and use it to classify the critical point.

   (c) Explain the economic meaning of the Lagrange multiplier in this problem. Use the appropriate envelope theorem.

   (d) Let $w = r$. Would it be right to conclude that if the wage rate goes up by 1% and the rental rate goes down by the same 1% under the fixed budget the output will not change? How can you prove this mathematically?

2. It is well known that a perfectly competitive firm operating in the long-run produces at the minimum point of its average costs curve, where $p = AC = MC$. Let the AC curve be U-shaped. If we assume that this particular firm uses only labor, equation $AC(y, w) = MC(y, w)$ may be used to find $y = y(w)$ as an implicit function of wages (y denotes the output). Moreover, the second-order condition that guarantees the profit maximization is supposed to hold.

   (a) Show that Implicit Function Theorem can be applied, the function $y = y(w)$ exists and find its derivative.

(b) If we know that at long-run equilibrium $\frac{\partial MC}{\partial w} > \frac{\partial AC}{\partial w}$ what can we say about new long-run equilibrium when the market adjusts to a small rise in wage w? Will the equilibrium price go up? What about the output of the firm?