Variant 1. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)

SECTION A

1. Consider the function $f(x,y,z) = x^5 + 2xyz - 3z^3$. Using the total differential find the approximate value of $f(1.01, 0.99, 1.01)$.

2. Consider the function $f(x,y) = 2x^4 - (x+y)^3$.

   (a) Find the Hesse matrix. Clearly state the Young theorem even if you don't use it.

   (b) Find the definiteness (positive definite, positive semidefinite, etc) of the Hesse matrix at the point $(1,2)$.

3. Let the function $f(x,y)$ be defined by the formula

$$f(x,y) = \begin{cases} -1, \text{ if } x > y \\ 1, \text{ if } x \leq y \end{cases}$$

   (a) Find the limits $\lim_{x\to\infty} \lim_{y\to\infty} f(x,y)$ and $\lim_{y\to\infty} \lim_{x\to\infty} f(x,y)$

   (b) Does the limit $\lim_{x\to\infty, y\to\infty} f(x,y)$ exist?

4. The functions f and g are given: $f(x,y) = x^2 + 2xy + y^4$, $g(x,y) = -5x^2 - xy - 2y^4$. Find at least one direction from the point $(1,1)$ in which both functions will grow.

5. The function z is defined by the formula $z(x,y) = f(x^3 - y^2)$. Simplify the expression $2y\frac{\partial z}{\partial x} + 3x^2\frac{\partial z}{\partial y}$.

6. Consider the function $f(x,y) = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \ldots}}}}$.

   (a) Find the value of $(f^2(x,y) - x)^2 - y - f(x,y)$

   (b) Find $\partial f / \partial x$ and $\partial f / \partial y$ at the point $(1,1)$

SECTION B

7. (20 points) Let S_1 and S_2 be two sets from $\mathbb{R}^2$: $S_1 = \{(x,y) \in \mathbb{R}^2 | xy = 1\}$, $S_2 = \{(x,y) \in \mathbb{R}^2 | xy = -1\}$ and $S = S_1 + S_2$. We denote by the sum $S_1 + S_2$ the set

$$S_1 + S_2 = \{(x,y) \in \mathbb{R}^2 | (x,y) = (x_1,y_1) + (x_2,y_2),\ (x_1,y_1) \in S_1,\ (x_2,y_2) \in S_2\}$$

   (a) Are the sets S_1 and S_2 closed? Justify your answer.

   (b) Does the origin belongs to the set S?

   (c) Is the set S closed?

8. (20 points) Consider a Cournot duopoly of the two identical firms that compete by choosing outputs y_1 and y_2 simultaneously. Marginal costs of these firms are constant $MC_1 = MC_2 = c > 0$. When the outputs y_1 and y_2 are set, the price of a good can be found by the formula $p = a - b(y_1 + y_2)$, where $a > c$, $b > 0$.

   (a) Find equations of the level curves for the profits of the firms $\pi_1(y_1, y_2)$ and $\pi_2(y_1, y_2)$.

   (b) It is known that the point of equilibrium outputs (y_1^*, y_2^*) in the coordinate plane (y_1, y_2) can be found by drawing tangent lines to the level curves and these tangents should be parallel to the axes. Then (y_1^*, y_2^*) is the point of intersection of the tangents. By finding corresponding gradients of $\pi_1(y_1, y_2)$, $\pi_2(y_1, y_2)$ and using the hint stated above, find (y_1^*, y_2^*) in terms of a, b and c.

Variant 2. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)

SECTION A

1. Consider the function $f(x, y, z) = 2x^5 + 2xyz - 3z^3$. Using the total differential find the approximate value of $f(1.02, 0.98, 1.01)$.

2. Consider the function $f(x, y) = 3x^4 - (x + y)^3$.

   (a) Find the Hesse matrix. Clearly state the Young theorem even if you don't use it.

   (b) Find the definiteness (positive definite, positive semidefinite, etc) of the Hesse matrix at the point $(1, 3)$.

3. Let the function $f(x, y)$ be defined by the formula

$$f(x, y) = \begin{cases} -2, \text{ if } x > y \\ 2, \text{ if } x \leq y \end{cases}$$

   (a) Find the limits $\lim_{x\to\infty} \lim_{y\to\infty} f(x, y)$ and $\lim_{y\to\infty} \lim_{x\to\infty} f(x, y)$

   (b) Does the limit $\lim_{x\to\infty, y\to\infty} f(x, y)$ exist?

4. The functions f and g are given: $f(x, y) = x^2 + 2xy + y^4$, $g(x, y) = -6x^2 - xy - 2y^4$. Find at least one direction from the point $(1, 1)$ in which both functions will grow.

5. The function z is defined by the formula $z(x, y) = f(x^3 - 2y^3)$. Simplify the expression $6y^2 \frac{\partial z}{\partial x} + 3x^2 \frac{\partial z}{\partial y}$.

6. Consider the function $f(x, y) = \sqrt{x + \sqrt{2y + \sqrt{x + \sqrt{2y + \ldots}}}}$.

   (a) Find the value of $(f^2(x, y) - x)^2 - 2y - f(x, y)$

   (b) Find $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(1, 1)$

SECTION B

7. (20 points) Let S_1 and S_2 be two sets from $\mathbb{R}^2$: $S_1 = \{(x, y) \in \mathbb{R}^2 | xy = 1\}$, $S_2 = \{(x, y) \in \mathbb{R}^2 | xy = -1\}$ and $S = S_1 + S_2$. We denote by the sum $S_1 + S_2$ the set

$$S_1 + S_2 = \{(x, y) \in \mathbb{R}^2 | (x, y) = (x_1, y_1) + (x_2, y_2),\ (x_1, y_1) \in S_1,\ (x_2, y_2) \in S_2\}$$

   (a) Are the sets S_1 and S_2 closed? Justify your answer.

   (b) Does the origin belongs to the set S?

   (c) Is the set S closed?

8. (20 points) Consider a Cournot duopoly of the two identical firms that compete by choosing outputs y_1 and y_2 simultaneously. Marginal costs of these firms are constant $MC_1 = MC_2 = c > 0$. When the outputs y_1 and y_2 are set, the price of a good can be found by the formula $p = a - b(y_1 + y_2)$, where $a > c$, $b > 0$.

   (a) Find equations of the level curves for the profits of the firms $\pi_1(y_1, y_2)$ and $\pi_2(y_1, y_2)$.

   (b) It is known that the point of equilibrium outputs (y_1^*, y_2^*) in the coordinate plane (y_1, y_2) can be found by drawing tangent lines to the level curves and these tangents should be parallel to the axes. Then (y_1^*, y_2^*) is the point of intersection of the tangents. By finding corresponding gradients of $\pi_1(y_1, y_2)$, $\pi_2(y_1, y_2)$ and using the hint stated above, find (y_1^*, y_2^*) in terms of a, b and c.

Variant 3. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)

SECTION A

1. Consider the function $f(x, y, z) = x^5 + 3xyz - 2z^3$. Using the total differential find the approximate value of $f(1.03, 0.99, 1.01)$.

2. Consider the function $f(x, y) = 2x^4 - 2(x + y)^3$.
   (a) Find the Hesse matrix. Clearly state the Young theorem even if you don't use it.
   (b) Find the definiteness (positive definite, positive semidefinite, etc) of the Hesse matrix at the point $(2, 2)$.

3. Let the function $f(x, y)$ be defined by the formula

$$f(x,y) = \begin{cases} -3, \text{ if } x > y \\ 3, \text{ if } x \le y \end{cases}$$

   (a) Find the limits $\lim_{x\to\infty} \lim_{y\to\infty} f(x, y)$ and $\lim_{y\to\infty} \lim_{x\to\infty} f(x, y)$
   (b) Does the limit $\lim_{x\to\infty, y\to\infty} f(x, y)$ exist?

4. The functions f and g are given: $f(x, y) = x^2 + 2xy + y^4$, $g(x, y) = -7x^2 - xy - 2y^4$. Find at least one direction from the point $(1, 1)$ in which both functions will grow.

5. The function z is defined by the formula $z(x, y) = f(x^3 - y^5)$. Simplify the expression $5y^4 \frac{\partial z}{\partial x} + 3x^2 \frac{\partial z}{\partial y}$.

6. Consider the function $f(x,y) = \sqrt{x + \sqrt{3y + \sqrt{x + \sqrt{3y + \ldots}}}}$.
   (a) Find the value of $(f^2(x, y) - x)^2 - 3y - f(x, y)$
   (b) Find $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(1, 1)$

SECTION B

7. (20 points) Let S_1 and S_2 be two sets from $\mathbb{R}^2$: $S_1 = \{(x, y) \in \mathbb{R}^2 | xy = 1\}$, $S_2 = \{(x, y) \in \mathbb{R}^2 | xy = -1\}$ and $S = S_1 + S_2$. We denote by the sum $S_1 + S_2$ the set

$$S_1 + S_2 = \{(x, y) \in \mathbb{R}^2 | (x, y) = (x_1, y_1) + (x_2, y_2),\ (x_1, y_1) \in S_1,\ (x_2, y_2) \in S_2\}$$

   (a) Are the sets S_1 and S_2 closed? Justify your answer.
   (b) Does the origin belongs to the set S?
   (c) Is the set S closed?

8. (20 points) Consider a Cournot duopoly of the two identical firms that compete by choosing outputs y_1 and y_2 simultaneously. Marginal costs of these firms are constant $MC_1 = MC_2 = c > 0$. When the outputs y_1 and y_2 are set, the price of a good can be found by the formula $p = a - b(y_1 + y_2)$, where $a > c$, $b > 0$.
   (a) Find equations of the level curves for the profits of the firms $\pi_1(y_1, y_2)$ and $\pi_2(y_1, y_2)$.
   (b) It is known that the point of equilibrium outputs (y_1^*, y_2^*) in the coordinate plane (y_1, y_2) can be found by drawing tangent lines to the level curves and these tangents should be parallel to the axes. Then (y_1^*, y_2^*) is the point of intersection of the tangents. By finding corresponding gradients of $\pi_1(y_1, y_2)$, $\pi_2(y_1, y_2)$ and using the hint stated above, find (y_1^*, y_2^*) in terms of a, b and c.

Variant 4. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)

SECTION A

1. Consider the function $f(x, y, z) = 2x^5 + 2xyz - 2z^3$. Using the total differential find the approximate value of $f(1.04, 0.99, 1.01)$.

2. Consider the function $f(x, y) = 3x^4 - (x + y)^3$.

   (a) Find the Hesse matrix. Clearly state the Young theorem even if you don't use it.

   (b) Find the definiteness (positive definite, positive semidefinite, etc) of the Hesse matrix at the point $(2, 1)$.

3. Let the function $f(x, y)$ be defined by the formula

$$f(x,y) = \begin{cases} -4, \text{ if } x > y \\ 4, \text{ if } x \leq y \end{cases}$$

   (a) Find the limits $\lim_{x\to\infty} \lim_{y\to\infty} f(x, y)$ and $\lim_{y\to\infty} \lim_{x\to\infty} f(x, y)$

   (b) Does the limit $\lim_{x\to\infty, y\to\infty} f(x, y)$ exist?

4. The functions f and g are given: $f(x, y) = x^2 + 2xy + y^4$, $g(x, y) = -8x^2 - xy - 2y^4$. Find at least one direction from the point $(1, 1)$ in which both functions will grow.

5. The function z is defined by the formula $z(x, y) = f(x^4 - y^2)$. Simplify the expression $2y\frac{\partial z}{\partial x} + 4x^3\frac{\partial z}{\partial y}$.

6. Consider the function $f(x, y) = \sqrt{x + \sqrt{4y + \sqrt{x + \sqrt{4y + \ldots}}}}$.

   (a) Find the value of $(f^2(x, y) - x)^2 - 4y - f(x, y)$

   (b) Find $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(1, 1)$

SECTION B

7. (20 points) Let S_1 and S_2 be two sets from $\mathbb{R}^2$: $S_1 = \{(x, y) \in \mathbb{R}^2 | xy = 1\}$, $S_2 = \{(x, y) \in \mathbb{R}^2 | xy = -1\}$ and $S = S_1 + S_2$. We denote by the sum $S_1 + S_2$ the set

$$S_1 + S_2 = \{(x, y) \in \mathbb{R}^2 | (x, y) = (x_1, y_1) + (x_2, y_2),\ (x_1, y_1) \in S_1,\ (x_2, y_2) \in S_2\}$$

   (a) Are the sets S_1 and S_2 closed? Justify your answer.

   (b) Does the origin belongs to the set S?

   (c) Is the set S closed?

8. (20 points) Consider a Cournot duopoly of the two identical firms that compete by choosing outputs y_1 and y_2 simultaneously. Marginal costs of these firms are constant $MC_1 = MC_2 = c > 0$. When the outputs y_1 and y_2 are set, the price of a good can be found by the formula $p = a - b(y_1 + y_2)$, where $a > c$, $b > 0$.

   (a) Find equations of the level curves for the profits of the firms $\pi_1(y_1, y_2)$ and $\pi_2(y_1, y_2)$.

   (b) It is known that the point of equilibrium outputs (y_1^*, y_2^*) in the coordinate plane (y_1, y_2) can be found by drawing tangent lines to the level curves and these tangents should be parallel to the axes. Then (y_1^*, y_2^*) is the point of intersection of the tangents. By finding corresponding gradients of $\pi_1(y_1, y_2)$, $\pi_2(y_1, y_2)$ and using the hint stated above, find (y_1^*, y_2^*) in terms of a, b and c.