Variant 1. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)

SECTION A

1. Consider the set $A = \cup_{i=1}^{\infty} \left\{1 - \frac{1}{i}\right\} \subset \mathbb{R}$.
   (a) Is the set A bounded? Open? Closed? Compact?
   (b) Roughly sketch the set $A \times A$

2. Consider the system of equations
$$\begin{cases} xyz^2 + x^3y + 3y^3z^3 - 2x = 3 \\ 2x^2yz + z^2 + 3xy + 5yx^3 = -5 \end{cases}$$
   (a) Are the function $x(z)$ and $y(z)$ defined around the point $A = (-1, 1, 1)$?
   (b) Find df/dz, where $f(z) = x(z)y(z)$

3. Consider the functions $f(x, y) = x^2 + 2x + y^2 + 6y + 7$ and $g(x, y) = x^2 - 8x + y^2 - 10y + 9$. Find all the points where the gradients are parallel.

4. Find and classify all the local extrema of the function $g(x, y) = y^3 - 12y + x^2e^y$. Which of them are global?

5. It is known that the point $(1, 0)$ is the constrained local maximum of the function $f(x, y) = 5x - ky - 3x^2 + 2xy - 5y^2$ subject to $x + y = 1$.
   (a) Find the value of k and the maximum value of the function f
   (b) Using Envelope theorem find the new value of maximum if k will increase by 0.1

6. Use Lagrange multipliers to find the height and radius of a cylinder with the least possible surface area among those with a volume of 6π m^3. Make sure you check the second order condition for minimisation.

SECTION B

7. In perfectly competitive agricultural industry a typical firm uses labor, capital and land. Its short-run costs can be found by the formula $C_{sr}(y, K^*, T^*) = \frac{y^3}{K^*T^*} + K^* + T^*$, where y is the output, K^* and T^* are fixed quantities of capital and land, respectively. The long run costs $C_{lr}(y)$ can be found by minimizing C_{sr} with respect to K^* and T^*.
   (a) Find $C_{lr}(y)$
   (b) The profits of the firm in both short-run and long-run can be found by $\pi_{sr} = py - C_{sr}$ and $\pi_{lr} = py - C_{lr}$, where p is price. When the profits are maximized with respect to the output, the maximum values of these are denoted by $\tilde{\pi}_{sr}(p, K^*, T^*)$ and $\tilde{\pi}_{lr}(p)$, respectively. Let $p = 3$. Show that $\tilde{\pi}_{sr}(3, K^*, T^*) \leq 0$ for all values of $K^* > 0$ and $T^* > 0$.
   (c) Use Envelope Theorem to evaluate $\frac{\partial \tilde{\pi}_{sr}}{\partial K^*}$ and $\frac{\partial \tilde{\pi}_{sr}}{\partial T^*}$ for $p = 3$. Is it possible that these derivatives turn zero simultaneously for $T^* \neq K^*$?

8. Let $f(x)$ be twice continuously differentiable function whose second-order derivative is $f''(x) > 0$ for all x. Consider a constrained minimization problem $F = \sum_{i=1}^{n} f(x_i) \to \min$ subject to $\sum_{i=1}^{n} x_i = 1$, $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Using the first-order conditions find the critical point. By checking bordered Hessian or otherwise show that the found point is minimum.

Variant 2. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)

SECTION A

1. Consider the set $A = \cup_{i=1}^{\infty} \{2 - \frac{1}{i}\} \subset \mathbb{R}$.
   (a) Is the set A bounded? Open? Closed? Compact?
   (b) Roughly sketch the set $A \times A$
2. Consider the system of equations
$$\begin{cases} xyz^2 + x^3y + 4y^3z^3 - 2x = 4 \\ 2x^2yz + z^2 + 3xy + 5yx^3 = -5 \end{cases}$$
   (a) Are the function $x(z)$ and $y(z)$ defined around the point $A = (-1, 1, 1)$?
   (b) Find df/dz, where $f(z) = x(z)y(z)$
3. Consider the functions $f(x, y) = x^2 - 2x + y^2 + 6y + 7$ and $g(x, y) = x^2 - 8x + y^2 - 10y + 9$. Find all the points where the gradients are parallel.
4. Find and classify all the local extrema of the function $g(x, y) = y^3 - 12y + 2x^2e^y$. Which of them are global?
5. It is known that the point $(1, 0)$ is the constrained local maximum of the function $f(x, y) = 5x - ky - 3x^2 + 2xy - 5y^2$ subject to $x + y = 1$.
   (a) Find the value of k and the maximum value of the function f
   (b) Using Envelope theorem find the new value of maximum if k will increase by 0.2
6. Use Lagrange multipliers to find the height and radius of a cylinder with the least possible surface area among those with a volume of 9π m^3. Make sure you check the second order condition for minimisation.

SECTION B

7. In perfectly competitive agricultural industry a typical firm uses labor, capital and land. Its short-run costs can be found by the formula $C_{sr}(y, K^*, T^*) = \frac{y^3}{K^*T^*} + K^* + T^*$, where y is the output, K^* and T^* are fixed quantities of capital and land, respectively. The long run costs $C_{lr}(y)$ can be found by minimizing C_{sr} with respect to K^* and T^*.
   (a) Find $C_{lr}(y)$
   (b) The profits of the firm in both short-run and long-run can be found by $\pi_{sr} = py - C_{sr}$ and $\pi_{lr} = py - C_{lr}$, where p is price. When the profits are maximized with respect to the output, the maximum values of these are denoted by $\tilde{\pi}_{sr}(p, K^*, T^*)$ and $\tilde{\pi}_{lr}(p)$, respectively. Let $p = 3$. Show that $\tilde{\pi}_{sr}(3, K^*, T^*) \leq 0$ for all values of $K^* > 0$ and $T^* > 0$.
   (c) Use Envelope Theorem to evaluate $\frac{\partial \tilde{\pi}_{sr}}{\partial K^*}$ and $\frac{\partial \tilde{\pi}_{sr}}{\partial T^*}$ for $p = 3$. Is it possible that these derivatives turn zero simultaneously for $T^* \neq K^*$?
8. Let $f(x)$ be twice continuously differentiable function whose second-order derivative is $f''(x) > 0$ for all x. Consider a constrained minimization problem $F = \sum_{i=1}^{n} f(x_i) \to \min$ subject to $\sum_{i=1}^{n} x_i = 1$, $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Using the first-order conditions find the critical point. By checking bordered Hessian or otherwise show that the found point is minimum.

Variant 3. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)

SECTION A

1. Consider the set $A = \cup_{i=1}^{\infty} \{3 - \frac{1}{i}\} \subset \mathbb{R}$.

   (a) Is the set A bounded? Open? Closed? Compact?

   (b) Roughly sketch the set $A \times A$

2. Consider the system of equations

$$\begin{cases} xyz^2 + x^3y + 5y^3z^3 - 2x = 5 \\ 2x^2yz + z^2 + 3xy + 5yx^3 = -5 \end{cases}$$

   (a) Are the function $x(z)$ and $y(z)$ defined around the point $A = (-1, 1, 1)$?

   (b) Find df/dz, where $f(z) = x(z)y(z)$

3. Consider the functions $f(x, y) = x^2 + 2x + y^2 - 6y + 7$ and $g(x, y) = x^2 - 8x + y^2 - 10y + 9$. Find all the points where the gradients are parallel.

4. Find and classify all the local extrema of the function $g(x, y) = y^3 - 12y + 3x^2e^y$. Which of them are global?

5. It is known that the point $(1, 0)$ is the constrained local maximum of the function $f(x, y) = 5x - ky - 3x^2 + 2xy - 5y^2$ subject to $x + y = 1$.

   (a) Find the value of k and the maximum value of the function f

   (b) Using Envelope theorem find the new value of maximum if k will increase by 0.3

6. Use Lagrange multipliers to find the height and radius of a cylinder with the least possible surface area among those with a volume of 12π m^3. Make sure you check the second order condition for minimisation.

SECTION B

7. In perfectly competitive agricultural industry a typical firm uses labor, capital and land. Its short-run costs can be found by the formula $C_{sr}(y, K^*, T^*) = \frac{y^3}{K^*T^*} + K^* + T^*$, where y is the output, K^* and T^* are fixed quantities of capital and land, respectively. The long run costs $C_{lr}(y)$ can be found by minimizing C_{sr} with respect to K^* and T^*.

   (a) Find $C_{lr}(y)$

   (b) The profits of the firm in both short-run and long-run can be found by $\pi_{sr} = py - C_{sr}$ and $\pi_{lr} = py - C_{lr}$, where p is price. When the profits are maximized with respect to the output, the maximum values of these are denoted by $\tilde{\pi}_{sr}(p, K^*, T^*)$ and $\tilde{\pi}_{lr}(p)$, respectively. Let $p = 3$. Show that $\tilde{\pi}_{sr}(3, K^*, T^*) \leq 0$ for all values of $K^* > 0$ and $T^* > 0$.

   (c) Use Envelope Theorem to evaluate $\frac{\partial \tilde{\pi}_{sr}}{\partial K^*}$ and $\frac{\partial \tilde{\pi}_{sr}}{\partial T^*}$ for $p = 3$. Is it possible that these derivatives turn zero simultaneously for $T^* \neq K^*$?

8. Let $f(x)$ be twice continuously differentiable function whose second-order derivative is $f''(x) > 0$ for all x. Consider a constrained minimization problem $F = \sum_{i=1}^{n} f(x_i) \to \min$ subject to $\sum_{i=1}^{n} x_i = 1$, $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Using the first-order conditions find the critical point. By checking bordered Hessian or otherwise show that the found point is minimum.

Variant 4. Please, don't forget to write you variant number. Sections A and B will make up 60% and 40% of the exam grade, respectively. Total duration of the exam is 120 min. Good luck! :)

SECTION A

1. Consider the set $A = \cup_{i=1}^{\infty} \left\{4 - \frac{1}{i}\right\} \subset \mathbb{R}$.
   (a) Is the set A bounded? Open? Closed? Compact?
   (b) Roughly sketch the set $A \times A$

2. Consider the system of equations
$$\begin{cases} xyz^2 + x^3y + 6y^3z^3 - 2x = 6 \\ 2x^2yz + z^2 + 3xy + 5yx^3 = -5 \end{cases}$$
   (a) Are the function $x(z)$ and $y(z)$ defined around the point $A = (-1, 1, 1)$?
   (b) Find df/dz, where $f(z) = x(z)y(z)$

3. Consider the functions $f(x, y) = x^2 - 2x + y^2 - 6y + 7$ and $g(x, y) = x^2 - 8x + y^2 - 10y + 9$. Find all the points where the gradients are parallel.

4. Find and classify all the local extrema of the function $g(x, y) = y^3 - 12y + 4x^2e^y$. Which of them are global?

5. It is known that the point $(1, 0)$ is the constrained local maximum of the function $f(x, y) = 5x - ky - 3x^2 + 2xy - 5y^2$ subject to $x + y = 1$.
   (a) Find the value of k and the maximum value of the function f
   (b) Using Envelope theorem find the new value of maximum if k will increase by 0.4

6. Use Lagrange multipliers to find the height and radius of a cylinder with the least possible surface area among those with a volume of 15π m^3. Make sure you check the second order condition for minimisation.

SECTION B

7. In perfectly competitive agricultural industry a typical firm uses labor, capital and land. Its short-run costs can be found by the formula $C_{sr}(y, K^*, T^*) = \frac{y^3}{K^*T^*} + K^* + T^*$, where y is the output, K^* and T^* are fixed quantities of capital and land, respectively. The long run costs $C_{lr}(y)$ can be found by minimizing C_{sr} with respect to K^* and T^*.
   (a) Find $C_{lr}(y)$
   (b) The profits of the firm in both short-run and long-run can be found by $\pi_{sr} = py - C_{sr}$ and $\pi_{lr} = py - C_{lr}$, where p is price. When the profits are maximized with respect to the output, the maximum values of these are denoted by $\tilde{\pi}_{sr}(p, K^*, T^*)$ and $\tilde{\pi}_{lr}(p)$, respectively. Let $p = 3$. Show that $\tilde{\pi}_{sr}(3, K^*, T^*) \leq 0$ for all values of $K^* > 0$ and $T^* > 0$.
   (c) Use Envelope Theorem to evaluate $\frac{\partial \tilde{\pi}_{sr}}{\partial K^*}$ and $\frac{\partial \tilde{\pi}_{sr}}{\partial T^*}$ for $p = 3$. Is it possible that these derivatives turn zero simultaneously for $T^* \neq K^*$?

8. Let $f(x)$ be twice continuously differentiable function whose second-order derivative is $f''(x) > 0$ for all x. Consider a constrained minimization problem $F = \sum_{i=1}^{n} f(x_i) \to \min$ subject to $\sum_{i=1}^{n} x_i = 1$, $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Using the first-order conditions find the critical point. By checking bordered Hessian or otherwise show that the found point is minimum.