1. Word-Level Neural Bigram Language Model

0.a Gradient of Softmax + CE

- Denote SCE - Softmax Cross Entropy function:

$SCE(\theta, y) = CE(\hat{y}, y) = CE(softmax(\theta), y)$
$= -\sum_i y_i log(\hat{y}_i)$
$if\ k\ -\ true\ label:$
$= -log(\hat{y}_i) = -log(softmax(\theta)_k) = -log(\frac{exp(\theta_k)}{\sum_j exp(\theta_j)})$
$= log(\sum_j exp(\theta_j)) - \theta_k$
$\Longrightarrow SCE(\theta, y) = log(\sum_j exp(\theta_j)) - \theta_k$

- Now let's calculate it's derivative with respect to some θ_i :

$\frac{\delta SCE}{\delta \theta_i} = \frac{log(\sum_j exp(\theta_j))}{\delta \theta_i} - \frac{\delta \theta_k}{\delta \theta_i}$
$\downarrow$
$denote:\ f\ = \sum_j exp(\theta_j)$
$\downarrow$
$\frac{log(\sum_j exp(\theta_j))}{\delta \theta_i} - \frac{\delta \theta_k}{\delta \theta_i}$
$= \frac{log(f)}{\delta f} \frac{\delta f}{\delta \theta_i} - \frac{\delta \theta_k}{\delta \theta_i}$
$= \frac{1}{f} \sum_j \frac{exp(\theta_j)}{exp(\theta_i)} - \frac{\delta \theta_k}{\delta \theta_i}$
$= \frac{1}{f} exp(\theta_i) - \frac{\delta \theta_k}{\delta \theta_i}$
$= \frac{exp(\theta_i)}{\sum_j exp(\theta_j)} - \frac{\delta \theta_k}{\delta \theta_i}$
$= \frac{exp(\theta_i)}{\sum_j exp(\theta_j)} - \mathbb{1}(i = k)$
$= softmax(\theta)_i - \mathbb{1}(i = k)$
$\Longrightarrow \nabla_\theta SCE = softmax(\theta) - y = \hat{y} - y$

- Basically the gradient of this function is prediction softmax vector minus true label one-hot vector.

0.b Gradients of NN

- In this section we use:

$h = \sigma(xW_1 + b_1)$
$\theta = hW_2 + b_2$
$\hat{y} = softmax(\theta)$
$CE(y, \hat{y}) = -\sum_i y_i log(\hat{y}_i)$
$\nabla_\theta SCE = \hat{y} - y$

- Also notice that Jacobian matrix of a vector function sigmoid is:

$$da(\sigma(a)) = diag(\sigma'(a)) = \begin{bmatrix} \sigma'(a_1) & 0 & 0 & 0 \\ 0 & \sigma'(a_2) & 0 & 0 \\ 0 & 0 & \sigma'(a_i) & 0 \\ 0 & 0 & 0 & \sigma'(a_n) \end{bmatrix}\ ;\ \sigma'(a_i) = \sigma(a_i)(1 - \sigma(a_i))$$

- Let's write a formula for each gradient using differentials:

- $dx(SCE) = dx(SCE(hW_2 + b_2))$

$= \nabla_\theta SCE\ dx(hW_2 + b_2)$
$= \nabla_\theta SCE\ dx(hW_2)$
$= \nabla_\theta SCE\ W_2^T\ dx(h)$
$= \nabla_\theta SCE\ W_2^T\ dx(\sigma(xW_1 + b_1))$
$= \nabla_\theta SCE\ W_2^T\ diag(\sigma'(xW_1 + b_1))\ dx(xW_1 + b_1)$
$= \nabla_\theta SCE\ W_2^T\ diag(\sigma'(xW_1 + b_1))\ W_1^T\ dx$
$\Longrightarrow \nabla_x SCE = \nabla_\theta SCE\ W_2^T\ diag(\sigma'(xW_1 + b_1))\ W_1^T$
$\Longrightarrow \nabla_x SCE = (\hat{y} - y)\ W_2^T\ diag(\sigma'(xW_1 + b_1))\ W_1^T$

This formula won't work for batched data, but we can write it in the more generic way using Hadamard product, i.e. elementwise multiplication - $\circ$:

$\Longrightarrow \nabla_x SCE = \{(\hat{y} - y)\ W_2^T\ \circ\ \sigma'(xW_1 + b_1)\}\ W_1^T$

- Let's sanity check it's dimentions:

$x - 2 \times 10$ (batch of two vectors)
$W_1 - 10 \times 7$
$b_1 - 1 \times 7$
$\sigma(xW_1 + b_1) - 2 \times 7$ (bias is broadcasted)
$\sigma'(xW_1 + b_1) - 2 \times 7$ (bias is broadcasted)
$W_2 - 7 \times 5$
$b_2 - 1 \times 5$
$y - 2 \times 5$ (batch of two vectors)

$\Longrightarrow \nabla_x SCE = \{(\hat{y} - y)\ W_2^T\ \circ\ \sigma'(xW_1 + b_1)\}\ W_1^T$
$\Longrightarrow\ 2\ \times\ 10\ = \{(2 \times 5)\ (5 \times 7)\ \circ\ (2 \times 7)\}\ (7 \times 10)$

All dimensions match.

- $dW_1(SCE) = dW_1(SCE(hW_2 + b_2))$

$= \nabla_\theta SCE\ dW_1(hW_2 + b_2)$
$= \nabla_\theta SCE\ dW_1(hW_2)$
$= \nabla_\theta SCE\ W_2^T\ dW_1(h)$
$= \nabla_\theta SCE\ W_2^T\ dW_1(\sigma(xW_1 + b_1))$
$= \nabla_\theta SCE\ W_2^T\ diag(\sigma'(xW_1 + b_1))\ dW_1(xW_1 + b_1)$
$= x^T\ \nabla_\theta SCE\ W_2^T\ diag(\sigma'(xW_1 + b_1))\ dW_1$
$\Longrightarrow \nabla_{W_1} SCE = x^T\ \nabla_\theta SCE\ W_2^T\ diag(\sigma'(xW_1 + b_1))$
$\Longrightarrow \nabla_{W_1} SCE = x^T\ (\hat{y} - y)\ W_2^T\ diag(\sigma'(xW_1 + b_1))$

Same trick with Hadamard product, i.e. elementwise multiplication - $\circ$:
$\Longrightarrow \nabla_{W_1} SCE = x^T\ \{(\hat{y} - y)\ W_2^T\ \circ\ \sigma'(xW_1 + b_1)\}$

- $db_1(SCE) = db_1(SCE(hW_2 + b_2))$

$= \nabla_\theta SCE\ db_1(hW_2 + b_2)$
$= \nabla_\theta SCE\ db_1(hW_2)$
$= \nabla_\theta SCE\ W_2^T\ db_1(h)$
$= \nabla_\theta SCE\ W_2^T\ db_1(\sigma(xW_1 + b_1))$
$= \nabla_\theta SCE\ W_2^T\ diag(\sigma'(xW_1 + b_1))\ db_1(xW_1 + b_1)$
$= \nabla_\theta SCE\ W_2^T\ diag(\sigma'(xW_1 + b_1))\ 1\ db_1$
$\implies \nabla_{b_1} SCE = \nabla_\theta SCE\ W_2^T\ diag(\sigma'(xW_1 + b_1))$
$\implies \nabla_{b_1} SCE = (\hat{y} - y)\ W_2^T\ diag(\sigma'(xW_1 + b_1))$

Same trick with Hadamard product, i.e. elementwise multiplication - $\circ$:
$\implies \nabla_{b_1} SCE \approx \{(\hat{y} - y)\ W_2^T\}\ \circ\ \sigma'(xW_1 + b_1)$

Notice that dims don't match, because bias was broadcasted. To match the dims, we sum the gradient along batch dimension.

$\implies \nabla_{b_1} SCE = \sum_{batch=0}^{N} \{\{(\hat{y} - y)\ W_2^T\}\ \circ\ \sigma'(xW_1 + b_1)\}_{batch}$

- $dW_2(SCE) = dW_2(SCE(hW_2 + b_2))$

$= \nabla_\theta SCE\ dW_2(hW_2 + b_2)$
$= \nabla_\theta SCE\ dW_2(hW_2)$
$= \nabla_\theta SCE\ h\ dW_2$
$= \sigma(xW_1 + b_1)^T\ \nabla_\theta SCE\ dW_2$
$\implies \nabla_{W_2} SCE = \sigma(xW_1 + b_1)^T\ \nabla_\theta SCE$
$\implies \nabla_{W_2} SCE = \sigma(xW_1 + b_1)^T\ (\hat{y} - y)$

- $db_2(SCE) = db_2(SCE(hW_2 + b_2))$

$= \nabla_\theta SCE\ db_2(hW_2 + b_2)$
$= \nabla_\theta SCE\ 1\ dW_2$
$\implies \nabla_{b_2} SCE = \nabla_\theta SCE$
$\implies \nabla_{b_2} SCE \approx (\hat{y} - y)$

Adjust dims:
$\implies \nabla_{b_2} SCE = \sum_{batch=0}^{N} (\hat{y} - y)_{batch}$

0.c

0.d Perplexity

Dev perplexity : 112.889

Section 2 - Theoretical Inquiry of a Simple RNN Language Model

(a) Some notation: Denote the elements of the matrices $\mathbf{H}, \mathbf{I}, \mathbf{U}, \mathbf{L}$ by H_{ij}, I_{ij}, U_{ij}, L_{ij}. Let $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ (Kroenecker delta). Also note that $\mathbf{L}_{\mathbf{x}^{(t)}} = \mathbf{e}^{(t)}$

We will also omit $|_{(t)}$ for notational simplicity but it is assumed when needed.

Let $\boldsymbol{\theta}_1^{(t)} = \mathbf{h}^{(t-1)}\mathbf{H} + \mathbf{e}^{(t)}\mathbf{I} + \mathbf{b}_1$ (logit vector), and denote its k-th element by $\theta_{1,k}^{(t)}$. Calculate its derivatives with respect to the various model parameters:

$$
\begin{aligned}
\frac{\partial \theta_{1,k}^{(t)}}{\partial H_{ij}} &= \frac{\partial}{\partial H_{ij}} \left(\sum_\ell h_\ell^{(t-1)} H_{\ell k} \right) = h_i^{(t-1)} \delta_{jk} \\
\frac{\partial \theta_{1,k}^{(t)}}{\partial I_{ij}} &= \frac{\partial}{\partial I_{ij}} \left(\sum_\ell e_\ell^{(t)} I_{\ell k} \right) = e_i^{(t)} \delta_{jk} \\
\frac{\partial \theta_{1,k}^{(t)}}{\partial b_{1,i}} &= \delta_{ik} \\
\frac{\partial \theta_{1,k}^{(t)}}{e_i^{(t)}} &= I_{ik} \\
\frac{\partial \theta_{1,k}^{(t)}}{\partial h_i^{(t-1)}} &= \frac{\partial}{\partial h_i^{(t-1)}} \left(\sum_\ell h_\ell^{(t-1)} H_{\ell k} \right) = H_{ik}
\end{aligned}
$$

We have $\mathbf{h}^{(t)} = \sigma(\boldsymbol{\theta}_1^{(t)})$. Defining $\boldsymbol{\theta}_2^{(t)} = \mathbf{h}^{(t)}\mathbf{U} + \mathbf{b}_2$ (logit vector) and denoting its k-th element by $\theta_{2,k}^{(t)}$, by the chain rule we have

$$
\begin{aligned}
\frac{\partial \theta_{2,k}^{(t)}}{\partial H_{ij}} &= \frac{\partial}{\partial H_{ij}} \left(\sum_\ell h_\ell^{(t)} U_{\ell k} \right) \\
&= \sum_\ell \frac{\partial h_\ell^{(t)}}{\partial H_{ij}} U_{\ell k}
\end{aligned}
$$

$$\begin{aligned}
&= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) \frac{\partial \theta_{1,\ell}^{(t)}}{\partial H_{ij}} U_{\ell k} \\
&= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) h_i^{(t-1)} \delta_{j\ell} U_{\ell k} \\
&= \sigma'(\theta_{1,k}^{(t)}) h_i^{(t-1)} U_{jk}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \theta_{2,k}^{(t)}}{\partial I_{ij}} &= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) \frac{\partial \theta_{1,\ell}^{(t)}}{\partial I_{ij}} U_{\ell k} \\
&= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) e_i^{(t)} \delta_{j\ell} U_{\ell k} \\
&= \sigma'(\theta_{1,j}^{(t)}) e_i^{(t)} U_{jk}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \theta_{2,k}^{(t)}}{\partial b_{1,i}} &= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) \frac{\partial \theta_{1,\ell}^{(t)}}{\partial b_{1,i}} U_{\ell k} \\
&= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) \delta_{i\ell} U_{\ell k} \\
&= \sigma'(\theta_{1,i}^{(t)}) U_{ik}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \theta_{2,k}^{(t)}}{\partial e_i^{(t)}} &= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) \frac{\partial \theta_{1,\ell}^{(t)}}{\partial e_i^{(t)}} U_{\ell k} \\
&= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) I_{i\ell} U_{\ell k}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \theta_{2,k}^{(t)}}{b_{2,i}} &= \delta_{ik} \\
\frac{\partial \theta_{2,k}^{(t)}}{U_{ij}} &= \frac{\partial}{\partial U_{ij}} \left(\sum_\ell h_\ell^{(t)} U_{\ell k} \right) \\
&= h_i^{(t)} \delta_{jk}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \theta_{2,k}^{(t)}}{h_i^{(t-1)}} &= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) \frac{\partial \theta_{1,\ell}^{(t)}}{\partial h_i^{(t-1)}} U_{\ell k} \\
&= \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) H_{i\ell} U_{\ell k}
\end{aligned}$$

Recall that $\sigma'(t) = \sigma(t)(1-\sigma(t))$.

Now since $J^{(t)} = \mathrm{CE}(\mathbf{y}^{(t)}, \hat{\mathbf{y}}^{(t)})$ and $\hat{\mathbf{y}}^{(t)} = \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)})$, by the result from problem 1a $\nabla_{\boldsymbol{\theta}_2^{(t)}} J^{(t)} = \mathbf{y}^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)})$, i.e. $\frac{\partial J^{(t)}}{\theta_{2,k}^{(t)}} = y_k^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)})_k$. Therefore we can calculate the derivatives of J with respect to the various parameters by the chain rule:

$$\begin{aligned}
\frac{\partial J^{(t)}}{\partial H_{ij}} &= \sum_k \frac{\partial J^{(t)}}{\partial \theta_{2,k}^{(t)}} \frac{\partial \theta_{2,k}^{(t)}}{\partial H_{ij}} \\
&= \sum_k (y_k^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)})_k)\sigma'(\theta_{1,k}^{(t)}) h_i^{(t-1)} U_{jk} \\
\frac{\partial J^{(t)}}{\partial \mathbf{H}} &= \mathbf{h}^{(t-1)T}(\mathbf{y}^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)}))\boldsymbol{\Sigma}(\boldsymbol{\theta}_1^{(t)})\mathbf{U}^T
\end{aligned}$$

where we define $\boldsymbol{\Sigma}(\mathbf{v})$ as in problem 1a to be the diagonal matrix with i-th diagonal element $\sigma'(v_i) = \sigma(v_i)(1 - \sigma(v_i))$.

$$\begin{aligned}
\frac{\partial J^{(t)}}{\partial I_{ij}} &= \sum_k \frac{\partial J^{(t)}}{\partial \theta_{2,k}^{(t)}} \frac{\partial \theta_{2,k}^{(t)}}{\partial I_{ij}} \\
&= \sum_k (y_k^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)})_k)\sigma'(\theta_{1,j}^{(t)}) e_i^{(t)} U_{jk} \\
\frac{\partial J^{(t)}}{\partial \mathbf{I}} &= \mathbf{e}^{(t)T}(\mathbf{y}^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)}))\mathbf{U}^T\boldsymbol{\Sigma}(\boldsymbol{\theta}_1^{(t)})
\end{aligned}$$

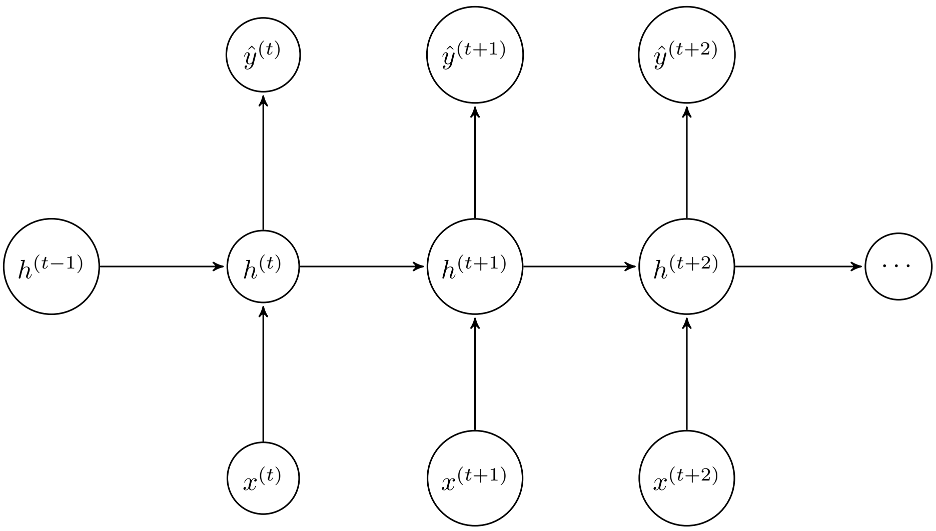
$$\begin{aligned}
\frac{\partial J^{(t)}}{\partial b_{1,i}} &= \sum_k \frac{\partial J^{(t)}}{\partial \theta_{2,k}^{(t)}} \frac{\partial \theta_{2,k}^{(t)}}{\partial b_{1,i}} \\
&= \sum_k (y_k^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)})_k)\sigma'(\theta_{1,i}^{(t)}) U_{ik} \\
\frac{\partial J^{(t)}}{\partial \mathbf{b}_1} &= (\mathbf{y}^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)}))\mathbf{U}^T\boldsymbol{\Sigma}(\boldsymbol{\theta}_1^{(t)})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial J^{(t)}}{\partial e_i^{(t)}} &= \sum_k \frac{\partial J^{(t)}}{\partial \theta_{2,k}^{(t)}} \frac{\partial \theta_{2,k}^{(t)}}{\partial e_i^{(t)}} \\
&= \sum_k (y_k^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)})_k) \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) I_{i\ell} U_{\ell k} \\
\frac{\partial J^{(t)}}{\partial \mathbf{L}_{\mathbf{x}^{(t)}}} &= \frac{\partial J^{(t)}}{\partial \mathbf{e}^{(t)}} = (\mathbf{y}^{(t)} - \mathrm{softmax}(\boldsymbol{\theta}_2^{(t)}))\mathbf{U}^T\boldsymbol{\Sigma}(\boldsymbol{\theta}_1^{(t)})\mathbf{I}^T
\end{aligned}$$

$$\begin{aligned}
\frac{\partial J^{(t)}}{\partial b_{2,i}} &= \sum_k \frac{\partial J^{(t)}}{\partial \theta_{2,k}^{(t)}} \frac{\partial \theta_{2,k}^{(t)}}{\partial b_{2,i}} \\
&= \sum_k (y_k^{(t)} - \text{softmax}(\boldsymbol{\theta}_2^{(t)})_k)\delta_{ik} \\
&= (y_i^{(t)} - \text{softmax}(\boldsymbol{\theta}_2^{(t)})_i) \\
\frac{\partial J^{(t)}}{\partial \mathbf{b}_2} &= \mathbf{y}^{(t)} - \text{softmax}(\boldsymbol{\theta}_2^{(t)})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial J^{(t)}}{\partial U_{ij}} &= \sum_k \frac{\partial J^{(t)}}{\partial \theta_{2,k}^{(t)}} \frac{\partial \theta_{2,k}^{(t)}}{\partial U_{ij}} \\
&= \sum_k (y_k^{(t)} - \text{softmax}(\boldsymbol{\theta}_2^{(t)})_k) h_i^{(t)} \delta_{jk} \\
&= (y_j^{(t)} - \text{softmax}(\boldsymbol{\theta}_2^{(t)})_j) h_i^{(t)} \\
\frac{\partial J^{(t)}}{\partial \mathbf{U}} &= \mathbf{h}^{(t)T}(\mathbf{y}^{(t)} - \text{softmax}(\boldsymbol{\theta}_2^{(t)}))
\end{aligned}$$

$$\begin{aligned}
\frac{\partial J^{(t)}}{\partial h_i^{(t-1)}} &= \sum_k \frac{\partial J^{(t)}}{\partial \theta_{2,k}^{(t)}} \frac{\partial \theta_{2,k}^{(t)}}{\partial h_i^{(t-1)}} \\
&= \sum_k (y_k^{(t)} - \text{softmax}(\boldsymbol{\theta}_2^{(t)})_k) \sum_\ell \sigma'(\theta_{1,\ell}^{(t)}) H_{i\ell} U_{\ell k} \\
\frac{\partial J^{(t)}}{\partial \mathbf{h}^{(t-1)}} &= (\mathbf{y}^{(t)} - \text{softmax}(\boldsymbol{\theta}_2^{(t)}))\mathbf{U}^T \boldsymbol{\Sigma}(\boldsymbol{\theta}_1^{(t)})\mathbf{H}^T
\end{aligned}$$



(b)

First we calculate the following auxiliary quantities, using all the notation and results from part (a) above:

$$
\begin{aligned}
\frac{\partial J^{(t)}}{\partial \boldsymbol{\theta}_2^{(t)}} &= \mathbf{y}^{(t)} - \hat{\mathbf{y}}^{(t)} \\
\frac{\partial J^{(t)}}{\partial h_i^{(t)}} &= \sum_j \frac{\partial J^{(t)}}{\partial \theta_{2,j}^{(t)}} \frac{\partial \theta_{2,j}^{(t)}}{\partial h_i^{(t)}} \\
&= \sum_j (y_j^{(t)} - \hat{y}_j^{(t)}) \frac{\partial \theta_{2,j}^{(t)}}{\partial h_i^{(t)}} \\
&= \sum_j (y_j^{(t)} - \hat{y}_j^{(t)}) \frac{\partial}{\partial h_i^{(t)}} (\sum_k h_k^{(t)} U_{kj} + b_{2,j}) \\
&= \sum_j (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \\
\frac{\partial J^{(t)}}{\partial \mathbf{h}^{(t)}} &= (\mathbf{y}^{(t)} - \hat{\mathbf{y}}^{(t)}) \mathbf{U}^T \\
\frac{\partial J^{(t)}}{\partial \theta_{1,i}^{(t)}} &= \sum_j \frac{\partial J^{(t)}}{\partial h_j^{(t)}} \frac{\partial h_j^{(t)}}{\partial \theta_{1,i}^{(t)}} \\
&= \sum_{j,k} (y_k^{(t)} - \hat{y}_k^{(t)}) U_{jk} \frac{\partial h_j^{(t)}}{\partial \theta_{1,i}^{(t)}} \\
&= \sum_{j,k} (y_k^{(t)} - \hat{y}_k^{(t)}) U_{jk} \frac{\partial}{\partial \theta_{1,i}^{(t)}} \sigma(\theta_{1,j}^{(t)}) \\
&= \sum_{j,k} (y_k^{(t)} - \hat{y}_k^{(t)}) U_{jk} \delta_{ij} \sigma'(\theta_{1,i}^{(t)}) \\
&= \sum_k (y_k^{(t)} - \hat{y}_k^{(t)}) U_{ik} \sigma'(\theta_{1,i}^{(t)}) \\
\frac{\partial J^{(t)}}{\partial \boldsymbol{\theta}_1^{(t)}} &= (\mathbf{y}^{(t)} - \hat{\mathbf{y}}^{(t)}) \mathbf{U}^T \boldsymbol{\Sigma}(\boldsymbol{\theta}_1^{(t)})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial h_i^{(t)}}{\partial e_j^{(t)}} &= \frac{\partial}{\partial e_j^{(t)}} \sigma(\sum_k h_k^{(t-1)} H_{ki} + \sum_k e_k^{(t)} I_{ki} + b_{1,i}) \\
&= I_{ji} \sigma'(\sum_k h_k^{(t-1)} H_{ki} + \sum_k e_k^{(t)} I_{ki} + b_{1,i}) \\
&= I_{ji} \sigma'(\theta_{1,i}^{(t)}) \\
\frac{\partial h_i^{(t)}}{\partial H_{jk}} &= \frac{\partial}{\partial H_{jk}} \sigma(\sum_\ell h_\ell^{(t-1)} H_{\ell i} + \sum_\ell e_\ell^{(t)} I_{\ell i} + b_{1,i}) \\
&= h_j^{(t-1)} \delta_{ik} \sigma'(\theta_{1,i}^{(t)}) \\
\frac{\partial h_i^{(t)}}{\partial I_{jk}} &= \frac{\partial}{\partial I_{jk}} \sigma(\sum_\ell h_\ell^{(t-1)} H_{\ell i} + \sum_\ell e_\ell^{(t)} I_{\ell i} + b_{1,i}) \\
&= e_j^{(t)} \delta_{ik} \sigma'(\theta_{1,i}^{(t)}) \\
\frac{\partial h_i^{(t)}}{\partial b_{1,j}} &= \frac{\partial}{\partial b_{1,j}} \sigma(\sum_\ell h_\ell^{(t-1)} H_{\ell i} + \sum_\ell e_\ell^{(t)} I_{\ell i} + b_{1,i}) \\
&= \delta_{ij} \sigma'(\theta_{1,i}^{(t)})
\end{aligned}
$$

Now we can calculate the desired quantities:

$$
\begin{aligned}
\frac{\partial J^{(t)}}{\partial L_{\mathbf{x}^{(t-1)},k}} &= \frac{\partial J^{(t)}}{\partial e_k^{(t-1)}} = \sum_i \frac{\partial J^{(t)}}{\partial \theta_{1,i}^{(t)}} \frac{\partial \theta_{1,i}^{(t)}}{\partial e_k^{(t-1)}} \\
&= \sum_{i,j} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) \frac{\partial \theta_{1,i}^{(t)}}{\partial e_k^{(t-1)}} \\
&= \sum_{i,j} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) \frac{\partial}{\partial e_k^{(t-1)}} (\sum_\ell h_\ell^{(t-1)} H_{\ell i}) \\
&= \sum_{i,j,\ell} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{\ell i} \frac{\partial h_\ell^{(t-1)}}{\partial e_k^{(t-1)}} \\
&= \sum_{i,j,\ell} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{\ell i} I_{k\ell} \sigma'(\theta_{1,\ell}^{(t-1)}) \\
\frac{\partial J^{(t)}}{\partial \mathbf{L}_{\mathbf{x}^{(t-1)}}} &= (\mathbf{y}^{(t)} - \hat{\mathbf{y}}_k^{(t)}) \mathbf{U}^T \mathbf{\Sigma}(\boldsymbol{\theta}_1^{(t)}) \mathbf{H}^T \mathbf{\Sigma}(\boldsymbol{\theta}_1^{(t-1)}) \mathbf{I}^T \\
\left.\frac{\partial J^{(t)}}{\partial H_{mn}}\right|_{(t-1)} &= \sum_{i,j,\ell} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{\ell i} \left.\frac{\partial h_\ell^{(t-1)}}{\partial H_{mn}}\right|_{(t-1)} \qquad \text{(by the same logic as above)} \\
&= \sum_{i,j,\ell} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{\ell i} h_m^{(t-2)} \delta_{\ell n} \sigma'(\theta_{1,\ell}^{(t-1)}) \\
&= \sum_{i,j} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{ni} h_m^{(t-2)} \sigma'(\theta_{1,n}^{(t-1)}) \\
\left.\frac{\partial J^{(t)}}{\partial \mathbf{H}}\right|_{(t-1)} &= \mathbf{h}^{(t-2)T} (\mathbf{y}^{(t)} - \hat{\mathbf{y}}_k^{(t)}) \mathbf{U}^T \mathbf{\Sigma}(\boldsymbol{\theta}_1^{(t)}) \mathbf{H}^T \mathbf{\Sigma}(\boldsymbol{\theta}_1^{(t-1)}) \\
\left.\frac{\partial J^{(t)}}{\partial I_{mn}}\right|_{(t-1)} &= \sum_{i,j,\ell} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{\ell i} \left.\frac{\partial h_\ell^{(t-1)}}{\partial I_{mn}}\right|_{(t-1)} \qquad \text{(by the same logic as above)} \\
&= \sum_{i,j,\ell} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{\ell i} e_m^{(t-1)} \delta_{\ell n} \sigma'(\theta_{1,\ell}^{(t-1)}) \\
&= \sum_{i,j} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{ni} e_m^{(t-1)} \sigma'(\theta_{1,n}^{(t-1)}) \\
\left.\frac{\partial J^{(t)}}{\partial \mathbf{I}}\right|_{(t-1)} &= \mathbf{e}^{(t-1)T} (\mathbf{y}^{(t)} - \hat{\mathbf{y}}_k^{(t)}) \mathbf{U}^T \mathbf{\Sigma}(\boldsymbol{\theta}_1^{(t)}) \mathbf{H}^T \mathbf{\Sigma}(\boldsymbol{\theta}_1^{(t-1)})
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial J^{(t)}}{\partial b_{1,k}}\right|_{(t-1)} &= \sum_{i,j,\ell} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{\ell i} \left.\frac{\partial h_\ell^{(t-1)}}{\partial b_{1,k}}\right|_{(t-1)} \\
&= \sum_{i,j,\ell} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{\ell i} \delta_{\ell k} \sigma'(\theta_{1,\ell}^{(t-1)}) \\
&= \sum_{i,j} (y_j^{(t)} - \hat{y}_j^{(t)}) U_{ij} \sigma'(\theta_{1,i}^{(t)}) H_{ki} \sigma'(\theta_{1,k}^{(t-1)}) \\
\left.\frac{\partial J^{(t)}}{\partial \mathbf{b}_1}\right|_{(t-1)} &= (\mathbf{y}^{(t)} - \hat{\mathbf{y}}_k^{(t)}) \mathbf{U}^T \mathbf{\Sigma}(\boldsymbol{\theta}_1^{(t)}) \mathbf{H}^T \mathbf{\Sigma}(\boldsymbol{\theta}_1^{(t-1)})
\end{aligned}
$$

3. GRU question

Advantage Smaller discrete space - There are about 100 English-language characters in common usage if we include all punctuation marks. By contrast, a vocabulary is many thousands of words. For char-based model we need about 100 embeddings to represent all possible tokens.

Disadvantage Char-level models can generate unusual words. Word-level models can't generate mistyped words as these are not in their vocabulary.

4. Perplexity

Write $p_i = p(s_i|s_1, \ldots, s_{i-1})$. Then for any $b > 0$,

$$b^{-\frac{1}{M}\sum_{i=1}^{M}\log_b p_i} = (b^{\sum_{i=1}^{M}\log_b p_i})^{-1/M} = (\prod_{i=1}^{M} b^{\log_b p_i})^{-1/M} = (\prod_{i=1}^{M} p_i)^{-1/M}$$

Therefore this expression has the same value for any such b, and in particular for $b = 2$ and $b = e$ which are the two given expressions.