PAVIT SAXENA Assingment → 4

CSE-0S (B)

Q.1 $f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$

$U = -x^2 + xy + y^2$, $V = ax^2 + bxy + cy^2$

$\frac{\partial u}{\partial x} = -2x + y + 0$, $\frac{\partial v}{\partial x} = 2ax + by$

$\frac{\partial u}{\partial y} = x + 2y$, $\frac{\partial v}{\partial y} = bx + 2cy$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad , \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$x + 2y = -2ax - by$, $-2x + y = bx + 2$

on Comparing,

$a = -\frac{1}{2}$, $b = -2$, $c = \frac{1}{2}$

$f(x) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$

$-x^2 + xy + y^2 + i\left(-\frac{1}{2}x^2 - 2xy + \frac{1}{2}y^2\right)$

$f(z) = U + iV$

$f'(z) = \frac{\partial U}{\partial x} + \frac{i\partial v}{\partial x}$

$= -2x + y + i(-x - 2y)$

$= -2x + y - ix - 2iy$

$= 2i^2x + y - ix - 2iy$

$= 2i^2x + 2iy + y - ix$

$= -\frac{1}{2}(2 + i)z^2$

Q.2 $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$f(z) = u \pm iv$

$f(re^{i\theta}) = u + iv$ —— (i)

differentiate this eqn w.r. to 'r'

$e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$ —— (ii)

diff. eqn. (i) w.r. to 'θ'

$i r e^{i\theta}(f'(re^{i\theta})) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

Keeping value of $e^{i\theta}(f'(re^{i\theta}))$ from (i)

$ir\left(\frac{\partial v}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

$ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

on Comparing

$-r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta}$, $r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$

Hence proved

Q.3 show that the function - - - - - - - - origin.

$f(z) = \frac{x^3y^5(x+iy)}{x^6+y^{10}}$, $z \neq 0$

$f(z) = \frac{x^3y^5 \cdot x}{x^6+y^{10}} + \frac{x^3y^6 i}{x^6+y^{10}}$

$= \frac{x^4y^5}{x^6+y^{10}} + \frac{x^3y^6 i}{x^6+y^{10}}$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - v(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{0 - 0}{x} \Rightarrow 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$\lim_{y \to 0} \frac{0}{y} \Rightarrow 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} \Rightarrow 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} \Rightarrow 0$$

$$\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right) = 0$$

The function is analytic but we have to check its neighbourhood points too.

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\frac{x^3 y^5 (x+iy)}{x^6 + y^{10}} - 0}{(x+iy)}$$

$$= \frac{x^3 y^5}{x^6 + y^{10}}$$

let, $x^3 = my^5$

$$\lim_{y \to 0} \frac{my^5 y^5}{m^2 y^{10} + y^{10}} \Rightarrow \frac{m}{m^2+1}$$

$f'(0)$ exist and differentiable at origin, so it is analytic at the origin.

Q.4 $\quad U(x,y) = x^3 - 4xy - 3xy^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 4y - 3y^2 \quad , \quad \frac{\partial^2 u}{\partial x^2} = 6x - 0 - 0 \Rightarrow 6x$$

$$\phi_1(z,0) = 3z^2$$

$$\frac{\partial u}{\partial y} = -4x - 6xy \quad , \quad \frac{\partial^2 u}{\partial y^2} = -6$$

$$\phi_2(z,0) = -4z$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x \Rightarrow 0$$

Hence it is harmonic

$$f(z) = \int (3z^2 + i4z)\,dz = \frac{3z^3}{3} + i\,2z^2 + c$$

$$f(z) = z^3 + 2iz^2 + c$$

$$= (x+iy)^3 + 2i(x+iy)^2$$

$$= x^3 + (-1)iy^3 + 3xyi(x+iy) + 2i(x^2+y^2+2xyi)$$

$$= x^3 - iy^3 + 3x^2yi - 3xy^2 + 2ix^2 - 2iy^2 - 4xy.$$

$$= \underbrace{x^3 - 3xy^2 - 4xy}_{u} + \underbrace{(2x^2 - 2y^2 + 3x^2y - y^3)}_{v}\, i$$

$\cancel{v = 2x^3 - 2y^2}$ $\quad \therefore [u + iv]$

$$v = 2x^2 - 2y^2 + 3x^2y - y^3$$

Q.5 show that - - - - - - - - $f(z) = u + iv$.

$V = e^x(\sin y)$, $\frac{\partial v}{\partial x} = e^x \sin y$, $\frac{\partial^2 v}{\partial x^2} = e^x \sin y$

$\frac{\partial v}{\partial y} = e^x \cos y$, $\frac{\partial^2 v}{\partial y^2} = -e^x \sin y$

$\phi_1(z,0) = e^z$, $\phi_2(z,0) = e^z \cdot 0 \Rightarrow 0$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x \sin y - e^x \sin y = 0$

Hence the function is harmonic

$f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz$

$= \int e^z dz \Rightarrow e^z + c$

$f(z) = e^z + c$

$f(z) = e^{x + iy} + c$

$= e^x \cdot e^{iy}$

$= e^x [\cos y + i \sin y]$

$U = e^x \cos y$

Q. 6 $(x-y)(x^2+4xy+y^2)$

$$U = x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 + 8xy + y^2 - 2xy - 4y^2$$

$$\phi_1(z,0) = 3z^2$$

$$\frac{\partial u}{\partial y} = 4x^2 + 2xy - x^2 - 8xy - 3y^2$$

$$\phi_2(z,0) = 3z^2$$

$$f(z) = \int(\phi_1 - \phi_2 i)\,dz + c$$

$$= \int(3z^2 - 3z^2 i)\,dz + c$$

$$\Rightarrow z^3 - z^3 i$$

$$\Rightarrow z^3(1-i)$$

Q.7. $v = e^x(x\sin y + y\cos y)$

$$\frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x(\sin y)$$

$$= e^x[x\sin y + y\cos y + \sin y]$$

$$\frac{\partial v}{\partial y} = e^x[x\cos y + \cos y + (-y\sin y)]$$

$$\phi_1(z,0) \Rightarrow e^z[z\cos 0 + \cos 0 - 0\sin 0]$$

$$= e^z[z + 1 - 0]$$

$$\Rightarrow e^z[z+1]$$

$$\phi_2(z,0) = e^z[z\sin 0 + 0\cos 0 + \sin 0]$$

$$\Rightarrow 0$$

By Milnes theorem method.

$$f(z) = \int[\phi_1(z,0) + i\phi_2(z,0)]\,dz + c$$

$$= \int e^z(z+1)\,dz + c$$

$$= e^z \cdot z + e^z \cdot 0 + c$$

$$\Rightarrow e^z \cdot z + c$$

Q. 6 find the image of - - - - $w = \frac{1}{z}$

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$x + iy = \frac{1}{u + iv}$$

$$x + iy = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{v^2 + u^2}$$

$$|z - ia| = r$$

$$|z - 2i| = 2$$

$$|x + iy - 2i| = 2$$

$$|x + i(y-2)| = 2$$

$$x^2 + (y-2)^2 = 4$$

$$\frac{u^2}{(u^2+v^2)^2} + \left(\frac{-v}{u^2+v^2} - 2\right)^2 = 4$$

$$\frac{u^2}{(u^2+v^2)^2} + \left(\frac{-v - 2(u^2+v^2)}{(u^2+v^2)}\right)^2 = 4$$

$$u^2 + v^2 + 4(u^2+v^2)^2 + 4v(u^2+v^2) = 4(u^2+v^2)^2$$

$$u^2 + v^2 (1 + 4v) = 0$$

$$u^2 + v^2 \neq 0 \;, \quad \& \quad 1 + 4v = 0$$

$$4v + 1 = 0$$

Q. 9 find the points - - - - - - $w = \frac{2z+3}{z+2}$

$$w = \frac{2z+3}{z+2}$$

$$z = \frac{2z+3}{z+2}$$

$$z^2 + 2z = 2z + 3$$

$$\boxed{z = \pm\sqrt{3}}$$

Q. 10 $z = 1, -i, -1$, $w = i, 0, -i$

The bilinear transformation mapping $z = 1, -i, -1$ into $w = i, 0, -1$ respectively

$$\frac{(w-i)(0+i)}{(w+i)(-i)} = \frac{(z-1)(1-i)}{(z+1)(-i-1)}$$

$$\frac{i-w}{i+w} = \frac{(z-1)(i-1)}{(z+1)(i+1)}$$

$$\frac{1-w}{1+w} = \frac{(i-1)(z+1-i)}{(i+i)(z+1+i)}$$

$$\frac{2i}{-2w} = \frac{2iz+z}{-2z-2i}$$ [applying componedo and devidededo].

$$\frac{i}{-w} = \frac{iz+1}{-(z+1)}$$

$$w = \frac{i(z+i)}{iz+1} \Rightarrow \frac{iz-1}{iz+1}$$