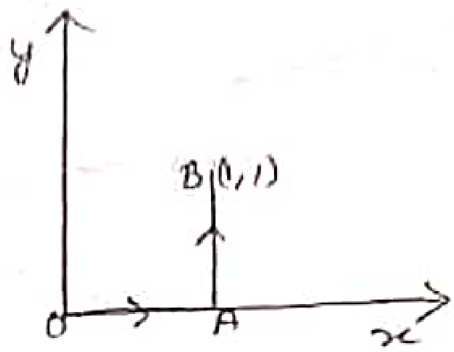
ASSIGNMENT-5

Sol^n 1) $I = \int_0^{1+i} (x-y-ix^2)\,dz$

$I = I_1 + I_2$

$(OA) + (OB)$



For OA :-

$y = 0$

$dy = 0$

$I_1 = \int_0^1 (x - 0 + ix^2)(dx)$

$= \int_0^1 (x + ix^2)\,dx$

$= \left[\frac{x^2}{2}\right]_0^1 + i\left[\frac{x^3}{3}\right]_0^1$

$= \frac{1}{2} + i\frac{1}{3}$

For AB

$x = 1$

$dx = 0$

$I_2 = \int_0^1 (1 - y + i)(i\,dy)$

$= i\int_0^1 dy - i\int_0^1 y\,dy - \int_0^1 dy$

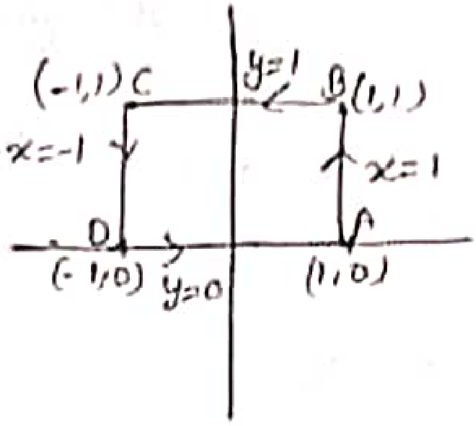
$= i - \frac{i}{2} - 1$

$I_2 = \frac{1}{2}i - 1$

$I = I_1 + I_2$

$= -\frac{1}{2} + \frac{5}{6}i$

(2) $f(z)=z^3$



for AB

$x=1$

$dx=0$

$f(z) = (x+iy)^3(dx+idy)$

$I_1 = \int_0^1 f(z) = \int_0^1 (1+iy)^3 (0+idy)$

$= (1+iy^3+3iy-3y^2)\, idy$

$= i\int_0^1 dy + \int_0^1 y^3 dy - 3\int_0^1 y\, dy - 3i\int_0^1 y^2 dy$

$= \cancel{i} + \frac{1}{4} - \frac{3}{2} - \cancel{i}$

$= \frac{-5}{4}$

for BC $y=1$

$dy=0$

$I_2 = \int_1^1 (x+i)^3 (dx)$

$= \int_1^{-1} (x^3-i+3x^2i-3x)\,dx$

$= \int_1^{-1} x^3 dx - i\int_1^{-1} dx + 3i\int_1^{-1} 3x^2 dx - 3\int_1^{-1} x\, dx$

$= \frac{1}{4} - \cancel{\frac{1}{4}} - i[-1-1] + i[-1-1] - \frac{3}{2}\cancel{[1-1]}$

$= i + i - i - i$

$= 0$

for CD $x=-1$

$dx=0$

$I_3 = \frac{5}{4}$

for DA $y=0$

$dy=0$

$I_4 = \int_{-1}^1 x^3 dx$

$I_4 = 1-1$

$I_4 = 0$

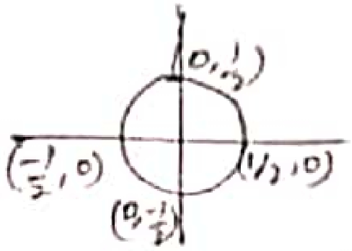
$\therefore I = I_1+I_2+I_3+I_4$

$I = 0$

Hence cauchy's theorem is verified.

③ $\oint_c \frac{z^2-z+1}{z-1}\,dz \qquad |z|=\frac{1}{2}$

$|z|$ represent a circle with centre $(0,0)$ & radius $\frac{1}{2}$



$z=1$ is a singular point which lies outside the circle therefore

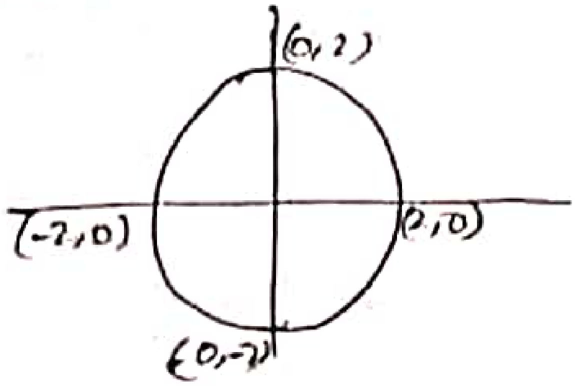
By cauchy's integral formula

$$\int f(z)\,dz = 0$$

④ $\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-3)} \qquad |z|=2$

Sol^n

$|z|$ represent the circle with centre $(0,0)$ & radius 2.



$z=1,3$ are the singular points but $z=1$ is lie inside the circle.

So, $I = \oint_c \frac{\frac{\sin\pi z^2+\cos\pi z^2}{z-3}}{z-1}\,dz$

$f(z) = \frac{\sin\pi z^2 + \cos\pi z^2}{z-3}$

$\therefore I = 2\pi i f(1)$

$= 2\pi i\left(\frac{\sin\pi + \cos\pi}{-2}\right)$

$\boxed{I = \pi i}$

⑤ $\oint_c \frac{e^{-2z}}{(z+1)^3}\,dz$

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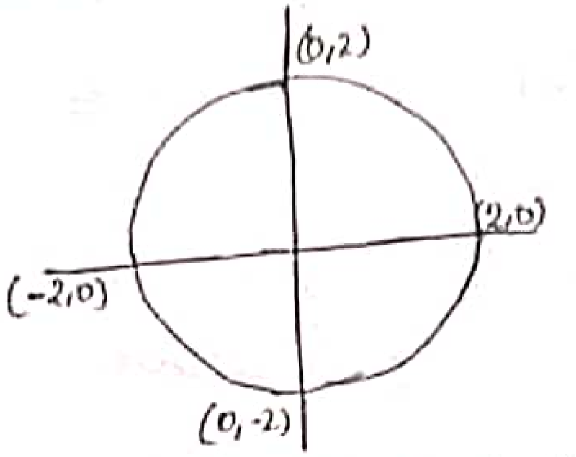
$|z|=2$ is a circle with centre $(0,0)$ & radius 2.



$z=-1$ is a singular point of order 3 so it comes under the circle.

$$\therefore \oint \frac{e^{-2z}}{(z+1)^3}\,dz = \frac{2\pi i}{2!} f^{2}(-1)$$

$f(z) = e^{-2z}$ $\qquad = \frac{2\pi i}{2!} 4e^{-2}$

$f'(z) = -2e^{-2z}$

$f''(z) = 4e^{-2z}$ $\qquad = 2\pi i e^{-2}$

⑥ $f(z) = \sin z$ $\qquad z = \frac{\pi}{4}$ $\qquad f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z$ $\qquad f'(\pi/4) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z$ $\qquad f''(\pi/4) = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z$ $\qquad f''(\pi/4) = -\frac{1}{\sqrt{2}}$

$$\sin z = \frac{1}{\sqrt{2}} + \left(z-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - \frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\frac{1}{\sqrt{2}} - \frac{1}{3!}\left(z-\frac{\pi}{4}\right)^3\frac{1}{\sqrt{2}}$$

$$f(z) = \frac{7z-2}{z(z+1)(z+2)}$$

$$= \frac{9}{z+2} - \frac{8}{z+1} - \frac{1}{z}$$

$1 < |z+1| < 3$ $\Rightarrow$

let $z+1 = t$ $\Rightarrow$ $1 < t < 3$

$z = t-1$

$$= \frac{9}{t+1} - \frac{8}{t} - \frac{1}{t-1}$$

$$= \frac{9}{t[1+\frac{1}{t}]} - \frac{8}{t} - \frac{1}{t[1-\frac{1}{t}]}$$

$$= \frac{9}{t}[1+\frac{1}{t}]^{-1} - \frac{8}{t} - \frac{1}{t}[1-\frac{1}{t}]$$

$$= \frac{9}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{(z+1)}\right)(-1)^n - \frac{8}{z+1} - \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n$$

(8) $f(z) = \sin\frac{1}{1-z}$ at $z=1$

$\lim_{z\to 1} \sin\frac{1}{1-z} = \sin\infty$ does not defined

so ~~this~~ essential singularity.

(9) $$f(z) = \frac{2z+1}{z^2-z+2}$$

$$= \frac{2z+1}{(z+1)(z-2)}$$

$z=-1$, and $z=2$ are the poles

Residue at $z=-1$

$$\lim_{z\to -1} (z+1)\frac{2z+1}{(z+1)(z-2)} = \frac{-1}{-3} = \frac{1}{3}$$

Residue at $z=2$

$$\lim_{z\to 2} (z-2)\frac{2z+1}{(z+1)(z-2)}$$

$$= \frac{5}{3}$$

(9b) $\int_0^{2\pi} \frac{d\theta}{5-3\sin\theta}$

$$I = \oint_c \frac{dz}{iz\left[5-\frac{3}{2i}\left(z-\frac{1}{z}\right)\right]}$$

$$= \oint_c \frac{dz}{iz\left[5-\frac{3}{2i}\left(\frac{z^2-1}{z}\right)\right]}$$

$$= \oint \frac{dz}{iz\left[\frac{10iz-3(z^2-1)}{2iz}\right]}$$

$$= \frac{2}{i}\oint_c \frac{dz}{10iz-3z^2+3}$$

$$= \frac{2}{i}\oint \frac{dz}{z^2-\frac{10i}{3}z-1}$$

$$= \quad z = 3i, \frac{i}{3}$$

$z = \frac{i}{3}$ is a pole inside 'c'.

∴ Residue at $\frac{i}{3}$

$$R_1 = \lim_{z\to\frac{i}{3}} \left(z-\frac{i}{3}\right)\frac{1}{2i(z-3i)\left(z-\frac{i}{3}\right)}$$

$$R_1 = \frac{3}{2i}$$

$$I = 2\pi i \times \frac{3}{2i}$$

$$\boxed{I = 6\pi}$$