Analysis of Algorithms

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Discussion 14 — University of Southern California

Approximation and Linear Programming

Problem 1 (Bin Packing)

You have an infinite supply of bins, each of which can hold M maximum weight. You also have n objects, each of which has a weight w_i (w_i is at most M). Our goal is to partition the objects into bins, such that we use as few bins as possible. This problem in general is **NP**-hard.

Devise an approximation algorithm and compute its approximation ratio.

Solution (first-fit approach)

1. process items in the original order
2. if an item doesn't fit the first bin, put it into the next bin
3. if an item does not fit into any bins, open a new bin

Runtime-? $O(n^2)$

Example,

items = {40, 20, 35, 15, 25, 5, 30, 10}

M = 45

Bin 1	Bin 2	Bin 3	Bin 4	Bin 5
40, 5	20,15,10	35	25	30

Solution (proof)

Suppose we use m bins.

Let m* denote the optimal number of bins

Clearly, $m^* \geq (\sum w_i)/M$,

On the other hand, $(\sum w_i)/M > (m-1)/2$

This follows form the fact that at least (m-1) bins are more than half full. Why?

Because, if there are two bins less than half full, items in the second bin will be placed into the first by our algorithm.

We have showed, $m^* \geq (\sum w_i)/ M > (m-1)/2$

Comparing the left and right hand sides, we derive

$$2m^* > m-1 \qquad \text{or,} \qquad \boxed{m \leq 2m^*}$$

Theorem (1976)

This is a 17/10-approximation algorithm.

Solution (sorted first-fit)

1. sort the input in descending order
2. apply the first-fit algorithm

Theorem.
This is a 11/9 - approximation algorithm.

Example,

items = {40, 20, 35, 15, 25, 5, 30, 10}

M = 45

Bin 1	Bin 2	Bin 3	Bin 4
40, 5	35,10	30,15	25, 20

Problem 2

A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported, and the cargo company may choose to carry any amount of each, up to the maximum available limits given below.

	Density	Volume	Price
Material 1	2 tons/m^3	40 m^3	\$1,000 per m^3
Material 2	1 tons/m^3	30 m^3	\$2,000 per m^3
Material 3	3 tons/m^3	20 m^3	\$12,000 per m^3

Write a linear program that optimizes revenue within the constraints. You do not need to solve the linear program.

Solution

	Density	Volume	Price
Material 1	2 tons/m^3	40 m^3	\$1,000 per m^3
Material 2	1 tons/m^3	30 m^3	\$2,000 per m^3
Material 3	3 tons/m^3	20 m^3	\$12,000 per m^3

Let x_1, x_2, x_3 denote the volume of the materials we're going to transport.

We want to maximize the profit:

$$\max 1000\, x_1 + 2000\, x_2 + 12000\, x_3$$

Subject to these constraints:

$x_1 + x_2 + x_3 \le 60$ // 60m^3 space available

$2\, x_1 + x_2 + 3\, x_3 \le 100$ // 100 tons weight capacity

$0 \le x_1 \le 40$ // maximum amount of material 1

$0 \le x_2 \le 30$ // maximum amount of material 2

$0 \le x_3 \le 20$ // maximum amount of material 3

Solution

LP in standard form.

$$\max 1000\, x_1 + 2000\, x_2 + 12000\, x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\le 60 \\ 2\, x_1 + x_2 + 3\, x_3 &\le 100 \\ x_1 &\le 40 \\ x_2 &\le 30 \\ x_3 &\le 20 \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 &\ge 0 \end{aligned}$$

$$\boxed{\begin{aligned} &\max\, (c^T x) \\ &A\, x \le b \\ &x \ge 0 \end{aligned}}$$

MATLAB

https://www.mathworks.com/help/optim/ug/linprog.html

linprog

Linear programming solver

Finds the minimum of a problem specified by

$$\min_x f^T x \text{ such that } \begin{cases} A \cdot x \le b, \\ Aeq \cdot x = beq, \\ lb \le x \le ub. \end{cases}$$

f, *x*, *b*, *beq*, *lb*, and *ub* are vectors, and *A* and *Aeq* are matrices.

Description

`x = linprog(f,A,b)` solves min `f'*x` such that `A*x ≤ b`.

`x = linprog(f,A,b,Aeq,beq)` includes equality constraints `Aeq*x = beq`. Set `A = []` and `b = []` if no inequalities exist.

`x = linprog(f,A,b,Aeq,beq,lb,ub)` defines a set of lower and upper bounds on the design variables, `x`, so that the solution is always in the range `lb ≤ x ≤ ub`. Set `Aeq = []` and `beq = []` if no equalities exist.

Problem 3

Convert the LP to standard form.

max $5x_1 - 2x_2 + 9x_3$

subject to

$3x_1 + x_2 + 4x_3 = 8$

$2x_1 + 7x_2 - 6x_3 \le 4$

$x_1 \le 0,\ x_3 \ge 1$

Solution

$3x_1 + x_2 + 4x_3 = 8$

$2x_1 + 7x_2 - 6x_3 \le 4$

$x_1 \le 0$

$x_3 \ge 1$

x_2 - free

$3x_1 + x_2 + 4x_3 - z_5 \le 8$

$z_5 \ge 0$

$x_1 = -z_1$

$z_1 \ge 0$

$x_3 = z_3 + 1$

$z_3 \ge 0$

$x_2 = z_2 - z_4$

$z_2 \ge 0$

$z_4 \ge 0$

Solution

max $5x_1 - 2x_2 + 9x_3$

$3x_1 + x_2 + 4x_3 - z_5 \le 8$

$2x_1 + 7x_2 - 6x_3 \le 4$

$x_1 \le 0,\ x_3 \ge 1,\ z_5 \ge 0$

$x_1 = -z_1$

$x_3 = z_3 + 1$

$x_2 = z_2 - z_4$

Here is the original LP in standard form:

max $-5z_1 - 2(z_2 - z_4) + 9z_3$

$-3z_1 + (z_2 - z_4) + 4z_3 - z_5 \le 4$

$-2z_1 + 7(z_2 - z_4) - 6z_3 \le 10$

$z_1 \ge 0,\ z_2 \ge 0,\ z_3 \ge 0,\ z_4 \ge 0,\ z_5 \ge 0$