On the computational content of the Bolzano-Weierstrass principle

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The paper under review presents a direct computational interpretation of the Bolzano-Weierstrass principle (BW), via a combination of the negative translation and Gödel's dialectica interpretation – in the form of Shoenfield's interpretation. It is know in reverse mathematics that BW is equivalent to arithmetical comprehension. The goal of the paper is to find the precise complexity of the functional needed for the interpretation, in particular, when limited induction is available and a single instance of the BW principle is used in a proof of a Π^0_2-theorem (Theorem 3.12).

By naively looking at the proof of BW, one might think that Π^0_2 comprehension is needed even when a single instance of BW is used in a proof. The clever analysis in the paper shows that in fact all one needs is Σ^0_1 comprehension plus weak König's lemma (and WKL can be easily dealt with via the monotone interpretation). In fact, the authors work essentially directly with Σ^0_1-WKL (i.e. weak König's lemma for Σ^0_1-definable trees).

The authors show that a single instance of BW to prove a Σ^0_1 formula over Σ^0_n-IA raises the complexity of the realiser by one, namely, from $\mathcal{T}_{n-1}$ to $\mathcal{T}_n$ (even if quantifier-free choice is allowed). The fact that this jump is optimal is argued by showing that a single instance of BW over Σ^0_{n-1}-IA is all one needs to get Σ^0_n-IA (Proposition 3.14). This exploits the connection between BW and arithmetical comprehension, and standard results of Parson's on the complexity of induction for Σ^0_n formulas. The monotone functional interpretation of (a single instance of) BW is also considered, resulting in much simpler realisers.

The paper is thecnically sound, and very well written. It also gives a nice overview of all the related issues around not only BW, but also WKL, majorisability, dialectica interpretation, comprehension and bar recursion. Therefore, I would recommend that the paper be accepted for publication.

Here is a list of typos and small corrections/clarifications suggested for the final version:

- the authors use $[s]$ for the infinite extension of a finite sequence s with 0's. Isn't $\hat{s}$ a more standard notation for this? I can see you are using instead $\hat{f}$ for the mapping turning any type one function into a Cauchy sequence with fixed rate of convergence.
- Page 5, paragraph before Def 1.3. Why is "Numbers" in capital letter in the expression "rational Numbers" and "real Numbers"?
- Definition 1.5. There is a typo in $\psi^{Sh}(\underline{b}) :\equiv \forall \underline{v} \exists \underline{y} \varphi_{Sh}(\underline{v}, \underline{y}, \underline{b})$. The "$\varphi_{Sh}$ should be "ψ_{Sh}".
- Wouldn't the soundness theorem go through by simply taking (S7*) as the definition? If not, explain what is going on, why do you need both (S7)

and (S7*)?

- Top of page 8, there is no need to give the definition of Φ again, it is enough to say that it is the case when $\rho = 0$.
- $\mathsf{B'}_2$ is defined but does not seem to be used, at least not explicitly. I think it's used implicitly in Theorem 2.4. Is that correct?
- Please include proof of Lemma 1.12. This is a crucial lemma. This is a particular case of the more general bar recursive interpretation of comprehension given by Spector. Alternatively, you could also point precisely to where this more general proof is given (maybe Kohlenbach's recent book?).
- Page 12. Please give a bit of intuition as to what the three constructions on the top of page 12 do.
- Page 13, definition of BW, I think $d^\omega(xk, a)$ should have been $d^\omega(xl, a)$.
- Page 16, section 3.2. Beginning of second paragraph should be "We introduce the following notations..."
- Page 18, Theorem 3.6. When you write "(we use the notation from 3.2 and 3.3)", do you mean "Sections 3.2..." or "Definition 3.2..." or "Theorem 3.2..."?
- Page 18, Theorem 3.6. The functional B^{WKL} is actually only defined on page 12 (not 11) and in Theorem 2.4 (not Def 2.3).
- Page 20. "Suppose namely (7)..." should simply be "Supose (7)..."