A quantitative nonlinear strong ergodic theorem for Hilbert spaces

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Abstract

We give a quantitative version of a strong nonlinear ergodic theorem for (a class of possibly even discontinuous) selfmappings of an arbitrary subset of a Hilbert space due to R. Wittmann and outline how the existence of uniform bounds in such quantitative formulations of ergodic theorems can be proved by means of a general logical metatheorem. In particular these bounds depend neither on the operator nor on the starting point of the averaging process. Furthermore, we extract such uniform bounds in our quantitative formulation of Wittmann's theorem, implicitly using the proof-theoretic techniques on which the metatheorem is based. However, we present our result and its proof in analytic terms without any reference to logic as such. Our bounds turn out to involve nested iterations of relatively low computational complexity. While in theory these kind of iterations ought to be expected, so far this seems to be the first occurrence of such a nested use observed in practice.

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1. Introduction

The Riesz version of the von Neumann mean ergodic theorem [33] asserts that for any linear operator T on a Hilbert space X, which is nonexpansive, i.e.

$$\forall u, v \in X \ \big(\|Tu - Tv\| \leq \|u - v\|\big),$$

the sequence of the Cesàro means

$$A_n x := \frac{1}{n+1} \sum_{i=0}^{n} T^i x,$$

converges in norm for any starting point x. It follows from an example by Genel and Lindenstrauss [6] that there is a nonexpansive operator on the unit ball of ℓ_2, for which

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the sequence of the Cesàro means does not converge strongly (see also [25]). So in comparison with von Neumann's linear mean ergodic theorem, in nonlinear ergodic theory one obtains either a weaker conclusion (such as weak convergence or convergence of a different iteration scheme instead of the Cesàro means) or one has to add additional requirements (to preserve at least some linearity).

Let H be a Hilbert space, C a subset of H and $T : C \to C$ a (possibly nonlinear) mapping. In 1975, Baillon [2] showed that if C is convex and closed, and T is nonexpansive and has a fixed point, then the sequence of the Cesàro means is weakly convergent to a fixed point of T. A year later, Baillon [3] also proved that if in addition T is *odd*, i.e.

$$-C = C \text{ and } \forall u \in C \ \big(T(-u) = -Tu\big),$$

then the sequence of the Cesàro means converges to a fixed point in norm. Shortly after this, Brézis and Browder [4] showed that Baillon's first result is also true for a more general averaging process than the usual Cesàro means and that Baillon's second result remains valid if $0 \in C$ and T is not necessarily odd but satisfies the following, weaker condition:

$$\exists c \in \mathbb{R}\ \forall u, v \in C\big(\|Tu + Tv\|^2 \le \|u + v\|^2 + c(\|u\|^2 - \|Tu\|^2 + \|v\|^2 - \|Tv\|^2)\big). \quad \text{(BB)}$$

On the other hand, in 1979, Hirano and Takahashi [11] showed that Baillon's weak convergence result remains true if the mapping is just *asymptotically* nonexpansive, i.e.

$$\forall u, v \in C\ \forall n \in \mathbb{N}\ \big(\|T^n u - T^n v\| \le \alpha_n \|u - v\|\big),$$

for some sequence $(\alpha_n)_n$ of nonnegative real numbers which converges to 1. Moreover, an odd and nonexpansive mapping satisfies the following condition

$$\forall u, v \in C\ (\|T^n u + T^n v\| \le \|u + v\|) \qquad (W)$$

and analogously an odd and asymptotically nonexpansive mapping satifies the assymptotic version

$$\forall n \in \mathbb{N}\ \forall u, v \in C\ (\|T^n u + T^n v\| \le \alpha_n \|u + v\|), \qquad (W^-)$$

for some sequence $(\alpha_n)_n$ of nonnegative real numbers which converges to 1. In 1990, Wittmann [34] proved a generalization of Baillon's strong convergence theorem to an arbitrary C and a mapping satisfying the condition (W^-) (see also Theorem 2.2 in [34] and Theorem 3.1 below). Two years later, Wittmann [35] also showed that for a nonexpansive T which has a fixed point, and a convex and closed C, the averaging series $A_n^\alpha x$, first defined by Halpern [9] as

$$A_0^\alpha x := 0, \quad A_{n+1}^\alpha x := \alpha_{n+1} x + (1 - \alpha_{n+1}) T(A_n^\alpha x),$$

converges to the closest fixed point of T in norm. The Halpern iteration coincides with the Cesàro means for linear maps. We depict this development in Figure 1 (the references in parentheses refer to quantitative versions of the respective theorems, which we discuss below).

[3]While the results were essentially available on arxive since 2007, the paper as such was submitted in 2008. Thereafter Kohlenbach and Leuştean extended the result to uniformly convex spaces and gave a better bound.

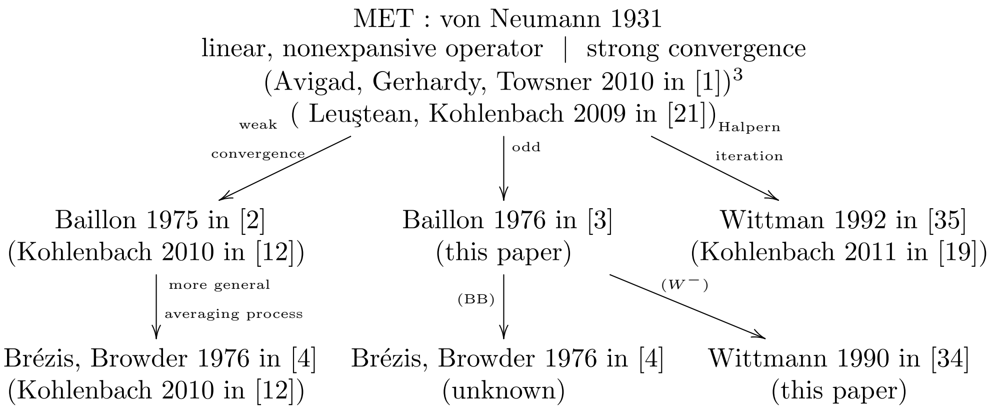

Figure 1: Some nonlinear ergodic theorems (for Hilbert spaces) and their finitisations.

There are many further results and generalizations in the field of nonlinear ergodic theory (regarding different spaces see e.g. [5, 10], even weaker "linearity" conditions see e.g. [26, 27], and other improvements) and it is subject to ongoing research.
In this paper we investigate the computational content of Wittmann's nonlinear strong ergodic theorems. Although in general the sequence of the ergodic averages does not have a computable rate of convergence (even for the von Neumann's mean ergodic theorem for a separable space and computable x and T), as was shown by Avigad, Gerhardy and Towsner in [1], the so called metastable version nevertheless has a primitive recursive bound. In our case this means that given the assumptions from Wittmann's strong ergodic theorem, the following holds

$$\forall b, l \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}, x \in S\ \exists m \leq M(l, g, b, K)\big(\|x\| \leq b \to \|A_m x - A_{m+g(m)} x\| \leq 2^{-l}\big),$$

for a primitive recursive M, where K is a rate of convergence for the sequence α in the assumption (W^-). We will not only prove the existence of such an M but also give such a bound explicitly in Corollary 5.12. For the specific case where $(\alpha_n)_n$ is a constant 1 sequence (i.e. T satisfies (W) rather than (W^-), see also Theorem 2.2 in [34] and Corollary 5.13 below), M can be defined as follows:

$$\begin{aligned}
M(l, g, b) &:= (N(2l+7, g^M) + P(2l+7, g^M, b))b2^{2l+8} + 1,\\
P(l, g, b) &:= P_0(l, F(l, g, N(l, g), b), b),\\
F(l, g, n, b)(p) &:= p + n + \tilde{g}((n+p)b2^{l+1}),\\
N(l, g, b) &:= \big(H(l, g, b)\big)^{b^2 2^{l+2}}(0),\\
H(l, g, b)(n) &:= n + P_0(l, F(l, g, n, b)) + \tilde{g}((n + P_0(l, F(l, g, n, b)))b2^{l+1}),
\end{aligned}$$

where

$$P_0(l, f, b) := \tilde{f}^{b^2 2^l}(0), \quad \tilde{g}(n) := n + g(n),\ g^M(n) := \max_{i \leq n+1} g(i).$$

Note that apart from the counterfunction g and the precision l, this bound depends only on b and not on S, T or x. For another quantitative result on operators satisfying the condition (W) see [20].
These results, along with those by Avigad, Gerhardy, Towsner [1] and Kohlenbach, Leuştean [21] for the finitary version of the von Neumann ergodic theorem as well as Kohlenbach's bounds for the finitary versions of Baillon's weak ergodic theorem [12] and Wittmann's convergence result for Halpern means [19] , can be seen as instances of 'hard analysis' in the sense of T. Tao; see [32, 31], where he discusses the uses and benefits of (the existence of) uniform bounds for such finitary formulations of well-known theorems. It is one of the goals of this paper to demonstrate that there are proof-theoretic means to systematically obtain such uniform bounds. In fact, for many theorems the existence of a uniform bound is guaranteed by Kohlenbach's metatheorems introduced in [17] and refined in [8]. Additionally, proof theoretic methods such as Kohlenbach's monotone functional interpretation (see [13]) can be used to systematically obtain these effective bounds. The paper at hand is a case study in applying such proof mining techniques.
We improve results in the area of nonlinear generalizations of the mean ergodic theorem and their corresponding finitisations (see Figure 1). Moreover, we have here a rare example of an application of these techniques to not necessarily continuous operators. In logical terms this amounts to the subtlety that only a weak version of extensionality is available. Also, for the first time, we obtain a bound which in fact makes use of nested iteration. One can see this quickly on the term M above. While F as a function is defined via iteration of the counterfunction g, it itself is being iterated by P. This is a direct consequence of the logical form of Wittmann's original proof.
It is a surprising observation that so far for all metastable versions of strong ergodic theorems primitive recursive bounds could be obtained.
We discuss the application of general logical metatheorems in more detail in Section 2 which is not necessary to understand and verify our main results. We present these, namely the explicit bounds for all three theorems in Wittmann's paper [34], in Section 5. We explain how to obtain the explicit bounds in Section 4, after formalizing Wittmann's proof in Section 3. Both of these sections, though inspired by proof theoretic methods, require no facts from logic.

2. A general bound existence theorem

The main result of this paper, Corollary 5.12, is a quantitative version of a nonlinear strong ergodic theorem for operators satisfying Wittmann's condition (W^-) on an arbitrary subset of a Hilbert space. In this section we outline how for this type of theorems the existence of such uniform bounds can be obtained by means of a general logical metatheorem. This sort of metatheorems was developed in [17] and [8] (see also [18]) and are applicable to many theorems concerning a wide range of classes of maps and abstract spaces. For example, they were successfully applied to the ergodic theorems mentioned in Figure 1 or to asymptotic regularity theorems in metric fixed point theory [22]. In the last mentioned example, as in this paper, the authors infer from the metatheorems that uniform bounds exist and derive them explicitly.
To apply the metatheorems, we need the analyzed theorem to meet only two conditions:

1. The proof does not use axioms or rules which are too strong.

2. The analyzed theorem in its logical form is not too complex in terms of quantification.

To formalize the first condition we start with a logical system for so called full classical analysis introduced by Spector in [30].[4] Kohlenbach extended this system by an additional basic type and its defining axioms representing a given abstract space and its properties. Kohlenbach also considers cases, where a specific subset of such a space (or rather its characteristic function) has to exist as a constant. For instance in [18] Kohlenbach defines such systems for the theory of metric, hyperbolic, normed, uniformly convex or Hilbert spaces – if required – together with a (bounded) convex subset.[5] In our case (i.e. simply for a pre-Hilbert space) this extended system is denoted by $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$. In general, the system can be extended to arbitrary Hilbert spaces, however it turns out that the completeness is not necessary for the proof that we analyze.
The second condition has to be investigated for each theorem specifically, depending on the given theorem and the metatheorem we wish to use. Examples are metastable versions of formulas expressing the convergence or fixed point properties of nonexpansive, Lipschitz, weakly quasi-nonexpansive or uniformly continuous functions even simply functions which are majorizable (see Corollary 6.6 in [8] and Theorem 2.1 below). The metatheorem applicable in our scenario follows from Corollary 6.6.7) in [8]. In particular with the theory $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$ with an additional parameter for an arbitrary subset S of the abstract Hilbert space X.[6]

Theorem 2.1 (Gerhardy-Kohlenbach [8] - specific case 1)**.** *Let $\varphi_\forall$, resp. $\psi_\exists$, be $\forall$- resp. $\exists$-formulas that contain only x, z, f free, resp. x, z, f, v free. Assume that $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$ proves the following sentence:*

$$\forall x \in \mathbb{N}^\mathbb{N}, z \in S, f \in S^S \big(\varphi_\forall(x, z, f) \to \exists v \in \mathbb{N}\ \psi_\exists(x, z, f, v)\big).$$

Then there is a computable functional $F : \mathbb{N}^\mathbb{N} \times \mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}$ s. t. the following holds in all non-trivial (real) inner product spaces $(X, \langle\cdot,\cdot\rangle)$ and for any subset $S \subseteq X$

$$\begin{aligned}&\forall x \in \mathbb{N}^\mathbb{N}, z \in S, b \in \mathbb{N}, f \in S^S, f^* \in \mathbb{N}^\mathbb{N}\\ &\big(\mathrm{Maj}(f^*, f)\ \wedge\ \|z\| \le b\ \wedge\ \varphi_\forall(x, z, f) \to \exists v \le F(x, b, f^*)\ \psi_\exists(x, z, f, v)\big),\end{aligned}$$

where

$$\mathrm{Maj}(f^*, f) :\equiv \forall n \in \mathbb{N} \forall z \in S\big(\|z\| \le_\mathbb{R} n \to \|f(z)\| \le_\mathbb{R} f^*(n)\big).$$

The theorem holds analogously for finite tuples.

Consider the following metastable version of Wittmann's Theorem 2.1 [34]:

[4] In particular this system covers full comprehension over numbers, including also full second order arithmetic.

[5] In any such abstract space, its metric plays a major role as two objects are defined to be equal, if and only if their distance is zero.

[6] This is analogous to the case where we add C to the theory for normed space, but this time without any additional axioms.

Theorem 2.2 (Theorem 2.1 in [34]). *Let S be a subset of a Hilbert space and $T : S \to S$ be a mapping satisfying*

$$\forall x, y \in S \ (\|Tx + Ty\| \leq \|x + y\|). \tag{W}$$

Then for any $x \in S$ the sequence of the Cesàro means,

$$A_n x := \frac{1}{n+1} \sum_{i=0}^{n} T^i x,$$

is norm convergent.

This theorem has the following form:

$$\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}},\ x \in S, T \in S^S \ \big(\mathrm{W}(T) \to \exists m \in \mathbb{N} \ (\|A_m x - A_{m+g(m)} x\| < 2^{-l}) \big). \quad (+)$$

Obviously the conclusion, i.e. $\exists m \left(\|A_m x - A_{m+g(m)} x\| < 2^{-l} \right)$, has the form $\exists m \ \psi_\exists(m, l, g)$ and the assumption $\mathrm{W}(T)$, i.e. $\forall x, y \in S \big(\|Tx + Ty\| \leq \|x + y\| \big)$, has the form $\varphi_\forall(T)$. Moreover, $\mathrm{W}(T)$ already implies $\mathrm{Maj}(\mathrm{id}, T)$ (here id stands simply for the identity function on $\mathbb{N}$), since $\mathrm{W}(T)$ applied to $x = y = z$ implies $\forall z \in S \big(\|T(z)\| \leq \|z\| \big)$.
Hence we can apply Theorem 2.1 to (+) by setting

$$\underline{x} :=_{\mathbb{N} \times \mathbb{N}^{\mathbb{N}}} l, g, \ \ z :=_S x, \ \ f :=_{S \to S} T, \ \ f^* :=_{\mathbb{N} \to \mathbb{N}} \mathrm{id},$$

and

$$\varphi_\forall(x, z, f) :\equiv \mathrm{W}(T), \ \ \exists v \in \mathbb{N} \ \psi_\exists(x, z, f, v) :\equiv \exists m \in \mathbb{N} \ \left(\|A_m x - A_{m+g(m)} x\| < 2^{-l} \right),$$

to obtain that there is a computable bound $M : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$, s.t.

$$\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, x \in S, T \in S^S \\ \big(\mathrm{W}(T) \wedge \|x\| \leq b \ \to \exists m \leq_{\mathbb{N}} M(l, g, b) \ (\|A_m x - A_{m+g(m)} x\| \leq 2^{-l}) \big).$$

It is rather easy to see that the proof can be formalized in $\mathcal{A}^\omega[X, \langle \cdot, \cdot \rangle, S]$, except for the question of the use of the axiom of extensionality (full extensionality is in general unavailable in any proof-theoretic extraction of computational bounds). Generally, one can avoid the use of full extensionality in proofs of statements about continuous objects. Note that in particular any nonexpansive operator is also continuous. However, in our case, the operator T may be discontinuous. Fortunately, Wittmann proves his main results as a consequence of a statement about a simple sequence of elements in S, which as such is independent of T (see Theorem 2.3 in [34] or Theorem 2.3 below), whereby all relevant equalities are provable directly. Therefore the rule of extensionality suffices to formalize his proof.
Hence the existence of a *uniform computable bound* for the metastable version can be inferred from the metatheorem in [8]. Furthermore, since the metatheorem is established by proof-theoretic reasoning, it provides not only the existence of a uniform bound but also a procedure for its extraction.
Now, in general such a bound might need so called bar-recursion (BR), which is required to interpret the schema of full comprehension over numbers in Spector's system (see [30]).

However, once more due to the way how Wittmann proved the analyzed theorem, it is easy to see that the only proof-theoretically non-trivial principles needed in the proof are the existence of the infimum/supremum of bounded sequences and the principle of convergence for bounded monotone sequences. Both of these principles need only bar-recursion restricted to numbers and functions ($\mathsf{BR}_{0,1}$) and not full BR. (Kohlenbach shows in [18, 16] that both principles are provable from arithmetical comprehension which is interpreted in $\mathcal{T}_0 + \mathsf{BR}_{0,1}$.) Moreover, since the bound itself has only functions and numbers as arguments, it follows from [28, 15] that the bound is not only computable, but that the *bound is a primitive recursive functional in the sense of Gödel's* $\mathcal{T}$.
These observations can be made a priori, without any in-depth analysis of the proof. In addition, one more conclusion can be drawn before one actually extracts the bound. In general, it is helpful, and sometimes even necessary, to simplify (in the sense of proof-theoretic strength) the analyzed proof. We do so in Section 3. As one can see, in fact the proof uses only arithmetical versions of the non-trivial principles (which can be proved by Σ^0_1-IA) and therefore we know that the use of $\mathsf{BR}_{0,1}$ can be eliminated as well. Hence, we can infer that there is actually *an ordinary primitive recursive bound* (a bound in $\mathcal{T}_0$) which we give explicitly in Section 5, Corollary 5.13.
We should point out that the original Corollary in [8] can be used in a more general context than the particular example we just discussed. For instance, it can be applied to both, Theorem 2.2 and Theorem 2.3 in [34]. Take for example the metastable formulation of Theorem 2.3 in [34]:

Theorem 2.3. *Let $X_{(\cdot)}$ be a sequence in a Hilbert space s.t.*

$$\forall m, n, k \in \mathbb{N} \quad \big(\|X_{n+k} + X_{m+k}\|^2 \leq \|X_n + X_m\|^2 + \delta_k\big),$$

and $\exists K \in \mathbb{N}^{\mathbb{N}} \forall n \in \mathbb{N} \forall i \geq Kn \quad (\delta_i \leq 2^{-n})$. *Then*

$$\forall l \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}\ \exists m \quad \big(\|A_m x - A_{m+g(m)} x\| \leq 2^{-l}\big).$$

For simplicity, let us here assume that the sequence $\delta_{(\cdot)}$ is in the real unit interval. In this case we have the following additional parameters: $K : \mathbb{N} \to \mathbb{N}$, $\delta : \mathbb{N} \to [0, 1]$ and a sequence $z_{(\cdot)} := X_{(\cdot)}$ (rather than a starting point $z := x$). We can apply Corollary 6.6.7) in [8] for the theory $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$ again. Additionally we use point 6.6.3). On the other hand, this time we don't need the function parameter f:

Theorem 2.4 (Gerhardy-Kohlenbach [8] - specific case 2)**.** *Let P be a $\mathcal{A}^\omega$-definable Polish space and let $\varphi_\forall$, resp. $\psi_\exists$, be a $\forall$- resp. an $\exists$-formula that contains only x, z, f free, resp. x, z, f, v free. Assume that $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle, S]$ proves the following sentence:*

$$\forall x \in \mathbb{N}^{\mathbb{N}}, y \in P, z_{(\cdot)} \in \mathbb{N}^S \big(\varphi_\forall(x, z) \to \exists v \in \mathbb{N}\ \psi_\exists(x, z, v)\big),$$

Then there is a computable functional $F : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ s. t. the following holds in all non-trivial (real) inner product spaces $(X, \langle\cdot,\cdot\rangle)$ and for any subset $S \subseteq X$

$$\begin{aligned} &\forall x \in \mathbb{N}^{\mathbb{N}}, z_{(\cdot)} \in \mathbb{N}^S, b_{(\cdot)} \in \mathbb{N}^{\mathbb{N}} \\ &\qquad \big(\forall n \in \mathbb{N}\ (\|z_n\| \leq b_n) \wedge \varphi_\forall(x, z) \to \exists v \leq F(x, b_{(\cdot)}) \psi_\exists(x, z, v)\big). \end{aligned}$$

The theorem holds analogously for finite tuples.

Given any rate of convergence for the sequence $\delta_{(\cdot)}$, the metastable version of the assumptions in Theorem 2.3 is purely universal:

$$\forall m, n, k, j \in \mathbb{N}\ \forall i \geq K(j)\ \big(\|X_{n+k} + X_{m+k}\|^2 \leq \|X_n + X_m\|^2 + \delta_k\ \wedge\ \delta_i \leq 2^{-n}\big). \quad (\mathrm{W}')$$

Moreover, the Polish space $[0,1]^{\mathbb{N}}$ is naturally definable in $\mathcal{A}^\omega$ so we obtain by Theorem 2.4 that

$$\begin{aligned}\forall l \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, K \in \mathbb{N}^{\mathbb{N}},\ \delta_{(\cdot)} \in [0,1]^{\mathbb{N}}, X_{(\cdot)} \in S^{\mathbb{N}}, b_{(\cdot)} \in \mathbb{N}^{\mathbb{N}} \\ \big(\ \forall i \in \mathbb{N}\ (X_i \leq b_i)\ \wedge\ \mathrm{W}'(X, K, \delta)\ \rightarrow \\ \exists m \leq_{\mathbb{N}} M(l, g, b_{(\cdot)}, K)\ (\|A_m x - A_{m+g(m)} x\| \leq 2^{-l})\ \big).\end{aligned}$$

Similarly as before, W$'$ implies $\forall i \in \mathbb{N}\ (X_i \leq b_i)$ for a suitable $b_{(\cdot)}$. We give such a bound explicitly in Section 5, Theorem 5.1. To be precise, in Theorem 5.1 we give a bound $M(l, g, b_0, b_1, \ldots, b_{K(0)}, K)$ for an arbitrary sequence $\delta_{(\cdot)} \in \mathbb{R}^{\mathbb{N}}$ converging to zero with the rate K. In the simplified case for the unit interval, it is straightforward to see that the bound also simplifies to an even more uniform bound $M(l, g, b_0, K)$.
To repeat, these are very specific scenarios. We should emphasize that the Corollaries in [8], and the metatheorem(s) even more so, have a much wider range of application.

3. Arithmetizing Wittmann's proof

The first step of a proof mining process is to investigate the proof of the theorem we want to analyze. For a discussion on proof mining techniques in connection with ergodic theory see [7] and the last section of [1] and for an in-depth analysis of applied proof theory see [18]. The nonconstructive, or ineffective, content of Wittmann's proof are the principle of convergence for bounded monotone sequences of real numbers and the existence of infimum for bounded sequences of real numbers. Formulated in the usual way, both principles state the existence of a real number, which we represent as fast converging Cauchy sequences of rationals[7] encoded as number theoretic functions (i.e. functions in $\mathbb{N}^{\mathbb{N}}$). However, for a given sequence, both principles can be replaced by weaker statements about natural numbers only (as opposed to statements about objects in $\mathbb{N}^{\mathbb{N}}$). In the presence of arithmetical comprehension, these weaker (arithmetical) statements are equivalent to the original (analytical) principles.[8] For the convergence we work with the arithmetic Cauchy property and for infimum we give for any precision an approximate infimum.

1. Arithmetized convergence of a monotone bounded sequence $a_{(\cdot)}$:

$$\forall l \exists n \forall m \geq n \quad \big(|a_n - a_m| \leq 2^{-l}\big).$$

[7]We represent rational numbers as pairs of natural numbers. For further details on representations in this context, see [18].

[8]While the analytical principles are actually known to be equivalent to arithmetic comprehension (see [29] and – for more detailed results – [16]), the arithmetic versions are equivalent to Σ^0_1-induction and hence have a functional interpretation by ordinarily primitive recursive functionals (see [18]).

2. Arithmetized existence of the infimum of a bounded sequence $a_{(\cdot)}$:

$$\forall l \exists n \forall m \quad \big(a_n - a_m \leq 2^{-l}\big).$$

Of course, in this way we don't get a single point which *is* the limit point or infimum. Therefore we have to analyze the proof and see whether such points are actually needed or whether these arithmetical versions suffice. Here, fortunately, it turns out that the latter is the case (see [14] for a general discussion of this point).
Following [18], we show that we can rewrite Wittmann's proof (see [34]) in the language of $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$, carefully using only weak (arithmetized) principles at the relevant places.

Theorem 3.1 (Wittmann 1990, [34]). *Let $X_{(\cdot)}$ be a sequence in a Hilbert space s.t. for all $m, n, k \in \mathbb{N}$*

$$||X_{n+k} + X_{m+k}||^2 \leq ||X_n + X_m||^2 + \delta_k \tag{i}$$

with

$$\lim_{k\to\infty} \delta_k = 0. \tag{ii}$$

Then the sequence $A_{(\cdot)}$ defined by

$$A_n := \frac{1}{n}\sum_{i=1}^{n} X_i,$$

is norm convergent.

We follow Wittmann's notation and use $X_{(\cdot)}$ to denote the sequence in the Hilbert space (not to be confused with the Hilbert space itself, which might be implied by the notation $\mathcal{A}^\omega[X, \langle\cdot,\cdot\rangle]$). Also, to keep the proof more readable, we refer the more technical steps to later sections. There we need to do a thorough analysis of those steps to obtain the precise bounds of the realizers, while here we can settle for their existence.
We formulate conditions (i) and (ii) from the theorem as arithmetical statements as follows (except for the variable K, which we will treat as a given parameter):

$$\forall m, n, k \in \mathbb{N} \quad \big(||X_{n+k} + X_{m+k}||^2 \leq ||X_n + X_m||^2 + \delta_k\big), \tag{1}$$

$$\exists K \in \mathbb{N}^{\mathbb{N}} \forall n \in \mathbb{N} \forall i \geq K(n) \quad \big(|\delta_i| \leq 2^{-n}\big). \tag{2}$$

From now on, let K always denote a rate of convergence of the sequence $\delta_{(\cdot)}$ (i.e. a function satisfying (2)) and B the upper bound of $X_{(\cdot)}$ (such a bound can be defined primitive recursively in $K(0)$ and some elements of $X_{(\cdot)}$, see Theorem 5.1).
It is easy to show that the sequence $(||X_n||)_n$ is a Cauchy sequence (see proof of Lemma 5.5 below). Such an arithmetical formulation of the convergence of $(||X_n||)_n$ is sufficient to infer that in particular we have that

$$\forall l^0 \exists n^0 \forall m_2^0 \geq m_1^0 \geq n \quad \big(\, ||X_{m_1}||^2 \leq ||X_{m_2}||^2 + 2^{-(l+1)} \,\big). \tag{3}$$

The Skolem function realizing n^0 would correspond to n_ϵ in Wittmann's proof (where he used the standard convergence), however we work only with the fact that such an n exists for any given precision l. From 3 we infer that

$$\forall l \exists n_l \forall m, n \geq n_l \; \forall k \geq K(l) \quad \big(\, \langle X_{n+k}, X_{m+k}\rangle \leq \langle X_n, X_m\rangle + 2^{-l} \,\big), \tag{W1}$$

the same way as Wittmann in [34], see also the end of the proof of Lemma 5.7 in Section 6. From now on, by n_l we always denote an n_l which satisfies (W1).
Since the norm of any convex combination of the elements of the sequence is bounded from below by 0, the set of all such convex combinations has an infimum. To give an arithmetic formulation of this fact, it is useful to have a primitive recursive functional C which gives us a 2^{-l} approximation of the smallest convex combination of $X_n, X_{n+1}, \ldots, X_p$ (more precisely, of the square of the infimum of the set of the norms of these convex combinations). Clearly, there are many ways to define such a functional. We do so in a straightforward way in Definition 5.2 below. Having such a C in place, the following arithmetical formula

$$\forall l \exists p_l \forall p \geq p_l \quad \left(\, C(l, n_l, p_l) \leq C(l, n_l, p) + 2^{-l} \, \right), \tag{W2}$$

states the existence of an approximate infimum of all convex combinations of $X_{(\cdot)}$ (see also Lemma 5.3). Similarly as with n_l, from now on by p_l we mean a number satisfying (W2). Wittmann introduces a specific point $z_{\epsilon,m}$, which we denote by $Z(l,n,p,m)$ and define using our notation in Definition 5.8 below. However, here the precise definition is not as important as the properties of this point. Firstly, we show that for any two natural numbers i and j, the distance between $Z(l,n_l,p_l,i)$ and $Z(l,n_l,p_l,j)$ is arbitrary small for sufficiently large l. Together with the convexity of the square function we have (again, see [34] or the proof of Lemma 5.9.(2) below) that

$$\forall l^0, m^0 \quad \left(\, ||Z(l, n_l, p_l, m)||^2 \leq C(l, n_l, p_l) + 2^{-l} \, \right). \tag{W3}$$

From (W2) we can infer (see the proof of Lemma 5.9.(1) below) that

$$\forall l, i, j \ \Big(\big\| \frac{1}{2}(Z(l, n_l, p_l, i) + Z(l, n_l, p_l, j)) \big\|^2 + 2^{-l} \geq C(l, n_l, p_l) \Big),$$

and together with (W3) and the parallelogram identity we can conclude that

$$\forall l, i, j \ \Big(\big\| Z(l, n_l, p_l, i) - Z(l, n_l, p_l, j) \big\|^2 \leq 2^{-l+4} \Big).$$

Secondly, it follows from the definition of Z (yet again see [34] or Section 6, proof of Lemma 5.10) that

$$\forall l \forall m \geq p_l \ \Big(\| Z(l, n_l, p_l, m) - A_{m+1} \| \leq \frac{2(p_l + n_l + K(l)) \sup_{n \in \mathbb{N}} \|X_n\|}{m+1} \Big),$$

which means that we can make the distance between A_{m+1} and $Z(l,n_l,p_l,m)$ arbitrarily small by choosing m sufficiently large.
In particular we have shown that the distance between any A_i and any A_j is arbitrarily small once i and j are sufficiently large. Note that we can choose an arbitrarily large l first and then we are still free to choose sufficiently large i and j after n_l and p_l are fixed. This concludes the sketch of the proof that $A_{(\cdot)}$ is a Cauchy sequence.

4. Obtaining a bound

The goal of this section is to roughly describe how to find a bound for m in

$$\forall l, g \exists m \left(\|A_m - A_{m+g(m)}\| \leq 2^{-l} \right).$$

Similarly as before, for better understanding we leave some technical details for later sections and disregard the monotonicity of the bounds as well as some small corrections needed in the exponent of 2^{-l}. We handle these two aspects very carefully in the sections 5 and 6, where we make also all of the following steps more explicit.
Let the assumptions in the proof of Theorem 3.1 hold and let K, B and C be as before as well.
Furthermore, we assume that N_0 and P_0 are the witnessing terms for (W1) (note that in fact this means rather that we have a witness for Kreisel's no-counterexample interpretation – n.c.i. see [23, 24] – of the convergence of $\|X_n\|$ in the first place) and (W2) as given in [18], i.e. we have that:

$$\forall l, h \forall m, n \in [N_0(l,h); N_0(l,h) + g(N_0(l,h))] \left(\langle X_{m+k}, X_{n+k} \rangle \leq \langle X_m, X_n \rangle + 2^{-l} \right)$$

and $\forall l, f \left(C(l, n_l, P_0(l,f)) \leq C(f(l, n_l, P_0(l,f))) + 2^{-l} \right)$ for any n_l satisfying (W1). This structure accounts for the already mentioned nested iteration. Eventually, we have to specify a counterfunction f s.t. it is sufficient that this inequality holds for that particular f. This f has to depend on n_l, which will obviously be defined as an iteration and f itself will be iterated by P_0. To obtain a bound for m, we follow the proof from the last section backwards. We define

$$M_0(l,n,p) := 2(p + n + K(l))B2^l,$$

since then we have that $\|Z(l, n_l, p_l, M_0(l, n_l, p_l)) - A_{m+1}\| \leq 2^{-l}$, for the right values of l, n_l and p_l (which we don't know yet).
From the proof we can infer that the largest p needed in (W2) is $K(l) + m + g(m) + n_l + p_l$ (for details see proof of Lemma 5.9.(1)), therefore we need that

$$f(P_0(l,f)) = K(l) + m + g(m) + n_l + P_0(l,f),$$

(where m and $m + g(m)$ correspond to the m and n in (W1)).
Moreover, we can see that the largest m or n needed in (W1) is $n_l + p_l + m + g(m) + K(l)$ (for details see proof of Lemma 5.7 and its application in proof of Lemma 5.9.(2)). So we need that $h(N_0(h,l)) = p_l + m + g(m) + K(l)$.
Keeping in mind that $N_0(l,h)$ corresponds to n_l (and to n below) and $P_0(l,f)$ to p_l (and to p below) we obtain[9]

$$\begin{aligned} h(l,g) = \lambda n \ . \ & \underbrace{P_0(l, \lambda p \ . \ K(l) + M_0(l,n,p) + g(M_0(l,n,p)) + p + n)}_{p_l} + \\ & + M_0(p_l, n, l) + g(M_0(p_l, n, l)) + K(l), \end{aligned}$$

[9]Here and in the following, when considering a complex term $t[a_1, \dots, a_n]$ in several variables $a_1, \dots, a_n$, the notation $\lambda a_i . t[a_1, \dots, a_n]$ defines (as usual in logic) the function $a_i \mapsto t[a_1, \dots, a_n]$.

and define

$$N(l,g) := N_0(l,h(l,g)).$$

Given $N(l,g)$, which corresponds to n_l, and again keeping in mind that $P_0(l,f)$ corresponds to p_l (and to p below), we can define $P(l,g)$ as well:

$$P(l,g) := P_0(l,f(l,g)),$$

where $f(l,g) = \lambda p\ .\ K(l) + M_0(p,N(l,g),l) + g(M_0(p,N(l,g),l)) + p + N(l,g)$. Finally we can define the desired witness for m as follows:

$$M(l,g) := M_0(l,N(l,g),P(l,g)).$$

5. Uniform bounds for Wittmann's ergodic theorems

We give a bound for a finitary version of Wittmann's convergence result for a general series in a Hilbert space satisfying a suitable formulation of the condition (W^-) first (see Theorem 5.1 and Corollary 5.11) to derive the bounds for the finitary versions of the actual ergodic theorems later (see Corollary 5.12 and Corollary 5.13).
We already discussed how to obtain the first bound in Sections 4 and 3, here we concentrate on a formal and precise definition of the bounds as well as on the proof that these bounds are correct.

Theorem 5.1 (Finitary version of Theorem 2.3 in [34])**.** *Let K be a function and $X_{(\cdot)}$ a sequence in a Hilbert space s.t. for all $m,n,k \in \mathbb{N}$*

$$\|X_{n+k} + X_{m+k}\|^2 \le \|X_n + X_m\|^2 + \delta_k,$$

with $\forall l \in \mathbb{N} \forall n \ge K(l)\ \big(|\delta_n| < 2^{-l}\big)$. Then the sequence $A_{(\cdot)}$, defined by $A_n := \frac{1}{n}\sum_{i=1}^{n} X_i$, is a Cauchy sequence and we have that

$$\forall l, g \exists m \le M'(l,g^M)\ \big(\|A_m - A_{m+g(m)}\| \le 2^{-l}\big),$$

with $g^M(n) := \max_{i \le n} g(n)$ and M' defined as follows:

$$\begin{aligned}
M'(l,g) &:= M(2l+6, \lambda n\ .\ g(n+1)) + 1,\\
M(l,g) &:= M_0(P(l,g),N(l,g),l),\\
P(l,g) &:= P_0(l,F(l,g,N(l,g))),\\
F(l,g,n) &:=_1 \lambda p\ .\ p + n + K^M(l) + M_0(l,n,p) + g(M_0(l,n,p)),\\
N(l,g) &:= N_0(l+1,H(l,g)),\\
H(l,g) &:=_1 \lambda n\ .\ H_0(l,g,n,P_0(l,F(l,g,n))),
\end{aligned}$$

where

$$
\begin{aligned}
H_0(l,g,n,p) &:= p + M_0(l,n,p) + g(M_0(l,n,p)) + K^M(l),\\
M_0(l,n,p) &:= (2n+2p+2K^M(l))B2^l,\\
P_0(l,f) &:= \tilde{f}^{B^2 2^l}(0), \quad \tilde{f}(n) := n + f(n),\\
N_0(l,h) &:= R(l+1, U(l,h)) + K^M(l+1),\\
R(l,u) &:= \tilde{u}^{\lceil \|X_0\| \rceil^2 2^l}(0),\\
U(l,h) &:=_1 \lambda n\ .\ (n + K^M(l+1)) + h^M(n + K^M(l+1)),\\
B &:= \max_{1 \le i \le K^M(0)} \lceil \|X_i\| \rceil + 1.
\end{aligned}
$$

From now on, we assume that the assumptions of the theorem hold and use the terms as they are defined in the theorem. We denote the set of natural numbers $\{x \in \mathbb{N} : a \le x \wedge x \le b\}$ by $[a;b]$ and define a specific primitive recursive 2^{-l} approximation of the square of the norm of the smallest convex combination of $X_n, X_{n+1}, \ldots, X_p$ as follows:

Definition 5.2 (C)**.** Let $C'(\underline{s}, l, n, p, X) := \|\sum_{i=0}^{p} \tilde{\underline{s}}(i) X_{n+i}\|^2$ to define

$$C(l,n,p) := C(l,n,p,X) := \min_{\underline{s} \in S_{p,l}} \{C'(\underline{s}, l, n, p, X)\},$$

where

$$S_{p,l} := \left\{ (s_0, \ldots, s_p) \;\Big|\; \sum_{i=0}^{p} s_i = 1 \;\wedge \forall i \in [0;p]\ \exists k_i \le_{\mathbb{N}} \left\lceil \frac{pB^2}{2^{-(l+1)}} \right\rceil \; \left(s_i = k_i \frac{2^{-(l+1)}}{pB^2} \right) \right\},$$

$$\widetilde{s_0, \ldots, s_m}(n) := \begin{cases} s_n & \text{if } n < m \wedge \quad 0 \le_{\mathbb{Q}} s_n \wedge s_n + \sum_{i=0}^{n-1} \tilde{\underline{s}}(i) \le_{\mathbb{Q}} 1,\\ 0 & \text{if } n < m \wedge \neg(0 \le_{\mathbb{Q}} s_n \wedge s_n + \sum_{i=0}^{n-1} \tilde{\underline{s}}(i) \le_{\mathbb{Q}} 1),\\ s_m & \text{if } n = m \wedge \quad 0 \le_{\mathbb{Q}} s_m \wedge s_m + \sum_{i=0}^{m-1} \tilde{\underline{s}}(i) =_{\mathbb{Q}} 1,\\ 1 - \sum_{i=0}^{m-1} \tilde{\underline{s}}(i) & \text{if } n = m \wedge \neg(0 \le_{\mathbb{Q}} s_m \wedge s_m + \sum_{i=0}^{m-1} \tilde{\underline{s}}(i) =_{\mathbb{Q}} 1),\\ 0 & \text{else.} \end{cases}$$

Lemma 5.3 (C approximates the smallest convex combination)**.**

$$\forall l, n, p, f \forall \underline{s} \, \big(C'(\underline{s}, l, n, p, f) + 2^{-l} \ge C(l,n,p) \big).$$

We will also use that M majorizes itself and that N_0 and P_0 are the right witnesses for the two main assumptions needed in Wittmann's proof. For better readability, we prove these lemmas in Section 6 at the end of the paper.

Lemma 5.4 (M is a majorant)**.** *Each of the terms M,P,N,M_0,P_0,N_0,R majorizes itself. In particular we have:*

$$\forall l' \ge l \forall h', h(\forall n(h(n) \ge h'(n)) \to N_0(l', h^M) \ge N_0(l, h'))$$

and

$$\forall l, g \forall n \leq N(l, g^M) \forall p \leq P(l, g^M)$$
$$(P(l, g^M) \geq P_0(l, F(l, g^M, n)) \ \wedge \ M(l, g^M) \geq M_0(l, n, p)).$$

Lemma 5.5 (N_0 is correct)**.** *The sentence*

$$\forall l, h \exists n \forall i, j \in [n; n + h(n)] \ i \leq j \to \|X_i\|^2 - \|X_j\|^2 \leq 2^{-l}$$

is witnessed by an $n \leq N_0(l, h)$.

Lemma 5.6 (P_0 is correct)**.** $\forall l, f, n \exists p \leq P_0(l, f) \ \big(C(l, n, p) \leq C(l, n, f(p)) + 2^{-l}\big)$.

Next three lemmas give a quantitative analysis of the original proof in [34]. Again, for better readability, we give the proofs in Section 6.

Lemma 5.7 (The scalar product increase is bounded)**.** *For any l and any g, consider $h := H(l, g^M)$. Let n be a witness for Lemma 5.5, i.e.*

$$n \leq N(l, h) \ \wedge \ \forall i, j \in [n; n + h(n)] \ \big(i \leq j \to \|X_i\|^2 - \|X_j\|^2 \leq 2^{-l-1}\big). \tag{N}$$

Moreover let $f := F(l, g^M, n)$, p be a number smaller than $P_0(l, f)$ and $m := M_0(l, n, p)$. Then we have that

$$\langle X_{a+k}, X_{b+k} \rangle \leq \langle X_a, X_b \rangle + 2^{-l}$$

holds for all k, a, b s.t. $K^M(l) \leq k \leq K^M(l) + m + g^M(m)$ and $n \leq a, b \leq n + p$.

Analogously to Wittmann [34] we define

Definition 5.8 (Z)**.** $Z(l, n, p, m) := \frac{1}{m+1} \sum_{k=K^M(l)}^{K^M(l)+m} \sum_{i=0}^{p} \underset{\sim}{\widetilde{s}}_i X_{n+k+i}$, with $\underline{s}$ corresponding to the tuple in the definition of $C(l, n, p)$ (see Definition 5.2 above).

Lemma 5.9 (Zs are close)**.** *For any l and any g, consider $h := H(l, g^M)$. Let n be a witness for Lemma 5.5, i.e.*

$$n \leq N(l, g^M) \ \wedge \ \forall i, j \in [n; n + h(n)] \ \big(i \leq j \to \|X_i\|^2 - \|X_j\|^2 \leq 2^{-l-1}\big). \tag{N}$$

Moreover, let $m := M_0(l, n, p)$, $f := F(l, g^M, n)$ and p be a witness for Lemma 5.6, i.e.

$$p \leq P_0(l, f) \ \wedge \ (C(l, n, p) \leq C(l, n, f(p)) + 2^{-l}), \tag{P}$$

Then we have that $\|Z(l, n, p, m) - Z(l, n, p, m + g^M(m))\|^2 \leq 2^{-l+4}$.

Lemma 5.10 (Zs and As are close)**.** *For any l and any g, consider $h := H(l, g^M)$. Let n be a witness for Lemma 5.5, i.e.*

$$n \leq N(l, g^M) \ \wedge \ \forall i, j \in [n; n + h(n)] \ \big(i \leq j \to \|X_i\|^2 - \|X_j\|^2 \leq 2^{-l-1}\big). \tag{N}$$

Moreover let $f := F(l, g^M, n)$, p be a witness for Lemma 5.6, i.e.

$$p \leq P_0(l, f) \ \wedge \ (C(l, n, p) \leq C(l, n, f(p)) + 2^{-l}), \tag{P}$$

and $m := M_0(l,n,p)$, $m' := m + g(m)$. *Then we have that*

$$\|A_{m+1} - Z(l,n,p,m)\| \le \frac{1}{m+1}(2n+2p+2K(l))B + 2^{-l}$$

and

$$\|A_{m'+1} - Z(l,n,p,m')\| \le \frac{1}{m'+1}(2n+2p+2K(l))B + 2^{-l}.$$

Proof of Theorem 5.1. Fix arbitrary l and g. Set $h := H(l, g^M)$. By Lemma 5.5 we know there is an n s.t.

$$n \le N(l, g^M) \ \wedge\ \forall i,j \in [n; n+h(n)] \ \big(i \le j \to \|X_i\|^2 - \|X_j\|^2 \le 2^{-l-1}\big).$$

Let $f := F(l, g^M, n)$. By Lemma 5.6 we know that there is a p s.t.

$$p \le P_0(l,f) \ \wedge\ (C(l,n,p) \le C(l,n,f(p)) + 2^{-l}).$$

Note that by Lemma 5.4 we have that $p \le P(l, g^M)$. We set $m := M_0(l,n,p)$. By Lemma 5.4 we get that $m \le M(l, g^M)$. Finally, it follows from lemmas 5.9 and 5.10 that

$$\begin{aligned}
\|A_{m+1} - A_{m+g(m)+1}\| &\le \|Z(l,n,p,m) - Z(l,n,p,m+g(m))\| \\
&\quad + 2\Big(\frac{1}{m+1}(2n+2p+2K(l))B + 2^{-l}\Big) \\
&\le \sqrt{2^{-l+4}} + 2^{-l+1} + \frac{2(2n+2p+2K(l))B}{m+1} \\
&= \sqrt{2^{-l+4}} + 2^{-l+1} + \frac{2(2n+2p+2K(l))B}{(2n+2p+2K(l))B2^l + 1} \\
&< \sqrt{2^{-l+4}} + 2^{-l+1} + 2^{-l+1} \le 2^{-\frac{l}{2}+3}.
\end{aligned}$$

This proves

$$\forall l, g \exists m \le M(l, g^M) \ \big(\|A_{m+1} - A_{m+g(m)+1}\| \le 2^{-\frac{l}{2}+3}\big),$$

from which the claim follows immediately by the definition of M'. □

Sometimes it is useful to work with the following version of the previous theorem, though both these formulations are equivalent (even in weaker systems than we used to formalize the original proof itself).

Corollary 5.11 (Finitary version of Theorem 2.3 in [34] for intervals)**.** *Let* K *be a function and* $X_{(\cdot)}$ *a sequence in a Hilbert space s.t. for all* $m,n,k \in \mathbb{N}$

$$\|X_{n+k} + X_{m+k}\|^2 \le \|X_n + X_m\|^2 + \delta_k,$$

with

$$\forall l \in \mathbb{N} \forall n \ge K(l) \ \big(|\delta_n| < 2^{-l}\big).$$

Then the sequence $A_{(\cdot)}$ *defined by*

$$A_n := \frac{1}{n}\sum_{i=1}^{n} X_i,$$

is a Cauchy sequence and we have that

$$\forall l, g \exists m \leq M'(l+1, g^M) \forall i, j \in [m; m+g(m)] \; \big(\|A_i - A_j\| \leq 2^{-l}\big),$$

with M' defined as in Theorem 5.1.

Proof. Given any l and g, apply Theorem 5.1 to the number $l+1$ and to the function

$$\begin{aligned} h(n) := \min\Big\{ & \; i \in [0; g(n)] \text{ s.t.} \\ & \forall j \in [0; g(n)] \quad \Big(\big|\|A_{n+i}\| - \|A_n\|\big| \geq \big|\|A_{n+j}\| - \|A_n\|\big|\Big)\Big\}. \end{aligned}$$

It follows that

$$\exists m \leq M'(l+1, h^M) \; \big(\|A_m - A_{m+h(m)}\| \leq 2^{-l-1}\big).$$

We fix such an m and conclude (by the triangle inequality) that

$$\forall i, j \in [m; m+g(m)] \; \big(\|A_i - A_j\| \leq 2^{-l}\big).$$

Moreover, since $\forall n \in \mathbb{N} \; (h^M(n) \leq g^M(n))$ we have that

$$m \leq M'(l+1, g^M)$$

due to Lemma 5.4. $\square$

Now, we obtain the bound for the metastable version of Theorem 2.2 in Wittmann's paper [34] as a simple conclusion:

Corollary 5.12 (Finitary version of Theorem 2.2 in [34])**.** *Let S be a subset of a Hilbert space and $T : S \to S$ be a mapping satisfying*

$$\begin{aligned} &\forall n \in \mathbb{N} \forall x, y \in S \; \big(\|T^n x + T^n y\| \leq \alpha_n \|x + y\|\big), \\ &\forall l \in \mathbb{N} \forall n \geq K'(l) \; \big(|1 - \alpha_n| < 2^{-l}\big). \end{aligned}$$

Then for any $x \in S$ the sequence of the Cesàro means

$$A_n x := \frac{1}{n+1} \sum_{i=0}^{n} T^i x$$

is norm convergent and the following holds:

$$\forall l, g \exists m \leq M'(l+1, g^M) \forall i, j \in [m; m+g(m)] \; \big(\|A_i x - A_j x\| \leq 2^{-l}\big),$$

with M' defined as in Theorem 5.1.

Proof. Fix an arbitrary $x \in S$ and set $B' := \max_{i \leq K'(0)}(T^i x) + 1$. The claim follows from Corollary 5.11 with $X_i := T^i x$, $\delta_n := 4B'^2(\alpha_n^2 - 1)$, $K(l) := K'(l + 2\lceil \log_2 B' \rceil + 2)$. $\square$

The following corollary follows immediately:

Corollary 5.13 (Finitary version of Theorem 2.1 in [34]). *Let S be a subset of a Hilbert space and $T : S \to S$ be a mapping satisfying*

$$\forall x, y \in S\ \big(\|Tx + Ty\| \le \|x + y\|\big).$$

Then for any $x \in S$ the sequence of the Cesàro means $A_n x := \frac{1}{n+1} \sum_{i=0}^{n} T^i x$ is norm convergent and the following holds:

$$\forall l, g \exists m \le M(l, g^M) \forall i, j \in [m; m + g(m)]\ \big(\|A_i x - A_j x\| \le 2^{-l}\big),$$

with M defined as follows:

$$\begin{aligned}
M(l, g) &:= (N(2l + 7, g) + P(2l + 7, g)) \lceil \|x\| \rceil 2^{2l+8} + 1,\\
P(l, g) &:= P_0(l, F(l, g, N(l, g))),\\
F(l, g, n) &:=_1 \lambda p\ .\ p + n + \tilde{g}((n + p)\lceil \|x\| \rceil 2^{l+1}),\\
N(l, g) &:= \big(\lambda n\ .\ n + P_0(l, F(l, g, n)) + \tilde{g}((n + P_0(l, F(l, g, n)))\lceil \|x\| \rceil 2^{l+1})\big)^{\lceil \|x\| \rceil^2 2^{l+2}}(0),
\end{aligned}$$

where $P_0(l, f) := \tilde{f}^{\lceil \|x\| \rceil^2 2^l}(0)$, $\tilde{f}(n) := n + f(n)$, $f^M(n) := \max_{i \le n+1} f(i)$.

Note that (due to Lemma 5.4) the bound for m depends only on a bound for the norm of the parameter x and not directly on the starting point.

6. Lemmas

Here, we assume that the assumptions of Theorem 5.1 hold and use the terms as they are defined in that theorem. We use also the definitions 5.2 and 5.8 for C and Z. Moreover, w.l.o.g we can assume $K^M = K$, since the original assumption implies

$$\forall l \in \mathbb{N} \forall n \ge K^M(l)\ \big(\|\delta_n\| < 2^{-l}\big).$$

We prove Lemma 5.3 first and then the two lemmas which show that N_0 and P_0 are the right witnesses for the two main assumptions needed in Wittmann's proof. The fact that M majorizes itself (and therefore so does M') follows simply from its definition in Theorem 5.1.

Recall Lemma 5.3 (C approximates the smallest convex combination).

$$\forall l, n, p, f \forall \underline{s}\ \big(C'(\underline{s}, l, n, p, f) + 2^{-l} \ge C(l, n, p)\big).$$

Proof. Given $\underline{s}$ choose $\underline{s}' \in S_{p,l}$ s.t. $|s'_i - \tilde{s}_i| \le \frac{2^{-(l+1)}}{pB^2}$. Then we have that

$$\left\| \sum_{i=0}^{p} \widetilde{s}_i X_{n+i} - \sum_{i=0}^{p} s'_i X_{n+i} \right\| = \left\| \sum_{i=0}^{p} |\widetilde{s}_i - s'_i| X_{n+i} \right\| \le \frac{2^{-(l+1)}}{pB^2} pB = \frac{2^{-(l+1)}}{B},$$

and therefore also that $|\ \|\sum_{i=0}^{p} \widetilde{s}_i X_{n+i}\| - \|\sum_{i=0}^{p} s'_i X_{n+i}\|\ | \le \frac{2^{-(l+1)}}{B}$, so finally we get that

$$\left|\ \left\| \sum_{i=0}^{p} \widetilde{s}_i X_{n+i} \right\|^2 - \left\| \sum_{i=0}^{p} s'_i X_{n+i} \right\|^2 \right| \le \frac{2^{-(l+1)}}{B} (B + \frac{2^{-(l+1)}}{B}) \le 2^{-l}.$$

$\square$

Recall Lemma 5.5 (N_0 is correct). *The sentence*

$$\forall l, h \exists n \forall i, j \in [n; n+h(n)] \, \big(i \leq j \to \|X_i\|^2 - \|X_j\|^2 \leq 2^{-l}\big)$$

is witnessed by an $n \leq N_0(l,h)$.
Proof. The sequence $(\|X_n\|)$ is bounded from below therefore we have:

$$\forall l \exists r \forall k \, \big(\|X_k\|^2 + 2^{-l} \geq \|X_r\|^2\big).$$

For any given l we fix such an r. The following statements imply that $(\|X_n\|)$ is a Cauchy sequence:

1. $\exists k_0 \forall k \geq k_0 \quad \big(\|X_k\|^2 \leq \|X_r\|^2 + 2^{-l}\big)$, and
2. $\forall k \quad \big(\|X_k\|^2 + 2^{-l} \geq \|X_r\|^2\big)$,

since (1)$\wedge$(2) means that there is an index from which on the norm of any of the remaining elements of the sequence is 2^{-l} close to a fixed number, namely $\|X_r\|$, and therefore the norms of any two such elements can't differ form each other by more than 2^{-l+1}. While the second condition follows immediately, to prove the first condition, assume towards contradiction

$$\forall k_0 \exists k \geq k_0 \quad \big(\|X_k\|^2 > \|X_r\|^2 + 2^{-l}\big).$$

Applied to $k_0 = r + K(l)$ this implies

$$\exists k \geq r + K(l) \quad \big(\|X_k\|^2 > \|X_r\|^2 + 2^{-l}\big),$$

which is a contradiction to (1) applied to $m = n$ and (2):

$$\forall n^0, k^0 \quad \big(\|X_{n+k}\|^2 \leq \|X_n\|^2 + \frac{\delta_k}{4}\big) \quad \wedge \quad \forall n^0 \forall i^0 \geq K(n) \quad \big(\delta_i \leq 2^{-n}\big).$$

This concludes the proof that $(\|X_n\|)$ is a Cauchy sequence:

$$\forall l \exists k_0 \forall k \geq k_0 \quad \Big(\big| \, \|X_k\|^2 - \|X_{k_0}\|^2 \big| \leq 2^{-l} \Big).$$

We rewrite this using the n.c.i. Let R denote a bound for the n.c.i. of the existence of the approximate infimum of the sequence, i.e.:

$$\forall l, u \exists r \leq R(l,u) \forall i \leq u(r) \quad \big(\|X_i\|^2 + 2^{-l} \geq \|X_r\|^2\big), \tag{R}$$

as it is defined in [18] (note that since R does not depend on the sequence, it does not matter, whether we consider $(\|X_n\|)$ or $(\|X_n\|^2)$, except that we have to consider a bound for $(\|X_0\|^2)$ rather than $(\|X_0\|)$). We have $(N_0(l,h) = R(l+1,u) + K(l)$ with $u :\equiv \lambda n.(n + K(l)) + h(n + K(l)))$

$$\forall l, h \exists n \leq N_0(l,h) \quad \Big(\big| \, \|X_{n+h(n)}\|^2 - \|X_n\|^2 \big| \leq 2^{-l} \Big), \tag{N0}$$

since the following holds (here $N_0'(l,h,r) = r + K(l)$):

$$\forall l, h \exists r \leq R(l+1,u) \ \big(\|X_{N_0'(l,h,r)+h(N_0'(l,h,r))}\|^2 \leq \|X_r\|^2 + 2^{-l-1} \ \wedge \\ \|X_{N_0'(l,h,r)+h(N_0'(l,h,r))}\|^2 + 2^{-l-1} \geq \|X_r\|^2\big).$$

The second inequality follows from (R) (for u as above) since

$$\begin{aligned} u(r) &= (\lambda n.(n + K(l)) + h(n + K(l)))(r) = r + K(l) + h(r + K(l)) \\ &= N_0'(l,h,r) + h(N_0'(l,h,r)). \end{aligned}$$

The first condition follows from

$$\begin{aligned} \|X_{N_0'(l,h,r)+h(N_0'(l,h,r))}\|^2 &= \|X_{r+K(l)+h(N_0'(l,h,r))}\|^2 \\ &\leq \|X_r\|^2 + \frac{\delta_{K(l)+h(N_0'(l,h,r))}}{4} \leq \|X_r\|^2 + 2^{-l-1}. \end{aligned}$$

Note that for all $r \leq R(l,u)$ we have that $N_0'(l,h,r) \leq N_0'(l,h,R(l,u)) = N_0(l,h)$. Finally, given any h in the claim, we can define

$$h'(n) := \min\Big\{i \in [0;h(n)] \,\Big|\, \forall j \in [0;h(n)] \ \Big(\big|\, \|X_{n+i}\|^2 - \|X_n\|^2\big| \geq \big|\, \|X_{n+j}\|^2 - \|X_n\|^2\big|\Big)\Big\}.$$

Now the claim follows from (N0) applied to h', the triangle inequality and the fact that we actually prove not only that $\|X_i\|^2 - \|X_j\|^2 \leq 2^{-l}$ but also $|\ \|X_i\|^2 - \|X_j\|^2| \leq 2^{-l}$, and $N_0(l,h') \leq N_0(l,h^M)$, which follows from lemma 5.4 (we discuss a similar argument in the proof of Corollary 5.11 in more detail).

□

Recall Lemma 5.6 (P_0 is correct).

$$\forall l, f, n \exists p \leq P_0(l,f) \ (C(l,n,p) \leq C(l,n,f(p)) + 2^{-l}).$$

Proof. Given any n and any l, the sequence (a_i) defined by $a_i := C(i,n,l)$ is monotone. Therefore the claim follows from Proposition 2.26 in [18]. □

Next we prove the three lemmas, which give a quantitative analysis of the original proof in [34].

Recall Lemma 5.7 (The scalar product increase is bounded). *For any l and any g, consider $h := H(l, g^M)$. Let n be a witness for Lemma 5.5, i.e.*

$$n \leq N(l,h) \ \wedge \ \forall i,j \in [n; n+h(n)] \ \big(i \leq j \to \|X_i\|^2 - \|X_j\|^2 \leq 2^{-l-1}\big). \tag{N}$$

Moreover let $f := F(l, g^M, n)$, p be a number smaller than $P_0(l,f)$ and $m := M_0(l,n,p)$. Then we have that

$$\langle X_{a+k}, X_{b+k}\rangle \leq \langle X_a, X_b\rangle + 2^{-l}$$

holds for all k, a, b s.t. $K(l) \leq k \leq K(l) + m + g^M(m)$ and $n \leq a, b \leq n + p$.
Proof. We have

$$\|X_{a+k} + X_{b+k}\|^2 \leq \|X_a + X_b\|^2 + 2^{-l} \tag{1}$$

since $k \geq K(l)$. Moreover we can infer

$$\|X_{a+k}\|^2 \geq \|X_a\|^2 - 2^{-l-1} \wedge \|X_{b+k}\|^2 \geq \|X_b\|^2 - 2^{-l-1} \tag{2}$$

from (N), $a \geq n$, $b \geq n$, and

$$\begin{aligned} a+k, b+k &\leq n+p+m+g^M(m)+K(l) \\ &= n+p+M_0(l,n,p)+g^M(M_0(l,n,p))+K(l) \\ &\leq n+H(l,g^M)(n) = n+h(n). \end{aligned}$$

Therefore

$$\begin{aligned} \langle X_{a+k}, X_{b+k} \rangle &= \frac{1}{2}(\|X_{a+k}+X_{b+k}\|^2 - \|X_{a+k}\|^2 - \|X_{b+k}\|^2) \\ &\leq \frac{1}{2}(\|X_a+X_b\|^2 + 2^{-l} - \|X_a\|^2 + 2^{-l-1} - \|X_b\|^2 + 2^{-l-1}) \\ &= \langle X_a, X_b \rangle + 2^{-l}. \end{aligned}$$

□

Recall Lemma 5.9 (Zs are close). *For any l and any g, consider $h := H(l, g^M)$. Let n be a witness for Lemma 5.5, i.e.*

$$n \leq N(l,g^M) \ \wedge \ \forall i,j \in [n; n+h(n)] \ \big(i \leq j \to \|X_i\|^2 - \|X_j\|^2 \leq 2^{-l-1}\big). \tag{N}$$

Moreover let $m := M_0(l,n,p)$, $f := F(l,g^M,n)$ and p be a witness for Lemma 5.6, i.e.

$$p \leq P_0(l,f) \ \wedge \ (C(l,n,p) \leq C(l,n,f(p)) + 2^{-l}), \tag{P}$$

Then we have that $\|Z(l,n,p,m) - Z(l,n,p,m+g(m))\|^2 \leq 2^{-l+4}$.

Proof. Firstly, we will show that

$$\|\frac{1}{2}(Z(l,n,p,m)+Z(l,n,p,m+g(m)))\|^2 + 2^{-l+1} \geq C(l,n,p). \tag{1}$$

Since $\frac{1}{2}(Z(l,n,p,m)+Z(l,n,p,m+g(m)))$ is a convex combination of

$$X_{n+K(l)}, \ldots, X_{n+K(l)+p+m+g(m)},$$

we obtain by Lemma 5.3 that

$$\left\|\frac{1}{2}(Z(l,n,p,m)+Z(l,n,p,m+g(m)))\right\|^2 + 2^{-l} \geq \\ C(l,n,n+K(l)+p+m+g^M(m)).$$

Now, because of

$$\begin{aligned} f(p) &= p+n+K(l)+M_0(l,n,p)+g^M(M_0(l,n,p)) \\ &= n+K(l)+p+m+g^M(m), \end{aligned}$$

it follows from (P) that

$$C(l,n,n+K(l)+p+m+g^M(m)) \geq C(l,n,p)-2^{-l},$$

which concludes the proof of (1). Secondly, we will show that

$$\forall o \leq m+g(m)\ \left(\left\|Z(l,n,p,o)\right\|^2 \leq C(l,n,p)+2^{-l}\right). \tag{2}$$

Let $\underline{s}$ be the tuple corresponding to the tuple in the definition of $C(l,n,p)$ (note that $\tilde{\underline{s}} = \underline{s}$). By Lemma 5.7 we have

$$\begin{aligned}\left\|\sum_{i=0}^{p} s_i X_{n+k+i}\right\|^2 &= \sum_{i,j=0}^{p} s_i s_j \langle X_{n+k+i}, X_{n+k+j}\rangle \\ &\leq \sum_{i,j=0}^{p} s_i s_j \langle X_{n+i}, X_{n+j}\rangle + \sum_{i,j=0}^{p} s_i s_j 2^{-l} = \left\|\sum_{i=0}^{p} s_i X_{n+i}\right\|^2 + 2^{-l},\end{aligned}$$

for all $K(l) \leq k \leq K(l)+m+g^M(m)$, since $n \leq n+i, n+j \leq n+p$. Together with the convexity of the square function (and the definition of Z) this implies (2).
Finally, the claim follows from (1) and (2) by the parallelogram identity:

$$\begin{aligned}&\left\|Z(l,n,p,m)-Z(l,n,p,\tilde{g}(m))\right\|^2 = \\ &\quad = 2\left\|Z(l,n,p,m)\right\|^2 + 2\left\|Z(l,n,p,\tilde{g}(m))\right\|^2 - \left\|Z(l,n,p,m)+Z(l,n,p,\tilde{g}(m))\right\|^2 \\ &\quad \leq 4(C(l,n,p)+2^{-l}) - 4(C(l,n,p)-2^{-l+1}) = 2^{-l+2}+2^{-l+3} \leq 2^{-l+4}.\end{aligned}$$

□

Recall Lemma 5.10 (Zs and As are close). *For any l and any g let $h := H(l,g^M)$ and n be a witness for Lemma 5.5, i.e.*

$$n \leq N(l,g^M)\ \wedge\ \forall i,j \in [n;n+h(n)]\ \left(i \leq j \to \|X_i\|^2 - \|X_j\|^2 \leq 2^{-l-1}\right). \tag{N}$$

Moreover let $f := F(l,g^M,n)$, p be a witness for Lemma 5.6, i.e.

$$p \leq P_0(l,f)\ \wedge\ (C(l,n,p) \leq C(l,n,f(p))+2^{-l}), \tag{P}$$

and set $m := M_0(l,n,p)$, $m' := m+g(m)$. Then we have that

$$\left\|A_{m+1}-Z(l,n,p,m)\right\| \leq \frac{1}{m+1}(2n+2p+2K(l))B+2^{-l}$$

and

$$\left\|A_{m'+1}-Z(l,n,p,m')\right\| \leq \frac{1}{m'+1}(2n+2p+2K(l))B+2^{-l}.$$

Proof. From the definition of Z we see that (note that $m, m' \geq p$):

$$(m+1)Z(l,n,p,m) - \sum_{i=n+p+K(l)}^{n+K(l)+m} X_i = \sum_{i=0}^{p-1} t_i X_{n+K(l)+i} + \sum_{i=l}^{p} r_i X_{n+K(l)+m+i},$$

for suitable $\underline{t}$ and $\underline{r}$ with $0 \leq t_i, r_i \leq 1$. Hence (note that $m, m' \geq K(l) + n + p$)

$$\begin{aligned}
(m+1)\big\|Z(l,n,p,m) - A_{m+1}\big\| &= \Bigg\| \sum_{k=K(l)}^{K(l)+m} \sum_{i=0}^{p} \widetilde{\underline{s}}_i X_{n+k+i} - \sum_{i=1}^{m+1} X_i \Bigg\| \\
&= \Bigg\| \sum_{i=0}^{p-1} t_i X_{n+K(l)+i} + \sum_{i=1}^{p} r_i X_{n+K(l)+m+i} + \sum_{i=n+p+K(l)}^{n+K(l)+m} X_i - \sum_{i=1}^{m+1} X_i \Bigg\| \\
&= \Bigg\| \sum_{i=0}^{p-1} t_i X_{n+K(l)+i} + \sum_{i=1}^{p} r_i X_{n+K(l)+m+i} + \sum_{i=m+2}^{n+K(l)+m} X_i - \sum_{i=1}^{n+p+K(l)-1} X_i \Bigg\| \\
&\leq \Bigg\| \sum_{i=0}^{p-1} t_i X_{n+K(l)+i} - \sum_{i=1}^{n+p+K(l)-1} X_i \Bigg\| + \Bigg\| \sum_{i=1}^{p} r_i X_{n+K(l)+m+i} \Bigg\| + \Bigg\| \sum_{i=m+2}^{n+K(l)+m} X_i \Bigg\| \\
&\leq (n+p+K(l)-1)B + pB + (n+K(l)-1)B \leq (2n+2p+2K(l))B.
\end{aligned}$$

Obviously, same holds for m'.

□

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