Referee report on the paper
A quantitative nonlinear strong ergodic theorem for Hilbert spaces
by Pavol Safarik

This paper represents a significant contribution to the program of "proof mining" in nonlinear analysis, which aims to extract effective information hidden in the proofs.

Applying methods coming from mathematical logic to results obtained by Wittmann [1], the author extracts effective rates of metastability (in the sense of Terence Tao) for nonlinear generalizations of the classical von Neumann mean ergodic theorem.

The main result of the paper is the following:

Theorem 1. *(Corollary 5.11 in the paper) Let (X_n) be a sequence in a Hilbert space and (δ_k) be a sequence of real numbers with $\lim_{k\to\infty} \delta_k = 0$ and assume that for all $n, m, k \in \mathbb{N}$*

$$\|X_{n+k} + X_{m+k}\|^2 \leq \|X_n + X_m\|^2 + \delta_k.$$

Define for all $n \in \mathbb{N}$, $A_n := \frac{1}{n}\sum_{i=1}^{n} X_i$. Then

$$\forall l \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \leq \Phi(l, g, K, B)\, \forall i, j \in [N, N+g(N)]\ (\|A_i - A_j\| \leq 2^{-l}),$$

where $K : \mathbb{N} \to \mathbb{N}$ is a rate of convergence of (δ_k) towards 0, $K^M : \mathbb{N} \to \mathbb{N}$ is defined by $K^M(n) = \max_{i\leq n+1} K(i)$ and $B = \max_{1\leq i\leq K^M(0)} \lceil \|X_i\| \rceil + 1$.

The rate of metastability $\Phi(l, g, K, B)$ is guaranteed by logical metatheorems and is extracted in the paper. As an immediate consequence the author gets a finitary version of the mean ergodic theorem for nonlinear mappings $T : X \to X$ satisfying $\|Tx + Ty\| \leq \|x + y\|$ for all $x, y \in X$.

Remarks.

1. The author has three sections (2.A general bound existence theorem, 3.Arithmetizing Wittmann's proof and 4.Obtaining a bound) explaining the logical methods used to get the effective bounds. While this

is probably of great help for the logic community, it could be very obscure for a paper published in an analysis journal as this one, as many notions from logic are and can not be explained here. So, I think that instead of helping the researchers in analysis to understand better the methods used by him, the effect will be the opposite. I advice the author to write a short (let us say 2 pages) logical discussion, where he could present the logical metatheorem that guarantees their results (as I understand, this is Theorem 2.4) and to give a hint on the methods he uses to get the bounds (the arithmetization of the convergence of a monotone bounded sequence and of the existence of the infimum of a bounded sequence)

2. In Section 5.Uniform bounds for Wittmann's ergodic theorem, the author gives the main results of the paper and states only the lemmas used to obtain them. The proofs of these lemmas are given in Section 6.Lemmas, where he recalls the statement of each lemma. The reason he does this is "better readability". At least in my case, the effect is again the opposite. I spent a lot of time going from one section to the other. I advice the author to have two sections:

   (a) one with the main results of the paper: Corollaries 5.11, 5.12 and 5.13. By the way, I think that the main result of the paper is Corollary 5.11 (the author says this, in fact, on the first line of Section 2), as this is the usual formulation of metastability. Hence, Corollary 5.11 should be a theorem, while the actual Theorem 5.1 is just an intermediate step.

   (b) another one with the proof of Corollary 5.11, where he should give the proofs of each lemma immediately after it is stated.

3. The author uses heavily the λ-notations for functions. This is very unusual in an analysis journal, so I advice him to avoid using it.

4. The proofs and the statements of the main results and lemmas are sometimes written without sufficient care:

   (a) In many cases, the author uses notations from logic: $F(l, g, n) =_1 \ldots$ (on page 12, line 3 from below), $\leq_{\mathbb{N}}, \leq_{\mathbb{Q}}$ more times in Definition 5.2, $\forall n^0, k^0$ (on page 18), etc...

(b) In the statements of the main results and lemmas, the author should write the domains of the variables. For example, in Corollary 5.11, one should write $\forall l \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N}$, and so on.

(c) The notion that M **majorizes** itself (see Lemma 5.4) is not defined in the paper.

Introduction

1. page 2, line 7 from below: I think it should be "averaging sequence" instead of "averaging series".

2. page 2, lines 5-7 from below: The notation $A_n^\alpha x$ for the Halpern iteration is a little strange. I think simply (x_n) would be good enough. The author should also say that (α_n) is a sequence in $[0,1]$. Furthermore, the definition of the iteration seems inaccurate. The Halpern iteration (x_n) is defined by

$$x_0 = x, \quad x_{n+1} = \alpha_{n+1}x + (1-\alpha_{n+1})Tx_n$$

   and Halpern defined it for $x = 0$. This is not the same thing with what the author writes.

3. page 2, line 3-4 from below: The Halpern iteration coincides with the Cesaro mean for linear maps and $\alpha_{\mathbf{n}} = \frac{\mathbf{1}}{\mathbf{n+1}}$.

4. page 2, footnote: add "Banach" between "uniformly convex" and "spaces".

5. page 3: The author writes on lines 4-5 "Wittmann's nonlinear strong ergodic theorems", after a few lines "from Wittmann's strong ergodic theorem". It is not clear to what theorem(s) he refers, so I advice him to give the statement of the theorem in the introduction. Furthermore, in the equation presenting the metastable version, "$x \in S$" should be replaced with "$x \in C$".

6. page 3: I think that there are some inaccuracies in the lines defining the bound $M(l, g, b)$:

   (a) in the definition of P, N should have 3 arguments.

   (b) in the definition of H, P_0 should have 3 arguments.

(c) Maybe it is a good idea in the definition of P_0 to specify that $f : \mathbb{N} \to \mathbb{N}$, to make things clear.

7. page 3, last line: the author defines $g^M(n) = \max_{i \le n+1} g(i)$. Is it so, or should it be $g^M(n) = \max_{i \le n} g(i)$?

8. page 4, line 2: S should be C.

Theorem 5.1

1. in the statement of Theorem 5.1, one should write "Let $K : \mathbb{N} \to \mathbb{N}$ be a function". In fact K is a rate of convergence of (δ_k) towards 0, so even stating this in the theorem could be helpful.

2. The author uses the notations M', M for the bounds and in the same time for g^M. I think this is unfortunate.

3. The rate of metastability depends on l, g, K and B, so I advise the author to express this dependency by writing $M'(l, g, K, B)$ in the metastability statement.

4. It is very difficult to read the definition of M'. Maybe a good idea would be to avoid using l, g as arguments for the different functionals. Furthermore, it would be well to avoid defining too many functionals. Of course, when this is possible.

5. The definition of K^M should be recalled here.

Definition 5.2, Lemmas 5.3, 5.6 and their proofs

The definition of C is very difficult to understand and contains some typos.

1. I see that $\underline{s} = s_0, \ldots, s_{p-1}$ is a sequence of length p that is also an element of the set $S_{p,l}$. What is the reason for taking $\tilde{\underline{s}}$ in the definition of C'?

2. In the definition of $\tilde{\underline{s}}(n)$, why does the author takes m instead of p?

3. In Definition 5.2, X is an argument of C', while in Lemma 5.3, one has f as an argument of C' instead of X. What is this f?

4. I do not understand the statement of Lemma 5.3 and its proof. Since $C(l,n,p)$ is the minimal value, we have obviously that $C(l,n,p) \leq C'(\underline{s},l,n,p)$ for all $\underline{s} \in S_{p,l}$. Furthermore, in the proof the author writes everywhere $\sum_{i=0}^{p} \tilde{s}_i, s'_i$, while the $\tilde{\underline{s}}, \tilde{\underline{s'}}$, used in the definition of C' do not appear.

5. In the proof of Lemma 5.6, the author writes that the sequence $a_i := C(i,n,l)$ is monotone. Shouldn't we have $a_i := C(n,l,i)$ instead? Is the sequence nonincreasing or nondecreasing and why?

6. The author should state Proposition 2.26 in [18] in the paper and explain how it is used in Lemma 5.6.

Lemma 5.5 and its proof

1. Lemma 5.5 is one of the most important in the paper and it is stated using terms coming from logic: "sentence", "witnessed". I advise the author to avoid using these terms and state it in the usual mathematical way.

2. Isn't it true that we get even $\forall ...(|\|X_i\|^2 - \|X_j\|^2| \leq 2^{-l})$ instead of the weaker $\forall ...(i \leq j \to \|X_i\|^2 - \|X_j\|^2 \leq 2^{-l})$?

The proof of the lemma is very unclear and contains typos:

1. The author tries to explain a lot in words the idea of the proof, but the result is that the proof is confused. I think that a straight mathematical proof with all the details would be better.

2. Replace everywhere $(\|X_n\|)$ with $(\|X_n\|^2)$, as this is the sequence used in the lemma.

3. What is the "n.c.i."?

4. The author again refers to [18] without stating the result used in the proof. He says that R is defined as in [18]. Is this the same R with the one defined in the statement of Theorem 5.1?

5. The discussion on page 18, line 4-5 from below (“it does not matter, whether we consider ...”) can be avoided by taking everywhere $(\|X_n\|^2)$, as I suggested above.

6. The definition of N_0 in the proof differs from the definition of N_0 in the statement of Theorem 5.1.

Corollary 5.12

Please give more details in the proof.

Typos

There are a number of typos:

1. page 1, line 3 from below: delete “.” after $T^i x$.

2. page 2, line 11: write “in norm” immediately after “converges“

3. page 2, first line after (W): replace ”satifies“ with ”satisfies”; replace “assymptotic” with ”asymptotic“.

4. page 2, footnote: replace ”arxive“ with ”http://arxiv.org/“.

5. page 3, line 12 from below: replace α with (α_n).

Summary As I said above, I think that the paper is a significant contribution to proof mining in nonlinear analysis. The proofs are involved, show a high level of originality and apply to operators which do not even need to be continuous. My final conclusion is that the paper should be accepted after a major revision, taking into account the above remarks.

References

[1] R. Wittmann, Mean ergodic theorems for nonlinear operators, Proc. Amer. Math. Soc., 108 (1990), 781-788.