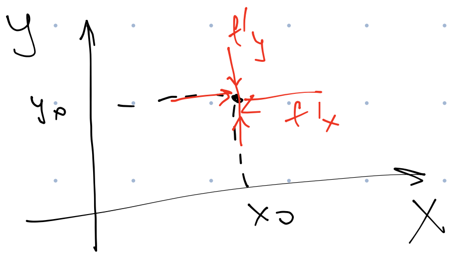
14.03

Частные производные



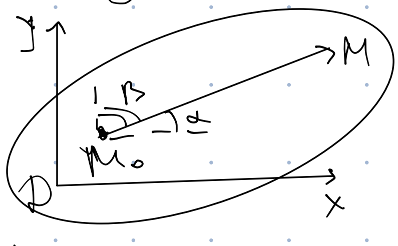
$$\Delta_x f = f(x_0+\Delta x, y_0) - f(x_0, y_0)$$

$$f'_x = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta_x f}{\Delta x}$$

$$\Delta_y f = f(x_0, y_0+\Delta y) - f(x_0, y_0)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta_y f}{\Delta y}$$

Производная по направлению



$\overrightarrow{M_0M}$ задаёт направление S

$$\frac{\partial f}{\partial S} = \frac{df}{dx}\cos\alpha + \frac{df}{dy}\cos\beta$$

Задача 1

Найдите все частные производные первого порядка

$\frac{df(x)}{dx}$ – реальная дробь $\qquad \frac{\partial f}{\partial y}$ – обозн.

а) $f(x,y) = \frac{x-y}{x+y}$

б) $f(x,y) = x^5 \sin(3x - 5y^2)$

в) $f(x,y) = e^{\sqrt{\sin^2 x + 2\cos 3x}}$

а) $f(x,y) = \frac{x-y}{x+y}$

$f'_x = \frac{(x-y)'(x+y) - (x-y)(x+y)'_x}{(x+y)^2} = \frac{x+y-x+y}{(x+y)^2} = \frac{2y}{(x+y)^2}$

$f'_y = \frac{(x-y)'(x+y) - (x-y)(x+y)'}{(x+y)^2} = \frac{-x-y-x+y}{(x+y)^2} = \frac{-2x}{(x+y)^2}$

б) $f'_x = \underbrace{(x^5)'}_{5x^4} \cdot \sin(3x-5y^2) + x^5 \cdot \underbrace{(\sin(3x-5y^2))'}_{3\cos(3x-5y^2)} =$

$= \dots$

$f'_y = -5x^5 \cos(3x-5y^2)$

в) $f'_x = e^{\sqrt{\sin^2 x + 2\cos 3x}} \cdot \frac{2\sin x \cos x - 2\sin 3x \cdot 3}{2\sqrt{\sin^2 x + 2\cos 3x}}$?

2) $u(x,y,z) = x^{y^z}$

$\boxed{(u(v))' = u'(v) \cdot v'}$

пр сл. ф.

$u'_x = y^z \cdot x^{y^z - 1}$

$u'_y = x^{y^z} \ln x \cdot z \cdot y^{z-1}$

$u'_z = x^{y^z} \cdot \ln x \cdot (y^z)' = x^{y^z} \cdot \ln x \cdot y^z \cdot \ln z$

5) $u(x,y,z) = \sqrt{x^2+2y^2+3z^2}$

$f'_x = (x^2+2y^2+3z^2)^{\frac{1}{2}\,'} =$

$= \frac{1}{2} \cdot (x^2+2y^2+3z^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+2y^2+3z}}$

Задача 2

Найд. все част. пр-е перв. пор. в (.) M_0

а) $z = \dfrac{xy}{x+y}$; $M_0(4;-5)$

$$z'_x = \frac{(xy)'(x+y) - xy\overset{=0}{(x+y)'}}{(x+y)^2} = \frac{y(x+y)-xy}{(x+y)^2} = \frac{yx+y^2-xy}{(x+y)^2} =$$

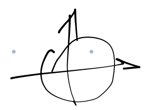
$$= \frac{y^2}{(x+y)^2} \qquad z'_x\Big| = \frac{25}{(-1)^2} = 25$$

$$z'_y = \frac{x(x+y)-xy\cdot 1}{(x+y)^2} = \frac{x^2}{(x+y)^2} \qquad z'_y\Big| = \frac{16}{(4-5)^2} = 16$$

б) $u = z^4 \cdot \dfrac{\cos xy}{1-x\,\mathrm{tg}\,y}$; $M_0(1;0;-1)$

$$u_x = z^4\left(\frac{(\cos xy)'(1-x\,\mathrm{tg}\,y) - \cos xy\,(1-x\,\mathrm{tg}\,y)'}{(1-x\,\mathrm{tg}\,y)^2}\right) =$$

$$= z^4\,\frac{-y\sin xy\,(1-x\,\mathrm{tg}\,y) + \cos xy\,(\mathrm{tg}\,y)}{\dots}$$



$$u'_x\Big| = 0$$

$$u'_y = z^4\cdot\left(\frac{-x\sin xy\,(1-x\,\mathrm{tg}\,y) + \cos xy\left(\frac{x}{\cos^2 y}\right)}{\dots}\right)$$

$$u'_y\Big|_{M_0} = \frac{-1\cdot\overset{0}{\sin 0}\,(1-\mathrm{tg}\,0) + \overset{1}{\cos 0}\cdot\frac{1}{\cos^2 0}}{(1-\underset{0}{\mathrm{tg}\,0})^2} = \qquad \sin 0 = 0,\ \cos 0 = 1$$

$$= \frac{1}{1} = 1$$

$u'_z = 4z^3 \cdot \frac{\cos xy}{1-\cancel{x}\operatorname{tg} y}$

$u'_z \Big|_{M_0} = -4$

Задача 3

Показать, что функция $u = \ln(e^x + e^y + e^z)$ уд. ур.-ю: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 1$

$u'_x = \ln(e^x + \underbrace{e^y + e^z}_{\text{-const}})' = \frac{e^x \cancel{\ln(e^x+}}{e^x + e^y + e^z}$

$u'_y = \frac{e^y}{e^x \ldots}$

$u'_z = \frac{e^z}{e^x + \ldots}$

$\frac{e^x}{t} + \frac{e^y}{t} + \frac{e^z}{t} = 1$

$\frac{t}{t} = 1$

Дифференциал

Утв: $z = z(x, y)$ — дифф-ма в (.) $M(x, y) \overset{def}{\Longleftrightarrow} \Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + O(\rho)$ $\quad O(\rho)$ — б.м. б. высш пор чем $\rho = \sqrt{\Delta x^2 + \Delta y^2}$

dz — лин часть отн Δx и Δy приращ.-ф.

$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

Задача 5

Найти дифференциал функции

а) $z = e^{xy^3}$

$\cancel{\frac{dz}{dy} = e^{xy^3} dx =}$

$$z'_x = (e^{xy^3})' = y^3 e^{xy^3}$$

$$z'_y = x \cancel{e^y} e^{xy^3} \cdot 3xy^2$$

$$dz = \left(e^{xy^3} y^3\right) dx + \left(e^{xy^3} \cdot 3xy^2\right) dy$$

б)

$$u'_x = \frac{(x-z)'(z+y) - (x-z)(z+y)'^{=0}}{(\ldots)^2} =$$

$$= \frac{x\cdot z + xy - \cancel{x(z+y)}}{(z+y)^2} = \frac{1}{z+y}$$

в) $u'_y = -\frac{x-z}{(z+y)^2}$ $\qquad du = \frac{1}{z+y}dx + \frac{z-x}{(z+y)^2}dy + \frac{y+x}{(y+z)^2}dz$

$$u'_z = \frac{\cancel{-z}-y - x + \cancel{z}}{(y+z)^2} \qquad du\Big|_{M_0} = \frac{1}{5}dx + \frac{2}{25}dy + \frac{-3}{25}dz$$

Есть теор Шварца о смеш произв.

$$\begin{tikzcd}
 & & z(x,y) \arrow[dl] \arrow[dr] & & \\
 & z'_x \arrow[dl] \arrow[d] \arrow[dr] & & z'_y \arrow[dl] \arrow[d] \arrow[dr] & \\
\underline{z''_{xx}} & \underline{z''_{xy}} & z''_{yx} \quad z''_{yx} & z''_{xy} & \underline{z''_{yy}}
\end{tikzcd}$$

Задача 6

Найти все производные второго порядка функции

а) $z = x^2 \cdot \ln(xy^2)$

$z'_x =$