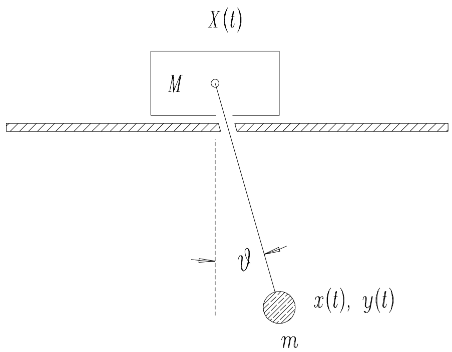
PHYS 321 Homework Assignment #7: Solutions

1. The system shown below consists of a block of mass M that can slide without friction, in the x-direction along a horizontal air track. A pendulum hangs from a pivot attached to the block, in such a way that it swings in the $x-y$ plane ($y =$ vertical). The pendulum consists of a massless rod of (fixed) length r and a bob of mass m.



Write the Lagrangian of the system in terms of the coordinates $\theta(t)$ (the angle of the rod relative to the vertical) and $X(t)$ (the position of the block along the air track). Derive the equations of motion. Find the constants of the motion (there are two of them.) Solve the equations of motion, assuming θ is "small", in the usual sense, and thereby determine the frequencies of oscillation of the system. Are the motions of block and pendulum independent?

Solution:
The partial solution is found in the **Solutions to the Midterm**, Prob. 3. The Lagrangian is

$$L = \frac{1}{2}(M+m)\dot{X}^2 + mr\cos\vartheta\dot{X}\dot{\vartheta} + \frac{1}{2}mr^2\dot{\vartheta}^2 + mgr\cos\vartheta\,.$$

The Euler-Lagange equations are

$$\begin{aligned}
\frac{\partial L}{\partial \dot{X}} &= (M+m)\dot{X} + mr\cos\vartheta\dot{\vartheta} = \text{constant} \\
0 &= \frac{d}{dt}\left(mr\cos\vartheta\dot{X}\right) + mr^2\ddot{\vartheta} + mgr\sin\vartheta - mr\sin\vartheta\dot{\vartheta}\dot{X}\,.
\end{aligned}$$

We also have a second constant of the motion, the energy,

$$H = \frac{1}{2}(M+m)\dot{X}^2 + mr\cos\vartheta\dot{X}\dot{\vartheta} + \frac{1}{2}mr^2\dot{\vartheta}^2 - mgr\cos\vartheta = \text{constant}\,.$$

For small displacements we have

$$L \approx \frac{1}{2}(M+m)\dot{X}^2 + mr\dot{X}\dot{\vartheta} + \frac{1}{2}mr^2\dot{\vartheta}^2 + mgr\left(1 - \frac{\vartheta^2}{2}\right)$$

where we have kept no terms more than quadratic in either coordinates or velocities. The Euler-Lagrange equations then become

$$\begin{aligned} \frac{\partial L}{\partial \dot{X}} &= (M+m)\dot{X} + mr\dot{\vartheta} = \text{constant} \\ 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vartheta}}\right) - \frac{\partial L}{\partial \vartheta} = mr\ddot{X} + mr^2\ddot{\vartheta} + mgr\vartheta\,. \end{aligned}$$

Since

$$\ddot{X} + \frac{mr}{M+m}\ddot{\vartheta} = 0$$

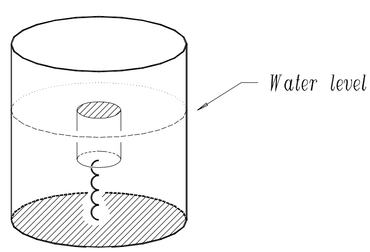
we have

$$mr^2\ddot{\vartheta}\left(1 - \frac{m}{M+m}\right) + mgr\vartheta = 0$$

—that is, the equation reduces to a simple harmonic oscillator of frequency

$$\omega = \sqrt{\frac{g}{r}\left(\frac{M+m}{M}\right)}\,.$$

2. A beaker of water has a cork in it, held partially submerged by a stretched spring attached to the bottom, as shown below.



If the beaker is dropped, what happens to the cork, relative to the bottom of the beaker?

a) It does not move. (The distance to the bottom remains unchanged.)

b) It rises higher relative to the surface of the water. (The distance to the bottom increases.)

c) It sinks. (The distance to the bottom decreases.)

Explain your answer. A letter indicating an outcome is an insufficient answer, be it never so correct.

Solution:
The answer is "c) It sinks." The reason is that in free fall there is no weight, hence no buoyancy. The spring tension was in equilibrium with the buoyancy but in free fall it is the only force acting on the cork, so it pulls the cork toward the bottom.

3. A beaker of water at the surface of the Earth spins about its axis of symmetry (locally vertical axis!) with angular velocity ω. Find the effective potential experienced by a test particle in the beaker's rest frame (this is *not* an inertial frame), and hence find the equation of the water's surface.

Solution:
The equation of motion of a mass point in the accelerated frame is

$$\frac{d^2\vec{s}}{dt^2} = -g\hat{z}+2\frac{d\vec{s}}{dt}\times\vec{\Omega}+\vec{\Omega}\times\left(\vec{s}\times\vec{\Omega}\right)$$

where $\vec{\Omega} = \Omega\hat{z}$. Dotting this with the velocity and integrating we find

$$\frac{1}{2}\left(\frac{d\vec{s}}{dt}\right)^2+gz-\frac{1}{2}\Omega^2 b^2, \quad b^2 = x^2+y^2$$

so we identify the potential as

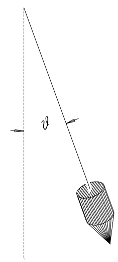
$$U_{eff} = gz - \frac{1}{2}\Omega^2 b^2$$

per unit mass. The equipotential surfaces therefore have the form

$$z = z_0 + \left(\frac{\Omega^2}{2g}\right) b^2\,,$$

a parabola that is concave upwards.

4. A suspended plumb bob is used to indicate the local direction of gravity, as shown below.



Imagine two situations:

a) you are in a train going up a $30°$ grade at the Earth's surface (at constant speed);

b) the train is moving on the level but accelerating at $g/\sqrt{3}$ along the track.

Either situation will cause the bob to deviate from the vertical (relative to the car) by $30°$. Devise a simple experiment using the plumb bob that can discriminate between the two cases, without looking out of the train windows.

Solution:
To distinguish the two, set the bob swinging. In Case a), the acceleration is still g, so the period will be $\sqrt{\ell/g}$. In Case b), the net acceleration is $g\sqrt{1+1/3} = 2g/\sqrt{3}$. The period will then be shorter by the factor $\sqrt{\sqrt{3}/2}$.

5. Problem 7-1 on p. 278 in Barger & Olsson. (My paraphasing.)

> A particle of mass m moves in a smooth straight horizontal tube that rotates with angular velocity ω about a vertical axis that intersects the tube. Set up the equations of motion in cylindrical coordinates and derive an expression for the distance of the particle from the axis of rotation as a function of time. If the particle is at $r = r_0$ at $t = 0$, what initial velocity v_0 must it have in order that it will be very close to the rotation axis after a very long time?

Solution:
The coordinates of the particle are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$$

The Lagrangian therefore becomes

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right)$$

where $\dot{z} = 0$ and $\dot{theta} = \omega$ because the particle is confined to the rotating tube. We can set the potential energy (Earth's gravitational potential) to a constant because the tube is horizontal. Therefore the equation of motion is

$$\frac{\partial L}{\partial \rho} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) = 0$$

i.e.

$$\rho\omega^2 = \ddot{\rho}\,.$$

Integrating once using the integrating factor $\dot{\rho}$ we find

$$\frac{1}{2}\dot{\rho}^2 - \frac{1}{2}\rho^2\omega^2 = \frac{1}{2}v_0^2 - \frac{1}{2}\rho_0^2\omega^2 = \text{constant}\,.$$

This can be solved for $\dot{\rho}$, giving

$$\dot{\rho} = \pm\sqrt{v_0^2 + \omega^2\left(\rho^2 - \rho_0^2\right)}\,.$$

We want the negative sign because the radial velocity must be negative for the particle to approach the axis of rotation against the centrifugal force. This equation can be separated:

$$\frac{-d\rho}{\sqrt{v_0^2 + \omega^2\left(\rho^2 - \rho_0^2\right)}} = dt\,,$$

and the result expressed as an integral:

$$t = \int_{\rho_{\min}}^{\rho_0} \frac{d\rho}{\sqrt{v_0^2 + \omega^2\left(\rho^2 - \rho_0^2\right)}} = \frac{1}{\omega}\left[\sinh^{-1}\left(\frac{\rho_0}{\rho_1}\right) - \sinh^{-1}\left(\frac{\rho_{\min}}{\rho_1}\right)\right]$$

where

$$\rho_1 = \sqrt{\frac{v_0^2}{\omega^2} - \rho_0^2}\,.$$

In other words, we have two cases:

1) if $\rho_1 > 0$ and $\rho_{\rm min} \to 0$, t remains finite, and is just $\omega^{-1}\sinh^{-1}(\rho_0/\rho_1)$.
2) if $\rho_{\rm min} = \rho_1$ and $\rho_1 \to 0$ then $t \to \infty$ but only as the logarithm of ρ_0/ρ_1.

6. Problem 7-9 on p. 279. (My paraphasing.)

> A particle moves with velocity v on a smooth (frictionless!!) horizontal plane at some co-latitude (angle from the North Pole) θ. Show that the particle will move in a circle owing to the Earth's axial rotation. Find the radius of the circle.

(Imagine the object is a hocky puck on very smooth ice, on the surface of a very large Canadian lake in midwinter.)

Solution:
Following the discussion in B&O, p. 237ff, we choose the point in the moving frame (the ice rink) to be $\vec{r}\,' = \vec{R} + \vec{s}$, where $\vec{R}$ is the position of the origin of coordinates in the moving frame relative to the center of the Earth. That is, we refer all points to the Earth's center. Since $\vec{R}$ is constant in the moving frame, the equation of motion is

$$\begin{aligned}\frac{d^2\vec{s}}{dt^2} &= \frac{1}{m}\vec{f} - g\hat{R} + 2\frac{d\vec{s}}{dt}\times\vec{\Omega} + \vec{\Omega}\times\left[\left(\vec{R}+\vec{s}\right)\times\vec{\Omega}\right] \\ &\approx \frac{1}{m}\vec{f} - g\hat{R} + 2\frac{d\vec{s}}{dt}\times\vec{\Omega} + \vec{\Omega}\times\left(\vec{R}\times\vec{\Omega}\right),\end{aligned}$$

where we have used the fact that $s \ll R$. Now the centrifugal term $\vec{\Omega}\times\left(\vec{R}\times\vec{\Omega}\right)$ is just a (small) constant acceleration in the moving frame. Taking θ as the co-latitude, the centrifugal force has both vertical and horizontal components relative to the ice rink:

$$\vec{\Omega}\times\left(\vec{R}\times\vec{\Omega}\right) = R\Omega^2\left[\hat{R}\sin^2\theta - \left(\hat{\Omega} - \hat{R}\cos\theta\right)\right].$$

We expect the vertical component (in the $\hat{R}$ direction) to slightly reduce the weight of the puck, but that is compensated by the reaction force $\vec{f}$. The horizontal component will cause a slow southward drift, which we disregard. We are left then with the approximate equation of motion in the ice frame

$$\frac{d^2\vec{s}}{dt^2} \approx 2\frac{d\vec{s}}{dt} \times \vec{\Omega}$$

which we see is the same as the problem of a charged particle in a magnetic field. As we saw, that problem led to circular orbits. The equations of motion are

$$\begin{aligned}
\ddot{z} &= \frac{1}{m}f - g + R\Omega^2 \sin^2\theta - 2\Omega\dot{y}\sin\theta = 0 \\
\ddot{x} &= 2\Omega\dot{y}\cos\theta \\
\ddot{y} &= -2\Omega\dot{x}\cos\theta - 2\Omega\dot{z}\sin\theta = -2\Omega\dot{x}\cos\theta
\end{aligned}$$

Thus we find

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} v\cos(2\Omega t\cos\theta) \\ -v\sin(2\Omega t\cos\theta) \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{v}{2\Omega\cos\theta}\begin{pmatrix} \sin(2\Omega t\cos\theta) \\ \cos(2\Omega t\cos\theta) \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The radius of the circle is therefore

$$\rho = \frac{v}{2\Omega\cos\theta}\,.$$