10

Asymptotes

10·1 Definition

A straight line at a fixed distance from the origin, is said to be an *asymptote to* an infinite branch of a curve, if the perpendicular distance of a point P on the curve from this straight line approaches zero, as the point P moves to infinity along the curve.

10·2 Determination of Asymptotes

The equation of a straight line not parallel to y-axis is of the form,

$$y=mx+c \qquad \ldots(1)$$

Excluding at present the case of asymptotes parallel to y-axis, it is obvious from (1) that as x approaches infinity, m and c must both tend to finite limits for asymptotes to exist. Let p be the perpendicular distance of any point $P(x, y)$ on an infinite branch of a given curve from the line (1), then

$$p=\frac{y-mx-c}{\sqrt{1+m^2}}$$

If line (1) is to be an asymptote to a given curve, than as $x\to\infty$, $p\to 0$

$$\therefore \qquad \operatorname*{Lt}_{x\to\infty}(y-mx-c)=0$$

or

$$\operatorname*{Lt}_{x\to\infty}(y-mx)=c \qquad \ldots(2)$$

Also from (1), we have

$$\frac{y}{x}=m+\frac{c}{x}$$

Taking limits on both sides as $x\to\infty$, we have

$$\operatorname*{Lt}_{x\to\infty}\left(\frac{y}{x}\right)=\operatorname*{Lt}_{x\to\infty}\left(m+\frac{c}{x}\right)=m$$

$$\therefore \qquad m=\operatorname*{Lt}_{x\to\infty}\left(\frac{y}{x}\right) \qquad \ldots(3)$$

Thus from (3) and (2), we get the values of m and c and hence the equation of the asymptote.

Note. A given curve may have more than one infinite branches then it is possible that each branch may have separate asymptotes. Hence a given curve may have more than one asymptote.

Example 1. *Find the asymptotes of the curve*

$$x^3+y^3=3ax^2.$$

Sol. Here the equation of the given curve is

$$x^3+y^3=3ax^2 \qquad ..(1)$$

In order to determine the asymptotes, we have to evaluate m and c given by $\underset{x\to\infty}{\text{Lt}}\ (y/x)$ and $\underset{x\to\infty}{\text{Lt}}\ (y-mx)$ respectively, then $y=mx+c$ will be the asymptote.

Dividing (1) by x^3, we have

$$1+\left(\frac{y}{x}\right)^3-\frac{3a}{x}=0$$

Taking limits as $x\to\infty$, we have

$$1+m^3=0 \qquad \left[\because \underset{x\to\infty}{\text{Lt}}\ (y/x)=m\right.$$

$$=(1+m)(1+m^2-m)=0$$

$\therefore$ $m=-1$, as the roots of $1+m^2-m=0$ are not real.

Now $c=\underset{x\to\infty}{\text{Lt}}\ (y-mx)$

$$=\underset{x\to\infty}{\text{Lt}}\ (y+x) \qquad [\because m=-1]$$

Let $y+x=K$, such that as $x\to\infty$, $K\to c$

Putting $y=(K-x)$ in (1), we have

$$x^3+(K-x)^3=3ax^2$$

or $$3(K-a)x^2-3K^2x+K^3=0$$

Dividing throughout by x^2, we get

$$3(K-a)-\frac{3K^2}{x}+\frac{K^3}{x^2}=0$$

Taking limits as $x\to\infty$ and $K\to c$

$$3(c-a)=0$$

or $$c=a$$

The asymptote to the curve is given by

$$y=mx+c$$

i.e. $$y=-x+a$$

or $$y+x=a$$

is the required asymptote.

Note. The method used to determine asymptotes in the above example is not convenient. The following methods are much easier and quicker to obtain the asymptotes.

10·3. The Asymptotes of the general Algebraic Curve

Let the equation to the curve be

$$(a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\dots+a_ny^n)+(b_0x^{n-1}+b_1x^{n-2}y$$
$$+b_2x^{n-3}y^2+\dots+b_ny^{n-1})+(c_0x^{n-2}+c_1x^{n-3}y$$
$$+c_2x^{n-4}y^2+\dots+c_ny^{n-2})+\dots\dots=0 \qquad \dots(1)$$

Further let $y=mx+c$ be an asymptote to the curve (1)

Since each expression in the brackets is homogeneous, (1) may be written as

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\dots=0 \qquad \dots(2)$$

where $\phi_r\left(\frac{y}{x}\right)$ is an expression of rth degree in $\frac{y}{x}$

Dividing (2) by x^n, we get

$$\phi_n\left(\frac{y}{x}\right)+\frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right)+\dots=0 \qquad \dots(3)$$

Now taking limits as $x\to\infty$

and $\qquad \underset{x\to\infty}{\text{Lt}}\left(\frac{y}{x}\right)=m$, we have

$$\phi_n(m)=0, \qquad \dots(4)$$

which determines the slopes of the asymptotes.

Substituting $y=mx+c$ in (2), we get

$$x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)$$
$$+x^{n-2}\phi_{n-2}\left(m+\frac{c}{x}\right)+\dots=0$$

Expanding $\phi_n\left(m+\frac{c}{x}\right)$, $\phi_{n-1}\left(m+\frac{c}{x}\right)$ etc. by Taylor's theorem, we have

$$x^n\left[\phi_n(m)+\frac{c}{x}\phi_n'(m)+\frac{1}{2!}\frac{c^2}{x^2}\phi_n''(m)+\frac{1}{3!}\frac{c^3}{x^3}\phi_n'''(m)+\dots\right.$$
$$+x^{n-1}\left[\phi_{n-1}(m)+\frac{c}{x}\phi'_{n-1}(m)+\frac{c^2}{2!\,x^2}\phi''_{n-1}(m)+\dots\right]$$
$$+x^{n-2}\left[\phi_{n-2}(m)+\frac{c}{x}\phi'_{n-2}(m)+\frac{c^2}{2!\,x^2}\phi''_{n-2}(m)+\dots\right]$$
$$+\dots=0$$

Arranging the terms in descending powers of x,

$$x^n\phi_n(m)+x^{n-1}[c\phi_n'(m)+\phi_{(n-1)}(m)]$$
$$+x^{n-2}\left[\frac{c^2}{2!}\phi_n''(m)+c\phi'_{(n-1)}m+\phi_{(n-2)}(m)\right]$$

$$+x^{n-3}\left[\frac{c^3}{3\,!}\phi_n'''(m)+\frac{c^2}{2\,!}\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)\right]+\ldots=0$$

Putting $\phi_n(m)=0$ by (4) and dividing by x^{n-1}, we get

$$[c\phi_n'(m)+\phi_{(n-1)}m]+\frac{1}{x}\left[\frac{c^2}{2\,!}\phi_n''(m)+c\phi'_{(n-1)}(m)+\phi_{n-2}(m)\right]$$
$$+\frac{1}{x^2}\left[\frac{c^3}{3\,!}\phi_n'''(m)+\frac{c^2}{2\,!}\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)\right]$$
$$+\ldots\ldots=0. \qquad \ldots(5)$$

Taking limits as $x\to\infty$, we have

$$c\phi'_n(m)+\phi_{n-1}(m)=0$$

or

$$c=-\frac{\phi_{n-1}(m)}{\phi_n'(m)} \text{ [provided } \phi'_n(m)\neq 0] \qquad \ldots(6)$$

Now equation (4) is of the nth degree in m, giving n values of m say $m_1, m_2, \ldots, m_n$. The corresponding values of c say $c_1, c_2 \ldots, c_n$ are given by the equation (6). Hence the asymptotes are

$$y=m_1x+c_1,$$
$$y=m_2x+c_2, \text{ etc.}$$

10·4. Parallel Asymptotes

In case $\phi_n(m)=0$ has two equal roots say $m_1=m_2$, then $\phi'_n(m_1)$ and $\phi_{n-1}(m_1)$ both become zero.

Now
$$c=-\frac{\phi_{n-1}(m_1)}{\phi_n'(m_1)}.$$

Thus c takes the indeterminate form 0/0.

In this case c is obtained from equation (5) of the Art. 10·3 as

$$\frac{c^2}{2\,!}\phi_n''(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0 \qquad [\phi'_n(m)=\phi_{n-1}(m)=0]$$

Thus we obtain two values of c say c_1, c_2 corresponding to $m=m_1$ (a repeated root).

Hence the asymptotes are $y=m_1x+c_1$. $y=m_1x+c_2$, *i.e.* parallel asymptotes.

Similarly in case there are three parallel asymptotes, c is obtained by the equation

$$\frac{c^3}{3\,!}\phi_n'''(m)+\frac{c^2}{2\,!}\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)=0.$$

This being cubic in c will give three values of c, corresponding to three repeated values of m.

10·5. Working Rule

(1) Substitute $y=mx+c$ in the equation of the curve.
(2) Equate the coefficients of two highest powers of x to zero.
(3) These give m and c and hence the asymptotes.

Example 1. *Find the asymptotes of the curve,*

$$x^3+2x^2y-xy^2-2y^3+xy-y^2-1=0.$$

Sol. Putting $y=mx+c$ in the given equation, we have

$$x^3+2x^2(mx+c)-x(mx+c)^2-2(mx+c)^3+x(mx+c)-(mx+c)^2-1=0$$

or
$$(1+2m-m^2-2m^3)x^3+(2c-2mc-6m^2c+m-m^2)x^2+\ldots=0$$

Equating co-efficients of x^3 and x^2 to zero, we get

$$1+2m-m^2-2m^3=0=(1+2m)(m+1)(m-1) \qquad \ldots(1)$$

and
$$2c-2mc-6m^2c+m-m^2=0$$

or
$$c=\frac{m^2-m}{2-2m-6m^2} \qquad \ldots(2)$$

From (1), we have $m=-\frac{1}{2},\ -1,\ 1$

From (2), we get

(*i*) when $m=-\frac{1}{2}$, $c=\frac{1}{2}$,
(*ii*) when $m=-1$, $c=-1$, and
(*iii*) when $m=1$, $c=0$.

Hence asymptotes, are

$$y=-\frac{1}{2}x+\frac{1}{2},$$

$$y=-x-1 \quad \text{and} \quad y=x$$

or
$$2y+x=1,$$

$$y+x+1=0 \quad \text{and} \quad y=x.$$

10·6. Shorter Method

The polynomials $\phi_n(m)$, $\phi_{n-1}(m)$, $\phi_{n-2}(m)$ etc. can be easily obtained by putting $x=1$ and $y=m$ in the nth degree, $(n-1)$th degree, $(n-2)$th degree, etc. terms respectively. The following examples show that this method gives asymptotes much quicker.

Example 1. *Find the asymptotes of the curve*

$$y^3-x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2x+2y+1=0$$

[*A.M.I.E. 1974, 75*]

Sol. Putting $x=1$ and $y=m$ in the third and second degree terms we get

$$\phi_3(m)=m^3-m-2m^2+2$$

and
$$\phi_2(m)=-7m+3m^2+2$$

The slopes of asymptotes are given by

$$\phi_3(m)=m^3-m-2m^2+2=0$$

$\therefore$

or $$(m+1)(m-1)(m-2)=0$$

Now $$m=-1, 1, 2.$$

Again $$\phi_3'(m)=3m^2-4m-1$$

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}$$

or $$c=-\frac{-7m+3m^2+2}{3m^2-4m-1}$$

(i) when $m=-1$, $c=-2$,

(ii) when $m=1$, $c=-1$ and

(iii) when $m=2$, $c=0$

Hence the asymptotes are

$$y=-x-2, \qquad y=x-1 \quad \text{and} \quad y=2x$$

or $y+x+2=0$, $\quad y+1=x \quad$ and $\quad y=2x$.

Example 2. *Find the asymptotes of the curve*

$$x^3+4x^2y+5xy^2+2y^3+2x^2+4xy+2y^2-x-9y+2=0$$

Sol. Putting $x=1$ and $y=m$ in the third degree terms, we get

$$\phi_3(m)=1+4m+5m^2+2m^3$$

Also $$\phi_2(m)=2+4m+2m^2$$

$$\phi_1(m)=-1-9m.$$

The slopes of asymptotes are given by

$$\phi_3(m) \quad 1+4m+5m^2+2m^3=0$$

or $$(2m+1)(m+1)^2=0$$

$\therefore$ $$m=-\tfrac{1}{2}, m=-1, -1$$

Now when $$m=-\tfrac{1}{2}$$

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}=-\frac{2+4m+2m^2}{4+10m+6m^2}=-1$$

Hence the asymptote corresponding to $m=-\frac{1}{2}$ **is**

$$y=-\tfrac{1}{2}x-1 \quad \text{or} \quad 2y+x+2=0$$

when $m=-1$, c is obtained from the following relation

$$\frac{c^2}{2!}\phi_3''(m)+c\phi_2'(m)+\phi_1(m)=0.$$

$$\left[\because \begin{array}{l}\phi_3'(m)=0\\ \phi_2(m)=0\end{array}\right]$$

$$\frac{c^2}{2}(10+12m)+c(4+4m)+(-9m-1)=0$$

$$c^2(5-6)+c.0+(9-1)=0 \qquad [\because m=-1]$$

$$c^2=8 \qquad \therefore \qquad c=\pm 2\sqrt{2}$$

Thus the asymptotes are

$$y=-x+2\sqrt{2} \quad \text{and} \quad y=-x-2\sqrt{2}$$

or

$$y+x=2\sqrt{2} \quad \text{and} \quad y+x+2\sqrt{2}=0$$

Hence asymptotes are

$$2y+x+2=0, \quad y+x=2\sqrt{2} \quad \text{and} \quad y+x+2\sqrt{2}=0.$$

Example 3. *Find the asymptotes of*

$$y^4-2xy^3+2x^3y-x^4-3x^3+3x^2y+3xy^2-3y^3-2x^2+2y^2-1=0.$$

Sol. Putting $x=1$, $y=m$ in the highest degree terms, we get

$$\phi_4(m)=m^4-2m^3+2m-1$$
$$\phi_3(m)=-3+3m+3m^2-3m^3$$
$$\phi_2(m)=-2+2m^2$$
$$\phi_1(m)=0$$
$$\phi_0(m)=-1.$$

The slopes of the asymptotes are given by

$$\phi_4(m)=m^4-2m^3+2m-1=0$$

or $(m^4-1)-2m(m^2-1)=0$

or $(m^2-1)(m^2+1)-2m(m^2-1)=0$

or $(m^2-1)(m^2-2m+1)=0$

or $(m-1)(m+1)(m-1)^2=0$

or $(m-1)^3(m+1)=0$

or $m=1, 1, 1, -1.$

When $m=1$ (thrice repeated root), c is obtained from the equation

$$\frac{c^3}{3!}\phi_4'''(m)+\frac{c^2}{2!}\phi_3''(m)+c\,\phi_2'(m)+\phi_1(m)=0.$$

or $\frac{c^3}{6}(24m-12)+\frac{c^2}{2}(6-18m)+c(4m)+0=0$

or $c^3(4-2)+c^2(3-9)+4c=0 \quad (\because m=1)$

or $2c^3-6c^2+4c=0$

or $2c(c^2-3c+2)=0$

or $2c(c-1)(c-2)=0$

$\therefore c=0, 1, 2$

$\phi_4'(m)=4m^3-6m^2+2$

$\phi_4''(m)=12m^2-12m$

$\phi_4'''(m)=24m-12$

Also

$\phi_3'(m)=3+6m-9m^2$

$\phi_3''(m)=6-18m$

and

$\phi_2'(m)=4m$

Hence asymptotes are $y=x$, $y=x+1$, $y=x+2$

For $\quad m=-1$

$$c=-\frac{\phi_3(m)}{\phi'_4(m)}$$

$$=0 \text{ (for } m=-1)$$

Hence asymptote is $y=-x$ or $y+x=0$

Thus asymptotes are

$$y=x,\ y=x+1,\ y=x+2,\ y+x=0$$

EXERCISE 10 (*a*)

Find the asymptotes of the following curves:

1. $x^2y+xy^2+xy+y^2+3x=0.$ (*D.U. 1979*)
2. $y^3-6xy^2+11x^2y-6x^3+y+x=0.$ (*D.U. 1977*)
3. $y^3-xy^2-x^2y+x^3+x^2-y^2-1=0.$ (*D.U. 1976*)
4. $x^3+2x^2y-xy^2-2y^3+xy-y^2+1=0.$ (*D.U. 1975*)
5. $x^3+2x^2y+xy^2-x^2-xy+2=0.$ (*D.U. 1974*)
6. $3x^3+2x^2y-7xy^2+2y^3-14xy+7y^2+4x+5y=0.$
7. $y(x-y)^2=x+y.$
8. $x^2y-xy^2+xy+y^2+y-1=0.$
9. $y^3+xy^3+2xy^2+y+1=0.$
10. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

10·7 Asymptotes Parallel to the Co-ordinate Axes

So far we have been excluding the case of the asymptotes parallel to y-axis. The reason being that the slope m of such asymptotes is infinite. Now we shall determine asymptotes parallel to y-axis.

The general equation of the curve of nth degree {equation (1) of art. 10·3}

arranged in descending powers of y is

$$y^n\,\phi(x)+y^{n-1}\,\phi_1(x)+y^{n-2}\,\phi_2(x)+\ldots=0 \qquad \ldots(1)$$

where $\quad \phi(x),\ \phi_1(x),\ \phi_2(x)$ etc. are polynomials in x.

Dividing (1) by y^n, we get

$$\phi(x)+(1/y)\,\phi_1(x)+(1/y^2)\,\phi_2(x)+\ldots=0 \qquad \ldots(2)$$

Now as $y\to\infty$, let $x\to k$.

From equation (2), we get

$$\phi(k)=0$$

Thus k is a root of $\phi(x)=0$

Let $\phi(x)=0$ have the roots k_1, k_2, k_3 etc. Then the asymptotes parallel to y-axis are

$$x=k_1,\ x=k_2,\ x=k_3 \text{ etc.}$$

Thus the asymptotes parallel to y-axis are obtained by equating to zero the coefficient of highest power of y (if not a constant) in the equation of the curve. However, if the coefficient of highest power of y is a constant or has imaginary factors, no asymptotes parallel to y-axis exist.

Now asymptotes parallel to x-axis are easily determined by the previous methods when $m=0$. However, it is convenient to determine these asymptotes separately, by the following rule.

The asymptotes parallel to x-axis are easily determined by equating to zero the coefficient of highest power of x (if not a constant) in the equation of the curve. However if the coefficient of highest power of x is a constant or has imaginary factors, no asymptotes parallel to x-axis exist.

Example 1. *Find the asymptotes parallel to the coordinate axes of the curve,*

$$y^2x-a^2(x+a)=0.$$

Sol. The equation of the curve can be written as

$$(y^2-a^2)\,x-a^3=0$$

Here highest power of x is one and its coefficient is y^2-a^2. So asymptotes parallel to x-axis are $y^2-a^2=0$ or $y=\pm a$.

Also highest power of y is 2 and its co-efficient is x. So asymptote parallel to y-axis $x=0$.

Hence the asymptotes parallel to co-ordinate axes are

$$y=\pm a \text{ and } x=0$$

EXERCISE 10 (*b*)

Find the asymptotes, which are parallel to either axis, of the following curves.

1 $x^2y^2-a(x^2+y^2)=0$ (*D.U. 1978*)

2. $(x^2-a^2)y^2=x^2(x^2-4a^2)$

3. $y^2(a^2+x^2)=x^3$

4. $x^2y^3+x^3y^2=x^3+y^3$.

5. $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$.

10·8. Important Deductions (*From Art. 10·3*)

(*i*) The maximum number of asymptotes of an algebraic curve of nth degree cannot exceed n.

We have seen that the slopes of the asymptotes of the curve which are not parallel to y-axis, are given by the roots of $\phi_n(m)=0$, which is of degree n and hence can't give more than n values of m

If there are asymptotes parallel to y axis, then it is obvious that degree of $\phi_n(m)$ will be less than n by at least the same number.

(*ii*) If some of the roots of $\phi_n(m)=0$, are imaginary, the asymptotes corresponding to these roots are called imaginary asymptotes.

(*iii*) There may be cases when even corresponding to real roots of $\phi_n(m)=0$, no asymptotes exist. For example, the parabola $y^2=4ax$ has no asymptotes even though $\phi_n(m)=0$ has real roots.

10·9. Other Methods of Determining Asymptotes

(1) The asymptotes of an algebraic curve are parallel to the lines obtained by equating to zero, the linear factors of the highes degree terms in its equation.

Let $(y-ax)$ be a non-repeated factor of the nth degree terms of the equation of the curve. The equation of the curve can be written as

$$(y-ax)\,F_{n-1}+U_{n-1}=0, \qquad \ldots(1)$$

where F_{n-1} is a polynomial of degree $(n-1)$ and U_{n-1} contains terms of various degrees, not higher than $(n-1)$.

It is evident that one of the roots of $\phi_n(m)=0$ is $m=a$ and hence an asymptote of the curve is

$$y-ax=c$$

where c is to be determined.

By definition we know $\underset{x\to\infty}{\text{Lt}}\,(y-ax)=c$, where (x, y) lie on (1).

$\therefore$ From (1), we have

$$y-ax=-\frac{U_{n-1}}{F_{n-1}}.$$

Taking limits on both sides as $x\to\infty$, we have

$$\underset{x\to\infty}{\text{Lt}}\,(y-ax)=-\underset{x\to\infty}{\text{Lt}}\,U_{n-1}/F_{n-1}$$

$$\therefore \qquad c=-\underset{x\to\infty}{\text{Lt}}\,U_{n-1}/F_{n-1}$$

Hence c is determined.

(2) Let the terms of the highest degree in the equation of the curve have a repeated factor $(y-ax)^2$. Then the equation of the curve can be written as

$$(y-ax)^2\,F_{n-2}+(y-ax)\,U_{n-2}+V_{n-2}=0 \qquad \ldots(1)$$

where F_{n-2} and U_{n-2} contain terms of degree $(n-2)$ and V_{n-2} contains terms of all degrees none of which is higher than $(n-2)$. It is obvious that $\phi_n(m)=0$ has a repeated root $m=a$.

Hence there are asymptotes of the curve

$$y-ax=c_1$$
$$y-ax=c_2.$$

The values of c_1 and c_2 are determined by the method given above, using the fact

$$\underset{x\to\infty}{\text{Lt}}\ y/x=a$$

The following examples illustrate the use of these methods.

Example 1. *Find the asymptotes of the curve*

$(x^2-y^2)(y^2-4x^2)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0.$

Sol. The curve has no asymyptotes parallel to the coordinate axes, as the co-efficients of highest powers of x and y are both constants.

Factorizing the highest degree terms, we get

$$(x-y)(x+y)(y-2x)(y+2x)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0 \quad \ldots(1)$$

Hence the curve has asymptotes parallel to the lines

$$x-y=0, \qquad x+y=0$$
$$y-2x=0 \quad \text{and} \quad y+2x=0.$$

Now from (1), we have

$$x-y=\frac{6x^3-5x^2y-3xy^2+2y^3+x^2-3xy+1}{(x+y)(y-2x)(y+2x)}$$

$$=\frac{6-5(y/x)-3(y/x)^2+2(y/x)^3+1/x-3(y/x)\,.\,1/x+1/x^3}{(1+y/x)(y/x-2)(y/x+2)}$$

[Dividing N^r and D^r by x^3]

Now taking limits on both sides as $x\to\infty$, then

$$\underset{x\to\infty}{\text{Lt}}\ (y-x)=c$$

or $$\underset{x\to\infty}{\text{Lt}}\ (x-y)=-c$$

and $$\underset{x\to\infty}{\text{Lt}}\ y/x=1, \text{ we get} \qquad (\because \text{ Slope of line is } 1)$$

$$-c=\frac{6-5-3+2}{(1+1)(1-2)(1+2)}$$

$\therefore \quad c=0.$

Hence the asymptote corresponding to the factor $x-y$ is

$$x-y=0 \qquad \text{or} \qquad y=x$$

Similarly we can find the value of c corresponding to other factors. These values of c are -1, 0 and -1 corresponding to the factors $(y+x)$, $y-2x$ and $y+2x$ respectively.

Hence the asymptotes are

$$y=x, \qquad y+x+1=0.$$
$$y=2x \quad \text{and} \quad y+2x+1=0.$$

Example 2. *Find the asymptotes of*

$$(x-y)(x-2y)^2+x^2-3xy+2y^2-7=0 \qquad ...(1)$$

Sol. There are no asymptotes parallel to the axes as the coefficients of highest powers of x and y are constant.

The equation of the curve may be written as

$$(x-y)(x-2y)^2+(x-y)(x-2y)-7=0 \qquad ...(2)$$

The curve has asymptotes parallel to the lines $x-y=0$ and $x-2y=0$. As $(x-2y)$ is a repeated factor, there will be two asymptotes parallel to the line $x-2y=0$.

Now $$x-y=\frac{7-(x-y)(x-2y)}{(x-2y)^2}$$

$$=\frac{\frac{7}{x^2}-\left(1-\frac{y}{x}\right)\left(1-\frac{2y}{x}\right)}{\left(1-\frac{2y}{x}\right)^2} \qquad ..(3)$$

[Dividing N^r and D^r by x^2]

Now taking limits as $x\to\infty$ then

$$\underset{x\to\infty}{\text{Lt}}(y-x)=c$$

or $$\underset{x\to\infty}{\text{Lt}}(x-y)=-c$$

and $$\underset{x\to\infty}{\text{Lt}}\, y/x=1, \quad \text{we get}$$

$$y-x=0$$

$\therefore$ $y=x$ is an asymptote.

Now the curve has two asymptotes parallel to the line

$$x-2y=0$$

We have to find $\underset{x\to\infty}{\text{Lt}}(y-\tfrac{1}{2}x)=c$

or $$\underset{x\to\infty}{\text{Lt}}(x-2y)=-2c$$

Dividing (2), by $(x-y)$, we have

$$(x-2y)^2+(x-2y)-\frac{7}{x(1-y/x)}=0.$$

Taking limits as $x\to\infty$,

$$(x-2y)\to-2c \text{ and } y/x=\tfrac{1}{2}$$

$$\therefore \qquad 4c^2-2c=0$$

$$2c(2c-1)=0$$

$$\therefore \qquad c=0$$

$$c=\tfrac{1}{2}.$$

Hence the asymptotes are

$$y=\tfrac{1}{2}x+0$$

and $$y=\tfrac{1}{2}x+\tfrac{1}{2}.$$

or $$x=2y$$

or $$x-2y+1=0.$$

Thus the asymptotes are

$$y=x, \quad x=2y, \quad x-2y+1=0.$$

Note. It may be noted that the asymptotes of this curve in these examples could also be found by the previous method of Art. 10·3 or 10·6

10·10. Asymptotes by Inspection

If the equation of a curve of nth degree can be put in the form

$$u_n+u_{n-2}=0$$

where u_{n-2} is a polynomial of degree not higher than $(n-2)$ then every linear factor of u_n, when equated to zero will give an asymptote, provided that no two straight lines so obtained are parallel or coincident.

Example 1. *Find the asymptotes of*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Sol. Here the equation of the curve can be written as

$$u_n+u_{n-2}=0$$

where $$u_n=\frac{x^2}{a^2}-\frac{y^2}{b^2}.$$

$$u_{n-2}=-1.$$

The asymptotes are given by

$$u_n=0$$

$$\therefore \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

$$\therefore \qquad y=\pm\frac{b}{a}x$$

Hence asymptotes are

$$y=\frac{b}{a}x,$$

and $$y=-\frac{b}{a}x.$$

Example 2. *Find the asymptotes of the curve*

$$y^3-6xy^2+11x^2y-6x^3+y+x=0.$$

Sol. The given equation is of the form $u_n+u_{n-2}=0$

Hence asymptotes are given by

$$y^3-6xy^2+11x^2y-6x^3=0$$

or $$(y-x)(y-2x)(y-3x)=0$$

or $$y=x, \quad y=2x,$$

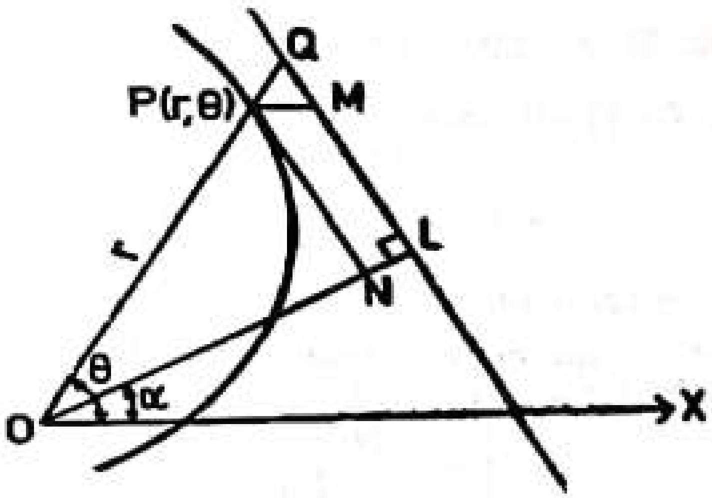
and $$y=3x.$$

10·11. Asymptotes of Polar Curves

Before proceeding with the determination of asymptotes of polar curves, we shall obtain the equation of a straight line in polar form.

Theorem 1. The polar equation of a straight line is

$$p=r\cos(\theta-\alpha)$$



where p is the perpendicular distance from the pole to the line and α the angle which this perpendicular makes with the initial line.

We know that the equation of a straight line can be written as

$$x\cos\alpha+y\sin\alpha=p \qquad \ldots(1)$$

where p is the length of the perpendicular from origin (pole) on the line and α the angle which this perpendicular makes with x-axis (initial line).

Putting $x=r\cos\theta$, $y=r\sin\theta$ in (1) we have the polar form of the line as

$$r(\cos\theta\cos\alpha+\sin\theta\sin\alpha)=p$$

$$r\cos(\theta-\alpha)=p$$

or $$p=r\cos(\theta-\alpha) \qquad \ldots(2)$$

Theorem 2. The line $r \sin(\theta-\gamma)=\dfrac{1}{f'(\gamma)}$ is an asymptote of the curve

$$\frac{1}{r}=f(\theta)$$

where γ is a root of the equation $f(\theta)=0$.

Let $P(r, \theta)$ be any point on the curve and OL a perpendicular on the line (2). Further let PM and PN be perpendiculars on the line (2) and OL respectively.

Now $$PM=LN=OL-ON$$

or $$PM=p-r\cos(\theta-\alpha) \quad ...(3)$$

Now $r\to\infty$ as the point moves to infinity along the curve.

Let $\theta\to\alpha$, when $r\to\infty$

Now if line (2) is to be asymptote to the curve

$$\frac{1}{r}=f(\theta)$$

then $$PM\to 0$$

Hence from (3), we have as $r\to\infty$.

$$\underset{\theta\to\gamma}{\text{Lt}} \cos(\theta-\alpha)=0=\cos\frac{\pi}{2}$$

$$\therefore \quad \alpha=\gamma-\frac{\pi}{2}$$

This gives the value of α.

Again $p=OL$ is the polar subtangent to the curve at infinity, *i.e.* at the point where the asymptote touches the curve

$$\therefore \quad p=\left[r^2\frac{d\theta}{dr}\right]_{\text{at } \theta=\gamma} \quad ...(4)$$

Let $$u=\frac{1}{r}=f(\theta)$$

$$\therefore \quad -\frac{1}{r^2}\frac{dr}{d\theta}=\frac{du}{d\theta}$$

Thus from (4), we get

$$p=\left[-\frac{d\theta}{du}\right]_{\text{at } \theta=\gamma}$$

Therefore the equation of asymptote is

$$\left[-\frac{d\theta}{du}\right]_{\text{at } \theta=\gamma}=r\cos\left(\theta-\gamma+\frac{\pi}{2}\right)$$

or $$-\frac{1}{f'(\gamma)}=-r\sin(\theta-\gamma)$$

Hence the equation of asymptote is

$$r \sin(\theta-\gamma)=\frac{1}{f'(\gamma)}.$$

Example 1. *Find the asymptotes of the curve*

$$r=a \tan\theta \qquad \textit{(D.U. 1971, 76, 1979)}$$

Sol. Therefore $\frac{1}{r}=\frac{1}{a}\cot\theta=f(\theta)$ (say) ...(1)

Now $\qquad f(\theta)=0$

when $\qquad \cot\theta=0$

or

$$\theta=n\pi+\frac{\pi}{2}=\gamma \text{ (say)}$$

Thus from (1), we have

$$f'(\theta)=-\frac{1}{a}\operatorname{cosec}^2\theta$$

$$\therefore \qquad f'(\gamma)=-\frac{1}{a} \qquad \left[\because \operatorname{cosec}^2\left(n\pi+\frac{\pi}{2}\right)=1\right]$$

Now the asymptotes are $r\sin(\theta-\gamma)=\frac{1}{f'(\gamma)}$

when $\gamma=\frac{\pi}{2}$, the asymptote is

$$r\sin\left(\theta-\frac{\pi}{2}\right)=-a \text{ or } r\cos\theta=a$$

when $\gamma=\frac{3\pi}{2}$, the asymptote is

$$r\sin\left(\theta-\frac{3\pi}{2}\right)=-a \text{ or } r\cos\theta=-a.$$

Now when $\theta=\frac{5\pi}{2}$, the asymptote is

$$r\sin\left(\theta-\frac{5\pi}{2}\right)=-a$$

or

$$r\cos\theta=a,$$

which is same as corresponding to

$$\gamma=\frac{\pi}{2}$$

Hence there are two asymptotes only

$$r\cos\theta=a$$

and

$$r\cos\theta=-a.$$

Example 2. *Find the asymptotes of the curve* $r\theta=a$.

Sol. Here the curve is $r = a/\theta$

or $$\frac{1}{r} = \frac{1}{a}\,\theta$$

$\therefore$ $$\frac{1}{r} = \frac{1}{a}\,\theta = f(\theta) \qquad \cdots (1)$$

Now $$f(\theta) = 0$$

when $$\theta = 0 = \gamma$$

$$f'(\theta) = \frac{1}{a}$$

or $$f'(\gamma) = \frac{1}{a}$$

Equation of asymptote is

$$r \sin(\theta - \gamma) = \frac{1}{f'(\gamma)}$$

or $$r \sin\theta = a.$$

EXERCISE 10 (*c*)

Find the asymptotes of the following curves :

1. $r \sin 2\theta = a \cos 3\theta$
2. $r \sin n\theta = a$
3. $r = a \sec\theta + b \tan\theta$
4. $r = a \log\theta$
5. $r = \dfrac{a}{\frac{1}{2} - \cos\theta}$
6. $r = \dfrac{3a \sin\theta \cos\theta}{\sin^3\theta + \cos^3\theta}$

4. $\frac{3}{2}$ 5. $\frac{a}{2}$

7. $\frac{a}{4}$

Exercise 9 (c) (Page 233)

1. $\frac{a}{2}$ 2. $\frac{a}{2}$.

3. $\frac{2}{3}\sqrt{2\,ar}$ 4. $\frac{a^n}{(n+1)\,r^{n-1}}$

6. $\frac{a^2b^2}{p^3}$

Exercise 9 (d) (Page 240)

1. $\left(-\frac{11}{2}, \frac{16}{3}\right)$ 2. $(-2, 3)$

3. $\left\{\frac{a^2+b^2}{a^4}\alpha^3, -\frac{(a^2+b^2)}{b^4}\beta^3\right\}$

4. $\{\alpha+3\alpha^{1/3}\beta^{2/3}, \beta+3\alpha^{2/3}\beta^{1/3}\}$

5. $\left\{-4a(20+a^2), \quad b+\left(\frac{81+4a^2}{18}\right)\right\}$

6. $\left\{\frac{1}{e}, \frac{e^2-1}{e}\right\}$

Exercise 9 (e) (Page 246)

2. $\left\{\frac{2+\sqrt{2}}{2}, \frac{3+2\sqrt{2}}{2} e^{-(2+\sqrt{2})}\right\}$

and $\left\{\frac{2-\sqrt{2}}{2}, \frac{3-2\sqrt{2}}{2} e^{-(2-\sqrt{2})}\right\}$

3. $(0, 0), \left(\frac{\pi}{3}, \pi-\frac{3\sqrt{3}}{2}\right)$ for concavity

10. $\left(\pm\frac{1}{\sqrt{2}}, e^{-1/2}\right)$, interval $\frac{-1}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}$, convex outside this interval.

Exercise 10 (a) (Page 255)

1. $y+x=0, y=0, x+1=0$
2. $y=x+1, y+3=2x, y-3x=2$
3. $y=x, y=x+1, y+x=0$
4. $y=x, y+x+1=0, 2y+x=1$
5. $y=0, y+x=0, y+x=1$
6. $y-x+\frac{7}{6}=0, \; y-3x+\frac{3}{2}=0, \; 2y+x+\frac{5}{3}=0$
7. $y=0, y+x=0$ 8. $y=0, x=1, y=x+2$

9. $y+x=\pm 1,\ y=0$ 10. $y=\pm\dfrac{b}{a}\,x.$

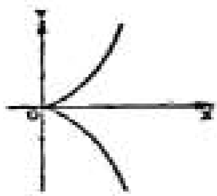
Exercise 10 (b) (Page 256)

1. $x=\pm\sqrt{a},\ y=\pm\sqrt{a}$ 2. $x=\pm a$
3. $y=\pm 1$ 4. $x=\pm 1,\ y=\pm 1$
5. $x=\pm a,\ y=\pm b.$

Exercise 10 (c) (Page 264)

1. $2r\sin\theta=a,\ 2\theta=\pi$
2. $r\sin\left(\theta-\dfrac{k\pi}{n}\right)=\dfrac{a}{n\cos k\pi}$, where k is an integer.
3. $r\cos\theta\pm b=a$ 4. $\theta=0$
5. $r\sin\left(\theta-\dfrac{\pi}{3}\right)=\dfrac{2a}{\sqrt{3}}$
6. $a+\sqrt{2}\,r\sin\left(\theta+\dfrac{\pi}{4}\right)=0.$

Exercise 11 (a) (Page 276)

1.


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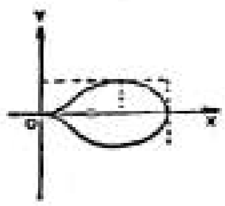

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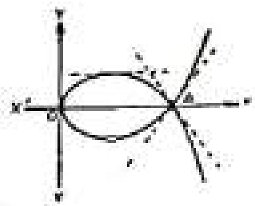

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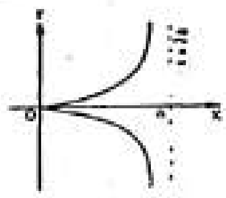

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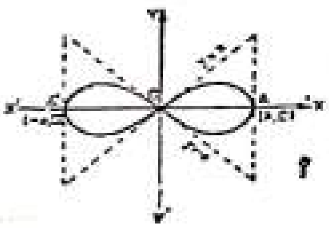

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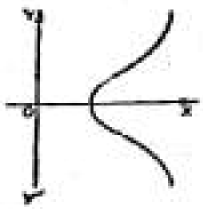

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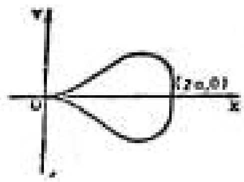

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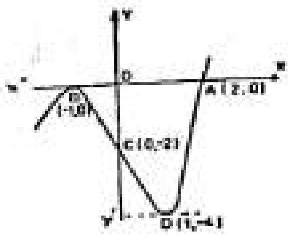

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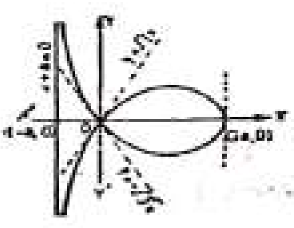

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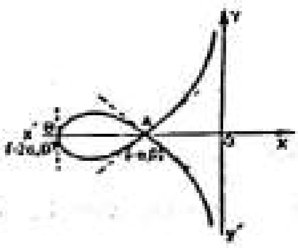

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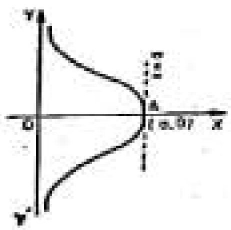

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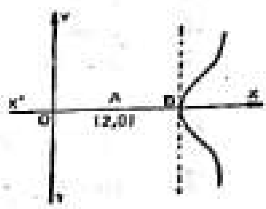


2

Infinite Series

2·1. Definitions

A series containing an infinite number of terms $u_1, u_2, u_3, \ldots, u_n, \ldots$ that occur according to some definite law, is called *infinite series* and is denoted by

$$u_1 + u_2 + u_3 + \ldots u_n + \ldots \text{ or } \sum_{n=1}^{\infty} u_n \text{ or simply } \Sigma u_n.$$

The sum of the first n terms of the series is denoted by S_n.

$\therefore$ $$S_n = u_1 + u_2 + u_3 + \ldots + u_n.$$

Convergence and Divergence of Series

Consider an infinite series

$$\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

and let $$S_n = u_1 + u_2 + u_3 + \ldots + u_n.$$

The given series Σu_n is said to be *convergent* if $\underset{n \to \infty}{\text{Lt}} S_n$ is finite quantity and the given series is said to be divergent if $\underset{n \to \infty}{\text{Lt}} S_n$ tends to infinity.

However if $\underset{n \to \infty}{\text{Lt}} S_n$ does not tend to a definite limit the given series is known as oscillatory, sometimes called non-convergent series.

Example 1. *Discuss the convergence of the series,*

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$$

Sol. Here $$S_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^{n-1}}$$

$$= \frac{1 \cdot (1 - 1/3^n)}{1 - 1/3} \qquad \text{(Sum of G.P.)}$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$$

$$= \frac{3}{2} - \frac{1}{2 \cdot 3^{n-1}}$$

Now $$\underset{n \to \infty}{\text{Lt}} S_n = \underset{n \to \infty}{\text{Lt}} \left(\frac{3}{2} - \frac{1}{2 \cdot 3^{n-1}}\right) = \frac{3}{2}$$

Since $\underset{n\to\infty}{\text{Lt}} S_n$ is a finite quantity, the given series is convergent.

Example 2. *Investigate the series for convergence or divergence,*

$$1+2+3+\ldots+n+\cdots$$

Sol. Here
$$S_n=1+2+3+\ldots+n$$
$$=\frac{n(n+1)}{2}$$
(Sum of first n natural numbers)

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} S_n=\underset{n\to\infty}{\text{Lt}} \frac{n(n+1)}{2}\to\infty$$

Since $\underset{n\to\infty}{\text{Lt}} S_n$ is infinite, the given series is divergent.

Example 3. *Discuss the nature of the series*

$$1-1+1-1+\cdots$$

Sol. Obviously S_n is 1 or 0 according as n is odd or even. Therefore, S_n cannot tend to a definite limit, hence the given series is oscillatory.

EXERCISE 2 (*a*)

Discuss the nature of the following series.

1. $1+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$
2. $1^2+2^2+3^2+\ldots$
3. $1^3+2^3+3^3+\ldots$
4. $2+6+18+\ldots$
5. $3+5+7+\ldots$
6. $k-k+k-k+\ldots$

2.2. Fundamental Theorem on Infinite Series

The convergence or divergence of an infinite series remains unaffected by removal or addition of a finite number of terms.

Since the sum of a finite number of terms is finite, so their removal from the series or their addition to the series will not change the convergence or divergence of the series.

2.3. Convergence of a Geometric Series

Show that the geometric series

$$a+ar+ar^2+\ldots\ldots+ar^{n-1}+\ldots$$

(*i*) Converges if $|r|<1$
(*ii*) Diverges if $r\geq 1$
(*iii*) Oscillates if $r\leq -1$

Let $S_n = a + ar^2 + \cdots + ar^{n-1}$ ...(1)

Case I. When $|r| < 1$.

Now $$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} S_n = \operatorname*{Lt}_{n\to\infty} \left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right)$$

Since $|r| < 1$, therefore, $\operatorname*{Lt}_{n\to\infty} r^n = 0$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} S_n = \frac{a}{1-r} \quad (a \text{ finite quantity})$$

Since $\operatorname*{Lt}_{n\to\infty} S_n$ is a finite quantity, the given series converges if $|r| < 1$.

Case II When $r \geq 1$

We first consider the case when $r > 1$

Here $$S_n = \frac{a(r^n - 1)}{r-1}$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} S_n = \operatorname*{Lt}_{n\to\infty} \left(\frac{ar^n}{r-1} - \frac{a}{r-1}\right)$$

Since $r > 1$, $\operatorname*{Lt}_{n\to\infty} r^n \to \infty$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} S_n \to \infty$$

Since $\operatorname*{Lt}_{n\to\infty} S_n$ tends to infinity the given series diverges if $r > 1$.

When $r = 1$, we have from (1),

$$S_n = a + a + a + \cdots n \text{ times}$$

$$= na$$

$$\therefore \operatorname*{Lt}_{n\to\infty} S_n = \operatorname*{Lt}_{n\to\infty} na \to \infty.$$

Hence the given series is divergent when $r = 1$.

Thus the given series diverges if $r \geqslant 1$,

Case III. When $r \leqslant -1$.

We first consider the case when $r = -1$, we have the given series from (1),

$$S_n = a - a + a - \cdots n \text{ times}$$

Obviously S_n oscillates between a and 0, according as n is odd or even.

When $r < -1$, let $r = -K$ (where $K > 1$)

$$\therefore \qquad S_n = \frac{a(1-r^n)}{1-r} = \frac{a[1-(-K)^n]}{1+K}$$

$$= \frac{a[1+(-1)^{n+1} K^n]}{1+K}$$

$\underset{n\to\infty}{\text{Lt}} S_n = \underset{n\to\infty}{\text{Lt}} \frac{a}{1+K} [1+(-1)^{n+1} K^n]$ which oscillates between $-\infty$ and $+\infty$, according as n is even or odd.

Therefore, the given series oscillates when $r \leqslant -1$

2·4. Convergence or Divergence of p-Series

Show that the series

$$\Sigma n^{-p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

(*i*) Converges if $p > 1$

(*ii*) Diverges if $p \leqslant 1$

(*i*) **Case I.** When $p > 1$

Let the terms of the given series be grouped in such a manner that first, second, third .. groups contain 1 term, 4 terms, 8 terms,...respectively. Thus the given series may be written as

$$\frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots$$

Now $\qquad \frac{1}{1^p} = 1$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \left(\frac{1}{2}\right)^{p-1} \quad \left[\because \frac{1}{3^p} < \frac{1}{2^p}\right]$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p}$$

$$= \left(\frac{1}{2}\right)^{2(p-1)}$$

Similarly $\frac{1}{8^p} + \frac{1}{9^p} + \ldots + \frac{1}{15^p} < \frac{8}{8^p} = \left(\frac{1}{2}\right)^{3(p-1)}$ and so on.

Adding corresponding sides, we get

$$\frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$
$$+ \left(\frac{1}{8^p} + \frac{1}{9^p} + \ldots \frac{1}{15^p}\right) + \ldots < 1 + \left(\frac{1}{2}\right)^{p-1} + \left(\frac{1}{2}\right)^{2(p-1)}$$
$$+ \left(\frac{1}{2}\right)^{3(p-1)} + \ldots$$

or $\qquad \Sigma n^{-p} < 1 + (\tfrac{1}{2})^{p-1} + (\tfrac{1}{2})^{2(p-1)} + (\tfrac{1}{2})^{3(p-1)} + \ldots$

The series on the right hand side is a geometric series with common ratio $(\frac{1}{2})^{p-1}<1$ (when $p>1$), hence convergent.

Thus the given series is convergent when $p>1$.

(*ii*) **Case II.** When $p=1$

The given series becomes,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots$$

$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots$$

$$1+\frac{1}{2}=1+\frac{1}{2}$$

Now $$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

Similarly $$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\ldots+\frac{1}{16}>\frac{1}{16}+\frac{1}{16}$$

$$+\ldots 8 \text{ times}=\frac{1}{2} \text{ and so on.}$$

Adding corresponding sides, we get

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$$

$$+\ldots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$$

The series on the right hand side (ignoring the first term) is a geometric series with common ratio unity hence divergent.

Thus the given series is divergent when $p=1$.

Case III. When $p<1$.

$$\frac{1}{1^p}=1$$

$$\frac{1}{2^p}>\frac{1}{2}$$

$$\frac{1}{3^p}>\frac{1}{3}$$

$$\frac{1}{4^p}>\frac{1}{4} \text{ and so on.}$$

Adding corresponding sides, we get

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots>1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

The series on right hand side is divergent (by case II)

Thus the given series is divergent when $p<1$.

2·5. Positive Term Series

An infinite series in which all the terms from and after some term are positive, is called a positive term infinite series or **positive term series**. For example $-1-2-3+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$, is a positive term series, because from 4th term onward all the terms of the series are positive.

Convergence or Divergence of a positive term Series

We have so far discussed the convergence or divergence of a series by evaluating the sum of first n terms of the series. However, it is not easy to calculate the sum of first n terms of every series. Thus it is not possible to determine the nature of every series by direct application of definition. Hence various other methods have been derived to determine the convergence of a series.

Necessary Condition for Convergence of a Series

Theorem. If Σu_n is a convergent series of positive terms, then $\underset{n\to\infty}{\text{Lt}}\ u_n=0$.

Let $S_n=u_1+u_2+u_3+u_4+\ldots+u_n$

Since Σu_n is convergent,

$$\underset{n\to\infty}{\text{Lt}}\ S_n=K \text{ (say)}$$

Also $\underset{n\to\infty}{\text{Lt}}\ S_{n-1}=K$

Now $u_n=S_n-S_{n-1}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ u_n=\underset{n\to\infty}{\text{Lt}}\ (S_n-S_{n-1})$$

$$=\underset{n\to\infty}{\text{Lt}}\ S_n-\underset{n\to\infty}{\text{Lt}}\ S_{n-1}$$

$$=K-K=0$$

or $$\underset{n\to\infty}{\text{Lt}}\ u_n=0$$

Hence the condition.

Note 1. It must be noted carefully that converse of the above theorem is not true, *i.e.* even if $\underset{n\to\infty}{\text{Lt}}\ u_n=0$, the series Σu_n may not converge.

For example, let $\Sigma u_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$

Here $$u_n=\frac{1}{n}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ u_n=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}=0$$

But we have seen that the given series is divergent.

Note 2. The result can effectively be applied to show that the given series is divergent if $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$.

For example, let $\Sigma u_n = 1+2+3+\ldots+n+\ldots$

Here $\qquad u_n = n$

$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ n \neq 0$

Hence the given series is divergent.

2·6. Tests for Convergence and Divergence of a Series

Comparison Test

If Σu_n and Σv_n be two positive term series, such that from and after some particular term $\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = k$ (a non-zero finite quantity), then Σu_n and Σv_n either both converge or both diverge.

Proof. Let the series from and after the particular term be, $u_1+u_2+u_3+\ldots+u_n+\ldots$

Now for all values of n,

let $\qquad \underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = k$

Therefore by definition of limit, there exists a positive number ϵ, such that

$$\left|\frac{u_n}{v_n} - k\right| < \epsilon \text{ for all } n$$

i.e. $$-\epsilon < \frac{u_n}{v_n} - k < \epsilon$$

or $$k-\epsilon < \frac{u_n}{v_n} < k+\epsilon \qquad \ldots (1)$$

Case I. When Σv_n is convergent.

From (1), we have

$$\frac{u_n}{v_n} < k+\epsilon \qquad \text{for all } n.$$

or $$u_n < (k+\epsilon)\, v_n$$
or $$\Sigma u_n < \Sigma (k+\epsilon)\, v_n$$
or $$\Sigma u_n < (k+\epsilon)\, \Sigma v_n$$

Each term of the series Σu_n is term by term less than the corresponding terms of $(k+\epsilon)\, \Sigma v_n$. Since Σv_n is convergent, therefore, Σu_n is also convergent.

Case II. When Σv_n is divergent

From (1), we have

$$\frac{u_n}{v_n} > (k-\epsilon) \text{ for all } n.$$

$$\therefore \qquad u_n > (k-\epsilon)\, v_n$$

or
$$\Sigma\, u_n > (k-\epsilon)\, \Sigma\, v_n$$

Each term of Σu_n is term by term greater than the corresponding terms of $(k-\epsilon)\,\Sigma v_n$. Since Σv_n is divergent, therefore, $\Sigma\, u_n$ is also divergent.

Note 1. This test is very useful when degree of n, in u_n can readily be determined.

Note 2. To select auxiliary series $\Sigma\, v_n = \Sigma\, \frac{1}{n^p}$, it should be noted that p=Difference in degree of n in denominator and numerator of u_n.

Example 1. *Test for convergence of the series,*

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\ldots+\frac{1}{n^2+1}+\ldots$$

Sol. Here $\qquad u_n = \dfrac{1}{n^2+1}$

Let $\qquad \Sigma v_n = \Sigma\, \dfrac{1}{n^2}$ be an auxiliary series,

(See note 2, Art. 2·6)

$$\therefore \qquad v_n = \frac{1}{n^2}$$

Now, $\qquad \dfrac{u_n}{v_n} = \dfrac{n^2}{n^2+1} = \dfrac{1}{\left(1+\dfrac{1}{n^2}\right)}$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{\left(1+\dfrac{1}{n^2}\right)}$$

$$=1 \qquad \text{(a non-zero, finite quantity)}$$

Hence Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma\, \frac{1}{n^2}$ is convergent (being p series with $p>1$), therefore, Σu_n is also convergent.

Example 2. *Test the convergence or divergence of the series,*

$$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots\infty$$

Sol. Here $\qquad u_n = \dfrac{n+1}{n^p}$

$\therefore$ Now let $\qquad \Sigma v_n = \Sigma\, \dfrac{1}{n^{p-1}}$ be the auxiliary series,

$\therefore \qquad v_n = \frac{1}{n^{p-1}}$

$$\frac{u_n}{v_n} = \frac{n+1}{n^p} \cdot n^{p-1}$$

$$= \frac{n+1}{n} = \left(1 + \frac{1}{n}\right)$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left(1 + \frac{1}{n}\right)$$

$= 1$ [a non-zero finite quantity]

Hence Σu_n and Σv_n converge or diverge together.

But $\qquad \Sigma v_n = \Sigma \frac{1}{n^{p-1}}$ is,

(*i*) Convergent if $p-1 > 1$ $\quad$ *i.e.* $p > 2$

(*ii*) Divergent if $p-1 \leqslant 1$ $\quad$ *i.e.* $p \leqslant 2$

$\therefore$ Σu_n converges if $p > 2$ and diverges if $p \leqslant 2$

Example 3. *Discuss the nature of the series,*

$$\Sigma u_n = \Sigma \left(\frac{2}{5n+1} \right)$$

Sol. Here $u_n = \frac{2}{5n+1}$

Let $\qquad \Sigma v_n = \Sigma \frac{1}{n}$

$$v_n = \frac{1}{n}$$

Now $\qquad \frac{u_n}{v_n} = \frac{2n}{5n+1}$

$$= \frac{2}{\left(5 + \frac{1}{n}\right)}$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \frac{2}{\left(5 + \frac{1}{n}\right)}$$

$= \frac{2}{5}$ [a non-zero finite quantity]

Hence Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n}$ is known to be divergent, therefore, the given series is also divergent.

2.7. Comparison of Ratios

If Σu_n and Σv_n be two positive term series such that from and after some particular term,

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

and if Σv_n is convergent, then Σu_n is also convergent.

Proof. Let the two series Σu_n and Σv_n, from and after the particular term be,

$u_1+u_2+u_3+\dots$ and $v_1+v_2+v_3+\dots$, respectively.

Since $$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

$$\frac{u_1}{u_2} > \frac{v_1}{v_2}, \quad \frac{u_2}{u_3} > \frac{v_2}{v_3}, \quad \frac{u_3}{u_4} > \frac{v_3}{v_4} \text{ and so on}$$

or $$\frac{u_2}{u_1} < \frac{v_2}{v_1}, \quad \frac{u_3}{u_2} < \frac{v_3}{v_2}, \quad \frac{u_4}{u_3} < \frac{v_4}{v_3} \text{ and so on} \quad \dots(1)$$

Now $$u_1+u_2+u_3+\dots = u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\dots\right)$$

$$= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\dots\right)$$

$$< u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\dots\right)$$

$$= \frac{u_1}{v_1}(v_1+v_2+v_3+\dots)$$

$$\therefore \quad u_1+u_2+u_3+\dots < \frac{u_1}{v_1}(v_1+v_2+v_3+\dots)$$

Now series Σv_n is convergent, therefore, Σu_n is also convergent.

Note. Similarly we can show Σu_n is divergent if

(*i*) Σv_n is divergent.

and (*ii*) from and after some particular term,

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

2.8. D' Alembert's Ratio Test

If Σu_n be a positive term series such that from and after some particular term,

$$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = k, \text{ then } \Sigma u_n$$

(*i*) Converges, if $k < 1$

(*ii*) Diverges, if $k > 1$.

Proof. Let the series from and after the particular term be

$$u_1+u_2+u_3+\ldots+u_n+\cdots$$

Case I. When $k < 1$.

By definition of a limit, a positive number $\lambda (k < \lambda < 1)$ can be found such that

$$\frac{u_{n+1}}{u_n} < \lambda, \qquad \text{for all } n \qquad \ldots(1)$$

Thus $\quad \dfrac{u_2}{u_1} < \lambda, \dfrac{u_3}{u_2} < \lambda, \dfrac{u_4}{u_3} < \lambda, \ldots$and so on. $\qquad \ldots(2)$

Now $u_1+u_2+u_3+\cdots+u_n+\cdots$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\cdots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\right)$$

$$<u_1(1+\lambda+\lambda^2+\lambda^3+\cdots) \qquad \text{[By (2)]}$$

$$=\frac{u_1}{1-\lambda}, \text{ a finite quantity.}$$

Hence $\Sigma\, u_n$ is convergent.

Case II. When $k>1$.

By definition of limit, we have

$$\frac{u_{n+1}}{u_n} > 1.$$

Now $\qquad S_n=u_1+u_2+u_3+\ldots+u_n$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\cdots+\frac{u_n}{u_1}\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\cdots+\frac{u_n}{u_{n-1}}\cdot\frac{u_{n-1}}{u_{n-2}}\cdots\frac{u_2}{u_1}\right)$$

$\therefore \qquad S_n > u_1(1+1+1+\cdots n \text{ terms})$

or $\qquad S_n > nu_1$

Now $\lim\limits_{n\to\infty} S_n > \lim\limits_{n\to\infty} nu_1$ which tends to infinity.

Hence Σu_n is divergent.

Note. If $\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}=1$, the test fails to give any information about convergence or divergence of the series and therefore further tests are needed. For example let

$$\Sigma u_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$$

Here $\quad u_n=\frac{1}{n}, \qquad u_{n+1}=\frac{1}{(n+1)}$

$$\therefore \quad \frac{u_{n+1}}{u_n}=\frac{n}{n+1}=\frac{1}{(1+1/n)}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}=\underset{n\to\infty}{\text{Lt}} \frac{1}{(1+1/n)}=1$$

The series Σu_n is known to be divergent, being p-series with $p=1$.

Further, let $\Sigma u_n=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$

Here $\quad u_n=\frac{1}{n^2}$ and $u_{n+1}=\frac{1}{(n+1)^2}$

$$\therefore \quad \frac{u_{n+1}}{u_n}=\frac{n^2}{(n+1)^2}=\frac{1}{(1+1/n)^2}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}=1$$

But the given series is convergent, being p-series with $p>1$.

Thus we see that when

$$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}=1,$$

D' Alembert's ratio test fails.

Example 1. *Test for convergence the series,*

$$\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\ldots$$

Sol. Here $\quad u_n=\frac{n}{1+2^n}$

$$\therefore \quad u_{n+1}=\frac{n+1}{1+2^{n+1}}$$

$$\therefore \quad \frac{u_{n+1}}{u_n}=\frac{n+1}{1+2^{n+1}}\times\frac{1+2^n}{n}$$

$$=\frac{n+1}{n}\cdot\frac{1+2^n}{1+2^{n+1}}$$

$$=\left(1+\frac{1}{n}\right)\cdot\frac{\left(1+\frac{1}{2^n}\right)}{\left(2+\frac{1}{2^n}\right)}$$

$$\therefore \underset{n\to\infty}{\text{Lt}}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{\text{Lt}}\frac{\left(1+\frac{1}{n}\right)\cdot\left(1+\frac{1}{2^n}\right)}{\left(2+\frac{1}{2^n}\right)}=\frac{1}{2}<1$$

Hence the given series is convergent.

Example 2. *Test for convergence the series*

$$1+3x+5x^2+7x^3+\dots \qquad (x>0)$$

Sol. Here $u_n=(2n-1)x^{n-1}$

$$u_{n+1}=(2n+1)x^n$$

$$\therefore \qquad \frac{u_{n+1}}{u_n}=\frac{(2n+1)}{(2n-1)}\frac{x^n}{x^{n-1}}$$

$$=\frac{(2+1/n)}{(2-1/n)}\cdot x$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{\text{Lt}}\frac{(2+1/n)}{(2-1/n)}x=x$$

The given series converges if $x<1$ and diverges for $x>1$.

If $x=1$, the test fails and we apply further tests.

When $x=1$, we have

$$u_n=(2n-1)$$

$$\underset{n\to\infty}{\text{Lt}}u_n=\underset{n\to\infty}{\text{Lt}}(2n-1)\neq 0$$

Since $\underset{n\to\infty}{\text{Lt}}u_n\neq 0$, the given series diverges for $x=1$.

Hence the given series converges for $x<1$ and diverges for $x>1$.

Example 3. *Discuss the convergence and divergence of the series*

(i) $\Sigma[\sqrt[3]{(n+1)}-\sqrt[3]{n}]$

(ii) $\Sigma \sin \dfrac{1}{n}$.

Sol. (i) Here $u_n=(n+1)^{1/3}-n^{1/3}$

$$=n^{1/3}\left(1+\frac{1}{n}\right)^{1/3}-n^{1/3}$$

$$=n^{1/3}\left[\left(1+\frac{1}{n}\right)^{1/3}-1\right]$$

$$=n^{1/3}\left[\left\{1+\frac{1}{3}\cdot\frac{1}{n}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^2}+\ldots\right\}-1\right]$$

$$=n^{1/3}\left(\frac{1}{3n}-\frac{1}{9n^2}+\ldots\right)$$

$$=\frac{1}{n^{2/3}}\left(\frac{1}{3}-\frac{1}{9n}+\ldots\right)$$

Let $\qquad v_n=\dfrac{1}{n^{2/3}}$. Applying the comparison test, we have

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\underset{n\to\infty}{\text{Lt}}\frac{1}{n^{2/3}}\left(\frac{1}{3}-\frac{1}{9n}+\ldots\right)\Big/\frac{1}{n^{2/3}}$$

$$=\underset{n\to\infty}{\text{Lt}}\left(\frac{1}{3}-\frac{1}{9n}+\ldots\right)=\frac{1}{3},$$

which is a non-zero finite quantity. Therefore, both the series Σu_n and Σv_n either converge or diverge together. But $\Sigma v_n (p=\frac{2}{3}<1)$ is known to be divergent. As such Σu_n is also divergent.

(ii) Here $\quad u_n=\sin\dfrac{1}{n}$.

Let $\qquad v_n=\dfrac{1}{n}$

Now $\qquad \underset{n\to\infty}{\text{Lt}}\dfrac{u_n}{v_n}=\underset{n\to\infty}{\text{Lt}}\left(\sin\dfrac{1}{n}\Big/\dfrac{1}{n}\right)$

$$=\underset{x\to0}{\text{Lt}}\frac{\sin x}{x}=1$$

where $\qquad \dfrac{1}{n}=x.$

But Σv_n is a divergent series ($p=1$).

Therefore $\Sigma u_n = \Sigma \sin \frac{1}{n}$ is also a divergent series.

EXERCISE 2 (*b*)

Test for convergence of the series.

1. $\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\ldots+\frac{n}{n^2+1}+\ldots$
2. $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{4}}+\ldots+\sqrt{\frac{n}{n+1}}+\ldots$ $\frac{u_n}{1/n}$
3. $\frac{1}{2^2}+\frac{1}{5^2}+\frac{1}{8^2}+\ldots$
4. $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\ldots$ $\frac{u_n}{v_n}$
5. $\frac{3}{2^2.3^2}+\frac{5}{3^2.4^2}+\frac{7}{4^2.5^2}+\ldots$ "
6. $\frac{1}{2}+\frac{\sqrt[3]{2}}{3\sqrt{2}}+\frac{\sqrt[3]{3}}{4\sqrt{3}}+\ldots$ "
7. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$ "
8. $\frac{1}{2}+\frac{\sqrt{2}}{5}+\frac{\sqrt{3}}{10}+\frac{\sqrt{4}}{17}+\ldots$ "
9. $\frac{1}{2}+\frac{1}{2\cdot2^2}+\frac{1}{3\cdot2^3}+\ldots\infty$
10. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\ldots+\frac{n^2-1}{n^2+1}x^n+\ldots$ $(x>0)$
11. $\frac{x}{1\cdot2}+\frac{x^2}{2\cdot3}+\frac{x^3}{3\cdot4}+\ldots$
12. $x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\infty$
13. $1+3x+5x^2+7x^3+\ldots$
14. $1^2+2^2.x+3^2x^2+\ldots$ $(0<x<1)$
15. $1+\frac{2^p}{2\,!}+\frac{3^p}{3\,!}+\frac{4^p}{4\,!}+\ldots\infty$
16. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\ldots\infty$
17. $\frac{1}{3}+\frac{2^2}{3^2}+\frac{3^2}{3^3}+\ldots\infty$
18. $\frac{2}{1}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\ldots$

19. $1+\frac{1^2.2^2}{1.3.5}+\frac{1^2.2^2.3^2}{1.3.5.7.9}+\ldots$

20. $\frac{1}{e}+\frac{8}{e^3}+\frac{27}{e^5}+\ldots$

Discuss the nature of the following seaies

21. $\sum\frac{1}{\sqrt{n+1}+\sqrt{n}}$

22. $\sum\frac{(n+1)}{n^p}$

23. $\Sigma(\sqrt{n^4+1}-\sqrt{n^4-1})$

24. $\Sigma(\sqrt{n^2+1}-n)$

25. $\sum\frac{n}{n^2+1}$

26. $\sum\frac{1}{\sqrt{n}}\tan\frac{1}{n}$

$\left(\textbf{Hint.}\quad \text{Let } v_n=\frac{1}{n^{3/2}}\right)$

27. $\sum_{n=2}^{\infty}\frac{1}{\log n}$

28. $\sum\sin\left(\frac{1}{n^2}\right)$

29. $\sum\left(1-\cos\frac{\pi}{n}\right)$

30. Show that the following series

(*a*) $\frac{x}{x+1}+\frac{x^2}{x+2}+\frac{x^3}{x+3}+\ldots$

(*b*) $\frac{x}{a\,.\,1^2+b}+\frac{2x^2}{a\,.\,2^2+b}+\frac{3x^3}{a\,.\,3^2+b}+\ldots$

converge when $x<1$ and diverge when $x\geqslant 1$.

2·9. Raabe's Test. [*Higher Ratio Test*]

If Σu_n be a positive term series, such that from and after some particular term

$$\operatorname*{Lt}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=k, \text{ then } \Sigma u_n,$$

(*i*) Converges if $k>1$

(*ii*) Diverges if $k<1$

Proof. Let the series from and after the particular term be

$$u_1+u_2+u_3+\ldots+u_n+\ldots$$

Case I. When $k>1$

Let p be a positive number such that $k>p>1$ and compare the given series Σu_n with an auxiliary series.

$$\Sigma v_n = \Sigma \frac{1}{n^p},$$

which is convergent when $p>1$. Now Σu_n converges if,

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \qquad \text{(See Art. 2·7)}$$

i.e. if
$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p \qquad \left[\because \; v_n = \frac{1}{n^p}\right]$$

or if
$$\frac{u_n}{u_{n+1}} > 1 + p \cdot \frac{1}{n} + \frac{p(p-1)}{1\,.\,2} \cdot \left(\frac{1}{n}\right)^2 + \ldots$$

or if
$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{1\,.\,2} \cdot \frac{1}{n} + \ldots\ldots$$

or if
$$\underset{n\to\infty}{\mathrm{Lt}}\; n\left(\frac{u_n}{u_{n+1}} - 1\right) > \underset{n\to\infty}{\mathrm{Lt}}\left[p + \frac{p(p-1)}{2n} + \ldots\right]$$

or if
$$\underset{n\to\infty}{\mathrm{Lt}}\; n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$$

i.e. if $k>p$, which is true.

$$\left[\because \; \underset{n\to\infty}{\mathrm{Lt}}\; n\left(\frac{u_n}{u_{n+1}} - 1\right) = k\right]$$

Hence the given series Σu_n is convergent.

Case II. When $k<1$.

Let p be a positive number such that $k<p<1$ and compare the given series with an auxiliary series

$$\Sigma v_n = \Sigma \frac{1}{n^p},$$

which is divergent when $p<1$.

Now Σu_n diverges if

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

or if
$$\frac{u_n}{u_{n+1}} < \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$$

or if
$$\frac{u_n}{u_{n+1}} < 1 + \frac{p}{n} + \frac{p(p-1)}{1\,.\,2} \cdot \left(\frac{1}{n}\right)^2 +$$

or if
$$n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{1\,.\,2} \cdot \frac{1}{n} + \ldots$$

or if $\underset{n\to\infty}{\text{Lt}}\; n\left(\frac{u_n}{u_{n+1}}-1\right) < \underset{n\to\infty}{\text{Lt}} \left[p+\frac{p(p-1)}{1.2}\left(\frac{1}{n}\right)+\ldots\right]$

or if $k<p$ which is true.

Hence the given series Σu_n diverges.

Note 1. The higher ratio test fails if $k=1$

Note 2. This test is applied when the ratio test fails.

Example 1. *Test for convergence the series,*

$$\frac{2^2}{3.4}+\frac{2^2.4^2}{3.4.5.6}+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}+\ldots\infty$$

Sol. Here $$u_n=\frac{2^2.4^2.6^2\ldots\ldots(2n)^2}{3.4.5\ldots\ldots(2n+2)}$$

and $$u_{n+1}=\frac{2^2.4^2.6^2\ldots\ldots(2n)^2.(2n+2)^2}{3.4.5\ldots\ldots(2n+2)(2n+3)(2n+4)}$$

$$\therefore \quad \frac{u_n}{u_{n+1}}=\frac{2^2.4^2.6^2\ldots(2n)^2}{3.4.5\ldots(2n+2)}\times\frac{3.4.5\ldots(2n+2)(2n+3)(2n+4)}{2^2.4^2.6^2\ldots(2n)^2(2n+2)^2}$$

$$=\frac{(2n+3)(2n+4)}{(2n+2)^2}$$

$$=\frac{(2+3/n)(2+4/n)}{(2+2/n)^2}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{(2+3/n)(2+4/n)}{(2+2/n)^2}=1$$

Hence the ratio test fails and we apply higher ratio test.

Now $$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2}-1\right]$$

$$=\frac{6n^2+8n}{(2n+2)^2}$$

$$=\frac{(3+4/n)}{2(1+1/n)^2}$$

$$\therefore \underset{n\to\infty}{\text{Lt}}\; n\left(\frac{u_n}{u_{n+1}}-1\right)=\underset{n\to\infty}{\text{Lt}}\frac{(3+4/n)}{2(1+1/n)^2}=\frac{3}{2}\;(>1)$$

Hence the given series converges.

Example 2. *Discuss the convergence of the series.*

$$\frac{1}{3}x+\frac{1.2}{3.5}x^2+\frac{1.2.3}{3.5.7}x^3+\ldots\infty \quad (x>0)$$

Sol. Here $$u_n=\frac{1.2.3\ldots n}{3.5.7\ldots(2n+1)}x^n$$

and $$u_{n+1}=\frac{1.2.3\ldots n.(n+1)}{3.5.7\ldots(2n+1)(2n+3)}x^{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}}=\frac{1.2.3\ldots n}{3.5.7\ldots(2n+1)}x^n\times\frac{3.5.7\ldots(2n+1)(2n+3)}{1.2.3\ldots n(n+1)}\times\frac{1}{x^{n+1}}$$

$$=\frac{(2n+3)}{(n+1)}\cdot\frac{1}{x}$$

$$=\frac{\left(2+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{\left(2+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x}$$

$$=\frac{2}{x}$$

The given series converges if $\frac{2}{x}>1$, *i.e.* $2>x$ or $x<2$ and diverges if $\frac{2}{x}<1$ or $2<x$ or $x>2$.

The test fails at $x=2$

When $x=2$,

$$\frac{u_n}{u_{n+1}}=\frac{(2n+3)}{(n+1)}\cdot\frac{1}{2}$$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(2n+3)}{2(n+1)}-1\right]$$

$$=\frac{n}{2n+2}$$

$$=\frac{1}{\left(2+\frac{2}{n}\right)}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right)=\underset{n\to\infty}{\text{Lt}}\frac{1}{\left(2+\frac{2}{n}\right)}=\frac{1}{2}\,(<1)$$

Thus the series diverges when $x=2$.

Hence the given series converges when $x < 2$ and diverges for $x>2$.

2·10. Logarithmic Test

If Σu_n be a positive term series, such that from and after some particular term,

$$\operatorname*{Lt}_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = k, \text{ then } \Sigma u_n$$

(*i*) converges if $k > 1$

(*ii*) diverges if $k < 1$.

Proof. Let from and after the particular term the series be

$$u_1+u_2+u_3+\ldots\ldots+u_n+\ldots\ldots$$

Case I. When $k > 1$.

Let p be a positive number such that $k > p > 1$ and compare the given series with an auxiliary series

$$\Sigma v_n = \Sigma \frac{1}{n^p},$$

which is convergent when $p > 1$.

Now Σu_n is convergent if

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

or if
$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$$

or if
$$\log \frac{u_n}{u_{n+1}} > p \log\left(1+\frac{1}{n}\right)$$

$$\left[\because \log\left(1+\frac{1}{n}\right)^p = p \log\left(1+\frac{1}{n}\right)\right]$$

or if
$$\log \frac{u_n}{u_{n+1}} > p\left(\frac{1}{n}-\frac{1}{2n^2}+\ldots\ldots\right)$$

or if
$$n \log \frac{u_n}{u_{n+1}} > p-\frac{p}{2n}+\ldots$$

or if
$$\operatorname*{Lt}_{n\to\infty} n \log \frac{u_n}{u_{n+1}} > p$$

or if $k > p$ which is true.

Hence the given series is convergent.

Case II. The case when $k < 1$ can be proved similary.

Note 1. The above test fails when $k=1$.

Example 1. *Test the convergence of the series*

$$\sum \frac{(n-1)^{n-1}}{n^n}$$

Sol. Here $u_n = \dfrac{(n-1)^{n-1}}{n^n}$

$$u_{n+1} = \frac{n^n}{(n+1)^{n+1}}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n-1)^{n-1}}{n^n} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{n^{n-1}\left(1-\frac{1}{n}\right)^{n-1} \cdot n^{n+1} \cdot \left(1+\frac{1}{n}\right)^{n+1}}{n^{2n}}$$

$$= \left(1-\frac{1}{n}\right)^{n-1} \cdot \left(1+\frac{1}{n}\right)^{n+1}$$

or
$$\frac{u_n}{u_{n+1}} = \left(1-\frac{1}{n}\right)^{n-1} \cdot \left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right)$$

$$\therefore \; n \log \frac{u_n}{u_{n+1}} = n\left[(n-1)\log\left(1-\frac{1}{n}\right) + \log\left(1+\frac{1}{n}\right)^n + \log\left(1+\frac{1}{n}\right)\right]$$

$$= n\left[(n-1)\left(-\frac{1}{n}-\frac{1}{2n^2}-\frac{1}{3n^3}+\dots\right) + \log\left(1+\frac{1}{n}\right)^n + \left(\frac{1}{n}-\frac{1}{2n^2}+\dots\right)\right]$$

$$= n\left[\left(-1+\frac{1}{n}-\frac{1}{2n}+\dots\right)\right] + \log\left(1+\frac{1}{n}\right)^n + \left(\frac{1}{n}-\frac{1}{2n^2}+\dots\right)\Big]$$

$$\therefore \quad \underset{n\to\infty}{\mathrm{Lt}}\, n \log \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\mathrm{Lt}}\, n \left[\left(-1+\frac{1}{n}-\frac{1}{2n}+\ldots\right)\right.$$

$$\left. +\log\left(1+\frac{1}{n}\right)^n+\left(\frac{1}{n}-\frac{1}{2n^2}+\ldots\right)\right]$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\, n \left[\left(-1+\frac{1}{2n}-\ldots\right)\right.$$

$$\left. +\log e+\left(\frac{1}{n}-\frac{1}{2n^2}+\ldots\right)\right]$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\, n \left[-1+\frac{3}{2n}+1+\ldots\right]$$

$$=\frac{3}{2}>1. \qquad [\because \log e=1]$$

Hence the given series converges.

Example 2. *Test the convergence of the series*

$$1+\frac{2x}{2\,!}+\frac{3^2x^2}{3\,!}+\frac{4^3x^3}{4\,!}+\frac{5^4x^4}{5\,!}+\ldots$$

Sol. Here $\qquad u_n=\dfrac{n^{n-1}\,x^{n-1}}{n\,!}$

and $\qquad u_{n+1}=\dfrac{(n+1)^n\,x^n}{(n+1)\,!}$

$$\therefore \qquad \frac{u_{n+1}}{u_n}=\frac{(n+1)^n}{n^{n-1}}\cdot\frac{x}{n+1}$$

$$=\left(1+\frac{1}{n}\right)^n\cdot\frac{n}{n+1}\cdot x.$$

$$\text{Thus } \underset{n\to\infty}{\mathrm{Lt}} \frac{u_{n+1}}{u_n}=\underset{n\to\infty}{\mathrm{Lt}}\left[\left(1+\frac{1}{n}\right)^n\cdot\frac{1}{1+\frac{1}{n}}\cdot x\right]=ex$$

Therefore, the series is convergent when $ex<1$ and divergent when $ex>1$. When $ex=1$ or $x=1/e$ we apply further test.

$$\underset{n\to\infty}{\mathrm{Lt}}\, n\log\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\mathrm{Lt}}\, n\log\left\{\frac{n^{n-1}}{(n+1)^n}\cdot e(n+1)\right\}$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\, n\log\left[\frac{e}{\left(1+\frac{1}{n}\right)^{n-1}}\right)$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\, n\left\{\log e-(n-1)\log\left(1+\frac{1}{n}\right)\right\}$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\ n\left\{1-(n-1)\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\cdots\right)\right\}$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\left\{n-(n-1)+\frac{(n-1)}{2n}-\frac{(n-1)}{3n^2}+\ldots\right\}$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\left\{\frac{3}{2}-\frac{1}{2n}-\frac{(n-1)}{3n^2}+\ldots\right\}=\frac{3}{2}>1$$

As such the given series is convergent when $x=1/e$.

EXERCISE 2 (*c*)

Test the following series for convergence or divergence.

1. $\frac{1}{3}x+\frac{1.2}{3.5}x^2+\frac{1.2.3}{3.5.7}x^3+\ldots\infty$ $\quad(x>0)$

2. $\frac{2}{5}x+\frac{2.4}{5.8}x^2+\frac{2.4.6}{5.8.11}x^3+\ldots\infty$ $\quad(x>0)$

3. $\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\ldots\infty$ $\quad(x>0)$

4. $\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots\infty$

5. $\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3}+\ldots\infty$

6. $1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma.(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3\,\gamma(\gamma+1)(\gamma+2)}x^3+\ldots$

2·11. Cauchy Root Test

If Σu_n be a positive term series, such that from and after some particular term,

$$\underset{n\to\infty}{\mathrm{Lt}}\ (u_n)^{1/n}=k, \text{ then } \Sigma u_n$$

(*i*) converges if $k<1$

(*ii*) diverges if $k>1$.

Proof. Let the series from and after the particular term be

$$u_1+u_2+u_3+\ldots+u_n+\ldots$$

Case. I. When $k<1$.

By definition of a limit a positive l $(k<l<1)$ can be found such that

$$(u_n)^{1/n}<l$$

$$\therefore \qquad (u_n) < l^n$$

or $$\Sigma u_n < \Sigma l^n$$

The series on the right hand side is a geometric series with common ratio l (< 1), hence convergent.

$\therefore$ Σu_n is also convergent.

Case II. When $k > 1$.

By definition of limit

$$(u_n)^{1/n} > 1$$

i.e. $$u_1 > 1,\ u_2 > 1,\ u_3 > 1,\ \dots,\ u_n > 1$$

$$\therefore \quad u_1+u_2+u_3+\dots u_n+\dots > 1+1+1+\dots+1+\dots$$

The series on the right hand sides is obviously divergent.

Therefore the given series Σu_n is also divergent.

Note. The test fails if $k=1$.

Example 1. *Discuss the convergence of the series,*

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\dots+\frac{1}{n^n}+\dots$$

Sol. Here $\quad u_n=\dfrac{1}{n^n} \quad$ or $\quad (u_n)^{1/n}=\dfrac{1}{n}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n}=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}=0<1$$

Hence the given series is convergent.

Example 2. *Discuss the convergence of the series.*

$$\sum\left(1+\frac{1}{n}\right)^{n^2}$$

Sol. Here . $\quad u_n=\left(1+\dfrac{1}{n}\right)^{n^2}$

$$\therefore \quad (u_n)^{1/n}=\left(1+\frac{1}{n}\right)^n$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n}=\underset{n\to\infty}{\text{Lt}}\ \left(1+\frac{1}{n}\right)^n=e>1$$

Therefore the given series is divergent.

Example 3. *Test the series for convergence and divergence,*

$$\frac{1}{2}\cdot x+\left(\frac{2}{3}\right)^4\cdot x^2+\left(\frac{3}{4}\right)^9\cdot x^3+\dots+\left(\frac{n}{n+1}\right)^{n^2}\cdot x^n+\dots$$

Sol. Here $$u_n=\left(\frac{n}{n+1}\right)^{n^2}.x^n$$

$\therefore$ $$(u_n)^{1/n}=\left(\frac{n}{n+1}\right)^n.x$$
$$=\left(\frac{1}{1+1/n}\right)^n.x$$
$$=\frac{x}{(1+1/n)^n}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n}=\underset{n\to\infty}{\text{Lt}}\ \frac{x}{(1+1/n)^n}=\frac{x}{e}$$

The given series converges if $\frac{x}{e}<1$, *i.e.* $x<e$ and diverges if $x/e>1$, *i.e.* $x>e$.

The above test fails at $x=e$.

When $x=e$
$$u_n=\left(\frac{n}{n+1}\right)^{n^2}.e^n$$

Now $\underset{n\to\infty}{\text{Lt}}\ u_n\neq 0$, therefore, the given series diverges at $x=e$.

Hence the given series converges for $x<e$ and diverges for $x\geqslant e$.

EXERCISE 2 (*d*)

Discuss the convergence and divergence of the following series :

1. $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\dots+\left(\frac{n}{2n+1}\right)^n+\dots$
2. $\frac{1}{2}+\left(\frac{2}{5}\right)^3+\left(\frac{3}{8}\right)^5+\dots$
3. $\frac{3}{4}+\left(\frac{5}{7}\right)^2+\left(\frac{7}{10}\right)^3+\dots$
4. $x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\dots$
5. $(x+3)+2^2(x+3)^2+3^2(x+3)^3+\dots$

2·12. Alternating Series

An infinite series in which from and after some particular term, the terms are alternately positive and negative is called an alternating series.

The general form of the series is

$$u_1 - u_2 + u_3 - u_4 + \cdots \qquad [u_n > 0]$$

For example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is an alternating series.

Leibnit'z Test for Convergence of Alternating Series

An alternating series

$$u_1 - u_2 + u_3 - u_4 + \cdots \qquad [u_n > 0]$$

is convergent, if for all values of n,

(i) $u_{n+1} < u_n$

(ii) $\underset{n \to \infty}{\text{Lt}} \; u_n = 0$

Proof. Let the sum of first $2n$ terms of the series be denoted by S_{2n}.

$$\therefore \; S_{2n} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots + (u_{2n-1} - u_{2n}) \qquad \ldots(1)$$

Since $\qquad u_n > u_{n+1}$,

$$u_1 > u_2 > u_3 > \ldots \qquad \ldots(2)$$

From (1) and (2) it follows that the expression in each of the brackets in (1) is positive.

Hence the sum S_{2n} is positive, *i e.* $S_{2n} > 0$

Now $S_{2n+1} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \cdots + (u_{2n} - u_{2n+1})]$

The expression in each of the parenthesis is positive and subtracted from u_1.

$$\therefore \qquad S_{2n+1} \leqslant u_1$$

Also $\qquad S_{2n+1} = S_{2n} + u_{2n+1}$

$$\therefore \qquad \underset{n \to \infty}{\text{Lt}} S_{2n+1} = \underset{n \to \infty}{\text{Lt}} S_{2n} + \underset{n \to \infty}{\text{Lt}} u_{2n+1} \qquad \ldots(3)$$

The given series is convergent if

$$\underset{n \to \infty}{\text{Lt}} S_{2n+1} = \underset{n \to \infty}{\text{Lt}} S_{2n} \qquad \ldots(4)$$

Therefore, from (3) and (4), we have

$$\underset{n \to \infty}{\text{Lt}} u_{2n+1} = 0 \text{ for all } n.$$

Replacing $(2n+1)$ by n, we have

or $$\underset{n \to \infty}{\text{Lt}} u_n = 0$$

Thus the given series is convergent if

(*i*) $u_1 > u_2 > u_3 > \ldots$

(*ii*) $\underset{n \to \infty}{\text{Lt}} \; u_n = 0$

Note 1. It must be noted carefully that both the conditions are satisfied simultaneously for convergence of an alternating series.

Note 2. An alternating series if not convergent is oscillatory.

Example 1. *Discuss the convergence of the series,*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Sol. The terms of the series are alternately positive and negative, and also each term is numerically less than the preceding term,

i.e. $$u_n > u_{n+1} \qquad \text{for all } n \qquad \ldots(1)$$

Also $$\underset{n \to \infty}{\text{Lt}} \; u_n = \underset{n \to \infty}{\text{Lt}} \; \frac{1}{n} = 0 \qquad \ldots(2)$$

From the conditions (1) and (2), the given series converges.

Example 2. *Test the convergence of the series,*

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$$

Sol. The terms of the series are alternately positive and negative and also each term is numerically less than the preceding term, *i.e.*,

$$u_n > u_{n+1} \qquad \text{for all } n \qquad \ldots(1)$$

Also $$u_n = \frac{1}{(2n-1)^2}$$

$\therefore$ $$\underset{n \to \infty}{\text{Lt}} \; u_n = \underset{n \to \infty}{\text{Lt}} \; \frac{1}{(2n-1)^2} = 0 \qquad \ldots(2)$$

From the conditions (1) and (2), the given series converges.

Example 3. *Test the convergence of the series,*

$$x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots$$

Sol. Here we shall apply D' Alembert test,

$$u_n = (-1)^{n-1} \frac{x^n}{n^2}$$

and $$u_{n+1} = (-1)^n \frac{x^{n+1}}{(n+1)^2}$$

$$\frac{u_{n+1}}{u_n}=-\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}$$

$$=-\frac{1}{(1+1/n)^2}\cdot x$$

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{-1}{(1+1/n)^2}\cdot x\right|=\frac{1}{(1+1/n)^2}\,|x|$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\left|\frac{u_{n+1}}{u_n}\right|=\underset{n\to\infty}{\text{Lt}}\left[\frac{1}{(1+1/n)^2}\right]|x|=|x|$$

Therefore the given series converges, when $|x|<1$ and diverges for $|x|>1$.

When $|x|=1$, D' Alembert's ratio test fails.

Now when $|x|=1$, the series can be shown to be convergent by Leibnitz's test.

Hence the given series is convergent for $|x|\leq 1$ and divergent for $|x|>1$.

EXERCISE 2 (*e*)

Examine each of the following series for convergence or divergence :

1. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$
2. $\frac{1}{2}-\frac{2}{3}\cdot\frac{1}{2^3}+\frac{3}{4}\cdot\frac{1}{3^3}-\frac{4}{5}\cdot\frac{1}{4^3}+\ldots$
3. $\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\ldots$
4. $\frac{1}{2}-\frac{1}{2\,.\,3}+\frac{1}{2\,.\,3\,.\,4}-\frac{1}{2\,.\,3\,.\,4\,.\,5}+\ldots$
5. $\sum\frac{(-1)^{n-1}}{n\,!}$
6. $\sum\frac{(-1)^{n-1}}{\log n}$
7. $1-\frac{x}{1^2}+\frac{x^2}{2^2}-\frac{x^3}{3^2}+\ldots$
8. $\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\ldots$

2·13. Absolute Convergence

A series Σu_n is said to be absolutely convergent if Σu_n and $\Sigma\,|\,u_n\,|$ both converge.

For example let

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots \infty$$

Here Σu_n is convergent by Leibnitz's rule.

Also $$\Sigma \mid u_n \mid = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \infty$$

is convergent. being a geometric series with common ratio $\frac{1}{2}$ (<1)

Hence the given series is absolutely convergent.

2·14. Conditional Convergence

A series Σu_n is said to be conditionally convergent if Σu_n converges but $\Sigma \mid u_n \mid$ does not converge.

For example let

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty$$

Here Σu_n is convergent by Leibnitz's rule

Now $\Sigma \mid u_n \mid = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \infty$, is a divergent series being a p-series with $p=1$.

Thus Σu_n converges but $\Sigma \mid u_n \mid$ diverges.

Hence the given series Σu_n is conditionally convergent.

EXERCISE 2 (*f*)

Find out which of the following series converge absolutely or conditionally :

1. $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \ldots$

2. $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \ldots$

3. $\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2^2} + \frac{1}{3} \cdot \frac{1}{2^3} - \ldots$

4. $\frac{1}{\sqrt[5]{2}} - \frac{1}{\sqrt[5]{3}} + \frac{1}{\sqrt[5]{4}} - \ldots + (-1)^n \frac{1}{\sqrt[5]{n}} + \ldots$

5. $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$

6. $\sum \frac{(-1)^{n-1}\, n}{n^2+1}$

7. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$

Exercise 1 (c) (Page 34—35)

4. (i) $\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

(ii) $\dfrac{\sin 2\alpha}{\cosh 2\beta-\cos 2\alpha}+i\dfrac{\sinh 2\beta}{\cosh 2\beta-\cos 2\alpha}$

(iii) $2\left[\dfrac{\sin\alpha\cosh\beta}{\cosh 2\beta-\cos 2\alpha}-i\dfrac{\cos\alpha\sinh\beta}{\cosh 2\beta-\cos 2\alpha}\right]$

(iv) $\cosh\alpha\cos\beta+i\sinh\alpha\sinh\beta$

(v) $\dfrac{\sinh 2\alpha}{\cosh 2\alpha+\cos 2\beta}+i\dfrac{\sin 2\beta}{\cosh 2\alpha+\cos 2\beta}$

(vi) $e^{\cosh x\cos y}[\cos(\sinh x\sin y)+i\sin(\sinh x\sin y)]$

Exercise 1 (f) (Page 42—43)

1. (i) $\cos^{-1}(\sqrt{\sin\theta})+i\log\{\sqrt{\sin\theta}+\sqrt{1+\sin\theta}\}$

(ii) $\sin^{-1}(\sqrt{\sin\theta})+i\log\{\sqrt{1+\sin\theta}-\sqrt{\sin\theta}\}$

(iii) $\dfrac{1}{2}\tanh^{-1}\left\{\dfrac{2x}{(1+x^2+y^2)}\right\}+\dfrac{i}{2}\tan^{-1}\left\{\dfrac{2y}{(1-x^2-y^2)}\right\}$

12. $e^{2n\pi+\alpha}[\cos(\log\sec\alpha)-i\sin(\log\sec\alpha)]$

13. $\operatorname{Exp}\left(-\dfrac{\pi^2}{8}\right)\cos\left(\dfrac{\pi}{4}\log 2\right)$

Exercise 1 (g) (Page 48)

1. $\dfrac{4\sin\alpha}{5-4\cos\alpha}$.

2. $\dfrac{1-x\cos y}{1-2x\cos y+y^2}$.

3. $\cot\alpha$

4. $e^{x\cos\beta}.\cos(\alpha+x\sin\beta)$.

5. $e^{-\cos\theta}\sin(\sin\theta)$.

6. $\frac{1}{2}\tan^{-1}\left[\dfrac{2c\sin\alpha}{1-c^2}\right]$, $(c\neq 1,\ \alpha\neq n\pi)$.

7. $\dfrac{\pi}{4}$ or $\dfrac{-\pi}{4}$ according as $\cos\alpha$ is positive or negative.

8. $\left(2\sin\dfrac{\alpha}{2}\right)^{-n}\cos\left(\dfrac{n\pi}{2}-\dfrac{n\alpha}{2}\right)$

9. $\dfrac{x\sin\theta+(-1)^{n+1}x^{n+1}\{x\sin n\theta+\sin(n+1)\theta\}}{x^2+2x\cos\theta+1}$

10. $1+\left(2\cos\dfrac{\theta}{2}\right)^n\cdot\sin\dfrac{n\theta}{2}$.

Exercise 2 (a) (Page 50)

1. Convergent
2. Divergent
3. Divergent
4. Divergent
5. Divergent
6. Oscillatory

Exercise 2 (b) Page (63—64)

1. Divgt.
2. Divgt.

3. Convgt.
4. Convgt.
5. Convgt.
6. Convgt.
7. Convgt.
8. Convgt.
9. Convgt.
10. Convgt. for $x<1$; Divgt. for $x\geqslant 1$
11. Convgt. for $x\leqslant 1$; Divgt. for $x>1$.
12. Convgt. for $x<1$; Divgt. for $x\geqslant 1$.
13. Convgt. for $x<1$; Divgt. for $x\geqslant 1$.
14. Convgt.
15. Convgt.
16. Convgt.
17. Convgt.
18. Convgt.
19. Divgt.
20. Congvt.
21. Divgt.
22. Convgt if $p>2$ and Divgt. for $p\leqslant 2$.
23. Convgt.
24. Divergent.
25. Divgt.
26. Convgt.
27. Divgt.
28. Convgt.
29. Convgt.

Exercise 2. (*c*) (Page 71)

1. Convgt. for $x<2$ and Divgt. for $x\geqslant 2$
2. Convgt. for $x<\frac{3}{4}$ and Divgt. for $x\geqslant \frac{3}{4}$
3. Convgt. for $x\leqslant 1$ and Divgt. for $x>1$
4. Convgt. for $x\leqslant 1$ and Divgt. for $x>1$
5. Convgt.
6. Convgt for $x<1$, Divgt. for $x>1$; when $x=1$, Convgt. if $\gamma-\alpha-\beta>0$ and Divgt. for $\gamma-\alpha-\beta\leqslant 0$,

Exercise 2 (*d*) (Page 73)

1. Convgt.
2. Convgt.
3. Convgt.
4. Convgt. for $x\leqslant 1$ and Divgt. for $x>1$.
5. Divgt.

Exereise 2 (*e*) (Page 76)

1. Convgt.
2. Convgt.
3. Convgt.
4. Convgt.
5. Convgt.
6. Convgt.
7. Convgt. for $|x|\leqslant 1$ and Divgt. for $|x|>1$.
8. Divgt.

Exercise 2 (*f*) (Page 77)

1. Abs. convgt.
2. Conditionally convgt.
3. Abs. convgt.
4. Conditionally convgt.
5. Conditionally convgt.
6. Conditionally convgt.
7. Conditionally convgt.

Exercise 3 (Page 84—85)

1. $\frac{4}{2x-3}-\frac{1}{x+1}$
2. $1-\frac{2}{x-3}+\frac{6}{x-4}$
3. $\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x+4}$
4. $\frac{3}{5(s-1)}-\frac{4}{15(s+4)}-\frac{1}{3(s-2)}$

6

Expansions of Functions and Indeterminate Forms

6·1. The student is already familiar with expansions of elementary functions using Binomial Theorem. In this chapter we shall expand the given function as an infinite convergent series in the form $a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$, known as power series. It is assumed that all the functions dealt here possess finite and continuous derivatives of all orders for the values of variables under consideration and are capable of expansions as power series.

6·2. Maclaurin's Theorem

If a function $f(x)$ can be expanded as an infinite convergent series of positive integral power of x, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\ldots+\frac{x^n}{n\,!}f^n(0)+\ldots$$

where $f^n(0)$ stands for nth derivative of $f(x)$ at $x=0$.

Proof. Since $f(x)$ is capable of being expanded as an infinite series, let

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots \qquad \ldots(1)$$

By successive differentiation, we get

$$f'(x)=a_1+2.a_2x+3.a_3x^2+4a_4x^3+\ldots \qquad \ldots(2)$$

$$f''(x)=2.a_2+3.2.a_3x+4.3.a_4x^2+\ldots \qquad \ldots(3)$$

$$f'''(x)=3.2.a_3+4.3.2.a_4x+\ldots \qquad \ldots(4)$$

Substituting $x=0$, successively in (1), (2), (3) and (4), we get

$$f(0)=a_0 \quad \text{or} \quad a_0=f(0)$$

$$f'(0)=a_1 \quad \text{or} \quad a_1=f'(0)$$

$$f''(0)=2.a_2 \quad \text{or} \quad a_2=\frac{f''(0)}{2\,!}$$

$$f'''(0)=3.2.a_3 \quad \text{or} \quad a_3=\frac{f'''(0)}{3\,!} \text{ and so on}$$

Substituting the values of a_0, a_1, a_2, a_3, etc. in (1), we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots+\frac{x^n}{n!}f^n(0)+\ldots$$

The series on the R.H.S. is known as Maclaurin's Series.

Note 1. Another useful form of the above series for the function $y=f(x)$ is,

$$f(x)=y=(y)_0+(y_1)_0x+(y_2)_0\frac{x^2}{2!}+\ldots+(y_n)_0\frac{x^n}{n!}+\ldots$$

where $(y_n)_0$ stands for the nth derivative of y at $x=0$.

6·3. Expansion of sin x

Let $f(x)=\sin x$	$\therefore\ f(0)=0$
$f'(x)=\cos x$	$f'(0)=1$
$f''(x)=-\sin x$	$f''(0)=0$
$f'''(x)=-\cos x$	$f'''(0)=-1$
$f^{iv}(0)=\sin x$	$f^{iv}(0)=0$ and so on.

The values of derivatives at $x=0$ are repeated in cycles of 0, 1, 0, -1.

By Malcaurin's Theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\ldots$$

$$\therefore\quad \sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

Aliter

Here $f(x)=\sin x$

$$\therefore\quad f^n(x)=\sin\left(x+\frac{n\pi}{2}\right)$$

Putting $x=0$ on both sides, we have

$$f^n(0)=\sin\frac{n\pi}{2}$$

Substituting $n=0, 1, 2, 3, \ldots$, we get

$f(0)=0$

$'(0)=\sin\frac{\pi}{2}=1$

$f''(0)=\sin\pi=0$

$f'''(0)=\sin\frac{3\pi}{2}=-1$

$f^{iv}(0)=\sin 2\pi=0$ and so on.

Hence by Maclaurin's Theorem, we have

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

6·4. Expansion of a^x

Let $f(x)=a^x$ $\qquad\therefore\ f(0)=a^0=1$

$f'(x)=a^x \log a$ $\qquad f'(0)=\log a$

$f''(x)=a^x (\log a)^2$ $\qquad f''(0)=(\log a)^2$

$f'''(x)=a^x (\log a)^3$ $\qquad f'''(0)=(\log a)^3$

Proceeding in this manner, we have, $f^n(0)=(\log a)^n$

By Maclaurin's theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots+\frac{x^n}{n\,!}f^n(0)+\ldots$$

$$\therefore\ a^x=1+(x\log a)+\frac{1}{2\,!}(x\log a)^2+\frac{1}{3\,!}(x\log a)^3+\ldots$$

$$+\frac{1}{n\,!}(x\log a)^n+\ldots$$

Note. The expansion of e^x can be obtained by putting $a=e$, in the above result so that $\log a=\log e=1$.

$$\therefore\ e^x=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots+\frac{x^n}{n\,!}+\ldots$$

6·5. Expansion of log $(1+x)$

Let $f(x)=\log(1+x)$ $\qquad\therefore\ f(0)=\log 1=0$

$f'(x)=\dfrac{1}{1+x}$ $\qquad f'(0)=1$

$f''(x)=-\dfrac{1}{(1+x)^2}$ $\qquad f''(0)=-1$

$f'''(x)=\dfrac{(-1)(-2)}{(1+x)^3}$ $\qquad f'''(0)=2\,!$

$f^{iv}(x)=\dfrac{(-1)(-2)(-3)}{(1+x)^4}$ $\qquad f^{iv}(0)=-3!$ and so on.

By Maclaurin's Theorem, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\frac{x^4}{4\,!}f^{iv}(0)+\ldots$$

$$\therefore\ \log(1+x)=x+\frac{x^2}{2\,!}(-1)+\frac{x^3}{3\,!}(2\,!)+\frac{x^4}{4\,!}\{-(3!)\}+\ldots$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

Note. Expansion of $\log(1-x)$ can be obtained by replacing x by $(-x)$, in the above result,

$$\therefore\ \log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots$$

Example 1. *Show that*

$$\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \dots$$

Hence find the value of π.

Sol. Let $y = \sin^{-1} x$ ...(1)

Differentiating both sides w.r.t. x,

$$y_1 = \frac{1}{\sqrt{1-x^2}} \qquad \text{...(2)}$$

or $y_1^2(1-x^2) = 1$

Differentiating again w.r.t. x, we get

$2y_1y_2(1-x^2) - 2y_1^2.x = 0$

or $y_2(1-x^2) - y_1x = 0$ ...(3)

Differentiating both sides of (3), n times using Leibnitz's Theorem.

$$[y_{n+2}(1-x^2) + {}^nC_1.y_{n+1}(-2x) + {}^nC_2.y_n(-2)] - [y_{n+1}x + {}^nC_1.y_n.1] = 0$$

or $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$ ...(4)

Putting $x = 0$ in (1), (2), (3) and (4) we get

$(y)_0 = 0, \ (y_1)_0 = 1, \ (y_2)_0 = 0$ ...(5)

and $(y_{n+2})_0 - n^2(y_n)_0 = 0$

or $(y_{n+2})_0 = n^2(y_n)_0$ ...(6)

Substituting $n = 1, 2, 3, 4, 5$, etc. in (6), we get

$(y_3)_0 = 1^2 (y_1)_0 = 1^2$ $[\because (y_1)_0 = 1]$

$(y_4)_0 = 2^2.(y_2)_0 = 0$ $[\because (y_2)_0 = 0]$

$(y_5)_0 = 3^2.(y_3)_0 = 3^2.1^2$

$(y_6)_0 = 4^2.(y_4)_0 = 0$

$(y_7)_0 = 5^2.(y_5)_0 = 5^2.3^2.1^2.$

By Maclaurin's Theorem,

$$y = (y_0) + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \frac{x^6}{6!}(y_6)_0 + \frac{x^7}{7!}(y_7)_0 + \dots$$

$$\therefore \quad \sin^{-1} x = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}.1^2 + \frac{x^4}{4!}.0 + \frac{x^5}{5!}1^2.3^2 + \frac{x^6}{6!}.0 + \frac{x^7}{7!}.1^2.3^2.5^2 + \dots$$

or

$$\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \dots$$

Putting $x=\frac{1}{2}$ on both sides, we get

$$\sin^{-1}\frac{1}{2}=\frac{1}{2}+\frac{1}{6}\left(\frac{1}{2}\right)^3+\frac{3}{40}\left(\frac{1}{2}\right)^5+\cdots$$

$$\therefore \quad \frac{\pi}{6}=0\cdot5000+0\cdot0208+0\cdot0023=0\cdot5231$$

$$\therefore \quad \pi=3\cdot1386=3\cdot14 \text{ (approximately)}$$

Example 2. *Expand* $\tan\left(\frac{\pi}{4}+x\right)$ *in ascending powers of* x. *Hence find the value of* $\tan 45° \, 30'$ *to four places of decimals.*

Sol. Let $f(x)=\tan\left(\frac{\pi}{4}+x\right)$ $\quad\therefore\quad f(0)=\tan\left(\frac{\pi}{4}\right)=1$

$$f'(x)=\sec^2\left(\frac{\pi}{4}+x\right) \quad \therefore\ f'(0)=\sec^2\frac{\pi}{4}=2$$

$$f''(x)=2\sec^2\left(\frac{\pi}{4}+x\right)\tan\left(\frac{\pi}{4}+x\right)$$

$$=2\left\{1+\tan^2\left(\frac{\pi}{4}+x\right)\right\}\tan\left(\frac{\pi}{4}+x\right)$$

$$=2\left\{\tan\left(\frac{\pi}{4}+x\right)+\tan^3\left(\frac{\pi}{4}+x\right)\right\}$$

$$\therefore\quad f''(0)=4$$

$$f'''(x)=2\left\{\sec^2\left(\frac{\pi}{4}+x\right)+3\tan^2\left(\frac{\pi}{4}+x\right)\sec^2\left(\frac{\pi}{4}+x\right)\right\}$$

$$=2\sec^2\left(\frac{\pi}{4}+x\right)\left\{1+3\tan^2\left(\frac{\pi}{4}+x\right)\right\}$$

$$=2\left\{1+\tan^2\left(\frac{\pi}{4}+x\right)\right\}\left\{1+3\tan^2\left(\frac{\pi}{4}+x\right)\right\}$$

$$=2\left\{1+4\tan^2\left(\frac{\pi}{4}+x\right)+3\tan^4\left(\frac{\pi}{4}+x\right)\right\}$$

$$\therefore\quad f'''(0)=16$$

$$f^{iv}(x)=2\left\{8\tan\left(\frac{\pi}{4}+x\right)\sec^2\left(\frac{\pi}{4}+x\right)+12\tan^3\left(\frac{\pi}{4}+x\right)\sec^2\left(\frac{\pi}{4}+x\right)\right\}$$

$$\therefore\quad f^{iv}(0)=80$$

By Maclaurin's Theorem

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\ldots$$

$$\therefore \quad \tan\left(\frac{\pi}{4}+x\right)=1+2x+\frac{x^2}{2\,!}(4)+\frac{x^3}{3\,!}(16)+\frac{x^4}{4\,!}(80)+\dots$$

$$=1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\dots$$

Putting $\quad x=30'=\frac{\pi}{360}$, we get

$$\tan 45^\circ\,30'=1+2\cdot\frac{\pi}{360}+2\cdot\left(\frac{\pi}{360}\right)^2+\dots$$

$$=1+0{\cdot}1745+0{\cdot}00015=1{\cdot}0176 \text{ (approximately).}$$

Example 3. *Expand* $\log\,[1-\log\,(1-x)]$ *in powers of* x *by Maclaurin's Theorem upto the term of* x^3 *and deduce the expansion of* $\log\,[1+\log\,(1+x)]$.

Sol. Let $f(x)=\log\,[1-\log\,(1-\log x)]$

We know, $\log\,(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots$

$$\therefore \qquad f(x)=\log\left[1-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\right)\right]$$

$$=\log\left[1+\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)\right]$$

Let $x+\frac{x^2}{2}+\frac{x^3}{3}+\dots=z$

$$\therefore \qquad f(x)=\log\,(1+z)$$

$$=z-\frac{z^2}{2}+\frac{z^3}{3}+\dots$$

$$=\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)-\frac{1}{2}\left(x+\frac{x^2}{2}+\frac{x^3}{3}-\dots\right)^2$$

$$+\frac{1}{3}\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)^3-\dots$$

$$=x+\frac{x^3}{6}+\dots$$

$$\therefore \qquad \log\,[1-\log\,(1-x)]=x+\frac{x^3}{6}+\dots \qquad \dots(1)$$

Replacing x by $\frac{x}{1+x}$ in both sides of (1), we get

$$\log\,[1+\log\,(1+x)]=x(1+x)^{-1}+\tfrac{1}{6}x^3(1+x)^{-3}$$

$$=x(1-x+x^2-\dots)+\tfrac{1}{6}x^3(1-3x+\dots)+\dots$$

$$\therefore \log[1+\log(1+x)]=x-x^2+\frac{7x^3}{6}+\dots$$

Example 4. *Apply Maclaurin's Theorem to obtain the expansion of the function $e^{ax} \sin bx$ in an infinite series of powers of x, giving the general term.*

Sol. Let $f(x)=e^{ax} \sin bx$

then $$f^n(x)=(a^2+b^2)^{n/2}.e^{ax} \sin (bx+n\theta) \quad ...(1)$$

where $$\tan \theta=\frac{b}{a} \quad [\S\ 5{\cdot}2(f)]$$

$$\therefore \quad \sin \theta=\frac{b}{\sqrt{a^2+b^2}} \text{ and } \cos \theta=\frac{a}{\sqrt{a^2+b^2}}$$

Now $f^n(0)=(a^2+b^2)^{n/2} \sin (n\theta)$ (from 1)

Putting $n=1, 2, 3, ...$ successively, we get

$$f'(0)=(a^2+b^2)^{1/2} . \sin \theta$$
$$=(a^2+b^2)^{1/2} . \frac{b}{\sqrt{a^2+b^2}}=b$$

$$f''(0)=(a^2+b^2). \sin 2\theta$$
$$=(a^2+b^2) . 2 \sin \theta \cos \theta$$
$$=(a^2+b^2) . \frac{2ab}{(a^2+b^2)}=2ab.$$

$$f'''(0)=(a^2+b^2)^{3/2} . \sin 3\theta$$
$$=(a^2+b^2)^{3/2} . (3 \sin \theta-4 \sin^3 \theta)$$
$$=(a^2+b^2)^{3/2}.\left[\frac{3b}{\sqrt{a^2+b^2}}-\frac{4b^3}{(a^2+b^2)^{3/2}}\right]$$
$$=b(3a^2-b^2)$$

Also $f(0)=0$

By Maclaurin's theorem,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!} f''(0)+\frac{x^3}{3!} f'''(0)+...$$

$$\therefore \quad e^{ax} \sin bx=bx+abx^2+\frac{b(3a^2-b^2)}{3!} x^3+...$$

6·6. Failure of Maclaurin's Theorem

It should be clearly understood that every function cannot be expanded by Maclaurin's Theorem. This theorem is not applicable in the following cases.

(*i*) The function $f(x)$ or any of its successive derivatives do not exist finitely at $x=0$.

(*ii*) The infinite series obtained by expansion does not converge. For example Maclaurin's Theorem cannot be applied to obtain the expansion of functions like $\cot x$, $\log x$ etc.

67. Taylor's Theorem

If a function $f(x+h)$ can be expanded as an infinite convergent series of positive integral powers of h, then

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots+\frac{h^n}{n!}f^n(x)+\ldots$$

where $f^n(x)$ stands for the nth derivative of $f(x+h)$ with respect to $(x+h)$, when $(x+h)$ is replaced by x.

Proof. Since $f(x+h)$ is capable of being expanded as an infinite series in powers of h,

$$f(x+h)=a_0+a_1h+a_2h^2+a_3h^3+a_4h^4+\ldots \quad \ldots(1)$$

Let us find derivative of $f(x+h)$.

$$\text{Now } \frac{d}{dh}f(x+h)=\frac{d}{d(x+h)}\cdot[f(x+h)]\cdot\frac{d}{dh}(x+h)$$

$$=f'(x+h)=f'(x+h)$$

$$\text{Also } \frac{d}{d(x+h)}f(x+h)=f'(x+h)$$

Hence differentiation of $f(x+h)$ with respect to $(x+h)$ or h gives the same results.

Differentiating (1) successively, with respect to h, we get

$$f'(x+h)=a_1+2.\,a_2h+3.a_3h^2+4.a_4h^3+\cdots \quad \ldots(2)$$

$$f''(x+h)=2.a_2+3.2.a_3h+4.3.a_4h^2+\cdots \quad \ldots(3)$$

$$f'''(x+h)=3.2.a_3+4.3.2\,a_4h+\cdots \quad \ldots(4)$$

Putting $h=0$, in (1), (2), (3) and (4) etc., we get

$$f(x)=a_0 \qquad \therefore \quad a_0=f(x)$$

$$f'(x)=a_1 \qquad a_1=f'(x)$$

$$f''(x)=2a_2 \qquad a_2=\frac{f''(x)}{2!}$$

$$f'''(x)=3.2.a_3 \qquad a_3=\frac{f'''(x)}{3!}$$

Substituting these values of a_0, a_1, a_2 and a_3 etc. in (1), we obtain

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\cdots+\frac{h^n}{n!}f^n(x)+\ldots$$

The series on R.H.S. is also known as Taylor's Series.

Note 1. A function $f(x)$ may be expanded in powers of $(x-a)$ by Taylor's Theorem by putting $h=x-a$.

$$\therefore f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$$

Note 2. Putting $x=0$ and $h=x$ in Taylor's series, we obtain Maclaurin's Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$$

Thus Maclaurin's Series can be obtained as a particular case of Taylor's Series.

Example 1. *Expand $\log_e (x+h)$ in powers of h by Taylor's Theorem.*

Sol. Let $f(x+h)=\log (x+h)$

$$\therefore \quad f(x)=\log x$$

$$f'(x)=\frac{1}{x}$$

$$f''(x)=-\frac{1}{x^2}=(-1)\cdot\frac{1}{x^2}$$

$$f'''(x)=\frac{(-1)(-2)}{x^3}=(-1)^2\cdot\frac{2!}{x^3}$$

$$\ldots\ldots\ldots\ldots$$

$$f^n(x)=(-1)^{n-1}\cdot\frac{(n-1)!}{x^n}$$

By Taylor's Theorem,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots+\frac{h^n}{n!}f^n(x)+\ldots$$

$$\therefore \log (x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}+\ldots+(-1)^{n-1}\frac{h^n}{nx^n}+\ldots$$

Example 2. *Expand $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$.*

Sol. Sin x may be written as $\sin\left[\frac{\pi}{2}+\left(x-\frac{\pi}{2}\right)\right]$. Here x is $\frac{\pi}{2}$ and h is $x-\frac{\pi}{2}$.

$$\text{Now } f(x)=\sin x \qquad \therefore \quad f\left(\frac{\pi}{2}\right)=1$$

$$f'(x)=\cos x \qquad f'\left(\frac{\pi}{2}\right)=0$$

$$f''(x)=-\sin x \qquad f''\left(\frac{\pi}{2}\right)=-1$$

$$f'''(x)=-\cos x \qquad f'''\left(\frac{\pi}{2}\right)=0$$

$$f^{iv}(x)=\sin x \qquad f^{iv}\left(\frac{\pi}{2}\right)=1$$

By Taylor's Theorem,

$$f(x)=f\left(\frac{\pi}{2}\right)+\left(\frac{\pi}{2}-x\right)f'\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2\,!}f''\left(\frac{\pi}{2}\right)$$

$$+\frac{\left(x-\frac{\pi}{2}\right)^3}{3\,!}f'''\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^4}{4\,!}f^{iv}\left(\frac{\pi}{2}\right)+\ldots$$

$$\sin x=1+0-\frac{\left(x-\frac{\pi}{2}\right)^2}{2\,!}+0+\frac{\left(x-\frac{\pi}{2}\right)^4}{4\,!}-\ldots$$

or $$\sin x=1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2\,!}+\frac{\left(x-\frac{\pi}{4}\right)^4}{4\,!}-\ldots$$

Example 3. *Prove by Taylor's Theorem.*

$$\tan^{-1}(x+h)=\tan^{-1}x+(h.\sin\alpha).\sin\alpha-(h\sin\alpha)^2.\frac{\sin 2\alpha}{2}$$

$$+(h\sin\alpha)^3.\frac{\sin 3\alpha}{3}+\ldots$$

where $\alpha=\cot^{-1}x.$

Sol. Here $f(x+h)=\tan^{-1}(x+h)$

A $f(x)=\tan^{-1}x$

$$f'(x)=\frac{1}{1+x^2}=\frac{1}{1+\cot^2\alpha}$$

$$=\frac{1}{\operatorname{cosec}^2\alpha}=\sin^2\alpha$$

$$f''(x)=-\frac{2x}{(1+x^2)^2}=-\frac{2\cot\alpha}{(1+\cot^2\alpha)^2}$$

$$=-\frac{2\cot\alpha}{\operatorname{cosec}^4\alpha}=-2\cot\alpha\sin^4\alpha$$

$$=-\sin^2\alpha.\sin 2\alpha$$

$$f'''(x)=-\frac{2(1-3x^2)}{(1+x^2)^3}=-\frac{2(1-3\cot^2\alpha)}{(1+\cot^2\alpha)^3}$$

$$=-2(\sin^2\alpha-3\cos^2\alpha)\sin^4\alpha$$

$$=-2(4\sin^2\alpha-3)\sin^4\alpha$$

$$=2(3\sin\alpha-4\sin^3\alpha)\sin^3\alpha$$

$$=2.\sin 3\alpha.\sin^3\alpha$$

$(\because 3\sin\alpha-4\sin^3\alpha=\sin 3\alpha)$

By Taylor's Theorem,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2\,!}f''(x)+\frac{h^3}{3\,!}f'''(x)+\ldots$$

$\therefore \quad \tan^{-1}(x+h) = \tan^{-1} x + (h \sin \alpha) \sin \alpha$

$$-(h \sin \alpha)^2 \cdot \frac{\sin 2\alpha}{2} + (h \sin \alpha)^3 \cdot \frac{\sin 3\alpha}{3} + \ldots$$

Aliter

$$f(x) = \tan^{-1} x$$

$\therefore \quad f^n(x) = (-1)^{n-1} \cdot (n-1)! \sin^n \alpha \sin n\alpha$

Substituting $\quad n = 1, 2, 3, \ldots$, we get

$$f'(x) = \sin \alpha \cdot \sin \alpha \qquad (\because 0! = 1)$$

$$f''(x) = -\sin^2 \alpha \cdot \sin 2\alpha$$

$$f'''(x) = 2! \sin^3 \alpha \cdot \sin 3\alpha$$

By Taylor's Theorem,

$$\tan^{-1}(x+h) = \tan^{-1} x + (h \sin \alpha) \cdot \sin \alpha$$

$$-(h \sin \alpha)^2 \cdot \frac{\sin 2\alpha}{2} + (h \sin \alpha)^3 \cdot \frac{\sin 3\alpha}{3} + \ldots$$

Example 4. *Apply Taylor's Theorem to calculate the value of $f\left(\frac{11}{10}\right)$, where $f(x) = x^3 + 3x^2 + 15x - 10$.*

Sol. By Taylor's Theorem, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

Put $\quad x = 1$ and $h = \frac{1}{10}$,

$$\therefore \quad f\left(1 + \frac{1}{10}\right) = f(1) + \frac{1}{10} f'(1) + \frac{1}{2}\left(\frac{1}{10}\right)^2 f''(1)$$

$$+ \frac{1}{6} \cdot \left(\frac{1}{10}\right)^3 f'''(1) + \ldots \quad \ldots(1)$$

Now $\quad f(x) = x^3 + 3x^2 + 15x - 10 \qquad \therefore \quad f(1) = 9$

$$f'(x) = 3x^2 + 6x + 15 \qquad f'(1) = 24$$

$$f''(x) = 6x + 6 \qquad f''(1) = 12$$

$$f'''(x) = 6 \qquad f'''(1) = 6$$

All other derivatives of $f(x)$ vanish.

Substituting these values in (1), we get

$$f\left(\frac{11}{10}\right) = 9 + \frac{1}{10} \cdot 24 + \frac{1}{2 \cdot 10^2}(12) + \frac{1}{6 \cdot 10^3}(6)$$

$$= 9 + 2 \cdot 4 + 0 \cdot 06 + 0 \cdot 001$$

$$= 11 \cdot 461$$

Example 5. *Given $\sin 30° = \frac{1}{2}$, use Taylor's Theorem to evaluate $\sin 31°$ correct to four significant figures.* ($\cos 30° = 0{\cdot}8660$)

Sol. Let $f(x+h) = \sin(x+h)$

$$\therefore \quad f(x) = \sin x$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x \text{ and so on}$$

By Taylor's Theorem, we have

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2}\sin x - \ldots \qquad \ldots(1)$$

Putting $x = \frac{\pi}{6}$ and $h = 1° = \frac{\pi}{180}$ radians in (1), we get

$$\sin 31° = \sin\frac{\pi}{6} + \frac{\pi}{180}\cos\frac{\pi}{6} - \frac{1}{2}\left(\frac{\pi}{180}\right)^2 \cdot \sin\frac{\pi}{6} - \ldots$$

$$= 0{\cdot}5 + 0{\cdot}0175 \times 0{\cdot}866 - \tfrac{1}{2}(0{\cdot}0175)^2 \times 0{\cdot}5 - \ldots$$

$$= 0{\cdot}5 + 0{\cdot}01515 - 0{\cdot}000076 - \ldots$$

$= 0{\cdot}5151$ upto four places of decimal.

6·8. Expansion by Differentiation and Integration of a known Series

These methods are useful, if the series for a given function is known and it is required to obtain a series for its derivative or integrals. The following examples illustrate the use of these methods.

Example 1. *Using the series for $\sin x$, obtain the series for $\cos x$.*

Sol. We know $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Differentiating both sides with respect to x,

$$\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Example 2. *Find by integration the series for.*

(i) $\log_e(1+x)$ (ii) $\tan^{-1} x$.

Sol. Now $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

Integrating both sides with respect to x, between limits 0 and x, we get

$$\int_0^x \frac{1}{1+x}\,dx=\int_0^x (1-x+x^2-x^3+\ldots)dx$$

or
$$\log_e (1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

(*ii*)
$$\frac{1}{1+x^2}=(1+x^2)^{-1}=1-x^2+x^4-x^6+\ldots$$

Integrating both sides with respect to x, between limits 0 and x, we get

$$\int_0^x \frac{1}{1+x^2}\,dx=\int_0^x (1-x^2+x^4-x^6+\ldots)dx$$

or
$$\tan^{-1} x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$$

6.9. Approximate Calculations

Let x and y be two variables related to each other by the relation $y=f(x)$. It is often required to find the change Δy in y when x changes by a small amount Δx. This can be obtained by Taylor's Theorem.

Now $y+\Delta y=f(x+\Delta x)$

$\therefore$
$$\Delta y=f(x+\Delta x)-y$$
$$=f(x+\Delta x)-f(x) \qquad [y=f(x)]$$

$\therefore$
$$\Delta y=[f(x)+f'(x)\Delta x+\frac{1}{2!}f''(x)(\Delta x)^2+\ldots]-f(x)$$

[By Taylor's Theorem]

Now as Δx is small, its square and higher powers may be neglected.

$\therefore$
$$\Delta y=f'(x)\,\Delta x$$

or
$$\Delta y=\frac{dy}{dx}\cdot\Delta x$$

If Δx is error in x, then $\frac{\Delta x}{x}$ is called the relative error and $\frac{\Delta x}{x}\times 100$ is called the percentage error.

Example 1. *A circular plate expands under the influence of heat so that its radius increases from 5 cm. to 5·06 cm. Find the approximate increase in the area.*

Sol. Let r be the radius of the circular plate and A its area, then
$$A=\pi r^2$$

$\therefore$
$$\Delta A=\frac{dA}{dr}\cdot\Delta r=2\pi r\,\Delta r$$

Here $\quad r=5$ cm. ; $\Delta r=0{\cdot}06$ cm.

$$\Delta A=2\pi\,(5)\,(0{\cdot}06)=0{\cdot}6\,\pi=1{\cdot}88 \text{ cm}^2.$$

Example 2. *What error in common logarithm of a number will be produced by an error of 1% in the number.*

Sol. Let $\quad y=\log_{10} x$

$$=\log_e x \,.\, \log_{10} e \quad \text{(Base changing formula)}$$

$$\therefore \quad \Delta y=\frac{dy}{dx}\,.\,\Delta x=\left(\frac{1}{x}\,.\,\Delta x\right)\log_{10} e$$

or
$$\Delta y=\frac{100\,\Delta x}{x}\,.\,\frac{\log_{10} e}{100}=\frac{1\times 0{\cdot}4343}{100}$$

$$=0{\cdot}004343.$$

Hence error in common logarithm is $0{\cdot}004343$.

Example 3. *The quantity Q of the water flowing over a V-notch is given by the formula $Q=CH^{5/2}$, where H is the head of water and C is a constant. Find the error in Q if the error in H is 1·5 per cent.*

Sol. Let error in Q be ΔQ. Then $\Delta Q=\dfrac{dQ}{dH}\,.\,\Delta H.$

Now $\dfrac{dQ}{dH}=\dfrac{5}{2}\,CH^{3/2}$. The percentage error in Q is given by

$$\frac{\Delta Q}{Q}\times 100=\frac{dQ}{dH}\,.\,\frac{\Delta H}{Q}\times 100$$

$$=\frac{5}{2}CH^{3/2}\times\frac{\Delta H}{CH^{5/2}}\times 100$$

$$=\frac{5}{2}\left(\frac{\Delta H}{H}\times 100\right)$$

$$=\frac{5}{2}(1{\cdot}5)=\frac{7{\cdot}5}{2}=3{\cdot}75.$$

Example 4. *The area of a triangle is calculated from the angles A and C and the side b. If a small error δA is made in measuring A, show that the percentage error in the area is about*

$$100\,\delta A\,.\,\sin C/\{\sin A\,.\,\sin(A+C)\}.$$

Sol. Let Δ be the area of the triangle and $\delta\Delta$ be the error in Δ. Then

$$\Delta=\tfrac{1}{2}\,bc\sin A$$

$$=\frac{1}{2}b^2\,\frac{\sin C\sin A}{\sin(A+C)}$$

since $\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$ or $c=\dfrac{b\sin C}{\sin B}=\dfrac{b\sin C}{\sin(180-A-C)}$

$$=\frac{b\sin C}{\sin(A+C)}.$$

$$\therefore \quad \frac{d\Delta}{dA}=\frac{1}{2}\, b^2 \sin C$$
$$\times\left[\frac{\cos A \,.\, \sin (A+C)-\cos (A+C) \sin A}{\sin^2 (A+C)}\right]$$
$$=\frac{b^2 \sin^2 C}{2 \sin^2 (A+C)}.$$

Now
$$\frac{\delta\Delta}{\Delta}\times 100=\left(\frac{d\Delta}{dA}\,.\,\delta A\right)\times\frac{100}{\Delta}$$
$$=\frac{b^2 \sin^2 C \,.\, \delta A}{2 \sin^2 (A+C)}\times\frac{100\times 2 \sin (A+C)}{b^2 \sin C \sin A}$$
$$=100\, \delta A \,.\, \sin C/\{\sin A \,.\, \sin (A+C)\}.$$

EXERCISE 6 (*a*)

1. Apply Maclaurin's Theorem to expand
(*i*) $\log \sec x$ (*ii*) $\cos x$ (*iii*) $\log (1+\sin x)$.

Prove the following by Maclaurin's Theorem.

2. $(1+x)^n=1+nx+\dfrac{n(n-1)}{1.2}\, x^2+\dfrac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\, x^3+\ldots$

3. $e^{x \cos x}=1+x+\dfrac{x^2}{2}-\dfrac{x^3}{3}+\ldots$

4. $e^{a \sin^{-1} x}=1+ax+\dfrac{(ax)^2}{2\,!}+\dfrac{a(1^2+a^2)}{3\,!}\, x^3$
$$+\frac{a^2(2^2+a^2)}{4\,!}\, x^4+\ldots\ldots$$

and hence show that
$$e^{\theta}=1+\sin \theta+\frac{\sin^2 \theta}{2\,!}+\frac{2}{3\,!}\sin^3 \theta+\frac{5}{4\,!}\sin^4 \theta+\ldots$$

5. Expand $\sin (m \sin^{-1} x)$ in ascending powers of x.

6. If $y=\sin \log (x^2+2x+1)$, then prove that
$$y=2x-x^2-\frac{2}{3}x^3+\frac{3}{2}x^4-\frac{5}{3}x^5+\frac{3}{2}\, x^6+\ldots$$

7. Expand $\log_e (x+\sqrt{x^2+1})$ up to first four terms by Maclaurin's theorem ; by putting $x=0{\cdot}75$ in the expansion, calculate the value of $\log_e 2$ to four places of decimals and find the percentage error if any.

8. Expand $\log_e \cos x$ by Maclaurin's theorem as far as the term x^4 and calculate $\log_{10} \cos \pi/12$ up to three places of decimal.

9. Calculate the approximate value of $\sqrt{10}$ to four places of decimal by taking the first four terms of an appropriate expansion.

[**Hint.** Expand $(1+x)^{1/2}$ by Maclaurin's theorem and put $x=1/9$.]

10. Expand $\log_e \sin(x+h)$ in ascending powers of h. Hence find the value of $\log_e \sin 32^\circ$ to four places of decimal. (Given $\log 2 = 0.69315$)

Expand the following functions by Taylor's theorem.

11. $\tan(x+h)$ in powers of h.

12. $\sin^{-1}(x+h)$ in powers of x.

13. $\log \sin x$ in powers of $x-2$.

14. $\tan^{-1} x$ in powers of $x-\frac{\pi}{4}$.

15. Expand the polynomial $f(x)=x^3-2x^2+3x+5$ in positive integral powers of $(x-2)$.

16. Apply Taylor's theorem to calculate the value of

$$f(2.001) \text{ if } f(x)=x^3-2x+5.$$

17. Evaluate (*i*) $\sin 30^\circ 30'$ (*ii*) $\sin 1^\circ 15'$.

18. Show that $\sin(a+h)$ differs from $\sin a + h\cos a$ by not more than $\frac{h^2}{2}$.

19. A heavy string suspended is in the form $y=a\cosh x/a$. Show that for small $|x|$ the shape of the string is approximately expressed by the parabola

$$y=a+\frac{x^2}{2a}.$$

20. Show that for $|x|<a$, to with in $\left(\frac{x}{a}\right)^2$, we have the approximate equality

$$e^{\frac{x}{a}} \approx \sqrt{\frac{a+x}{a-x}}.$$

21. Find the change in the total surface of a right circular cone when

(*a*) the radius remains constant while the altitude changes by small amount δh.

(*b*) the altitude remains constant while the radius changes by small amount δr.

22. A soap bubble of radius of 2 cm shrinks to radius 1.9 cm. Approximate the decrease in (*i*) volume (*ii*) surface area.

23. If an aviator flies around the world at a distance of 3 km above the equator, how many more kilometres will he travel than a persons who travels along the equator.

24. If $pv=20$ and p is measured as 5 ± 0.01, find v.

25. If $T=2\pi\sqrt{\frac{l}{g}}$, find the error in T corresponding to an error of 2 per cent in l, where g is a constant.

26. The pressure p and the volume v of a gas are connected by $pv^{1\cdot4}=c$. Find the percentage increase in the pressure corresponding to a decrease of $\frac{1}{2}$ per cent in the volume.

27. $ABCD$ is a rectangular protractor in which $AB=6$ cm, $BC=2$ cm and O is the mid-point of AB. An angle BOP is indicated by a mark P on the edge CD. If in setting off an angle θ degrees, a mark is made $\frac{1}{100}$ of a cm along the edge from the correct spot, show that the error in the angle is

$$\left(\frac{9\sin^2\theta}{10\pi}\right) \text{ degrees.}$$

28. If A is the area of a triangle having sides equal to a, b, c, and s its semi-perimeter, prove that the error in A resulting from a small error in the measurement of c is given by

$$\triangle A=\frac{1}{4}A\left\{\frac{1}{s}+\frac{1}{(s-a)}+\frac{1}{(s-b)}-\frac{1}{(s-c)}\right\}\triangle c$$

29. The power P required to propel a ship of length l moving with a velocity v is given by

$$P=kv^3 l^2.$$

Prove that a 3% increase in velocity and 4% increase in length requires a 17% increase in power.

30. The resistance R of a circuit having a battery with an e.m.f. E is given by the formula

$$R=E/I,$$

where I is the current in the circuit. If possible errors in E and I are 20% and 10% respectively, what is the error in R.

6·10. Indeterminate Forms

If $F(a)=f(a)=0$, then $\underset{x\to a}{\text{Lt}}\frac{F(x)}{f(x)}$ takes the form $\frac{0}{0}$, which is known as an **Indeterminate Form.** The other indeterminate forms are $\frac{\infty}{\infty}$, $\infty-\infty$, $0\times\infty$, 0^0, ∞^0, 1^∞ etc. These forms are evaluated by the use of Taylor's Theorem.

6·11. The Form $\frac{0}{0}$

Let $F(x)$ and $f(x)$ be functions that can be expanded by Taylor's Theorem in the neighbourhood of a point $x=a$ and also let $F(a)=f(a)=0$, then

$$\underset{x\to a}{\text{Lt}}\frac{F(x)}{f(x)}=\underset{x\to a}{\text{Lt}}\frac{F'(x)}{f'(x)}$$

Now $$\underset{x\to a}{\text{Lt}}\frac{F(x)}{f(x)}=\underset{x\to a}{\text{Lt}}\frac{F(a+x-a)}{f(a+x-a)}$$

By Taylor's Theorem, we get

$$\underset{x\to a}{\text{Lt}} \frac{F(x)}{f(x)} = \underset{x\to a}{\text{Lt}} \frac{F(a)+(x-a)\,F'(a)+\frac{(x-a)^2}{2\,!}F''(a)+\ldots}{f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2\,!}f''(a)+\ldots}$$

Since $F(a)=f(a)=0$, we have

$$\underset{x\to a}{\text{Lt}} \frac{F(x)}{f(x)} = \underset{x\to a}{\text{Lt}} \frac{(x-a)F'(a)+\frac{(x-a)^2}{2\,!}F''(a)+\ldots}{(x-a)f'(a)+\frac{(x-a)^2}{2\,!}f''(a)+\ldots}$$

$$= \underset{x\to a}{\text{Lt}} \frac{F'(a)+\frac{(x-a)}{2\,!}F''(a)+\ldots}{f'(a)+\frac{(x-a)}{2\,!}f''(a)+\ldots}$$

$$\therefore \quad \underset{x\to a}{\text{Lt}} \frac{F(x)}{f(x)} = \underset{x\to a}{\text{Lt}} \frac{F'(x)}{f'(x)}$$

This result is known as **De-L'Hospital's Rule.**

Now if $F'(a)=f'(a)=0$, then it follows that the value of the limit is $\frac{F''(a)}{f''(a)}$, provided $F''(a)$ and $f''(a)$ are not zero simultaneously.

In general if $F(a)=F'(a)=F''(a)=F'''(a)=\ldots=F^{n-1}(a)=0$ and also $f(a)=f'(a)=\ldots=f^{n-1}(a)=0$ but $F^n(a)$ and $f^n(a)$ are not both zero, then

$$\underset{x\to a}{\text{Lt}} \frac{F(x)}{f(x)} = \underset{x\to a}{\text{Lt}} \frac{F^n(x)}{f^n(x)}$$

Working Rule

(*i*) Put the given function in the form $\frac{0}{0}$.

(*ii*) Differentiate numerator and denominator separately.

(*iii*) Find the value of the function for the given value of the variable.

(*iv*) If still it is indeterminate *i.e* of the form $\frac{0}{0}$, repeat the above process, till a definite value is obtained.

Note. The students must note that numerator and denominator are to be differentiated **separately** and not as a single function.

Example 1 *Evaluate* $\underset{x\to 3}{Lt} \frac{x^3-3x}{x^2-9}$.

Sol. Here when $x\to 3$, both numerator and denominator approach zero, the quotient takes the form $\frac{0}{0}$.

Now, $\quad \underset{x\to 3}{\text{Lt}} \frac{x^3-3x}{x^2-9} \qquad \left(\frac{0}{0}\right)$

$$= \underset{x \to 3}{\text{Lt}} \frac{2x-3}{2x} = \frac{6-3}{6} = \frac{1}{2}.$$

Example 2. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{\log(1-x^2)}{\log \cos x}$

Sol. Here $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1-x^2)}{\log \cos x}$ $\qquad (0/0)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\dfrac{-2x}{1-x^2}}{\dfrac{-\sin x}{\cos x}}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2x}{(1-x^2)\tan x} \qquad (0/0)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2}{(1-x^2)\sec^2 x - 2x \tan x} = 2.$$

Example 3. *The current i in a coil containing a resistance R, an inductance L and an electromotive force E at time t is given by,* $i = \dfrac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$. *Obtain a suitable formula to evaluate i when R is negligible.*

Sol. Here we have to evaluate i as $R \to 0$.

$$\therefore \underset{R \to 0}{\text{Lt}}\, i = \underset{R \to 0}{\text{Lt}} \frac{E\left(1 - e^{-\frac{R}{L}t}\right)}{R} \qquad (0/0)$$

$$= \underset{R \to 0}{\text{Lt}} \frac{E\left(\frac{t}{L}\, e^{-\frac{R}{L}t}\right)}{1} \qquad \text{(Differentiation w.r.t. } R)$$

$$= \frac{Et}{L}$$

Thus the required formula is $\dfrac{Et}{L}$

Example 4. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{xe^x - \log(1+x)}{x^2}$

Sol. Here $\underset{x \to 0}{\text{Lt}} \dfrac{xe^x - \log(1+x)}{x^2}$ $\qquad (0/0)$

$$= \underset{x \to 0}{\text{Lt}} \frac{x.e^x + e^x - \dfrac{1}{1+x}}{2x} \qquad (0/0)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x.e^x + 2e^x + \dfrac{1}{(1+x)^2}}{2} = \frac{3}{2}.$$

Aliter. The above problem can easily be solved by using series expansions of e^x and $\log(x+1)$

$$\therefore \quad \underset{x \to 0}{\text{Lt}} \frac{xe^x - \log(1+x)}{x^2} \qquad (0/0)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)}{x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{3}{2}x^2+\frac{1}{6}x^3+\ldots}{x^2} = \frac{3}{2}$$

Example 5. *Find the value of* $\underset{x \to 0}{Lt} \frac{\tan 3x}{\sin 2x}$

Sol. Here $\underset{x \to 0}{\text{Lt}} \frac{\tan 3x}{\sin 2x} \qquad (0/0)$

$$= \underset{x \to 0}{\text{Lt}} \frac{3\sec^2 3x}{2\cos 2x} = \frac{3}{2}$$

Aliter. This problem could also be solved easily by using

$$\underset{x \to 0}{\text{Lt}} \frac{\tan x}{x} = 1 \text{ and } \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1.$$

i.e. $\underset{x \to 0}{\text{Lt}} \frac{\tan 3x}{\sin 2x}$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left(\frac{\tan 3x}{3x}\right)}{\left(\frac{\sin 2x}{2x}\right)} \times \frac{3}{2} = \frac{3}{2}.$$

6·12. Form $\frac{\infty}{\infty}$.

Let $F(x)$ and $f(x)$ be functions such that $F(a)$ and $f(a)$ both approach infinity, then to evaluate $\underset{x \to a}{\text{Lt}} \frac{F(x)}{f(x)}$;

let $\underset{x \to a}{\text{Lt}} \frac{F(x)}{f(x)} = K \qquad \ldots(1)$

Case 1. $K \neq 0$ and also $K \neq \infty$.

Now $\underset{x \to a}{\text{Lt}} \frac{F(x)}{f(x)} \qquad (\infty/\infty)$

$$= \underset{x \to a}{\text{Lt}} \frac{\frac{1}{f(x)}}{\frac{1}{F(x)}} \qquad (0/0)$$

Applying L-Hospital's rule, we get

$$\underset{x\to a}{\mathrm{Lt}} \frac{F(x)}{f(x)} = \underset{x\to a}{\mathrm{Lt}} \frac{\frac{-f'(x)}{[f(x)]^2}}{\frac{-F'(x)}{[F(x)]^2}}$$

$$= \underset{x\to a}{\mathrm{Lt}} \frac{f'(x)}{F'(x)} \cdot \underset{x\to a}{\mathrm{Lt}} \frac{[F(x)]^2}{[f(x)]^2}$$

$\therefore$
$$K = \underset{x\to a}{\mathrm{Lt}} \frac{f'(x)}{F'(x)} \cdot K^2 \qquad \text{(By 1)}$$

or
$$\frac{1}{K} = \underset{x\to a}{\mathrm{Lt}} \frac{f'(x)}{F'(x)}$$

$\therefore$
$$K = \underset{x\to a}{\mathrm{Lt}} \frac{F'(x)}{f'(x)}$$

or
$$\underset{x\to a}{\mathrm{Lt}} \frac{F(x)}{f(x)} = \underset{x\to a}{\mathrm{Lt}} \frac{F'(x)}{f'(x)}$$

Case II. When $K=0$.

From (1) we have
$$K+1 = \underset{x\to a}{\mathrm{Lt}} \frac{F(x)}{f(x)} + 1$$

$$= \underset{x\to a}{\mathrm{Lt}} \frac{F(x)+f(x)}{f(x)} \qquad (\infty/\infty)$$

$$= \underset{x\to a}{\mathrm{Lt}} \frac{F'(x)+f'(x)}{f'(x)} \qquad \text{(By case I)}$$

$$= \underset{x\to a}{\mathrm{Lt}} \frac{F'(x)}{f'(x)} + 1$$

$\therefore$
$$K = \underset{x\to a}{\mathrm{Lt}} \frac{F'(x)}{f'(x)}$$

or
$$\underset{x\to a}{\mathrm{Lt}} \frac{F(x)}{f(x)} = \underset{x\to a}{\mathrm{Lt}} \frac{F'(x)}{f'(x)}$$

Case III. K is infinity.

$$\underset{x\to a}{\mathrm{Lt}} \frac{1}{\frac{F(x)}{f(x)}} = \underset{x\to a}{\mathrm{Lt}} \frac{f(x)}{F(x)} \qquad (\infty/\infty)$$

$$= \underset{x\to a}{\mathrm{Lt}} \frac{f'(x)}{F'(x)} \qquad \text{(By case II)}$$

$$\therefore \qquad \underset{x \to a}{\text{Lt}} \frac{F(x)}{f(x)} = \underset{x \to a}{\text{Lt}} \frac{F'(x)}{f'(x)}.$$

Thus we see $\underset{x \to a}{\text{Lt}} \dfrac{F(x)}{f(x)} = \underset{x \to a}{\text{Lt}} \dfrac{F'(x)}{f'(x)}$

Note. It must be noted that while evaluating the form $\dfrac{\infty}{\infty}$, we should try to put in the form $\dfrac{0}{0}$ as soon as possible, for quick solution of the problem.

Example 1. *Evaluate* $\underset{x \to 1}{Lt} \dfrac{\log (I-x)}{\cot \pi x}$

Sol. Here
$$\underset{x \to 1}{\text{Lt}} \frac{\log (1-x)}{\cot \pi x} \qquad (\infty/\infty)$$

$$= \underset{x \to 1}{\text{Lt}} \frac{\dfrac{-1}{1-x}}{-\pi \operatorname{cosec}^2 \pi x}$$

$$= \frac{1}{\pi} \underset{x \to 1}{\text{Lt}} \frac{\sin^2 \pi x}{1-x} \qquad (0/0)$$

$$= \frac{1}{\pi} \underset{x \to 1}{\text{Lt}} \frac{2 \pi \sin \pi x \cos \pi x}{-1} = 0$$

Example 2. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{\cot x}{\cot 2x}$

Sol. Now $\underset{x \to 0}{\text{Lt}} \dfrac{\cot x}{\cot 2x} \qquad (\infty/\infty)$

$$= \underset{x \to 0}{\text{Lt}} \frac{\tan 2x}{\tan x} \qquad (0/0)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sec^2 2x}{\sec^2 x} = 2.$$

Note. It may be seen that the function has been put in the form $\dfrac{0}{0}$ at the earliest opportunity.

6·13. The Forms $0 \times \infty$ and $\infty - \infty$

These forms are easily reducible to the forms $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, as is evident from the following examples.

Example 1. *Evaluate* $\underset{x \to 0}{Lt}\ x \log \sin x$.

Sol. Here $\underset{x \to 0}{\text{Lt}}\ x \log \sin x \qquad (0 \times \infty)$

$$[\because \log 0 \to -\infty]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log \sin x}{\frac{1}{x}} \qquad (\infty/\infty)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}} = -\underset{x \to 0}{\text{Lt}}\ x^2 \cot x$$

$$= -\underset{x \to 0}{\text{Lt}} \frac{x^2}{\tan x} \qquad (0/0)$$

$$= -\underset{x \to 0}{\text{Lt}} \frac{2x}{\sec^2 x} = 0$$

Example 2. *Evaluate* $\underset{x \to 1}{Lt} \left(\frac{2}{x^2-1} - \frac{1}{x-1} \right)$.

Sol. Here $\underset{x \to 1}{\text{Lt}} \left(\frac{2}{x^2-1} - \frac{1}{x-1} \right) \qquad (\infty - \infty)$

$$= \underset{x \to 1}{\text{Lt}} \frac{1-x}{x^2-1} \qquad (0/0)$$

$$= \underset{x \to 1}{\text{Lt}} \left(-\frac{1}{2x} \right) = -\frac{1}{2}.$$

Example 3. *Evaluate* $\underset{x \to 0}{Lt} \left(\frac{1}{x^2} - \cot^2 x \right)$

Sol. Here $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \cot^2 x \right) \qquad (\infty - \infty)$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x} \right) \qquad (0/0)$$

$$= \underset{x \to 0}{\text{Lt}} \left[\frac{\sin^2 x - x^2 \cos^2 x}{x^4 \cdot \left(\frac{\sin x}{x} \right)^2} \right] = \underset{x \to 0}{\text{Lt}} \frac{\sin^2 x - x^2 \cos^2 x}{x^4} \quad (0/0)$$

$$\left(\because \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1 \right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \right)^2 - x^2 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\cdots \right)^2}{x^4}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\left(x^2-\frac{x^4}{3}+\ldots\right)-x^2(1-x^2+\ldots)}{x^4}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\left(-\frac{1}{3}+1\right)x^4+\ldots}{x^4}=\frac{2}{3}.$$

6·15. The Forms 0^0 and ∞^0.

These forms can be reduced to either of the two forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking logs or by transformations, as illustrated in the following examples.

Example 1. *Evaluate* $\underset{x\to 0}{\mathrm{Lt}} (\sinh x)^{\tanh x}$

Sol. Let $\quad y= \underset{x\to 0}{\mathrm{Lt}} (\sinh x)^{\tanh x} \qquad (0^0).$

$\therefore \quad \log y= \underset{x\to 0}{\mathrm{Lt}} \tanh x \log \sinh x \qquad (0\times\infty)$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\log \sinh x}{\coth x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\frac{1}{\sinh x}.\cosh x}{-\operatorname{cosech}^2 x}$$

$$=- \underset{x\to 0}{\mathrm{Lt}} \frac{\sinh^2 x \cosh x}{\sinh x}$$

$$=- \underset{x\to 0}{\mathrm{Lt}} \sinh x \cosh x=0 \quad (\because \sinh 0=0)$$

$\therefore \quad y=e^0=1$

Hence $\quad \underset{x\to 0}{\mathrm{Lt}} (\sinh x)^{\tanh x}=1.$

Example 2. *Evaluate* $\underset{x\to a}{\mathrm{Lt}} \left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Sol. Let $\quad y= \underset{x\to a}{\mathrm{Lt}} \left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} \qquad (1^\infty)$

$\therefore \quad \log y= \underset{x\to a}{\mathrm{Lt}} \tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right) \qquad (\infty\times 0)$

$$= \underset{x\to a}{\mathrm{Lt}} \frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left(\frac{0}{0}\right)$$

$$= \underset{x \to a}{\text{Lt}} \frac{\frac{1}{(2-x/a)} \cdot \left(-\frac{1}{a}\right)}{-\frac{\pi}{2a} \cdot \operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)} = \frac{2}{\pi}$$

$\therefore$ $$y = e^{\frac{2}{\pi}}$$

Hence $$\underset{x \to a}{\text{Lt}} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} = e^{\frac{2}{\pi}}$$

Example 3. *Evaluate* $\underset{x \to \frac{\pi}{2}}{Lt} (\tan x)^{\cos x}$

Sol. Let $$y = \underset{x \to \frac{\pi}{2}}{\text{Lt}} (\tan x)^{\cos x} \qquad (\infty^0)$$

$\therefore$ $$\log y = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \cos x \log \tan x \qquad (0 \times \infty)$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\log \tan x}{\sec x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\sec x \cdot \tan x}$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\cos x}{\sin^2 x} = 0$$

$\therefore$ $$y = e^0 = 1$$

Hence $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\tan x)^{\cos x} = 1.$

6·16. Miscellaneous Problems

Example. *Evaluate*

(i) $\underset{x \to 0}{Lt} \dfrac{(1+x)^{\frac{1}{x}} - e + \frac{1}{2}ex}{x^2}$

(ii) $\underset{x \to 0}{Lt} \left[\dfrac{\pi}{4x} - \dfrac{\pi}{2x(e^{\pi x}+1)}\right]$

(iii) $\underset{x\to 0}{Lt}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$

(iv) *Find a and b such that*

$$\underset{x\to 0}{Lt}\ \frac{x(1+a\cos x)-b\sin x}{x^3}=1$$

Sol. (*i*) Here $\underset{x\to 0}{\text{Lt}}\ \dfrac{(1+x)^{\frac{1}{x}}-e+\frac{1}{2}ex}{x^2}$ $\left(\dfrac{0}{0}\right)$

$$\left(\because\ \underset{x\to 0}{\text{Lt}}\ (1+x)^{\frac{1}{x}}=e\right)$$

First we shall evaluate $(1+x)^{\frac{1}{x}}$.

Let $y=(1+x)^{\frac{1}{x}}$

$$\log y=\frac{1}{x}\ \log(1+x)=\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)$$

$$=1-\frac{x}{2}+\frac{x^2}{3}-\ldots$$

$$\therefore\quad y=e^{1-\frac{x}{2}+\frac{x^2}{3}-\ldots}$$

$$=e^1\cdot e^z,\qquad \text{where } z=-\frac{x}{2}+\frac{x^2}{3}-\ldots$$

$$y=e\cdot\left(1+z+\frac{z^2}{2!}+\ldots\right)$$

$$=e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)^2+\ldots\right]$$

$$=e\left[1-\frac{x}{2}+\left(\frac{1}{3}+\frac{1}{8}\right)x^2+\ldots\right]$$

$$=e-\frac{ex}{2}+\frac{11}{24}ex^2+\ldots$$

$$\therefore\ \underset{x\to 0}{\text{Lt}}\ \frac{(1+x)^{\frac{1}{x}}-e+\frac{1}{2}ex}{x^2}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{e-\frac{ex}{2}+\frac{11}{24}ex^2+\ldots-e+\frac{1}{2}ex}{x^2}=\frac{11}{24}e.$$

Hence $$\underset{x\to 0}{\text{Lt}} \frac{(1+x)^{\frac{1}{x}}-e+\frac{1}{2}ex}{x^2}=\frac{11}{24}e.$$

(ii) $$\underset{x\to 0}{\text{Lt}} \left\{\frac{\pi}{4x}-\frac{\pi}{2x\,(e^{\pi x}+1)}\right\} \qquad (\infty-\infty)$$

$$=\underset{x\to 0}{\text{Lt}}\ \pi\left\{\frac{e^{\pi x}-1}{4x(e^{\pi x}+1)}\right\} \qquad \left(\frac{0}{0}\right)$$

$$=\pi\ \underset{x\to 0}{\text{Lt}}\left\{\frac{\pi.e^{\pi x}}{4x(\pi e^{\pi x})+4(e^{\pi x}+1)}\right\}=\frac{\pi^2}{8}.$$

Hence $$\underset{x\to 0}{\text{Lt}}\left\{\frac{\pi}{4x}-\frac{\pi}{2x(e^{\pi x}+1)}\right\}=\frac{\pi^2}{8}.$$

(iii) $$\underset{x\to 0}{\text{Lt}}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} \qquad (1^\infty)$$

Let $$y=\underset{x\to 0}{\text{Lt}}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} \qquad (1^\infty)$$

$$\therefore\ \log\ y=\underset{x\to 0}{\text{Lt}}\ \frac{1}{x^2}\log\frac{\tan x}{x} \qquad \left(\frac{0}{0}\right)$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{\left(\frac{x}{\tan x}\right).\ \frac{x\sec^2 x-\tan x}{x^2}}{2x}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{x\sec^2 x-\tan x}{2x^3} \qquad \left(\frac{0}{0}\right)$$

$$\left[\because\ \underset{x\to 0}{\text{Lt}}\left(\frac{x}{\tan x}\right)=1\right]$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{2x\sec^2 x\tan x+\sec^2 x-\sec^2 x}{6x}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{1}{3}\sec^2 x.\left(\frac{\tan x}{x}\right)$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{1}{3}\sec^2 x \qquad \left[\because\ \underset{x\to 0}{\text{Lt}}\ \frac{\tan x}{x}=1\right]$$

$$=\frac{1}{3}$$

$$\therefore\ y=e^{1/3}$$

Hence $$\underset{x\to 0}{\text{Lt}}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}=e^{1/3}.$$

Note. The above problem can be solved by using series expansion for $\tan x$ etc.

(*iv*) $\underset{x \to 0}{\text{Lt}} \dfrac{x(1+a\cos x)-b\sin x}{x^3} \qquad \left(\dfrac{0}{0}\right)$

$$= \underset{x \to 0}{\text{Lt}} \frac{x\left\{1+a\left(1-\dfrac{x^2}{2\,!}+\dfrac{x^4}{4\,!}\cdots\right)\right\}-b\left(x-\dfrac{x^3}{3\,!}+\dfrac{x^5}{5\,!}\cdots\right)}{x^3}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(1+a-b)x+\left(-\dfrac{a}{2}+\dfrac{b}{6}\right)x^3+\cdots}{x^3}$$

Since the given limit is equal to 1, we must have

$$1+a-b=0 \qquad \ldots(i)$$

and
$$-\frac{a}{2}+\frac{b}{6}=1 \qquad \ldots(ii)$$

Solving (*i*) and (*ii*), we get

$$a=-\frac{5}{2}, \quad b=-\frac{3}{2}.$$

EXERCISE 6 (*b*)

Evaluate

1. $\underset{x \to 0}{\text{Lt}} \dfrac{x^4-256}{x^2-16}$ 2. $\underset{x \to 0}{\text{Lt}} \dfrac{xe^x}{1-e^x}$

3. $\underset{x \to \frac{\pi}{4}}{\text{Lt}} \dfrac{\sec^2 x-2\tan x}{1+\cos 4x}$

4. $\underset{x \to 0}{\text{Lt}} \dfrac{a^x-b^x}{x}$ 5. $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \dfrac{\log \tan x}{\sec x}$

6. $\underset{x \to \frac{1}{2}}{\text{Lt}} \dfrac{\sec \pi x}{\tan 3\pi x}$ 7. $\underset{x \to 0}{\text{Lt}} \dfrac{\log \sin 2x}{\log \sin x}$

8. $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sec x-\tan x)$

9. $\underset{x \to 0}{\text{Lt}} \left\{\dfrac{1}{x}-\dfrac{1}{x^2}\log(1+x)\right\}$ (*Delhi 1983*)

10. $\underset{x \to 0}{\text{Lt}} \left\{\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right\}$

11. $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \left\{\dfrac{1}{\cos x}-\dfrac{1}{1-\sin x}\right\}$

12. $\underset{x\to 0}{\text{Lt}} \ (\cot x)^{\sin x}$ 13. $\underset{x\to\infty}{\text{Lt}} \ 2^x . \sin\left(\frac{a}{2^x}\right)$

14. (*i*) $\underset{x\to 0}{\text{Lt}} \ (\cos 2x)^{\frac{3}{x^2}}$ (*ii*) $\underset{x\to 0}{\text{Lt}} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

15. $\underset{x\to\infty}{\text{Lt}} \left(1+\frac{a}{x}\right)^x$ 16. $\underset{x\to 0}{\text{Lt}} \left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}}$

17. If the limit $(\sin 2x+a \sin x)/x^3$ be finite, find the value of a and the limit.

Prove the following :

18. $\underset{x\to 0}{\text{Lt}} \dfrac{e^x-e^{-x}-2x}{x-\sin x}=2$ 19. $\underset{x\to 0}{\text{Lt}} \dfrac{x-\sin x}{x^3}=\dfrac{1}{6}$

20. $\underset{x\to \frac{\pi}{2}}{\text{Lt}} \ (\sin x-\cos x)^{\tan x}=\dfrac{1}{e}$.

21. $\underset{x\to 0}{\text{Lt}} \left(\dfrac{1}{x}\right)^{\tan x}=1$.

22. $\underset{x\to 0}{\text{Lt}} \dfrac{\sin x-\tan^{-1} x}{x^2 \log (1+x)}=\dfrac{1}{6}$. 23. $\underset{x\to 0}{\text{Lt}} \ x^x=1$.

24. $\underset{x\to\infty}{\text{Lt}} \dfrac{x-\sin x}{x+\sin x}=1$. 25. $\underset{x\to 0}{\text{Lt}} \ \log_{\tan x} \tan 2x=1$

Exercise 5 (b) (Page 131-132)

1. (a) $[9x^2+6nx+n(n-1)]\,e^{3x}.3^{n-2}$.

(b) $x^4 \,.\, 3^n \sin\left(3x+\frac{n\pi}{2}\right)+4nx^3 \,.\, 3^{n-1}\sin\left\{3x+(n-1)\frac{\pi}{2}\right\}$
$+\ldots+n(n-1)(n-2)(n-3)3^{n-4}\sin\left\{3x+(n-4)\frac{\pi}{2}\right\}$

(c) $e^x[x^n+{}^nC_1.nx^{n-1}+{}^nC_2.n(n-1)x^{n-2}+\ldots+n\,!]$

(d) $e^x\left[\log x+\frac{{}^nC_1}{x}-\frac{{}^nC_2}{x^2}+\frac{{}^nC_3\,2\,!}{x^3}+\ldots+\frac{(-1)^{n-1}(n-1)!}{x^n}\right]$.

5. $y_n(0)=0$, when n is even

$y_n(0)=m(1-m^2)(3^2-m^2)\ldots\ldots[(n-2)^2-m^2]$,
when n is odd.

6. $f^n(0)=m^2(2^2+m^2)(4^2+m^2)\ldots\ldots[(n-2)^2+m^2]\,e^{\frac{m\pi}{2}}$ n is even

$=-m(1+m^2)(3^2+m^2)\ldots.[(n-2)^2+m^2]\,e^{\frac{m\pi}{2}}$ n is odd

10. $y_n(0)=(n-2)^2\,.\,(n-4)^2\ldots.4^2.2^2.2$, if n is even
$y_n(0)=0$, if n is odd.

11. $f^{n+1}(0)=n^2(n-2)^2(n-4^2)\ldots4^2.2^2$ when n is even
$=0$, when n is odd.

Exercise 6 (a) (Page 147-149)

1. (i) $\frac{x^2}{2\,!}+\frac{2x^4}{4\,!}+\frac{16x^6}{6\,!}+\ldots$

(ii) $1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\ldots$

(iii) $x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\frac{x^5}{24}-\ldots$

5. $mx+m(1^2-m^2)\cdot\frac{x^3}{3\,!}+m(1^2-m^2)(3^2-m^2)\,\frac{x^5}{5\,!}+\ldots$

7. $x-\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7}+\ldots$; 0·6899 ; 0·46

8. $-\frac{x^2}{2}-\frac{x^4}{12}+\ldots\ldots$; $\bar{1}\cdot954$

9. 3·1629

10. $\log\sin x+h\cot x-\frac{h^2}{2}\operatorname{cosec}^2 x+\frac{h^3}{3}\cot x\operatorname{cosec}^2 x+\ldots$;
$\bar{1}\cdot36486$

11. $\tan x+h\sec^2 x+h^2\sec^2 x\tan x+\frac{h^3}{3}(1+3\tan^2 x).\sec^2 x+\ldots$

12. $\sin^{-1}(h)+x(1-h^2)^{-1/2}+\frac{x^2}{2!}h(1-h^2)^{-3/2}+\ldots$

13. $\log\sin 2+(x-2)\cot 2-\frac{1}{2}(x-2)^2\operatorname{cosec}^2 2+\frac{1}{3}(x-2)^3\operatorname{cosec}^2 2\cot 2+\ldots$

14. $\tan^{-1}\frac{\pi}{4}+\left(x-\frac{\pi}{4}\right)\cdot\frac{1}{1+\left(\frac{\pi^2}{16}\right)}-\frac{\left(x-\frac{\pi}{4}\right)^2\pi}{4\left(1+\frac{\pi^2}{16}\right)^2}+\ldots$

15. $11+7(x-2)+4(x-2)^2+(x-2)^3$

16. 9·01 (app.)

17. (*i*) 0·50725 (*ii*) 0·02178

21. (*a*) $\frac{\pi rh\,\delta h}{\sqrt{r^2+h^2}}$ (*b*) $\frac{\pi(2r^2+2r\sqrt{r^2+h^2}+h^2)}{\sqrt{r^2+h^2}}\delta r.$

22. (*i*) $-5·024$ cu. cm. (*ii*) $-5·024$ sq. cm, **23.** 6π km.

24. $4\pm0·008$ **25.** 1%

26. 0·71 **30.** 10%.

Exercise 6 (*b*) (Page 160—161)

1. 16 **2.** (-1)
3. $\frac{1}{2}$ **4.** $(\log a-\log b)$
5. 0 **6.** 3. **7.** 1
8. 0 **9.** $\frac{1}{2}$ **10.** $-\frac{1}{3}$ **11.** $-\infty$
12. 1 **13.** a **14.** (*i*) e^{-6} (*ii*) ∞. **15.** e^a **16.** $e^{\frac{1}{3}}$
17. $a=-2$; -1

Exercise 7 (*a*) (Page 165—167)

3. $\frac{i_0R}{L}e^{-Rt/L}$ **4.** $\frac{3}{8\pi}$ cm per minute

5. $\frac{1}{20}$ radian per second. **6.** $\frac{90}{\sqrt{34}}$ m per second

8. 120 kg per square cm decreasing.

11. $\frac{8\sqrt{3}}{3}$ m per second **14.** 8π km/minute

Exercise 7 (*b*) (Page 175—179)

1. $\frac{2\sqrt{3}}{9}$ **4.** $t=\sqrt{(\mu^2+1)}-\mu$

5. $\frac{a}{2}$; $\frac{16}{25\sqrt{5}a^4}$ **6.** $\frac{3K}{4}$ when $x=4$.

14

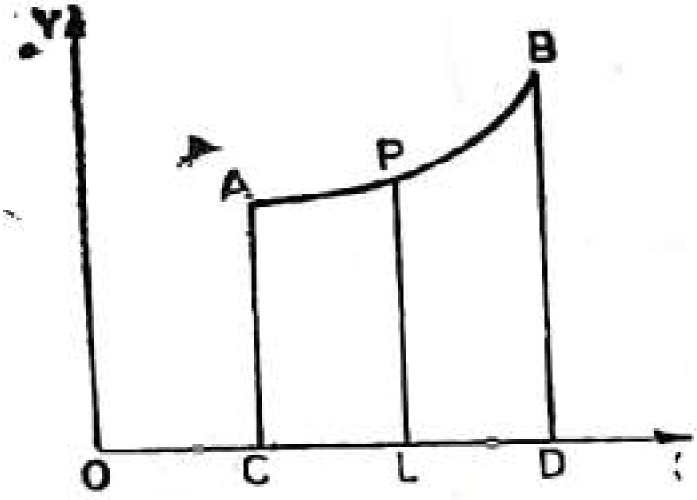
Length of Curves (Rectification)

14·1. Definition

The process of finding the length of the arc of a curve is called **rectification.**

14·2. The length s of the arc of the curve y=f(x) between the points, where x=a, x=b is given by

$$s=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$



where y and $\frac{dy}{dx}$ are continuous and single valued functions in the interval $a \leqslant x \leqslant b$ and the integrand does not change sign in the this interval.

Let AB be the arc of the curve $y=f(x)$; CA and DB the ordinates at the points A and B, whose abcissae are a and b respectively, *i.e.* $x=a$ and $x=b$ respectively.

Let $P(x, y)$ be a point on the curve and LP its ordinate. If s denotes the length of the arc AP, measured from a fixed point A, then clearly s is a function of x.

We know

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \ldots(i)$$

(From Differential Calculus)

Integrating both sides with respect to x, between the limits $x=a$, $x=b$, we have

$$\int_a^b \frac{ds}{dx}\,dx=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

or

$$\Big[\,s\,\Big]_a^b=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

or (Value of s at B) $-$ (Value of s at A)

$$=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\text{arc } AB-0=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\text{Hence arc } AB=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

From (i), we have

$$s=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

14·3. The length s of the arc of a curve x=f (y), between the points, where y=c, y=d is given by

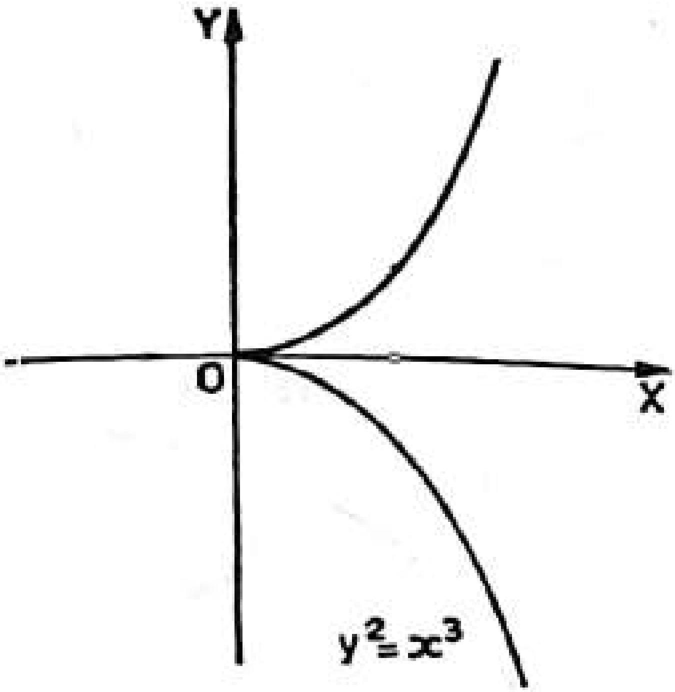
$$\boxed{\mathbf{s}=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,\mathbf{dy}}$$

where x and $\frac{dx}{dy}$ are continuous and single valued functions in the interval $c \leqslant y \leqslant d$ and the integrand does not change sign in this interval.

The proof is similar to the Article 14·2.

Example 1. *Find the length of curve $y^2=x^3$ from origin to the point (1, 1).*

Sol. The curve can easily be traced and its shape is shown in the figure.



The equation of the curve is

$$y^2=x^3 \qquad \ldots (1)$$

$\therefore$ $$2y\frac{dy}{dx}=3x^2$$

or $$\frac{dy}{dx}=\frac{3x^2}{2y}$$

$$=\frac{3x^2}{2x^{3/2}}=\frac{3}{2}x^{1/2} \qquad \text{[from (1)]}$$

Now $$s=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^1\sqrt{\left(1+\frac{9x}{4}\right)}\,dx=\frac{1}{2}\int_0^1\sqrt{4+9x}\,dx$$

$$=\frac{1}{2}\left[\frac{1}{9}\times\frac{2}{3}(4+9x)^{3/2}\right]_0^1$$

$$=\frac{1}{27}\left[(13)^{3/2}-(4)^{3/2}\right]$$

$$=\frac{1}{27}\left[13\sqrt{13}-8\right].$$

Example 2. *Find the length of the arc of the parabola* $y^2=4ax$

(a) cut off by the latus rectum

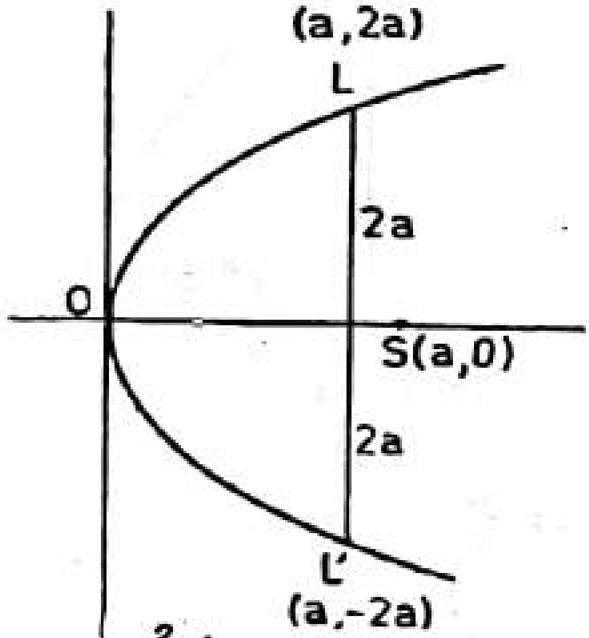
(b) cut off by the line $3y=8x$.

Sol. The equation of the curve is $y^2=4ax$ ...(i)

(a) The latus rectum is the straight line LL'.

The required length$=2\times OL$.

From (i), we have $x=\frac{y^2}{4a}$



$y^2=4ax$

$$\frac{dx}{dy}=\frac{2y}{4a}=\frac{y}{2a}$$

Now
$$OL=\int_0^{2a}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_0^{2a}\sqrt{}$$

$$=\frac{1}{2a}\int_0^{2a}\sqrt{4a^2+y^2}\,dy$$

$$=\frac{1}{2a}\left[\frac{y\cdot\sqrt{4a^2+y^2}}{2}\right.$$

$$\left.+\frac{4a^2}{2}\log\left(\frac{y+\sqrt{4a^2+y^2}}{2a}\right)\right]_0^{2a}$$

$$=\frac{1}{2a}\left[\frac{2a\,.\,\sqrt{4a^2+4a^2}}{2}\right.$$

$$\left.+2a^2\log\left(\frac{2a+\sqrt{4a^2+4a^2}}{2a}\right)\right]$$

$$=a[\sqrt{2}+\log(1+\sqrt{2})].$$

The required length $=2\times OL$

$$=2a[\sqrt{2}+\log(1+\sqrt{2})].$$

(*b*) To find the length of the arc of the parabola $y^2=4ax$ cut off by the line $3y=8x$, we solve these equations by eliminating x between them.

$$\therefore \qquad y^2=4a\quad\frac{3}{8}\,y$$

$$2y^2-3ay=0$$

or
$$y(2y-3a)=0$$

or
$$\therefore \qquad y=0,\ y=3a/2$$

Required length of the arc

$$=\int_0^{3a/2}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=\int_0^{3a/2}\sqrt{1+\frac{y^2}{4a^2}}\,dy$$

$$=\frac{1}{2a}\int_0^{3a/2}\sqrt{4a^2+y^2}\,dy$$

$$=\frac{1}{2a}\left[\frac{y\sqrt{4a^2+y^2}}{2}+\frac{4a^2}{2}\log\left(\frac{y+\sqrt{4a^2+y^2}}{2a}\right)\right]_0^{3a/2}$$

$$=\frac{1}{2a}\left[\frac{\frac{3a}{2}\sqrt{4a^2+\frac{9a^2}{4}}}{2}+2a^2\log\left(\frac{\frac{3a}{2}+\sqrt{4a^2+\frac{9a^2}{4}}}{2a}\right)\right]$$

$$=\frac{1}{2a}\left[\frac{15a^2}{8}+2a^2\log\left(\frac{\frac{3a}{2}+\frac{5a}{2}}{2a}\right)\right]$$

$$=\frac{1}{2a}\left[\frac{15a^2}{8}+2a^2\log 2\right]$$

$$=\left(\frac{15}{16}+\log 2\right)a.$$

Example 3. *Show that whole length of the loop of the curve* $3ay^2=x(x-a)^2$ *is* $\frac{4}{\sqrt{3}}a$.

Sol. The curve can easily be traced and its shape is shown in the figure of Example 2 (page 378).

For the loop of the curve x varies from 0 to a.

The required length $=2\times$ Length of the loop of the curve in the first quadrant.

The equation of the curve is

$$3ay^2=x(x-a)^2$$

or
$$y=\frac{1}{\sqrt{3a}}\sqrt{x}\,(x-a)$$

(Taking +ve sign only)

$$=\frac{1}{\sqrt{3a}}(x^{3/2}-ax^{1/2})$$

$\therefore$
$$\frac{dy}{dx}=\frac{1}{\sqrt{3a}}\left(\frac{3}{2}x^{1/2}-\frac{1}{2}ax^{-1/2}\right)$$

$$=\frac{3x-a}{2\sqrt{3ax}}$$

Now length of the loop in the first quadrant

$$=\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^a\sqrt{1+\frac{(3x-a)^2}{12ax}}\,dx$$

$$=\int\sqrt{\frac{12ax+(3x-a)^2}{12ax}}\,dx$$

$$=\int_0^a\sqrt{\frac{(3x+a)^2}{12ax}}\,dx$$

$$=\frac{1}{2\sqrt{3a}}\int_0^a\frac{3x+a}{\sqrt{x}}\,dx$$

$$=\frac{1}{2\sqrt{3a}}\int_0^a \left(3x^{1/2}+ax^{-1/2}\right)dx$$

$$=\frac{1}{2\sqrt{3a}}\left[3\cdot\frac{2}{3}x^{3/2}+2ax^{1/2}\right]_0^a$$

$$=\frac{1}{\sqrt{3a}}\left[a^{3/2}+a^{3/2}\right]$$

$$=\frac{2}{\sqrt{3}}a$$

Therefore the required length

$$=2\times\frac{2}{\sqrt{3}}a=\frac{4a}{\sqrt{3}}.$$

Example 4. *If s be the length of the arc of a catenary*

$$y=c\cosh x/c$$

from its lowest point to any point $P(x, y)$ *on the curve, show that*

$$y^2=c^2+s^2.$$

Sol. For shape of the curve see example 2 (page 294).

The equation of the curve is

$$y=c\cosh x/c \qquad \text{...}(i)$$

$$\therefore \quad \frac{dy}{dx}=c.\frac{1}{c}.\sinh x/c$$

$$=\sinh x/c.$$

Now

$$s=\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^x \sqrt{1+\sinh^2 x/c}\,dx$$

$$=\int_0^x \cosh x/c\,dx \qquad [\because \cosh^2 x/c=1+\sinh^2 x/c]$$

$$=c\left[\sinh x/c\right]_0^x$$

$$\therefore \quad s=c\sinh x/c \qquad \text{...}(ii)$$

From (*i*), we have

$$y^2 = c^2 \cosh^2 x/c$$
$$= c^2(1+\sinh^2 x/c)$$
$$= c^2(1+s^2/c^2) \qquad \text{[from (ii)]}$$
$$= c^2+s^2$$

$\therefore \qquad y^2 = c^2+s^2.$

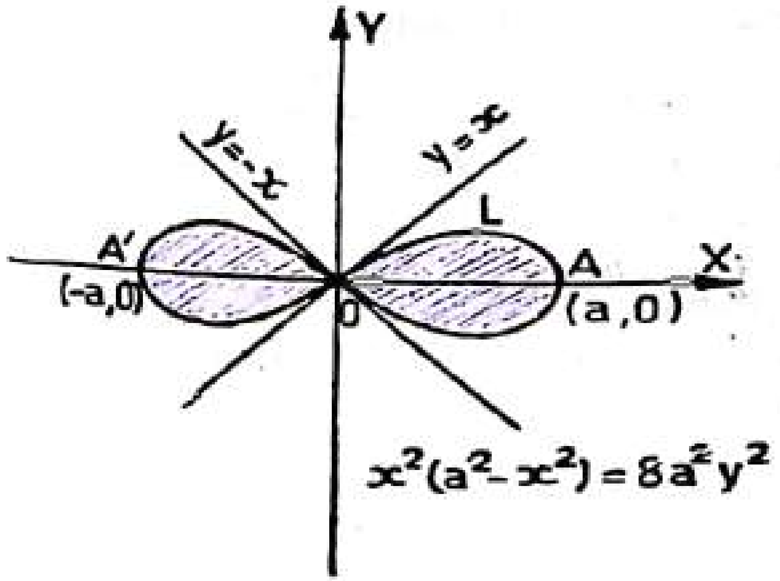
Example 5. *Show that the whole length of the curve*

$$x^2(a^2-x^2)=8a^2y^2 \text{ is } \pi a\sqrt{2}.$$

Sol. The shape of the curve is shown in the figure.

For one loop of the curve x varies from 0 to a.

The required length of the whole curve $=4\times OLA$.



The equation of the curve is

$$8a^2y^2 = x^2(a^2-x^2)$$

$\therefore$
$$y = \frac{1}{2\sqrt{2}\,a}\, x.\sqrt{a^2-x^2} \qquad \text{(Taking +ve sign only)}$$

$\therefore$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{2}a}\left[\sqrt{a^2-x^2}\,.\,1+\frac{x\,.\,(-2x)}{2\sqrt{a^2-x^2}}\right]$$
$$= \frac{1}{2\sqrt{2}a}\left[\frac{a^2-x^2-x^2}{\sqrt{a^2-x^2}}\right]$$
$$= \frac{1}{2\sqrt{2}a}\left[\frac{a^2-2x^2}{\sqrt{a^2-x^2}}\right]$$

Now

$$OLA = \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$=\int_0^a \sqrt{1+\frac{(a^2-2x^2)^2}{8a^2(a^2-x^2)}}\,dx$$

$$=\int_0^a \sqrt{\frac{8a\,(a^2-x^2)+(a^2-2x^2)^2}{8a^2(a^2-x^2)}}\,dx$$

$$=\int_0^a \sqrt{\frac{4x^4-12a^2x^2+9a^4}{8a^2(a^2-x^2)}}\,dx$$

$$=\int_0^a \sqrt{\frac{(3a^2-2x^2)^2}{8a^2(a^2-x^2)}}\,dx$$

$$=\frac{1}{2\sqrt{2}a}\int_0^a \frac{3a^2-2x^2}{\sqrt{a^2-x^2}}\,dx$$

$$\left[\begin{array}{l}\text{Put } x=a\sin\theta,\ dx=a\cos\theta\,d\theta \\ \text{when } x=0,\ \theta=0, \text{ when } x=a,\ \theta=\pi/2\end{array}\right]$$

$$=\frac{1}{2\sqrt{2}a}\int_0^{\pi/2} \frac{(3a^2-2a^2\sin^2\theta)a\cos\theta}{a\cos\theta}\,d\theta$$

$$=\frac{a^2}{2\sqrt{2}a}\left[\left\{3\,\theta\right\}_0^{\pi/2}-2\,.\,\frac{1}{2}\,\frac{\pi}{2}\right]$$

$$=\frac{a}{2\sqrt{2}}\left[3\,.\,\frac{\pi}{2}-\frac{\pi}{2}\right]=\frac{\pi a}{2\sqrt{2}}$$

Therefore the required length

$$=4\times\frac{\pi a}{2\sqrt{2}}$$

$$=\pi a\sqrt{2}.$$

EXERCISE 14 (a)

1. Find the length of the arc of the semi-cubical parabola $ay^2=x^3$ from the origin to the point (a, a).

2. Find the length of the arc of the parabola $x^2=4ay$ extending from the vertex to one extremity of the latus rectum.

3. In the evolute of a parabola, *viz.* $4(x-2a)^3=27\,ay^2$, show that the length of the curve from its cusp $(x=2a)$ to the point where it meets the parabola $y^2=4ax$ is $2a(3\sqrt{3}-1)$.

4. Find the perimeter of the loop of the following curves.

(i) $ay^2=x^2(a-x)$

(ii) $9ay^2=(x-2a)(x-5a)^2$.

5. Show that the length of an arc of the curve

$$x^2=a^2(1-e^{y/a})$$

measured from the origin to any point (x, y) is

$$a \log \frac{a+x}{a-x}-x$$

6. Find the length of the curve

$$y^2=(2x-1)^3 \text{ cut off by the line } x=4.$$

7. The chain of a suspension bridge has the form of a curve

$$x^2=\frac{b^2}{h}y.$$

Write down the length of the curve in the form of an integral.

Show that when h is very much smaller than b, the radical under the integral sign can be expanded by binomial theorem and that the length of the chain is approximately

$$2b+\frac{4h^2}{3b}.$$

8. A man walks along the curve

$$20y=3(4x^2-20x+9)$$

between the points where $x=\frac{1}{2}$ and $x=9/2$. Prove that he walks

$$\frac{5}{6}\left(\frac{156}{25}+\log 5\right) \text{ units of length}$$

14·4. The length s of the arc of the curve $x=f(t)$, $y=\phi(t)$ between the points where $t=t_1$, $t=t_2$ is given by

$$s=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

Let AB be the curve $x=f(t)$, $y=\phi(t)$, CA and DB the ordinates at the points A and B, where $t=t_1$, $t=t_2$ respectively (see figure Art. 14·2).

Let $P(x, y)$ be a point on the curve and LP its ordinate.

If s denotes the length of the arc AP, measured from a fixed point A, then s is clearly a function of t.

We know

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \qquad \ldots(i)$$

(From Differential calculus)

Integrating both sides of (*i*) with respect to t, between the limits $t=t_1$, and $t=t_2$ we have

$$\int_{t_1}^{t_2} \frac{ds}{dt}\, dt=\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

$$\Big[s \Big]_{t_1}^{t_2}=\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

(Value of s at B)—(value of s at A)

$$=\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

$$\text{arc } AB-(0)=\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

Hence arc AB

$$=\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

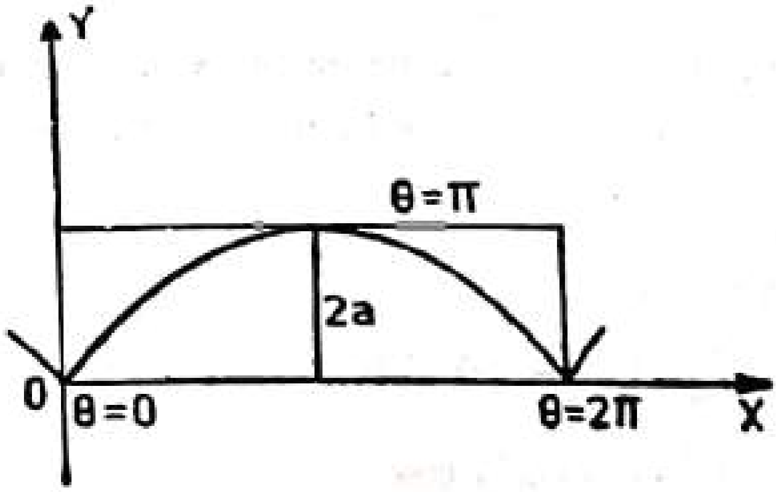
From (*i*), we have

$$s=\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

Example 1. *Find the length of the one arch of the cycloid*

$$x=a(\theta-\sin\theta),\ y=a(1-\cos\theta)$$

Sol. The shape of the curve is shown in the figure.



$$x=a(\theta-\sin\theta),\ \ y=a(1-\cos\theta)$$

$$\therefore \qquad \frac{dx}{d\theta}=a(1-\cos\theta),\ \ \frac{dy}{d\theta}=a\sin\theta$$

Now
$$s=\int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\ d\theta$$

$$=\int_0^{2\pi}\sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}\ d\theta$$

$$=a\int_0^{2\pi}\sqrt{4\sin^4\theta/2+4\sin^2\theta/2\cos^2\theta/2}\ d\theta$$

$$=2a\int_0^{2\pi}\sin\theta/2\sqrt{\sin^2\theta/2+\cos^2\theta/2}\ d\theta$$

$$=2a\int_0^{2\pi}\sin\theta/2\ .\ d\theta$$

$$=4a\Big[-\cos\theta/2\Big]_0^{2\pi}$$

$$=-4a\,[\cos\pi-\cos 0]$$

$$=-4a\,[-1-1]=8a$$

$$=8a$$

Example 2. *Find the whole length of the hypocycloid*

$$x=a\cos^3 t,\ y=b\sin^3 t.$$

Sol. For shape of the curve see figure of example 5 (page 273)

Required length of the curve$=4\times$Length of the curve in the first quadrant.

The parametric equations of the curve are

$$x=a\cos^3 t,\ y=b\sin^3 t$$

$$\therefore \qquad \frac{dx}{dt}=-3a\cos^2 t\sin t$$

$$\frac{dy}{dt}=3b\sin^2 t\cos t$$

Length of the curve in the first quadrant

$$s=\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\int_0^{\pi/2}\sqrt{9a^2\cos^4 t\sin^2 t+9b^2\sin^4 t\cos^2 t}\;dt$$

$$=3\int_0^{\pi/2}\sin t\cos t\sqrt{a^2\cos^2 t+b^2\sin^2 t}\,dt$$

Putting $a^2\cos^2 t+b^2\sin^2 t=z^2$

$\therefore\quad 2(b^2-a^2)\cos t\sin t\,dt=2z\,dz$

$\therefore\qquad \sin t\cos t\,dt=\dfrac{z\,dz}{(b^2-a^2)}$

when $\qquad t=0,\quad z=a$

when $\qquad t=\pi/2,\quad z=b$

or $\qquad s=3\displaystyle\int_a^{\pi/2}\frac{z.\,z\,dz}{(b^2-a^2)}=\frac{3}{(b^2-a^2)}\int_a^b z^2\,dz$

$$=\frac{3}{(b^2-a^2)}\left[\frac{z^3}{3}\right]_a^b$$

$$=\frac{b^3-a^3}{b^2-a^2}=\frac{(b-a)(b^2+ab+a^2)}{(b-a)(b+a)}$$

$$=\frac{a^2+ab+b^2}{a+b}$$

Therefore the required length of the curve

$$=4.\frac{a^2+ab+b^2}{a+b}.$$

See where y gets 0.

Example 3. *Show that the perimeter of the ellipse*

$$x=a\cos t,\qquad y=b\sin t,\ \textit{is}$$

$$2a\pi\left[1-\frac{1}{2^2}e^2-\frac{1^2.3}{2^2.4^2}e^4-\frac{1^2.3^2.5}{2^2\ 4^2.6^2}e^6-\dots\infty\right]$$

Sol. The parametric equations of the ellipse are

$$x=a\cos t,\qquad y=b\sin t$$

$\therefore\qquad \dfrac{dx}{dt}=-a\sin t$

$\qquad\qquad \dfrac{dy}{dt}=b\cos t$

Perimeter of the ellipse $= 4 \times$ Length of the ellipse in the first quadrant

Length of the ellipse in the first quadrant

$$=\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\int_0^{\pi/2}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,dt$$

$$=\int_0^{\pi/2}\sqrt{a^2\sin^2 t+a^2(1-e^2)\cos^2 t}\,dt$$

$$[\because\ b^2=a^2(1-e^2)]$$

$$=\int_0^{\pi/2}\sqrt{a^2(\sin^2 t+\cos^2 t)-a^2e^2\cos^2 t}\,dt$$

$$=a\int_0^{\pi/2}\sqrt{1-e^2\cos^2 t}\,dt$$

$$=a\int_0^{\pi/2}(1-e^2\cos^2 t)^{1/2}\,dt$$

$$=a\int_0^{\pi/2}\Big[1+\frac{1}{2}(-e^2\cos^2 t)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}(-e^2\cos^2 t)^2$$

$$+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}(-e^2\cos^2 t)^3+\ldots\infty\Big]\,dt$$

[Expansion by Binomial Theorem for any index, as $e^2\cos^2 t<1$]

$$=a\int_0^{\pi/2}\Big[1-\frac{1}{2}e^2\cos^2 t-\frac{1}{2.4}e^4\cos^4 t-\frac{1.3}{2.4.6}e^6\cos^6 t-\ldots\infty\Big]dt$$

$$=a\Big[\int_0^{\pi/2}1\,dt-\frac{1}{2}e^2\int_0^{\pi/2}\cos^2 t\,dt-\frac{1}{2.4}e^4\int_0^{\pi/2}\cos^4 t\,dt$$

$$-\frac{1.3}{2.4.6}e^6\int_0^{\pi/2}\cos^6 t\,dt-\ldots\infty\Big]$$

$$=a\left[\frac{\pi}{2}-\frac{1}{2}e^2\cdot\frac{\pi}{4}-\frac{1}{2.4}e^4\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2}-\frac{1.3}{2.4.6}e^6\cdot\frac{5.3.1}{6.4.2}\frac{\pi}{2}-\ldots\infty\right]$$

[Using reduction formula]

$$=\frac{\pi a}{2}\left[1-\frac{1}{2^2}\cdot e^2-\frac{1^2.3}{2^2.4^2}e^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}e^6-\ldots\infty\right]$$

Therefore the required length

$$=4\times\frac{\pi a}{2}\left[1-\frac{1}{2^2}\cdot e^2-\frac{1^2.3}{2^2.4^2}e^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}e^6-\ldots\infty\right]$$

$$=2a\left[1-\frac{1}{2^2}e^2-\frac{1^2.3}{2^2.4^2}e^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}e^6-\ldots\infty\right]$$

EXERCISE 14 (*b*)

1. Find the length of the one arch of the cycloid

$$x=a(\theta+\sin\theta),\quad y=a(1-\cos\theta).$$

In the same curve show that the length of the arc from the vertex to the point (x, y) is $\sqrt{(8ay)}$.

2. Find the whole length of the astroid

$$x=a\cos^3 t,\quad y=a\sin^3 t.$$

Also show that s varies as x^2, s being the length of the curve measured from the point on the curve which lies on y-axis.

3. Find the length of the curve

$$x=t^2,\quad y=t^3 \text{ from } t=0 \text{ to } t=2.$$

4. Find the length of the loop of the curve

$$x=t^2,\quad y=t-\tfrac{1}{3}t^3.$$

5. Show that in the epicycloid, for which

$$x=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta$$

$$y=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta$$

the length of the arc measured from the point where $\theta=\pi b/a$ is

$$\frac{4b\,(a+b)}{a}\cos\left(\frac{a}{2b}.\theta\right).$$

6. Find the length of any arc of the curve

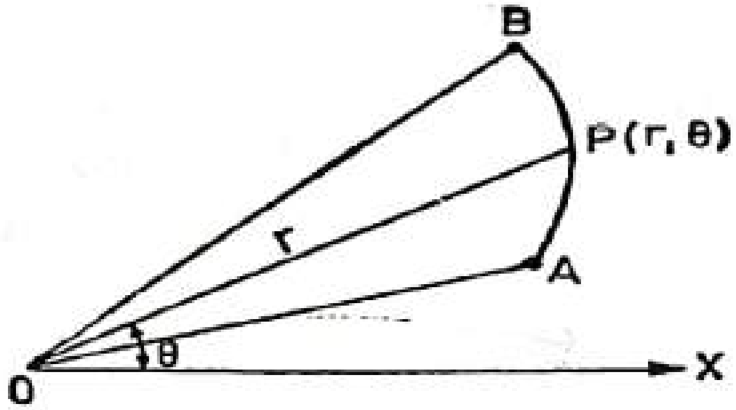
$$x^{2/3}-y^{2/3}=a^{2/3}.$$

14·5. The length s of the arc of the curve $r=f(\theta)$ between the points where $\theta=\alpha$, $\theta=\beta$ is given by

$$s=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Let AB be the arc of the curve $r=f(\theta)$ and A, B the points where $\theta=\alpha$, $\theta=\beta$ respectively.

Let P (r, θ) be a point on the curve.



If s denotes the length of the arc AP measured from a fixed point A, then clearly s is a function of θ.

We know

$$\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \quad \ldots(i)$$

(From Differential Calculus)

Integrating both sides with respect to θ, between the limits $\theta=\alpha$, $\theta=\beta$, we have

$$\int_\alpha^\beta \frac{ds}{d\theta}\, d\theta=\int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\Big[s\Big]_\alpha^\beta=\int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

or (Value of s at B)—(Value of s at A)

$$=\int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\text{arc } AB-0=\int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Hence arc AB

$$=\int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

From (*i*), we have

$$s=\int_{\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Note. If the equation of the curve is of the form $\theta=f(r)$, the length of the arc of the curve between the points where $r=r_1$, $r=r_2$, is given by

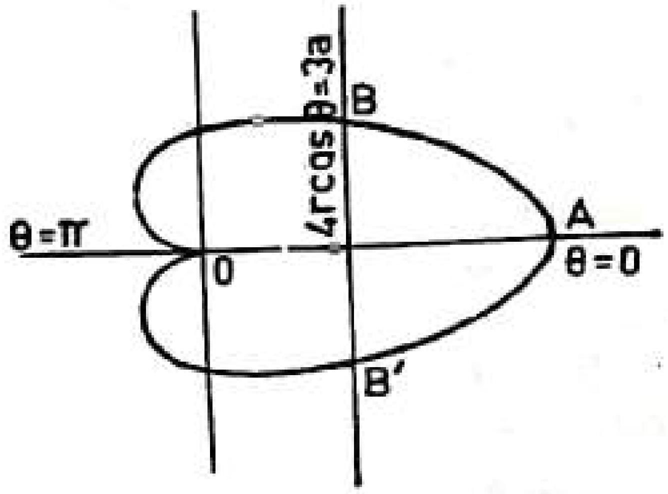
$$s=\int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr.$$

Example 1. *Find the perimeter of the curve*

$$r=a(1+\cos\theta)$$

Also show that the length of the part of the curve which lies on the side of the line $4r=3a\sec\theta$ remote from the pole is equal to $4a$.

Sol. The shape of the curve is shown in the figure.



The required length of the curve

=2×Length of the upper half of the curve

For the upper half of the curve θ varies from 0 to π.

The equation of the curve is

$$r=a(1+\cos\theta) \qquad \cdots(i)$$

$$\therefore \qquad \frac{dr}{d\theta}=-a\sin\theta$$

Length of the upper half of the curve

$$=\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\ d\theta$$

$$=\int_0^{\pi} a\sqrt{2(1+\cos\theta)}\ d\theta$$

$$=a\int_0^{\pi} \sqrt{2.2\cos^2\theta/2}\ d\theta$$

$$=2a\int_0^{\pi} \cos\theta/2\ d\theta \qquad (ii)$$

$$=2a\Big[2\sin\theta/2\Big]_0^{\pi}$$

$$=4a\left[\sin\frac{\pi}{2}-0\right]$$

$$=4a.$$

Therefore the perimeter of the curve

$$=2\times 4a=8a$$

The equation of the line

$$4r=3a\sec\theta$$

may be written as,

$$4r\cos\theta=3a$$

This line cuts the curve (i), where

$$a(1+\cos\theta)=\frac{3a}{4\cos\theta}$$

$\therefore$ $4\cos\theta\,(1+\cos\theta)=3$

or $4\cos^2\theta+4\cos\theta-3=0$

$\therefore$ $$\cos\theta=\frac{-4\pm\sqrt{16+48}}{8}=\frac{1}{2},\ -\frac{3}{2}$$

or $$\cos=\tfrac{1}{2}$$

$$\left(\text{The value } -\frac{3}{2} \text{ being inadmissible}\right)$$

$\therefore$ $$\theta=\frac{\pi}{3},\ -\frac{\pi}{3}.$$

The line $4r\cos\theta=3a$, meets the curve at the points B and B'. The length of the arc on the side of the line

$$4r\cos\theta=3a,$$

remote from the pole is $B'AB$.

$$\text{Arc } AB=\text{Arc } AB'$$

$\therefore$ Length of the Arc AB

$$=\int_0^{\pi/3}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{\pi/3} 2a\cos\theta/2\,d\theta \qquad \text{[by (ii)]}$$

$$=4a\left[\sin\frac{\theta}{2}\right]_0^{\pi/3}$$

$$=4a\left[\sin\frac{\pi}{6}-0\right]$$

$$=4a\left[\tfrac{1}{2}-0\right]=2a.$$

Therefore the length of the arc $B'AB$

$$=2\times 2a=4a.$$

Example 2. *Show that the length of the arc of the equiangular spiral*

$$r=ae^{\theta\cot\alpha}$$

between the points for which the radii vectors are r_1 and r_2 is $(r_2-r_1)\sec\alpha$.

Sol. The equation of the curve is

$$r=ae^{\theta\cot\alpha}$$

$$\therefore \qquad \frac{dr}{d\theta}=a\,.\,e^{\theta\cot\alpha}\cot\alpha$$

$$=r\cot\alpha$$

Now required length s of the arc from $r=r_1$ to $r=r_2$

$$s=\int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr$$

$$=\int_{r_1}^{r_2}\sqrt{1+r^2\cdot\frac{1}{r^2\cot^2\alpha}}\,dr \qquad \left[\because \frac{d\theta}{dr}=\frac{1}{r\cot\alpha}\right]$$

$$=\int_{r_1}^{r_2}\sqrt{1+\tan^2\alpha}\,dr=\int_{r_1}^{r_2}\sec\alpha\,.\,dr$$

$$=\sec\alpha\left[r\right]_{r_1}^{r_2}=(r_2-r_1)\sec\alpha.$$

EXERCISE 14 (c)

1. Find the perimeter of the following curves

 (i) $r=a\cos\theta$

 (ii) $r=a\sin\theta$

 (iii) $r=a(\cos\theta+\sin\theta)$.

2. (a) Find the perimeter of the cardioid

$$r=a(1-\cos\theta)$$

(b) Show that arc of the upper half of the above cardioid is bisected by the line $\theta=\frac{2\pi}{3}$.

3. Show that the upper half of the cardioid $r=a(1+\cos\theta)$ is bisected by $\theta=\frac{\pi}{3}$.

4. Find the length of any arc of the cissoid,

$$r=a\,\frac{\sin^2\theta}{\cos\theta}$$

5. Prove that perimeter of the lamicon $r=a+b\cos\theta$, if b/a be small, is approximately

$$2\pi a\left(1+\frac{1}{4}\,\frac{b^2}{a^2}\right)$$

6. Find the length of arc of the parabola

$$\frac{2a}{r}=1+\cos\theta$$

cut off by the latus rectum.

7. Prove that the length of the arc of the hyperbolic spiral $r\theta=a$, taken from the point $r=a$ to $r=2a$ is

$$a\left[\sqrt{5}-\sqrt{2}+\log\left(\frac{2+\sqrt{8}}{1+\sqrt{5}}\right)\right]$$

8. Find the whole length of the curve

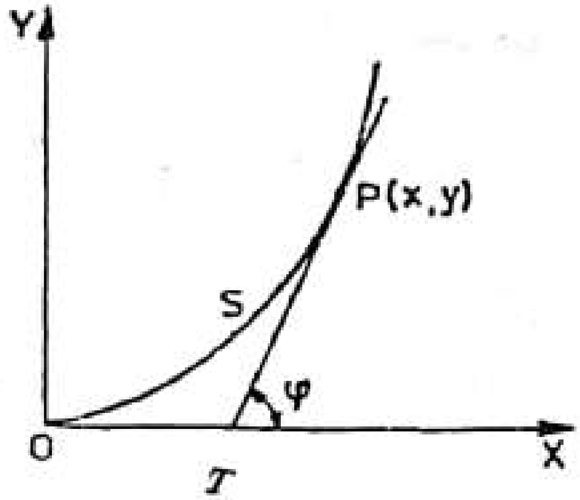
$$r^2=a^2\cos 2\theta.$$

9. Find the length of the one loop of the curve

$$r^3=\sin 3\theta.$$

14·6. Intrinsic Equation

The *intrinsic* equation of a curve is a relation beween the length s of a curve measured from a fixed point O on it to any point P,



and ψ the angle which the tangent at P makes with the tangent at O or with any other fixed line in the plane of the curve.

14·7. (*a*) **To find the intrinsic equation of a curve $y=f(x)$.**

Let the origin $O(0,0)$ be a fixed point on the curve from which the arc is measured, x-axis being tangent to the curve at the point O.

Let $P(x, y)$ be any point on the curve such that arc $OP=s$.

If PT is the tangent at P and meet the x-axis in T, so that

$$\angle XTP=\psi.$$

Now $$\frac{dy}{dx}=f'(x)$$

or $$\tan\psi=f'(x) \qquad \ldots(i)$$

Also, $$s=\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

or $$s=\int_0^x \sqrt{1+\{f'(x)\}^2}\; dx \qquad \ldots(ii)$$

Eliminating x from (*i*) and (*ii*) (after integration), we get a relation between s and ψ, which is the required intrinsic equation.

(*b*) **To find the intrinsic equation of a curve $x=(t)$, $y=\phi(t)$**

The parametric equations of the curve are

$$x=f(t), \quad y=\phi(t) \qquad \ldots(i)$$

From (*i*), we have

$$\frac{dx}{dt}=f'(t), \quad \frac{dy}{dt}=\phi'(t)$$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\phi'(t)}{f'(t)}$$

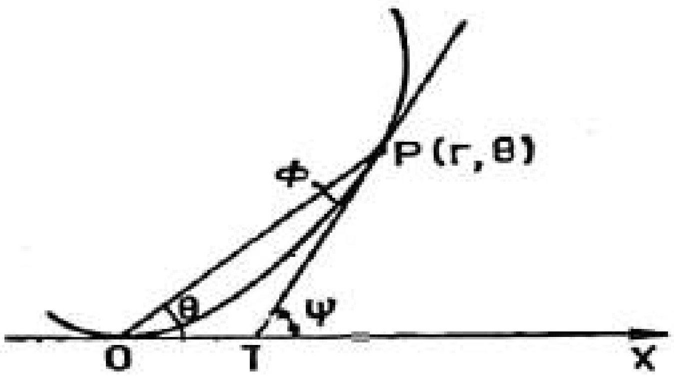
$\therefore$ $$\tan\psi=\frac{\phi'(t)}{f'(t)} \qquad \ldots(ii)$$

Now $$s=\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \qquad \ldots(iii)$$

Eliminating t from (ii) and (iii) (after integration), we have a relation between s and ψ, the intrinsic equation of the curve.

(c) Intrinsic equation of a polar curve $r=f(\theta)$

Let O be the pole and $P(r, \theta)$ any point on the curve. Choose OX, the line tangent to the curve at the pole, as the initial line. Let PT be the tangent to the curve at P meeting the initial line at T.



Let further

$$\text{Arc } OP=s$$

and $$\angle XTP=\psi$$

The equation of the curve is

$$r=f(\theta)$$

$\therefore$ $$\frac{dr}{d\theta}=f'(\theta)$$

or $$s=\int_0^\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

[Putting $r=f(\theta)$]

or $$s=\int_0^\theta \sqrt{\{f(\theta)\}^2+\{f'(\theta)\}^2}\,d\theta \qquad \ldots(i)$$

Also let ϕ be the angle between the radius vector OP and tangent PT at the point $P(r, \theta)$, then

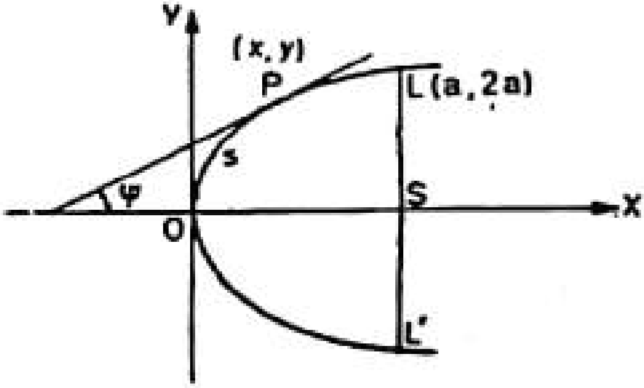
$$\tan\phi=r\,\frac{d\theta}{dr}=\frac{r}{\left(\frac{dr}{d\theta}\right)}$$

or
$$\phi=\frac{f(\theta)}{f'(\theta)} \qquad \ldots(ii)$$

Also
$$\psi=\theta+\phi \qquad \ldots(iii)$$

Eliminating θ, ϕ from (i), (ii) and (iii) we have a relation between s and ψ, the intrinsic equation of the curve.

Example 1. *Find the intrinsic equation of the parabola $y^2=4ax$, the fixed point being the vertex. Hence obtain the length of the arc intercepted between the vertex and one extremity of the latus rectum.*



Sol. The equation of the curve is

$$y^2=4ax$$

or
$$x=\frac{y^2}{4a}$$

$\therefore$
$$\frac{dx}{dy}=\frac{2y}{4a}=\frac{y}{2a} \qquad \ldots(i)'$$

Let $P(x, y)$ be any point on the parabola and tangent at P make an angle ψ with x axis.

If arc $OP=s$, then

$$s=\int_0^y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_0^y\sqrt{1+\frac{y^2}{4a^2}}\,dy$$

$$=\frac{1}{2a}\int_0^y\sqrt{4a^2+y^2}\,dy$$

$$=\frac{1}{2a}\left[\frac{y\sqrt{4a^2+y^2}}{2}+\frac{4a^2}{2}\log\left(\frac{y+\sqrt{4a^2+y^2}}{2a}\right)\right]_0^y$$

$\therefore \quad s=\frac{1}{2a}\left[\frac{y\sqrt{4a^2+y^2}}{2}+2a^2 \log\left(\frac{y+\sqrt{4a^2+y^2}}{2a}\right)\right] \quad ...(ii)$

From (*i*) $\frac{dy}{dx}=\frac{2a}{y}=\tan\psi$

$\therefore \quad y=2a\cot\psi \quad ...(iii)$

Substituting this value of y in (*ii*), we get

$$s=\frac{1}{2a}\left[\frac{2a\cot\psi\sqrt{4a^2+4a^2\cot^2\psi}}{2}\right.$$
$$\left.+2a^2\log\left(\frac{2a\cot\psi+\sqrt{4a^2+4a^2\cot^2\psi}}{2a}\right)\right]$$

or $\quad s=a\cot\psi\operatorname{cosec}\psi+a\log(\cot\psi+\operatorname{cosec}\psi) \quad ...(iv)$

is the required intrinsic equation of the parabola.

Now if LSL' is the latus rectum, then at the extremity L, $y=2a$

$\therefore$ From (*iii*),

$\cot\psi=1,$

$\psi=\frac{\pi}{4}.$

Putting $\quad \psi=\frac{\pi}{4}$ in (*iv*), we get the required length

$\therefore \quad s=a\cot\frac{\pi}{4}\operatorname{cosec}\frac{\pi}{4}+a\log\left(\cot\frac{\pi}{4}+\operatorname{cosec}\frac{\pi}{4}\right)$

$=a\sqrt{2}+a\log(1+\sqrt{2})$

$=a[\sqrt{2}+\log(1+\sqrt{2})].$

Example 2. *Find the instrinsic equation of the cycloid*

$x=a(t+\sin t),\ y=a(1-\cos t)$

and also show that $s^2+\rho^2=16a^2$, *where* ρ *is the radius of the curvature at any point* t.

Sol. For shape of the curve see example 1 (Art. 13·6).

The equation of the cycloid is

$x=a(t+\sin t),\ y=a(1-\cos t)$

$\therefore \quad \frac{dx}{dt}=a(1+\cos t),\quad \frac{dy}{dt}=a\sin t$

$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{a\sin t}{a(1+\cos t)}=\frac{2a\sin t/2\cos t/2}{2a\cos^2 t/2}=\tan\frac{t}{2}$

$\therefore \quad \tan\psi=\tan\frac{t}{2}$

or $\quad \psi=\frac{t}{2}$ or $t=2\psi \quad ...(i)$

Also
$$s=\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

$$=\int_0^t \sqrt{a^2\,(1+\cos t)^2+a^2 \sin^2 t}\; dt$$

$$=a\int_0^t \sqrt{2+2\cos t}\; dt$$

$$=a\int_0^t \sqrt{2(1+\cos t)}\; dt$$

$$=2a\int_0^t \sqrt{2.2\cos^2\frac{t}{2}}\; dt$$

$$=2a\int_0^t \cos\frac{t}{2}\, dt=2a\,.\left[2\sin\frac{t}{2}\right]_0^t=4a\sin\frac{t}{2}$$

$\therefore$
$$s=4a\sin\frac{t}{2} \qquad \ldots (ii)$$

Eliminating t between (i) and (ii), we have
$$s=4a\sin\psi,$$

the required intrinsic equation.

For the second part, we have
$$\rho=\frac{ds}{d\psi}=4a\cos\psi$$

$$\therefore \quad s^2+\rho^2=(4a\sin\psi)^2+(4a\cos\psi)^2$$
$$=16a^2\,(\sin^2\psi+\cos^2\psi)$$

$$\therefore \quad s^2+\rho^2=16a^2$$

Example 3. *Find the intrinsic equation of the cardioid*
$$r=a(1+\cos\theta)$$
the arc being measured from the pole. Also prove that
$$s^2+9\rho^2=16a^2,$$
where ρ is the radius of curvature at any point.

Sol. The equation of the curve is
$$r=a\,(1+\cos\theta)$$

$\therefore \quad \frac{dr}{d\theta}=-a \sin \theta$

$\therefore \quad \tan \phi=\frac{r}{\frac{dr}{d\theta}}=\frac{a(1+\cos \theta)}{-a \sin \theta}=-\frac{2 \cos^2 \theta/2}{2 \sin \theta/2 \cos \theta/2}$

$=-\cot \frac{\theta}{2}=\tan \left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$\therefore \quad \phi=\frac{\pi}{2}+\frac{\theta}{2}$

Now $\quad \psi=\theta+\phi=\theta+\frac{\pi}{2}+\frac{\theta}{2}$

$\therefore \quad \psi=\frac{\pi}{2}+\frac{3\theta}{2}$ ...(i)

Now

$$s=\int_0^\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$=\int_0^\theta \sqrt{a^2(1+\cos \theta)^2+a^2 \sin^2 \theta}\, d\theta$$

$$=a\int_0^\theta \sqrt{2+2 \cos \theta}\, d\theta$$

$$=a\int_0^\theta \sqrt{2(1+\cos \theta)}\, d\theta=a\int_0^\theta \sqrt{2\times 2 \cos^2 \frac{\theta}{2}}\, d\theta$$

$$=2a \int_0^\theta \cos \frac{\theta}{2}\, d\theta=2a \left[2 \sin \frac{\theta}{2}\right]_0^\theta$$

$s=4a \sin \frac{\theta}{2}$ ...(ii)

Eliminating θ between (i) and (ii), we get

$$s=4a \sin \frac{1}{3}\left(\psi-\frac{\pi}{2}\right)=4a \sin \left(\frac{\psi}{3}-\frac{\pi}{6}\right),$$

the required intrinsic equation.

Now $\rho=\frac{ds}{d\psi}=4a \cos \left(\frac{\psi}{3}-\frac{\pi}{6}\right) \cdot \frac{1}{3}$

$\therefore \quad 3\rho=4a \cos \left(\frac{\psi}{3}-\frac{\pi}{6}\right)$

$$\therefore\ s^2+9\rho^2=16a^2\left[\sin^2\left(\frac{\psi}{3}-\frac{\pi}{6}\right)+\cos^2\left(\frac{\psi}{3}+\frac{\pi}{6}\right)\right]=16a^2$$

or $s^2+9\rho^2=16a^2$.

EXERCISE 14 (*d*)

Find the intrinsic equation of the following curves

1. The catenary $y=c\cosh x/c$.
2. The parabola $x^2=4ay$.
3. The cycloid
   $x=a(t+\sin t)$, $y=a(1+\cos t)$
4. The astriod
   $x=a\cos^3 t$, $y=a\sin^3 t$.
5. The cardioid $r=a(1-\cos\theta)$.
6. The equiangular spiral $r=ae^{\theta\cot\alpha}$.
7. The spiral $r=a\theta$

3. $\dfrac{a^2}{2}$.

4. (*i*) $\dfrac{\pi}{2}(a^2+b^2)$ (*ii*) $\dfrac{\pi l^2}{(1-e^2)^{3/2}}$

6. (*i*) $\dfrac{\pi a^2}{4}$ (*ii*) $\dfrac{5}{2}a^2$

(*iii*) $\frac{1}{2}$. (*iv*) $\dfrac{a^2}{2}(4-\pi)$.

7. $\dfrac{a^2}{8}(\pi-2)$ 8. $\dfrac{\pi a^2}{2}$

8. $\dfrac{a^2}{16}(15\sqrt{3}-8\pi)$.

Exercise 13 (d) (Page 413—414)

2. 97·2
3. 0·69315 4. 7·78
5. 6·71 π cubic mm. 6. 5·16 km.
7. 350·3 sq m. 8. 21820 cu. cm.
9. 561·89.

Exercise 14 (a) (Page 423)

1. $\dfrac{a}{27}(13\sqrt{13}-8)$

2. $a[\sqrt{2}+\log(1+\sqrt{2})]$

4. (*i*) $\dfrac{4a}{\sqrt{3}}$ (*ii*) $4\sqrt{3}a$

6. $\dfrac{1022}{27}$

Exercise 14 (b) (Page 429)

1. $8a$ 2. $6a$

3. $\dfrac{8}{27}[10\sqrt{10}-1]$ 4. $4\sqrt{3}$

6. $\frac{1}{2}[(x_2^{2/3}+y_2^{2/3})^{3/2}-(x_1^{2/3}+y_1^{2/3})^{3/2}]$

Exercise 14 (c) (Page 434)

1. (*i*) πa (*ii*) πa (*iii*) $\sqrt{2}\pi a$ 2. (*a*) $8a$

4. $a\sqrt{3}\Big[\log\{\sqrt{3}\cos\theta+\sqrt{3\cos^2\theta+1}\}+\sqrt{1+3\cos^2\theta}/\cos\theta\Big]_{\alpha}^{\beta}$

5. $2a[\sqrt{2}+\log(1+\sqrt{2})]$

8. $\sqrt{2}\pi a\left[1+\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\cdot\left(\frac{1}{2}\right)^2 +\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\cdot\left(\frac{1}{2}\right)^3+\ldots\right]$

9. $2\int_0^1 \frac{dr}{\sqrt{1-r^6}}$

Exercise 14 (d) (Page 441)

1. $s=a\tan\psi$
2. $s=a[\tan\psi\sec\psi+\log(\tan\psi+\sec\psi)]$
3. $s=a[\cos\psi\operatorname{cosec}\psi+\log(\cot\psi+\operatorname{cosec}\psi)]$
4. $s+\frac{3a}{4}\cos 2\psi=3$ or $s=\frac{3a}{2}\sin^2 2\psi$, according as s is measured from the vertex or the cusp on the x-axis.
5. $s=a\sin^3\psi/6$
6. $s=a\sec\alpha\,[e^{\psi\cot\alpha}-1]$
7. $s=\frac{a}{2}\left[\theta\sqrt{\theta^2+1}+\log(\theta+\sqrt{\theta^2+1})\right]$ and $\psi=\theta+\tan^{-1}\theta$

Exercise 15 (a) (Page 455—457)

1. $\pi h^2\left(a-\frac{h}{3}\right)$ 3. $2\pi^2a^2b$.

4. $2\pi ah^2$ 5. $\frac{4}{5}\pi a^3$ 6. $\frac{32}{15}\pi a^3$

7. $\frac{4}{3}\pi ab^2$, $\frac{4}{3}\pi a^2b$.

8. $2\pi^2$, π^2 10. (*a*) (*i*) $2\pi a^3\left(\log 2-\frac{2}{3}\right)$

(*ii*) $\frac{2\pi}{15}$ (*iii*) $\frac{\pi}{24}a^3$. (*b*) $\frac{8}{3}\pi a^3$

11. $\frac{9}{\pi^2}$.

12. $\frac{1536}{5}\pi$ cubic cm. 13. $\frac{5\pi}{28}$ 14. $\frac{4}{5}\pi a^3(5\pi+9)$

15. 9π 16. $2\pi^2$

18. $5\pi^2a^3$

3

Partial Fractions

3·1. Introduction

We can combine several fractions into a single fraction. For example

(*i*) $$\frac{1}{(x+2)}+\frac{1}{(x+3)}=\frac{2x+5}{(x+2)\,(x+3)}$$

(*ii*) $$\frac{1}{x+3}+\frac{1}{x-1}+\frac{1}{(x-1)^2}=\frac{2x^2+x+1}{(x+3)(x-1)^2}$$

Here we shall study the reverse process, *i.e.* to split a given fraction having two or more factors in the denominator into simpler fractions. The denominators of the simpler or partial fractions are given by the factors of the given fraction while numerators are unknown and as such they will be assumed by us. Then by simple algebraic methods, the unknown quantities in the numerator will be obtained. Thus we will assume

$$\frac{2x+5}{(x+2)(x+3)}=\frac{A}{x+2}+\frac{B}{x+3}$$

and then we will obtain the value of A and B.

In general, a rational algebraic fraction is of the type

$$\frac{(a_0\,x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_n)}{(b_0x^m+b_1x^{m-1}+b_2x^{m-2}+\ldots\ldots+b_m)}$$

where $a_0, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$ are constants. Here the degree of the numerator is n and the degree of the denominator is m. If $n \geqslant m$, the given fraction is called an improper fraction. In case $n<m$, the given fraction is a proper fraction. By actual division, an improper fraction can be written as the sum of an integral part and a proper fraction. Thus

$$\frac{x^3+3x+5}{(x-2)(x-3)}=(x+5)+\frac{22x-25}{(x-2)(x-3)}\,.$$

On the L.H.S. we have an improper fraction where $n=3$ and $m=2$ and after actual division, the R.H.S. consists of the integral part

$(x+5)$ and the proper fraction

$$\frac{22x-25}{(x-2)(x-3)}.$$

Before resolving into partial fractions, we must check that the given algebraic fraction is a proper fraction, otherwise we must divide the num. by the denom. to get a proper fraction.

Sometimes, we need not divide actually but by inspection, we write the partial fractions. Thus

$$\frac{x^3+2x+5}{(x-1)(x+2)(x-3)}=1+\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x-3)}$$

as we guess that on actual division, the quotient or integral part will be unity and the resulting proper fraction can then be resolved into three partial fractions, correpondine to the three factors in the denominator.

3·2. Methods of Resolving a Fraction into Partial Fractious

Type I. In case of non-repeated first degree factors of the denominator, we have

$$\frac{N(x)}{D(x)}\equiv\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)} \qquad ...(i)$$

where $\quad D(x)=(x-a)(x-b)(x-c)$

Therefore $\quad N(x)\equiv A(x-b)(x-c)+B(x-a)(x-c)$
$$+C(x-a)(x-b) \qquad ...(ii)$$

Now the values of A, B, C are found by putting $x=a$, $x=b$, $x=c$ respectively in the identity (ii). They can also be obtained by equating coefficients of x^2, x and constant terms on both sides of (ii) and then solving the three equations involving A, B, C.

Short Cut Method

Multiplying both sides of (i) by $(x-a)$, we have

$$(x-a)\frac{N(x)}{D(x)}=A+\frac{B(x-a)}{(x-b)}+\frac{C(x-a)}{(x-c)}.$$

$$\Rightarrow \frac{N(x)}{(x-b)(x-c)}=A+\frac{B(x-a)}{(x-b)}+\frac{C(x-a)}{(x-c)}$$

Now let $x=a$, then

$$\frac{N(a)}{(a-b)(a-c)}=A. \quad \text{Similarly} \quad \frac{N(b)}{(b-a)(b-c)}=B$$

and $$C=\frac{N(c)}{(c-a)(c-b)}$$

Example 1. *Resolve into partial fractions*

$$\frac{x^2-10x+13}{(x-1)(x-2)(x-3)}$$

Sol. Let $\dfrac{N(x)}{D(x)}=\dfrac{x^2-10x+13}{(x-1)(x-2)(x-3)}$

$$=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$$

$$=\frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

Equating Numerators, we have

$$x^2-10x+13=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

Put $x=1$, $\quad 4=A(1-2)(1-3)+0+0 \Rightarrow A=2$

Put $x=2$, $\quad -3=0+B(2-1)(2-3)+0 \Rightarrow B=3$

Put $x=3$, $\quad -8=0+0+C(3-1)(3-2) \Rightarrow C=-4$

Therefore the partial fractions are

$$\frac{2}{(x-1)}+\frac{3}{(x-2)}-\frac{4}{(x-3)}.$$

Short-cut Method

$$A=\left[(x-1)\ \frac{N(x)}{D(x)}\right]_{x=1}$$

$$=\left[\frac{x^2-10x+13}{(x-2)(x-3)}\right]_{x=1}$$

$$=\frac{1-10+13}{(1-2)(1-3)}=\frac{4}{2}=2.$$

$$B=\left[(x-2)\ \frac{N(x)}{D(x)}\right]_{x=2}$$

$$=\left[\frac{x^2-10x+13}{(x-1)(x-3)}\right]_{x=2}=3$$

$$C=\left[(x-3)\ \frac{N(x)}{D(x)}\right]_{x=3}$$

$$=\left[\frac{x^2-10x+13}{(x-1)(x-2)}\right]_{x=3}=-4.$$

Thus we get the unknown constants A, B, C easily and that leads to partial fractions.

Type II. In case of repeated first degree factors of the denominators, we have

$$\frac{N(x)}{D(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\frac{B}{(x-b)}+\frac{C}{(x-c)} \qquad ...(i)$$

where $\qquad D(x)=(x-a)^3\,(x-b)(x-c).$

Here $(x-a)$ is repeated thrice. Now equating numerators, we get

$$N(x) \equiv A_1(x-a)^2(x-b)(x-c)+A_2(x-a)(x-b)(x-c)$$
$$+A_3(x-b)(x-c)+B(x-a)^3(x-c)+C(x-a)^3(x-b) \quad ...(ii)$$

On putting $x=a$ in (ii), we get A_3. Similarly, on putting $x=b$ and $x=c$ in (ii), we get B and C respectively. To get A_1 and A_2, we equate any two of the co-efficients of x^4, x^3, x^2, x or the constant terms on both sides of (ii).

We also get A_1, A_2 etc. by putting $x=0$, 1, 2, etc. in the identity (ii) and then solving equations involving the unknown constants.

Example 2. *Resolve into partial fractions*

$$\frac{3x+2}{(x+1)^2(x-2)}.$$

Sol. Let $$\frac{N(x)}{D(x)}=\frac{3x+2}{(x+1)^2\,(x-2)}$$

$$\equiv\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x-2)}$$

or $$(3x+2)\equiv A(x+1)(x-2)+B(x-2)+C(x+1)^2 \quad ...(1)$$

Let $x=-1$, then $-1=A(0)+B(-3)+C(0) \Rightarrow B=\frac{1}{3}$.

Let $x=2$, then $8=A(0)+B(0)+C(9) \Rightarrow C=8/9$.

To obtain A, substituting $x=0$ in (1), we have

$$2=A(1)(-2)+B(-2)+C(1)^2$$

or $$-2A-2B+C=2$$

or $$-2A-\frac{2}{3}+\frac{8}{9}=2 \qquad \text{or} \qquad A=-\frac{8}{9}$$

Therefore the partial fractions are

$$-\frac{8}{9(x+1)}+\frac{1}{3(x+1)^2}+\frac{8}{9(x-2)}.$$

Shorter Method

$$B=\left[(x+1)^2\ \frac{N(x)}{D(x)}\right]_{x=-1}=\left[\frac{3x+2}{(x-2)}\right]_{x=-1}=\frac{1}{3}$$

$$C=\left[(x-2)\ \frac{N(x)}{D(x)}\right]_{x=2}=\left[\frac{3x+2}{(x+1)^2}\right]_{x=2}=\frac{8}{9}.$$

and we obtain A by the method given earlier, *i.e.* by putting $x=0$ in (1).

Example 3. *Resolve into partial fractions*

$$\frac{3x^2-13x+16}{(x-2)^3}.$$

Sol. In such cases, we put $x-2=t$ or $x=t+2$. Thus the given fraction reduces to

$$\frac{3(t+2)^2-13(t+2)+16}{t^3}$$

$$=\frac{3t^2-t+2}{t^3}=\frac{3}{t}-\frac{1}{t^2}+\frac{2}{t^3}$$

$$=\frac{3}{x-2}-\frac{1}{(x-2)^2}+\frac{2}{(x-2)^3}.$$

Example 4. *Resolve into partial fractions*

$$\frac{x^2+1}{(x+1)^3(x-2)}.$$

Sol. Here also, we can simplify the method by putting $x+1=t$ or $x=t-1$. Thus the given fraction reduces to

$$\frac{(t-1)^2+1}{t^3(t-3)}=\frac{t^2-2t+2}{t^3(t-3)}$$

$$=-\frac{1}{t^3}\left(\frac{2-2t+t^2}{3-t}\right)$$

$$=-\frac{1}{t^3}\left\{\frac{2}{3}-\frac{4t}{9}+\frac{5t^2}{27}+\frac{5t^3}{27(3-t)}\right\}$$

$$=-\frac{2}{3t^3}+\frac{4}{9t^2}-\frac{5}{27t}-\frac{5}{27(3-t)}$$

$$=-\frac{2}{3(x+1)^3}+\frac{4}{9(x+1)^2}-\frac{5}{27(x+1)}+\frac{5}{27(x-2)}.$$

which are the required partial fractions.

Type III. Denominator containing second degree non-repeated factors. Let

$$\frac{N(x)}{D(x)}=\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{dx^2+ex+f},$$

where $\quad D(x)=(ax^2+bx+c)(dx^2+ex+f).$

Equating Num. of both sides, we get

$$N(x)=(Ax+B)(dx^2+ex+f)+(Cx+D)(ax^2+bx+c) \quad \ldots (1)$$

Now equating coefficients of x^3, x^2, x and the constant terms and solving the four equations, we get A, B, C and D.

Sometimes, it is easier to put $x=0, 1,$ etc. in (1) and then obtain the values of A, B, C, D.

Example 5. *Resolve into partial fractions*

$$\frac{x^2+1}{x^4+x^2+1}.$$

Sol. Let
$$\frac{N(x)}{D(x)}=\frac{x^2+1}{(x^2+x+1)(x^2-x+1)}$$
$$=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$$

or
$$(x^2+1)=(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$$

Equating co-efficients of x^3, x^2, x etc., we get

$$0=A+C$$
$$1=-A+B+C+D$$
$$0=A-B+C+D$$
$$1=B+D.$$

Solving these equations, we get $A=0$, $B=\frac{1}{2}$, $C=0$ and $D=\frac{1}{2}$.

Thus the partial fractions are

$$\frac{1}{2(x^2+x+1)}+\frac{1}{2(x^2-x+1)}.$$

Example 6. *Resolve into partial fractions*

$$\frac{x^2+1}{(x^3-1)}.$$

Sol. Let
$$\frac{N(x)}{D(x)}=\frac{x^2+1}{(x-1)(x^2+x+1)}$$

or
$$\frac{x^2+1}{x^3-1}=\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+x+1)} \qquad \ldots(1)$$

Now
$$A=\left[(x-1)\,\frac{N(x)}{D(x)}\right]_{x=1}$$
$$=\left[\frac{x^2+1}{x^2+x+1}\right]_{x=1}=\frac{2}{3}$$

To calculate B and C,

let $x=0$ and $x=-1$ in (1), then

$$-1=-A+C \quad \Rightarrow C=-\tfrac{1}{3},$$

and
$$-1=-\tfrac{1}{2}A-B+C \Rightarrow B=\tfrac{1}{3}.$$

Thus the partial fractions are

$$\frac{2}{3(x-1)}+\frac{1}{3}\left(-\frac{x-1}{x^2+x+1}\right)$$

Example 7. *Resolve into partial fractions*

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}.$$

Sol. Let $x^2=t$,

then the given fraction reduces to

$$\frac{N(t)}{D(t)}=\frac{(t+1)(t+2)}{(t+3)(t+4)}=1+\frac{A}{(t+3)}+\frac{B}{(t+4)}$$

Now equating numerators, we have

$$(t+1)(t+2)=(t+3)(t+4)+A(t+4)+B(t+3)$$

Putting $t=-3$ and $t=-4$, we have $A=2$ and $B=-6$. Therefore, the partial fractions are

$$1+\frac{2}{(x^2+3)}-\frac{6}{(x^2+4)}.$$

EXERCISE 3

Resolve into partial fractions.

1. $\frac{2x-1}{(x-1)(2x-3)}$
2. $\frac{(x-1)(x-2)}{(x-3)(x-4)}$
3. $\frac{6x^2-x-10}{(x-1)(x-2)(x+4)}$
4. $\frac{s-4}{(s+4)(s^2-3s+2)}$
5. $\frac{6x^3+x^2-7}{3x_2-2x-1}$
6. $\frac{s(s+1)(s+2)}{(s+3)(s+4)(s+5)}$
7. $\frac{x^3-6x^2+13x-5}{(x-1)(x-2)(x-3)}$
8. $\frac{(1+2t)(1+3t)(1+4t)}{(1-2t)(1-3t)(1-4t)}$
9. $\frac{x^2+1}{(x^2+2)(x^2+3)(x^2+4)}$
10. $\frac{s^6+3s^4+3s^2+5}{(s^2+1)(s^2+2)}$
11. $\frac{3x^3-8x^2+10}{(x-1)^4}$
12. $\frac{2y^2+3}{(y-1)(y+1)^2}$
13. $\frac{x^2}{(x-1)^3(x+1)}$
14. $\frac{1}{(s^2+s)(s^2-1)}$
15. $\frac{x-4}{(x^2-3x+2)(x^2+4)}$
16. $\frac{s^2+1}{s^3+1}$
17. $\frac{x^4+1}{(x-1)(x^2+1)}$
18. $\frac{y+1}{(y^2+1)(y-1)^2}$

19. $\dfrac{1}{(x^2+1)^2(x^2+3)}$.

20. $\dfrac{s^2+4s}{(1-s)(1+s^2)^2}$

21. $\dfrac{1}{p^4-a^4}$

22. $\dfrac{s}{s^4+s^2+1}$

23. $\dfrac{s}{s^4+4a^4}$

24. $\dfrac{a(m^2-2a^2)}{m^4+4a^4}$

25. $\dfrac{1}{s^6-1}$.

[**Hint.**

$$\frac{1}{s^6-1}=\frac{1}{(s^3-1)(s^3+1)}$$

$$=\frac{1}{2}\left(\frac{1}{s^3-1}-\frac{1}{s^3+1}\right)$$

$$=\frac{1}{2\,(s-1)(s^2+s+1)}-\frac{1}{2(s+1)(s^2-s+1)}$$

Now let $\dfrac{1}{(s-1)(s^2+s+1)}=\dfrac{A}{s-1}+\dfrac{Bs+C}{s^2+s+1}$

and $\dfrac{1}{(s+1)(s^2-s+1)}=\dfrac{D}{s+1}+\dfrac{Es+F}{s^2-s+1}$]

3. Convgt.
4. Convgt.
5. Convgt.
6. Convgt.
7. Convgt.
8. Convgt.
9. Convgt.
10. Convgt. for $x<1$; Divgt. for $x\geqslant 1$
11. Convgt. for $x\leqslant 1$; Divgt. for $x>1$.
12. Convgt. for $x<1$; Divgt. for $x\geqslant 1$.
13. Convgt. for $x<1$; Divgt. for $x\geqslant 1$.
14. Convgt.
15. Convgt.
16. Convgt.
17. Convgt.
18. Convgt.
19. Divgt.
20. Congvt.
21. Divgt.
22. Convgt if $p>2$ and Divgt. for $p\leqslant 2$.
23. Convgt.
24. Divergent.
25. Divgt.
26. Convgt.
27. Divgt.
28. Convgt.
29. Convgt.

Exercise 2. (*c*) (Page 71)

1. Convgt. for $x<2$ and Divgt. for $x\geqslant 2$
2. Convgt. for $x<\frac{3}{2}$ and Divgt. for $x\geqslant\frac{3}{2}$
3. Convgt. for $x\leqslant 1$ and Divgt. for $x>1$
4. Convgt. for $x\leqslant 1$ and Divgt. for $x>1$
5. Convgt.
6. Convgt for $x<1$, Divgt. for $x>1$; when $x=1$, Convgt. if $\gamma-\alpha-\beta>0$ and Divgt. for $\gamma-\alpha-\beta\leqslant 0$,

Exercise 2 (*d*) (Page 73)

1. Convgt.
2. Convgt.
3. Convgt.
4. Convgt. for $x\leqslant 1$ and Divgt. for $x>1$.
5. Divgt.

Exereise 2 (*e*) (Page 76)

1. Convgt.
2. Convgt.
3. Convgt.
4. Convgt.
5. Convgt.
6. Convgt.
7. Convgt. for $|x|\leqslant 1$ and Divgt. for $|x|>1$.
8. Divgt.

Exercise 2 (*f*) (Page 77)

1. Abs. convgt.
2. Conditionally convgt.
3. Abs. convgt.
4. Conditionally convgt.
5. Conditionally convgt.
6. Conditionally convgt.
7. Conditionally convgt.

Exercise 3 (Page 84−85)

1. $\dfrac{4}{2x-3}-\dfrac{1}{x-1}$
2. $1-\dfrac{2}{x-3}+\dfrac{6}{x-4}$
3. $\dfrac{1}{x-1}+\dfrac{2}{x-2}+\dfrac{3}{x+4}$
4. $\dfrac{3}{5(s-1)}-\dfrac{4}{15(s+4)}-\dfrac{1}{3(s-2)}$

5. $2x+3+\frac{1}{x-1}+\frac{5}{3x+1}$

6. $1-\frac{3}{s+3}+\frac{24}{s+4}-\frac{30}{s+5}$

7. $1+\frac{3}{2(x-1)}-\frac{5}{x-2}+\frac{7}{2(x-3)}$

8. $-1+\frac{30}{1-2t}-\frac{70}{1-3t}+\frac{42}{1-4t}$

9. $\frac{2}{x^2+3}-\frac{1}{2(x^2+2)}-\frac{3}{2(x^2+4)}$

10. $s^2+\frac{4}{s^2+1}-\frac{3}{s^2+2}$

11. $\frac{3}{x-1}+\frac{1}{(x-1)^2}-\frac{7}{(x-1)^3}+\frac{5}{(x-1)^4}$

12. $\frac{5}{4(y-1)}+\frac{3}{4(y+1)}-\frac{1}{2(y+1)^2}$

13. $\frac{1}{8(x-1)}-\frac{1}{8(x+1)}+\frac{3}{4(x-1)^3}+\frac{1}{2(x-1)^2}$

14. $\frac{1}{4(s-1)}-\frac{1}{s}+\frac{3}{4(s+1)}+\frac{1}{2(s+1)^2}$

15. $\frac{1}{20}\left(\frac{12}{x-1}-\frac{5}{x-2}-\frac{7x+2}{x^2+4}\right)$

16. $\frac{2}{3(s+1)}+\frac{s+1}{3(s^2-s+1)}$

17. $x+1+\frac{1}{x-1}-\frac{x+1}{x^2+1}$

18. $\frac{y-1}{2(y^2+1)}-\frac{1}{2(y-1)}+\frac{1}{(y-1)^2}$

19. $\frac{1}{2(x^2+1)^2}-\frac{1}{4(x^2+1)}+\frac{1}{4(x^2+3)}$

20. $\frac{5}{4(1-s)}+\frac{5(s+1)}{4(1+s^2)}+\frac{1}{2}\,\frac{(3s-5)}{(1+s^2)^2}$

21. $\frac{1}{2a^2}\left[\frac{1}{2a}\left(\frac{1}{p-a}-\frac{1}{p+a}\right)-\frac{1}{p^2+a^2}\right]$

22. $\frac{1}{2}\left(\frac{1}{s^2-s+1}-\frac{1}{s^2+s+1}\right)$

23. $\frac{1}{4a}\left(\frac{1}{s^2-2as+2a^2}-\frac{1}{s^2+2as+2a^2}\right)$

24. $\frac{1}{2}\left(\frac{m-a}{m^2-2am+2a^2}-\frac{m+a}{m^2+2am+2a^2}\right)$

25. $\frac{1}{2}\left[\frac{1}{3(s-1)}-\frac{s+2}{3(s^2+s+1)}\right]-\frac{1}{2}\left[\frac{1}{3(s+1)}-\frac{s-2}{3(s^2-s+1)}\right]$

Exercise 4 (a) (Page 92–95)

1. $f(0)=9, f(2)=19, f(-3)=-51, f(x+1)=x^3+x^2+4x+8$

$f\left(\frac{2}{x}\right)=(9x^3+10x^2-8x+8)/x^3$

5. $\frac{-\pi}{2}$; 0 ; $\frac{\pi}{4}$

6. (*a*) $\frac{2}{3}$ (*b*) $\sqrt{2}$ (*c*) 2 (*d*) $p-q$

9. (*a*) $\frac{1}{4}$ (*b*) π (*c*) 1

11. (*i*) e^{-4} (*ii*) 0 (*iii*) 1 (*iv*) 1 (*v*) $\frac{1}{2}$

(*vi*) Does not exist.

13. Continuous at $x=2$ and $x=3$

14. Continuous at $x=0$ and $x=1$

15. Discontinuous at $x=0$.

Exercise 4(b) (page 106-108)

1 $b+2cx$

5. (*i*) $ma^m x^{m-1}-\frac{5}{2}b^{5/2}x^{-7/2}-\frac{c}{x^2}$

(*ii*) $3\sqrt{x}+\frac{3}{2\sqrt{x}}-\frac{2}{x\sqrt{x}}$

8. (*i*) $\frac{(x^4+2x^3-3x^2-2x-1)}{(x^2+x-1)^2}$ (*ii*) $\frac{x(3a^2-x^2)}{(a^2-x^2)^{3/2}}$

(*iii*) $\frac{(a^2+2x^2)}{(a^2-x^2)^{5/2}}$ (*iv*) $\frac{1}{(1-x)\sqrt{1-x^2}}$

(*v*) $1+\frac{x}{\sqrt{x^2-1}}$.

9. (*i*) $n\sin^{n-1}x\cos^{m+1}x-m\sin^{n+1}\cos^{m-1}x$

(*ii*) $\sin(ax^2+bx+c)+(2ax+b)x \, . \, x\cos(ax^2+bx+c)$

(*iii*) $\frac{2x-3x^2-x^4}{(1+x^3)^2}\cos\left(\frac{1+x^2}{1+x^3}\right)$ (*iv*) $\frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$

(*v*) $-\frac{2\sec x}{(\sec x+\tan x)^2}$

5

Successive Differentiation

5·1. Introduction

Let $y=f(x)=e^{6x}+5\sin 2x+4x^2-3x+7$, then on differentiating w.r.t. x, we get $y_1=f'(x)=6e^{6x}+10\cos 2x+8x-3$.

Here y_1 or $f'(x)$ is called the first derivative of y or $f(x)$ and it is a function of x. Again differentiating, we have

$$y_2=f''(x)=36e^{6x}-20\sin 2x+8.$$

This is the second derivative of y and it can be further differentiated to give

$y_3=f'''(x)=216\,e^{6x}-40\cos 2x$, which is the third derivative of y. It is denoted by $\dfrac{d^3y}{dx^3}$, D^3y or y'''.

Thus if we differentiate a function y, n-times successively, we will obtain nth derivative of the function y which we denote by

$$y_n, f^n(x), \quad \frac{d^ny}{dx^n} \text{ or } D^ny.$$

5·2. Standard Results

(a) If $\quad y=(ax+b)^m$, then

$$y_1=m.\,a(ax+b)^{m-1}$$
$$y_2=m(m-1).\,a^2(ax+b)^{m-2}$$
$$y_3=m(m-1)(m-2).\,a^3(ax+b)^{m-3}.$$

In general, $y_n=m(m-1)(m-2)\,.\,(m-n+1).\,a^n(ax+b)^{m-n}$

In particular (i) $y_n=\dfrac{m\,!}{(m-n)\,!}\,a^n(ax+b)^{m-n}$,

if m is a positive interger$>n$.

(ii) $y_n=0$, if m is a positive integer$<n$,

(iii) $y_n=n\,!\,a^n$, when $m=n$,

(iv) $y=(ax+b)^{-1}$, when $m=-1$,

$y_n=(-1)(-2)(-3)\,..(-n).\,a^n(ax+b)^{-1-n}$

$$=(-1)^n(n\,!)\,a^n.(ax+b)^{-(n+1)}$$

(*b*) If $y=e^{ax}$, then

$$y_1=ae^{ax},\ y_2=a^2\,e^{ax},\ y_3=a^3\,e^{ax}.$$

This can be generalised to

$$y_n=a^n\,e^{ax}.$$

(*c*) If $y=\log(ax+b)$, then

$$y_1=a/(ax+b)=a(ax+b)^{-1},$$
$$y_2=a^2.\,(-1)(ax+b)^{-2},$$
$$y_3=a^3(-1)(-2)(ax+b)^{-3},$$
$$y_4=a^4(-1)(-2)(-3)(ax+b)^{-4}$$
$$=a^4(-1)^3\,3\,!\,(ax+b)^{-4}.$$

$\therefore\quad y_n=a^n(-1)^{n-1}(n-1)\,!\,(ax+b)^{-n}.$

(*d*) If $y=\sin(ax+b)$, then

$$y_1=a.\cos(ax+b)=a.\sin\left(ax+b+\frac{\pi}{2}\right),$$

on using $\cos\theta=\sin\left(\theta+\frac{\pi}{2}\right)$. Again differentiating, we get

$$y_2=a^2.\cos\left(ax+b+\frac{\pi}{2}\right)$$
$$=a^2.\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$
$$=a^2.\sin\left(ax+b+2.\,\frac{\pi}{2}\right).$$

Similarly, $y_3=a^3.\sin\left(ax+b+3.\,\frac{\pi}{2}\right).$

Continuing this process, we get

$$y_n=a^n\sin\left(ax+b+\frac{n\pi}{2}\right).$$

(*e*) If $y=\cos(ax+b)$, then proceeding exactly as above we get $y_n=a^n\cos\left(ax+b+\frac{n\pi}{2}\right).$

(*f*) If $y=e^{ax}\sin(bx+c)$, then

$$y_1=ae^{ax}.\sin(bx+c)+e^{ax}.\,b\cos(bx+c)$$
$$=e^{ax}\{a\sin(bx+c)+b\cos(bx+c)\}.$$

Now let $a=r\cos\phi$ and $b=r\sin\phi$, we get

$$y_1=e^{ax}\,.\,r\,\{\sin(bx+c)\cos\phi+\cos(bx+c)\sin\phi\}$$
$$=re^{ax}\sin(bx+c+\phi).$$

Similarly, again differentiating and simplifying, we have

$$y_2=r^2e^{ax}\sin(bx+c+2\phi)$$

and $$y_3=r^3e^{ax}\sin(bx+c+3\phi).$$

If this process is continued n times, we get

$$y_n=r^ne^{ax}\sin(bx+c+n\phi),$$

where $$r=\sqrt{(a^2+b^2)}\text{ and }\phi=\tan^{-1}\frac{b}{a}.$$

(g) If $y=e^{ax}\cos(bx+c)$, then proceeding as above we have

$$y_n=r^n\,e^{ax}\cos(bx+c+n\phi).$$

where r and ϕ are given earlier.

Example 1. *Find the nth derivative of* $\dfrac{x^4}{(x-1)(x-2)}$

Sol. Let $$y=\frac{x^4}{(x-1)(x-2)}$$

$$=x^2+3x+7+\frac{15x-14}{(x-1)(x-2)}$$

Resolving into partial fractions, we have

$$y=x^2+3x+7-\frac{1}{(x-1)}+\frac{16}{(x-2)}$$

Now differentiating n (>2) times and using the result 5·2(a) with $m=-1$, we have

$$y_n=(-1)^n\,(n\,!)\left\{\frac{16}{(x-2)^{n+1}}-\frac{1}{(x-1)^{n+1}}\right\}.$$

Example 2. *If $y=\sin^4 x$, find y_n.*

Sol. Here $$y=\sin^4 x$$

$$=(\sin x^2)^2=\left(\frac{1-\cos 2x}{2}\right)^2$$

$$=\tfrac14(1-2\cos 2x+\cos^2 2x)$$

$$=\tfrac14\{1-2\cos 2x+\tfrac12(1+\cos 4x)\}$$

$$=\tfrac18(3-4\cos 2x+\cos 4x)$$

Now differentiating n times w. r. t. x and using the result 5·2 (e), we have

$$y_n=\tfrac18\left\{-4.2^n\cos\left(2x+\frac{n\pi}{2}\right)+4^n\cos\left(4x+\frac{n\pi}{2}\right)\right\}$$

$$=2^{n-1}\left\{2^{n-2}\cos\left(4x+\frac{n\pi}{2}\right)-\cos\left(2x+\frac{n\pi}{3}\right)\right\}$$

Example 3. *If $y=e^{3x}\cos x\cos 2x\sin x$, find y_n.*

Sol. Here $y=\frac{1}{2}e^{3x}\,(2\cos x\ \sin x)\cos 2x$

$$=\tfrac{1}{2}\,e^{3x}\sin 2x\cos 2x$$

$$=\tfrac{1}{4}\,e^{3x}\sin 4x$$

Now using the result 5·2 (*d*), we have

$$y_n=\tfrac{1}{4}\,.\,5^n\,e^{3x}\sin\left(4x+n\tan^{-1}\frac{4}{3}\right).$$

Example 4. *Find y_n when $y=\tan^{-1}\left(\frac{x}{a}\right)$.*

Sol. Differentiating y w. r. t. x, we have

$$y_1=\frac{a}{x^2+a^2}=\frac{a}{(x-i\,a)(x+i\,a)}$$

$$=\frac{1}{2i}\left\{\frac{1}{(x-i\,a)}-\frac{1}{(x+i\,a)}\right\}$$

Differentiating $(n-1)$ times, we have

$$y_n=\frac{(-1)^{n-1}\,(n-1)\,!}{2\,i}\left\{\frac{1}{(x-i\,a)^n}-\frac{1}{(x+i\,a)^n}\right\}$$

Now y_n can be simplified by using De Moivre's theorem. Let $x=r\cos\phi$ and $a=r\sin\phi$.

Then $(x-i\,a)^{-n}=(r\cos\phi-i\,r\sin\phi)^{-n}$

$$=r^{-n}\,(\cos\phi-i\sin\phi)^{-n}=r^{-n}\,(\cos n\phi+i\sin n\phi).$$

Similarly $(x+i\,a)^{-n}=r^{-n}\,(\cos n\phi-i\sin n\phi)$.

$$\therefore\qquad y_n=\frac{(-1)^{n-1}\,(n-1)\,!}{2\,i}\,r^{-n}\,\{(\cos n\phi+i\sin n\phi)$$

$$-(\cos n\phi-i\sin n\phi)\}$$

$$=(-1)^{n-1}\,(n-1)\,!\,r^{-n}\sin n\phi$$

$$=(-1)^{n-1}\,(n-1)\,!\,a^{-n}\sin^n\phi\sin n\phi$$

for $\qquad a=r\sin\phi\ \Rightarrow\ \dfrac{1}{r}=\dfrac{1}{a}\sin\phi$

or $\qquad r^{-n}=a^{-n}\sin^n\phi.$

Example 5. *Find the nth derivative of* $\dfrac{1}{(1+x+x^2)}$

Sol. Let $y=\dfrac{1}{x^2+x+1}=\dfrac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}=\dfrac{1}{t^2+a^2}$,

where $\qquad t=x+\dfrac{1}{2}$ and $a=\dfrac{\sqrt{3}}{2}$

Now applying the method of Example 4, we get

$$y_n=\frac{(-1)^n\,n\,!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\,\theta$$

where $\qquad \tan\theta=\dfrac{\sqrt{3}}{2x+1}$, $a=\dfrac{\sqrt{3}}{2}$.

EXERCISE 5 (a)

1. If $y=\left(\dfrac{1+x}{1-x}\right)^n$, show that $\dfrac{dy}{dx}=\dfrac{2ny}{(1-x^2)}$ and
$$\frac{d^2y}{dx^2}=\frac{2(n+x)}{(1-x^2)}\cdot\frac{dy}{dx}$$

2. Find $\dfrac{d^2y}{dx^2}$ where $x=a\,(\theta+\sin\theta)$, $y=a(1-\cos\theta)$

3. If $y=a\,e^{-kt}\cos(pt+c)$, show that
$$\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+n^2y=0,\text{ where } n^2=p^2+k^2.$$

4. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$, prove that
$$p+\frac{d^2p}{d\theta^2}=\frac{a^2b^2}{p^3}$$

5. If $y=\sin kx+\cos kx$, prove that
$$y_n=k^n\,\{1+(-1)^n\sin 2\,kx\}^{1/2}.$$

6. Find the nth differential co-efficients of

(i) $\cos 2x\sin 3x$ (ii) $3\cos 5x\cos 3x+x^n$

(iii) $\cos x\cos 2x\cos 3x$ (iv) $\sin^2 x\cos^3 x$

(v) $\cos^4 x$ (vi) $e^x\sin x\sin 2x$

7. Find the nth derivatives of

(i) $\dfrac{2x-1}{(x-2)(x+1)}$ (ii) $(1-5x+6x^2)^{-1}$

(iii) $\dfrac{4x}{(x-1)^2(x+1)}$ (iv) $\tan^{-1}\left\{\left(\dfrac{1+x}{1-x}\right)\right\}$.

8. If $y=\tan^{-1}x$, show that
$$y_n=(n-1)!\cos\left\{ny+(n-1)\frac{\pi}{2}\right\}\cos^n y.$$

9. If $y=e^{x\cos\beta}\cos(x\sin\beta)$, show that
$$y_n=e^{x\cos\beta}\,.\,\cos(x\sin\beta+n\,\beta)$$

10. Prove that the value of the nth derivative of $\dfrac{x^3}{x^2-1}$ for $x=0$, is zero when n is even and $(-n\,!)$ when n is odd and greater than one.

11. If $y=\log\sqrt{\dfrac{2x+1}{x-2}}$, show that
$$y_n=\frac{1}{2}(-1)^{n-1}(n-1)!\left\{\frac{2^n}{(2x+1)^n}-\frac{1}{(x-2)^n}\right\}$$

12. If $y=\cosh 2x$, show that (*i*) $y_n=2^n \sinh 2x$ when n is odd and (*ii*) $y_n=2^n \cosh 2x$ when n is even.

[**Hint.** $y=\cosh 2x=\frac{1}{2}(e^{2x}+e^{-2x})$. Now find y_n.]

13. If $y=e^{ax} \sin bx$, prove that

(*i*) $y_n=(a \sec \theta)^n e^{ax} \sin\left(bx+n \tan^{-1}\frac{b}{a}\right)$

(*ii*) $y_{n+1}-2ay_n+(a^2+b^2)\,y_{n-1}=0$

14. If $f(x)=e^{-ax} \cos (bx+c)$ show that

$$f^n(x)=(-1)^n (a^2+b^2)^{\frac{n}{2}} e^{-ax} \cos\left(bx+c+n \tan^{-1}\frac{b}{a}\right).$$

Also find $f^n(0)$ when $a=b=1$ and $c=0$.

15. If $(1-x^2) \tan y=2x$, show that

$$y_n=2(-1)^{n-1}(n-1)!\,\sin^n w \sin nw$$

where $\cot w=x$.

16. If $y=x \log\left(\frac{x-1}{x+1}\right)$, prove that

$$y_n=(-1)^n (n-2)!\left[\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right]$$

17. If $y=x(a^2+x^2)^{-1}$, show that

$$y_n=(-1)^n n!\,a^{-n-1} \sin^{n+1}\phi \cos (n+1)\phi$$

where $\phi=\tan^{-1}\left(\frac{a}{x}\right)$.

18. If $y=x \log (x+1)$, prove that

$$y_n=\frac{(-1)^{n-2}(n-2)!\,(x+n)}{(x+1)^n}$$

5·3. Leibnitz's Theorem

If $y=u.v$, where u and v are functions of x having derivatives of nth order, then

$$y_n=u_n \cdot v+{}^nC_1 u_{n-1} \cdot v_1+{}^nC_2 \cdot u_{n-2} \cdot v_2+\ldots+{}^nC_r\, u_{n-r} \cdot v_r+\ldots+u \cdot v_n$$

where suffixes in u and v denote the order of differentiation with respect to x and ${}^nC_1$, ${}^nC_2$,...have their usual meanings.

This theorem is proved by Mathematical induction.

Step I. By differentiating $y=u \cdot v$ successively, we get

$$y_1=u_1 v+u \cdot v_1,$$

$$y_2=(u_2 \cdot v+u_1 \cdot v_1)+(u_1 \cdot v_1+u \cdot v_2)$$

$$=u_2 \cdot v+2u_1 \cdot v_1+u \cdot v_2$$

$$=u_2 \cdot v+{}^2C_1 u_1 v_1+u \cdot v_2$$

$$\begin{aligned} y_3 &= (u_3 . v + u_2 . v_1) + 2(u_2 . v_1 + u_1 . v_2) + (u_1 . v_2 + u . v_3) \\ &= u_3 . v + 3u_2 . v_1 + 3u_1 . v_2 + u . v_3 \\ &= u_3 . v + {}^3C_1\, u_2 . v_1 + {}^3C_2\, u_1 . v_2 + u . v_3. \end{aligned}$$

Thus the theorem is true for $n=1$, 2 and 3.

Step II. Now we assume that the theorem is true for a particular value of n. Differentiating y_n with respect to x once again, we have

$$\begin{aligned} y_{n+1} &= (u_{n+1} . v + u_n . v_1) + ({}^nC_1\, u_n . v_1 + {}^nC_1\, u_{n-1} . v_2) + \ldots\ldots \\ &\quad + ({}^nC_r\, u_{n-r+1} . v_r + {}^nC_r\, u_{n-r} . v_{r+1}) + \ldots + (u_1 . v_n + u . v_{n+1}) \\ &= u_{n+1} . v + (1 + {}^nC_1)\, u_n . v_1 + ({}^nC_1 + {}^nC_2)\, u_{n-1} . v_2 + \ldots \\ &\quad + ({}^nC_{r-1} + {}^nC_r)\, u_{n-r+1} . v_r + \ldots + u . v_{n+1} \\ &= u_{n+1} . v + {}^{n+1}C_1\, u_n . v_1 + {}^{n+1}C_2\, u_{n-1} . v_2 + \ldots \\ &\quad + {}^{n+1}C_r\, u_{n-r+1} . v_r + \ldots + u . v_{n+1}, \end{aligned}$$

since $({}^nC_{r-1} + {}^nC_r) = {}^{n+1}C_r$; ${}^nC_0 + {}^nC_1 = {}^{n+1}C_1$

or $\qquad 1 + {}^nC_1 = {}^{n+1}C_1$; ${}^nC_1 + {}^nC_2 = {}^{n+1}C_2$ etc.

Thus we see that if the theorem is true for a particular value of n, it is also true for $(n+1)$. But the theorem is true for $n=3$, so it is also true for $n=3+1=4$ and so on. Therefore, it must be true for every positive integral value of n.

Remark 1. If one of the functions out of the product is a polynomial function of x, we will generally take it as v while the function whose nth derivative is easily known, will be taken as u. For example, in case

$$y = (3x^2 - 7x + 4)\, e^{5x},$$

take $u = e^{5x}$ and $v = 3x^2 - 7x + 4$.

Then $\qquad u_n = 5^n e^{5x},$

$\qquad\qquad u_{n-1} = 5^{n-1} e^{5x},$

$\qquad\qquad u_{n-2} = 5^{n-2} e^{5x}$ etc.

and $\qquad v_1 = 6x - 7,\ v_2 = 6,\ v_3 = 0 = v_4 = v_5 = ..$ etc.

$$\begin{aligned} \therefore \quad y^n &= D^n\{e^{5x} \times (3x^2 - 7x + 4)\} \\ &= (5^n e^{5x}) \times (3x^2 - 7x + 4) + {}^nC_1 (5^{n-1} e^{5x}) \\ &\qquad \times (6x - 7) + {}^nC_2 (5^{n-2}\, e^{5x}) \times 6 \\ &= 5^{n-2}\, e^{5x}\, \{25(3x^2 - 7x + 4) + 5n(6x - 7) \\ &\qquad + 3n(n-1)\} \end{aligned}$$

Here we get the first three terms only as all the rest vanish.

Remark 2. The Leibnitz's theorem can also be stated as follows.

$$\begin{aligned} D^n(uv) &= D^n\, u . v + {}^nC_1 . D^{n-1}\, u . Dv + {}^nC_2\, D^{n-2}\, u . D^2 v + \ldots \\ &\quad + {}^nC_r\, D^{n-r}\, u . D^r v + \ldots + u . D^n v. \end{aligned}$$

Example 1. *If* $y = x^n/(x+1)$,

show that $$y_n = \frac{n!}{(x+1)^{n+1}}.$$

Sol. Let $$u = \frac{1}{(x+1)}$$
$$= (x+1)^{-1}$$

and $$v = x^n.$$

Then $$u_1 = -(x+1)^{-2},$$
$$u_2 = (-1)^2\, 2!\, (x+1)^{-3}$$
$$u_n = (-1)^n\, n!\, (x+1)^{-(n+1)}.$$

and $$v_1 = nx^{n-1},$$
$$v_2 = n(n-1)\, x^{n-2}, \ldots, v_n = n!$$

Now $$y_n = D^n\,(u \times v)$$
$$= (-1)^n\, n!\, (x+1)^{-(n+1)} \cdot x^n + {}^nC_1(-1)^{n-1} \times (n-1)!\,(x+1)^{-n} \cdot nx^{n-1} + {}^nC_2 \cdot (-1)^{n-2} \times (n-2)!\,(x+1)^{-(n-1)} \cdot n(n-1)x^{n-2} + \ldots + \frac{1}{x+1} \cdot n!$$
$$= \frac{(-1)^n\, n!}{(x+1)^{n+1}} \{x^n - {}^nC_1\, x^{n-1} \cdot (x+1) + {}^nC_2\, x^{n-2} \cdot (x+1)^2 - \ldots + (x+1)^n\}$$
$$= \frac{(-1)^n\, n!}{(x+1)^{n+1}} \{[x-(x+1)]^n\}$$
$$= \frac{(-1)^n\, n!}{(x+1)^{n+1}} \{(-1)^n\}$$
$$= n!/(x+1)^{n+1}.$$

Example 2. *If* $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$

prove that $x^2\, y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2\, y_n = 0.$

Sol. Simplifying $\cos^{-1}\left(\frac{y}{b}\right) = n \log\left(\frac{x}{n}\right)$
$$\Rightarrow \frac{y}{b} = \cos\left\{n \log\left(\frac{x}{n}\right)\right\}.$$

Differentiating, $$y_1 = -b \sin\left\{n \log\left(\frac{x}{n}\right)\right\} \times \frac{n}{x}$$
$$\Rightarrow y_1 \cdot x = -bn \sin\{n \log(x/n)\}.$$

Differentiating again and simplifying, we have
$$y_2 \cdot x + y_1 \cdot 1 = -bn \cos\{n \log(x/n)\} \cdot n/x$$
$$= -\frac{n^2}{x}\, y$$

$$y_2 \cdot x^2 + y_1 \cdot x + n^2 y = 0.$$

or Now differentiating n times by Leibnitz's theorem, we have

$$(y_{n+2} \cdot x^2 + {}^nC_1\, y_{n+1} \cdot 2x + {}^nC_2\, y_n \cdot 2) + (y_{n+1} \cdot x + {}^nC_1\, y_n \cdot 1) + n^2 y_n = 0$$

or $$\{x^2 y_{n+2} + 2nx\, y_{n+1} + n(n-1) y_n\} + (x\, y_{n+1} + n\, y_n) + n^2 y_n = 0$$

or $$x^2 y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2 y_n = 0.$$

Example 3. *If* $y = e^{a \sin^{-1} x}$, *prove that*

(i) $(1-x^2)\, y_2 - xy_1 - a^2 y = 0$

(ii) $(1-x^2) y_{n+2} - (2n+1)\, x\, y_{n+1} - (n^2+a^2)\, y_n = 0.$

Hence find the value of y_n *when* $x = 0$.

Sol. Differentiating y w.r.t. x, we get

$$y_1 = \frac{a e^{a \sin^{-1} x}}{\sqrt{(1-x^2)}} = \frac{ay}{\sqrt{1-x^2}} \quad \text{...(i)}$$

or $$y_1^2 (1-x^2) = a^2 y^2$$

Differentiating again, we have

$$2y_1 y_2 \cdot (1-x^2) + y_1^2 \cdot (-2x) = a^2 \cdot 2yy_1$$

or $$y_2 \cdot (1-x^2) - y_1 \cdot x - a^2 y = 0 \quad \text{...(ii)}$$

Differentiating (*ii*), n times by Leibnitz theorem, we have

$$\{y_{n+2} \cdot (1-x^2) + {}^nC_1\, y_{n+1} \cdot (-2x) + {}^nC_2 \cdot y_n \cdot (-2)\} - (y_{n+1} \cdot x + {}^nC_1\, y_n \cdot 1) - a^2 y_n = 0$$

or $$(1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} - (n^2+a^2) y_n = 0 \quad \text{...(iii)}$$

Now putting $x = 0$ in (*i*), (*ii*) and (*iii*), we get

$$(y_1)_0 = a,$$

$$(y_2)_0 = a^2$$

and $$(y_{n+2})_0 = (n^2+a^2)(y_n)_0 \quad \text{...(iv)}$$

Taking $n = 1, 3, 5, 7, \ldots (n-2)$ in (*iv*), we get

$$(y_3)_0 = (1^2+a^2)(y_1)_0 = (1+a^2) \cdot a$$

$$(y_5)_0 = (3^2+a^2)(y_3)_0 = (3^2+a^2)(1+a^2) \cdot a$$

$$(y_7)_0 = (5^2+a^2)(y_5)_0 = (5^2+a^2)(3^2+a^2)(1+a^2)a$$

$$\vdots \qquad\qquad \vdots$$

$$(y_n)_0 = \{(n-2)^2+a^2\}(y_{n-2})_0$$

$$= \{(n-2)^2+a^2\} \cdot \{(n-4)^2+a^2\}(y_{n-4})_0$$

$$= \{(n-2)^2+a^2\} \cdot \{(n-4)^2+a^2\} \ldots (3^2+a^2)(1+a^2)\, a.$$

This is when n is odd. When n is even, taking $n = 2, 4, 6, \ldots (n-2)$ in (*iv*), we have

$$(y_4)_0 = (2^2+a^2)(y_2)_0 = (2^2+a^2) \cdot a^2$$

$$(y_6)_0=(4^2+a^2)(y_4)_0=(4^2+a^2)\,.\,(2^2+a^2)\,.\,a^2$$

$$(y_8)_0=(6^2+a^2)(y_6)_0=(6^2+a^2)(4^2+a^2)(2^2+a^2)\,.\,a^2$$

$$\vdots \qquad \vdots$$

$$(y_n)_0=\{(n-2)^2+a^2\}(y_{n-2})_0$$

$$=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}(y_{n-4})_0$$

$$=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\ldots(4^2+a^2)(2^2+a^2)\,a^2,$$

when n is even.

Remark. If $y=f(x)$,

then
$$(y)_0=(y)_{x=0}=f(0),$$
$$(y_1)_0=(y_1)_{x=0}=f'(0),$$
$$(y_2)_0=(y_2)_{x=0}=f''(0),$$
$$(y_3)_0=(y_3)_{x=0}=f'''(0),$$
and in general
$$(y_n)_0=(y_n)_{x=0}=f^n(0).$$

EXERCISE 5 (*b*)

1. Find the nth derivative of

   (*a*) $x^3\, e^{2x}$ (*b*) $x^4 \sin 3x$ (*c*) $x^n e^x$ (*d*) $e^x \log x$

2. If $y=a\cos(\log x)+b\sin(\log x)$, show that

   (*i*) $x^2y_2+xy_1+y=0$

   (*ii*) $x^2y_{n+2}+(2n+1)x\,y_{n+1}+(n^2+1)y_n=0.$

3. If $y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x$, prove that

   $(x^2-1)\,y_{n+2}+(2n+1)x\,y_{n+1}+(n^2-m^2)\,y_n=0.$

4. If $y=(x^2-1)^n$, prove that

   $(x^2-1)\,y_{n+2}+2x\,y_{n+1}-n(n+1)\,y_n=0.$

5. If $y=\sin(m\sin^{-1}x)$, prove that

   $(1-x^2)\,y_{n+2}-(2n+1)\,x\,y_{n+1}-(n^2-m^2)\,y_n=0.$

   Also find the value of y_n when $x=0$. (*A.M.I.E. 1961, 66, 71, 74*)

6. If $f(x)=e^{m\cos^{-1}x}$, find $f^n(0)$ when n is even.

7. If $f(x)=\{\log(x+\sqrt{1+x^2})\}^2$, show that

   $f^{n+2}(0)=-n^2 f^n(0).$

8. If $x+y=1$, prove that

   $D^n(x^n y^n)=n\,!\,[y^n-({}^nC_1)^2\,y^{n-1}\,x+({}^nC_2)^2\,y^{n-2}\,.\,x^2-\ldots\ldots$
   $\ldots+(-1)^n\,x^n].$

9. If $y=\log(x+\sqrt{x^2+a^2})$, show that

   (*i*) $(a^2+x^2)y_2+xy_1=0$ (*ii*) $\lim\limits_{x\to 0}\left(\dfrac{y_{n+2}}{y_n}\right)=-\dfrac{n^2}{a^2}.$

10. If $y=(\sin^{-1} x)^2$, show that

$$(1-x^2)y_{n+2}-(2n+1)x\, y_{n+1}-n^2\, y_n=0.$$

Hence find y_n when $x=0$.

11. If $f(x)=\sin^{-1} x/\sqrt{(1-x^2)}$, show that

$$f^{n+1}(0)-n^2 f^{n-1}(0)=0.$$

Hence evaluate $f^n(0)$.

12. If $y=e^{\tan^{-1} x}$, prove that

$$(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0.$$

Hence calculate y_2, y_4, y_6 and y_8 when $x=0$.

13. Show that

$$D^n(x^{n-1} \log x)=\frac{(n-1)!}{x}$$

14. Show that

$$(1-x)\, y_{n+1}-(n+\alpha x).\, y_n-n\alpha y_{n-1}=0,$$

where $y=(1-x)^{-\alpha}\, e^{-\alpha x}$.

15. If $x=\cosh\left(\frac{1}{m} \log y\right)$, prove that

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$$

16. If $V_n=\frac{d^n}{dx^n}(x^n \log x)$, show that

$$V_n=nV_{n-1}+(n-1)!$$

Hence show that

$$V_n=n!\left(\log x+1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)$$

17. If $y=(\tan^{-1} x)^2$, prove that

(*i*) $(x^2+1)^2y_2+2x(x^2+1)y_1=2$

(*ii*) $(x^2+1)^2y_{n+2}+(4n+2)x(x^2+1)y_{n+1}+2n^2(3x^2+1)y_n$
$+2n(n-1)(2n-1)xy_{n-1}+n(n-1)^2(n-2)y_{n-2}=0.$

18. If $y=e^{\frac{1}{2}x^2}\cos x$, prove that

$$y_{2n+2}(0)-4ny_{2n}(0)+2n(2n-1)y_{2n-2}(0)=0$$

19. If $y=(1+x^2)^{\frac{m}{2}} \sin(m \tan^{-1} x)$, prove that

(*i*) $y_{2n}(0)=0$ (*ii*) $y_{2n+1}(0)=(-1)^n m(m-1)(m-2)\ldots(m-2n)$.

20. If $\sin^{-1} y=2\log(x+1)$, prove that

$$(x+1)^2\, y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2+4)y_n=0.$$

(c) $-\tan t$ (d) $\dfrac{x(y^2-x^2+2\sqrt{x^2+y^2})}{y(x^2-y^2+2\sqrt{x^2+y^2})}$

(e) $\dfrac{b(t^2-1)}{2at}$ (f) $y^2 \tan x/(y \log \cos x-1)$

7. (a) $\dfrac{1}{x\sqrt{(x+1)}}$ (b) $\dfrac{1}{2\sqrt{(x+a)(x+b)}}$

8. (a) $\left(\dfrac{x^3-1}{x^2-4}\right)$

(b) $x^x(1+\log x)+x^{\frac{1}{x}-2}(1-\log x)$.

13. $\dfrac{1}{2}$ 16. (a) $x^{\sin^{-1}x}\left\{\log x+\dfrac{\sin^{-1}x}{x}\cdot\sqrt{1-x^2}\right\}$

(b) $2\sqrt{t}\, e^t$.

Exercise 5 (a) (Page 126—127)

2. $\dfrac{1}{4a}\sec^4\dfrac{\theta}{2}$

6. (i) $\dfrac{1}{2}\left[5^n\sin\left(5x+\dfrac{n\pi}{2}\right)+\sin\left(x+\dfrac{n\pi}{2}\right)\right]$

(ii) $\dfrac{3}{2}\left[8^n\cos\left(8x+\dfrac{n\pi}{2}\right)+2^n\cos\left(2x+\dfrac{n\pi}{2}\right)\right]+n\,!$

(iii) $\dfrac{1}{4}\left[6^n\cos\left(6x+\dfrac{n\pi}{2}\right)+4^n\cos\left(4x+\dfrac{n\pi}{2}\right)\right.$

$\left.+2^n\cos\left(2x+\dfrac{n\pi}{2}\right)\right]$

(iv) $\dfrac{1}{16}\left[2\cos\left(x+\dfrac{n\pi}{2}\right)-3^n\cos\left(3x+\dfrac{n\pi}{2}\right)\right.$

$\left.-5^n\cos\left(5x+\dfrac{n\pi}{2}\right)\right]$

(v) $\dfrac{1}{8}\left[2^{n+2}\cos\left(2x+\dfrac{n\pi}{2}\right)+4^n\cos\left(4x+\dfrac{n\pi}{2}\right)\right]$

(vi) $\dfrac{1}{2}e^x\left[2^{n/2}\cos\left(x+\dfrac{n\pi}{4}\right)-10^{n/2}\cos(3x+n\tan^{-1}3)\right]$

7. (i) $(-1)^n\, n\,!\,[(x-2)^{-(n+1)}+(x+1)^{-(n+1)}]$

(ii) $\left[\dfrac{n\,!\,.\,3^{n+1}}{(1-3x)^{n+1}}-\dfrac{n\,!\,.\,2^{n+1}}{(1-2x)^{n+1}}\right]$

(iii) $(-1)^n\, n\,!\left[\dfrac{1}{(x-1)^{n+1}}-\dfrac{1}{(x+1)^{n+1}}+\dfrac{2(n+1)}{(x-1)^{n+2}}\right]$

(iv) $(-1)^{n-1}(n-1)\,!\,\sin^n\theta\sin n\theta$, where $\theta=\cot^{-1}x$.

Exercise 5 (b) (Page 131-132)

1. (a) $[9x^2+6nx+n(n-1)]\,e^{3x}.3^{n-2}$.

(b) $x^4 . 3^n \sin\left(3x+\frac{n\pi}{2}\right)+4nx^3 . 3^{n-1}\sin\left\{3x+(n-1)\frac{\pi}{2}\right\}$
$+\ldots+n(n-1)(n-2)(n-3)3^{n-4}\sin\left\{3x+(n-4)\frac{\pi}{2}\right\}$

(c) $e^x[x^n+{}^nC_1.nx^{n-1}+{}^nC_2.n(n-1)x^{n-2}+\ldots+n\,!]$

(d) $e^x\left[\log x+\frac{^nC_1}{x}-\frac{^nC_2}{x^2}+\frac{^nC_3\,2\,!}{x^3}+\ldots+\frac{(-1)^{n-1}(n-1)!}{x^n}\right]$.

5. $y_n(0)=0$, when n is even
$y_n(0)=m\,(1-m^2)(3^2-m^2)\ldots\ldots[(n-2)^2-m^2]$,
when n is odd.

6. $f^n(0)=m^2(2^2+m^2)(4^2+m^2)\ldots\ldots[(n-2)^2+m^2]\,e^{\frac{m\pi}{2}}$ n is even
$=-m(1+m^2)(3^2+m^2)\ldots.[(n-2)^2+m^2]\,e^{\frac{m\pi}{2}}$ n is odd

10. $y_n(0)=(n-2)^2 . (n-4)^2\ldots.4^2.2^2.2$, if n is even
$y_n(0)=0$, if n is odd.

11. $f^{n+1}(0)=n^2(n-2)^2(n-4^2)\ldots4^2.2^2$ when n is even
$=0$, when n is odd.

Exercise 6 (a) (Page 147-149)

1. (i) $\frac{x^2}{2\,!}+\frac{2x^4}{4\,!}+\frac{16x^6}{6\,!}+\ldots$

(ii) $1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\ldots$

(iii) $x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\frac{x^5}{24}-\ldots$

5. $mx+m(1^2-m^2)\cdot\frac{x^3}{3\,!}+m(1^2-m^2)(3^2-m^2)\,\frac{x^5}{5\,!}+\ldots$

7. $x-\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7}+\ldots$; 0·6899 ; 0·46

8. $-\frac{x^2}{2}-\frac{x^4}{12}+\ldots\ldots$; $\bar{1}\cdot954$

9. 3·1629

10. $\log\sin x+h\cot x-\frac{h^2}{2}\operatorname{cosec}^2 x+\frac{h^3}{3}\cot x\operatorname{cosec}^2 x+\ldots$;
$\bar{1}\cdot36486$

4. $\cos^5\theta \sin^7\theta = -\frac{1}{2^{11}}$ [sin $12\theta - 2\sin 10\theta - 4\sin 8\theta$
$+10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta$].

1·10. Exponential Function of a Complex Variable

We know

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.$$

Similarly we define an exponential function of a complex variable $z=(x+iy)$, as

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots,$$

or $e^{x+iy} = 1 + (x+iy) + \frac{1}{2!}(x+iy)^2 + \frac{1}{3!}(x+iy)^3 + \frac{1}{4!}(x+iy)^4 +$

Putting $x=0$, we have

$$e^{iy} = 1 + iy + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \frac{1}{4!}(iy)^4 + \dots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right)$$

$\therefore \quad e^{iy} = \cos y + i \sin y.$ .. (1)

Changing i to $-i$ in (1), we get

$e^{-iy} = \cos y - i \sin y.$ ...(2)

Adding (1) and (2), we get

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}. \quad \dots (3)$$

Substracting (2) from (1) we get

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}. \quad \dots (4)$$

The formulae (3) and (4) are known as **Euler's Formulae** for circular functions.

Circular functions of a complex variable

If z is a complex variable, then we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

and $$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

Evidently, $\cos z + i \sin z = e^{iz}$

The values of sin z and cos z in terms of exponential functions are also called **Euler's Formulae**.

From (1) and (2)

$$e^z = e^{z+2n\pi i}$$

Thus $2\pi i$ is period of e^z.

1·12. Hyperbolic Functions.

We know trigonometric functions like $\sin\theta$, $\cos\theta$ etc. are connected with a circle. Similarly, there are functions of e^x, which are connected with a hyperbola and these are known as **Hyperbolic Functions.** These functions have a great similarity to trigonometric functions and are defined as under.

(i) $\sinh x = \dfrac{e^x - e^{-x}}{2}$ (v) $\text{sech}\ x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(ii) $\cosh x = \dfrac{e^x + e^{-x}}{2}$ (vi) $\text{cosech}\ x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

(iii) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(iv) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

From (i) and (ii), we have

$$e^x = \cosh x + \sinh x$$

and

$$e^{-x} = \cosh x - \sinh x$$

The following table gives names and nomenclature of these hyperbolic functions.

Name	*Abbreviation*	*Pronunciation*
Hyperbolic sin of x	$\sinh x$	shin x
Hyperbolic cos of x	$\cosh x$	cosh x
Hyperbolic tan of x	$\tanh x$	than x
Hyperbolic sec of x	$\text{sech}\ x$	shec x
Hyperbolic cosec of x	$\text{cosech}\ x$	coshec x
Hyperbolic cot of x	$\coth x$	coth x

1·13 Relation Between Hyperbolic and Circular Functions

$$\cos ix = \tfrac{1}{2}\left(e^{i(ix)} + e^{-i(ix)}\right)$$

$$= \tfrac{1}{2}\left(e^{-x} + e^{x}\right) = \tfrac{1}{2}\left(e^x + e^{-x}\right) = \cosh x.$$

$\therefore$ $\quad$ $\mathbf{\cos ix = \cosh x}$

Similarly, $\quad$ $\mathbf{\sin ix = i \sinh x,}$

and $\quad$ $\mathbf{\tan ix = i \tanh x.}$

Also, $\quad$ $\cosh ix = \tfrac{1}{2}[e^{ix} + e^{-ix}] = \cos x$

$\sinh ix = \tfrac{1}{2}[e^{ix} - e^{-ix}] = i \sin x$

$\tanh ix = \dfrac{\sinh ix}{\cosh ix} = i \tan x$ and so on.

Example 1. *Using Euler's formulae, prove that*

(*i*) $\cos^2\theta+\sin^2\theta=1$, (*ii*) $\cos 2\theta=2\cos^2\theta-1$

(*iii*) $\sin 2\theta=2\sin\theta\cos\theta$,

(*iv*) $\sin(\theta\pm\phi)=\sin\theta\cos\phi\pm\cos\theta\sin\phi$.

Sol. (*i*) We know $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\therefore \quad \cos^2\theta+\sin^2\theta=\frac{1}{4}(e^{i\theta}+e^{-i\theta})^2+\frac{1}{4i^2}(e^{i\theta}-e^{-i\theta})^2$$

$$=\tfrac{1}{4}[(e^{i\theta}+e^{-i\theta})^2-(e^{i\theta}-e^{-i\theta})^2] \quad [\because i^2=-1]$$

$$=\tfrac{1}{4}\,4e^{i\theta}\,.\,e^{-i\theta}=1.$$

(*ii*)
$$\cos 2\theta=\frac{e^{2i\theta}+e^{-2i\theta}}{2}$$

$$=\frac{(e^{i\theta}+e^{-i\theta})^2-2e^{i\theta}\,.\,e^{-i\theta}}{2}$$

$$=\frac{4\cos^2\theta-2}{2}=2\cos^2\theta-1.$$

(*iii*)
$$\sin 2\theta=\frac{e^{2i\theta}-e^{-2i\theta}}{2i}$$

$$=2\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)$$

$$=2\sin\theta\cos\theta.$$

(*iv*)
$$e^{i(\theta\pm\phi)}=e^{i\theta}\,.\,e^{\pm i\phi}$$

$$=(\cos\theta+i\sin\theta)(\cos\phi\pm i\sin\phi)$$

$$\cos(\theta\pm\phi)+i\sin(\theta\pm\phi)=(\cos\theta\cos\phi\mp\sin\theta\sin\phi)$$
$$+i(\sin\theta\cos\phi\pm\cos\theta\sin\phi)$$

Equating imaginary parts on both sides, we get

$$\sin(\theta\pm\phi)=\sin\theta\cos\phi\pm\cos\theta\sin\phi.$$

1·11. Period of e^z, where z is a Complex Variable

Let
$$z=x+iy$$

$$e^z=e^{x+iy}=e^x(\cos y+i\sin y) \qquad \ldots(1)$$

Also $e^{z+2n\pi i}=e^{x+i(2n\pi+y)}=e^x[\cos(2n\pi+y)+i\sin(2n\pi+y)]$

$$=e^x(\cos y+i\sin y) \qquad \ldots(2)$$

From (1) and (2)

$$e^z = e^{z+2n\pi i}$$

Thus $2\pi i$ is period of e^z.

1·12. Hyperbolic Functions.

We know trigonometric functions like $\sin\theta$, $\cos\theta$ etc. are connected with a circle. Similarly, there are functions of e^x, which are connected with a hyperbola and these are known as **Hyperbolic Functions.** These functions have a great similarity to trigonometric functions and are defined as under.

(*i*) $\sinh x = \dfrac{e^x - e^{-x}}{2}$ (*v*) $\text{sech}\ x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(*ii*) $\cosh x = \dfrac{e^x + e^{-x}}{2}$, (*vi*) $\text{cosech}\ x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

(*iii*) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(*iv*) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

From (*i*) and (*ii*), we have

$$e^x = \cosh x + \sinh x$$

and

$$e^{-x} = \cosh x - \sinh x$$

The following table gives names and nomenclature of these hyperbolic functions.

Name	*Abbreviation*	*Pronunciation*
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1·13 Relation Between Hyperbolic and Circular Functions

$$\cos ix = \tfrac{1}{2}\left(e^{i(ix)} + e^{-i(ix)}\right)$$

$$= \tfrac{1}{2}\left(e^{-x} + e^{x}\right) = \tfrac{1}{2}\left(e^{x} + e^{-x}\right) = \cosh x.$$

$\therefore$ $\mathbf{\cos ix = \cosh x}$

Similarly, $\mathbf{\sin ix = i \sinh x}$,

and $\mathbf{\tan ix = i \tanh x}$.

Also, $\cosh ix = \tfrac{1}{2}[e^{ix} + e^{-ix}] = \cos x$

$\sinh ix = \tfrac{1}{2}[e^{ix} - e^{-ix}] = i \sin x$

$\tanh ix = \dfrac{\sinh ix}{\cosh ix} = i \tan x$ and so on.

1·14 Relations Between Hyperbolic Functions.

1. $\cosh^2 x - \sinh^2 x = 1$
2. $1 - \tanh^2 x = \text{sech}^2 x$
3. $\coth^2 x - 1 = \text{cosech}^2 x$
4. $\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
5. $\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
6. $\tanh (x \pm y) = \dfrac{\tanh x + \tanh y}{1 \pm \tanh x \tanh y}$
7. $\cosh 2x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$
8. $\sinh 2x = 2 \sinh x \cosh x$
9. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$
10. $\sinh x + \sinh y = 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$
11. $\sinh x - \sinh y = 2 \cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$
12. $\cosh x + \cosh y = 2 \cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$
13. $\cosh x - \cosh y = 2 \sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

Proofs of some of the above results are given below.

(1) We know $\cosh x = \dfrac{e^x + e^{-x}}{2}$

and $$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\therefore \quad \cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \tfrac{1}{4} \cdot 4 \cdot e^x \cdot e^{-x} = 1$$

or $$\cosh^2 x - \sinh^2 x = 1.$$

Aliter. $\cos^2 x + \sin^2 x = 1$

Changing x to ix, we have

$$(\cos ix)^2 + (\sin ix)^2 = 1$$

or $$\cosh^2 x + (i \sinh x)^2 = 1$$
$$\cosh^2 x - \sinh^2 x = 1.$$

(5) $$\cosh (x+y) = \tfrac{1}{2}\{e^{x+y} + e^{-(x+y)}\}$$
$$= \tfrac{1}{2}\{e^x \cdot e^y + e^{-x} \cdot e^{-y}\}$$
$$\tfrac{1}{4}[(\cosh x + \sinh x)(\cosh y + \sinh y) + (\cosh x - \sinh x)(\cosh y - \sinh y)]$$
$$= \tfrac{1}{4}[2 \cosh x \cosh y + 2 \sinh x \sinh y] \quad \text{(On simplification)}$$
$$= \cosh x \cosh y + \sinh x \sinh y$$

(7) $\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$

$$= \tfrac{1}{2}[(e^x + e^{-x})^2 - 2]$$
$$= \tfrac{1}{2}[4 \cosh^2 x - 2]$$
$$= 2 \cosh^2 x - 1$$
$$= 2(1 + \sinh^2 x) - 1$$
$$= 1 + 2 \sinh^2 x.$$

(10) $\sinh x + \sinh y = \frac{1}{i}(\sin ix + \sin iy)$

$$= \frac{1}{i}\left(2 \sin \frac{ix+iy}{2} \cos \frac{ix-iy}{2}\right)$$
$$= \frac{2i}{i} \sinh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right)$$

[$\because$ $\sin ix = i \sinh x$]

$$= 2 \sinh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right).$$

Similarly we can prove other formulae.

1·15. Series Expansions of sinh x and cosh x

We know $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\cdots\cdots\right)\right]$$
$$= \frac{1}{2}\left[2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\right]$$

$\therefore$ $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots$

Now $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right]$$
$$= \frac{1}{2}\left[2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\right]$$

$\therefore$ $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$

These expansions could easily be obtained by replacing x by ix in expansion of $\sin x$ and $\cos x$ respectively.

1·16. Periodicity of Hyperbolic Functions

Let n be an integer,

$$\therefore \sinh(\theta+2n\pi i)=\frac{e^{(\theta+2n\pi i)}-e^{-(\theta+2n\pi i)}}{2}$$

$$=\frac{e^{\theta}-e^{-\theta}}{2}=\sinh\theta$$

$$\left[\because \begin{array}{l} e^{2n\pi i}=\cos 2n\pi+i\sin 2n\pi=1 \\ e^{-2n\pi i}=\cos 2n\pi-i\sin 2n\pi=1 \end{array}\right]$$

Thus period of sinh θ is $2\pi i$.

Similarly it can be shown that period of cosh θ is also $2\pi i$.

Also $$\sinh(\theta+\pi i)=\frac{e^{(\theta+\pi i)}-e^{-(\theta+\pi i)}}{2}$$

$$=-\frac{(e^{\theta}-e^{-\theta})}{2}$$

$$\left[\because \begin{array}{l} e^{\pi i}=\cos\pi+i\sin\pi=-1 \\ e^{-\pi i}=\cos\pi-i\sin\pi=-1 \end{array}\right]$$

$$=-\sinh\theta$$

$$\cosh(\theta+\pi i)=\frac{e^{(\theta+\pi i)}+e^{-(\theta+\pi i)}}{2}$$

$$=\frac{-e^{\theta}-e^{-\theta}}{2}$$

$$=-\frac{(e^{\theta}+e^{-\theta})}{2}=-\cosh\theta$$

$$\therefore \quad \tanh(\theta+\pi i)=\frac{\sinh(\theta+\pi i)}{\cosh(\theta+\pi i)}$$

$$=\frac{-\sinh\theta}{-\cosh\theta}=\tanh\theta$$

Similarly $\coth(\theta+\pi i)=\coth\theta$

Thus the period of tanh θ and coth θ is πi.

Example 1. *Separate into real and imaginary parts.*

(iv) $\sin(x+iy)$ (ii) $\tan(x+iy)$ (iii) $\sec(x+iy)$

(iv) $\sinh(x+iy)$ (v) $e^{(x-iy)^2}$.

Sol. (i) $\sin(x+iy)=\sin x\cos iy+\cos x\sin iy$

$$=\sin x\cosh y+i\cos x\sinh y$$

(ii) $$\tan(x+iy)=\frac{\sin(x+iy)}{\cos(x+iy)}$$

Multiplying numerator and denominator on R.H.S. by $2\cos(x-iy)$, we have

$$\tan(x+iy)=\frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$=\frac{\sin 2x+\sin 2iy}{\cos 2x+\cos 2iy}$$

$$=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$

$$=\frac{\sin 2x}{\cos 2x+\cosh 2y}+i\,\frac{\sinh 2y}{\cos 2x+\cosh 2y}$$

(*iii*) $\sec(x+iy)=\dfrac{1}{\cos(x+iy)}$

$$=\frac{2\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

[Multiplying N^r and D^r by $2\cos(x-iy)$]

$$=\frac{2(\cos x\cosh y+i\sin x\sinh y)}{\cos 2x+\cosh 2y}$$

$$=\frac{2\cos x\cosh y}{\cos 2x+\cosh 2y}+i\,\frac{2\sin x\sinh y}{\cos 2x+\cosh 2y}$$

(*iv*) $\sinh(x+iy)=\dfrac{1}{i}\sin i(x+iy)$

$$=\frac{1}{i}\sin(ix-y)=\frac{1}{i}\Big[\sin ix\cos y-\cos ix\sin y\Big]$$

$$=\frac{1}{i}\Big[i\sinh x\cos y-\cosh x\sin y\Big]$$

$$=\sinh x\cos y+i\cosh x\sin y$$

$$\left[\because -\frac{1}{i}=i\right]$$

(*v*) $e^{(x-iy)^2}=e^{x^2-y^2-2ixy}=e^{x^2-y^2}\cdot e^{-2ixy}$

$$=e^{x^2-y^2}.(\cos 2xy-i\sin 2xy)$$

Example 3. *If* $\sin(\alpha+i\beta)=x+iy$, *prove that*

(*i*) $\dfrac{x^2}{\cosh^2\beta}+\dfrac{y^2}{\sinh^2\beta}=1$

(*ii*) $\dfrac{x^2}{\sin^2\alpha}-\dfrac{y^2}{\cos^2\alpha}=1.$

Sol. Here $\sin(\alpha+i\beta)=x+iy$

or $\sin\alpha\cosh\beta+i\cos\alpha\sinh\beta=x+iy$

Equating real and imaginary parts, we have

$$x=\sin\alpha\cosh\beta,$$
$$y=\cos\alpha\sinh\beta$$

or $\dfrac{x}{\cosh\beta}=\sin\alpha$

and $\dfrac{y}{\sinh\beta}=\cos\alpha$

$$\therefore \quad \frac{x^2}{\cosh^2\beta}+\frac{y^2}{\sinh^2\beta}=\sin^2\alpha+\cos^2\alpha$$

or, $$\frac{x^2}{\cosh^2\beta}+\frac{y^2}{\sinh^2\beta}=1$$

Also $$\frac{x}{\sin\alpha}=\cosh\beta$$

and $$\frac{y}{\cos\alpha}=\sinh\beta$$

or $$\frac{x^2}{\sin^2\alpha}-\frac{y^2}{\cos^2\alpha}=\cosh^2\beta-\sinh^2\beta$$

$$\therefore \quad \frac{x^2}{\sin^2\alpha}-\frac{y^2}{\cos^2\alpha}=1.$$

Example 3. *If $x+iy=\cosh(u+iv)$, then prove that*

(*i*) $$\frac{x^2}{\cosh^2 u}+\frac{y^2}{\sinh^2 u}=1$$

(*ii*) $$\frac{x^2}{\cos^2 v}-\frac{y^2}{\sin^2 v}=1.$$

Sol. Here $x+iy=\cosh(u+iv)=\cos(iu-v)$

$$=\cosh u\cos v+i\sinh u\sin v$$

Equating real and imaginary parts, we have

$$x=\cosh u\cos v \quad \ldots(1)$$

and $$y=\sinh u\sin v \quad \ldots(2)$$

Eliminating v from (1) and (2), we have

$$\frac{x^2}{\cosh^2 u}=\cos^2 v \text{ and } \frac{y^2}{\sinh^2 u}=\sin^2 v$$

$$\therefore \quad \frac{x^2}{\cosh^2 u}+\frac{y^2}{\sinh^2 u}=\cos^2 v+\sin^2 v$$

or $$\frac{x^2}{\cosh^2 u}+\frac{y^2}{\sinh^2 u}=1.$$

Again eliminating u from (1) and (2), we have

$$\frac{x^2}{\cos^2 v}-\frac{y^2}{\sin^2 v}=\cosh^2 u-\sinh^2 u$$

$$\therefore \quad \frac{x^2}{\cos^2 v}-\frac{y^2}{\sin^2 v}=1.$$

Example 4. *If $\tan(\theta+i\phi)=\tan\alpha+i\sec\alpha$, than show that*

$$2\theta=n\pi+\frac{\pi}{2}+\alpha$$

$$e^{2\phi}=\pm\cot(\alpha/2)$$

Sol. Here $\tan(\theta+i\phi)=\tan\alpha+i\sec\alpha$

$$\therefore \quad \tan(\theta-i\phi)=\tan\alpha-i\sec\alpha$$

Now $\tan 2\theta = \tan[(\theta+i\phi)+(\theta-i\phi)]$

$$= \frac{\tan(\theta+i\phi)+\tan(\theta-i\phi)}{1-\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{(\tan\alpha+i\sec\alpha)+(\tan\alpha-i\sec\alpha)}{1-(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}$$

$$= \frac{2\tan\alpha}{1-(\tan^2\alpha+\sec^2\alpha)} = -\frac{2\tan\alpha}{2\tan^2\alpha}$$

$$= -\cot\alpha = \tan\left[\left(n\pi+\frac{\pi}{2}\right)+\alpha\right]$$

$$\therefore \quad 2\theta = n\pi+\frac{\pi}{2}+\alpha$$

$\tan 2i\phi = \tan[(\theta+i\phi)-(\theta-i\phi)]$

$$= \frac{\tan(\theta+i\phi)-\tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{(\tan\alpha+i\sec\alpha)-(\tan\alpha-i\sec\alpha)}{1+(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}$$

$$= \frac{2i\sec\alpha}{1+(\tan^2\alpha+\sec^2\alpha)} = \frac{2i\sec\alpha}{2\sec^2\alpha} = i\cos\alpha$$

$$\therefore \quad i\tanh 2\phi = i\cos\alpha$$

$$\tanh 2\phi = \cos\alpha$$

or $$\frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}} = \frac{\cos\alpha}{1}$$

Applying componendo-dividendo, we have

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1+\cos\alpha}{1-\cos\alpha} = \frac{2\cos^2\alpha/2}{2\sin^2\alpha/2} = \cot^2\frac{\alpha}{2}$$

or $$e^{4\phi} = \cot^2\alpha/2$$

$$\therefore \quad e^{2\phi} = \pm\cot\alpha/2.$$

Example 5. *If* $\tan(x+iy) = \sin(u+iv)$, *then prove that*

$$\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$$

Sol. Here $\tan(x+iy) = \sin(u+iv)$

$$\frac{\sin 2x}{\cos 2x+\cosh 2y} + \frac{i\sinh 2y}{\cos 2x+\cosh 2y} = \sin u\cosh v + i\cos u\sinh v$$

[see example 1 (*ii*)]

Equating real and imaginary parts on both sides, we have

$$\frac{\sin 2x}{\cos 2x+\cosh 2y} = \sin u\cosh v \qquad \ldots(1)$$

and $$\frac{\sinh 2y}{\cos 2x+\cosh 2y} = \cos u\sinh v \qquad \ldots(2)$$

Dividing (1) by (2), we get

$$-\frac{\sin 2x}{\sinh 2y}=\frac{\sin u \cosh v}{\cos u \sinh v}$$

or
$$\frac{\sin 2x}{\sinh 2y}=\frac{\tan u}{\tanh v}.$$

Example 6. *$\sin(\theta+i\phi)=\tan\alpha+i\sec\alpha$, show that*
$$\cos 2\theta \cosh 2\phi=3.$$

Sol. Here $\sin(\theta+i\phi)=\tan\alpha+i\sec\alpha$

$\therefore\ \sin\theta\cosh\phi+i\cos\theta\sinh\phi=\tan\alpha+i\sec\alpha$

Equating real and imaginary parts, we have

$$\sin\theta\cosh\phi=\tan\alpha \qquad \ldots(1)$$

and
$$\cos\theta\sinh\phi=\sec\alpha \qquad \ldots(2)$$

Squaring and subtracting, we have

$$\sin^2\theta\cosh^2\phi-\cos^2\theta\sinh^2\phi=\tan^2\alpha-\sec^2\alpha=-1$$

or
$$\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cosh 2\phi}{2}\right)-\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\cosh 2\phi-1}{2}\right)=-1$$

Simplifying, we have

$$\cos 2\theta\cosh 2\phi=3.$$

EXERCISE 1 (*e*)

1. Using Euler's formulae, prove that

(*i*) $\sin^2 z+\cos^2 z=1$

(*ii*) $\cos 2z=\cos^2 z-\sin^2 z$

(*iii*) $\sin 2z=2\sin z\cos z$

2. Show that

(*i*) $\{\sin(\alpha+\theta)-e^{i\alpha}\sin\theta\}^n=\sin^n\alpha.\ e^{-n\theta i}$

(*ii*) $\{\sin(\alpha-\theta)+e^{\pm i\alpha}\sin\theta\}^n$
$$=\sin^{n-1}\alpha\{\sin(\alpha-n\theta)+e^{\pm i\alpha}\sin n\theta\}$$

3. Prove that (*i*) $\sinh 3\theta=3\sinh\theta+4\sinh^3\theta$

(*ii*) $\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$

(*iii*) $\tanh 3\theta=\dfrac{3\tanh\theta+\tanh^3\theta}{1+3\tanh^2\theta}$.

4. Separate into real and imaginary parts

(*i*) $\cos(\alpha+i\beta)$ (*ii*) $\cot(\alpha-i\beta)$

(*iii*) $\text{cosec}(\alpha+i\beta)$ (*iv*) $\cosh(\alpha+i\beta)$

(*v*) $\tanh(\alpha+i\beta)$ (*vi*) $e^{\cosh(x+iy)}$

5. If $\tan(\alpha+i\beta)=x+iy$, prove that

$$x^2+y^2+2x\cot 2\alpha=1$$

and $$x^2+y^2-2y\coth 2\beta=-1.$$

6. If $\tan(\theta+i\phi)=\sin(x+iy)$, show that
$\coth y\sinh 2\phi=\cot x\sin 2\theta$.

7. If $\sin(\theta+i\phi)=\rho(\cos\alpha+i\sin\alpha)$, prove that

$$\rho^2=\tfrac{1}{2}(\cosh 2\phi-\cos 2\theta)$$

and $$\tan\alpha=\tanh\phi\cot\theta.$$

8. If $u=\log\tan(\pi/4+\theta/2)$, prove that

(i) $\tanh\dfrac{u}{2}=\tan\dfrac{1}{2}\theta$

(ii) $\theta=-i\log\tan\left(\dfrac{\pi}{4}+\dfrac{iu}{2}\right)$.

9. If $\tan(\theta+i\phi)=\cos\alpha+i\sin\alpha$, prove that

$$\theta=\frac{n\pi}{2}+\frac{\pi}{4}$$

and $$\phi=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

10. If $\tan(x+iy)=\theta+i\phi$, prove that

$$\theta^2+\phi^2=\frac{\cosh^2 y-\cos^2 x}{\cosh^2 y-\sin^2 x}.$$

11. If $\cos(\theta+i\phi)=R(\cos\alpha+i\sin\alpha)$, prove that

$$\phi=\frac{1}{2}\log\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

12. If $\log\sin(\theta+i\phi)=\alpha+i\beta$, prove that

(i) $2\cos 2\theta=e^{2\phi}+e^{-2\phi}-4e^{2\alpha}$.

(ii) $\cos(\theta-\beta)=e^{2\phi}\cos(\theta+\beta)$.

13. If $\tan(\alpha+i\beta)=i$, where α and β are real, prove that α is indeterminate and β is infinite.

14. If $\sin(\theta+i\phi)=e^{i\alpha}$, prove that
$\cos^2\theta=\sinh^2\phi=\pm\sin\alpha$.

15. If $x=2\cos\theta\cosh\phi$, $y=2\sin\theta\sin h\phi$, prove that

$$\sec(\theta+i\phi)+\sec(\theta-i\phi)=\frac{4x}{x^2+y^2}.$$

1·17 Inverse Hyperbolic Functions

If $\sinh z = w$, then $z = \sinh^{-1} w$ is known as the inverse hyperbolic sin of w and pronounced as *shin inverse w*. Similarly other inverse hyperbolic functions are $\cosh^{-1} w$, $\tanh^{-1} w$ etc.

The inverse hyperbolic functions are many-valued functions but we shall consider only their principal values.

We shall express inverse hyperbolic functions in the logarithmic forms.

Let $\sinh^{-1} w = z$

than $$w = \sinh z = \frac{e^z - e^{-z}}{2}$$

or $$2w = e^z - \frac{1}{e^z}$$

or $$e^{2z} - 2we^z - 1 = 0$$

This equation is quadratic in e^z:

$\therefore$ $$e^z = \frac{2w \pm \sqrt{4w^2+4}}{2}$$

As e^z is always positive, we shall take positive sign only

$\therefore$ $$e^z = w + \sqrt{w^2+1}$$

or $$z = \log\left(w + \sqrt{w^2+1}\right)$$

or $$\sinh^{-1} w = \log\left(w + \sqrt{w^2+1}\right)$$

Similarly we can show,

$$\cosh^{-1} w = \log\left(w + \sqrt{w^2-1}\right)$$

Further, let $\tanh^{-1} w = z$

$\therefore$ $$w = \tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

or $$\frac{w}{1} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

Applying componendo-dividendo, we get

$$\frac{1+w}{1-w} = \frac{2e^z}{2e^{-z}} = e^{2z}$$

$\therefore$ $$2z = \log \frac{1+w}{1-w}.$$

or $$z = \frac{1}{2} \log \frac{1+w}{1-w}$$

or $$\tanh^{-1} w = \frac{1}{2} \log \frac{1+w}{1-w}.$$

1·18. General values of Inverse Hyperbolic Functions

The general values of $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1} x$ are denoted by $\text{Sinh}^{-1} x$, $\text{Cosh}^{-1} x$ and $\text{Tanh}^{-1} x$ respectively.

These general values are

$$\text{Sinh}^{-1} x = 2n\pi i + (-1)^n \log \{x + \sqrt{x^2+1}\}$$

$$\text{Cosh}^{-1} x = 2n\pi i \pm \log \{x + \sqrt{x^2-1}\}$$

$$\text{Tanh}^{-1} x = n\pi i + \frac{1}{2} \log \left(\frac{1+x}{1-x}\right).$$

Example 1. *Separate $\sin^{-1}(\alpha+i\beta)$ into real and imaginary parts.*

Sol. Let $\sin^{-1}(\alpha+i\beta) = x+iy$

$\therefore \quad \alpha+i\beta = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

Equating real and imaginary parts, we get

$$\alpha = \sin x \cosh y \qquad ...(i)$$

$$\beta = \cos x \sinh y \qquad ...(ii)$$

$$\therefore \quad \frac{\alpha^2}{\sin^2 x} - \frac{\beta^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1$$

or $\quad \alpha^2 \cos^2 x - \beta^2 \sin^2 x = \sin^2 x \cos^2 x$

$$\alpha^2(1-\sin^2 x) - \beta^2 \sin^2 x = \sin^2 x\,(1-\sin^2 x)$$

or $\quad \sin^4 x - (\alpha^2+\beta^2+1)\sin^2 x + \alpha^2 = 0$

This equation is quadratic in $\sin^2 x$,

$$\therefore \quad \sin^2 x = \tfrac{1}{2}[(\alpha^2+\beta^2+1) \pm \sqrt{(\alpha^2+\beta^2+1)^2 - 4\alpha^2}]$$

$$x = \sin^{-1}[\pm\sqrt{\tfrac{1}{2}\{\alpha^2+\beta^2+1\} \pm \sqrt{(\alpha^2+\beta^2+1)^2-4\alpha^2}}]$$

Similarly eliminating x from (*i*) and (*ii*), we have

$$\frac{\alpha^2}{\cosh^2 y} + \frac{\beta^2}{\sinh^2 y} = 1$$

or $\quad \alpha^2 \sinh^2 y + \beta^2 \cosh^2 y = \cosh^2 y \sinh^2 y$

$$\alpha^2 \sinh^2 y + \beta^2 (1+\sinh^2 y) = (1+\sinh^2 y)\sinh^2 y$$

or $\quad \sinh^4 y - (\alpha^2+\beta^2-1)\sinh^2 y - \beta^2 = 0$

$$\sinh^2 y = \tfrac{1}{2}[(\alpha^2+\beta^2-1) \pm \sqrt{(\alpha^2+\beta^2-1)^2+4\beta^2}]$$

$$\therefore \quad y = \sinh^{-1}[\pm\sqrt{\tfrac{1}{2}\{\alpha^2+\beta^2-1\} \pm \sqrt{(\alpha^2+\beta^2-1)^2+4\beta^2}}].$$

Example 2. *Separate into real and imaginary parts*

$$\tan^{-1}(\cos\theta + i\sin\theta).$$

Sol. Let $\tan^{-1}(\cos\theta + i\sin\theta) = A+iB$

$$\therefore \quad \cos\theta + i\sin\theta = \tan(A+iB) \qquad ...(1)$$

and $\quad \cos\theta - i\sin\theta = \tan(A-iB) \qquad ..(2)$

Now $\tan 2A = \tan \{(A+iB)+(A-iB)\}$

$$= \frac{\tan (A+iB) + \tan (A-iB)}{1-\tan (A+iB) \tan (A-iB)}$$

$$= \frac{(\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta)}{1-(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)}$$

[from (1) and (2)]

$$= \frac{2 \cos \theta}{1-(\cos^2 \theta + \sin^2 \theta)}$$

$$= \frac{2 \cos \theta}{0} = \infty \qquad \left[\because \tan \frac{\pi}{2} = \infty\right]$$

$$\tan 2A = \tan \left(n\pi + \frac{\pi}{2}\right)$$

$$\therefore \qquad A = \frac{n\pi}{2} + \frac{\pi}{4}$$

Now $\tan 2iB = \tan \{(A+iB)-(A-iB)\}$

$$= \frac{\tan (A+iB) - \tan (A-iB)}{1+\tan (A+iB) \tan (A-iB)}$$

$$= \frac{(\cos \theta + i \sin \theta) - (\cos \theta - i \sin \theta)}{1+(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)}$$

$$= \frac{2i \sin \theta}{1+(\cos^2 \theta + \sin^2 \theta)} = i \sin \theta$$

$$\therefore \quad i \tanh 2B = i \sin \theta$$

or $$\tanh 2B = \sin \theta$$

or $$\frac{e^{2B}-e^{-2B}}{e^{2B}+e^{-2B}} = \frac{\sin \theta}{1}$$

Applying componendo-dividendo,

$$\frac{2e^{2B}}{2e^{-2B}} = \frac{1+\sin \theta}{1-\sin \theta}$$

$$\therefore \quad e^{4B} = \frac{1+\cos\left(\frac{\pi}{2}-\theta\right)}{1-\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$$

$$= \cot^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\therefore \quad e^{2B} = \cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\therefore \quad B=\frac{1}{2}\log\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\therefore \quad \tan^{-1}(\cos\theta+i\sin\theta)=\frac{n\pi}{2}+\frac{\pi}{4}+\frac{i}{2}\log\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right).$$

Example 3. *If $\cos^{-1}(u+iv)=\alpha+i\beta$, prove that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation*

$$x^2-x(1+u^2+v^2)+u^2=0.$$

Sol. Now $\cos^{-1}(u+iv)=\alpha+i\beta$

$$u+iv=\cos(\alpha+i\beta)$$

$$=\cos\alpha\cosh\beta-i\sin\alpha\sinh\beta$$

$$u=\cos\alpha\cosh\beta \qquad \ldots(i)$$

and $$v=-\sin\alpha\sinh\beta$$

Now $$u^2+v^2=\cos^2\alpha\cosh^2\beta+\sin^2\alpha\sinh^2\beta$$

$$=\cos^2\alpha\cosh^2\beta+(1-\cos^2\alpha)(\cosh^2\beta-1)$$

$$=\cos^2\alpha+\cosh^2\beta-1$$

or $$1+u^2+v^2=\cos^2\alpha+\cosh^2\beta$$

also $$u^2=\cos^2\alpha\cosh^2\beta. \qquad \text{from } (i)$$

The equation which has roots $\cos^2\alpha$ and $\cosh^2\beta$ is,

$$x^2-(\cos^2\alpha+\cosh^2\beta)x+\cos^2\alpha\cosh^2\beta=0$$

or $$x^2-(1+u^2+v^2)x+u^2=0.$$

1·19. Logarithm of Complex Quantities

If $\quad z=x+iy$ and $w=e^z$, then z is defined as $\log_e w$

Now $$e^{z+2n\pi i}=e^z.\,e^{2n\pi i}$$

$$=e^z=w. \qquad [\because e^{2n\pi i}=1]$$

$$\therefore \quad z+2n\pi i=\log_e w \qquad \text{[By definition]}$$

Thus we see that $\log_e w$ is a multivalued function. Hence logarithm of a complex quantity is a multivalued function.

The value of $\log_e w$ corresponding to $n=0$, is called **principal value**. The general value of $\log_e w$ denoted by $\log_e w$, and $\log_e w=2n\pi i+\log_e w$, where $\log_e w$ denotes the principal value.

Example 1. *Separate $\log_e(x+iy)$ into real and imaginary parts and hence obtain the real and imaginary parts of $\log_e(x+iy)$.*

Sol. Let $\log_e(x+iy)=\alpha+i\beta$

$$\therefore \quad \alpha+iy=e^{\alpha+i\beta}=e^\alpha.e^{i\beta}$$

or $$x+iy=e^\alpha.(\cos\beta+i\sin\beta)$$

Equating real and imaginary parts, we get

$$x=e^\alpha.\cos\beta \qquad \ldots(1)$$

and $$y=e^\alpha.\sin\beta \qquad \ldots(2)$$

Squaring and adding (1) and (2), we have

$$x^2+y^2=e^{2\alpha}$$

$\therefore \quad 2\alpha = \log_e (x^2+y^2)$

or $\quad \alpha = \frac{1}{2} \log_e (x^2+y^2)$

Also from (1) and (2), we have

$$\tan \beta = \frac{y}{x}$$

or $\quad \beta = \tan^{-1} \frac{y}{x}$

$\therefore \quad \log_e (x+iy) = \frac{1}{2} \log_e (x^2+y^2) + i \tan^{-1} (y/x).$

Now $\quad \text{Log}_e (x+iy) = 2n\pi i + \log_e (x+iy)$

$$= \frac{1}{2} \log_e (x^2+y^2) + i\,(2n\pi + \tan^{-1} y/x).$$

Example 2. *Find the principal values of*
$\log(-1+i) - \log(-1-i)$.

Sol. Here $\log(-1+i) - \log(-1-i)$

Let $\quad -1 = r\cos\theta$ and $1 = r\sin\theta$

$\therefore \quad r = \sqrt{2}, \tan\theta = -1$ or $\theta = \frac{3\pi}{4}$

$\therefore \quad -1+i = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$= \sqrt{2}e^{\frac{3\pi i}{4}}$$

$$-1-i = r(\cos\theta - i\sin\theta) = re^{-i\theta}$$

$$= \sqrt{2}e^{-\frac{3\pi i}{4}}$$

$\therefore \log(-1+i) - \log(-1-i) = \log \frac{(-1+i)}{(-1-i)}$

$$= \log \frac{\sqrt{2}e^{\frac{3\pi i}{4}}}{\sqrt{2}e^{-\frac{3\pi i}{4}}}$$

$$= \log \frac{1}{e^{-\frac{3\pi i}{2}}} = -\log e^{-\frac{3\pi i}{2}}$$

$$= -\log\left(e^{2\pi i - \frac{3\pi i}{2}}\right) = -\log e^{\frac{\pi i}{2}}$$

$$= -\frac{\pi i}{2}, \text{ is the principal value.}$$

Example 3. *Show that (i) Log $i = \left(2n + \frac{1}{2}\right)\pi i$*

(ii) Log $(-x) = (2n+1)\pi i + \log x$.

Sol. (*i*) $i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=e^{\frac{\pi i}{2}}$

$$\log i=\log e^{\frac{\pi}{2}i}=\frac{\pi}{2}i$$

Now
$$\text{Log } i=2n\pi i+\log i$$
$$=2n\pi i+\frac{\pi i}{2}$$
$$=\left(2n+\frac{1}{2}\right)\pi i.$$

(*ii*)
$$-x=x(\cos\pi+i\sin\pi)=xe^{i\pi}$$
$$\log(-x)=\log x+\log e^{\pi i}$$
$$=\log x+\pi i$$

$\therefore$
$$\text{Log}(-x)=2n\pi i+\log(-x)$$
$$=2n\pi i+\log x+\pi i$$
$$=(2n+1)\pi i+\log x$$

Example 4. *Prove that* $\tan\left(i\log\frac{a-ib}{a+ib}\right)=\frac{2ab}{a^2-b^2}$

Sol. Let $a=r\cos\theta$, $b=r\sin\theta$

$\therefore$
$$r=\sqrt{a^2+b^2},\quad \tan\theta=\frac{b}{a}$$

$\therefore$
$$a+ib=r(\cos\theta+i\sin\theta)=re^{i\theta}$$
$$a-ib=r(\cos\theta-i\sin\theta)=re^{-i\theta}$$
$$\frac{a-ib}{a+ib}=\frac{re^{-i\theta}}{re^{i\theta}}=e^{-2i\theta}$$
$$\log\frac{a-ib}{a+ib}=\log e^{-2i\theta}=-2i\theta$$
$$i\log\frac{a-ib}{a+ib}=i(-2i\theta)=2\theta$$
$$=2\tan^{-1}\frac{b}{a}$$
$$=\tan^{-1}\frac{b}{a}+\tan^{-1}\frac{b}{a}$$

$\therefore$
$$\tan\left[i\log\frac{a-ib}{a+ib}\right]=\tan\left[\tan^{-1}\frac{b}{a}+\tan^{-1}\frac{b}{a}\right]$$
$$=\tan\left[\tan^{-1}\frac{\frac{b}{a}+\frac{b}{a}}{1-\frac{b^2}{a^2}}\right]$$

$$=\tan\left[\tan^{-1}\frac{2ab}{a^2-b^2}\right]$$

$$\therefore \tan\left[i\log\frac{a-ib}{a+ib}\right]=\frac{2ab}{a^2-b^2}$$

Example 5. *If* $i^{i^{i\cdots\infty}}=A+iB$, *prove that*

$$\tan\frac{\pi A}{2}=\frac{B}{A}$$

and $$A^2+B^2=e^{-\pi B}$$

Sol. Here $$i^{i^{i\cdots}}=A+iB$$

$$\therefore \quad i^{A+iB}=A+iB$$

Takings logs on both sides, we have

$$(A+iB)\log i=\log(A+iB)$$

$$(A+iB)\left[\frac{1}{2}\log 1+i\tan^{-1}\frac{1}{0}\right]=\log(A+iB)$$

$$\left[\because \log(A+iB)=\frac{1}{2}\log(A^2+B^2)+i\tan^{-1}\frac{B}{A}\right]$$

$$(A+iB)\left(\frac{\pi}{2}i\right)=\frac{1}{2}\log(A^2+B^2)+i\tan^{-1}\frac{B}{A}$$

Equating real and imaginary parts on both sides, we have

$$\therefore \quad \frac{-B\pi}{2}=\frac{1}{2}\log(A^2+B^2)$$

and $$\frac{\pi A}{2}=\tan^{-1}\frac{B}{A}$$

$$\therefore \quad A^2+B^2=e^{-\pi B}$$

and $$\tan\frac{\pi A}{2}=\frac{B}{A}.$$

EXERCISE 1 (*f*)

1. Separate into real and imaginary parts.
   (*i*) $\sin^{-1}(\cos\theta+i\sin\theta)$
   (*ii*) $\cos^{-1}(\cos\theta+i\sin\theta)$
   (*iii*) $\tanh^{-1}(x+iy)$.

2. Prove that $\tanh^{-1}x=\sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$.

3. Prove that $\coth^{-1}\frac{2}{x}=\sinh^{-1}\left(\frac{x}{\sqrt{4-x^2}}\right)$.

4. Show that $\sin^{-1}(ix)=2n\pi+i\log[\sqrt{1+x^2}+x]$ and hence show that

$$\sin^{-1}(i)=2n\pi+i\log(\sqrt{2}+1).$$

5. Prove that $\tan^{-1} i\left(\frac{x-a}{x+a}\right)=-\frac{i}{2}\log\left(\frac{a}{x}\right)$.

6. Show that $\sin^{-1}(\operatorname{cosec}\theta)$

$$=[2n+(-1)^n]\frac{\pi}{2}+i(-1)^n\log\cot\frac{\theta}{2}\cdot$$

7. Prove that $\log\sin(x+iy)$

$$=\frac{1}{2}\log\left(\frac{\cosh 2y-\cos 2x}{2}\right)+i\tan^{-1}(\cot x\tanh y).$$

8. Prove that $\log\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)=i\tan^{-1}(\sinh x)$

9. Prove that $\log\left(\frac{a+ib}{a-ib}\right)=2i\tan^{-1}\left(\frac{b}{a}\right)$.

10. If $\cosh^{-1}(x+iy)+\cosh^{-1}(x-iy)=\cosh^{-1}(a)$, show that $2(a-1)x^2+2(a+1)y^2=a^2-1$.

11. Prove the $\log(i^i)=-\frac{\pi}{2}$

12. Find the general value of $(1+i\tan\alpha)^{-i}$

13. Find the real part of the principal value of $(i)^{\log(1+i)}$

14. If $\log\sin(\theta+i\phi)=\alpha+i\beta$, prove that

$$2\cos 2\theta=e^{2\phi}+e^{-2\phi}-4e^{2\alpha}.$$ (*D.U. 1983*)

1·20. Summation of Trigonometric Series ($C+i\,S$ *method*)

This method is very useful in finding the sum of a series (*finite or infinite*) containing sines or cosines of multiple angles. The method consists in selecting another series, called the auxiliary series. The auxiliary series when combined with the given series, yields a new series which can be summed up easily. For example to find the sum of the series

$$S=a_0\sin\alpha+a_1\sin(\alpha+\beta)+a_2\sin(\alpha+2\beta)+ \qquad \ldots(i)$$

We select another series

$$C=a_0\cos\alpha+a_1\cos(\alpha+\beta)+a_2\cos(\alpha+2\beta)+\ldots\ldots,\ *$$

Then

$$C+iS=a_0(\cos\alpha+i\sin\alpha)+a_1[\cos(\alpha+\beta)+i\sin(\alpha+\beta)]+a_2[\cos(\alpha+2\beta)+i\sin(\alpha+2\beta)]+\ldots$$

or
$$C+i\,S=a_0e^{i\alpha}+a_1e^{i(\alpha+\beta)}+a_2e^{i(\alpha+2\beta)}+\ldots \qquad \ldots(ii)$$

The series on R.H.S. of (ii) can be summed up easily by using the following standard series.

Geometric Series

$$a+ar+ar^2+\ldots+ar^{n-1}=\frac{a(1-r^n)}{1-r}$$

---

*If the sum of the sine series is required, we choose auxiliary series as a corresponding cosine series and vice-versa. The sine series is denoted by S and the cosine series by C.

or $\quad a+ar+ar^2+\ldots\infty \quad = \dfrac{a}{1-r}$, where $|r| < 1$

(b) **Binomial Series**

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots\infty=(1+x)^n$$

(c) **Sine and Cosine Series**

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\infty=\sin x$$

and
$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\infty=\cos x$$

(d) **Sinh or Cosh Series**

$$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots\infty=\sinh x$$

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots\infty=\cosh x$$

(e) **Exponential Series**

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\infty=e^x$$

(f) **Logarithmic Series**

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty=\log(1+x)$$

or
$$-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots\infty=\log(1-x)$$

(g) **Gregory's Series**

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\infty=\tan^{-1}x$$

The following examples illustrate the method.

Example 1. *Find the sum of the series*

$$Sin\ \alpha+\frac{1}{3}\sin 3\alpha+\frac{1}{3^2}\sin 5\alpha+\ldots\infty$$

Sol. Let $\quad S=\sin\alpha+\dfrac{1}{3}\sin 3\alpha+\dfrac{1}{3^2}\sin 5\alpha+\ldots\infty$

and
$$C=\cos\alpha+\frac{1}{3}\cos 3\alpha+\frac{1}{3^2}\cos 5\alpha+\ldots\infty$$

$\therefore \quad C+iS=(\cos\alpha+i\sin\alpha)+\dfrac{1}{3}(\cos 3\alpha+i\sin 3\alpha)$

$$+\frac{1}{3^2}(\cos 5\alpha+i\sin 5\alpha)+\ldots\infty$$

$$=e^{i\alpha}+\frac{1}{3}e^{3i\alpha}+\frac{1}{3^2}e^{5i\alpha}+\dots\infty \qquad \dots(i)$$

The series on R.H.S. of (*i*) is a geometric series, with common ratio $\frac{1}{3}e^{2i\alpha}$

$$\therefore \quad C+i\,S=\frac{e^{i\alpha}}{1-\frac{1}{3}e^{2i\alpha}}=\frac{e^{i\alpha}\left(1-\frac{1}{3}e^{-2i\alpha}\right)}{\left(1-\frac{1}{3}e^{2i\alpha}\right)\left(1-\frac{1}{3}e^{-2i\alpha}\right)}$$

(conjugate of $1-\frac{1}{3}e^{2i\alpha}$ is $1-\frac{1}{3}e^{-2i\alpha}$)

$$=\frac{e^{i\alpha}-\frac{1}{3}e^{-i\alpha}}{1-\frac{1}{3}(e^{2i\alpha}+e^{-2i\alpha})+\frac{1}{9}}$$

$$=\frac{(\cos\alpha+i\sin\alpha)-\frac{1}{3}(\cos\alpha-i\sin\alpha)}{1-\frac{2}{3}\cos 2\alpha+\frac{1}{9}}$$

($\because \; e^{2i\alpha}+e^{-2i\alpha}=2\cos 2\alpha$)

$$=\frac{\frac{2}{3}\cos\alpha+\frac{i\,4}{3}\sin\alpha}{1-\frac{2}{3}\cos 2\alpha+\frac{1}{9}}=\frac{6\cos\alpha+i\,.\,12\sin\alpha}{10-6\cos 2\alpha}$$

or $$C+i\,S=\frac{3\cos\alpha+i\,6\sin\alpha}{5-3\cos 2\alpha}$$

Equating imaginary parts on both sides, we get

$$S=\frac{6\sin\alpha}{5-3\cos 2\alpha}, \text{ the required sum.}$$

Example 2. *Sum the series*

$$x\cos\theta-\frac{x^2}{2}\cos 2\theta+\frac{x^3}{3}\cos 3\theta-\dots\infty$$

Sol. Let $C=x\cos\theta-\frac{x^2}{2}\cos 2\theta+\frac{x^3}{3}\cos 3\theta-\dots\infty$

and $$S=x\sin\theta-\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta-\dots\infty$$

$$\therefore \quad C+i\,S=x(\cos\theta+i\sin\theta)-\frac{x^2}{2}(\cos 2\theta+i\sin 2\theta)$$
$$+\frac{x^3}{3}(\cos 3\theta+i\sin 3\theta)+\dots$$

$$=xe^{i\theta}-\frac{x^2e^{2i\theta}}{2}+\frac{x^3e^{3i\theta}}{3}\dots\infty$$

(The series on R.H.S. is a logarithmic series)

$$=\log(1+xe^{i\theta})=\log[1+x(\cos\theta+i\sin\theta)]$$

$$= \log [(1+x\cos\theta)+i\,.\,x\sin\theta]$$

$$= \tfrac{1}{2}\log[(1+x\cos\theta)^2+x^2\sin^2\theta] + i\tan^{-1}\left(\frac{x\sin\theta}{1+x\cos\theta}\right)$$

or $$C+iS=\tfrac{1}{2}\log(1+2x\cos\theta+x^2)+i\tan^{-1}\left(\frac{x\sin\theta}{1+x\cos\theta}\right)$$

Equating real parts on both sides, we get

$$C=\tfrac{1}{2}\log(1+2x\cos\theta+x^2)$$

Example 3. *Find the sum of the series*

$$\sin\alpha+x\sin(\alpha+\beta)+\frac{x^2}{2!}\sin(\alpha+2\beta)+\ldots\infty$$

Sol. Let $S=\sin\alpha+x\sin(\alpha+\beta)+\dfrac{x^2}{2!}\sin(\alpha+2\beta)+\ldots\infty$

and $$C=\cos\alpha+x\cos(\alpha+\beta)+\frac{x^2}{2!}\cos(\alpha+2\beta)+\ldots\infty$$

$\therefore$ $$C+iS=(\cos\alpha+i\sin\alpha)+x[\cos(\alpha+\beta)+i\sin(\alpha+\beta)] + \frac{x^2}{2!}[\cos(\alpha+2\beta)+i\sin(\alpha+2\beta)]+\ldots\infty$$

$$=e^{i\alpha}+xe^{i(\alpha+\beta)}+\frac{x^2}{2!}e^{i(\alpha+2\beta)}+\ldots\infty$$

$$=e^{i\alpha}\left[1+xe^{i\beta}+\frac{x^2}{2!}e^{2i\beta}+\ldots\infty\right]$$

(The series on R.H.S. is an exponential series)

$$=e^{i\alpha}\,.\,(e^{xe^{i\beta}})$$

$$=e^{i\alpha}\,.\,[e^{x(\cos\beta+i\sin\beta)}].$$

$$=e^{x\cos\beta}e^{i(\alpha+x\sin\beta)}$$

or $$C+iS=e^{x\cos\beta}[\cos(\alpha+x\sin\beta)+i\sin(\alpha+x\sin\beta)]$$

Equating imaginary parts on both sides, we get

$$S=e^{x\cos\beta}\sin(\alpha+x\sin\beta)$$

Example 4. *Sum to n terms the series*

$$1+x\cos\theta+x^2\cos 2\theta+x^3\cos 3\theta+\ldots \qquad (x<1).$$

Sol. Let $C=1+x\cos\theta+x^2\cos 2\theta+x^3\cos 3\theta+\ldots +x^{n-1}\cos(n-1)\theta \quad (x<1)$

and $$S=x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\ldots +x^{n-1}\sin(n-1)\theta.$$

$\therefore$ $$C+iS=1+x(\cos\theta+i\sin\theta)+x^2(\cos 2\theta+i\sin 2\theta) +x^3(\cos 3\theta+i\sin 3\theta)+\ldots+x^{n-1}[\cos(n-1)\theta +i\sin(n-1)\theta]$$

$$=1+xe^{i\theta}+x^2e^{2i\theta}+x^3e^{3i\theta}+\ldots\ldots+x^{n-1}e^{(n-1)i\theta}$$

(The series is a G.P.)

$$=\frac{1-x^ne^{ni\theta}}{1-xe^{i\theta}}=\frac{(1-x^ne^{ni\theta})(1-xe^{-i\theta})}{(1-xe^{i\theta})(1-xe^{-i\theta})}$$

$$=\frac{1-xe^{-i\theta}-x^n e^{ni\theta}+x^{n+1}e^{i(n-1)\theta}}{1-x(e^{i\theta}+e^{-i\theta})+x^2}$$

$$=\frac{1-x(\cos\theta-i\sin\theta)-x^n(\cos n\theta+i\sin n\theta)+x^{n+1}[\cos(n-1)\theta+i\sin(n-1)\theta]}{1-2x\cos\theta+x^2}$$

or $$C+iS=\frac{1-x\cos\theta-x^n\cos n\theta+x^{n+1}\cos(n-1)\theta}{1-2x\cos\theta+x^2}$$

$$+i\,.\,\frac{x\sin\theta-x^n\sin n\theta+x^{n+1}\sin(n-1)\theta}{1-2x\cos\theta+x^2}$$

Equating real parts on both sides, we get

$$C=\frac{1-x\cos\theta-x^n\cos n\theta+x^{n+1}\cos(n-1)\theta}{1-2x\cos\theta+x^2},$$

the required sum.

Example 5. *Sum the series*

$$\sin\alpha+\tfrac{1}{2}\sin 3\alpha+\frac{1\cdot 3}{2\cdot 4}\sin 5\alpha+\ldots\infty$$

Sol. Let $S=\sin\alpha+\tfrac{1}{2}\sin 3\alpha+\dfrac{1\cdot 3}{2\cdot 4}\sin 5\alpha+\ldots\infty$

and $$C=\cos\alpha+\tfrac{1}{2}\cos 3\alpha+\frac{1\cdot 3}{2\cdot 4}\cos 5\alpha+\ldots\infty$$

$\therefore$ $$C+iS=(\cos\alpha+i\sin\alpha)+\tfrac{1}{2}(\cos 3\alpha+i\sin 3\alpha)+\frac{1\cdot 3}{2\cdot 4}(\cos 5\alpha+i\sin 5\alpha)+\ldots\infty$$

$$=e^{i\alpha}+\tfrac{1}{2}e^{3i\alpha}+\frac{1\cdot 3}{2\cdot 4}e^{5i\alpha}+\ldots$$

$$=e^{i\alpha}\left[1+\tfrac{1}{2}e^{2i\alpha}+\frac{1\cdot 3}{2\cdot 4}e^{4i\alpha}+\ldots\right]$$

(The series with in brackets is a Binomial series)

$$=e^{i\alpha}(1-e^{2i\alpha})^{-1/2}$$

$$=(\cos\alpha+i\sin\alpha)(1-\cos 2\alpha-i\sin 2\alpha)^{-1/2}$$

$$=(\cos\alpha+i\sin\alpha).\,(2\sin^2\alpha-2i\sin\alpha\cos\alpha)^{-1/2}$$

$$=(2\sin\alpha)^{-1/2}(\cos\alpha+i\sin\alpha)\,.\,(\sin\alpha-i\cos\alpha)^{-1/2}$$

$$=(2\sin\alpha)^{-1/2}\,.\,(\cos\alpha+i\sin\alpha)\,.\left[\cos\left(\frac{\pi}{2}-\alpha\right)-i\sin\left(\frac{\pi}{2}-\alpha\right)\right]^{-1/2}$$

$$=(2\sin\alpha)^{-1/2}\,.\,(\cos\alpha+i\sin\alpha)\,.\left[\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right]$$

(By Demoivre's theorem)

or
$$C+iS=(2\sin\alpha)^{-1/2}\,.\left[\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right]$$

(Amplitudes are added on multiplication.)

Equating imaginary parts on both sides, we get

$$S=(2\sin\alpha)^{-1/2}\,.\,\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

EXERCISE 1 (*g*)

Sum the following series.

1. $\sin\alpha+\frac{1}{2}\sin 2\alpha+\frac{1}{2^2}\sin 3\alpha+\ldots\infty$
2. $1+x\cos y+x^2\cos 2y+\ldots\infty$
3. $\cos\alpha\sin\alpha+\cos^2\alpha\sin 2\alpha+\cos^3\alpha\sin 3\alpha+\ldots\infty$
   [**Hint.** $C=\cos\alpha\cos\alpha+\cos^2\alpha\cos 2\alpha+\cos^3\alpha\cos 3\alpha+\ldots$]
4. $\cos\alpha+x\cos(\alpha+\beta)+\frac{x^2}{2!}\cos(\alpha+2\beta)+\ldots\infty$
5. $\sin\theta-\frac{\sin 2\theta}{2!}+\frac{\sin 3\theta}{3!}-\ldots$
6. $c\sin\alpha+\frac{c^3}{3}\sin 3\alpha+\frac{c^5}{5}\sin 5\alpha+\ldots\infty$
7. $\cos\alpha-\frac{1}{3}\cos 3\alpha+\frac{1}{5}\cos 5\alpha-\ldots\infty$
8. $1+{}^nC_1\cos\alpha+{}^nC_2\cos 2\alpha+{}^nC_3\cos 3\alpha+\ldots\infty$
9. $x\sin\theta-x^2\sin 2\theta+x^3\sin 3\theta-\ldots$ to n terms
10. $1+n\sin\theta+\frac{n(n-1)}{2!}\sin 2\theta+\frac{n(n-1)(n-2)}{3!}\sin 3\theta+\ldots+\sin n\theta.$

Exercise 1 (e) (Page 34—35)

4. (i) $\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

(ii) $\dfrac{\sin 2\alpha}{\cosh 2\beta-\cos 2\alpha}+i\,\dfrac{\sinh 2\beta}{\cosh 2\beta-\cos 2\alpha}$

(iii) $2\left[\dfrac{\sin\alpha\cosh\beta}{\cosh 2\beta-\cos 2\alpha}-i\,\dfrac{\cos\alpha\sinh\beta}{\cosh 2\beta-\cos 2\alpha}\right]$

(iv) $\cosh\alpha\cos\beta+i\sinh\alpha\sinh\beta$

(v) $\dfrac{\sinh 2\alpha}{\cosh 2\alpha+\cos 2\beta}+i\,\dfrac{\sin 2\beta}{\cosh 2\alpha+\cos 2\beta}$

(vi) $e^{\cosh x\cos y}[\cos(\sinh x\sin y)+i\sin(\sinh x\sin y)]$

Exercise 1 (f) (Page 42—43)

1. (i) $\cos^{-1}(\sqrt{\sin\theta})+i\log\{\sqrt{\sin\theta}+\sqrt{1+\sin\theta}\}$

(ii) $\sin^{-1}(\sqrt{\sin\theta})+i\log\{\sqrt{1+\sin\theta}-\sqrt{\sin\theta}\}$

(iii) $\dfrac{1}{2}\tanh^{-1}\left\{\dfrac{2x}{(1+x^2+y^2)}\right\}+\dfrac{i}{2}\tan^{-1}\left\{\dfrac{2y}{(1-x^2-y^2)}\right\}$

12. $e^{2n\pi+\alpha}[\cos(\log\sec\alpha)-i\sin(\log\sec\alpha)]$

13. $\operatorname{Exp}\left(-\dfrac{\pi^2}{8}\right)\cos\left(\dfrac{\pi}{4}\log 2\right)$

Exercise 1 (g) (Page 48)

1. $\dfrac{4\sin\alpha}{5-4\cos\alpha}$.

2. $\dfrac{1-x\cos y}{1-2x\cos y+y^2}$.

3. $\cot\alpha$

4. $e^{x\cos\beta}.\cos(\alpha+x\sin\beta)$.

5. $e^{-\cos\theta}\sin(\sin\theta)$.

6. $\frac{1}{2}\tan^{-1}\left[\dfrac{2c\sin\alpha}{1-c^2}\right]$, $(c\neq 1,\ \alpha\neq n\pi)$.

7. $\dfrac{\pi}{4}$ or $\dfrac{-\pi}{4}$ according as $\cos\alpha$ is positive or negative.

8. $\left(2\sin\dfrac{\alpha}{2}\right)^{-n}\cos\left(\dfrac{n\pi}{2}-\dfrac{n\alpha}{2}\right)$

9. $\dfrac{x\sin\theta+(-1)^{n+1}x^{n+1}\{x\sin n\theta+\sin(n+1)\theta\}}{x^2+2x\cos\theta+1}$

10. $1+\left(2\cos\dfrac{\theta}{2}\right)^n\cdot\sin\dfrac{n\theta}{2}$.

Exercise 2 (a) (Page 50)

1. Convergent
2. Divergent
3. Divergent
4. Divergent
5. Divergent
6. Oscillatory

Exercise 2 (b) Page (63—64)

1. Divgt.
2. Divgt.

7

Applications of Differential Calculus

7·1. Derivative as a Rate Measure

Let $y=f(x)$ and let $\triangle x$ represent a small change in x and let $\triangle y$ be the corresponding change in y. Then $\triangle y/\triangle x$ gives the average rate of change in y w.r.t. x in the interval $(x, x+\triangle x)$. When $\triangle x \to 0$, the average rate $\triangle y/\triangle x$ becomes the instantaneous rate of change of y w.r.t. x and it is denoted by dy/dx.

Let a moving particle cover a distance s in time t and let $s=f(t)$. Then ds/dt is the rate of change of s with respect to t and as such it is the velocity of the particle at the instant t. Therefore $\dfrac{ds}{dt}=v$. Further, $\dfrac{dv}{dt}$ measures the rate of change of v w.r.t. t and as such it is acceleration a of the particle at an instant t. Thus

$$a=\frac{dv}{dt}=\frac{d}{dt}\left(\frac{ds}{dt}\right)=\frac{d^2s}{dt^2}.$$

Also $$a=\frac{dv}{dt}=\frac{dv}{ds}\cdot\frac{ds}{dt}=\frac{dv}{ds}\cdot v=v\frac{dv}{ds}.$$

Example 1. *The radius of a sphere (losing mass) is decreasing at the rate of 0·02 cm per mt. At what rate is (i) the surface (ii) the weight, varying when the radius is 15 cm and the material weighs 0·3 kg per c.c.*

Sol. Let the radius of the sphere be x.

(i) Then the surface s is given by

$$s=4\pi x^2.$$

Now differentiating both sides w.r.t. t we have,

$$\frac{ds}{dt}=4\pi\left(2x\frac{dx}{dt}\right)$$
$$=8\pi x\times(-0{\cdot}02),$$

since x is decreasing while t is increasing, dx/dt is negative. Therefore, when $x=15$ cm,

$$\frac{ds}{dt}=8\pi\times 15\times(-0{\cdot}02)=-2{\cdot}4\pi \text{ sq. cm/mt.}$$ *i.e.*

the surface is decreasing at the rate of $2{\cdot}4\pi$ sq. cm per minute when the radius is 15 cm.

(*ii*) The volume of the sphere is given by $\frac{4}{3}\pi x^3$

Now weight w of the sphere is

$$w=\frac{4}{3}\pi x^3\times 0{\cdot}3=0{\cdot}4\pi x^3$$

Therefore $\frac{dw}{dt}={\cdot}4\pi\times 3x^2\frac{dx}{dt}=0{\cdot}4\pi\times 3x^2\times(-0{\cdot}02)$

When $\quad x=15$ cm,

$$\frac{dw}{dt}={\cdot}4\pi\times 3\times 15^2\times(-0{\cdot}02)=-5{\cdot}4\pi \text{ kg/mt.}$$

Thus the weight of the sphere is decreasing at the rate of $5{\cdot}4\pi$ kg per minute when radius of the sphere is 15 cm.

Example 2. *If a particle falls the distance*

$$s=\frac{g}{k^2}\log\left(\frac{e^{kt}+e^{-kt}}{2}\right)$$

in time t, show its velocity v and acceleration α *satisfy the equation*
$\alpha=g-k^2v^2/g.$ (*A.M.I.E. May 1964*)

Sol. Differentiating s w.r.t. t, we have

$$\frac{ds}{dt}=\frac{g}{k^2}\cdot\frac{2}{(e^{kt}+e^{-kt})}\times\frac{1}{2}(ke^{kt}-ke^{-kt})$$

or
$$v=\frac{g}{k}\left(\frac{e^{kt}-e^{-kt}}{e^{kt}+e^{-kt}}\right)$$

Again differentiating w.r.t. t, we have

$$\frac{dv}{dt}=\frac{g}{k}\left[\frac{k(e^{kt}+e^{-kt})^2-k(e^{kt}-e^{-kt})^2}{(e^{kt}+e^{-kt})^2}\right]$$

or
$$\alpha=g\left[1-\left(\frac{e^{kt}-e^{-kt}}{e^{kt}+e^{-kt}}\right)^2\right]=g\left[1-\left(\frac{vk}{g}\right)^2\right]$$

$$=g\left(1-\frac{v^2k^2}{g^2}\right)=g-\frac{k^2v^2}{g}.$$

Example 3. *Water escapes at the rate of 8 cubic metre per minute from an inverted conical container whose depth is 12 m and whose base is a circle of radius 4 m. At what rate is the level of water sinking when the level of water is 6 m in the container.*

Sol. Let the depth of water be x m. and radius of the water surface be r m. at any instant t. Then $\frac{x}{r}=\frac{12}{4}$ or $r=\frac{1}{3}x$

Now volume of water in the cone is given by

$$v=\frac{1}{3}\pi r^2x=\frac{\pi}{27}x^3$$

$$\therefore \quad \frac{dv}{dt}=\frac{\pi}{27}\cdot 3x^2\frac{dx}{dt}$$

But $\frac{dv}{dt}=-8$ cu. m. per minute

Therefore $\frac{dx}{dt}=\frac{9}{\pi x^2}\cdot\frac{dv}{dt}$

$$=\frac{9\times 7}{22\times 6^2}\times(-8)$$

$$=-\frac{7}{11} \text{ m. per mt.}$$

7·2. Increasing and Decreasing Functions

A function $f(x)$ is said to be monotonic increasing function of x, if $f(x)$ increases when x increases or $f(x)$ decreases when x decreases *i.e.* both x and $f(x)$ increase or decrease simultaneously.

A function $f(x)$ is said to be a monotonic decreasing function of x if $f(x)$ decreases when x increases and *vice-versa.*

Theorem. (*i*) A function $f(x)$ is increasing at a point when its derivative $f'(x)$ is positive at that point (*ii*) $f(x)$ is decreasing at a point when its derivative $f'(x)$ is negative at that point.

Proof. (*i*) Let $f(x)$ be an increasing function of x. Let x increase by $\triangle x$, then $f(x)$ will also increase to $f(x+\triangle x)$. Then

$$f(x+\triangle x)>f(x)$$

or $$f(x+\triangle x)-f(x)>0$$

or $$\frac{f(x+\triangle x)-f(x)}{\triangle x}>0$$

Now taking limit as $\triangle x\to 0$, we have

$$\lim_{\triangle x\to 0}\left\{\frac{f(x+\triangle x)-f(x)}{\triangle x}\right\}>0$$

$$f'(x)>0$$

i.e. if $y=f(x)$ is an increasing function at a point x, then $\frac{dy}{dx}$ or $f'(x)$ is $+$ve at that point.

(*ii*) Let $f(x)$ be a decreasing function of x, and let x increase by $\triangle x$. Then $f(x)$ decreases to $f(x+\triangle x)$. Therefore

$$f(x+\triangle x)<f(x)$$

or $$f(x+\triangle x)-f(x)<0$$

or $$\frac{f(x+\triangle x)-f(x)}{\triangle x}<0$$

or $$\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}<0$$

or $$f'(x)<0$$

i.e. if $y=f(x)$ is a decreasing function at a point x, then $\frac{dy}{dx}$ or $f'(x)$ is $-$ve at that point.

Example 1. *Show that* $y=\log(1+x)-\frac{2x}{(2+x)}$ *is an increasing function of* x *for all values of* $x>-1$.

Sol. Differentiating y w.r.t. x and simplifying, we have

$$\frac{dy}{dx}=\frac{x^2}{(1+x)(2+x)^2}$$

Now $\frac{dy}{dx}$ is positive when $1+x>0$ or $x>-1$.

As such y is an increasing function of x when $x>-1$.

Example 2. *Prove that*

$$y=\frac{4\sin\theta}{(2+\cos\theta)}-\theta$$

is an increasing function of θ *in the range* $\theta=0$ *to* $\theta=\frac{\pi}{2}$.

Sol. Differentiating y w.r.t. θ and simplifying, we get

$$\frac{dy}{d\theta}=\frac{\cos\theta\,(4-\cos\theta)}{(2+\cos)^2}.$$

The sign of $\frac{dy}{d\theta}$ will depend on $\cos\theta$. But $\cos\theta$ is +ve in the first quadrant $0\leqslant\theta\leqslant\frac{\pi}{2}$. Therefore y is an increasing function of θ in the given range.

EXERCISE 7 (*a*)

1. If $s=\sqrt{(1+t)}$, where s is the displacement and t the time, show that acceleration is negative and proportional to the cube of the velocity.

2. The displacement of a body is given by $s=e^{-2t}\sin 5t$. Show that its velocity is given by $\sqrt{29}\ e^{-2t}\cos\left(5t+\tan^{-1}\frac{2}{5}\right)$ which vanishes when t is $\frac{1}{5}\tan^{-1}\left(\frac{5}{2}\right)$.

3. If $i=i_0(1-e^{-Rt/L})$, find the rate at which current i is changing, taking i_0, R and L as constants.

4. An inverted cone has a depth of 10 cm. and a base of radius 5 cm. Water is poured into it at the rate of $1\frac{1}{2}$ c.c. per minute. Find the rate at which the level of the water in the cone is rising when the depth is 4 cm.

5. A ladder 5 m long standing on a horizontal floor leans against a vertical wall. If the top slides downwards at the rate of 10 cm/sec, find the rate at which the angle between the floor and the

ladder is decreasing when the lower end of the ladder is 2 m from the wall.

6. A captive balloon 500 m. high is moving horizontally at a constant rate of 20 m. per sec. and is also rising at the rate of 6 m per sec. At what rate it is receding from a point over which it passed 15 sec. ago ?

7. The motion of the needle of a galvanometer is given by $\theta=6e^{-\frac{1}{2}t}$ sin $3t$, where θ is the angle in radians made by the needle with the zero position at the end of t sec. Find the angular velocity of the needle at time t, and show that the extreme excursions to the right and left of the zero position occur at intervals of $\frac{\pi}{3}$ sec., and that the angles corresponding to these extreme excursions form a G.P. of common ratio $(-e^{-\pi/6})$.

8. A certain mass of gas is contained in a vessel of volume v c.c. under a pressure p kg per sq. cm, where $pv=200$. If the volume increases at the rate of 40 c.c. per minute, find the rate of change of pressure when the volume is 20 c.c.

9. A crank OQ rotates round O with constant angular velocity w, and a connecting rod QP is hinged to it at one end Q, while the other end P moves along a fixed straight line OX. If PQ meets the perpendicular to OX through O in R, prove that the angular velocity of PQ is proportional to QR and the velocity of P is proportional to OR.

10. The distances of a moving point from two rectangular axes at the end of t seconds are given by $x=a+c\cos t$, $y=b+c\sin t$. Show that the resultant velocity and acceleration are constant.

11. A man standing on a dock 40 m above the water pulls a boat towards the dock by taking in rope at the rate of 4 m/sec. At what rate is the boat approaching the dock when 80 m of rope are paid out ?

12. A horizontal trough 3 m long has a vertical section in the shape of an isosceles right triangle. If water is poured into it at the rate of 3·6 litres sec, show that surface of the water is rising at the rate of 6 cm minute when the water is 60 cm deep.

13. At a certain instant the radius of a right circular cylinder is 6 cm and is increasing at the rate 0·3 cm/sec, while the altitude is 8 cm and is decreasing at the rate 0·4 cm/sec. Show that the time rate of change of the (*i*) volume is $4·8\pi$ c.c./sec and (*ii*) surface is $3·2\pi$ sq. cm/sec.

14. A revolving beacon light in a light house 0·5 km off-shore makes two revolutions per minute. If the shore line is a straight line, how fast is the ray of light moving along the shore when it passes a point 1 km from the light house. (*A.M.I.E. 1979 'S'*)

15. Show that the function $\phi(x)=x^3-6x^2+9x-1$ is always decreasing for $1<x<3$ but outside this interval it is an increasing function of x.

16. Show that $y = 2x - \tan^{-1} x - \log (x + \sqrt{1+x^2})$ is an increasing function for all real $x > 0$.

17. Prove that $\frac{d}{dx}\left[(x+1) \log \left(\frac{x+1}{x}\right)\right]$ is negative for $x > 0$ and deduce that $\left(1+\frac{1}{x}\right)^{x+1}$ is a decreasing function for $x > 0$.

18. Show that $\tan x - 4x$ is a decreasing function of x for $-\frac{\pi}{3} < x < 0$, while $\sin x - \frac{x}{2}$ is an increasing function for $-\frac{\pi}{3} < x < \frac{\pi}{3}$.

19. If $x > 0$ prove that

(*i*) $\frac{x}{1+x} < \log (1+x) < x$

(*ii*) $x - \frac{x^2}{3} < \log (1+x) < x - \frac{x^2}{2(1+x)}$

20. Prove that as x increases, $\frac{a \sin x + b \cos x}{c \sin x + d \cos x}$, either increases for all values of x or decreases for all values of x; a, b, c, d being constants.

7·3. Maxima and Minima

A function $f(x)$ is maximum at $x = a$ if $f(a)$ is greater than any other value that the function can assume for value of x in the interval $(a-h, a+h)$ where h is a sufficiently small positive number.

That is $f(a) > f(x)$ or $f(x) < f(a)$ or $\{f(x) - f(a)\}$ is $-$ve for every value of x in the immediate neighbourhood of a

A function $f(x)$ is minimum at $x = b$ if $f(b)$ is less than any other value that function can assume in the interval $(b-h, b+h)$.

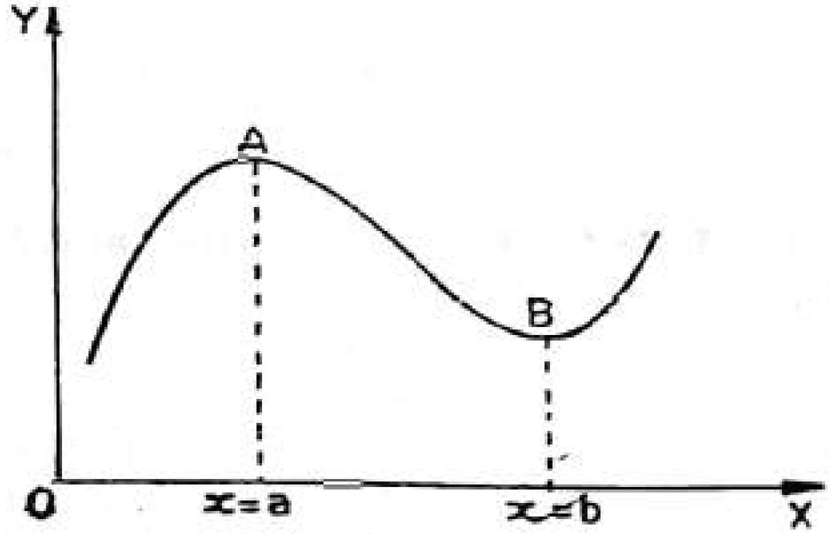

Fig. 7·1.

where h is a sufficiently small positive number.

That is $f(b) \lessdot f(x)$ or $f(x) > f(b)$ or $\{f(x)-f(b)\}$ is +ve for every value of x in the immediate neighbourhood of b.

The max. or min. value of a function is called extreme or stationary value of $f(x)$. A point where a max. or a min. occurs is called a turning point. In the above figure we have max. and min. at the turning points A and B respectively.

7.4. Conditions for Maxima and Minima

Expanding $f(x)$ by Taylor's theorem, we have

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)^2+\frac{(x-a)^3}{3!}f'''(a)+\ldots$$

or
$$f(x)-f(a)=(x-a)f'(a)+\frac{1}{2}(x-a)^2f''(a)+\frac{1}{6}(x-a)^3 \times f'''(a)+\ldots$$

For sufficiently small values of $(x-a)$, $f'(a)$ is numerically much large than the terms which follow. Thus the sign of $f(x)-f(a)$ depends on $(x-a)f'(a)$. It will have one sign when $x > a$ and another when $x < a$. But $f(x)-f(a)$ must have one definite sign (−ve for max. and +ve for min.) whether $(x-a)$ is +ve or −ve. As such no max. or min. is possible at $x=a$ unless

$$f'(a)=0$$

When $f'(a)=0$, we have

$$f(x)-f(a)=(x-a)^2\left\{\frac{1}{2}f''(a)+\frac{1}{6}(x-a)f'''(a)+\ldots\ldots\right\}$$

Again for sufficiently small values of $(x-a)$, $\frac{1}{2}f''(a)$ is much larger numerically than the terms which follow. Thus the sign of $f(x)-f(a)$ depends on $\frac{1}{2}(x-a)^2f''(a)$ or $f''(a)$, since $(x-a)^2$ is always positive. Therefore for max. value of $f(x)$ at $x=a$, $f''(a)$ must be −ve and for min. value of $f(x)$ at $x=a$ $f''(a)$ must be +ve.

In case $f'(a)=0$ and also $f''(a)=0$, then following the above argument, we should have $f'''(a)=0$ for max. or min. value of $f(x)$ at $x=a$. Then the sign of $f^{iv}(a)$ will decide whether $f(x)$ is max. or min. at $x=a$.

7.5. Working Rule for Finding Maxima and Minima

1. Denote the given function by $f(x)$.
2. Find $f'(x)$ and put it equal to zero.
3. Solve $f'(x)=0$. Let its roots be $x=a, b, c,\cdots$
4. Now find $f''(x)$ at $x=a, b, c, \ldots$ If $f''(a)$ is −ve, $f(x)$ is max. at $x=a$. If $f''(a)$ is +ve, $f(x)$ is min. at $x=a$. Similarly test at $x=b, c, \ldots$

5. In case $f''(a)=0$, find $f'''(x)$. If $f'''(a) \neq 0$, there is neither max. nor min. at $x=a$. If $f'''(a)=0$, find $f^{iv}(x)$. In case $f^{iv}(a)$ is $-$ve, $f(x)$ is max. at $x=a$ and if it is $+$ve, $f(x)$ is min. at $x=a$.

6. This procedure is continued till we arrive at some conclusion.

Aliter. We can obtain the conditions for extreme values of $f(x)$ by applying the knowledge of increasing and decreasing functions as follows.

We have max. value of $f(x)$ at $x=a$ when $f(x)$ stops to increase and begins to decrease at $x=a$ and min. value of $f(x)$ at $x=b$ when $f(x)$ stops to decrease and begins to increase at $x=b$.

Thus $f(x)$ is an increasing function for values of x which just precede 'a' and it is decreasing function for values of x which just follow 'a'. That is $\frac{dy}{dx}$ is $+$ve just before $x=a$ and $-$ve just after $x=a$. It means that $\frac{dy}{dx}$ changes sign from $+$ve to $-$ve at $x=a$. As $\frac{dy}{dx}$ is a continuous function of x, it must pass through zero value while changing sign from $+$ve to $-$ve. Hence $\frac{dy}{dx}=0$ at max. points such as $x=a$.

On similar lines, it can be shown that $\frac{dy}{dx}=0$ at min. points, such as $x=b$. It is a necessary condition for extreme or stationary values of $f(x)$.

Further, for max. value of $f(x)$ at A $(x=a)$, $\frac{dy}{dx}$ changes sign from $+$ve to $-$ve as x increases through a. Thus $\frac{dy}{dx}$ is a decreasing function of x in small neighbourhood of a. Hence its derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ must be negative at $x=a$. *i.e.*

$$\frac{d^2y}{dx^2}=-\text{ve}$$

Similarly, for a min. value of $f(x)$ at B $(x=b)$, $\frac{dy}{dx}$ changes sign from $-$ve to $+$ve *i.e.* $\frac{dy}{dx}$ is an increasing function of x so its derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ must be $+$ve at $x=b$.

Remark. Sometimes it is difficult to obtain $\frac{d^2y}{dx^2}$ at extreme points. In that case, we use the following results.

(*i*) The sign of $\frac{dy}{dx}$ changes from +ve to −ve as x passes (while increasing) through the value which makes y a max.

(*ii*) The sign of $\frac{dy}{dx}$ changes from −ve to +ve as x passes through the value which makes y a min.

Illustration. Let $y=\frac{A\cot^2 x}{(1+\operatorname{cosec} x)^2}$, $0<x<\frac{1}{2}$.

Now differentiating y w.r.t. x and simplifying, we have

$\frac{dy}{dx}=A(\operatorname{cosec} x\cot x)\frac{(\operatorname{cosec} x-3)}{(1+\operatorname{cosec} x)^3}$. For max. or min. value of x, $\frac{dy}{dx}=0 \Rightarrow (\operatorname{cosec} x-3)=0$ as the other values of x are inadmissble. Thus $\sin x=\frac{1}{3}$ or $x=\sin^{-1}\left(\frac{1}{3}\right)$.

To decide whether y is max. or min. at $x=\sin^{-1}\frac{1}{3}$, we observe that $\frac{dy}{dx}$ is +ve when $\operatorname{cosec} x>3$ or $x<\sin^{-1}\left(\frac{1}{3}\right)$ and $\frac{dy}{dx}$ is −ve when $\operatorname{cosec} x<3$ or $x>\sin^{-1}\left(\frac{1}{3}\right)$. Thus $\frac{dy}{dx}$ changes sign from +ve to −ve as it passes through the value $x=\sin^{-1}\frac{1}{3}$. Thus y is max. at $x=\sin^{-1}\frac{1}{3}$.

Example 1. *Find max. and min. values of the function*

$$f(x)=x^5-5x^4+5x^3-10.$$

Sol. Differentiating $f(x)$ w.r.t. x, we have

$$f'(x)=5x^4-20x^3+15x^2$$
$$=5x^2(x-1)(x-3)$$

Now $\qquad f'(x)=0 \Rightarrow x=0,\ 1,\ 3.$

Again differentiating $f'(x)$, we have

$$f''(x)=20x^3-60x^2+30x.$$

(*i*) When $x=0$, $f''(0)=0$. So we have to differentiate again.

Now $\qquad f'''(x)=60x^2-120x+30$

At $x=0$, $f'''(0)=30\neq 0$. Hence there is neither max. nor min. at $x=0$.

(*ii*) When $x=1$, $f''(1)=20-60+30=-10$.

Therefore $f(x)$ is max. at $x=1$.

(*iii*) When $x=3$, clearly $f''(3)$ is +ve. So $f(x)$ is min. at $x=3$.

Example 2. *The efficiency of a screw jack is given by*

$$E=\frac{\tan\theta}{\tan(\theta+\alpha)}$$

where α is a constant angle. Determine the value of θ for which the efficiency is maximum.

Find the value of maximum efficiency.

Sol. Differentiating E w.r.t. θ, we have

$$\frac{dE}{d\theta}=\frac{\tan(\theta+\alpha)\sec^2\theta-\tan\theta\sec^2(\theta+\alpha)}{\tan^2(\theta+\alpha)}$$

$$=\frac{\sin(\theta+\alpha)\cos(\theta+\alpha)-\sin\theta\cos\theta}{\sin^2(\theta+\alpha)\cos^2\theta}$$

$$=\frac{\sin\alpha\cos(2\theta+\alpha)}{\sin^2(\theta+\alpha)\cos^2\theta}$$

Now $\dfrac{dE}{d\theta}=0 \Rightarrow \cos(2\theta+\alpha)=0$ as $\sin\alpha\neq 0$, being a given constant.

$$\therefore \qquad (2\theta+\alpha)=\frac{\pi}{2} \text{ or } \theta=\frac{\pi}{4}-\frac{\alpha}{2}.$$

Writing $\qquad \dfrac{\sin\alpha}{\sin^2(\theta+\alpha)\cos^2\theta}=f(\theta)$

we have $\qquad \dfrac{dE}{d\theta}=f(\theta)\cos(2\theta+\alpha)$

Again differentiating w.r.t. θ, we have

$$\frac{d^2E}{d\theta^2}=f'(\theta)\cos(2\theta+\alpha)+f(\theta)\{-2\sin(2\theta+\alpha)\}$$

When $\qquad (2\theta+\alpha)=\dfrac{\pi}{2}$ or $\theta=\dfrac{\pi}{4}-\dfrac{\alpha}{2}$,

$$\frac{d^2E}{d\theta^2}=f'\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\frac{\pi}{2}-2f\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\sin\frac{\pi}{2}$$

$$=f'\left(\frac{\pi}{2}-\frac{\alpha}{2}\right).0-2f\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

$$=-2f\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).$$

As $f(\theta)$ is always +ve for every value of θ.

$\dfrac{d^2E}{d\theta^2}=-$ve. As such E is max. when $\theta=\dfrac{\pi}{4}-\dfrac{\alpha}{2}$

Now maximum efficiency is given by

$$E=\frac{\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}+\alpha\right)}=\frac{\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}$$

$$=\left(\frac{1-\tan\frac{\alpha}{2}}{1+\tan\frac{\alpha}{2}}\right)^2=\left(\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}\right)^2$$

$$=\frac{1-\sin\alpha}{1+\sin\alpha}.$$

Example 3. *When travelling x km per hour, a truck burns diesel oil at the rate of* $\left(\frac{900}{x}+x\right)\Big/300$ *litre per km. If diesel oil costs 40 paisa per litre and the driver is paid Rs 1·50 an hour, find the steady speed that will minimise the total cost of a trip of 500 km.*

(A.M.I.E. May, 1972)

Sol. Here speed of the truck $=x$ km. per hour. Now time taken for the trip of 500 km.

$$=\frac{\text{Distance}}{\text{Speed}}=\frac{500}{x}\text{ hours.}$$

Let y be the total cost of the trip in rupees, then

$$y=\text{Expenses on the driver}+\text{Cost of diesel oil}$$

$$=1{\cdot}5\times\frac{500}{x}+\left\{\frac{0{\cdot}4\times\left(\frac{900}{x}+x\right)}{300}\right\}\times 500$$

$$=\frac{1350}{x}+\frac{2x}{3}.$$

Differentiating y w.r.t. x

$$\frac{dy}{dx}=-\frac{1350}{x^2}+\frac{2}{3}=\frac{2x^2-4050}{3x^2}$$

Now $\qquad \frac{dy}{dx}=0 \Rightarrow 2x^2-4050=0$

or $\qquad x^2=2025, \qquad i.e. \quad x=45.$

Also $\qquad \frac{d^2y}{dx^2}=\frac{2700}{x^3}=+\text{ve for all } x>0.$

Hence the cost y is minimum when $x=45$ km/hr.

7·6. In applied problems, certain physical quantities are to be maximised or minimised. We express such quantities as a function of some variable x. In case it is a function of two variables, one of them has to be eliminated with the help a condition which connects the two variables.

Example 4. *Find the maximum value of the volume of a right circular cylinder situated symmetrically inside a sphere of radius a.*
(A.M.I.E. May 1961)

Sol. Let r be the radius and x be the height of the cylinder inscribed in the given sphere. Then volume V of the cylinder is given by

$$V=\pi r^2 x=f(r, x).$$

Now in $\triangle OAB$,

$$OB^2+BA^2=OA^2$$

or $$\left(\frac{x}{2}\right)^2+r^2=a^2$$

or $$r^2=a^2-\tfrac{1}{4}x^2,$$

which connects the two variables x and r.

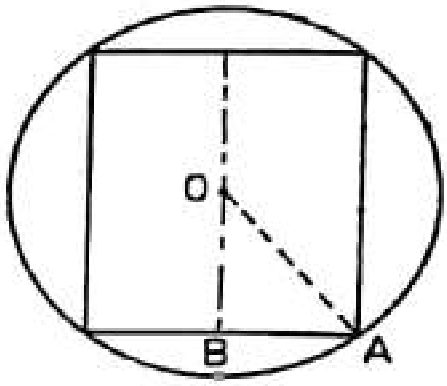

Fig. 7·2.

Thus $V=\pi\left(a^2-\frac{1}{4}x^2\right)x=f.x)$

or $$V=\pi\left(a^2x-\frac{1}{4}x^3\right).$$

$\therefore$ $$\frac{dV}{dx}=\pi\left(a^2-\frac{3}{4}x^2\right)$$

Now $$\frac{dV}{dx}=0 \Rightarrow \left(a^2-\frac{3}{4}x^2\right)=0$$

or $$x^2=\frac{4a^2}{3}.$$

Also $$\frac{d^2V}{dx^2}=-\frac{3\pi x}{2};$$

which is negative,

Thus the volume is maximum when

$$x=\frac{2a}{\sqrt{3}},$$

and then $$r=a\sqrt{\frac{2}{3}},$$

$\therefore$ Max. Volume$=\pi\times\dfrac{2a^2}{3}\times\dfrac{2a}{\sqrt{3}}=\dfrac{4\pi a^3}{3\sqrt{3}}.$

Example 5. *A lane runs at right angles of a road d metres wide. Find how many metres wide the lane must be if it is just possible to carry a pole p metres long ($p > d$) from the road into the lane keeping it horizontal.*

Sol. Let AOB be the pole and θ be the angle which the pole makes with the road. Let x metres be the width of the lane.

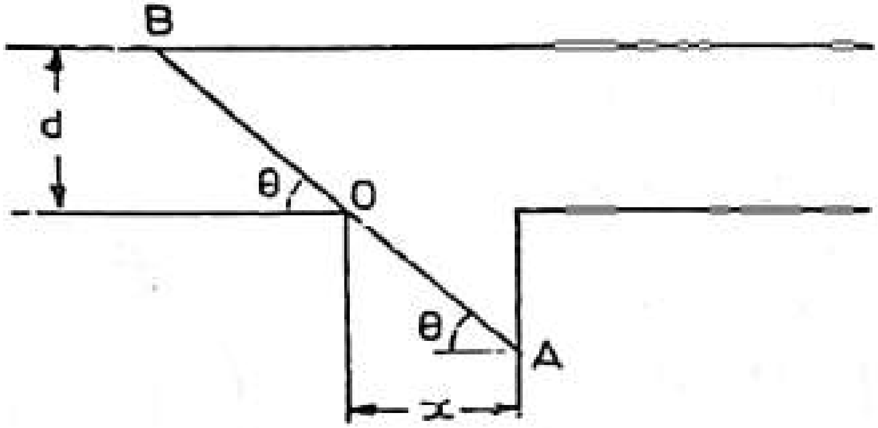

Fig. 7·3.

Then $AO = x \sec\theta$, $\quad OB = d \operatorname{cosec}\theta$.

$\therefore \quad p = AB = AO + OB = x \sec\theta + d \operatorname{cosec}\theta$

or $\quad x \sec\theta = p - d \operatorname{cosec}\theta$

or $\quad x = p\cos\theta - d\cot\theta$

Now $\quad \dfrac{dx}{d\theta} = -p\sin\theta + d\operatorname{cosec}^2\theta = 0$

$\Rightarrow \quad p\sin\theta = d\operatorname{cosec}^2\theta$

or $\quad \sin^3\theta = \dfrac{d}{p}$

or $\quad \sin\theta = \left(\dfrac{d}{p}\right)^{\frac{1}{3}}$

Further $\quad \dfrac{d^2x}{d\theta^2} = -p\cos\theta - 2d\operatorname{cosec}^2\theta\cot\theta$

$$= -(p\cos\theta + 2d\operatorname{cosec}^2\theta\cot\theta)$$

Thus the max. value of x so that the pole may just touch the road, the lane and their junction O is given by

$$x = p\sqrt{\left(1 - \frac{d^{2/3}}{p^{2/3}}\right)} - d\,\frac{\sqrt{(p^{2/3} - d^{2/3})}}{d^{1/3}}$$

$$= (p^{2/3} - d^{2/3})^{3/2}$$

Example 6. *If* $e = \dfrac{\{\tan x\,(1 - \mu\tan x)\}}{(\mu + \tan x)}$,

show that e is max. when $\tan x = \sqrt{(1 + \mu^2)} - \mu$

Sol. Differentiating e w.r.t. x and simplifying, we get

$$\frac{de}{dx}=-\frac{\mu \sec^2 x\,(\tan^2 x+2\mu \tan x-1)}{(\mu+\tan x)^2}$$

Now $$\frac{de}{dx}=0 \Rightarrow \tan^2 x+2\mu \tan x-1=0$$

or $$\tan x=-\mu\pm\sqrt{(\mu^2+1)}.$$

Choosing $\tan x=-\mu+\sqrt{(\mu^2+1)}=\sqrt{(1+\mu^2)}-\mu$, we find

$\frac{de}{dx}=+$ve when $\tan x$ is slightly less than $\sqrt{(1+\mu^2)}-\mu$

and $\frac{de}{dx}=-$ve when $\tan x$ is slightly greater than $\sqrt{(1+\mu^2)}-\mu$

Thus $\frac{de}{dx}$ changes sign from +ve to −ve as it passes through the value $\tan x=\sqrt{(1+\mu^2)}-\mu$.

$\therefore$ e is maximum when $\tan x=\sqrt{(1+\mu^2)}-\mu$.

EXERCISE 7 (b)

1. Find the maximum value of

$$(x-1)(x-2)(x-3).$$

2. Show that $\sin x\,(1+\cos x)$ is maximum when

$$x=\frac{\pi}{3}.$$

3. Show that $\sin^p \theta \cos^q \theta$ attains a maximum when

$$\theta=\tan^{-1}\sqrt{\frac{p}{q}}.$$

4. If $E=\frac{t(1-\mu t)}{\mu+t}$, find for what value of t, E is max., treating μ as a constant.

5. If $F=\frac{x}{(a^2+x^2)^{3/2}}$, find the value of x for which F is a max. Also find max. value of F.

6. What is the min. value of I, when

$$I=\frac{8K}{x^2}+\frac{K}{(6-x)^2},$$

where K is a constant.

7. $R=\frac{V^2}{54}-\frac{3(V-12)}{V+12}$, show that for minimum value of R, $V=6$.

8. Show that $f(x)=\dfrac{(x+1)^2}{(x+3)^3}$ has a maximum value $\dfrac{2}{27}$ and a minimum value zero.

9. Show that x^x is a minimum when $x=\dfrac{1}{e}$.

10. Show that the maximum radius vector 'r' of the curve

$$\frac{c^4}{r^2}=\frac{a^2}{\sin^2\theta}+\frac{b^2}{\cos^2\theta}$$

is $\dfrac{c^2}{(a+b)}$.

11. A figure consists of a semi-circle with a rectangle on its diameter. If perimeter of the figure is given, show that rectangle should be a square in order that the area of the figure may be maximum.

12. An open tank is to be constructed with a square base and vertical sides so as to contain a given quantity of oil. Show that the expenses of lining with lead will be least when the depth is made half of the width.

13. Assuming that the strength of a beam of rectangular cross-section varies as the breadth and as the cube of the depth, prove that for the strongest beam which can be cut from a given circular log breadth must be equal to the radius of the log.

14. A wire 3 metres long has to be bent into the form of a rectangle with an external loop at one corner and the rectangle is to have one side double the other. Show that the radius of the loop is one-third of the longest side of the rectangle for total area enclosed to be minimum.

15. The straight shore of a large lake runs east and west. A and B are two points on this shore 12 km apart. There is a town C, 9 km north of A and another town D, 15 km north of B. A single pumping station on the lake shore is to supply water to both the towns. Where should it be located in order that the sum of its distances from C to D may be minimum.

16. Show that the greatest triangle which can be inscribed in a circle is equilateral.

17. Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}$ of the diameter of the sphere.

18. A rectangular sheet of metal has four equal square portions removed at the corners, and the sides are then turned-up so as to form an open rectangular box. Show that when volume contained in the box is a maximum, the depth will be

$$\frac{1}{6}\{(p+q)-\sqrt{(p^2+q^2-pq)}\},$$

where p, q are the sides of the original rectangle.

19. A long piece of tin of width $4c$ metres is to be formed into a gutter of trapezoidal cross-section by bending up strips of width c metres along each edge. For getting maximum carrying capacity, show that $\cos\theta=\frac{\sqrt{3}-1}{2}$, where θ is the angle at which the strips should be bent.

20. The bending moment of a beam, supported at the ends and uniformly loaded at a distance x from one end is

$$M=\frac{wlx}{2}-\frac{wx^2}{2}$$

where w is the load on the beam per unit run.

Find the point at which the bending moment is maximum.

21. The section of a window consists of a rectangle surmounted by an equilateral triangle. If the perimeter be given as 16 m, find the width of the window in order that the max. amount of light may be admitted.

22. A given quantity of metal is to be cast into a half cylinder with a rectangular base and semi-circular ends. Show that in order that the total surface area may be minimum, the ratio of the length of the cylinder to the diameter of its semi-circular ends is $\frac{\pi}{(\pi+2)}$.

23. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.

[**Hint.** Perimeter$=2r\cot\theta+4\,(r\operatorname{cosec}\theta+r)\tan\theta$, where θ is half of the unequal angle.]

24. A circular cylinder is to be inscribed in a given sphere of radius R. If the total surface of the cylinder, including the two ends is to be min. show that

$$\frac{h^2}{R^2}=2\left(1-\frac{1}{\sqrt{5}}\right),$$

where h is the height of the cylinder.

25. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is

$$\frac{4}{27}\pi h^3\tan^2\alpha$$

26. The shape of a hole bored by a drill is a cone surmounting a cylinder. If the cylinder be of height h and radius r and the semi-vertical angle of the cone be α, where $\tan\alpha=\frac{h}{r}$, show that for a total fixed depth H of the hole, the volume removed is maximum if

$$h=\frac{1}{6}H(\sqrt{7}+1).$$

27. Two towns are to get their water supply from a river. They are situated on the same side of the river at a distance of 6 km and 18 km from a river bank. If the distance between the points on the river bank nearest to the two towns respectively be 10 km, find where a single pumping station may be located to require the least amount of pipe. How much pipe is needed ?

28. One corner of a long rectangular sheet of paper of width one metre is folded so as to reach the opposite edge of the sheet. Find the minimum length of the crease.

29. In a sub-marine telegraph cable, the speed of signalling varies as $x^2 \log\left(\frac{1}{x}\right)$, where x is the ratio of the radius of the core to that of the covering. Show that the greatest speed is attained when the ratio is $1 : \sqrt{e}$.

30. Assuming that the petrol burnt per hour in driving a motor boat varies as the cube of the velocity, show that the most economical speed when going against a current of c km. p.h. is $\frac{3}{2}c$ km. p.h.

31. A person being in a boat a km from the nearest point of the beach wishes to reach as quickly as possible a point b km from the point along the shore. The ratio of his rate of walking to his rate of rowing is $\sec \alpha$. Prove that he should land at a distance $(b - a \cot \alpha)$ km from the place to be reached.

32. A tent is to be made of cloth in the form of a right circular cone of given volume. Show that the ratio of the height to the radius of the base is $\sqrt{2}$ when the amount of cloth used is minimum.

33. A thin closed rectangular box is to have one edge n times the length of another edge and the volume of the box is given to be V. Prove that the least surface S is given by $n S^3 = 54 (n+1)^2 V^2$.

34. The function $y = \dfrac{(ax+b)}{(x-1)(x-4)}$ has a turning value at the point $(2, -1)$. Find a and b and show that the turning value is a maximum.

35. Buses are to be chartered for an excursion. The price per ticket is Rs. 30·00 for the first 200 tickets with 10 paise rebate for every ticket for each passenger in excess of 200. What number of passengers will produce the maximum gross income for the company.

36. A body is moving in a straight line such that the distance at any time t is given by $s = \frac{1}{4} t^4 - 2t^3 + 4t^2 - 7$.
Find when the velocity is max. and when its acceleration is min.

37. A tablet 7 m high is placed on a wall, with its base 9 m above the level of an observer's eye. How far from the wall should the observer stand so that the angle of vision subtended by the tablet may be maximum ?

[**Hint.** $\tan\alpha=\frac{9}{x}$, $\tan(\alpha+\theta)=\frac{16}{x} \Rightarrow \tan\theta=\frac{7x}{144+x^2}$. Now θ will be max. when $\frac{7x}{144+x^2}$ is max.]

38. The sum of the surfaces of a cube and a sphere is given. Show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube.

39. A sheet of tin of width a has to be bent into an open cylinderical channel. What should the central angle ϕ be so that the channel will have maximum capacity.

40. If $f(\phi)=(\sin\phi)^{\sin\phi}$, show that the maximum value of $f(\phi)$ is unity when $\phi=\frac{\pi}{2}$ and that the minimum value is $\left(\frac{1}{e}\right)^{\frac{1}{e}}$ when $\phi=\sin^{-1}\left(\frac{1}{e}\right)$.

11. $\tan x + h \sec^2 x + h^2 \sec^2 x \tan x + \frac{h^3}{3}(1+3\tan^2 x).\sec^2 x + \ldots$

12. $\sin^{-1}(h) + x(1-h^2)^{-1/2} + \frac{x^2}{2!}h(1-h^2)^{-3/2} + \ldots.$

13. $\log \sin 2 + (x-2)\cot 2 - \frac{1}{2}(x-2)^2 \operatorname{cosec}^2 2 + \frac{1}{3}(x-2)^3 \operatorname{cosec}^2 2 \cot 2 + \ldots$

14. $\tan^{-1}\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)\cdot\frac{1}{1+\left(\frac{\pi^2}{16}\right)} - \frac{\left(x-\frac{\pi}{4}\right)^2 \pi}{4\left(1+\frac{\pi^2}{16}\right)^2} + \ldots$

15. $11 + 7(x-2) + 4(x-2)^2 + (x-2)^3$

16. 9·01 (app.)

17. (*i*) 0·50725 (*ii*) 0·02178

21. (*a*) $\frac{\pi r h\,\delta h}{\sqrt{r^2+h^2}}$ (*b*) $\frac{\pi(2r^2+2r\sqrt{r^2+y^2+h^2})}{\sqrt{r^2+h^2}}\,\delta r.$

22. (*i*) $-5·024$ cu. cm. (*ii*) $-5·024$ sq. cm, **23.** 6π km.

24. $4 \pm 0·008$ **25.** 1%

26. 0·71 **30.** 10%.

Exercise 6 (*b*) (Page 160—161)

1. 16 **2.** (-1)

3. $\frac{1}{2}$ **4.** $(\log a - \log b)$

5. 0 **6.** 3. **7.** 1

8. 0 **9.** $\frac{1}{2}$ **10.** $-\frac{1}{3}$ **11.** $-\infty$

12 1 **13.** a **14.** (*i*) e^{-6} (*ii*) ∞. **15.** e^a **16.** $e^{\frac{1}{3}}$

17. $a=-2$; -1

Exercise 7 (*a*) (Page 165—167)

3. $\frac{i_0 R}{L} e^{-Rt/L}$ **4.** $\frac{3}{8\pi}$ cm per minute

5. $\frac{1}{20}$ radian per second. **6.** $\frac{90}{\sqrt{34}}$ m per second

8. 120 kg per square cm decreasing.

11. $\frac{8\sqrt{3}}{3}$ m per second **14.** 8π km/minute

Exercise 7 (*b*) (Page 175—179)

1. $\frac{2\sqrt{3}}{9}$ **4.** $t=\sqrt{(\mu^2+1)}-\mu$

5. $\frac{a}{2}$; $\frac{16}{25\sqrt{5a^4}}$ **6.** $\frac{3K}{4}$ when $x=4$.

15. 4·5 km from A.

20. $x=\frac{l}{2}$

21. $x=\frac{16(6+\sqrt{3})}{33}=3{\cdot}75$ m.

27. $\frac{5}{2}$ km · 26 km.

28. $\frac{3\sqrt{3}}{4}$

34. $a=1, \quad b=0$

35. 250

36. $2-\frac{2}{\sqrt{3}}$; 2

37. 12 m

39. π.

Exercise 8 (a) (Page 190 – 192)

1. (i) $\frac{Xx}{a^2}+\frac{Y.y}{b^2}=1$ (ii) $Xx-Y.y=7$

(ii) $Yy=2p(X+x)$ (iv) $Y-y=\left(\sinh\frac{x}{c}\right)(X-x)$

2. $9x-8y\pm26=0$ (ii) $8x+9x\pm7=0$

3. (a) (i) (2, 13), (2, −3) (ii) (1, 6), (−1, 4)

(b) (0, 0), $\left(2a, \frac{2a^3}{b^2}\right)$

4. (i) $y=(x-at)\tan t/2$; $y+(x-at)\cos(t/2)=0$

(ii) $\frac{x}{a\cos t}+\frac{y}{b\sin t}=1$; $by\sin t-ax\cos t$
$=b\sin^4 t-a\cos^4 t$.

5. Parallel at (0, 0), $\left[\frac{4a}{3}, \frac{(32)^{1/3}a}{3}\right]$,
perpendicular at (0, 0), $(2a, 0)$

7. $y=2a$; $x=0$

14. (i) $\frac{\pi}{2}$ at (0, 0) and $\tan^{-1}\frac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}$
at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

(ii) $\frac{\pi}{2}$ (iii) 0°

(iv) 45°.

Exercise 8 (b) (Page 198—199)

3. S.T.$=2a\sin^3\theta/2\sec\theta/2$; S.N.$=a\sin\theta$.
Normal$=2a\sin\theta/2$, tangent$=2a\sin\theta/2\tan\theta/2$.

Exercise 8 (c) (Page 202)

3. $\sqrt{1+\frac{(2a-3x)^2}{12a(x-a)}}$ 4. $\frac{e^{2a}+1}{e^{2a}-1}$.

Exercise 8 (d) (Page 208 – 209)

1. (i) $\frac{\pi}{2}-\frac{\theta}{2}$. (ii) $\frac{\pi}{2}+\frac{\theta}{2}$.

8

Tangents and Normals

8·1. Definition of Tangent

Let P and Q be any two points on a curve $y=f(x)$. Now as point Q approaches P along the curve, the chord PQ, in general, tends to a straight line PT, which is defined as the tangent to the curve at the point P.

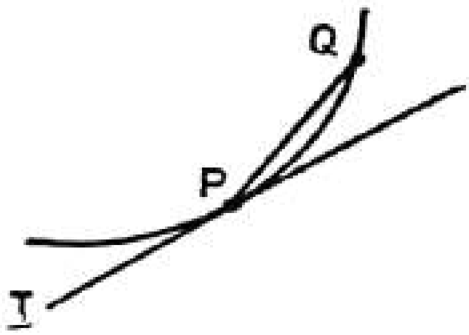

Fig. 8·1.

8·2. Geometrical Interpretation of the Derivative

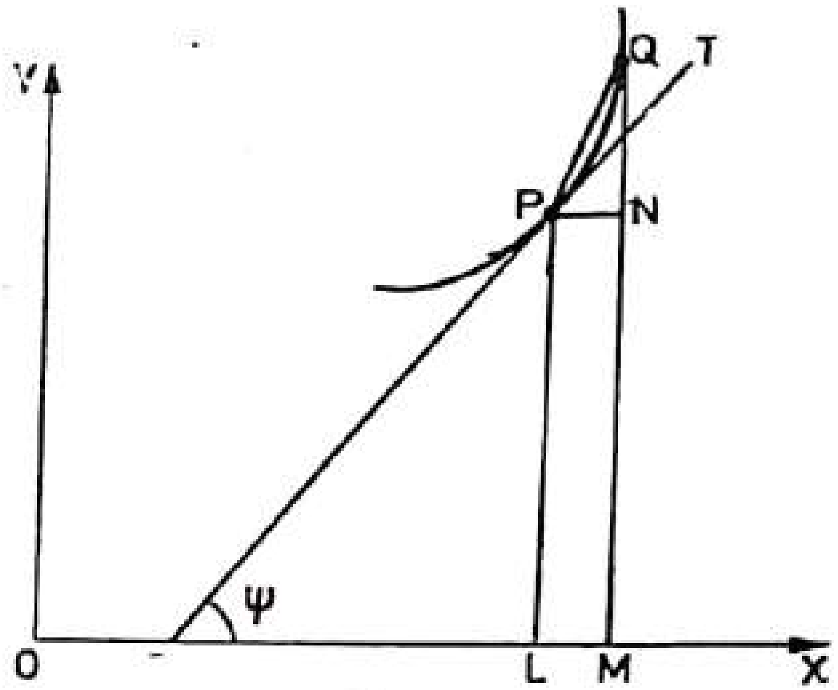

Fig. 8·2.

Let $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ be the points on a curve $y=f(x)$. Draw LP and MQ prependiculars on the x-axis and PN perpendicular to MQ.

We have $$LP=MN=y,$$
$$MQ=y+\delta y$$
and $$PN=\delta x$$
Now $$NQ=MQ-MN=MQ-LP$$
$$=(y+\delta y)-y=\delta y.$$

From $\triangle$ PQN, we have

$$\tan \angle QPN=\frac{NQ}{PN}=\frac{\delta y}{\delta x}$$
$$=\text{slope of chord } PQ.$$

Now as the point Q approaches the point P along the curve, the chord PQ tends to tangent PT and $\angle QPN$ approaches ψ, the angle which the tangent PT makes with the x-axis.

$$\therefore \quad \tan \psi=\underset{Q\to P}{\text{Lt}} \frac{\delta y}{\delta x}$$
$$=\underset{\delta x\to 0}{\text{Lt}} \frac{\delta y}{\delta x}=\frac{dy}{dx}.$$
$$\therefore \quad \mathbf{\tan \psi=\frac{dy}{dx}}.$$

Thus the derivative of a function at a point denotes the slope of the tangent to the curve at that point.

8·3. Equation of Tangent to a Curve

Let us find the equation of a tangent to the curve $y=f(x)$ at a point $P(x, y)$ on the curve. As already seen in previous article, the slope of tangent to the curve at (x, y) is $\frac{dy}{dx}$. Let (X, Y) be current co-ordinates of a point on the tangent, then its equation is given by,

$$\mathbf{Y-y=\frac{dy}{dx}(X-x).}$$

Example 1. *Find the equation of the tangent to the curve*

$y=x^3-2x^2+4$ at the point $(1, 3)$.

Sol. Here $$y=x^3-2x^2+4$$

$\therefore$ $$\frac{dy}{dx}=3x^2-4x$$

or $$\left(\frac{dy}{dx}\right)_{(1,\,3)}=-1.$$

Hence required equation of tangent is

$$y-3=-1(x-1)$$

or $$x+y=4.$$

Example 2. *Find the equation of the tangent to the curve*

$$y^2=2x$$

(i) parallel to the line $x+y+1=0$,

(ii) perpendicular to the line $x-2y+1=0$.

Sol. Here $$y^2=2x$$

$\therefore$ $$2y\frac{dy}{dx}=2$$

or $$\frac{dy}{dx}=\frac{1}{y}$$

(*i*) Let the tangent be parallel to the given line at the point (x_1, y_1) on the curve.

$\therefore$ $$\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}=\frac{1}{y_1}$$

Now slope of given line is -1

$\therefore$ $$\frac{1}{y_1}=-1$$

or $$y_1=-1.$$

Since the point (x_1, y_1) lies on the curve

$$y_1^2=2x_1$$

or $$1=2x_1$$

$\therefore$ $$x=1/2.$$

$\therefore$ Equation of tangent parallel to the given line is

$$y+1=-1(x-\tfrac{1}{2})$$

or $$x+y+\tfrac{1}{2}=0$$

or $$2x+2y+1=0.$$

(*ii*) Now $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)}=\frac{1}{y_1}$$

slope of the given line is $\frac{1}{2}$. Since the tangent at (x_1, y_1) is to be perpendicular to the given line,

$$\frac{1}{y_1}=-2$$

or $$y_1=-\tfrac{1}{2}$$

$\therefore$ $$x_1=\frac{1}{8}\cdot \qquad \text{(from the equation of curve)}$$

Thus, the equation of tangent perpendicular to the given line is

$$y+\frac{1}{2}=-2\left(x-\frac{1}{8}\right),$$

or $$2x+y+\tfrac{1}{4}=0,$$

or $$8x+4y+1=0.$$

Example 3. *Find the points on the curve $ax^2+2hxy+by^2=1$ at which tangent is parallel to x-axis.*

Sol. Here $ax^2+2hxy+by^2=1$ and let (x_1, y_1) be the required point.

Differentiating both sides with respect to x, we have

$$2ax+2h\left(x\frac{dy}{dx}+y\right)+2by\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=-\frac{ax+hy}{hx+by}$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)}=-\frac{ax_1+hy_1}{hx_1+by_1} \qquad ...(1)$$

Tangent is parallel to x-axis, if

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)}=0.$$

$\therefore$ From (1), we have

$$ax_1+hy_1=0,$$

or $$x_1=-\frac{h}{a}y_1. \qquad ...(2)$$

From equation of the curve, we have

$$ax_1^2+2hx_1y_1+by_1^2=1$$

$$a\cdot\frac{h^2}{a^2}y_1^2-\frac{2h^2}{a}y_1^2+by_1^2=1$$

$$\therefore \qquad (h^2-2h^2+ab)y_1^2=1$$

or $$y_1^2=\frac{1}{ab-h^2}$$

or $$y_1=\pm\frac{1}{\sqrt{ab-h^2}}$$

Therefore, from (2), $x_1=\mp\dfrac{h}{a\sqrt{ab-h^2}}$

Hence the required point is

$$\left(\mp\frac{h}{a\sqrt{ab-h^2}},\ \pm\frac{1}{\sqrt{ab-h^2}}\right)$$

Example 4. *Find the equation of the tangent at $t=\frac{\pi}{2}$ to the cycloid* $x=a(t-\sin t), \qquad y=a(1-\cos t).$

Sol. Here $x=a(t-\sin t), \qquad y=a(1-\cos t)$

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{a\sin t}{a(1-\cos t)}$$

$$=\frac{2a\sin\frac{t}{2}\cos\frac{t}{2}}{2a\sin^2\frac{t}{2}}=\cot t/2$$

$$\therefore \qquad \frac{dy}{dx}=\cot\frac{\pi}{4}=1 \quad \text{at } t=\frac{\pi}{2}.$$

Also, when $t=\frac{\pi}{2}$,

$$x=a\left(\frac{\pi}{2}-1\right),\ y=a$$

Therefore, the required equation of the tangent is

$$y-a=1.\left[x-a\left(\frac{\pi}{2}-1\right)\right]$$

or $$y-x=2a-\frac{\pi a}{2}.$$

Example 5. *Prove that the curve* $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ *touches the straight line* $\frac{x}{a}+\frac{y}{b}=2$ *at the point (a, b), whatever be the value of n.*

Sol. Here $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)=2$

$$\therefore \quad \frac{n}{a^n}x^{n-1}+\frac{n}{b^n}\cdot y^{n-1}\frac{dy}{dx}=0$$

or
$$\frac{dy}{dx}=-\frac{b^n}{a^n}\cdot\frac{x^{n-1}}{y^{n-1}}$$

or
$$\left(\frac{dy}{dx}\right)_{(a,\ b)}=-\frac{b}{a}$$

Therefore, the equation of the tangent to the given curve at (a, b) is

$$y-b=-\frac{b}{a}(x-a)$$

or
$$\frac{x}{a}+\frac{y}{b}=2.$$

Hence the line $\frac{x}{a}+\frac{y}{b}=2$, always touches the given curve at the point (a, b), whatever may be value of n.

8·4. Equation of Normal

The *Normal* to a curve $y=f(x)$ at a point $P(x, y)$ is a straight line passing through P and perpendicular to the tangent there at.

Since the normal is at right angle to the tangent, its slope is negative reciprocal of the slope of the tangent at $P(x, y)$.

Slope of normal at P

$$=-\frac{1}{\frac{dy}{dx}}.$$

Hence equation of normal at $P(x, y)$ is given by

$$Y-y=-\frac{1}{\frac{dy}{dx}}(X-x),$$

or
$$(\mathbf{X}-\mathbf{x})+\frac{\mathbf{dy}}{\mathbf{dx}}(\mathbf{Y}-\mathbf{y})=0$$

Example 1. *Find the equation of the normal to the curve.*
$4x^2+9y^2=25$ at $(2, 1)$.

Sol Here $4x^2+9y^2=25$

$\therefore$ $$8x+18y\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=-\frac{8x}{18y}=-\frac{4x}{9y}$$

or $$\left(\frac{dy}{dx}\right)_{at\ (2,\ 1)}=-\frac{8}{9}=m_1 \text{ (say)},$$

Therefore the slope of the normal

$$=-\frac{1}{m_1}=\frac{9}{8}.$$

Required equation of normal is

$$y-1=\frac{9}{8}(x-2)$$

or $$9x-8y=10$$

Example 2. *Find the equation of normal to the parabola $y^2=4ax$ in the form $y=mx-2am-am^3$, where m denotes the slope of the normal.*

Sol. Here $y^2=4ax$

$\therefore$ $$2y\frac{dy}{dx}=4a$$

or $$\frac{dy}{dx}=\frac{2a}{y}$$

Now slope of the normal is

$$=-\frac{y}{2a}=m \quad \text{(given)}.$$

$\therefore$ $$y=-2\,am$$

From the equation of the parabola

$$x=am^2.$$

Thus the point of contact is $(am^2, -2am)$.

Hence the equation of the normal is,

$$y+2am=m(x-am^2)$$
$$y=mx-2am-am^3.$$

Example 3. *If the normal to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is*

$$y\cos\phi-x\sin\phi=a\cos 2\phi.$$

Sol. Here $x^{2/3}+y^{2/3}=a^{2/3}$

$$\therefore \qquad \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

or
$$\frac{dy}{dx}=-\frac{y^{1/3}}{x^{1/3}}$$

Therefore, the slope of the normal is

$$-\frac{1}{\frac{dy}{dx}}=-\left(-\frac{x^{1/3}}{y^{1/3}}\right)=\frac{x^{1/3}}{y^{1/3}}$$

Slope of normal is $\tan\phi$, as normal makes an angle ϕ with the x-axis.

$$\therefore \qquad \frac{x^{1/3}}{y^{1/3}}=\tan\phi=\frac{\sin\phi}{\cos\phi}$$

$$\frac{x^{1/3}}{\sin\phi}=\frac{y^{1/3}}{\cos\phi}$$

$$=\frac{\sqrt{x^{2/3}+y^{2/3}}}{\sqrt{\sin^2\phi+\cos^2\phi}}=a^{1/3}$$

$$(\because \quad x^{2/3}+y^{2/3}=a^{2/3})$$

Hence $\qquad x=a\sin^3\phi, \qquad y=a\cos^3\phi$

Therefore, the equation of the normal is

$$y-a\cos^3\phi=\tan\phi(x-a\sin^3\phi),$$

or
$$y\cos\phi-x\sin\phi=a(\cos^4\phi-\sin^4\phi),$$
$$=a(\cos^2\phi-\sin^2\phi)(\cos^2\phi+\sin^2\phi),$$
$$=a\cos 2\phi.$$

or
$$y\cos\phi-x\sin\phi=a\cos 2\phi.$$

8·5. Tangents at the Origin

Sometimes it is not possible to evaluate $\frac{dy}{dx}$ at the origin, as it takes the form $\left(\frac{0}{0}\right)$. To find the equations of tangents in such cases, a different method is adopted. We know equation of any straight line through the origin is of the form $y=mx$ (exception being y-axis). So we substitute, $y=mx$ in the equation of the curve and evaluate m as x approaches zero. The following example illustrates the use of this method.

Example 1. *Find the equations of the tangents at the origin to the curves,*

(*a*) $y^2=x^2(1+x)$ $\qquad$ (*b*) $x^2=4ay$.

Sol. (*a*) Putting $y=mx$ in the equation of the curve, we have

$$m^2x^2=x^2(1+x)$$

$\therefore$ $$(m^2-1)x^2=x^3$$

or $$(m^2-1)=x$$

Now as $x\to 0$, $\quad m^2-1=0 \quad$ or $\quad m=\pm 1$.

Therefore, the equations of tangents are $y=x$ and $y=-x$.

(*b*) Putting $y=mx$ in the equation of the curve, we get

$$x^2=4am\,x$$

or $$x=4am$$

Now as $x\to 0$, $\quad m=0$;

Therefore the equation of tangent is $y=0$. *i.e.* the x-axis.

The above example shows that equation of the tangent at the origin can be obtained by equating the *lowest degree terms*, in the equation of the curve to zero.

8·6. Angle of Intersection of two Curves

The angle of intersection of two curves is defined as the angle between the tangents at their common point of intersection.

Let m_1 and m_2 be the slopes of tangents to the curves at their point of intersection and if θ be the angle between them, then

$$\tan\theta=\frac{m_1\sim m_2}{1+m_1m_2}$$

where $m_1\sim m_2$ means positive difference between m_1 and m_2.

The acute angle θ between the tangents is taken to be the angle of intersection of the two curves.

When $m_1m_2=-1$, $\theta=90°$, the curves intersect each other at right angles and are called *orthogonal*.

Example 1. *Find the angle of intersection of the circle $x^2+y^2=2a^2$ and the rectangular hyperbola $x^2-y^2=a^2$.*

Sol. Let the two curves intersect at (x_1, y_1).

$\therefore$ $$x_1^2+y_1^2=2a^2 \qquad \ldots(i)$$

and $$x_1^2-y_1^2=a^2 \qquad \ldots(ii)$$

From (*i*) and (*ii*), we get

$$x_1^2=\frac{3a^2}{2} \qquad \ldots(iii)$$

and $$y_1^2=\frac{a^2}{2} \qquad \ldots(iv)$$

From the first curve $\frac{dy}{dx}_{(x_1, y_1)} = -\frac{x_1}{y_1} = m_1$ (say),

and from the second curve $\frac{dy}{dx}_{(x_1, y_1)} = \frac{x_1}{y_1} = m_2$ (say),

Let θ be the angle between two curves, then

$$\tan \theta = \frac{m_1 \sim m_2}{1+m_1 m_2} = \frac{\frac{x_1}{y_1} + \frac{x_1}{y_1}}{1 - \frac{x_1^2}{y_1^2}}.$$

or
$$\tan \theta = \frac{2x_1 y_1}{x_1^2 - y_1^2}$$

From (*iii*) and (*iv*), we have

$$\tan \theta = \frac{2\sqrt{\frac{3}{2}} \cdot a \cdot \sqrt{\frac{1}{2}} \cdot a}{\frac{3a^2}{2} - \frac{a^2}{2}},$$

$$= \sqrt{3}$$

$\therefore$ $\theta = \frac{\pi}{6}$ is the required angle.

Example 2. *Find the condition that the conics,*

$ax^2 + by^2 = 1$ and $a_1 x^2 + b_1 y^2 = 1$ may cut each other orthogonally

Sol. Let the given curves intersect each other at (x_1, y_1)

$\therefore$ $$a x_1^2 + b y_1^2 = 1 \qquad \ldots(i)$$

and $$a_1 x_1^2 + b_1 y_1^2 = 1, \qquad \ldots(ii)$$

Solving (*i*) and (*ii*), we have

$$x_1^2 = \frac{b_1 - b}{ab_1 - a_1 b}, \quad y_1^2 = \frac{a - a_1}{ab_1 - a_1 b}. \qquad \ldots(iii)$$

For the first curve $\frac{dy}{dx}_{(x_1, y_1)} = -\frac{a x_1}{b y_1} = m_1$ (say).

For the second curve $\frac{dy}{dx}_{(x_1, y_1)} = -\frac{a_1 x_1}{b_1 y_1} = m_2$ (say).

If the curves intersects orthogonally, then

$$m_1 m_2 = -1,$$

$$\therefore \qquad \frac{aa_1x_1^2}{bb_1y_1^2} = -1. \qquad \ldots(iv)$$

From (*iii*), we have

$$\frac{x_1^2}{y_1^2} = \frac{b_1-b}{a-a_1} \qquad \ldots(v)$$

Substituting the value of $\frac{x_1^2}{y_1^2}$ from (*v*) into (*iv*), we have

$$\frac{aa_1}{bb_1} \cdot \frac{b_1-b}{a-a_1} = -1,$$

or $$aa_1(b_1-b) = -bb_1(a-a_1).$$

Dividing both sides by aa_1bb_1, we get

$$\frac{b_1-b}{bb_1} = -\frac{a-a_1}{aa_1},$$

or $$\frac{1}{b} - \frac{1}{b_1} = \frac{1}{a} - \frac{1}{a_1},$$ which is the required condition.

EXERCISE 8 (*a*)

1. Find the equations of the tangent and normal at the point (x, y) on each of the following curves.

(*i*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

(*ii*) $x^2 - y^2 = 7.$

(*iii*) $y^2 = 4px.$

(*iv*) $y = c \cosh \frac{x}{c}.$

2. Find the equations of the tangents to the curve

$$9x^2 + 16y^2 = 52$$

(*i*) parallel to the line $9x - 8y = 10$.

(*ii*) perpendicular to this line.

3. (*a*) Find the points at which the tangent to the curve $y = x^3 + 5$ is

(*i*) parallel to the line $12x - y = 17$,

(*ii*) perpendicular to the line $x + 3y = 2$.

(*b*) Find the points on the curve $b^2y = \frac{x^3}{3} - ax^2$ at which tangent is parallel to the x-axis.

4. Find the equations of tangent and normal at the point t on each of the following curves

(i) $x=a\,(t+\sin t)$, $y=a\,(1-\cos t)$,

(ii) $x=a\cos^3 t$ $y=b\sin^3 t$.

5. Find where the tangent is parallel to the axis of x and where it is perpendicular to that axis for the curve

$$y^3=x^2\,(2a-x)$$

6. Prove $\dfrac{x}{a}+\dfrac{y}{b}=1$ touches the curve $y=b\,e^{-\frac{x}{a}}$ at the point where the curve crosses the y-axis.

7. Find the equations of the tangent and normal to the curve

$$y=\frac{8a^3}{4a^2+x^2} \text{ at the point where it cuts the } y\text{-axis.}$$

8. Tangents are drawn from the origin to the curve $y=\sin x$.

Prove that their points of contact lie on the curve

$$x^2y^2=x^2-y^2.$$

9. Show that the condition for the line $x\cos\theta+y\sin\theta=p$ to touch the curve $x^my^n=a^{m+n}$

is $$p^{m+n}\,.\,m^m\,.\,n^n=(m+n)^{m+n}\,a^{m+n}\cos^m\theta\,\sin^n\theta.$$

10. Find the equation to the tangent at the point (a, a^3) on the curve $y=x^3$. Show that the tangent meets the curve again at the point $(-2a, -8a^3)$, and that the area included between the tangent and the curve is $\dfrac{27\,a^4}{4}$.

11. If OP_1 and OP_2 are the perpendiculars from O on the tangent and normal respectively at any point t on the curve $x=a\cos^3 t$ and $y=a\sin^3 t$, prove that

$$4\,.\,OP_1^2+OP_2^2=a^2.$$

12. Show that the tangents to the curve $x^3+y^3=3axy$ at the points other than origin where it meets the parabola $y^2=ax$ are parallel to the y-axis.

13. In the catenary $y=a\cosh x/a$, prove that the length of the portion of the normal intercepted between the curve and the x-axis is y^2/a.

14. Find the angle of intersection of the following curves

(i) $y^2=4ax$ and $x^2=4by$.

(ii) $x^2-4x+y^2=0$, $x^2+y^2=8$.

(iii) $y=x^2+2$ and $y=2x^2+2$.

(iv) $x^2+y^2=\sqrt{2}\,a^2$ and $x^2-y^2=a^2$.

15. Show that the curves

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1,$$

and

$$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1,$$

intersect orthogonally.

16. Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all points of the curve

$$\log (x^2+y^2)=k \tan^{-1}\left(\frac{y}{x}\right)$$

(A.M.I.E Nov. 1959, May 1963, 1965, 1969)

17. If α and β be the intercepts on the axes of x and y cut off by the tangent to the curve

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=1, \text{ tnen}$$

$$\left(\frac{a}{\alpha}\right)^{\frac{n}{n-1}}+\left(\frac{b}{\beta}\right)^{\frac{n}{n-1}}=1.$$

(I.E.T.E. Dec. 1973)

18. The tangents at any point of the curve $x^3+y^3=2a^3$ cut off lengths p and q on the co-ordinate axes. Show that

$$p^{-3/2}+q^{-3/2}=2^{-1/2}\, a^{-3/2}.$$

19. If x_1 and y_1 be the parts of the axes of x and y intercepted by the tangent at any point (x, y) on the curve

$$\left(\frac{x}{a}\right)^{3/2}+\left(\frac{y}{b}\right)^{3/2}=1,$$

show that

$$\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}=1.$$

(A.M.I.E. Nov. 1963)

20. Show that the ellipse $x^2+4y^2=8$ and the hyperbola $x^2-2y^2=2$, intersect orthogonally. *(A.M.I.E, May 1968)*

8·7. Length of Tangent, Normal, Subtangent and Subnormal

Let tangent PT and normal PN to a curve $y=f(x)$ at a point $P(x, y)$, meet the x-axis at the points T and N respectively. Now $PR=y$, is the ordinate of the point P.

Then TR is called *Subtangent* and RN the *Subnormal* of the curve. Let the tangent PT make an angle ψ with the positive direction of the x-axis, then

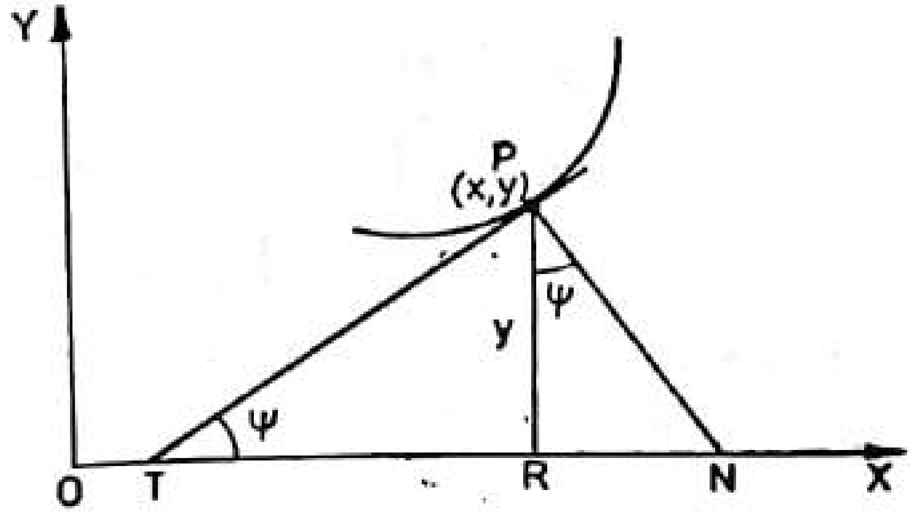

Fig. 8·3.

$$\angle PTR = \psi = \angle RPN$$

and
$$\tan\psi = \frac{dy}{dx} = y'.$$

(*i*) **Length of Tangent PT.**

From $\triangle TPR$, we have

$$\frac{PT}{RP} = \operatorname{cosec}\psi$$

$$\therefore \qquad PT = RP \operatorname{cosec}\psi$$

$$= y\sqrt{1+\cot^2\psi} \qquad [\because \; RP = y]$$

$$= y\,\frac{\sqrt{1+\tan^2\psi}}{\tan\psi}$$

$$= y \,.\, \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} = \frac{y}{y'}\,.\,\sqrt{1+y'^2}$$

$$\left[\because \; \tan\psi = \frac{dy}{dx} = y\right]$$

Hence length of tangent PT $= \dfrac{\mathbf{y}}{\mathbf{y'}}\sqrt{\mathbf{1+y'^2}}$.

(*ii*) **Length of Normal PN.** From ΔRPN, we have

$$\frac{PN}{RP}=\sec\psi$$

$$\therefore \quad PN=RP\sec\psi$$

$$=y\,.\,\sqrt{1+\tan^2\psi} \qquad [\because \; RP=y]$$

$$=y.\sqrt{1+\left(\frac{dy}{dx}\right)^2}=y\,.\,\sqrt{1+y'^2}$$

$$\left[\because \quad \tan\psi=\frac{dy}{dx}=y'\right]$$

Hence length of normal $\mathbf{PN=y\sqrt{1+y'^2}}$

(*iii*) **Length of subtangent TR.** From $\Delta\,TPR$, we get

$$\frac{TR}{RP}=\cot\psi$$

$$\therefore \quad TR=RP\cot\psi$$

$$=\frac{y}{\tan\psi} \qquad [\because \quad RP=y]$$

$$=\frac{y}{\frac{dy}{dx}}=\frac{y}{y'} \qquad \left[\because \tan\psi=\frac{dy}{dx}=y'\right]$$

Hence length of Subtangent $\mathbf{TR=\frac{y}{y'}}$.

(*iv*) **Length of Subnormal RN.** From $\Delta\ RPN$, we have

$$\frac{RN}{RP}=\tan\psi$$

$$\therefore \quad RN=RP\tan\psi$$

$$=y\,.\,\frac{dy}{dx}=yy'$$

$$\left[\because \quad RP=y \text{ and } \tan\psi=\frac{dy}{dx}\right]$$

Hence length of Subnormal $\mathbf{RN=yy'}$.

Example 1. *Find the lengths of the tangent, subtangent, normal and subnormal at the point (1, 3) of the curve* $y=x^2+x+1$.

Sol. Here $\quad y=x^2+x+1$

$$\frac{dy}{dx}=2x+1$$

$$\therefore \qquad \frac{dy}{dx}=3 \text{ at } (1, 3).$$

(*i*) Length of tangent

$$=\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \qquad [\S\ 8\cdot7\ (i)]$$

$$=\frac{3.\sqrt{1+3^2}}{3}=\sqrt{10}.$$

(*ii*) Length of subtangent

$$=\frac{y}{\frac{dy}{dx}}=\frac{3}{3}=1.$$

(*iii*) Length of normal

$$=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=3\sqrt{1+3^2}=3\sqrt{10}.$$

(*iv*) Length of subnormal

$$=y.\frac{dy}{dx}=3\times3=9.$$

Example 2. *Find the lengths of tangent, normal and subnormal at the point $t=\frac{\pi}{2}$ on the cycloid $x=a(t+\sin t)$; $y=a(1-\cos t)$.*

Sol. Here $x=a(t+\sin t)$, $\quad y=a(1-\cos t)$

$$\therefore \qquad \frac{dx}{dt}=a(1+\cos t), \quad \frac{dy}{dt}=a\sin t.$$

$$\text{Now} \quad y'=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{a\sin t}{a(1+\cos t)}$$

$$=\frac{2a\sin t/2\cos t/2}{2a\cos^2 t/2}=\tan t/2.$$

$$y'=\frac{dy}{dx}=1 \qquad \text{at } t=\frac{\pi}{2}$$

Also from the equation of the curve $y=a$ at $t=\pi/2$.

$\therefore$

(i) Length of tangent $=\frac{y}{y'}\sqrt{1+y'^2}$

$$=\frac{a}{1}\cdot\sqrt{1+1^2}.=a\sqrt{2}.$$

(ii) Length of subtangent $=\frac{y}{y'}=a.$

(iii) Length of normal $=y\sqrt{1+y'^2}=a\sqrt{2}$

(iv) Length of subnormal $=yy'=a.$

Example 3. *Show that the subtangent at any point of the curve $x^m y^n=a^{m+n}$, varies as the abscissa of the point of contact.*

Sol. Let the co-ordinates of the point of contact be (x, y).

We have to show that

$$\frac{\text{Subtangent}}{x}=\text{constant}$$

Here $x^m y^n=a^{m+n}$

Taking logs on both sides, we have

$$m\log x+n\log y=(m+n)\log a$$

Differentiating both sides w.r.t. x, we get

$$\frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=0,$$

$$\therefore\quad \frac{dy}{dx}=-\frac{my}{nx}.$$

Length of subtangent $=\frac{y}{\frac{dy}{dx}}=-\frac{y.nx}{my}=-\frac{n}{m}x$

$$\therefore\quad \frac{\text{Subtangent}}{x}=-\frac{n}{m}=\text{constant}.$$

Example 4. *Prove that the sum of intercepts of the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$, upon the co-ordinate axes is constant.*

Sol. Here $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

Differentiating both sides, w.r.t. x, we get

$$\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\frac{dy}{dx}=0,$$

$$\frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/2}$$

or

Let (X, Y) be current co-ordinates of a point on the tangent, then equation of the tangent at any point $P(x, y)$ is,

$$Y-y=-\left(\frac{y}{x}\right)^{1/2}(X-x)$$

$$\frac{Y}{y^{1/2}}-y^{1/2}=-\left(\frac{X}{x^{1/2}}-x^{1/2}\right)$$

or $$\frac{X}{\sqrt{x}}+\frac{Y}{\sqrt{y}}=\sqrt{x}+\sqrt{y}=\sqrt{a}. \qquad (\because \sqrt{x}+\sqrt{y}=\sqrt{a})$$

or $$\frac{X}{\sqrt{ax}}+\frac{Y}{\sqrt{ay}}=1$$

Intercepts made by this tangent on the co-ordinates axes are $\sqrt{ax}$ and $\sqrt{ay}$.

$\therefore$ Sum of intercepts$=\sqrt{ax}+\sqrt{ay}=\sqrt{a}(\sqrt{x}+\sqrt{y})$.

$=\sqrt{a}\,.\,\sqrt{a}=a$ (constant).

Example 5. *Prove that the perpendicular dropped from the foot of the ordinate to the tangent of a curve is of length* $\frac{y}{\sqrt{1+y'^2}}$ *and show that it is constant for the catenary* $y=c \cosh (x/c)$.

Sol. Let $P(x, y)$ be any point on the curve and N the foot of the ordinate PN. Further let NR be the perpendicular dropped from N on the tangent PT.

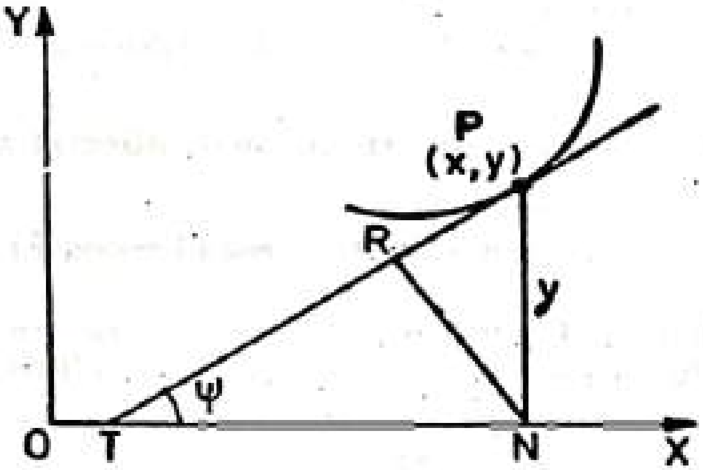

Fig. 8·4.

Now $TN = \text{subtangent} = \frac{y}{y'}$.

Also from $\triangle TRN$, we have

$$RN = TN \sin \psi$$

$$= \frac{y}{y'} \sin \psi$$

$$= y \cos \psi = \frac{y}{\sec \psi} \qquad (\because \quad y' = \tan \psi)$$

$$= \frac{y}{\sqrt{1+\tan^2 \psi}} = \frac{y}{\sqrt{1+y'^2}} \qquad \ldots(1)$$

In case of

$$y = c \cosh x/c \qquad \ldots(2)$$

$$y' = \sinh x/c \qquad \ldots(3)$$

$$\therefore \qquad RN = \frac{y}{\sqrt{1+y'^2}} \qquad \text{(from 1)}$$

$$= \frac{y}{\sqrt{1+\sinh^2 x/c}} \qquad \text{(from 3)}$$

$$= \frac{y}{\cosh x/c} = \frac{c \cosh x/c}{\cosh x/c} \qquad \text{(from 2)}$$

$$= c \text{ (constant).}$$

EXERCISE 8 (*b*)

1. Show that for the parabola $y^2 = 4ax$, subtangent at any point is twice the abcisse and the subnormal is of constant length.

2. Show that in the exponential curve $y = be^{x/a}$, the subnormal varies as the square of the ordinate and subtangent is of constant length.

3. Find the subtangent, subnormal, normal and tangent at the point θ on the cycloid.

$$x = a\ (\theta - \sin \theta),\ y = a\ (1 - \cos \theta)$$

4. Prove that the length of the normal varies inversely as perpendicular from the origin on the tangent to the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

5. Prove that in the curve $x^{m+n} = a^{m-n}\, y^{2n}$, mth power of subtangent varies as nth power of subnormal.

6. Prove that for the curve

$$x=a(\cos t+\log \tan t/2),\ y=a \sin t$$

the portion of the tangent intercepted between the curve and the axis of x is of constant length.

7. Show that in the curve

$$y=a \log (x^2-a^2)$$

sum of the lengths of tangent and subtangent varies as the product of the coördinates of the point of contact.

8. If the coordinates of a point on the curve

$$x^{2/3}+y^{2/3}=a^{2/3}$$

are represented as

$$x=a \sin^3 t,\ y=a \cos^3 t,$$

show that t is the angle which the perpendicular from the origin to the tangent makes with x-axis.

8·8. Derivative of Arc (Cartesian)

Let $P(x, y)$ be any point on a curve $y=f(x)$, s the length of the arc AP, where A is a fixed point on the given curve and ψ the angle that the tangent at P makes with x-axis. Further let $Q(x+\delta x, y+\delta y)$ be a neighbouring point to P on the curve and arc PQ and chord PQ be of length δs and δl respectively.

We shall assume that the arc PQ of the curve is concave to the chord PQ, then as $Q \to P$,

$$\frac{\text{arc } PQ}{\text{chord } PQ}=1 \text{ or } \underset{Q\to P}{\text{Lt}} \frac{\overset{\frown}{PQ}}{PQ}=1$$

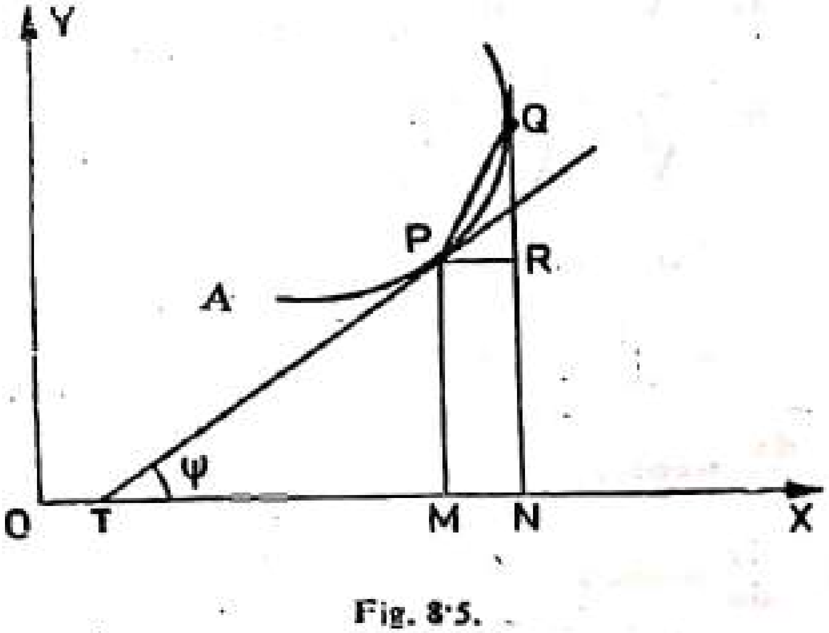

Fig. 8·5.

Draw PM and QN perpendiculars on x-axis from the points P and Q respectively and PR be perpendicular to QN.

Now $PR=MN=\delta x$ and $QR=QN-RN=(y+\delta y)-y=\delta y$.

Since $RN=PM=y$

We have from ΔPQR,

$$PQ^2=PR^2+QR^2$$

or
$$(\delta l)^2=(\delta x)^2+(\delta y)^2 \qquad (\because\ PQ=\delta l)$$

or
$$\left(\frac{\delta l}{\delta x}\right)^2=1+\left(\frac{\delta y}{\delta x}\right)^2$$

or
$$\frac{\delta l}{\delta x}=\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}$$

Now
$$\frac{\delta s}{\delta x}=\frac{\delta s}{\delta l}\ \frac{\delta l}{\delta x}=\frac{\delta s}{\delta l}\cdot\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2} \qquad \dots(1)$$

When $Q\to P$ *i.e.* $\delta x\to 0$, $\frac{\delta s}{\delta l}\to 1$, $\frac{\delta s}{\delta x}$ becomes $\frac{ds}{dx}$ and $\frac{\delta y}{\delta x}$ as $\frac{dy}{dx}$.

Then (1) becomes,

$$\frac{\mathbf{ds}}{\mathbf{dx}}=\sqrt{1+\left(\frac{\mathbf{dy}}{\mathbf{dx}}\right)^2}$$

Cartesian

Similarly we can show that

$$\frac{\mathbf{ds}}{\mathbf{dy}}=\sqrt{1+\left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^2}$$

Also
$$\frac{dx}{ds}=\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$=\frac{1}{\sqrt{1+\tan^2\psi}}$$

$$=\frac{1}{\sec\psi}=\cos\psi$$

$\therefore$
$$\frac{\mathbf{dx}}{\mathbf{ds}}=\mathbf{\cos}\ \psi,$$

and similarly $\frac{\mathbf{dy}}{\mathbf{ds}}=\mathbf{\sin}\ \psi.$

When equation of the curve is in parametric form $x=f(t)$ and $y=\phi(t)$, we know

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

then
$$\frac{ds}{dx}=\sqrt{1+\left\{\left(\frac{dy}{dt}\right)^2\Big/\left(\frac{dx}{dt}\right)^2\right\}}$$

$$\therefore \quad \frac{ds}{dt}=\frac{ds}{dx}\cdot\frac{dx}{dt}=\sqrt{1+\left(\frac{dy}{dt}\right)^2\Big/\left(\frac{dx}{dt}\right)^2}\cdot\frac{dx}{dt}$$

$$\therefore \quad \boxed{\frac{\mathbf{ds}}{\mathbf{dt}}=\sqrt{\left(\frac{\mathbf{dx}}{\mathbf{dt}}\right)^2+\left(\frac{\mathbf{dy}}{\mathbf{dt}}\right)^2}.}$$

Example 1. *Calculate* $\frac{ds}{dx}$ *for the curve* $y^2=4ax$.

Sol. Here $y^2=4ax$

$$\therefore \quad 2y\frac{dy}{dx}=4a$$

or
$$\frac{dy}{dx}=\frac{2a}{y}$$

We know $\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$=\sqrt{1+\frac{4a^2}{y^2}}$$

$$=\sqrt{1+\frac{4a^2}{4ax}} \qquad (\because \quad y^2=4ax)$$

or
$$\frac{ds}{dx}=\sqrt{1+\frac{a}{x}}\,.$$

Example 2. *Find* $\frac{ds}{dt}$ *for the astroid* $x=a\cos^3 t$, $y=a\sin^3 t$.

Sol. Here $x=a\cos^3 t$,

$$y=a\sin^3 t$$

$$\therefore \quad \frac{dx}{dt}=-3a\cos^2 t\sin t.$$

$$\frac{dy}{dt}=3a\sin^2 t\cos t$$

We know

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$=\sqrt{9a^2(\cos^4 t\sin^2 t+\sin^4 t\cos^2 t)}$$

$$=3a\sin t\cos t\sqrt{\cos^2 t+\sin^2 t}$$

$$\frac{ds}{dt}=3a\sin t\cos t.$$

EXERCISE 8 (c)

1. Prove that for the catenary

$$y=c\cosh\frac{x}{c},\qquad \frac{ds}{dx}=\frac{y}{a}.$$

2. Prove that for the ellipse

$$x=a\cos t,\ y=b\sin t,$$

$$\frac{ds}{dt}=a\sqrt{(1-e^2\cos^2 t)}.$$

Find $\dfrac{ds}{dx}$ for the following curves :

3. $3ay^2=x^2(a-x)$

4. $y=\log\dfrac{e^x-1}{e^x+1}$

5. If $x=a(t+\sin t)$, $y=a(1-\cos t)$, prove that

$$\frac{ds}{dy}=\sqrt{\frac{2a}{y}}\ \text{and}\ \frac{ds}{dt}=2a\cos\frac{t}{2}$$

8·9. Polar Coordinates

The position of a point $P(x, y)$ in a plane can also be fixed if its distance r from a fixed point O and the angle θ the line OP makes

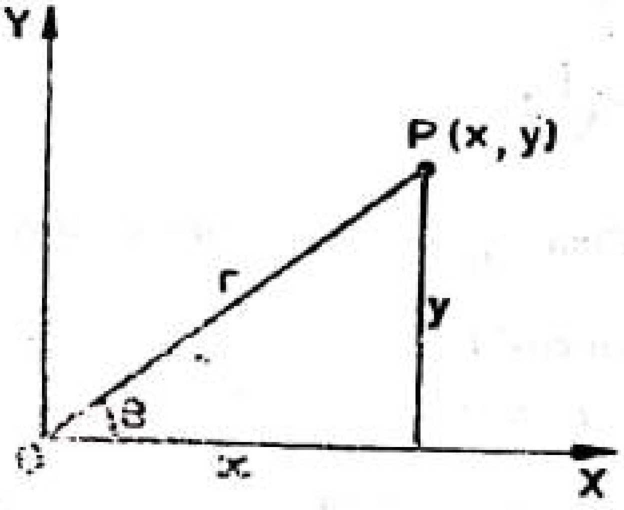

Fig. 8·6.

with the positive direction of a fixed line OX is known. The point O is called the *pole* and the line OX the *initial line.* The distance r is called *radius vector* and angle θ the *amplitude* of the point P. The coordinates (r, θ) are known as the polar coordinates of P.

Obviously, $x=r\cos\theta$ and $y=r\sin\theta$

To each pair (r, θ) there corresponds one and only one point in the plane

However the coordinates of P may be described as $(r, \theta\pm 2n\pi)$ and $[-r, \theta\pm(2n+1)\pi]$, where n is a positive integer. The polar coordinates of the pole may be given as $(0, \theta)$, where θ is arbitrary.

The equation of a curve in the form $r=f(\theta)$ or $\theta=f(r)$ is called the polar equation of the curve.

Example 1. *Find the equation of the curve* $(x^2+y^2)^2=a^2(x^2-y^2)$, *in the polar form.*

Sol. The equation of the given curve is

$$(x^2+y^2)^2=a^2(x^2-y^2). \qquad \text{.. } (i)$$

To change to polar form, we put

$$x=r\cos\theta,\ y=r\sin\theta,$$

then (i) reduce to

$$r^4=a^2r^2(\cos^2\theta-\sin^2\theta)$$

or

$$r^2=a^2\cos 2\theta.$$

8·10. Angle between Radius Vector and Tangent to a Polar Curve

Let $P(r, \theta)$ be a given point on the given curve and $Q(r+\delta r, \theta+\delta\theta)$ be an adjacent point to P on the curve. PN is perpendicular from P on OQ and PT the tangent at P, making an angle ψ with the positive direction of OX. Further let $\angle NQP$ be α and $\angle OPT=\phi$.

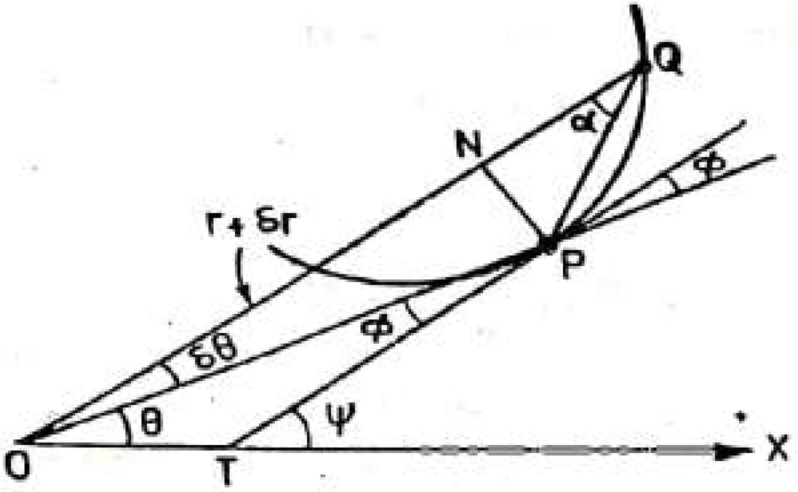

Fig. 8·7.

Now $\quad OP=r,$

$\quad OQ=r+\delta r.$

From $\triangle NPQ$, we have

$$\tan \alpha=\frac{PN}{NQ}=\frac{PN}{OQ-ON} \qquad ...(i)$$

Also from $\triangle OPN$,

$$PN=r \sin \delta\theta \text{ and } ON=r \cos \delta\theta.$$

Substituting these values of PN and ON in (i), we have

$$\tan \alpha=\frac{r \sin \delta\theta}{(r+\delta r)-r \cos \delta\theta} \qquad (\because \; OQ=r+\delta r)$$

$$=\frac{r\left(\frac{\sin \delta\theta}{\delta\theta}\right)}{r\left(\frac{1-\cos \delta\theta}{\delta\theta}\right)+\left(\frac{\delta r}{\delta\theta}\right)}$$

Now as $Q \to P$, along the curve, $\delta\theta \to 0$, $OQ \to OP$, $PQ \to PT$ and $\alpha \to \phi$.

Also $\quad \underset{\delta\theta \to 0}{\text{Lt}} \frac{\sin \delta\theta}{\delta\theta}=1$

and $\quad \underset{\delta\theta \to 0}{\text{Lt}} \frac{(1-\cos \delta\theta)}{\delta\theta}=0$

$$\therefore \quad \tan \phi=\frac{r \,.\, 1}{\frac{dr}{d\theta}}=r \frac{d\theta}{dr}$$

$$\therefore \quad \boldsymbol{\tan \phi=r \frac{d\theta}{dr}}$$

Example 1. *Find the angle between the radius vector and the tangent at a point (r, θ) for the cardiod, $r=a(1-\cos \theta)$.*

Sol. Here $\quad r=a(1-\cos \theta).$

$$\therefore \quad \frac{dr}{d\theta}=a \sin \theta$$

or $\quad \frac{d\theta}{dr}=\frac{1}{a \sin \theta}$

$$\therefore \quad r \frac{d\theta}{dr}=\frac{a(1-\cos \theta)}{a(\sin \theta)}$$

$$=\frac{2a \sin^2 \frac{\theta}{2}}{2a \sin \frac{\theta}{2} \cos \frac{\theta}{2}}=\tan \frac{\theta}{2}$$

$$\tan\phi = \tan\frac{\theta}{2} \qquad \left[\because \ r\frac{d\theta}{dr} = \tan\phi\right]$$

or $$\phi = \theta/2.$$

$\therefore$

8·11. Angle of Intersection of Two Curves (*Polar Curves*)

As already defined, by angle of intersection of two curves, we mean the angle between the tangents at their point of intersection.

Let the two curves intersect at the point $P(r, \theta)$ and tangents to these curves at P make angles ϕ_1 and ϕ_2 with the radius vector OP. Further let $\angle OPT' = \phi_1$ and $\angle OPT = \phi_2$.

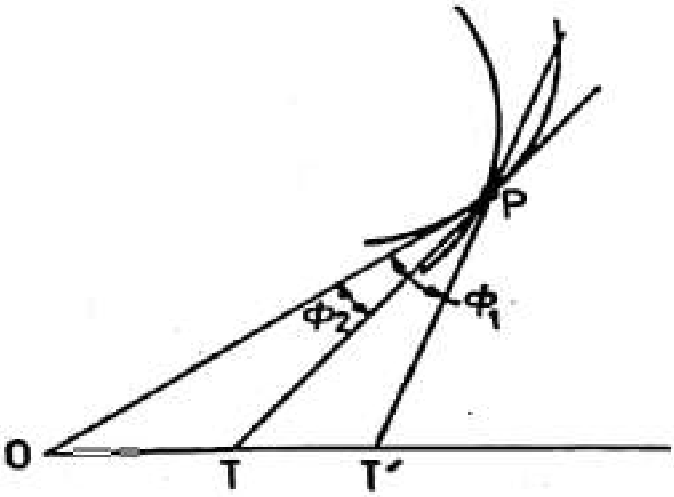

Fig. 8·8.

From figure it is obvious, that angle between the curves at P is

$$\phi_1 \sim \phi_2.$$

Further let $\tan\phi_1 = m_1$ and $\tan\phi_2 = m_2$,

Now $$\tan(\phi_1 \sim \phi_2) = \frac{\tan\phi_1 \sim \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2} = \frac{m_1 \sim m_2}{1 + m_1 m_2}$$

$\therefore$ $$\phi_1 \sim \phi_2 = \tan^{-1}\left(\frac{m_1 \sim m_2}{1 + m_1 m_2}\right)$$

Two curves intersect orthogonally if $\phi_1 \sim \phi_2 = 90°$,

then $$m_1 m_2 = -1.$$

Example 1. *Show that the curves* $r^m = a^m \cos m\theta$ *and* $r^m = a^m \sin m\theta$ *intersect orthogonally.*

Sol. For curve $r^m = a^m \cos m\theta$, taking logs, we have

$$m \log r = m \log a + \log \cos m\theta$$

Differentiating both sides with respect to θ, we get

$$\frac{m}{r}\frac{dr}{d\theta} = -m \tan m\theta$$

$$\therefore \quad r\frac{d\theta}{dr} = -\cot m\theta$$

or $$\tan\phi_1 = \tan\left(\frac{\pi}{2} + m\theta\right)$$

$$\therefore \quad \phi_1 = \frac{\pi}{2} + m\theta .$$

For the curve $\quad r^m = a^m \sin m\theta \qquad \ldots(1)$

taking logarithmic differentiation, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \cot m\theta$$

$$\therefore \quad r\frac{d\theta}{dr} = \tan m\theta$$

$$\therefore \quad \tan\phi_2 = \tan m\theta$$

$$\therefore \quad \phi_2 = m\theta$$

$$\therefore \quad \phi_1 \sim \phi_2 = \left(\frac{\pi}{2} + m\theta\right) - m\theta \qquad \ldots(2)$$

[from 1 and 2]

$$= \frac{\pi}{2} .$$

Hence the two curves intersect orthogonally.

Example 2. *Find the angle of intersection of the curves,*
$r = 3\cos\theta$ *and* $r = 1 + \cos\theta$.

Sol. The curves intersect where

$$3\cos\theta = 1 + \cos\theta$$

or $$\cos\theta = \tfrac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Hence the points of intersection are,

$$\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right).$$

Now for the curve $r = 3\cos\theta$,

$$\frac{dr}{d\theta} = -3\sin\theta$$

$$\therefore \quad \tan\phi_1 = r\frac{d\theta}{dr} = -\cot\theta$$

For the curve $\quad r = 1 + \cos\theta$,

$$\frac{dr}{d\theta} = -\sin\theta$$

$$\therefore \qquad \tan \phi_2 = r\frac{d\theta}{dr} = -\frac{1+\cos\theta}{\sin\theta} = -\cot\frac{\theta}{2}$$

At $\theta = \frac{\pi}{3}$, we have $\tan \phi_1 = -\cot \frac{\pi}{3} = -\frac{1}{\sqrt{3}}$

and
$$\tan \phi_2 = -\cot\frac{\pi}{6} = -\sqrt{3}$$

Now
$$\tan(\phi_1 \sim \phi_2) = \frac{\tan\phi_1 \sim \tan\phi_2}{1+\tan\phi_1 \tan\phi_2}$$

$$= \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{1+1} = \frac{1}{\sqrt{3}}$$

$$\therefore \qquad \phi_1 \sim \phi_2 = \pi/6,$$

Similarly, we can show that the angle of intersection at the point $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$ is $\frac{\pi}{6}$.

8·12. Polar Subtangent and Subnormal

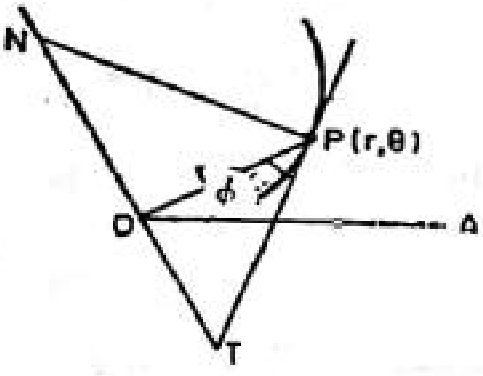

Fig. 8·9.

Let $P(r, \theta)$ be any point on a curve $r = f(\theta)$, and TON a straight line at right angles to radius vector OP. Further let tangent and normal at P meet TON at T and N respectively. Then OT is called the *polar subtangent* and ON the *polar subnormal* to the curve at P.

Now if angle between PT and OP is ϕ, and $OP = r$, then from $\triangle OPT$, we have

$$OT = r \tan\phi$$

$$= r \cdot \frac{r\,d\theta}{dr} \qquad \left[\because \tan\phi = r\frac{d\theta}{dr}\right]$$

or
$$OT = r^2 \frac{d\theta}{dr}$$

Hence length of the polar Subtangent $= r^2 \dfrac{d\theta}{dr}$.

Now from ΔOPN, we have

$$ON = r \tan\left(\frac{\pi}{2} - \phi\right) = r \cot \phi$$

$$ON = r . \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{dr}{d\theta} .$$

$$\left[\because \quad \cot \phi = \frac{1}{r} \frac{dr}{d\theta}\right]$$

Hence length of the polar subnormal $= \dfrac{dr}{d\theta}$.

Example 1. *Find the length of polar subtangent and subnormal of the cardioid $r = a(1 + \cos \theta)$.*

Sol. Here $\quad r = a(1 + \cos \theta)$

$$\frac{dr}{d\theta} = -a \sin \theta$$

$$\therefore \quad r^2 \frac{d\theta}{dr} = -a^2(1 + \cos \theta)^2 . \frac{1}{a \sin \theta}$$

$$= -a . 4 \cos^4 \frac{\theta}{2} \cdot \frac{1}{2 \sin \theta/2 \cos \theta/2}$$

$$= -2a \cos^3 \frac{\theta}{2} \operatorname{cosec} \frac{\theta}{2}$$

As length is always taken positive, the length of the subtangent

$$= 2a \cos^3 \frac{\theta}{2} \operatorname{cosec} \frac{\theta}{2} \cdot$$

Polar subnormal $= \dfrac{dr}{d\theta} = a \sin \theta$

[Taking positive sign]

EXERCISE 8 (*d*)

1. Find ϕ for the following curves :

(*i*) $\dfrac{2a}{r} = 1 + \cos \theta$, $\qquad$ (*ii*) $r = a(1 + \cos \theta)$,

(*iii*) $\dfrac{l}{r} = 1 + e \cos \theta$.

2. Prove that for the equiangular spiral

$$r = ae^{\cot \alpha . \theta}$$

ϕ is always constant and equal to α.

Find the angle of intersection of the following curves,

3. $r=a(1+\cos\theta)$ and $r=b(1-\cos\theta)$

4. $r=a\theta$ and $r\theta=a$.

5. $\frac{2a}{r}=1+\cos\theta$ and $\frac{2b}{r}=1-\cos\theta$.

6. $r=a$ and $r=a(1+\cos\theta)$.

7. $r=a\cos\theta$ and $r=a\sin\theta$.

8. $r^m=a^m\cos m\theta$ and $r^m=a^m\sin m\theta$.

Find the polar subtangent and subnormal for the following curves.

9. $r=a(1-\cos\theta)$. 10. $\frac{2a}{r}=1+\cos\theta$.

11. $r=a+b\cos\theta$. 12. $r=ae^{\cot\alpha\cdot\theta}$.

13. Show that for the sprial $r=a\theta$, the polar subnormal is constant and for $r\theta=a$, the polar subtangent is constant.

14. Show that for the curve $r=a$ exp. $(m\theta^2)$, the ratio of polar subnormal to polar subtangent is proportional to θ^2.

8·13. Length of the Perpendicular from Pole on a Tangent

Let $P(r, \theta)$ be any point on a curve $r=f(\theta)$ and ON perpendicular from pole O on the tangent PT. We have

$$\angle OPN=\phi \text{ and } OP=r \qquad \text{(Fig 8·9)}$$

From $\triangle OPN$, $\quad ON=OP\sin\phi$

or
$$\boxed{p=r\sin\phi.}$$

$\therefore$
$$\frac{1}{p^2}=\frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$=\frac{1}{r^2}(1+\cot^2\phi) \qquad \ldots(1)$$

We know $\tan\phi=r\frac{d\theta}{dr}$

and
$$\cot\phi=\frac{1}{r}\cdot\frac{dr}{d\theta}\cdot$$

Thus from (1), we have

$$\frac{1}{p^2}=\frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

or
$$\boxed{\mathbf{\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.}} \qquad \ldots(2)$$

Sometimes $\frac{1}{r}$ is written as u, then $u=\frac{1}{r}$ $\qquad r=\frac{1}{u}$

$$\frac{du}{d\theta}=-\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)$$

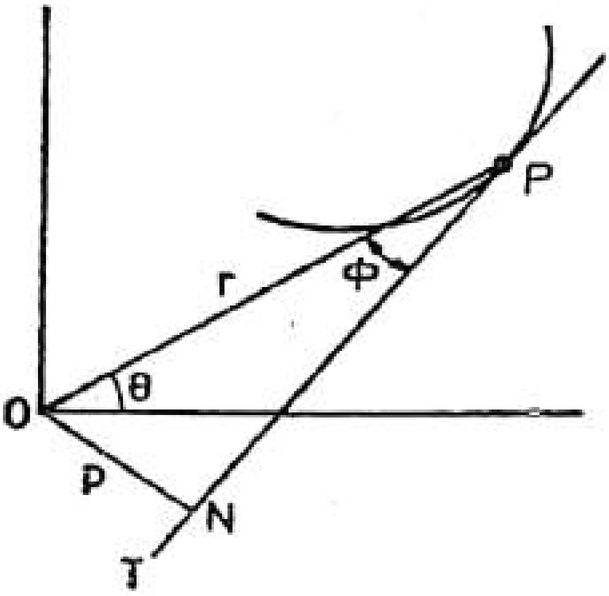

Fig. 8·9.

or $$\left(\frac{du}{d\theta}\right)^2=\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

From (2), we have

$$\frac{\mathbf{1}}{\mathbf{p^2}}=\mathbf{u^2}+\left(\frac{\mathbf{du}}{\mathbf{d\theta}}\right)^2$$

8·14. Pedal Equation

Pedal equation of a curve is defined as a relation between p and r, where p is the length of the perpendicular from the pole on the tangent and r the radius vector.

(1) To obtain pedal equation of a curve whose cartesian equation is given.

Let the equation of the curve be $f(x, y)=0$ ...(1)

Equation of the tangent to the curve (1) at a point (x, y) is given by

$$Y-y=\frac{dy}{dx}(X-x),$$

or $$Y-y-\frac{dy}{dx}(X-x)=0. \quad ...(2)$$

Let $p=$ length of perpendicular from the origin on the tangent given by (2), then

$$p=\frac{x\dfrac{dy}{dx}-y}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}, \quad ...(3)$$

Also $$r^2=x^2+y^2 \quad ...(4)$$

Eliminating x and y from (1), (3) and (4), we get a relationship between p and r, which is the pedal equation of the curve.

(*ii*) To obtain the pedal equation of a curve whose polar equation is given.

Let the equation of the curve be $r=f(\theta)$ ...(1)

We know $$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad ...(2)$$

Eliminating θ between (1) and (2), we get the required pedal equation.

Example 1. *Find the pedal equation of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Sol. The equation of the ellipse in parametric form is

$$x=a\cos\theta,\ y=b\sin\theta,$$

Now $$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=-\frac{b\cos\theta}{a\sin\theta}$$

Equation of a tangent to the ellipse at a point

$(a\cos\theta, b\sin\theta)$ is

$$y-b\sin\theta=-\frac{b}{a}\frac{\cos\theta}{\sin\theta}(x-a\cos\theta)$$

or $$b\cos\theta\, x+a\sin\theta\, y=ab. \quad ...(1)$$

Now length of the perpendicular from (0, 0) on (1) is

$$p=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

$$\therefore \quad \frac{1}{p^2}=\frac{b^2\cos^2\theta+a^2\sin^2\theta}{a^2b^2}.$$

or $$a^2\sin^2\theta+b^2\cos^2\theta=\frac{a^2b^2}{p^2} \quad ...(2)$$

Also $$r^2=x^2+y^2=a^2\cos^2\theta+b^2\sin^2\theta$$
$$=a^2(1-\sin^2\theta)+b^2(1-\cos^2\theta)$$
$$=a^2+b^2-(a^2\sin^2\theta+b^2\cos^2\theta)$$

or $$r^2=a^2+b^2-\frac{a^2b^2}{p^2}, \quad \text{[from 2]}$$

which is the required pedal equation.

Example 2. *Find the pedal equation of the parabola*

$$\frac{2a}{r}=1-\cos\theta.$$

Sol. Here $\dfrac{2a}{r}=1-\cos\theta$ ...(1)

Taking logs on both sides, we get

$$\log 2a-\log r=\log(1-\cos\theta)$$

Differentiating both sides with respect to θ, we get

$$-\frac{1}{r}\frac{dr}{d\theta}=\frac{\sin\theta}{1-\cos\theta}=\cot\frac{\theta}{2} \quad \text{...(2)}$$

Now $$r\frac{d\theta}{dr}=\tan\phi$$

or $$\frac{1}{r}\frac{dr}{d\theta}=\cot\phi$$

Therefore we get from (2),

$$-\cot\phi=\cot\frac{\theta}{2}$$

$\therefore$ $$\cot(\pi-\phi)=\cot\frac{\theta}{2}$$

$\therefore$ $$\phi=\pi-\frac{\theta}{2}.$$

Also $$p=r\sin\phi=r\sin\left(\pi-\frac{\theta}{2}\right),$$

$\therefore$ $$p=r\sin\frac{\theta}{2}$$

or $$p^2=r^2\sin^2\frac{\theta}{2}$$

$$=r^2\left(\frac{1-\cos\theta}{2}\right)=r^2\cdot\frac{a}{r} \quad \text{[from 1]}$$

$\therefore$ $p^2=ar$, the required pedal equation.

EXERCISE 8 (*e*)

Find the pedal equation of the following curves.

1. $y^2=4a(x+a)$ 2. $x^2+y^2=2ay$

3. $x=a\cos^3 t,\quad y=a\sin^3 t$

4. Show that the pedal equation of the leminscate $r^2=a^2\cos 2\theta$ is $r^3=a^2 p$.

Find the pedal equation of the following curves :

5. $r=a\theta$ 6. $r\theta=a$

7. $r=a(1+\cos\theta)$ 8. $r=a+b\cos\theta$

9. $r^m=a^m\cos m\theta$ 10. $\dfrac{l}{r}=1+e\cos\theta.$

8·15 Derivative of Arc (Polar Curves.)

Let $P(r, \theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be two adjacent points on a given curve $r=f(\theta)$. Further let s and $s+\delta s$ be the length of arcs AP and AQ respectively, where A is a fixed point on the curve and chord PQ be of length l. Draw PM perpendicular on OQ.

Now
$$OP=r,$$
$$OQ=r+\delta r,$$
and
$$\widehat{PQ}=\delta s.$$

From $\triangle MPQ$,
$$PQ^2=PM^2+MQ^2 \qquad \ldots(1)$$

From $\triangle OPM$,
$$PM=r\sin\delta\theta,\ OM=r\cos\delta\theta.$$

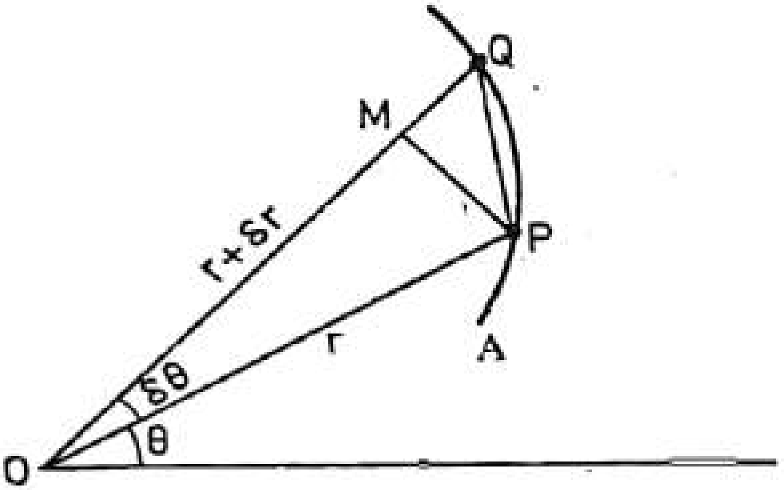

Fig. 8·10

Now, $MQ=(r+\delta r)-r\cos\delta\theta$

From (1), we get,
$$PQ^2=r^2\sin^2\delta\theta+[(r+\delta r)-r\cos\delta\theta]^2$$
$$=(r\sin\delta\theta)^2+[r(1-\cos\delta\theta)+\delta r]^2$$

$$\therefore \quad \left(\frac{PQ}{\delta\theta}\right)^2=r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2+\left[\frac{r(1-\cos\delta\theta)}{\delta\theta}+\frac{\delta r}{\delta\theta}\right]^2$$

or
$$\frac{PQ}{\delta\theta}=\frac{\delta l}{\delta\theta}$$
$$=\sqrt{r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2+\left[\frac{r(1-\cos\delta\theta)}{\delta\theta}+\frac{\delta r}{\delta\theta}\right]^2}$$

Now,
$$\underset{\delta\theta\to 0}{\text{Lt}}\ \frac{\delta s}{\delta\theta}=\underset{\delta\theta\to 0}{\text{Lt}}\ \frac{\delta s}{\delta l}\cdot\frac{\delta l}{\delta\theta}$$

$$=\underset{\delta\theta\to 0}{\text{Lt}}\frac{\delta s}{\delta l}\cdot\sqrt{r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2+\left[\frac{r(1-\cos\delta\theta)}{\delta\theta}+\frac{\delta r}{\delta\theta}\right]^2} \qquad \ldots(2)$$

As $\delta\theta \to 0$, we have

$$\frac{\delta s}{\delta l} \to 1, \qquad \frac{\sin \delta\theta}{\delta\theta} = 1, \text{ and } \frac{1-\cos \delta\theta}{\delta\theta} = 0. \quad \ldots (1$$

From (2) and (3), we get

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

Similarly we can show

$$\frac{ds}{dr} = \sqrt{1 + \left(r\,\frac{d\theta}{dr}\right)^2}.$$

Also we know

$$\tan \phi = r\,\frac{d\theta}{dr}$$

or

$$\cot \phi = \frac{1}{r}\,\frac{dr}{d\theta}$$

Now $\quad \dfrac{ds}{dr} = \sqrt{1 + \left(r\,\dfrac{d\theta}{dr}\right)^2} = \sqrt{1 + \tan^2 \phi} = \sec \phi$

$\therefore \quad \cos \phi = \dfrac{dr}{ds}$

Also $\quad \dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} = r\sqrt{1 + \left(\dfrac{1}{r}\,\dfrac{dr}{d\theta}\right)^2}$

$$= r\sqrt{1+\cot^2 \phi} = r \operatorname{cosec} \phi$$

$\therefore \quad \sin \phi = r\,\dfrac{d\theta}{ds}$

Example 1. *Find* $\dfrac{ds}{d\theta}$ *for the cardiode* $r = a(1+\cos\theta)$.

Sol. Here $\quad r = a(1+\cos\theta)$

$\therefore \quad \dfrac{dr}{d\theta} = -a\sin\theta$

Now $\quad \dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$

$$= \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{4\cos^4 \theta/2 + 4\sin^2 \theta/2 \cos^2 \theta/2}$$

$$= 2a\cos\theta/2\,\sqrt{\cos^2 \theta/2 + \sin^2 \theta/2}$$

Hence $\quad \dfrac{ds}{d\theta} = 2a\cos\dfrac{\theta}{2}.$

Example 2. *Show that for the spiral* $r\,\theta=a$,

$$\frac{ds}{dr}=\frac{\sqrt{r^2+a^2}}{r}.$$

Sol. Here $$\theta=\frac{a}{r}$$

$$\frac{d\theta}{dr}=-\frac{a}{r^2}$$

Now $$\frac{ds}{dr}=\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}=\sqrt{1+r^2\cdot\frac{a^2}{r^4}}$$

$$=\sqrt{1+\frac{a^2}{r^2}}=\frac{\sqrt{r^2+a^2}}{r}$$

Hence $$\frac{ds}{dr}=\frac{\sqrt{r^2+a^2}}{r}.$$

EXERCISE 8 (*f*)

Find $\frac{ds}{d\theta}$ for the following curves.

1. $r=a(1-\cos\theta)$
2. $r=a\,\theta$
3. $r^2=a^2\cos 2\theta$
4. $r=ae^{\theta\cot\alpha}$.

15. 4·5 km from A.

20. $x=\frac{l}{2}$ 21. $x=\frac{16(6+\sqrt{3})}{33}=3{\cdot}75$ m.

27. $\frac{5}{2}$ km · 26 km. 28. $\frac{3\sqrt{3}}{4}$

34. $a=1, \quad b=0$ 35. 250

36. $2-\frac{2}{\sqrt{3}}$; 2 37. 12 m

39. π.

Exercise 8 (*a*) (Page 190 – 192)

1. (*i*) $\frac{Xx}{a^2}+\frac{Y.y}{b^2}=1$ (*ii*) $Xx-Y.y=7$

(*ii*) $Yy=2p(X+x)$ (*iv*) $Y-y=\left(\sinh\frac{x}{c}\right)(X-x)$

2. $9x-8y\pm 26=0$ (*ii*) $8x+9x\pm 7=0$

3. (*a*) (*i*) (2, 13), (2, −3) (*ii*) (1, 6), (−1, 4)

(*b*) (0, 0), $\left(2a, \frac{2a^3}{b^2}\right)$

4. (*i*) $y=(x-at)\tan t/2$; $y+(x-at)\cos(t/2)=0$

(*ii*) $\frac{x}{a\cos t}+\frac{y}{b\sin t}=1$; $by\sin t-ax\cos t$
$=b\sin^4 t-a\cos^4 t$.

5. Parallel at (0, 0), $\left[\frac{4a}{3}, \frac{(32)^{1/3}a}{3}\right]$,
perpendicular at (0, 0), (2*a*, 0)

7. $y=2a$; $x=0$

14. (*i*) $\frac{\pi}{2}$ at (0, 0) and $\tan^{-1}\frac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}$
at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

(*ii*) $\frac{\pi}{2}$ (*iii*) 0°

(*iv*) 45°.

Exercise 8 (*b*) (Page 198 – 199)

3. S.T. $=2a\sin^3\theta/2\sec\theta/2$; S.N. $=a\sin\theta$.
Normal $=2a\sin\theta/2$, tangent $=2a\sin\theta/2\tan\theta/2$.

Exercise 8 (*c*) (Page 202)

3. $\sqrt{1+\frac{(2a-3x)^2}{12a(x-a)}}$ 4. $\frac{e^{2x}+1}{e^{2x}-1}$.

Exercise 8 (d) (Page 208 – 209)

1. (*i*) $\frac{\pi}{2}-\frac{\theta}{2}$. (*ii*) $\frac{\pi}{2}+\frac{\theta}{2}$.

(*iii*) $\cot^{-1}\left(\frac{e \sin \theta}{1+e \cos \theta}\right)$.

3. 90° (orthogonally) 4. 90° (orthogonally)
5. 90° (orthogonally) 6 90° (orthogonally)
7. 90° (orthogonally) 8. 90° (orthogonally)
9. (*i*) S.T.$=2a \sin^3 \theta/2 \sec \theta/2$ S.N.$=a \sin \theta$
10. S.T.$=2 \operatorname{cosec} \theta$, S.N.$=a \tan \theta/2 \sec^2 \theta/2$.
11. S.T. $=\frac{(a+b \cos \theta)^2}{b \sin \theta}$, S.N.$=b \sin \theta$
13. S.T.$=r \tan \alpha$, S.N.$=r \cot \alpha$.

Exercise 8 (e) (Page 212)

1. $p^2=ar$. 2. $r^3=2\ ap$.
3. $r^2+3p^2=a^2$ 5. $p^2=\frac{r^4}{r^2+a^2}$
7. $r^3=2\ ap^2$
8. $r^4=(b^2-a^2+2ar)\ p^2$ 9. $r^{m+1}=a^m p$.
10. $\frac{1}{p^2}=\frac{1}{l^2}\left[\frac{2l}{r}-1+e^2\right]$

Exercise 8 (f) (Page 215)

1. $2a \sin \theta/2$. 2. $a\sqrt{1+\theta^2}$
3. $a\sqrt{\sec 2\theta}$ 4. $r \operatorname{cosec} \alpha$

Exercise 9 (a) (Page 225–226)

1. $-\frac{\sqrt{5}}{25}$ 3. $\frac{5\sqrt{5}}{3}$
4. $2\sqrt{2}$ 5. (*i*) $\left(\frac{a}{4}, \frac{a}{4}\right)$
(*ii*) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\log 2\right)$ (*iii*) $\left(-\frac{1}{2}\log 2, \frac{\sqrt{2}}{2}\right)$
6. $\left(-\frac{1}{2}\log 2, \frac{\sqrt{2}}{2}\right)$ 7. $\frac{t(4+9t^2)^{3/2}}{6}$
8. $\frac{3 \sin t \cos t\,(a^2 \cos^2 t+b^2 \sin^2 t)^{3/2}}{ab}$
9. $4a \sin \frac{t}{2}$ 10. $a \cot t$. 19. 0·0069 radian/Km

Exercise 9 (b) (Page 228)

2. $2a$ 3. $2a$