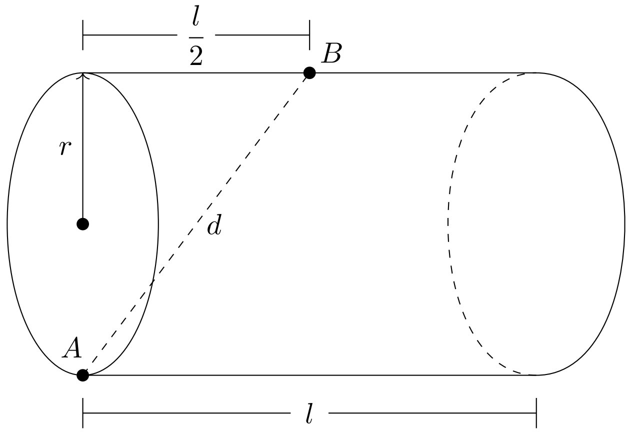
Essential Revision Problems – Set 1 – Calculus, Trigonometry, Plane Geometry

Question 1: 1992 Q4 c.



Consider a cylindrical barrel of length l and radius r. The point A is at one end of the barrel, at the very bottom of the rim. The point B is at the very top of the barrel, half-way along its length. The length of AB is d.

i) Show that the volume of the barrel is

$$V = \frac{\pi l}{4}\left(d^2 - \frac{l^2}{4}\right).$$

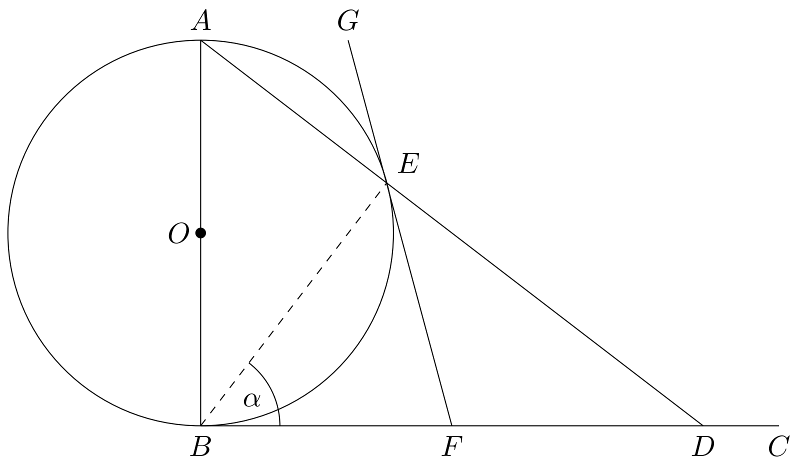
ii) Find l in terms of d if the barrel has a maximum volume for the given d.

Question 2: 1993 Q2 a.

Prove

$$\frac{\sin A}{\cos A + \sin A} + \frac{\sin A}{\cos A - \sin A} = \tan 2A.$$

Question 3: 1993 Q4 a.



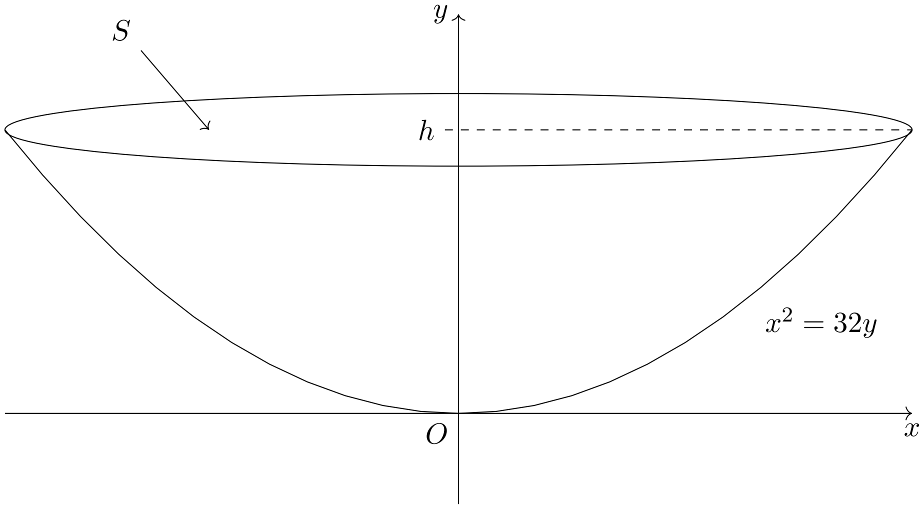
In the diagram, AB is a diameter of the circle, centre O, and BC is tangential to the circle at B. The line AED intersects the circle at E and BC at D. The tangent to the circle at E intersects BC at F. Let $\angle EBF = \alpha$.

i) Copy the diagram into your workbook.

ii) Prove that $\angle FED = 90 - \alpha$.

iii) Prove that $BF = FD$.

Question 4: 1994 Q5 c, i–ii.



The diagram represents the water in the dam on a farm. The curve is described by the equation $x^2 = 32y$. The depth of the water is h meters, the volume of water in the dam in cubic metres is given by $V = 16\pi h^2$ and the surface area of the water is S square metres. The water in the dam evaporates according to the rule

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -kS$$

where k is a positive constant and t is the time in hours.

i) Describe in words what the rule says about the rate of evaporation.

ii) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -k$.

Question 5*: 1994 Q7 c.

Uluru is a large rock on flat ground in Central Australia. Three tourists, A, B and C are observing Uluru from the ground. A is due North of Uluru, C is due East of Uluru and B is on the line-of-sight from A to C and between them[1]. The angles of elevation to the summit of Uluru from A, B and C are 26, 28°and 30°respectively.

Determine the bearing of B from Uluru.

[1] i.e. on the line from A to C.

Question 6: 1995 Q4 a–e.

Consider the real-valued function f such that

$$f(x) = \frac{e^x}{3 + e^x}.$$

Note that e^x is always positive, $f(x)$ is defined for all real x, and the following important facts:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x \qquad\qquad e^x = y \iff x = \log_e(y) = \ln(y).$$

a) Show that $f(x)$ has no stationary points. Hint: use the first important fact. (2)

b) Find the coordinates of the point of inflexion, given that

$$f''(x) = \frac{3e^x(3 - e^x)}{(3 + e^x)^3}.$$

Hint: use the second important fact. (1)

c) Show that $0 < f(x) < 1$ for all x. (2)

d) Describe the behaviour of $f(x)$ for very large positive and very large negative values of x, i.e. as $x \to \infty$ and $x \to -\infty$. (2)

e) Sketch the curve $y = f(x)$. (2)

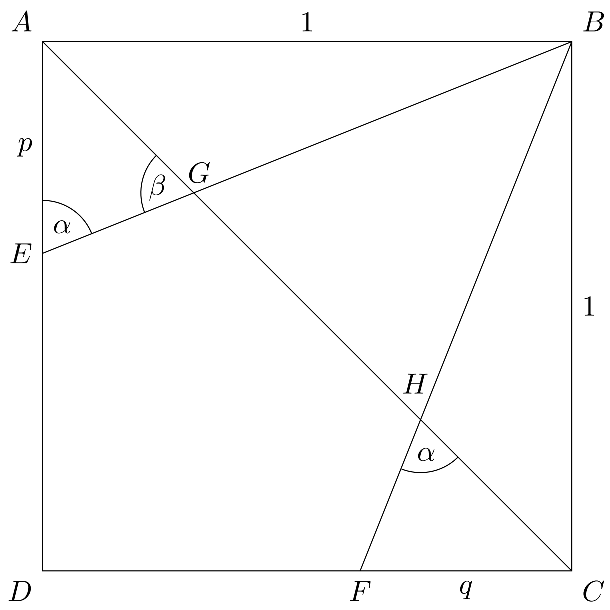
Question 7: 2000 Q3 a.

Use the definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

to find the derivative of x^3 where $x = a$. (2)

Question 8*: 2003 Q6 b.



In the diagram, $ABCD$ is a unit square. Points E and F are chosen on AD and DC respectively, such that $\angle AEG = \angle FHC$, where G and H are the points at which BE and BF respectively cut the diagonal AC.

Let $AE = p$, $FC = q$, $\angle AEG = \alpha$ and $\angle ABE = \beta$.

i) Express α in terms of p and β in terms of q. (2)

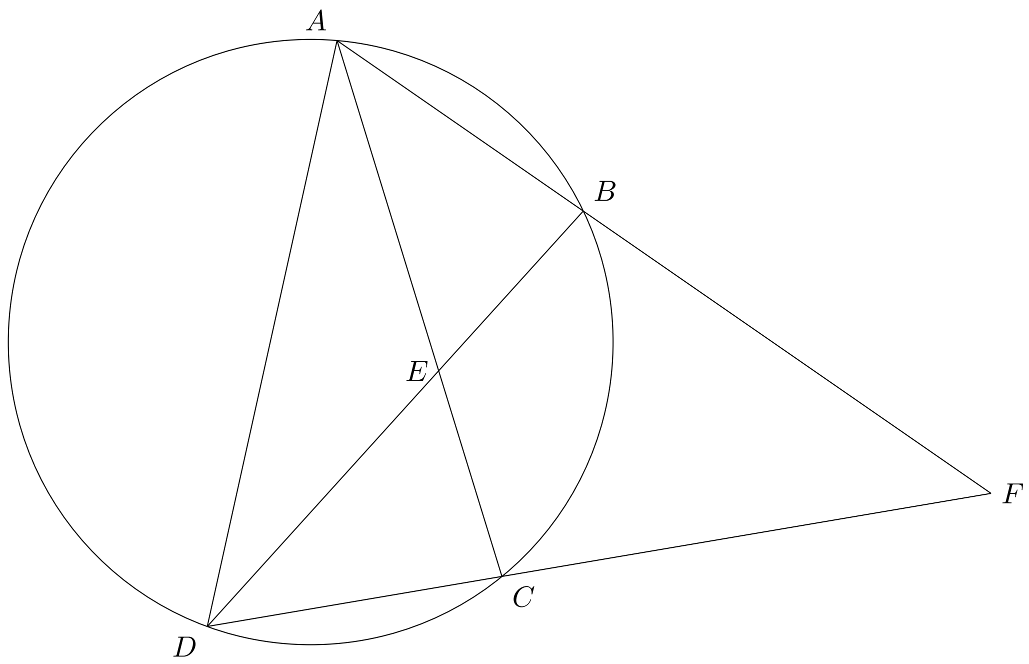
ii) Prove that $p + q = 1 - pq$. (2)

iii) Show that the area of the quadrilateral $EBFD$ is given by

$$1 - \frac{p}{2} + \frac{p-1}{2(p+1)}. \quad (1)$$

iv) What is the maximum value of the area of $EBFD$? (2)

Question 9: 2004 Q6 a.

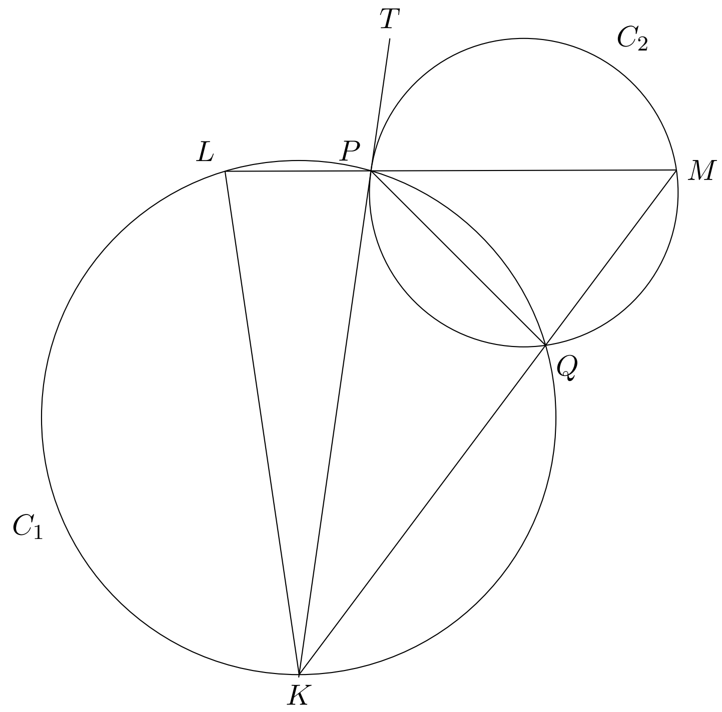


The points A, B, C and D are placed on a circle of radius r such that AC and BD meet at E. The lines AB and DC are produced to meet F and $BECF$ is a cyclic quadrilateral.

i) Find the size of $\angle DBF$, giving reasons for your answer. (2)

ii) Find an expression for the length of AD in terms of r. (1)

Question 10: 2008 Q5 c.



Two circles C_1 and C_2 intersect at P and Q as show in the diagram. The tangent TP to C_2 meets C_1 at K. The line KQ meets C_2 at M. The line MP meets C_1 at L.

Prove that $\triangle PKL$ is isosceles. (3)

Question 11: 2009 Q3 c.

i) Prove that

$$\tan^2 \theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

provided that $\cos 2\theta \neq -1$. (2)

ii) Hence, find the exact value of $\tan 22.5°$. (1)

Question 12: 2009 Q4 b.

Consider the function f given by $f(x) = \dfrac{x^4 + 3x^2}{x^4 + 3}$.

i) Show that $f(x)$ is an even function. (1)

ii) What is the equation of the horizontal asymptote to the graph $y = f(x)$? (1)

iii) Find the $x-$coordinates of all stationary points for the graph $y = f(x)$. (3)

iv) Sketch the graph $y = f(x)$. You are not required to find any points of inflexion. (2)