Now, recall from part ii we have equation (5). We note that the expression we are required to obtain involves only p, so we can use this previous result to eliminate q. In particular,

$$\begin{aligned} p + q &= 1 - pq \\ \iff q(1+p) &= 1 - p \\ \iff q &= \frac{1-p}{1+p}. \end{aligned} \tag{7}$$

Substituting equation (7) into equation (6), we have that

$$\begin{aligned} A &= 1 - \frac{p}{2} - \frac{1-p}{2(1+p)} \\ &= 1 - \frac{p}{2} + \frac{p-1}{2(p+1)} \end{aligned}$$

as required.

iv) The last part of the question mandates that we maximise the area found in the previous question. That is, we wish to maximise

$$A = 1 - \frac{p}{2} + \frac{p-1}{2(p+1)}. \tag{8}$$

We note initially

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}p} &= \frac{\mathrm{d}}{\mathrm{d}p}\left(1 - \frac{p}{2} + \frac{p-1}{2(p+1)}\right) \\ &= -\frac{1}{2} + \frac{p+1-(p-1)}{2(p+1)^2} \\ &= \frac{1}{(p+1)^2} - \frac{1}{2} \end{aligned}$$

through judicious application of the appropriate differentiation theorems. To determine the values of p for which extrema occur, we solve the equation $\frac{\mathrm{d}A}{\mathrm{d}p} = 0$. That is,

$$\begin{aligned} \frac{1}{(p+1)^2} - \frac{1}{2} &= 0 \\ (p+1)^2 &= 2 \\ p + 1 &= \sqrt{2} \\ p &= \sqrt{2} - 1 \end{aligned}$$

where the third line follows from the fact that p is a distance, so $p > 0$, thus $p + 1 > 0$.

In order to determine whether this is a local maximum or minimum, we check the value of the second derivative. We have that

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}p} &= (p+1)^{-2} - \frac{1}{2} \\ \implies \frac{\mathrm{d}^2A}{\mathrm{d}p^2} &= -\frac{2}{(p+1)^3} < 0 \text{ for all } p > 0. \end{aligned}$$

Since the second derivative is negative for all p, thus for $p = \sqrt{2} - 1$, we have a local maximum at $p = \sqrt{2} - 1$, as we expect.

Remark. Do not forget to determine the nature of the stationary point! The fact that the question tells you that you're supposed to find a maximum is not a sufficiently rigorous reason to state a point is a local maximum without further justification.