Essential Revision Problems – Set 2 – Locus and Parametrics, Combinatorics and Probability, and Polynomials

Question 1: 2001 Q6 b.

Consider the point $P(2at, at^2)$ on the parabola $x^2 = 4ay$.

i) Prove that the equation of the normal at P is $x + ty = 2at + at^3$.

ii) Find the coordinates of the point Q on the parabola such that the normal at Q is perpendicular to the normal at P.

iii) Show that the two normals of part ii) intersect at the point R with coordinates

$$x = a\left(t - \frac{1}{t}\right) \qquad y = a\left(t^2 + 1 + \frac{1}{t^2}\right).$$

iv) Find the Cartesian form of the equation of the locus of the point R.

Question 2: CBHS Lewisham 2014 Preliminary Final

Show that if $y = mx + b$ is a tangent to the curve $x^2 = 4ay$ then $am^2 + b = 0$.

(Yes, I still have my old preliminary exams lying around)

Question 3: 2014 Q13 c.

The point $P(2at, at^2)$ lies on the parabola $x^2 = 4ay$ with focus S. The point Q divides the interval PS internally in the ratio $t^2 : 1$.

i) Show that the coordinates of the point Q are given by

$$x = \frac{2at}{1 + t^2} \qquad y = \frac{2at^2}{1 + t^2}.$$

ii) Express the slope of OQ in terms of t.

iii) Using the result from part ii), or otherwise, show that Q lies on a fixed circle of radius a.

Question 4: Inspired by my morning commute

It is well known by frequent users of public transport that every carriage on a Sydney Trains train has a four digit serial number comprised of digits between 0 and 9 (inclusive!). The "Train Game" (or whatever students call it now) has the simple aim of combining the digits of the carriage serial number, using standard operations, to make ten[1]. Traditionally, it is stipulated that no number can be used more than once and all numbers must be used in the order in which they appear. For example, 1345: $(1 - 3 + 4) \times 5$.

I'm bad at basic arithmetic and my morning train ride isn't that long, so I prefer to rearrange the numbers into whatever order is convenient. Thus, suppose the order of the digits is NOT important (i.e. 1234 and 3421 are equivalent).

i) If serial numbers have no repetitions, how many different serial numbers are possible?

Serial numbers often DO have repeats, however. For example, a three digit serial number could comprise a triple (e.g. 111), a pair and a single (e.g. 112) or three singles (e.g. 123); there are three unique cases to consider.

ii) For a four digit serial number, how many unique cases are there? List them.

iii) How many different arrangements are possible, assuming order is unimportant (i.e. 1234 and 3421 are considered the same, as they both comprise 1, 2, 3 and 4) and repetitions are permitted?

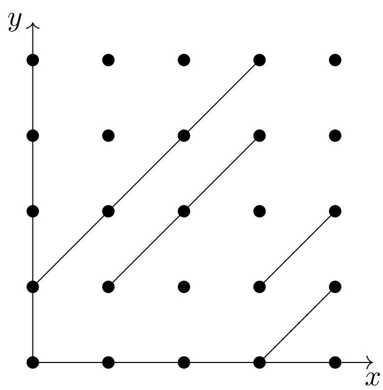
iv) Repeat parts i), ii) and iii), but for five digit serial numbers.

Suppose we insist that we use the numbers in order. That is, order is important (i.e. 1234 and 3421 are NOT equivalent).

v) Repeat parts i), ii), iii) and iv) assuming order is important.

Question 5: 2009 Q6 c.

Consider a square grid with n rows and n columns of equally spaced points.



The diagram illustrates such a grid. Several intervals of gradient 1, whose endpoints are a pair of points in the grid, are shown.

[1]Sometimes this is very easy; other times this is particularly difficult and sometimes it is impossible! An interesting problem to consider is this: what conditions must be imposed on a serial number in order to guarantee that there exists at least one way to make ten?

i) Explain why the number of such intervals on the line $y = x$ is equal to $\binom{n}{2}$.

ii) Explain why the total number, S_n, of such intervals in the grid is given by

$$S_n = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{n-1}{2} + \binom{n}{2} + \binom{n-1}{2} + \cdots + \binom{3}{2} + \binom{2}{2}.$$

iii) Given

$$\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r}$$

show that

$$S_n = \frac{n(n-1)(2n-1)}{6}.$$

Question 6: 2006 Q6 b.

In an endurance event, the probability that a competitor will complete the course is p and the probability that a competitor will not complete the course is $q = 1 - p$. Teams consist of either two or four competitors. A team scores points if at least half its members complete the course.

i) Show that the probability that a four-member team will have at least three of its members not complete the course is $4pq^3 + q^4$.

ii) Hence, or otherwise, find an expression in terms of q only for the probability that a four-member team will score points.

iii) Find an expression in terms of q only for the probability that a two-member team will score points.

iv) Hence, or otherwise, find the range of values of q for which a two-member team is more likely than a four-member team to score points.

Question 7: 1990 (4U) Q6 a.

i) Write down the relations which hold between the roots α, β and γ of the equation $ax^3 + bx^2 + cx + d = 0, a \neq 0$ and the coefficients a, b, c and d.

ii) Consider the equation $36x^3 - 12x^2 - 11x + 2 = 0$. You are given that the roots α, β and γ of this equation satisfy $\alpha = \beta + \gamma$. Use part i) to find α.

iii) Suppose that the equation $x^3 + px^2 + qx + r = 0$ has roots λ, μ and ν which satisfy $\lambda = \mu + \nu$. Show that $p^3 - 4pq + 8r = 0$.