CS 5800: Algorithms SEC 05 Summer Full 2024
Homework 7

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Question 1: Given the original graph on page 2, use the max flow algorithm to compute the max-flow network and min-cut. At each step, show the current flow network and the current residual network. Starting with the original residual network, identify a path to add to the flow network. Then show the updated edges and their weights on the flow graph and the updated residual graph. Show the final max flow graph, the final residual graph, and identify the min-cut(s).

Solution:

The below figure shows the flow graph for the following augmenting paths with the given bottleneck values:

1. $S \rightarrow A \rightarrow C \rightarrow T = min(10, 3, 5) = 3$
2. $S \rightarrow D \rightarrow F \rightarrow T = min(10, 3, 5) = 3$
3. $S \rightarrow A \rightarrow F \rightarrow T = min(7, 2, 5) = 2$
4. $S \rightarrow A \rightarrow T = min(5, 5) = 5$

Therefore, the maximum flow value = sum of bottleneck values.
$3 + 3 + 2 + 5 = 13$

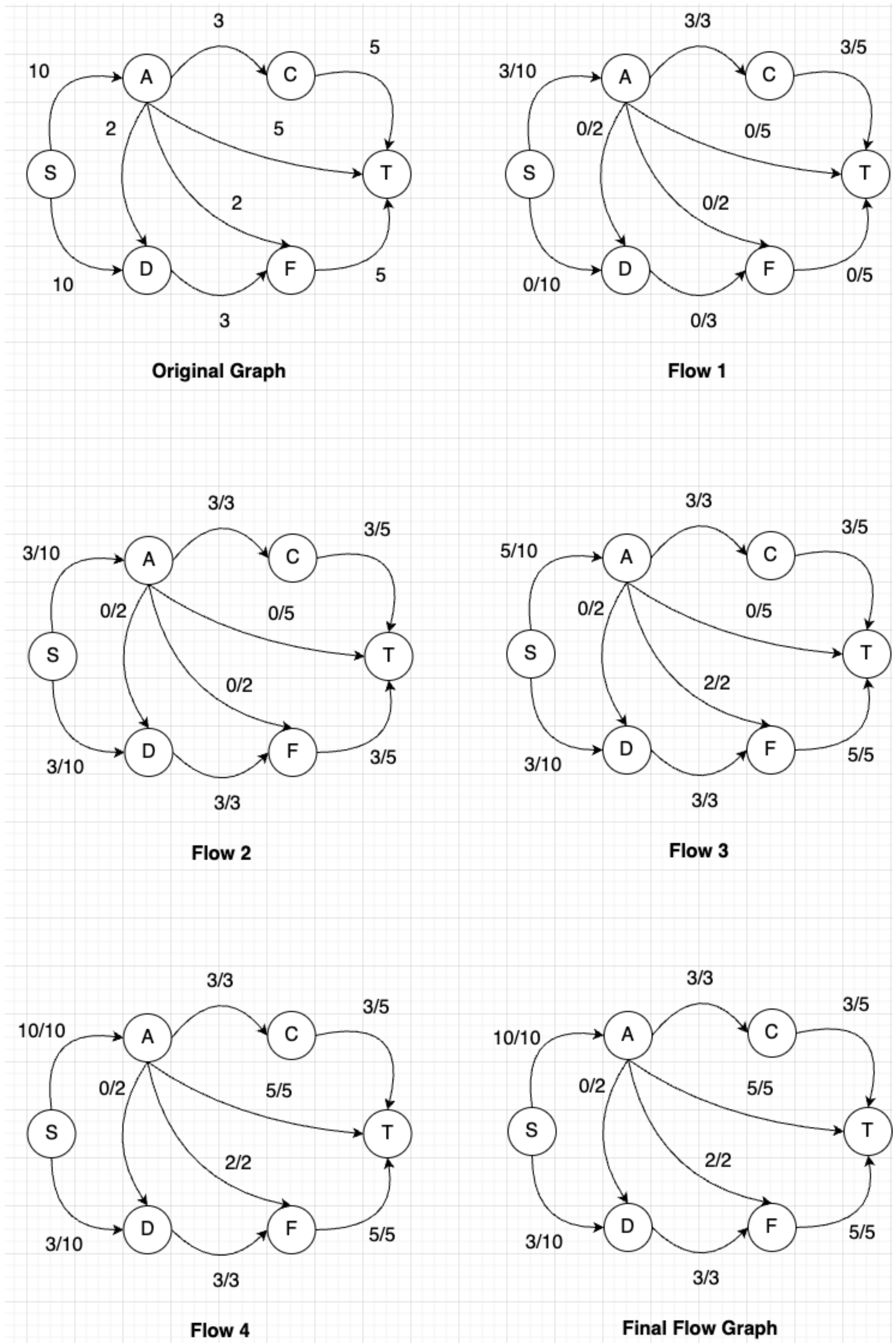

Figure 1: Flow Graph

Similarly, the residual graph figure is shown below that corresponds to an edge going backwards in G and undoes the flow that was pushed in the forward direction in the past.

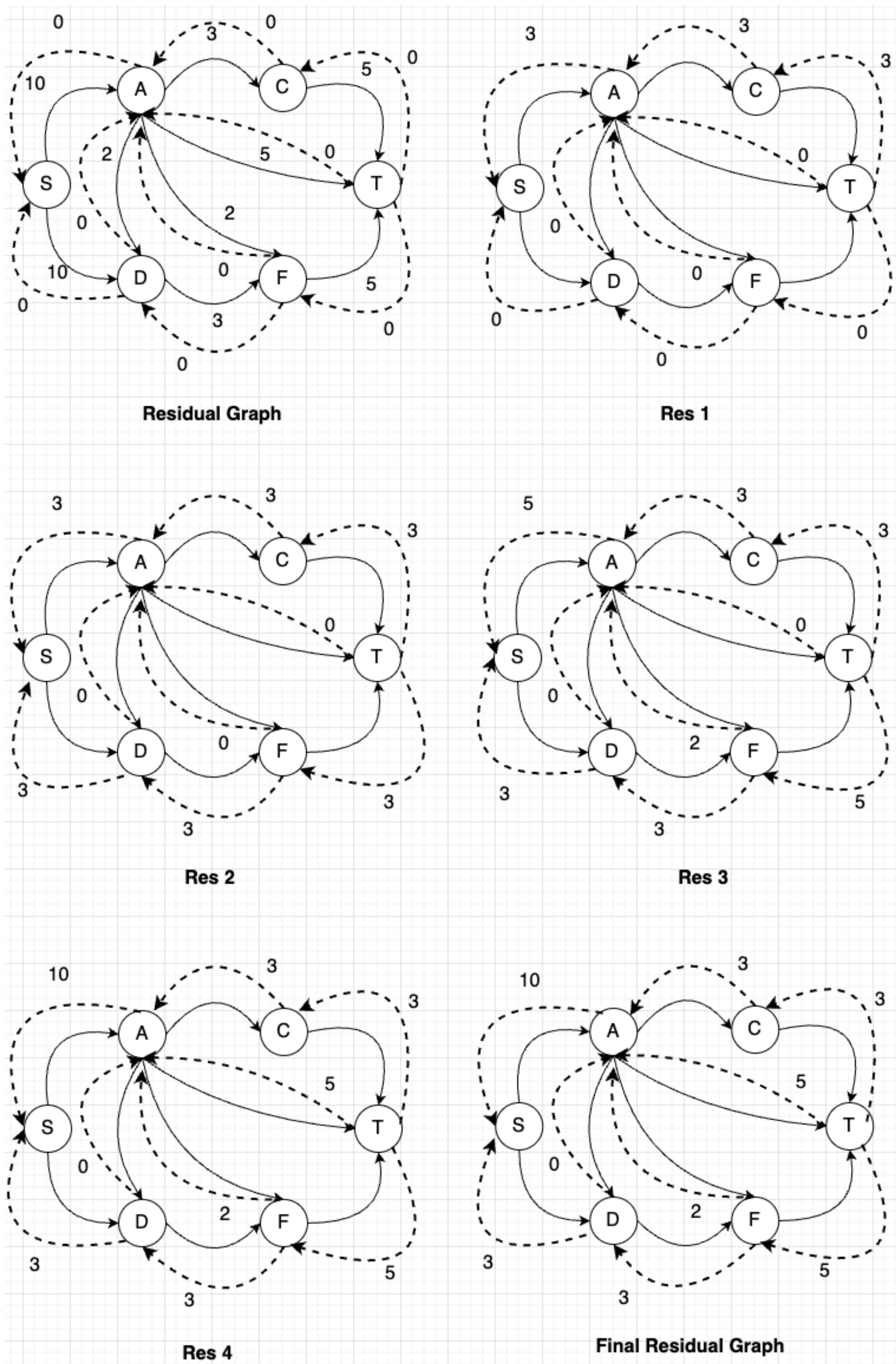

Figure 2: Residual Graph

In order to find the minimum cut, we need to identify a subset of vertices that divides the network into two distinct groups, namely the source group and the sink group. The minimum cut will be the minimum total capacity of all the edges connecting these two groups.

Upon running the algorithm, we obtain a maximum flow of 13 units, with a minimum cut on the edges (S, A), and (D, F). The vertices reachable from the source (S) in the residual graph form the minimum cut, which in this case gets divided into (S,D), and (A, C, F, T).

Question 2: How many constraints would be required to set up the flow problem from question 1 using a linear program? Explain the type of each constraint and how many of each type. You don't need to specify each one exactly.

Solution:

To set up the flow problem using linear programming, we need to define constraints based on the flow conservation and capacity limits.

1. **Flow Conservation Constraints:** For each vertex v (excluding the source S and sink T), the total incoming flow must be equal to the total outgoing flow. If f_{uv} represents the flow from vertex u to vertex v:
$$\sum(f_{vu}) = \sum(f_{uv}), \quad \forall v \neq S, T$$
2. **Capacity Constraints:** The flow on each edge must not exceed its capacity:
$$0 \leq f_{uv} \leq c_{uv}, \quad \forall (u, v) \text{ in the network}$$
3. **Source Constraint:** The total flow out of the source (S) must be equal to the maximum flow (F):
$$\sum(f_{Su}) = F$$
4. **Sink Constraint:** The total flow into the sink (T) must also be equal to the maximum flow (F):
$$\sum(f_{vT}) = F$$

Number of Constraints:

1. **Flow Conservation Constraints:** One for each vertex except S and T. If there are V vertices, we have $V - 2$ such constraints.
2. **Capacity Constraints:** One for each directed edge. If there are E directed edges, we have E such constraints.
3. **Source and Sink Constraints:** Two constraints, one for the source and one for the sink.

Thus, if V is the number of vertices and E is the number of edges, the total number of constraints is:

$$(V - 2) + E + 2$$

For the given graph, with 6 vertices (S, A, C, D, F, T) and 9 edges:
Flow Conservation $= 6 - 2 = 4$ constraints.
Capacity Constraints $= 9$ constraints.
Source and Sink = 2 constraints.

Total Constraints $= 4 + 9 + 2 = 15$

Question 3: Give an example of a linear program in two variables whose feasible region is infinite, but such that there is an optimum solution of bounded cost. Draw a diagram of the feasible region and explain the objective function for your linear program and why there is an optimum solution.

Solution:

Let's consider a linear programming problem with the following constraints and objective function:

Constraints:

1. $x + y \geq 1$
2. $x \geq 0$
3. $y \geq 0$

Objective Function:
Maximize $z = 3x + 2y$

1. **Constraints:** The first constraint $x + y \geq 1$ indicates that the sum of x and y should be at least 1.
   The second constraint $x \geq 0$ ensures that x is non-negative.
   The third constraint $y \geq 0$ ensures that y is non-negative.

2. **Feasible Region:**The feasible region corresponds to the area in the first quadrant of the Cartesian plane that satisfies all three constraints.
   It extends infinitely in the positive (x) and (y) directions because there are no upper bounds on (x) and (y).

3. **Objective Function:** We aim to maximize $z = 3x + 2y$.
   The objective function line can be visualized as moving upwards and to the right, increasing the value of z.

4. **Optimum Solution:** Although the feasible region is infinite, the optimal solution lies at the intersection of the constraints $x + y = 1$, $x \geq 0$, and $y \geq 0$. Increasing (x) or (y) further would violate the constraint $x + y \geq 1$.
   The intersection point occurs where $x + y = 1$. Let's find the points where this line intersects the axes:
   When $x = 0$, $y = 1$.
   When $x = 1$, $y = 0$.
   These points are $(1, 0)$ and $(0, 1)$.
   Since we are maximizing the function $z = 3x + 2y$, let's evaluate it at these points:
   At $(1, 0)$: $z = 3(1) + 2(0) = 3$.
   At $(0, 1)$: $z = 3(0) + 2(1) = 2$.
   The optimum solution is at $(1, 0)$ with the maximum value $z = 3$.

Next, we will plot the feasible region using the constraints provided.

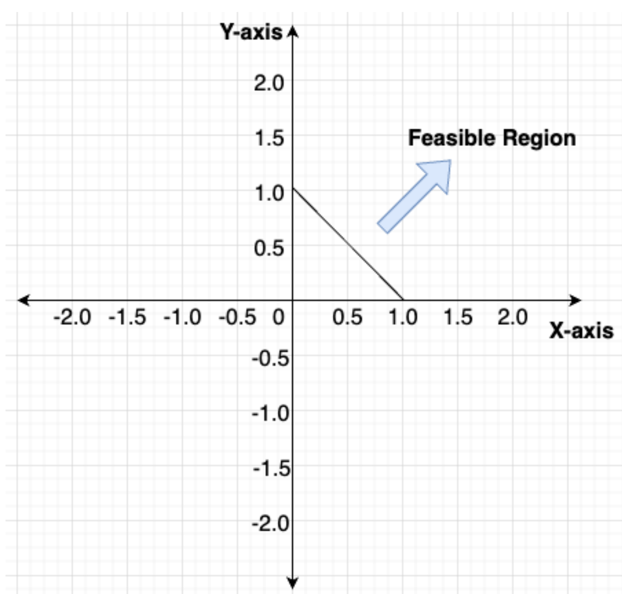

Figure 3: Plot showing feasible region

Now, we will see why there is an optimum solution.

1. The objective function $z = 3x + 2y$ increases as we move away from the origin.
2. However, due to the constraint $x + y \geq 1$, the feasible region is limited such that x and y cannot both be arbitrarily large in the feasible region.
3. The maximum value of the objective function within the feasible region occurs at the vertices of the region.
4. Evaluating the objective function at the vertices $(1, 0)$ and $(0, 1)$ shows that the maximum value is at $(1, 0)$ with $z = 3$.

Thus, even though the feasible region is infinite, the optimum solution of the linear program is bounded and occurs at the point $(1, 0)$.

Reflection:

In this exercise, I acquired valuable insights into several fundamental concepts. Firstly, I mastered the art of solving linear programming problems by carefully considering the provided constraints and objective function. Additionally, I delved into the intricacies of maximum flow graph problems, grasping the significance of residual graphs and the identification of min-cuts.

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4) https://web.mit.edu/15.053/www/AMP-Chapter-02.pdf: This document helped me with the understanding of the basics of Linear Programming.
5) https://www.scaler.com/topics/data-structures/maximum-flow-and-minimum-cut/: This website helped me in understanding the min-cut problem.