Assignment 3

1] given :- No. of Black Balls = 4
No. of Green Balls = 2
Total no. of Balls = 6
No. of balls drawn = 3

to find :- P.D (No. of Green Balls)
$m = 2, \quad n = 6.$

$$p = P(\text{Green}) = \frac{2}{6} = \frac{1}{3}$$

$$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

Using Bernoulli's formula,

formula $P_n(K) = C(n, K)\, p^K q^{n-K}$

where $n = 3, \quad K = 0, 1, 2, 3$

$$P(X = K) = (3 \text{ choose } K) * \left(\frac{1}{3}\right)^K * \left(\frac{2}{3}\right)^{3-K}$$

① $P(X = 0) = C(3, 0) * \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

② $P(X = 1) = C(3, 1) * \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

③ $P(X = 2) = C(3, 2) * \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1 = \frac{2}{9}$

④ $P(X = 3) = C(3, 3) * \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0 = \frac{1}{27}$

Therefore, the probability distribution is,

X	0	1	2	3
P(X)	8/27	4/9	2/9	1/27

2] $f(x) = \frac{1}{2000} \exp(-x/2000)$, $x \geq 0$

$f(x) = 0$, $x < 0$

Let, 'X' be the random variable which represents time to failure in hours.

a) By CDF.,

formula $F(x) = P(X \leq x)$

$= \int_{-\infty}^{x} f(t)\, dt$

$= \int_{0}^{x} \frac{1}{2000} e^{-1/2000}\, dt + \int_{-\infty}^{0} 0\, dt$

$= \left[-e^{t/2000}\right]_0^x + 0$

$= -e^{x/2000} + e^0$

$\boxed{F(x) = 1 - e^{-x/2000}}$, for all $x \geq 0$

$F(x) = 0$, for all $x < 0$

because f is a zero-function for $x < 0$

b) $P(X > 800) = 1 - P(X \leq 800)$

$= 1 - F(800)$

$= 1 - \left[1 - e^{-800/2000}\right]$

$= 1 - 1 + e^{-0.4}$

$\boxed{P(X > 800) = 0.671}$

c) $P(X<2000) = P(X \leq 2000)$

$= F(2000)$

$= 1 - e^{-2000/2000}$

$= 1 - \frac{1}{e}$

$$\boxed{P(X<2000) = 0.6321}$$

3] given :- Total weight of the box = 1 kg

weight of the creams = X

weight of the toffees = Y

$\therefore$ weight of the cordials = 1-X-Y

formula $P(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx$.

a) $P(1-X-Y > \frac{1}{2}) = P(X+Y < \frac{1}{2})$

$$= \int_0^{1/2}\int_0^{1/2-x} f(x,y)\, dy\, dx$$

$$= \int_0^{1/2}\int_0^{1/2-x} 24xy\, dy\, dx$$

$$= \int_0^{1/2} x\left[\int_0^{1/2-x} 24y\, dy\right] dx$$

$$= \int_0^{1/2} x\left[12y^2\right]_0^{1/2-x} dx$$

$$= \int_0^{1/2} x\left[12\left(\frac{1}{2}-x\right)^2\right] dx$$

$$= \int_0^{1/2} x\left[3x-12x^2+12x^3\right] dx$$

$$= \left[\frac{3x^2}{2} - \frac{12x^3}{3} + \frac{12x^4}{4}\right]_0^{1/2}$$

$$= \frac{3\left(\frac{1}{2}\right)^2}{2} - 4\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4$$

$$= \frac{3}{8} - \frac{1}{2} + \frac{3}{16}$$

$$\boxed{P(1-X-Y > \tfrac{1}{2}) = \frac{1}{16}}$$

b) Marginal Density for X

formula $\quad g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_0^{1-x} 24xy\,dy$$

$$= 24x\left[\frac{y^2}{2}\right]_0^{1-x}$$

$$= 24x\,\frac{(1-x)^2}{2}$$

$$\boxed{g(x) = 12x(1-x)^2}\ , \text{for } 0 \le x \le 1$$

c) We need to determine conditional distribution,

$$P\left(Y < \tfrac{1}{8} \,\middle|\, X = \tfrac{3}{4}\right) = P\left(0 < Y < \tfrac{1}{8} \,\middle|\, X = \tfrac{3}{4}\right)$$

$$= \int_0^{1/8} f\left(y \,\middle|\, \tfrac{3}{4}\right) dy$$

$$= \int_0^{1/8} \frac{f(3/4, y)}{g(3/4)}\,dy$$

$$= \int_0^{1/8} \frac{24 \cdot \frac{3}{4} y}{12 \cdot \frac{3}{4} \left(1-\frac{3}{4}\right)^2} \, dy$$

$$= \int_0^{1/8} \frac{18y}{9\left(\frac{1}{4}\right)^2} \, dy$$

$$= 32\left[\frac{y^2}{2}\right]_0^{1/8}$$

$$= \frac{32\left(\frac{1}{8}\right)^2}{2}$$

$$\boxed{P\left(Y<\frac{1}{8} \,\middle|\, X=\frac{3}{4}\right) = \frac{1}{4}}$$

4] Since X & Y are independent random variables,

formula $E(Z) = E(XY) = E(X)\,E(Y)$ —— (1)

$$E(X) = \int_x x\, g(x) = \int_2^{\infty} x\left(\frac{8}{x^3}\right) dx$$

$$= \lim_{t \to \infty} \int_2^t \frac{8}{x^2}\, dx$$

$$= \lim_{t \to \infty} \left(\frac{-8}{x}\right)_2^t$$

$$= \lim_{t \to \infty} \left(\frac{-8}{t} + 4\right)$$

$\therefore$ $E(X) = 4$

$$E(Y) = \int_y y\, g(y) = \int_0^1 y.(2y)\, dy$$

$$= \int_0^1 2y^2\, dy$$

$$= \frac{2}{3}\left[y^3\right]_0^1$$

$$\therefore E(Y) = \frac{2}{3}$$

Using eqⁿ ①, we have,

$$E(Z) = E(X)\,E(Y) = 4 \cdot \frac{2}{3}$$

$$\boxed{E(Z) = \frac{8}{3}}$$

5] Let X & Y be two random variables having joint pdf given by,

$$f(x,y) = \begin{cases} 2 & , 0 \leq x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

① Marginal pdf of X is given by,

formula $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

$$= \int_x^1 2\, dy$$

$$= 2(1-x)$$

② Marginal pdf of Y is given by,

formula $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_{0}^{y} 2\, dx$$

$$= 2y$$

③ Mean of X is given by,

formula $$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_{0}^{1} 2(1-x)\, dx$$

$$E(X) = \frac{1}{3}$$

formula $$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_{0}^{1} 2(1-x)\, x^2\, dx$$

$$E(X^2) = \frac{1}{6}$$

formula $$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{1}{6} - \left(\frac{1}{3}\right)^2$$

$$Var(X) = \frac{1}{18}$$

④ Mean of Y is given by,

formula $$E(Y) = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$= \int_0^1 2y^2 \, dy$$

$$E(Y) = \frac{2}{3}$$

formula $$E(Y^2) = \int_{-\infty}^{\infty} y^2 \, f_Y(y) \, dy$$

$$= \int_0^1 2y^3 \, dy$$

$$E(Y^2) = \frac{1}{2}$$

formula $$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$Var(Y) = \frac{1}{18}$$

formula $$Cov(X, Y) = E(X, Y) - E(X) E(Y)$$

$$E(X, Y) = \int_0^1 \int_0^y xy \, f(x, y) \, dx \, dy$$

$$= \int_0^1 \int_0^y 2xy \, dx \, dy$$

$$= \int_0^1 2y \left[\frac{x^2}{2}\right]_0^y dy$$

$$= \int_0^1 y^3 \, dy$$

$$= \left[\frac{y^4}{4}\right]_0^1$$

$$E(X,Y) = \frac{1}{4}$$

$$Cov(X,Y) = E(X,Y) - E(X)E(Y)$$

$$= \frac{1}{4} - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$$

$$= \frac{1}{36}$$

$\therefore$ The correlation coefficient is,

formula

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$= \frac{1/36}{\sqrt{1/18}\sqrt{1/18}}$$

$$= \frac{1/36}{1/18}$$

$$= \frac{18}{36}$$

$$\boxed{\rho = \frac{1}{2}}$$