I. Proof of Lemma 1

In this Section, we present the proof of Lemma 1. For every order-i conditional independence (CI) test, we have a privacy budget of ϵ_i. Given a CI test statistic $f(\mathcal{D})$ with l_1-sensitivity Δ, threshold T and margins (β_1, β_2), we perturb the test statistic by *Laplace noise* defined as $Z = Lap(\frac{\Delta}{\epsilon_i})$, and check for conditional independence between $(v_a, v_b) \in \mathcal{G}$ conditioned on S as:

(a) If $f(\mathcal{D}) + Z > T(1+\beta_2) \implies$ delete edge (v_a, v_b),
(b) If $f(\mathcal{D}) + Z < T(1-\beta_1) \implies$ keep edge (v_a, v_b),
(c) Else keep edge (v_a, v_b) with probability $\frac{1}{2}$.

For the simplicity of notations, we define $f_{v_a,v_b|S}(\mathcal{D}) \triangleq f(\mathcal{D})$.
Type-I Error: We now analyze the Type-I error relative to the unperturbed CI test, i.e., the private algorithm keeps the edge given that the unperturbed test statistic deletes the edge $f(\mathcal{D}) > T$. In other words, this can be written as follows:

$$\mathbb{P}(E_1^i) = \mathbb{P}(\text{Error}|f(\mathcal{D}) > T). \tag{1}$$

We next note that the error event occurs only for cases (b) and (c). We can bound the relative Type-I error as follows:

$$\begin{aligned}
&\mathbb{P}(E_1^i) = \mathbb{P}(\text{Error}|f(\mathcal{D}) > T) && (2)\\
&\le \frac{1}{2}\left(\mathbb{P}(f(\mathcal{D}) + Z \in [T(1-\beta_1), T(1+\beta_2)]|f(\mathcal{D}) > T)\right)\\
&\quad + \mathbb{P}(f(\mathcal{D}) + Z < T(1-\beta_1)|f(\mathcal{D}) > T) && (3)\\
&\le \frac{c_1}{2} + \mathbb{P}(f(\mathcal{D}) + Z < T(1-\beta_1)|f(\mathcal{D}) > T) && (4)\\
&\le \frac{c_1}{2} + \frac{1}{2}\exp\left(\frac{-T\beta_1\epsilon_i}{\Delta}\right). && (5)
\end{aligned}$$

where the last inequality follows from the Laplacian tail bound and using the fact that $f(\mathcal{D}) > T$; and we have defined c_1 as $c_1 \triangleq \mathbb{P}(f(\mathcal{D}) + Z \in [T(1-\beta_1), T(1+\beta_2)]|f(\mathcal{D}) > T)$.
Upper-bound on $\mathbb{P}[f(\mathcal{D}) + Z < T(1-\beta_1)|f(\mathcal{D}) > T]$ is obtained from *Laplace Tail bound* as:

$$\begin{aligned}
&\mathbb{P}[f(\mathcal{D}) + Z < T(1-\beta_1)|f(\mathcal{D}) > T]\\
&= \mathbb{P}[Z < T(1-\beta_1) - f(\mathcal{D})]\\
&= \frac{1}{2}\exp\left(\frac{T - T\beta_1 - f(\mathcal{D})}{\Delta/\epsilon_i}\right)\\
&\le \frac{1}{2}\exp\left(\frac{T - T\beta_1 - T}{\Delta/\epsilon_i}\right)\\
&= \frac{1}{2}\exp\left(\frac{-T\beta_1\epsilon_i}{\Delta}\right). && (6)
\end{aligned}$$

Type-II Error: Next we analyze the Type-II error relative to the unperturbed CI test, i.e., the differentially private algorithm deletes an edge given that the unperturbed CI test statistic keeps the edge, $f(\mathcal{D}) < T$. Mathematically,

$$\mathbb{P}[E_2^i] = \mathbb{P}(\text{Error}|f(\mathcal{D}) < T). \tag{7}$$

The type-II error occurs only for cases (a) and (c). Therefore, we can bound the Type-II error as:

$$\begin{aligned}
&\mathbb{P}(E_2^i) = \mathbb{P}(\text{Error}|f(\mathcal{D}) < T) && (8)\\
&\le \frac{1}{2}\left(P(f(\mathcal{D}) + Z \in [T(1-\beta_1), T(1+\beta_2)]|f(\mathcal{D}) < T)\right)\\
&\quad + \mathbb{P}(f(\mathcal{D}) + Z > T(1+\beta_2)|f(\mathcal{D}) < T) && (9)\\
&\le \frac{c_2}{2} + \mathbb{P}(f(\mathcal{D}) + Z > T(1+\beta_2)|f(\mathcal{D}) < T) && (10)\\
&\le \frac{c_2}{2} + \frac{1}{2}\exp\left(\frac{-T\beta_2\epsilon_i}{\Delta}\right). && (11)
\end{aligned}$$

where the last inequality follows from the Laplacian tail bound and using the fact that $f(\mathcal{D}) < T$; and we have defined c_2 as $c_2 \triangleq \mathbb{P}(f(\mathcal{D}) + Z \in [T(1-\beta_1), T(1+\beta_2)]|f(\mathcal{D}) < T)$. The probability $\mathbb{P}[f(\mathcal{D}) + Z > T(1+\beta_2)|f(\mathcal{D}) < T]$ can also be upper bounded as:

$$\begin{aligned}
&\mathbb{P}[f(\mathcal{D}) + Z > T(1+\beta_2)|f(\mathcal{D}) < T]\\
&\quad = \mathbb{P}[Z > T(1+\beta_2) - f(\mathcal{D})]\\
&\quad = \frac{1}{2}\exp\left(-\frac{T + T\beta_2 - f(\mathcal{D})}{\Delta/\epsilon_i}\right)\\
&\quad \le \frac{1}{2}\exp\left(-\frac{T + T\beta_2 - T}{\Delta/\epsilon_i}\right) && (12)\\
&\quad = \frac{1}{2}\exp\left(\frac{-T\beta_2\epsilon_i}{\Delta}\right). && (13)
\end{aligned}$$

II. Sensitivity Analysis of Weighted Kendall's τ:

Conditional independence (CI) tests in Causal Graph Discovery (CGD) measures the dependence of one variable (v_a) on other (v_b) conditioned on a set of variables. Let, the CI test statistic for connected variable pairs (v_a, v_b) in graph $\mathcal{G}$ is $\tau(\mathcal{D})$ for dataset $\mathcal{D}$ and $\tau(\mathcal{D}')$ for dataset $\mathcal{D}'$. For large samples, the test statistic $\tau(\cdot)$ follows a *Gaussian Distribution*. Therefore, the sensitivity can be bounded as:

$$\begin{aligned}
\Delta(\Phi(\tau(\mathcal{D}))) &= \sup_{\mathcal{D}\neq\mathcal{D}'} |\Phi(\tau(\mathcal{D})) - \Phi(\tau(\mathcal{D}'))|\\
&\le \Delta(\Phi(\cdot)) \cdot \Delta(\tau(\cdot))\\
&= \sup_{\mathcal{D}\neq\mathcal{D}'} \frac{|\Phi(\tau(\mathcal{D})) - \Phi(\tau(\mathcal{D}'))|}{|\tau(\mathcal{D}) - \tau(\mathcal{D}')|} \cdot |\tau(\mathcal{D}) - \tau(\mathcal{D}')|\\
&\le L_\Phi \cdot \sup |\tau(\mathcal{D}) - \tau(\mathcal{D}')|. && (14)
\end{aligned}$$

Here, $\sup_{\mathcal{D}\neq\mathcal{D}'} |\tau(\mathcal{D}) - \tau(\mathcal{D}')|$ is the l_1-sensitivity of the CI test statistic for dataset $\mathcal{D}$ and $\mathcal{D}'$, and Φ is the PDF of standard normal distribution. As, $\Phi(\cdot)$ is differentiable, therefore the Lipschitz constant (L_Φ) can be upper bounded as $\mathrm{L}_\Phi \le \frac{1}{\sqrt{2\pi}}$. Therefore, the sensitivity can easily be calculated with the sensitivity of the weighted test statistic.
l_1-sensitivity analysis: For large sample size ($n >> 1$), Kendall's τ test statistic follows Gaussian Distribution with zero mean and variance $\frac{2(2n+5)}{9n(n-1)}$ where n is the number of i.i.d. samples. Given a dataset $\mathcal{D}$ with d-features, the conditional dependence of between variables (v_a, v_b) conditioned on set S can be measured with Kendall's τ as a CI test statistic. For instance, the data is split according to the unique values of set S into k-bins. For each i^{th}-bin test statistic τ_i is calculated and

weighted average of all τ_i represents the test statistic for the entire dataset. The weighted average[8] is measures as follows:

$$\tau = \frac{\sum_{i=1}^{k} w_i \tau_i}{\sqrt{\sum_{i=1}^{k} w_i}}.$$

where w_i is the inverse of the variance $w_i = \frac{9n_i(n_i-1)}{2(2n_i+5)}$

For the scope of this paper, we consider the *Lipschitz Constant* of Gaussian distribution while calculating the sensitivity. The weighted average (τ) essentially follows the standard normal distribution, i.e., $\tau \sim \mathcal{N}(0,1)$. Hence, the l_1-sensitivity can be defined as:

$$\begin{aligned}\Delta &= |\Phi(\tau(\mathcal{D}) - \Phi(\tau(\mathcal{D}')| \\ &= \frac{|\Phi(\tau(\mathcal{D}) - \Phi(\tau(\mathcal{D}')|}{|\tau(\mathcal{D}) - \tau(\mathcal{D}')|} \cdot |\tau(\mathcal{D}) - \tau(\mathcal{D}')| \\ &\leq \mathrm{L}_{\Phi}|\tau(\mathcal{D}) - \tau(\mathcal{D}')| \leq \frac{1}{\sqrt{2\pi}}|\tau(\mathcal{D}) - \tau(\mathcal{D}')|. \end{aligned} \tag{15}$$

The sensitivity of the weighted Kendall's τ can be also upper-bounded as:

$$\begin{aligned}\Delta(\tau) &= \max_{|\mathcal{D}'-\mathcal{D}|\leq 1} |\tau(\mathcal{D}') - \tau(\mathcal{D})| \\ &\leq \Delta(\tau_i)\Delta(w_i)\end{aligned}$$

The sensitivity of τ_i depends upon the number of elements n_i and $\Delta(\tau_i) \leq \frac{2}{n_i-1}$[8]. The sensitivity of weights $\Delta(w_i)$ can be represented as follows:

$$\begin{aligned}\Delta(w_i) &\leq \left| \frac{w_i'}{\sqrt{\sum_{i\neq j}^{k} w_j + w_i'}} - \frac{w_i}{\sqrt{\sum_{j=1}^{k} w_j}} \right| \\ &\leq \left| \frac{\frac{9n_i(n_i+1)}{2(2(n_i+1)+5)}}{\sqrt{\sum_{j=1}^{k} w_j + w_i'}} \right| - \left| \frac{\frac{9n_i(n_i-1)}{2(2n_i+5)}}{\sqrt{\sum_{j=1}^{k} w_j}} \right|. \end{aligned} \tag{16}$$

Through triangle inequality we can provide an upper-bound on Equation (II) and the sensitivity of the weight can be bounded as:

$$\Delta(w_i) \leq \sqrt{\frac{2}{n}} \left(\left| \frac{9n_i(n_i+1)}{2(2(n_i+1)+5)} \right| - \left| \frac{9n_i^2}{2(2n_i+5)} \right| \right). \tag{17}$$

The sensitivity $\Delta(\tau)$ essentially depends upon the number of elements in the i^{th} bin (the bin that changed due to the addition or removal of a single user). For dataset with block size at least size c and $kc \approx n$, with Equation (II) and Equation (II), the overall sensitivity can be bounded as:

$$\begin{aligned}\Delta &\leq \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{n_i-1} \cdot \sqrt{\frac{2}{n}} \cdot \\ &\left(\left| \frac{9n_i(n_i+1)}{2(2(n_i+1)+5)} \right| - \left| \frac{9n_i^2}{2(2n_i+5)} \right| \right) \\ &= \frac{2}{\sqrt{n\pi}} \left(\frac{\left| \frac{9n_i(n_i+1)}{2(2(n_i+1)+5)} \right| - \left| \frac{9n_i^2}{2(2n_i+5)} \right|}{n_i-1} \right)\end{aligned}$$

$$\Delta \leq \frac{2}{\sqrt{n\pi}} \left(\frac{\left| \frac{9n_i(n_i+1)}{2(2(n_i+1)+5)} \right| - \left| \frac{9n_i^2}{2(2n_i+5)} \right|}{n_i-1} \right).$$

Therefor, the l_1 sensitivity Δ can be upper bounded as $\Delta \leq \frac{C}{\sqrt{n}}$ where the constant C can be derived by setting the permissible bin size (n_i). This concludes the sensitivity analysis of Kendall's τ test statistic.